Research Article

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Estimates for eigenvalues of the Neumann and Steklov problems

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Abstract: We prove Li-Yau-Kröger-type bounds for Neumann-type eigenvalues of the biharmonic operator on bounded domains in a Euclidean space. We also prove sharp estimates for lower order eigenvalues of a biharmonic Steklov problem and of the Laplacian, which directly implies two sharp Reilly-type inequalities for the corresponding first nonzero eigenvalue.

Keywords: Neumann eigenvalue problem, Steklov eigenvalue problem, biharmonic operator, eigenvalues, Fourier transform

MSC 2020: 35P15, 53C40, 58C40

1 Introduction

Throughout this article, let Ω be a bounded domain with smooth boundary $\partial\Omega$ in the Euclidean n-space $\mathbb{R}^n$. Consider the Neumann eigenvalue problem of the Laplacian Δ as follows:

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative on the boundary $\partial\Omega$ w.r.t. the outward unit normal vector ν. The system (1.1) can be used to describe the vibration of membrane and is also called the *free membrane problem.* It is well known that this problem has discrete spectrum $\{\mu_i\}_{i=1}^{\infty}$ diverging to infinity and satisfying

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots \uparrow +\infty.$$

In [1], Ashbaugh and Benguria conjectured that

$$\sum_{i=1}^{n} \frac{1}{\mu_{i+1}(\Omega)} \geq \frac{n}{\mu_2(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball}, \tag{1.2}$$

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where B_Ω is the ball of same volume as Ω, $\mu_i(\Omega)$ is the ith Neumann eigenvalue on Ω, and $\mu_2(B_\Omega)$ is the first nonzero Neumann eigenvalue on B_Ω. In [21], Wang and Xia proved that

$$\sum_{i=1}^{n-1}\frac{1}{\mu_{i+1}(\Omega)} \geq \frac{n-1}{\mu_2(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball}, \tag{1.3}$$

which supports the above conjecture of Ashbaugh and Benguria.

On the other hand, corresponding to the Li-Yau's classical result for Dirichlet eigenvalues of the Laplacian [15], Kröger [14] obtained the following inequality for the sum of the Neumann eigenvalues:

$$\sum_{j=1}^{k}\mu_j(\Omega) \leq (2\pi)^2\frac{n}{n+2}k^{\frac{n+2}{n}}\left(\frac{1}{\omega_n|\Omega|}\right)^{\frac{2}{n}}, \quad k \geq 1, \tag{1.4}$$

and the upper bound estimate for the $(k+1)$th Neumann eigenvalue

$$\mu_{k+1}(\Omega) \leq (2\pi)^2\left(\frac{n+2}{2\omega_n|\Omega|}\right)^{\frac{2}{n}}k^{\frac{2}{n}}, \quad k \geq 0, \tag{1.5}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega|$ represents the volume of Ω.

Consider a Neumann-type eigenvalue problem of the biharmonic operator Δ^2 as follows:

$$\begin{cases} \Delta^2 u - \tau\Delta u = \Lambda u & \text{in } \Omega, \\ (1-\sigma)\dfrac{\partial^2 u}{\partial \nu^2} + \sigma\Delta u = 0 & \text{on } \partial\Omega, \\ \tau\dfrac{\partial u}{\partial \nu} - (1-\sigma)\,\mathrm{div}_{\partial\Omega}(D^2u\cdot\nu)_{\partial\Omega} - \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\tau \geq 0$ and $\sigma \in (-1/(n-1), 1)$ are two constants, and $\mathrm{div}_{\partial\Omega}$ denotes the tangential divergence operator on $\partial\Omega$, D^2u is the Hessian matrix of u and $(D^2u\cdot\nu)_{\partial\Omega}$ stands for the projection of $D^2u\cdot\nu$ to the tangent bundle of $\partial\Omega$. In this setting, problem (1.6) has discrete spectrum and all the eigenvalues in the spectrum can be listed non-decreasingly as follows (e.g., [8, Proposition 4.1]):

$$0 = \Lambda_1(\Omega) \leq \Lambda_2(\Omega) \leq \Lambda_3(\Omega) \leq \cdots \leq \cdots \uparrow +\infty.$$

This problem is called the *eigenvalue problem of free plate under tension and with nonzero Poisson's ratio*, which for $n = 2$ can be used to describe the deformation of a planar material under compression, τ, σ denote a parameter related to the tension and a Poisson's ratio of the material, respectively. By the Rayleigh-Ritz characterization, the Neumann-type eigenvalues (if exist and with the abuse of terminology) of (1.6) are given by (e.g., [8,16] while [2,7] for the case $\sigma = 0$)

$$\Lambda_k(\Omega) = \inf_{0\neq u\in H^2(\Omega)}\left\{\frac{\int_\Omega[(1-\sigma)|D^2u|^2 + \sigma(\Delta u)^2 + \tau\,|\nabla u|^2]}{\int_\Omega u^2}\,\middle|\,\int_\Omega uu_j = 0,\ j = 1, \ldots, k-1\right\}, \tag{1.7}$$

where ∇ is the gradient operator, u_j is an eigenfunction corresponding to the eigenvalue $\Lambda_j(\Omega)$, and $|D^2u|^2 = \sum_{i,j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2$. For convenience, without specification, in the sequel we will drop the integral measure for all integrals.

Remark 1.1.

(1) In [7,8,16], the authors therein used the operator $\mathrm{Proj}_{\partial\Omega}[(D^2u)\nu]$ to denote the projection of $(D^2u)\nu$ onto the space tangent to $\partial\Omega$, which obviously has the same meaning as $(D^2u\cdot\nu)_{\partial\Omega}$ here.

(2) As before, let $B_\Omega \subset \mathbb{R}^n$ be the ball of same volume as Ω. When $\tau > 0$, $\sigma = 0$, Chasman [7] proved the following isoperimetric inequality:

$$\Lambda_1(\Omega) \leq \Lambda_1(B_\Omega), \quad \text{with equality if and only if } \Omega \text{ is a ball}.$$

When $\tau > 0$, $\sigma \in (-1/(n-1), 1)$, Chasman [8] conjectured that the above isoperimetric inequality is still true and successfully proved a weaker version of it. Moreover, she also gave numerical and analytic evidence to support this conjecture – see [8, Section 8] for details.

When $\tau \geq 0$, $0 \leq \sigma < 1$, for the eigenvalue problem (1.6), we can obtain the following:

Theorem 1.2. *Let* Ω, $|\Omega|$ *and* ω_n *be defined as before, and let* $\Lambda_j(\Omega)$ *be the jth eigenvalue of system* (1.6).

(*i*) *When* $\tau \geq 0$ *and* $0 \leq \sigma < 1$, *we have*

$$\sum_{j=1}^{k} \Lambda_j(\Omega) \leq (2\pi)^4 \frac{n}{(n+4)} k^{\frac{n+4}{n}} \left(\frac{1}{\omega_n |\Omega|} \right)^{\frac{4}{n}} + \tau (2\pi)^2 \frac{n}{(n+2)} k^{\frac{n+2}{n}} \left(\frac{1}{\omega_n |\Omega|} \right)^{\frac{2}{n}}, \quad k \geq 1; \tag{1.8}$$

(*ii*) *When* $\tau = 0$ *and* $0 \leq \sigma < 1$, *it holds*

$$\Lambda_{k+1}(\Omega) \leq (2\pi)^4 \left(\frac{n+4}{4\omega_n |\Omega|} \right)^{\frac{4}{n}} k^{\frac{4}{n}}, \quad k \geq 0; \tag{1.9}$$

(*iii*) *When* $\tau > 0$ *and* $0 \leq \sigma < 1$, *we have*

$$\Lambda_{k+1}(\Omega) \leq \min_{r > 2\pi \left(\frac{k}{\omega_n |\Omega|} \right)^{\frac{1}{n}}} \frac{n \omega_n |\Omega| \left(\frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right)}{\omega_n |\Omega| r^n - k (2\pi)^n}, \quad k \geq 0. \tag{1.10}$$

Remark 1.3.

(1) Recently, when $\tau \geq 0$, $\sigma = 0$, Brandolini et al. [2] have already obtained upper bounds for the sum of the first k eigenvalues $\Lambda_i(\Omega)$ and for the $(k+1)$th eigenvalue $\Lambda_{k+1}(\Omega)$. Inspired by this fact and our Theorem 1.2 here, together with the coercivity argument for the sesquilinear form shown in [8, Section 4], the corresponding author, Prof. J. Mao, and his another collaborator can also obtain the estimates (1.8)–(1.10) under a more general setting that $\tau \geq 0$, $\sigma \in (-1/(n-1), 1)$ – see [16, Theorem 1.1 and Corollary 1.2] for details. Although [16] has been published formally recently, we still prefer to remain Theorem 1.2 to emphasize and embody the origin and continuity of our thought.

(2) One might find that Theorem 1.2 can be seen as a generalization of those related eigenvalue estimates shown in [5, 19].

(3) Clearly, if $\tau = 0$ and $\sigma = 1$, then (1.6) degenerates into

$$\begin{cases} \Delta^2 u = \Lambda u & \text{in } \Omega, \\ \Delta u = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

At the end of [8, Section 4], Chasman showed that for the eigenvalue problem (1.11), all $H^2(\Omega)$ harmonic functions are eigenfunctions with eigenvalue zero, and one has at least an eigenvalue of infinite multiplicity. Based on this fact, we need to expel $\tau = 0$, $\sigma = 1$ in Theorem 1.2 here.

We also consider the following Steklov-type eigenvalue problem of the biharmonic operator

$$\begin{cases} D^2 u - \tau \Delta u = 0 & \text{in } \Omega, \\ (1-\sigma) \dfrac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0 & \text{on } \partial\Omega, \\ \tau \dfrac{\partial u}{\partial \nu} - (1-\sigma)\, \mathrm{div}_{\partial\Omega}(D^2 u \cdot \nu)_{\partial\Omega} - \dfrac{\partial \Delta u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $\tau, \sigma \in \mathbb{R}$ and other same symbols have the same meanings as those in (1.6).

Remark 1.4.

(1) Li and Mao [17, Theorem 2.1] showed clearly that if $\tau > 0$ and $\sigma \in (-1/(n-1), 1)$, the eigenvalue problem (1.12) has the discrete spectrum and its elements (i.e., eigenvalues) can be listed non-decreasingly as follows:

$$0 = \lambda_1(\Omega) < \lambda_2(\Omega) \le \lambda_3(\Omega) \le \cdots \le \lambda_k(\Omega) \le \cdots \uparrow \infty.$$

By means of variational principle, the Rayleigh-Ritz-type characterization of the kth eigenvalue $\lambda_k(\Omega)$ is given by

$$\lambda_k(\Omega) = \inf_{0 \neq u \in H^2(\Omega)} \left\{ \frac{\int_\Omega [(1-\sigma)|D^2u|^2 + \sigma(\Delta u)^2 + \tau\, |\nabla u|^2]}{\int_{\partial\Omega} u^2} \,\middle|\, \int_{\partial\Omega} u u_j = 0, j = 1, \ldots, k-1 \right\}, \tag{1.13}$$

where u_j is an eigenfunction corresponding to the eigenvalue $\lambda_j(\Omega)$. Besides, the eigenfunction u_1 of $\lambda_1(\Omega) = 0$ should be nonzero constant function.

(2) When $\tau > 0, \sigma = 0$, Buoso and Provenzano [6] proved an isoperimetric inequality for the fundamental tone $\lambda_2(\Omega)$ of system (1.12) which states that

$$\lambda_2(\Omega) \le \lambda_2(B_\Omega),$$

with equality if and only if Ω is a ball. Here, as before, $B_\Omega \subset \mathbb{R}^n$ is the ball of same volume as Ω. Recently, Li and Mao [17, Theorem 1.1] showed that the above isoperimetric inequality is still true for $\tau > 0$ and $\sigma \in (-1/(n-1), 1)$, and moreover, the inequality can be achieved when Ω is the ball B_Ω.

(3) For some other estimates for λ_i's, see, e.g., [3,4,6,12,22].

Our next result is a sharp lower bound for the sum of the reciprocals of the first n nonzero eigenvalues of problem (1.12).

Theorem 1.5. *Let Ω and $|\Omega|$ be defined as before, and let $\lambda_j(\Omega)$ be the jth eigenvalue of system* (1.12). *When $\tau > 0$ and $\sigma \in (-1/(n-1), 1)$, we have*

$$\sum_{j=1}^{n} \frac{1}{\lambda_{j+1}(\Omega)} \ge \frac{|\partial\Omega|^2}{\tau|\Omega| \int_{\partial\Omega} |\mathbf{H}|^2}, \tag{1.14}$$

where $\mathbf{H}$ is the mean curvature vector of $\partial\Omega$ in $\mathbb{R}^n$, $|\partial\Omega|$ denotes the area of $\partial\Omega$. Equality in (1.14) *holds if and only if Ω is a ball.*

Using the monotonicity of eigenvalues λ_i's and Theorem 1.5 immediately, we obtain

$$\frac{n}{\lambda_2(\Omega)} \ge \sum_{j=1}^{n} \frac{1}{\lambda_{j+1}(\Omega)} \ge \frac{|\partial\Omega|^2}{\tau|\Omega| \int_{\partial\Omega} |\mathbf{H}|^2},$$

which directly implies the following Reilly-type eigenvalue estimate.

Corollary 1.6. *Under the assumptions in Theorem* 1.5, *we have*

$$\lambda_2(\Omega) \le n\tau \frac{|\Omega|}{|\partial\Omega|^2} \int_{\partial\Omega} |\mathbf{H}|^2,$$

with equality holding if and only if Ω is a ball.

Remark 1.7. Clearly, when the Reilly-type eigenvalue estimate in Corollary 1.6 attains the equality case, one also has $\lambda_2(\Omega) = \lambda_3(\Omega) = \cdots = \lambda_{n+1}(\Omega)$.

Let M be an n-dimensional compact submanifold without the boundary, and the so-called closed eigenvalue problem of the Laplacian Δ on M is actually to find all possible real numbers such that

$$\Delta u + \eta u = 0 \quad \text{in } M$$

has non-trivial solutions. It is well-known that in this setting, Δ only has discrete spectrum and all the elements (i.e., eigenvalues) in this discrete spectrum can be listed non-decreasingly as follows:

$$0 = \eta_1(M) < \eta_2(M) \leq \eta_3(M) \leq \cdots \leq \eta_k(M) \leq \cdots \uparrow \infty.$$

The eigenspace of $\eta_i(M)$, which consists of all the eigenfunctions of $\eta_i(M)$, has finite dimension, and moreover, each $\eta_i(M)$ in the above sequence should repeat according to its multiplicity (i.e., the dimension of its eigenspace). It is easy to know that the eigenfunctions of the first trivial eigenvalue $\eta_1(M) = 0$ are nonzero constant functions. By using the variational principle (i.e., essentially, Rayleigh's theorem and Max-min theorem – see, e.g., [9, Chapter I]), one knows that the kth closed eigenvalue η_k can be characterized as follows:

$$\eta_k(M) = \inf_{0\neq u\in H^2(M)} \left\{ \frac{\int_M |\nabla u|^2}{\int_M u^2} \,\middle|\, \int_M u u_j = 0,\ j = 1, \ldots, k-1 \right\},$$

where u_j is an eigenfunction corresponding to the eigenvalue $\eta_j(M)$, and as usual, by the abuse of notations, ∇ is the gradient operator.

Our final result is a sharp lower bound for the sum of the reciprocals of the first n nonzero eigenvalues of the Laplacian on a closed submanifold immersed in a Euclidean space. Namely, we have:

Theorem 1.8. *Let M be an n-dimensional compact submanifold without the boundary isometrically immersed in $\mathbb{R}^N$ and let $\eta_j(M)$ be the jth closed eigenvalue of the Laplacian on M. We have*

$$\sum_{j=1}^{n} \frac{1}{\eta_{j+1}(M)} \geq \frac{|M|}{\int_M |\overline{\mathbf{H}}|^2}, \tag{1.15}$$

where $\overline{\mathbf{H}}$ is the mean curvature vector of M in $\mathbb{R}^N$. Moreover, when $n = N-1$, equality holds in (1.15) *if and only if M is a hypersphere of $\mathbb{R}^N$, and when $n < N-1$, if the equality holds in* (1.15)*, then M is a minimal submanifold of some hypersphere of $\mathbb{R}^N$.*

Using the monotonicity of nonzero closed eigenvalues η_i's of the Laplacian and Theorem 1.8 immediately, we obtain

$$\frac{n}{\eta_2(M)} \geq \sum_{j=1}^{n} \frac{1}{\eta_{j+1}} \geq \frac{|M|}{\int_M |\overline{\mathbf{H}}|^2},$$

which directly implies the following Reilly's eigenvalue estimate (i.e., the main result of the influential article [20]).

Corollary 1.9. *Under the assumptions in Theorem* 1.8, *we have*

$$\eta_2(M) \leq \frac{n}{|M|} \int_M |\overline{\mathbf{H}}|^2,$$

and moreover, the equality holds implying the rigidity described as in Theorem 1.8.

Remark 1.10.

(1) Clearly, when the Reilly-type eigenvalue estimate in Corollary 1.9 attains the equality case, one also has $\eta_2(M) = \eta_3(M) = \cdots = \eta_{n+1}(M)$, and if furthermore $n = N-1$, then $\eta_{n+3}(M) > \eta_{n+2}(M) = \eta_i(M)$ for $i = 2, 3, \ldots, n+1$, since the multiplicity of the first nonzero closed eigenvalue of the Laplacian on any n-sphere in $\mathbb{R}^{n+1}$ is $n+1$ and the corresponding eigenfunctions are the restrictions (to n-sphere) of $n+1$ coordinate functions of $\mathbb{R}^{n+1}$ (e.g., [9, Chapter 2] for this fact).

(2) Except Reilly's estimate for the first nonzero eigenvalue of the Laplacian (see [20] or Corollary 1.9 here) and our Reilly-type estimate for the first nonzero eigenvalue of (1.12) – the Steklov-type eigenvalue problem of

the biharmonic operator (Corollary 1.6), some interesting Reilly-type estimates for the first nonzero eigenvalue of other type have also been obtained. For instance, Ilias and Makhoul [13] have obtained the Reilly-type estimate for the first nonzero Steklov eigenvalue of the Laplacian on compact submanifolds (with boundary) isometrically immersed in a Euclidean space; Du and Mao [11] have obtained the Reilly-type estimate for the first nonzero closed eigenvalue of the nonlinear p-Laplacian ($1 < p < +\infty$) on compact submanifolds (without boundary) isometrically immersed into a Euclidean space, a unit sphere, or even a projective space.

For convenience, in the sequel, we prefer to simplify the notations for four types of eigenvalues discussed in this article, that is, we separately write $\Lambda_i(\Omega)$, $\lambda_i(\Omega)$, and $\eta_i(M)$ as Λ_i, λ_i and η_i. We also make an agreement that these notations would be written completely if necessary.

This article is organized as follows. In Section 2, we will prove Li-Yau-Kröger-type estimates for lower-order eigenvalues of the Neumann-type eigenvalue problem (1.6) of the biharmonic operator. Two sharp extrinsic lower bounds for the sum of the reciprocals of the first n nonzero eigenvalues of the Steklov-type eigenvalue problem (1.12) and for the sum of the reciprocals of the first n nonzero closed eigenvalues of the Laplacian will be separately proven in Section 3.

2 Li-Yau-Kröger-type estimates

In this section, inspired by [2,14,15], and using the method of Fourier transformation, together with the Rayleigh-Ritz type characterization (1.7), we can give the proof of Li-Yau-Kröger-type estimates (for the biharmonic operator) by appropriately constructing trial functions.

We have:

Proof of Theorem 1.2. Let $\{\psi_j\}_{j=1}^{\infty}$ be the set of orthonormal eigenfunctions of system (1.6), that is,

$$\begin{cases} \Delta^2\psi_j - \tau\Delta\psi_j = \Lambda_j\psi_j & \text{in } \Omega, \\ (1-\sigma)\dfrac{\partial^2\psi_j}{\partial\nu^2} + \sigma\Delta\psi_j = 0 & \text{on } \partial\Omega, \\ \tau\dfrac{\partial\psi_j}{\partial\nu} - (1-\sigma)\operatorname{div}_{\partial\Omega}(D^2\psi_j\cdot\nu)_{\partial\Omega} - \dfrac{\partial\Delta\psi_j}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \int\limits_{\Omega}\psi_j\psi_l = 0. \end{cases}$$

Define

$$\Phi(x,y) = \sum_{j=1}^{k}\psi_j(x)\psi_j(y),\ x,y\in\Omega,$$

and let

$$\hat{\Phi}(z,y) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\Omega}\Phi(x,y)e^{ix\cdot z}dx$$

be the Fourier transform of Φ in the variable x, where we have used the same definition (for Fourier transform) as that in Li-Yau's article [15]. Since one can check that

$$(2\pi)^{\frac{n}{2}}\hat{\Psi}(z,y) = \sum_{j=1}^{k}\psi_j(y)\int\limits_{\Omega}\psi_j(x)e^{ix\cdot z}dx$$

is the orthogonal projection of the function $h_z(y) = e^{iy\cdot z}$ onto the subspace of $L^2(\Omega)$ spanned by $\psi_1,\ldots,\psi_k$, we can use $\varphi(z,y) = h_z(y) - (2\pi)^{\frac{n}{2}}\hat{\Psi}(z,y)$ as a trial function for Λ_{k+1} to obtain

$$\Lambda_{k+1}\int_{\Omega}|\varphi(z,y)|^2\mathrm{d}y\mathrm{d}z \le \int_{\Omega}[(1-\sigma)|D_y^2\varphi(z,y)|^2+\sigma\|\Delta_y\varphi(z,y)\|^2+\tau\,|\nabla_y\varphi(z,y)|^2]\mathrm{d}y\mathrm{d}z.$$

Integrating both sides of the above inequality over $B_r = \{z \in \mathbb{R}^n \,|\, |z| < r\}$ yields

$$\Lambda_{k+1} \le \inf_r \left\{ \frac{\int_{B_r}\int_{\Omega}[(1-\sigma)|D_y^2\varphi(z,y)|^2+\sigma\|\Delta_y\varphi(z,y)\|^2+\tau\,|\nabla_y\varphi(z,y)|^2]\mathrm{d}y\mathrm{d}z}{\int_{B_r}\int_{\Omega}|\varphi(z,y)|^2\mathrm{d}y\mathrm{d}z} \right\}, \tag{2.1}$$

where $r > 2\pi\left(\frac{k}{\omega_n\,|\,\Omega\,|}\right)^{\frac{1}{n}}$.

Noticing $|h_z(y)| = 1$ and $\hat{\Phi}(z,y) = \sum_{j=1}^{k}\psi_j(y)\hat{\psi}_j(z)$, we have

$$\begin{aligned}
\iint_{B_r\,\Omega}|\varphi(z,y)|^2\mathrm{d}y\mathrm{d}z &= \iint_{B_r\,\Omega}\left|\,h_z(y)-(2\pi)^{\frac{n}{2}}\hat{\Phi}(z,y)\right|^2\,\mathrm{d}y\mathrm{d}z\\
&= \|h_z(y)\|^2 - 2(2\pi)^{\frac{n}{2}}\,\mathrm{Re}\left(\iint_{B_r\,\Omega}h_z(y)\overline{\hat{\Phi}(z,y)}\mathrm{d}y\mathrm{d}z\right) + (2\pi)^n\|\hat{\Phi}(z,y)\|^2\\
&= \omega_n|\Omega|r^n - 2(2\pi)^{\frac{n}{2}}\,\mathrm{Re}\left(\sum_{j=1}^{k}\iint_{B_r\,\Omega}\mathrm{e}^{\mathrm{i}y\cdot z}\psi_j(y)\overline{\hat{\psi}_j(z)}\mathrm{d}y\mathrm{d}z\right)\\
&\quad + (2\pi)^n\sum_{j,l=1}^{k}\iint_{B_r\,\Omega}\psi_j(y)\psi_l(y)\hat{\psi}_j(z)\overline{\hat{\psi}_l(z)}\mathrm{d}y\mathrm{d}z\\
&= \omega_n|\Omega|r^n - (2\pi)^n\sum_{j=1}^{k}\int_{B_r}|\hat{\psi}_j(z)|^2\mathrm{d}z,
\end{aligned} \tag{2.2}$$

where $\|f\|^2 = \int_{B_r}\int_{\Omega}|f|^2\mathrm{d}y\mathrm{d}z$.

Let

$$P = \iint_{B_r\,\Omega}[(1-\sigma)|D_y^2\varphi(z,y)|^2+\sigma\|\Delta_y\varphi(z,y)\|^2+\tau\,|\nabla_y\varphi(z,y)|^2]\mathrm{d}y\mathrm{d}z = P_1+P_2+P_3,$$

where

$$P_1 = \iint_{B_r\,\Omega}((1-\sigma)|D_y^2h_z(y)|^2+\sigma\,|\Delta_y h_z(y)|^2+\tau\,|\nabla_y h_z(y)|^2)\mathrm{d}y\mathrm{d}z,$$

$$P_2 = -2(2\pi)^{\frac{n}{2}}\,\mathrm{Re}\left\{\iint_{B_r\,\Omega}((1-\sigma)D_y^2h_z(y)\cdot\overline{D_y^2\hat{\Psi}(z,y)}+\sigma\Delta_y h_z(y)\overline{\Delta_y\hat{\Psi}(z,y)}+\tau\nabla_y h_z(y)\cdot\overline{\nabla_y\hat{\Psi}(z,y)})\mathrm{d}y\mathrm{d}z\right\},$$

$$P_3 = \iint_{B_r\,\Omega}((1-\sigma)|D_y^2\hat{\Psi}(z,y)|^2+\sigma\,|\Delta_y\hat{\Psi}(z,y)|^2+\tau\,|\nabla_y\hat{\Psi}(z,y)|^2)\mathrm{d}y\mathrm{d}z.$$

Since $|h_z(y)_{y_p}| = |z_p|$ and $|h_z(y)_{y_py_q}| = |z_p||z_q|$, then $|\Delta_y h_z(y)| = |z|^2$, $|\nabla_y h_z(y)| = |z|$, and

$$|D^2h_z(y)|^2 = \sum_{p,q=1}^{n}|h_z(y)_{y_py_q}|^2 = \sum_{p,q=1}^{n}|z_p|^2|z_q|^2 = |z|^4.$$

So, we have

$$P_1 = n\omega_n|\Omega|\left(\frac{r^{n+4}}{n+4}+\tau\frac{r^{n+2}}{n+2}\right). \tag{2.3}$$

Integrating by parts and noticing $\hat{\Psi}(z,y) = \sum_{j=1}^{k}\psi_j(y)\widehat{\psi_j}(z)$, it follows that

$$P_2 = -2(2\pi)^{\frac{n}{2}}\operatorname{Re}\left\{\int_{B_r}\int_{\Omega}((1-\sigma)h_z(y)\overline{\Delta_y^2\hat{\Psi}(z,y)} + \sigma h_z(y)\overline{\Delta_y^2\hat{\Psi}(z,y)} - \tau h_z(y)\overline{\Delta_y\hat{\Psi}(z,y)})\mathrm{d}y\mathrm{d}z\right\}$$
$$= -2(2\pi)^n\sum_{j=1}^{n}\Lambda_j\int_{B_r}|\widehat{\psi_j}(z)|^2\mathrm{d}z \tag{2.4}$$

and

$$P_3 = \int_{B_r}\int_{\Omega}((1-\sigma)|D_y^2\hat{\Psi}(z,y)|^2 + \sigma|\Delta_y\hat{\Psi}(z,y)|^2 + \tau\,|\nabla_y\hat{\Psi}(z,y)|^2)\mathrm{d}y\mathrm{d}z$$
$$= \int_{B_r}\int_{\Omega}\hat{\Psi}(z,y)\overline{(\Delta_y^2 - \tau\Delta_y)\hat{\Psi}(z,y)}\mathrm{d}y\mathrm{d}z \tag{2.5}$$
$$= (2\pi)^n\sum_{j=1}^{k}\Lambda_j\int_{B_r}|\widehat{\psi_j}(z)|^2\mathrm{d}z.$$

Combining (2.1)–(2.5), we have

$$\Lambda_{k+1} \le \inf_{r>2\pi\left(\frac{k}{\omega_n|\Omega|}\right)^{\frac{1}{n}}}\left\{\frac{\omega_n|\Omega|\left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right) - (2\pi)^n\sum_{j=1}^{k}\Lambda_j\int_{B_r}|\hat{\psi}_j(z)|^2\mathrm{d}z}{\omega_n|\Omega|r^n - (2\pi)^n\sum_{j=1}^{k}\int_{B_r}|\hat{\psi}_j(z)|^2\mathrm{d}z}\right\}. \tag{2.6}$$

Setting $c_j := \int_{B_r}|\hat{\psi}_j(z)|^2\mathrm{d}z,\ j = 1,\ldots,k$. By Plancherel's theorem, one has

$$c_j \le 1 \quad \text{for } j = 1,\ldots,k, \tag{2.7}$$

and we deduce from (2.6) that

$$\Lambda_{k+1}\left(\omega_n|\Omega|r^n - (2\pi)^n\sum_{j=1}^{k}c_j\right) \le n\omega_n|\Omega|\left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right) - (2\pi)^n\sum_{j=1}^{k}\Lambda_j c_j,$$

which by (2.7) implies that

$$\Lambda_{k+1}\omega_n|\Omega|r^n - n\omega_n|\Omega|\left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right) \le (2\pi)^n\sum_{j=1}^{k}(\Lambda_{k+1} - \Lambda_j)$$

with $r > 2\pi\left(\frac{k}{\omega_n\,|\,\Omega\,|}\right)^{\frac{1}{n}}$. Hence,

$$(2\pi)^n\sum_{j=1}^{k}\Lambda_j \le n\omega_n|\Omega|\left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right) + (k(2\pi)^n - \omega_n|\Omega|r^n)\Lambda_{k+1}, \quad r > 2\pi\left(\frac{k}{\omega_n|\Omega|}\right)^{\frac{1}{n}}.$$

Since $r > 2\pi\left(\frac{k}{\omega_n\,|\,\Omega\,|}\right)^{\frac{1}{n}}$, we infer from the above inequality that

$$\sum_{j=1}^{k}\Lambda_j \le \left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right)\frac{n\omega_n|\Omega|}{(2\pi)^n}, \quad r > 2\pi\left(\frac{k}{\omega_n|\Omega|}\right)^{\frac{1}{n}}.$$

One obtains (1.8) directly by letting $r \to 2\pi\left(\frac{k}{\omega_n\,|\,\Omega\,|}\right)^{\frac{1}{n}}$.

Combining (2.6) and (2.7), we have

$$\Lambda_{k+1} \le \frac{\left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right)\omega_n|\Omega|}{\omega_n|\Omega|r^n - k(2\pi)^n}, \quad \forall r > 2\pi\left(\frac{k}{\omega_n|\Omega|}\right)^{\frac{1}{n}}. \tag{2.8}$$

Consequently, we have

$$\Lambda_{k+1}(\Omega) \le \inf_{r>2\pi\left(\frac{k}{\omega_n|\Omega|}\right)^{\frac{1}{n}}} \frac{n\omega_n|\Omega|\left(\frac{r^{n+4}}{n+4} + \tau\frac{r^{n+2}}{n+2}\right)}{\omega_n|\Omega|r^n - k(2\pi)^n}, \quad k \ge 0.$$

For the case $\tau = 0$, solving $F'(r) = 0$ yields

$$r = 2\pi\left(\frac{(n+4)k}{4\omega_n|\Omega|}\right)^{\frac{1}{n}}.$$

Taking the above value of r into (2.8), we have (1.9). □

3 Reilly-type estimates

In the last section, by using the QR-factorization theorem and the variational principle, we can give the proofs of two sharp extrinsic lower bounds for the sum of the reciprocals of the first n nonzero eigenvalues (given in Theorems 1.5 and 1.8) by constructing appropriately trial functions. In fact, we have already used the method of QR-factorization (together with other approaches) to try to obtain estimates for the sum of the reciprocals of the first n nonzero eigenvalues of prescribed eigenvalue problems (see, e.g., [18]).

First, we have:

Proof of Theorem 1.5. Let $x_1, \dots, x_n$ be the coordinate functions in $\mathbb{R}^n$. Since Ω is a bounded domain in $\mathbb{R}^n$, we can regard $\partial\Omega$ as a closed hypersurface of $\mathbb{R}^n$ without boundary.

Let u_j be an eigenfunction corresponding to the eigenvalue λ_j such that $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2(\partial\Omega)$, that is,

$$\begin{cases} \Delta^2 u_j - \tau\Delta u_j = 0 & \text{in } \Omega, \\ (1-\sigma)\dfrac{\partial^2 u_j}{\partial \nu^2} + \sigma\Delta u_j = 0 & \text{on } \partial\Omega, \\ \tau\dfrac{\partial u_j}{\partial \nu} - (1-\sigma)\operatorname{div}_{\partial\Omega}(D^2u_j \cdot \nu)_{\partial\Omega} - \dfrac{\partial \Delta u_j}{\partial \nu} = -\lambda_j u_i & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_i u_j = \delta_{ij}. \end{cases}$$

Observe that $u_1 = 1/\sqrt{|\partial\Omega|}$ is a constant. By translating the origin appropriately, we can assume that

$$\int_{\partial\Omega} x_i = 0, \quad i = 1, \dots, n, \tag{3.1}$$

that is, $x_i \perp u_1$. Next, we will show that a suitable rotation of axes can be made so as to insure that

$$\int_{\partial\Omega} x_j u_{i+1} = 0, \tag{3.2}$$

for $j = 2, 3, \dots, n$ and $i = 1, 2 \dots, j-1$. To see this, define an $n \times n$ matrix $Q = (q_{ji})$, where $q_{ji} = \int_{\partial\Omega} x_j u_{i+1}$, for $i, j = 1, 2, \dots, n$. Using the orthogonalization of Gram and Schmidt (i.e., QR-factorization theorem), we know that there exist an upper triangle matrix $T = (T_{ji})$ and an orthogonal matrix $U = (a_{ji})$ such that $T = UQ$, i.e.,

$$T_{ji} = \sum_{k=1}^{n} x_{jk} q_{ki} = \int_{\partial\Omega} \sum_{k=1}^{n} a_{jk} x_k u_{i+1} = 0, \quad 1 \le i < j \le n.$$

Letting $y_j = \sum_{k=1}^{n} a_{jk} x_k$, we obtain

$$\int_{\partial\Omega} y_j u_{i+1} = \int_{\partial\Omega} \sum_{k=1}^{n} a_{jk} x_k u_{i+1} = 0, \quad 1 \le i < j \le n. \tag{3.3}$$

Since U is an orthogonal matrix, $y_1, y_2, \dots, y_n$ are also coordinate functions on $\mathbb{R}^n$. Therefore, denoting these coordinate functions still by $x_1, x_2, \dots, x_n$, one can obtain (3.2). From (3.1) and (3.2), one sees that $x_j \perp \{u_1, u_2, \dots, u_{j-1}, u_j\}$ in $L^2(\partial\Omega)$.

It follows from the variational characterization (1.13) that

$$\lambda_{j+1} \int_{\partial\Omega} x_j^2 \le \int_{\Omega} (|D^2 x_j|^2 + \tau |\nabla x_j|^2) = \tau|\Omega|, \quad j = 1, \dots, n,$$

which implies that

$$\sum_{j=1}^{n} \frac{1}{\lambda_{j+1}} \tau|\Omega| \ge \sum_{j=1}^{n} \int_{\partial\Omega} x_j^2 = \int_{\partial\Omega} |x|^2.$$

Multiplying both sides of the above inequality by $\int_{\partial\Omega} |\mathbf{H}|^2$, and using the Schwarz inequality, we obtain

$$\sum_{j=1}^{n} \frac{1}{\lambda_{j+1}} \tau|\Omega| \int_{\partial\Omega} |\mathbf{H}|^2 \ge \int_{\partial\Omega} |x|^2 \int_{\partial\Omega} |\mathbf{H}|^2 \ge \left(\int_{\partial\Omega} \langle x, \mathbf{H} \rangle \right)^2 = |\partial\Omega|^2, \tag{3.4}$$

which gives (1.14).

If equality holds in (1.14), then all the inequalities in (3.4) should be equalities, which implies that $x = \kappa\mathbf{H}$ holds on $\partial\Omega$ for some constant $\kappa \ne 0$. Thus, for any tangent vector field V on $\partial\Omega$, we have $V(|x|^2) = 2\langle V, x \rangle = 0$ and so $|x|$ and $|\mathbf{H}|$ are constants on $\partial\Omega$. Since $\partial\Omega$ is a closed hypersurface of $\mathbb{R}^n$, we conclude that $\partial\Omega$ is a round sphere. This completes the proof of Theorem 1.5. □

At the end, we also have:

Proof of Theorem 1.8. As before, by the abuse of notations, Δ and ∇ denote the Laplacian and the gradient operator on M, respectively. Without loss of generality, we can assume that M does not lie in a hyperplane of $\mathbb{R}^N$. Let $x = (x_1, \dots, x_N)$ be the position vector of M in $\mathbb{R}^N$, and let u_j be the normalized eigenfunction corresponding to the jth nonzero eigenvalue μ_j of the Laplacian of M. By a similar discussion as in the proof of Theorem 1.5, we can assume that $x_j \perp \{u_1, u_2, \dots, u_{j-1}, u_j\}$ in $L^2(M)$. Then, one has

$$\eta_{j+1} \int_M x_j^2 \le \int_M |\nabla x_j|^2, \ j = 1, \dots, N,$$

which implies that

$$\sum_{j=1}^{N} \frac{1}{\eta_{j+1}} \int_M |\nabla x_j|^2 \ge \sum_{j=1}^{N} \int_M x_j^2 = \int_M |x|^2.$$

By the derivation of (2.2) in [10], it is easy to know that

$$|\nabla x_j|^2 \le 1, \quad \sum_{j=1}^{N} |\nabla x_j|^2 = n,$$

and then we have

$$\begin{aligned} \sum_{j=1}^{N} \frac{1}{\eta_{j+1}} |\nabla x_j|^2 &\le \sum_{j=1}^{n} \frac{1}{\eta_{j+1}} |\nabla x_j|^2 + \frac{1}{\eta_{n+2}} \sum_{A=n+1}^{N} |\nabla x_A|^2 \\ &= \sum_{j=1}^{n} \frac{1}{\eta_{j+1}} |\nabla x_j|^2 + \frac{1}{\eta_{n+2}} \left(n - \sum_{i=1}^{n} |\nabla x_j|^2 \right) \\ &\le \sum_{j=1}^{n} \frac{1}{\eta_{j+1}} |\nabla x_j|^2 + \sum_{i=1}^{n} \frac{1}{\eta_{i+1}} (1 - |\nabla x_i|^2) \\ &= \sum_{j=1}^{n} \frac{1}{\eta_{j+1}}, \end{aligned} \tag{3.5}$$

which gives

$$\sum_{j=1}^{n} \frac{1}{\eta_{j+1}} |M| \geq \int_M |x|^2. \tag{3.6}$$

Multiplying both sides of the above inequality by $\int_M |\mathbf{H}|^2$, and using the Schwarz inequality, we have

$$\sum_{j=1}^{n} \frac{1}{\eta_{j+1}} |M| \int_M |\overline{\mathbf{H}}|^2 \geq \int_M |x|^2 \int_M |\overline{\mathbf{H}}|^2 \geq \left(\int_M \langle x, \overline{\mathbf{H}} \rangle \right)^2 = |M|^2, \tag{3.7}$$

which implies that (1.15) is true.

If equality holds in (1.15), then equalities hold in all of the above inequalities, which implies that

$$\eta_2 = \cdots = \eta_N = \eta_{N+1} \equiv C,$$

$$\Delta x_j = -C x_j, \ \ j = 1, \ldots, N, \ \text{ on } M,$$

and $x = \kappa \overline{\mathbf{H}}$ hold on M for some constant $\kappa \neq 0$. From these facts, we know that $|x|$ and $|\overline{\mathbf{H}}|$ are constants on M. Therefore, when $n = N - 1$, M is a hypersphere, and when $n < N - 1$, M is a minimal submanifold of some hypersphere of $\mathbb{R}^N$. □

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