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Stationary solutions to a degenerate logistic equation with superlinear or asymptotically linear nonlinearity

by

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Aos oito dias do mês de maio do ano de dois mil e vinte e quatro, instalou-se a banca examinadora de Tese De Doutorado da aluna Flávia Elisandra Magalhães Furtado, matrícula 200065670. A banca examinadora foi composta pelos professores Dra. MAYRA SOARES COSTA RODRIGUES (membro interno - MAT/UnB), Dr. JOÃO RODRIGUES DOS SANTOS JÚNIOR (membro externo - UFPA), Dr. MARCOS LEANDRO MENDES CARVALHO (membro externo - UFG), Dr. WILLIAN CINTRA DA SILVA (Suplente - MAT/UnB) e Dra. LILIANE DE ALMEIDA MAIA, orientadora/presidente. A discente apresentou o trabalho intitulado "Stationary solutions to a degenerate logistic equation with superlinear or asymptotically linear nonlinearity".

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I dedicate this work to my family, in particular, to my dear and beloved brother, Ricardo, and to my sister, Hortência.

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Resumo

Neste trabalho estamos interessados em resolver o problema logístico estacionário com termo superlinear

$$\begin{cases} -\Delta u = \lambda u - b(x)f(u) & \text{em } \Omega \\ u = 0 & \text{em } \partial\Omega \end{cases} \tag{P}$$

em que $\Omega \subset \mathbb{R}^N$ é domínio aberto limitado com bordo $\partial\Omega$ suave, λ é um parâmetro real positivo, $b : \overline{\Omega} \to \mathbb{R}$ é uma função contínua em $L^\infty(\Omega)$ tal que $b(x)$ é não negativa com $\Omega_0 = \{x \in \Omega : b(x) = 0\}$ subconjunto conexo, regular e com medida de Lebesgue $|\Omega_0| > 0$. Sob essas condições, juntamente com a variedade de Nehari e o Teorema do Passo da Montanha, mostramos primeiramente, no caso em que $f(s)$ é superlinear e subcrítica quando s tende a $\pm\infty$, que o problema (P) possui uma solução positiva e uma solução que muda de sinal em $u \in H_0^1(\Omega)$.

Além disso, no segundo caso em que $f(s)$ é assintoticamente linear no infinito o termo linear λu em (P) é substituído por um termo mais geral $\lambda a(x)u$, com $a : \overline{\Omega} \to \mathbb{R}$ função em $L^\infty(\Omega)$ com $a(x) > 0$ q.t.p. em Ω, mostraremos também a existência de solução positiva única e uma solução que muda de sinal, utilizando os mesmos métodos anteriores e a teoria espectral com peso.

Palavras - chave: Superlinear, Assintoticamente Linear, Teorema do Passo da Montanha e Variedade de Nehari.

Título da tese: Soluções estacionárias para uma equação logística degenerada com não-linearidade superlinear ou assintoticamente linear.

Abstract

In this work we are interested in solving the stationary logistic problem with a superlinear nonlinearity

$$\begin{cases} -\Delta u = \lambda u - b(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is bounded open domain with $\partial\Omega$ smooth, λ is a positive real parameter, $b : \overline{\Omega} \to \mathbb{R}$ is a function in $L^\infty(\Omega)$ such that $b(x)$ is non-negative with $\Omega_0 = \{x \in \Omega : b(x) = 0\}$ is a connected, regular subset and with Lebesgue measure $|\Omega_0| > 0$. Under these conditions, along with Nehari's manifold and the Mountain Pass Theorem, we first show, in the case where $f(s)$ is superlinear and subcritical as s tends to $\pm\infty$, that the problem (P) has a positive solution and a solution that changes sign at $u \in H_0^1(\Omega)$.

Furthermore, in the second case where $f(s)$ is asymptotically linear at infinity, the linear term λu in (P) is replaced by a more general term $\lambda a(x)u$, with $a : \overline{\Omega} \to \mathbb{R}$ a function in $L^\infty(\Omega)$ with $a(x) > 0$ a.e. in Ω. We will also show the existence of a unique positive solution and a solution that changes sign, using the same previous methods and spectral theory with weight.

Keywords: Superlinear, Asymptotically Linear, Mountain Pass Theorem and Nehari Manifold.

Contents

Introduction

The study of elliptic and parabolic partial differential equations has grown significantly due to applications in various areas, for example, biology, ecology and physics. In this sense, we were motivated to study the population dynamics that can be modeled by the parabolic semilinear equation

$$\begin{cases} \partial_t u = \Delta u + \lambda u - b(x)|u|^{\nu-1}u, & (x,t) \in \Omega \times (0,+\infty) \\ u|_{\partial\Omega} = 0, \quad t \in (0,\infty) \\ u|_{t=0} = u_0(x), \quad x \in \Omega \end{cases} \tag{0.1}$$

where Ω is an open smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$, λ is a real positive parameter, $1 < \nu < 2^* - 1$, $2^* = +\infty$ if $N = 2$ or $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and b is an $L^\infty(\overline{\Omega})$ function satisfying $b(x) \geq 0$ and $b(x) = 0$ in a subset of its own Ω_0 de Ω with positive Lebesgue measure and smooth boundary.

In [13], Fernandes and Maia studied the parabolic equation mentioned in the previous paragraph. They also exploited the stationary case and did a variational approach. In the stationary case, they ensured the existence of positive solution and sign-changing solution. In [19], the author conducts a detailed study on the parabolic logistic problem.

In this work, we want to solve the stationary logistic problem with a nonlinear term which may be superlinear or asymptotically linear at infinity in a bounded domain.

In the first chapter, our goal is to investigate the nonexistence of positive solution, the existence of positive, as well as sign-changing, steady-state solutions of the degenerate logistic problem with a general superlinear nonlinearity

$$\begin{cases} -\Delta u = \lambda u - b(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is an open smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, λ is a real positive parameter, $b : \overline{\Omega} \to \mathbb{R}$ is an $L^\infty(\overline{\Omega})$ function on $\overline{\Omega}$ satisfying $b(x) \geq 0$ and $b(x) = 0$ in a connected subset $\Omega_0 \subset\subset \Omega$ with positive Lebesgue measure and smooth boundary, in other words, $\partial\Omega_0 \in C^2$.

These problems have attracted a lot of interest in the recent years, since the inclusion of the refuge Ω_0 brought about completely different results from those obtained in the case where b is a positive function. Basically, in the latter case, the main results related to existence, uniqueness, stability of the positive solution of (P) and (0.1) are well-understood. However, less is known with regard to the degenerate case with a non-homogeneous nonlinear term, and even fewer results exist in the literature for sign-changing solutions.

The stationary problem with pure power nonlinearity $f(u) = u^p$, when looking for non-negative solutions u, and $2 \leq p < \dfrac{N+2}{N-2}$, has been extensively investigated through degree theory, variational and topological methods, see for example, [9], [10] and [25]. Alama and Tarantelo [1] studied the problem with nonlinear term $b(x)f(u)$ with $b(x)$ changing sign, where they showed the existence of positive solution and multiplicity of solution for the logistic equation in bounded domain using the bifurcation method and some variational techniques.

Ouyang [25] studied the logistic problem with the nonlinearity $f(u) = u^p$ for $p > 1$ and demonstrated the existence, non-existence, and uniqueness of positive solutions on a compact Riemannian manifold using the method of sub and super-solutions and bifurcation.

Brown and Zhang [4] investigated the problem (P) with the nonlinear term $b(x)|u|^{p-1}$ for $1 < p < 2^* - 1$, and $b(x)$ changing sign. They demonstrated the existence and non-existence of positive solutions depending on the parameter λ using the Nehari manifold and variational techniques.

Delpino and Felmer [10] proved that the logistic problem with nonlinearity $b(x)|u|^{p-1}$, where $p > 1$ and $b(x)$ is non-negative, has a unique positive solution and multiple sign-changing solutions. The authors employed a variational approach to prove these results.

Concerning to solutions of problem (P) which change sign, very few studies are found in the literature for the stationary logistic equation. In the recent years the study of existence and multiplicity of nodal solutions for the one-dimensional degenerate model (P) has been extensively investigated in a series of works by López-Gómez and Rabinowitz [20–23] and also developed in [7].

In population dynamics, problem (P) can be interpreted as the population density of a single species $u(x)$ at equilibrium in a heterogeneous environment, as discussed in [11] and [5] and their references. The parameter λ can be interpreted as the growth rate of the species $u(x)$, and the function $b(x)$ as the evolution of the species and its carrying capacity. Additionally, we can obtain information about the influence of environmental conditions on the species within the region $\mathbb{R}^N \setminus \Omega_0$. There may be more habitable or inhabitable regions, denoted by the same Ω_0, depending on the possibility of greater availability of resources. The function b also provides information that exponential growth of the species may occur, i.e., when b approaches zero. Conversely, when there are constraints on population growth, we can understand that the function b is large. A detailed and comprehensive study on this evolutionary dynamics can be found in [17]. The existence of sign-changing solutions to Problem (P) can be interpreted as two populations in the same environment with diffusion and significant interaction. For detailed information, see [13].

In this work we focused at using variational methods to tackle the stationary elliptic problem, particularly finding solutions of the Euler equation via the Nehari manifold [24] associated to the problem (P), depending on the parameter λ. The main obstacles in order to apply such tools are dealing with a general non-linearity, not necessarily a pure power of u, and the difficulties imposed by a degenerate weight $b(x)$, which may be trivial in a subset of positive measure. We generalize previous important results for positive solutions found in the literature, such as Ouyang [25], Alama and Tarantello [1], Brown and Zhang [4], Del Pino and Felmer [10], and obtain new results on sign-changing solutions for dimensions higher or equal to 2, by applying the Mountain Pass theorem on the Nehari manifold. To this end, a Ghoussoub [14] theorem will be used, which is essential to guarantee the validity of the mountain pass theorem over the Nehari manifold.

Henceforth, the main purpose of this article is to shed some light on the existence of positive and sign-changing solutions for problems comprehending the nonlinearities already found in the literature, as well as to new non-homogeneous nonlinear terms as in problem (P). This type of result may be of interest in applications, in order to consider more realistic models of populations and their diffusion.

In problem (P), the f function of class $C^1(\mathbb{R})$ is odd and satisfies the following conditions:

(f_1) $\lim_{s \to 0} \dfrac{f(s)}{s} = 0;$

(f_2) let $F(s) = \int_0^s f(t)dt$,

$$\lim_{s\to\infty} \frac{F(s)}{s^2} = +\infty;$$

(f_3) $\dfrac{f(s)}{s} < f'(s)$ for $s > 0$;

(f_4) there are $2 < q < 2^*$, $a_1 > 0$ and an integer $k \in \{0,1\}$ such that $|f^{(k)}(s)| \leq a_1(1 + |s|^{q-(k+1)})$.

In addition theses hypotheses, let us consider $\lambda_1(\Omega)$ and $\lambda_1(\Omega_0)$ the first eigenvalues of $-\Delta$, with Dirichlet boundary conditions, in Ω and Ω_0 respectively.

Henceforth, it is possible to find a range of the parameter λ for which the nonexistence of a positive solution occurs.

Theorem 1. *Problem (P) does not admit a positive solution for any* $\lambda \geq \lambda_1(\Omega_0)$ *or* $0 < \lambda < \lambda_1(\Omega)$.

Moreover, our result obtained for problem (P) on the existence of a positive solution is as follows.

Theorem 2. *Assume* $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ *and* $(f_1)-(f_4)$. *There is a non-negative solution* $u \in H_0^1(\Omega)$ *of problem (P). Moreover, if* $b \in C^{0,\alpha}(\Omega)$, *for some* $\alpha \in (0,1]$, *then* u *is the positive unique solution of problem (P).*

Furthermore, considering the effect of the spectrum of the Laplacian operator in Ω, we were able to find a sign-changing solution for the problem.

Theorem 3. *Assume* $\lambda_1(\Omega) < \lambda_2(\Omega) < \lambda < \lambda_1(\Omega_0)$ *and* f *satisfies* $(f_1)-(f_4)$. *Then, there exists* $u^* \in H_0^1(\Omega)$ *a non-trivial solution of problem (P), different from the non-negative solution. Moreover, if* $b \in C^{0,\alpha}(\Omega)$, *for some* $\alpha \in (0,1]$, *then* u^* *is a sign-changing solution of problem (P).*

In the second chapter, the objective is to study the logistic problem in a bounded domain and understand what happens when we assume asymptotically linear $f(s)$ as $|s|$ goes to infinity, that is, we want to show the existence of a non-trivial solution to the problem

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P'}$$

where $\Omega \subset \mathbb{R}^N$ is domain bounded by a boundary $\partial\Omega$ smooth, λ is a real positive parameter, $a, b : \overline{\Omega} \to \mathbb{R}$ are functions in $L^\infty(\Omega)$ such that $a(x) > 0$ *a.e.* and $b(x)$ is non-negative and $b(x) = 0$ in a connected subset $\Omega_0 \subset\subset \Omega$ with positive Lebesgue measure and smooth boundary, in other words, $\partial\Omega_0 \in C^2$.

We are going to assume that f satisfies $(f_1), (f_3), (f_4)$,

(f_2)' $\lim_{s\to\infty} \frac{f(s)}{s} = l_\infty > 0$.

and

(f_5)' For $F(s) = \int_0^s f(t)dt$,

$$f(s)s - 2F(s) > 0 \text{ if } s \neq 0, \text{ and}$$
$$\lim_{|s|\to\infty} f(s)s - 2F(s) = +\infty.$$

In the investigation of the existence of a solution, we encounter various difficulties, among them, the major obstacles are to verify that the Nehari manifold is non-empty and bounded. To achieve this goal, we make an abstract assumption using the eigenvalue problem. The hypothesis is given by

(b_1)

$$b(x) > \frac{\lambda a(x) - \lambda_1(\Omega)}{l_\infty}.$$

where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta)$ in Ω with Dirichlet boundary condition. We will comment on hypothesis (b_1) in Remark 2.6. Constrained to the Nehari manifold, we are able to minimize the functional associated with the problem (P) over the manifold. Further, applying the Mountain Pass Theorem and Spectral Theory [16], it is possible to ensure the existence of a sign-changing solution.

Our major contribution was to conduct a study of the logistic problem with asymptotically linear nonlinearity at infinity. This is noteworthy as we performed a literature review and found few works on this subject under this type of nonlinear growth.

Therefore, the main results of this chapter concern the nonexistence of positive solution, the existence of positive solutions and sign-changing solutions in this novel case. Before stating these results, consider the following eigenvalue problems

$$\begin{cases} -\Delta u = \lambda a(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

where $\lambda_1(a)$ is the first eigenvalue associated with the problem (P_1). We denote by $\lambda_1^0(a)$ the first eigenvalue associated with the problem (P_1) restricted to Ω_0.

We also prove that, depending on the parameter λ, the problem (P') does not have a positive solution, as stated in the following theorem.

Theorem 4. *Problem (P') does not admit a positive solution for any* $\lambda \geq \lambda_1^0(a)$ *or* $0 < \lambda < \lambda_1(a)$.

A positive solution is obtained on the Nehari manifold leading to the following result.

Theorem 5. *Assume* $\lambda_1(a) < \lambda < \lambda_1^0(a)$, *b satisfies* (b_1) *and f satisfies* $(f_1), (f_2)', (f_3), (f_4), (f_5)'$. *There is a non-negative solution* $u \in H_0^1(\Omega)$ *of (P'). Moreover, if* $b \in C^{0,\alpha}(\Omega)$, *for some* $\alpha \in (0,1]$, *then u is the positive unique solution of (P').*

Consider the eigenvalue problem

$$\begin{cases} -\Delta u + b(x) l_\infty u = \lambda a(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By [18], there exists the first eigenvalue $\lambda_1^{a,b} = \lambda_1(-\Delta + b l_\infty; a)$.

Remark 0.1. *Assume* (b_1), *then it holds that* λ *in Theorem 5 is in the range* $\lambda_1(\Omega) < \lambda < \lambda_1^{a,b}$. *This will be shown the proof in Claims 2.1 and 2.2.*

Finally, we obtain the theorem that guarantees the existence of a sign-changing solution. We antecipate that it heavily relies on Spectral Theory and the Hopf's Lemma.

Theorem 6. *Assume* $\lambda_1(a) < \lambda_2(a) < \lambda < \lambda_1^0(a)$, *b satisfies* (b_1) *and f satisfies* $(f_1), (f_2)', (f_3), (f_4), (f_5)'$. *Then, there exists* $u^* \in H_0^1(\Omega)$ *a non-trivial solution of problem (P'), different from the non-negative solution. Moreover, if* $b \in C^{0,\alpha}(\Omega)$, *for some* $\alpha \in (0,1]$, *then* u^* *is a sign-changing solution of problem (P').*

Finally, in the Appendix, we conducted a brief study on the eigenvalue problem with weight in a bounded domain, since the properties of this investigation were crucial for the completion of this study.

Chapter 1

The Superlinear Problem

In this chapter, we study problem (P) when the nonlinear term is superlinear. We want to look for solutions in the space $H_0^1(\Omega)$ and introduce the Nehari manifold $\mathcal{N}$ defined as the union of subsets $\mathcal{N}^+$, $\mathcal{N}^-$ and $\mathcal{N}^0$. We study the spectral theory for the operator $-\Delta$ in Ω, study the properties of the subsets of the Nehari manifold, and are able to show that $\mathcal{N}^+$ is non-empty and bounded. Thus, we minimize the functional I associated to problem (P) problem and ensure the existence of a positive and unique solution for a given parameter λ. In addition, we check the condition (PS) and and applying the Mountain Pass Theorem restricted to Nehari and the Spectral Theory, we can guarantee the existence of a solution that changes sign for the problem (P).

Consider the problem

$$\begin{cases} -\Delta u = \lambda u - b(x) f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded smooth, λ is a positive real parameter, $b : \overline{\Omega} \to \mathbb{R}$ is a function in $L^\infty(\overline{\Omega})$ such that $b(x)$ is non-negative with $\Omega_0 = \{x \in \Omega : b(x) = 0\}$, a connected subset $\Omega_0 \subset\subset \Omega$ such that the Lebesgue measure $|\Omega_0| > 0$.

Example 1.1. *The function $f(s) = |s|^{p-2}s$ for $2 < p < 2^*$ satisfies the hypotheses $(f_1), (f_2), (f_3)$ and (f_4).*

Indeed,

(f_1) $\lim_{s\to 0} \frac{f(s)}{s} = \lim_{s\to 0} |s|^{p-2} = 0$;

(f_2) $\lim_{s\to\infty} \frac{F(s)}{s^2} = \lim_{s\to\infty} \frac{|s|^{p-2}}{p} = \infty$;

(f_3) $\frac{f(s)}{s} = |s|^{p-2} < (p-1)|s|^{p-2} = f'(s)$;

(f_4) $|f(s)| = |s|^{p-1} \leq a_1(1 + |s|^{p-1})$ and $|f'(s)| = (p-1)|s|^{p-2} \leq a_1(1 + |s|^{p-2})$.

Other examples of functions which satisfy our hypotheses are the non-homogeneous functions $F(s) = s^2 \ln(1 + s^2)$, or $f(s) = s^p/(1 + s^{(p-(q-1))})$ with $2 < q < 2^*$.
Immediate considerations can be stated based on our initial hypotheses.

Remark 1.1. *By hypothesis (f_3) and integration by parts*

$$f(s)s - 2F(s) > 0 \textit{ if } s \neq 0.$$

Remark 1.2. *Let* $g(s) := \frac{F(s)}{s^2}$ *for* $s \in \mathbb{R}$, *then*

$$g'(s) = \frac{f(s)s - 2F(s)}{s^3} > 0, \quad \textit{for} \quad s > 0$$

by Remark 1.1. Suppose $\limsup_{s\to\infty} f(s)s - 2F(s) < M$ *for a positive constant* M, *then*

$$\limsup_{s\to\infty} s^3 g'(s) < M,$$

which implies $s^3 g'(s) < M$. *Integrating the inequality*

$$\int_1^s g'(t)dt < \int_1^s \frac{M}{t^3}dt,$$

we obtain

$$g(s) - g(1) < M\left(1 - \frac{2}{s^2}\right)$$

that is,

$$\frac{F(s)}{s^2} < g(1) + M\left(1 - \frac{2}{s^2}\right).$$

Taking the limit as $s \to \infty$, *by* (f_2),

$$+\infty = \lim_{s\to+\infty} \frac{F(s)}{s^2} \leq g(1) + M,$$

giving an absurd. The same is true for $s < 0$. *Hence*

$$\limsup_{|s|\to\infty} f(s)s - 2F(s) = +\infty.$$

Remark 1.3. *Hypothesis* (f_3) *imply that* $f(s)s > 2F(s)$ *if* $s \neq 0$, *and dividing by* s^2,

$$\frac{f(s)}{s} > 2\frac{F(s)}{s^2}.$$

Thus, taking the limit when s *goes to infinity it results*

$$\lim_{s\to+\infty} \frac{f(s)}{s} \geq 2 \lim_{s\to+\infty} \frac{F(s)}{s^2}.$$

Using also the hypothesis (f_2), *we conclude that*

$$\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty. \tag{1.1}$$

Remark 1.4. *Since* f *is odd, the hypothesis* (f_3) *holds for any* $s \neq 0$. *Indeed, suppose that* $-s < 0$ *and* $\frac{f(s)}{s}$ *is an even function, then*

$$f'(-s) = f'(s) > \frac{f(s)}{s} = \frac{f(-s)}{-s} \; \textit{for} \; -s < 0.$$

Remark 1.5. *Hypothesis* (f_1), *implies that given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$|f(s)| \leq \varepsilon|s| \tag{1.2}$$

for all $|s| \leq \delta$. *By* (f_4), *it follows that for* $|s| > R > 1$,

$$|f(s)| \leq 2a_1|s|^{q-1}. \tag{1.3}$$

Adding up the inequalities (1.2) *and* (1.3), *we have*

$$|f(s)| \leq \varepsilon|s| + M|s|^{q-1}. \tag{1.4}$$

Therefore,

$$|F(s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{M}{q}|s|^q \tag{1.5}$$

for all $s \in \mathbb{R}$.

We will work in the Hilbert space $H_0^1(\Omega)$, with the standard norm

$$\|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

The functional $I : H_0^1(\Omega) \to \mathbb{R}$ associated with problem (P) is given by

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)F(u)dx, \tag{1.6}$$

which is of class C^2, and its derivative is

$$I'(u)v = \int_\Omega (\nabla u \nabla v - \lambda uv)dx + \int_\Omega b(x)f(u)vdx,\ u, v \in H_0^1(\Omega).$$

Another functional $J : H_0^1(\Omega) \to \mathbb{R}$ which is related with problem (P) is given by

$$J(u) := I'(u)u = \int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)f(u)udx,$$

J is of class C^1 and defines the Nehari manifold as the following set

$$\mathcal{N} = \{u \in H_0^1(\Omega) : I'(u)u = 0\}.$$

Note that, $u \in \mathcal{N}$ if and only if

$$\int_\Omega (|\nabla u|^2 - \lambda u^2)dx = -\int_\Omega b(x)f(u)udx. \tag{1.7}$$

There is a fibrering map associated with I defined by

$$\phi_u(t) := I(tu) = \frac{t^2}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)F(tu)dx, \tag{1.8}$$

and its derivative in the variable t is given by

$$\phi_u'(t) = t\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)f(tu)u\ dx. \tag{1.9}$$

It follows that if $u \in H_0^1(\Omega) \setminus \{0\}$ and $t > 0$. Then $tu \in \mathcal{N}$ if and only if $\phi_u'(t) = 0$. The second derivative of ϕ_u is

$$\phi_u''(t) = \int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)f'(tu)u^2\ dx, \tag{1.10}$$

and we consider the subsets

$$\mathcal{N}^+ = \left\{ u \in \mathcal{N} : \int_\Omega (|\nabla u|^2 - \lambda u^2) dx + \int_\Omega b(x) f'(u) u^2 dx > 0 \right\},$$
$$\mathcal{N}^- = \left\{ u \in \mathcal{N} : \int_\Omega (|\nabla u|^2 - \lambda u^2) dx + \int_\Omega b(x) f'(u) u^2 dx < 0 \right\},$$
$$\mathcal{N}^0 = \left\{ u \in \mathcal{N} : \int_\Omega (|\nabla u|^2 - \lambda u^2) dx + \int_\Omega b(x) f'(u) u^2 dx = 0 \right\}.$$

The sets $\mathcal{N}^+, \mathcal{N}^-, \mathcal{N}^0$ correspond to local minimum points, local maximum points and inflection points of $\phi_u(t)$ for $t = 1$, respectively.

The following lemma relates the Nehari manifold to the first and second derivatives of the fibrering map.

Lemma 1.1. *If $u \in \mathcal{N}$, then $\phi'_u(1) = 0$. In addition, i) if $\phi''_u(1) > 0$, then $u \in \mathcal{N}^+$; ii) if $\phi''_u(1) < 0$, then $u \in \mathcal{N}^-$; iii) if $\phi''_u(1) = 0$, then $u \in \mathcal{N}^0$.*

Proof. For the first part, consider $u \in \mathcal{N}$. By equalities (1.7) and (1.8), we have

$$\phi'_u(1) = -\int_\Omega b(x) f(u) u dx + \int_\Omega b(x) f(u) u dx = 0.$$

The second part follows from the definition of the sets above. Indeed, if $\phi''_u(1) > 0$, then

$$\int_\Omega (|\nabla u|^2 - \lambda u^2) dx + \int_\Omega b(x) f'(u) u^2 dx > 0,$$

in other words, $u \in \mathcal{N}^+$, and this proves item (i). The other items follow in a similar way. □

Finally, we have the following characterization of the subsets $\mathcal{N}^+, \mathcal{N}^-$ and $\mathcal{N}^0$.

$$\begin{aligned} \mathcal{N}^+ &= \left\{ u \in \mathcal{N}; \int_\Omega (|\nabla u|^2 - \lambda u^2) dx + \int_\Omega b(x) f'(u) u^2 \, dx > 0 \right\} \\ &= \left\{ u \in \mathcal{N}; -\int_\Omega b(x) f(u) u \, dx + \int_\Omega b(x) f'(u) u^2 \, dx > 0 \right\} \\ &= \left\{ u \in \mathcal{N}; \int_\Omega b(x) (f'(u) u^2 - f(u) u) dx > 0 \right\} \\ &= \left\{ u \in \mathcal{N}; \int_\Omega b(x) \left(f'(u) - \frac{f(u)}{u} \right) u^2 \, dx > 0 \right\}. \end{aligned}$$

Similarly, the sets

$$\mathcal{N}^0 = \left\{ u \in \mathcal{N}; \int_\Omega b(x) \left(f'(u) - \frac{f(u)}{u} \right) u^2 \, dx = 0 \right\},$$
$$\mathcal{N}^- = \left\{ u \in \mathcal{N}; \int_\Omega b(x) \left(f'(u) - \frac{f(u)}{u} \right) u^2 \, dx < 0 \right\}.$$

Moreover, we also define the following sets

$$L^+ = \left\{ u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda u^2) dx > 0 \right\},$$
$$L^0 = \left\{ u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda u^2) dx = 0 \right\},$$
$$L^- = \left\{ u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda u^2) dx < 0 \right\},$$

and with respect to the nonlinear term, we define

$$B^{+} = \left\{ u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega b(x)f(u)u\ dx > 0 \right\},$$
$$B^{0} = \left\{ u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega b(x)f(u)u\ dx = 0 \right\},$$
$$B^{-} = \left\{ u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega b(x)f(u)u\ dx < 0 \right\}.$$

Let us consider the subsets $\Omega_+ = \{x \in \Omega : b(x) > 0\}$ and $\Omega_0 = \{x \in \Omega : b(x) = 0\}$. Note that if b is continuous, then $\Omega_0 = \overline{\Omega_0}$ is closed in $\mathbb{R}^N$.

Remark 1.6. *The hypotheses (f_3) and $b(x) \geq 0$ ensure that $\mathcal{N}^- = \emptyset$. In addition, if $\lambda_1(\Omega) < \lambda$, we have that $L^- \neq \emptyset$. Indeed, taking ϕ_1 the first eigenfunction of $-\Delta$ in Ω, associated with the first eigenvalue $\lambda_1(\Omega)$, then $\int_\Omega (|\nabla \phi_1|^2 - \lambda \phi_1^2)dx = \int_\Omega (\lambda_1(\Omega) - \lambda)\phi_1^2 dx < 0$.*

On the other hand, if $\lambda < \lambda_1(\Omega_0)$, $L^+ \neq \emptyset$ because taking ϕ_1^0 as the eigenfunction of $-\Delta$ in Ω_0, define

$$\varphi(x) = \begin{cases} \phi_1^0(x), & x \in \Omega_0 \\ 0, & x \in \Omega \backslash \overline{\Omega_0}. \end{cases}$$

We have that $\|\varphi\| = \|\phi_0^1\| = 1$. Thus,

$$\begin{aligned} \int_\Omega (|\nabla \varphi|^2 - \lambda \varphi^2)dx &= \int_{\Omega_0} (|\nabla \varphi|^2 - \lambda \varphi^2)dx + \int_{\Omega \backslash \overline{\Omega_0}} (|\nabla \varphi|^2 - \lambda \varphi^2)dx \\ &= \int_{\Omega_0} (|\nabla \phi_1^0|^2 - \lambda (\phi_1^0)^2)dx \\ &= \int_{\Omega_0} (\lambda_1(\Omega_0) - \lambda)(\phi_1^0)^2 dx > 0. \end{aligned}$$

Then, $\varphi \in L^+$.

Remark 1.7. *If $0 < \lambda < \lambda_1(\Omega)$, then $\mathcal{N}^+ = \emptyset$, as we will see in Section 2, $\int_\Omega (|\nabla u|^2 - \lambda u^2)dx$ will be a norm, and by hypothesis (f_1), $\frac{f(s)}{s}$ increases for $s > 0$ and $b(x) \geq 0$, then the equality (1.7) will not occur. On the other hand, if $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ then Lemma 2.9 will guarantee that $\mathcal{N}^+ \neq \emptyset$.*

Lemma 1.2. *If $0 < \lambda < \lambda_1(\Omega_0)$ then $\overline{L^-} \cap B^0 = \emptyset$.*

Proof. By contradiction, suppose that there is $u \in \overline{L^-} \cap B^0$. Then $\|u\| = 1$ and

$$0 = \int_\Omega b(x)f(u)udx = \int_{\Omega^+} b(x)f(u)udx.$$

Since $f(s)s \geq 0$ and $b(x) \geq 0 (\not\equiv 0)$, then

$$\int_\Omega b(x)f(u)udx = 0 \text{ implies that } b(x)f(u(x))u(x) = 0 \text{ a.e in } \Omega^+.$$

Consider $x \in \Omega_+$, in other words, $b(x) > 0$, then $f(u(x))u(x) = 0$ *a.e.* in Ω^+, and by the hypotheses (f_1) and (f_3), we have that $u(x) = 0$ *a.e.* Ω_+. Therefore, u has support in $\overline{\Omega_0}$.

In addition, since $u \in \overline{L^-}$, and $supp\{u\} \subset \overline{\Omega_0}$, in case $0 < \lambda < \lambda_1(\Omega_0)$, then

$$\begin{aligned} 0 \leq \int_{\Omega_0} (\lambda_1(\Omega_0) - \lambda)u^2 dx &\leq \int_{\Omega_0} (|\nabla u|^2 - \lambda u^2)dx \\ &= \int_{\Omega} (|\nabla u|^2 - \lambda u^2)dx \\ &\leq 0. \end{aligned}$$

Therefore $u \equiv 0$, which is a contradiction with $\|u\| = 1$. It follows that $\overline{L^-} \cap B^0 = \emptyset$.

□

Lemma 1.3. *Suppose that* $0 < \lambda < \lambda_1(\Omega_0)$, *then* $\mathcal{N}^0 = \{0\}$.

Proof. Suppose by contradiction there is $u_0 \in \mathcal{N}^0 \backslash \{0\}$, i.e., u_0 satisfies

$$\int_\Omega b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx = 0.$$

Thus

$$\int_{\Omega_0} b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 + \int_{\Omega\backslash\Omega_0} b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx = 0,$$

which implies that

$$\int_{\Omega\backslash\Omega_0} b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx = 0.$$

If $supp\{u_0\} \cap \Omega \backslash \overline{\Omega_0} \neq \emptyset$, by (f_3), we have a contradiction. Thus $supp\{u_0\} \subset \overline{\Omega_0}$. It follows that

$$\int_\Omega b(x) f(u_0) u_0 dx = \int_{\Omega_0} b(x) f(u_0) u_0 dx = 0,$$

and so, $\frac{u_0}{\|u_0\|} \in B^0$. In addition, since $u_0 \in \mathcal{N}$, then

$$0 = -\int_{\Omega_0} b(x) f(u_0) u_0 dx = \int_{\Omega_0} (|\nabla u_0|^2 - \lambda u_0^2) dx,$$

which implies $\frac{\int_{\Omega_0} |\nabla u_0|^2 dx}{\int_{\Omega_0} u_0^2 dx} = \lambda$ and $\lambda_1(\Omega_0) \leq \frac{\int_{\Omega_0} |\nabla u_0|^2 dx}{\int_{\Omega_0} u_0^2 dx} = \lambda$, giving is a contradiction. □

Lemma 1.4. *For all* $u \in \mathcal{N}$, *the inequality* $I(u) \leq 0$ *holds.*

Proof. Let $u \in \mathcal{N}$, then applying (1.7) and Remark 1.1 we have

$$\begin{aligned} I(u) &= \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda a(x) u^2) dx + \int_\Omega b(x) F(u) dx \\ &= -\frac{1}{2} \int_\Omega b(x) f(u) u dx + \int_\Omega b(x) F(u) dx \\ &= \frac{1}{2} \int_\Omega b(x) [2F(u) - f(u)u] dx \leq 0. \end{aligned} \tag{1.11}$$

Moreover, $I(u) = 0$ if and only if $u = 0$ or $\text{supp}\{u\} \subset \overline{\Omega_0}$. □

Lemma 1.5. *Suppose that* $0 < \lambda < \lambda_1(\Omega_0)$ *and* f *satisfies* $(f_1) - (f_4)$. *Then* $\mathcal{N}^+$ *is bounded.*

Proof. Suppose otherwise, then there exists a sequence $(u_n) \in \mathcal{N}^+$ such that $\|u_n\| \to +\infty$. Define $v_n := \frac{u_n}{\|u_n\|}$, up to a subsequence, (v_n) is bounded in $H_0^1(\Omega)$, $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ in $L^p(\Omega)$ for $2 \leq p < 2^*$ and $v_n(x) \to v(x)$ a.e. in Ω. Since $\frac{f(s)}{s}$ is increasing if $s > 0$, by hypothesis (f_3), $\frac{f(s)}{s}$ is even and applying (f_1) we obtain

$$b(x)\frac{f(\|u_n\|v_n(x))}{\|u_n\|v_n(x)}v_n^2(x) \geq 0.$$

Initially, we note that as $u_n \in \mathcal{N}^+$, then $\Omega_n = supp\{u_n\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$ and $|\Omega_n| > 0$, and hence, $supp\{v_n\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$. Thus, for all $n \in \mathbb{N}$

$$\int_{\Omega_n} b(x)f(u_n)u_n \, dx > 0 \Longrightarrow \int_{\Omega} b(x)\frac{f(\|u_n\|v_n)}{\|u_n\|v_n}v_n^2 dx > 0.$$

We claim that $v \neq 0$. Since $(u_n) \in \mathcal{N}^+$ and $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, then dividing $J(u_n)$ by $\|u_n\|^2$, we have

$$\begin{aligned} 0 &= \int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)dx + \int_\Omega b(x)f(u_n)u_n dx \\ &= \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx + \int_\Omega b(x)\frac{f(\|u_n\|v_n)}{\|u_n\|v_n}v_n^2 dx \\ &> \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx. \end{aligned} \tag{1.12}$$

Taking the limit as n goes to infinity in (1.12) and by the compact embeddeding of $H_0^1(\Omega)$ in $L^2(\Omega)$, we obtain

$$0 \geq 1 - \lambda \int_\Omega v^2 dx.$$

If $v = 0$, then this yields an absurd, and so $v \neq 0$. This way, we have two possibilities: either $supp\{v\} \subset \overline{\Omega_0}$ or $supp\{v\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$ with positive measure. If $supp\{v\} \subset \overline{\Omega_0}$, and since $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, by (1.12) we have

$$\begin{aligned} 0 &\geq \liminf_{n\to\infty} \left\{ \int_\Omega |\nabla v_n|^2 dx - \int_\Omega \lambda v_n^2 dx \right\} \\ &\geq \int_\Omega |\nabla v|^2 dx - \int_\Omega \lambda v^2 dx \\ &\geq \int_{\Omega_0} \lambda_1(\Omega_0) v^2 dx - \lambda \int_{\Omega_0} v^2 dx \\ &= (\lambda_1(\Omega_0) - \lambda) \int_\Omega v^2 dx > 0, \end{aligned}$$

which is a contradiction. We conclude that $supp\{v\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$ with positive measure.

Moreover, due to the fact that $u_n \in \mathcal{N}^+$, $v_n \rightharpoonup v \neq 0$ and by Lemma 1.4, we obtain

$$\begin{aligned}
0 \geq \frac{I(u_n)}{\|u_n\|^2} &= \frac{I(\|u_n\| v_n)}{\|u_n\|^2} \\
&= \frac{\|u_n\|^2}{2\|u_n\|^2} \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx + \frac{1}{\|u_n\|^2} \int_\Omega b(x) F(\|u_n\| v_n) dx \\
&= \frac{1}{2} \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx + \int_\Omega b(x) \frac{F(\|u_n\| v_n)}{\|u_n\|^2} dx \\
&= \frac{1}{2} \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx + \int_\Omega b(x) \frac{F(\|u_n\| v_n(x))}{\|u_n\|^2} \frac{v_n^2(x)}{v_n^2(x)} dx \\
&= \frac{1}{2} \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx + \int_\Omega b(x) \frac{F(\|u_n\| v_n(x))}{(\|u_n\| v_n(x))^2} v_n^2(x) dx.
\end{aligned}$$

We know that $v \neq 0$ in Ω, so there exists $\tilde{\Omega} \subset \Omega$ such that $v(x) \neq 0$, for almost every x in $\tilde{\Omega}$ and $|\tilde{\Omega}| > 0$. Since $\lim_{n\to\infty} v_n(x) = v(x)$, then $u_n(x) = \|u_n\| v_n(x)$ and hence $\lim_{n\to\infty} u_n(x) = \pm\infty$, for almost every x in $\tilde{\Omega}$. Define $r_n(x) = \|u_n\| v_n(x)$. As $v \neq 0$ and $\|u_n\| \to \infty$, we obtain $r_n(x) \to \pm\infty$. Thus, using (f_2)

$$\lim_{n\to\infty} \frac{F(r_n(x))}{r_n^2(x)} = +\infty, \quad \text{for almost every} \quad x \in \tilde{\Omega},$$

and applying Fatou's lemma, it follows that

$$\liminf_{n\to\infty} \int_\Omega b(x) \frac{F(r_n(x))}{r_n^2(x)} v_n^2(x) dx \geq \int_{\tilde{\Omega}} \liminf_{n\to\infty} b(x) \frac{F(r_n(x))}{r_n^2(x)} v_n^2(x) dx = +\infty.$$

Therefore,

$$\begin{aligned}
0 \geq \frac{I(u_n)}{\|u_n\|^2} &= \liminf_{n\to\infty} \left\{ \frac{1}{2} \int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)dx + \int_\Omega b(x) \frac{F(r_n(x))}{r_n^2(x)} v_n^2(x) dx \right\} \\
&\geq 1 - \int_\Omega \lambda v^2 dx + \int_{\tilde{\Omega}} b(x) \liminf_{n\to\infty} \frac{F(r_n(x))}{r_n^2(x)} v_n^2(x) dx = +\infty,
\end{aligned}$$

which is absurd. Thus $\mathcal{N}^+$ is bounded and this concludes the proof. □

Lemma 1.6. *Suppose that u_0 is a critical point of I restricted to $\mathcal{N}$ such that $u_0 \notin \mathcal{N}^0$, then $I'(u_0) = 0$ in $H^{-1}(\Omega)$.*

Proof. If u_0 is a critical point for I restricted to $\mathcal{N}$, then u_0 is a minimizer of $I(u)$ subject to the constraint $J(u) = 0$. Hence, by the Theorem of Lagrange Multiplier, there exists $\mu \in \mathbb{R}$ such that $I'(u_0) = \mu J'(u_0)$. Thus,

$$\langle I'(u_0), u_0 \rangle = \mu \langle J'(u_0), u_0 \rangle. \tag{1.13}$$

Replacing $J(u_0) = 0$ into (1.13), and using (1.7), we have

$$\begin{aligned}
\langle J'(u_0), u_0 \rangle &= 2 \int_\Omega (|\nabla u_0|^2 - \lambda u_0^2)dx + \int_\Omega b(x) f'(u_0) u_0^2 dx + \int_\Omega b(x) f(u_0) u_0 dx \\
&= \int_\Omega b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx.
\end{aligned}$$

Since $u_0 \notin \mathcal{N}^0$, it follows that $\langle J'(u_0), u_0 \rangle \neq 0$ and applying (1.13) we obtain $\mu = 0$, that is $I'(u_0) = 0$. □

Now, we prove Theorem 1, which shows the non-existence of a solution to the problem (P) in a range of the parameters.

Proof of Theorem 1. Suppose by contradiction that there exists a positive solution u of problem (P). Multiplying the first equation of the problem (P) by ϕ_1^0, the positive first eigenfunction of $-\Delta$ in Ω_0,

$$-\Delta u\phi_1^0 = \lambda u\phi_1^0 - b(x)f(u)\phi_1^0,$$

and integrating by parts in this set with smooth boundary $\partial\Omega_0$, we obtain

$$\begin{aligned} 0 &= \int_{\Omega_0} \nabla u \nabla \phi_1^0 dx - \lambda \int_{\Omega_0} u\phi_1^0 dx + \int_{\partial\Omega_0} u\frac{\partial \phi_1^0}{\partial \eta} dx + \int_{\Omega_0} b(x)f(u)\phi_1^0 dx \\ &= -(\lambda - \lambda_1(\Omega_0)) \int_{\Omega_0} u\phi_1^0 dx + \int_{\partial\Omega_0} u\frac{\partial \phi_1^0}{\partial \eta} dx \end{aligned}$$

where η is the exterior unitary normal vector on $\partial\Omega_0$. On the other hand, by Hopf lemma

$$\int_{\partial\Omega_0} u\frac{\partial \phi_1^0}{\partial \eta} dx < 0 \quad \text{and since} \quad (\lambda - \lambda_1(\Omega_0)) \int_{\Omega_0} u\phi_1^0 dx > 0,$$

and this yields a contradiction. □

Definition 1.1. *A sequence $(u_n) \subset H_0^1(\Omega)$ is called a Palais-Smale sequence for the functional I at $c \in \mathbb{R}$, denoted by $(PS)_c$ sequence, if $I(u_n) \to c$ and $\|I'(u_n)\| \to 0$. The functional I satisfies the $(PS)_c$ condition if any (PS) sequence $\{u_n\}$ has a convergent subsequence.*

We will now present some consequences of Lebesgue dominated convergence theorem that will be used throughout this chapter. Consider $u_n \in \mathcal{N}^+$, by Lemma 1.5, the sequence (u_n) is bounded and, up to a subsequence, we have $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \to u_0$ in $L^p(\Omega)$ for $2 \leq p < 2^*$ and $u_n(x) \to u_0(x)$ *a.e.* $x \in \Omega$. Using the continuity of F, we have

$$b(x)F(u_n(x)) \to b(x)F(u_0(x)) \ a.e. \ x \in \Omega.$$

It follows from the fact that $\|u_n - u_0\|_p \to 0$ in $L^p(\Omega)$ that there is $\psi \in L^p(\Omega)$ such that $|u_n(x)| \leq \psi$. Then, there are $\psi_1 \in L^2(\Omega)$ and $\psi_2 \in L^q(\Omega)$, $2 \leq q < 2^*$ such that $|u_n(x)|^2 \leq \psi_1^2(x)$ and $|u_n(x)|^q \leq \psi_2^q(x)$

$$\begin{aligned} |b(x)F(u_n(x))| &\leq \|b\|_\infty |F(u_n(x)| \\ &\leq \|b\|_\infty a_1 |u_n(x)|^2 + \|b\|_\infty a_1 |u_n(x)|^q \\ &\leq \|b\|_\infty a_1 \psi_1^2(x) + \|b\|_\infty a_1 \psi_2^q(x) \in L^1(\Omega). \end{aligned}$$

By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega b(x)F(u_n)dx = \int_\Omega b(x)F(u_0)dx. \tag{1.14}$$

Similarly, we have that

$$b(x)f(u_n(x))u_n(x) \to b(x)f(u_0(x))u_0(x) \ q.t.p. \ x \in \Omega$$

and also

$$\begin{aligned} |b(x)f(u_n(x))u_n(x)| &\leq \|b\|_\infty |f(u_n(x)||u_n(x)| \\ &\leq \|b\|_\infty a_1 |u_n(x)|^2 + \|b\|_\infty a_1 |u_n(x)|^q \\ &\leq \|b\|_\infty a_1 \psi_1^2(x) + \|b\|_\infty a_1 \psi_2^q(x) \ \in L^1(\Omega). \end{aligned}$$

By Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_\Omega b(x)f(u_n)u_n\,dx=\int_\Omega b(x)f(u_0)u_0\,dx. \tag{1.15}$$

In addition, we have

$$b(x)f'(u_n(x))u_n^2(x)\to b(x)f'(u_0(x))u_0^2(x)\ \ q.t.p.\ x\in\Omega,$$

and also by (f_4)

$$|b(x)f'(u_n(x))u_n^2(x)|\le \|b\|_\infty a_1+\|b\|_\infty a_1\psi_2^{q-1}(x)\ \in L^1(\Omega).$$

$$\lim_{n\to\infty}\int_\Omega b(x)f'(u_n)u_n^2dx=\int_\Omega b(x)f'(u_0)u_0^2dx. \tag{1.16}$$

The following lemma is analogous to the lemma that is found in [6]. Hereafter C will denote a positive constant, not necessarily the same. Let $\nabla I(u_n)$ and $\nabla J(u_n)$ be the vectors in $H_0^1(\Omega)$ which represent, respectively, the linear functionals $I'(u_n)$ and $J'(u_n)$ in $H^{-1}(\Omega)$ by Riesz theorem. Then there exists $\mu_n\in\mathbb{R}$ such that $\nabla_{\mathcal{N}^+}I(u_n):=\nabla I(u_n)-\mu_n\nabla J(u_n)$, the orthogonal projection of $\nabla I(u_n)$ onto $\mathcal{T}_{u_n}\mathcal{N}^+$, which is the tangent space of $\mathcal{N}^+$ at u_n.

Lemma 1.7. *Assume $0<\lambda<\lambda_1(\Omega_0)$. Every $(PS)_c$ sequence (u_n) for I restricted to $\mathcal{N}^+$, with $c\le 0$, contains a subsequence which is a $(PS)_c$ sequence for I in $H_0^1(\Omega)$.*

Proof. Let $(u_n)\subset\mathcal{N}^+$ be a $(PS)_c$ sequence for I restricted to $\mathcal{N}^+$. By Lemma 1.5, (u_n) is bounded in $H_0^1(\Omega)$. We may write

$$\nabla I(u_n)=\nabla_{\mathcal{N}^+}I(u_n)+\mu_n\nabla J(u_n). \tag{1.17}$$

We will proceed with the continuation of the demonstration in two cases: the first case when $c<0$ and the second case when $c=0$. If $c<0$, since $\mathcal{N}^+$ is bounded in $H_0^1(\Omega)$, then, up to a subsequence, $u_n\rightharpoonup u$ in $H_0^1(\Omega)$ and $u_n\to u$ in $L^p(\Omega)$ for $2\le p<2^*$. By Lemma 1.4 we have $I(u_n)<0$, then

$$\frac{1}{2}\int_\Omega|\nabla u_n|^2dx-\frac{\lambda}{2}\int_\Omega a(x)u_n^2dx+\int_\Omega b(x)F(u_n)dx<0.$$

If $u_n\rightharpoonup u=0$ in $H_0^1(\Omega)$, then $u_n\to 0$ in $L^p(\Omega)$, $2\le p<2^*$. Through weak convergence, Sobolev embedding, and Lebesgue's Dominated Convergence Theorem in the inequality above

$$0\le\liminf_{n\to\infty}\int_\Omega|\nabla u_n|^2dx\le 0.$$

Thus,

$$\lim_{n\to\infty}\|u_n\|=0.$$

Since I is continuous, then

$$c=\lim_{n\to\infty}I(u_n)=I(u)=0$$

which is a contradiction. Therefore, $u\neq 0$.

By (f_4), for any $v\in H_0^1(\Omega)$, it holds

$$\left|\int_\Omega[f'(u_n)u_n+f(u_n)]vdx\right|\le C(1+\|u_n\|^{q-1})\|v\|\le C\|v\|.$$

Therefore,

$$|\langle \nabla J(u_n), v\rangle| \;=\; \left|2\langle u_n, v\rangle - \int_\Omega [f'(u_n)u_n + f(u_n)]v dx\right| \le C\|v\|, \quad \forall v \in H_0^1(\Omega).$$

This shows that the sequence $(\nabla J(u_n))$ is bounded in $H_0^1(\Omega)$.

Since $|J'(u_n)u_n| \le \|\nabla J(u_n)\|\|u_n\| < C$, after passing to a subsequence, we have that $|J'(u_n)u_n| \to \rho \ge 0$. Let us show that $\rho > 0$. Since $\lim_{n\to\infty} I(u_n) = c \le 0$, $u_n \in \mathcal{N}$, (u_n) bounded, $u_n \rightharpoonup u$, and by (1.14) and (1.15)

$$0 \ge c \;=\; \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \int_\Omega b(x)\Big[F(u_n) - \frac{1}{2}f(u_n)u_n\Big]dx = \int_\Omega b(x)\Big[F(u) - \frac{1}{2}f(u)u\Big]dx.$$

If $\text{supp}\{u\} \cap (\Omega\backslash\overline{\Omega_0}) \ne \emptyset$ with positive measure, applying (1.16) and (f_3),

$$\begin{aligned} \rho &= \lim_{n\to\infty} |\langle \nabla J(u_n), u_n\rangle| \\ &= \lim_{n\to\infty} \int_\Omega b(x)[f'(u_n)u_n^2 - f(u_n)u_n]dx \\ &= \int_\Omega b(x)[f'(u) - \frac{f(u)}{u}]u^2 dx > 0. \end{aligned}$$

Taking the inner product of (1.17) with $u_n \in \mathcal{N}^+$, we obtain

$$0 = \langle I'(u_n), u_n\rangle \;=\; \langle \nabla_{\mathcal{N}^+} I(u_n), u_n\rangle + \mu_n\langle J(u_n), u_n\rangle = o_n(1) + \mu_n\langle \nabla J(u_n), u_n\rangle.$$

It follows that $\mu_n \to 0$, since $(\nabla J(u_n))$ is bounded in $H_0^1(\Omega)$ and $|J'(u_n)u_n| \to \rho > 0$. Therefore, by (1.17)

$$\nabla I(u_n) = \nabla_{\mathcal{N}^+} I(u_n) + o_n(1)$$

and taking the limit it implies that $I'(u_n) \to 0$.

If $supp\{u\} \subset \overline{\Omega_0}$, then

$$0 \ge c = \liminf_{n\to\infty} I(u_n) \ge \frac{1}{2}\|\nabla u\|^2 - \frac{\lambda}{2}\|u\|^2 \ge \frac{1}{2}(\lambda_1(\Omega_0) - \lambda)\|u\|^2,$$

which is an absurd.

On the other hand, if $c = 0$, since $(u_n) \subset \mathcal{N}$ is bounded, we have that $u_n \rightharpoonup u$, by (1.14) and (1.15)

$$\begin{aligned} 0 = c &= \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) - \frac{1}{2}I'(u_n)u_n \\ &= \lim_{n\to\infty} \int_\Omega b(x)(F(u_n) - \frac{1}{2}f(u_n)u_n)dx \\ &= \int_\Omega b(x)(F(u) - \frac{1}{2}f(u)u)dx. \end{aligned}$$

It follows that $u = 0$ or $\text{supp}\{u\} \subset \overline{\Omega_0}$. If $\text{supp}\{u\} \subset \overline{\Omega_0}$, then by the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ and by Lebesgue's Dominated Convergence Theorem

$$\begin{aligned} 0 &= \liminf_{n\to\infty} I(u_n) \ge \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx \\ &\ge (\lambda_1(\Omega_0) - \lambda)\int_{\Omega_0} u^2 dx. \end{aligned}$$

Thus, we have that $u = 0$, that is, $u_n \rightharpoonup 0$ when $n \to \infty$. By the compact embedding and Lebesgue's Dominated Convergence Theorem

$$0 = \lim_{n\to\infty} I(u_n) \geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \geq \int_\Omega |\nabla u|^2 \geq 0$$

implies $\|u_n\| \to 0$ when $n \to \infty$. Since I is a functional of class C^1, then

$$I(u_n) \to I(0) = 0 \quad \text{and} \quad I'(u_n) \to I'(0) = 0$$

when $n \to \infty$, and the proof of the lemma is complete. □

1.1 Regularity of the solution

In this section, we will assume that $b \in C^{0,\alpha}(\Omega)$. Let

$$-\Delta u = \lambda u - b(x)f(u) = k(x)(1 + |u|) = g(x, u)$$

where

$$k(x) = \frac{\lambda u - b(x)f(u)}{1 + |u|}.$$

We will show that $k \in L^{\frac{N}{2}}_{loc}(\Omega)$. Note that

$$\begin{aligned} |k(x)| &= \left| \frac{\lambda u - b(x)f(u)}{1 + |u|} \right| \\ &\leq \frac{\lambda|u| + \|b\|_\infty |f(u)|}{1 + |u|} \\ &\leq \frac{\lambda|u|}{1 + |u|} + \frac{\|b\|_\infty a_1(1 + |u|^{q-1})}{1 + |u|} \\ &\leq \lambda + \frac{\|b\|_\infty a_1}{1 + |u|} + \frac{\|b\|_\infty a_1 |u|^{q-2}|u|}{1 + |u|} \\ &\leq \lambda + \|b\|_\infty a_1 + \|b\|_\infty a_1 |u|^{q-2} =: c_1 + c_2|u|^{q-2}, \end{aligned}$$

with c_1, c_2 positive constants. Thus

$$\int_\Omega |k(x)|^{\frac{N}{2}} dx \leq c_1|\Omega| + c_2 \int_\Omega |u|^{(q-2)\frac{N}{2}} dx < \infty,$$

Since $2 < q < 2^*$, we have

$$0 < q - 2 < 2^* - 2 \implies 0 < (q-2)\frac{N}{2} < (2^* - 2)\frac{N}{2} \implies$$

$$(2^* - 2)\frac{N}{2} = \left(\frac{2N}{N-2} - 2\right)\frac{N}{2} = \left(\frac{2N - 2N + 4}{N-2}\right)\frac{N}{2} = \left(\frac{4}{N-2}\right)\frac{N}{2} = 2^*.$$

Therefore, $k \in L^{\frac{N}{2}}_{loc}(\Omega)$. Using Brezis-Kato's theorem [Lemma B.3, [26]], $u \in L^s(\Omega)$, for any $1 \leq s < +\infty$. Then, $-\Delta u = \lambda u - b(x)f(u) \in L^s(\Omega)$,

$$\begin{aligned}
\int_\Omega |\lambda u - b(x)f(u)|^s dx &\leq \int_\Omega (\lambda|u| + \|b\|_\infty |f(u)|)^s dx \\
&\leq \int_\Omega (\lambda|u| + \|b\|_\infty a_1(1+|u|^{q-1})]^s dx \\
&\leq 2^{s-1}\lambda^s \int_\Omega |u|^s dx + 2^{s-1}(\|b\|_\infty a_1)^s \int_\Omega (1+|u|^{q-1})^s dx \\
&= C\|u\|_s^s + C|\Omega| + C\|u\|^{(q-1)s} < \infty.
\end{aligned}$$

Since $q-1>1$, $(q-1) \leq (q-1)s < +\infty$ and $u \in L^s(\Omega)$, taking $\overline{s} := (q-1)s$, with $1 < \overline{s} < +\infty$, we have $\int_\Omega |\lambda u - b(x)f(u)|^{\overline{s}} dx < \infty$.

Therefore, $u \in W^{2,\overline{s}}(\Omega)$ for all $1 < \overline{s} < \infty$, and by Sobolev embedding, $W^{2,\overline{s}}(\Omega) \hookrightarrow C^{1,\alpha}(\bar{\Omega})$, $u \in C^{1,\alpha}(\bar{\Omega})$. It follows from this and the hypotheses that $f \in C^1(\Omega)$ and $b \in C^{0,\alpha}(\Omega)$ that $\lambda u - b(x)f(u) \in C^{0,\alpha}(\Omega)$. Finally, by Schauder's estimates $u \in C^{2,\alpha}(\Omega)$, and this shows, that $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ is a classical solution of problem (P).

1.2 The case $0 < \lambda < \lambda_1(\Omega)$

In this case, the norm $\|u\|_\lambda := \int_\Omega (|\nabla u|^2 - \lambda u^2)dx$ is equivalent to the standard norm of $H_0^1(\Omega)$. Indeed,

$$\|u\|_\lambda^2 = \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \leq \int_\Omega |\nabla u|^2 dx = \|u\|^2,$$

therefore, $\|u\|_\lambda \leq \|u\|$. On the other hand, by the Poincare's inequality, we have

$$\begin{aligned}
\|u\|_\lambda^2 &= \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx \geq \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{\lambda_1(\Omega)} \int_\Omega |\nabla u|^2 dx \\
&= \left(1 - \frac{\lambda}{\lambda_1(\Omega)}\right) \int_\Omega |\nabla u|^2 dx = \left(\frac{\lambda_1(\Omega) - \lambda}{\lambda_1(\Omega)}\right) \|u\|^2,
\end{aligned}$$

in other words, $\|u\|_\lambda^2 \geq C\|u\|^2$ and so $\|u\|_\lambda \geq C\|u\|$, for C is a positive constant. Therefore, $\|\cdot\|_\lambda$ and $\|\cdot\|$ are equivalent. Furthermore,

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)F(u)dx = \frac{1}{2}\|u\|_\lambda^2 + \int_\Omega b(x)F(u)dx \geq \frac{C}{2}\|u\|^2 \to +\infty,$$

as $\|u\| \to \infty$, implying that the functional I is coercive and I is bounded from below. In fact, since $0 < \lambda < \lambda_1(\Omega)$,

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)F(u)dx = \frac{1}{2}\|u\|_\lambda^2 + \int_\Omega b(x)F(u)dx \geq 0$$

and so, there exists $u_0 \in H_0^1(\Omega)$ such that $I(u_0) = \inf_{u \in H_0^1(\Omega)} I(u) \geq 0$.

Since $I(0) = 0$, then $u_0 = 0$ is a minimizer for I on $H_0^1(\Omega)$. Suppose that there exists a positive solution $u_1 \in H_0^1(\Omega)$ to problem (P), then $u_1 \in \mathcal{N}^+$. However, by Remark 1.6, it holds

$$0 < \int_\Omega (|\nabla u_1|^2 - \lambda u_1^2)dx = -\int_\Omega b(x)f(u_1)u_1 dx \leq 0,$$

which is an absurd. Therefore, in this case $0 < \lambda < \lambda_1(\Omega)$ the problem (P) does not have any positive solution.

1.3 The case $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$

The condition $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ implies that $\overline{L^-} \cap B^0 = \emptyset$, by the Lemma 1.2. Moreover, by Lemma 1.3 it follows that $\mathcal{N}^0 = \{0\}$.

Lemma 1.8. *Assume $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$. If $u \in \mathcal{N}^+$, then $supp\{u\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$.*

Proof. Let $u \in \mathcal{N}^+$, that is

$$\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)f(u)udx = 0,$$

and suppose by contradiction that $supp\{u\} \cap (\Omega \setminus \overline{\Omega_0}) = \emptyset$, i.e., $supp\{u\} \subset \overline{\Omega_0}$. Then,

$$\int_\Omega b(x)f(u)udx = 0, \tag{1.1}$$

which implies $\dfrac{\int_{\Omega_0} |\nabla u|^2 dx}{\int_{\Omega_0} u^2 dx} = \lambda$ and $\lambda_1(\Omega_0) \leq \dfrac{\int_{\Omega_0} |\nabla u|^2 dx}{\int_{\Omega_0} u^2 dx} = \lambda$, giving is a contradiction. □

Lemma 1.9. *Assume $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ and $(f_1) - (f_4)$. Let $u \in H_0^1(\Omega)\backslash\{0\}$, then there exists a real number $t = t(u) > 0$ such that $tu \in \mathcal{N}^+$ if, and only if, $\frac{u}{\|u\|} \in L^-$. Moreover $t(u)$ is unique.*

Proof. As seen in Remark 1.6, we have $L^- \neq \emptyset$, because $\lambda_1(\Omega) < \lambda$. Suppose that $v = \frac{u}{\|u\|} \in L^-$, that is, $\int_\Omega (|\nabla u|^2 - \lambda u^2)dx < 0$. If $supp\{u\} \subset \overline{\Omega_0}$, then $supp\{v\} \subset \overline{\Omega_0}$, so $\int_\Omega b(x)f(v)vdx = 0$, which implies $v = \frac{u}{\|u\|} \in \overline{L^-} \cap B^0$ leading to an absurd by Lemma 1.2.

On the other hand, if $supp\{u\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$ with positive measure, then we can take $t \to 0^+$ and using the hypothesis (f_1), given $\varepsilon > 0$, there is δ such that, if $0 < t < \delta$,

$$\left| b(x)\frac{f(tu)}{tu}u^2 \right| \leq \|b\|_\infty \left| \frac{f(tu)}{tu} \right| u^2 \leq \varepsilon \|b\|_\infty u^2 \in L^1(\Omega).$$

By Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{t \to 0^+} \int_\Omega b(x)\frac{f(tu)}{tu}u^2 dx = 0, \tag{1.2}$$

thus,

$$\lim_{t \to 0^+} \frac{\phi_u'(t)}{t} = \lim_{t \to 0^+} \left\{ \int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)\frac{f(tu)}{tu}u^2 dx \right\} = \int_\Omega (|\nabla u|^2 - \lambda u^2)dx < 0. \tag{1.3}$$

On the other hand, since $\frac{f(s)}{s}$ is increasing for $s > 0$ and f is odd, then $b(x)\frac{f(tu)}{tu}u^2 \geq 0$. Applying Fatou's lemma, Lemma 1.8 and then using Remark 1.3, it follows that,

$$\liminf_{t \to \infty} \int_\Omega b(x)\frac{f(tu)}{tu}u^2 dx \geq \int_\Omega b(x) \liminf_{t \to \infty} \frac{f(tu)}{tu}u^2 dx = +\infty.$$

Thus,

$$\begin{aligned} \liminf_{t \to \infty} \frac{\phi_u'(t)}{t} &= \liminf_{t \to \infty} \left\{ \int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)\frac{f(tu)}{tu}u^2 dx \right\} \\ &\geq \int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x) \liminf_{t \to \infty} \frac{f(tu)}{tu}u^2 dx = +\infty. \end{aligned} \tag{1.4}$$

It follows from (1.3) and (1.4) that there exists t_1 such that $t_1 u \in \mathcal{N}^+$.

Now assume that $t_1 u \in \mathcal{N}^+$. By Lemma 1.8 we have $supp\{t_1 u\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$ and by (1.7) it holds

$$t_1^2 \int_\Omega (|\nabla u|^2 - \lambda u^2) dx = -\int_\Omega b(x) f(t_1 u) t_1 u \; dx < 0,$$

which implies that $\dfrac{u}{\|u\|} \in L^-$.

Finally, we will show that the projection on Nehari manifold is unique. Suppose there are $0 < t_1 < \tilde{t_1}$ such that $t_1 u, \; \tilde{t_1} u \in \mathcal{N}^+$. It follows that

$$t_1^2 \int_\Omega (|\nabla u|^2 - \lambda u^2) dx = -\int_\Omega b(x) f(t_1 u) t_1 u \; dx, \tag{1.5}$$

$$\tilde{t_1}^2 \int_\Omega (|\nabla u|^2 - \lambda u^2) dx = -\int_\Omega b(x) f(\tilde{t_1} u) \tilde{t_1} u \; dx. \tag{1.6}$$

Dividing equation (1.5) by t_1^2 and equation (1.6) by $\tilde{t_1}^2$ yields

$$\int_\Omega (|\nabla u|^2 - \lambda u^2) dx = -\int_\Omega b(x) \frac{f(t_1 u)}{t_1 u} u^2 dx, \tag{1.7}$$

$$\int_\Omega (|\nabla u|^2 - \lambda u^2) dx = -\int_\Omega b(x) \frac{f(\tilde{t_1} u)}{\tilde{t_1} u} u^2 dx. \tag{1.8}$$

Subtracting equation (1.7) from equation (1.8) it follows

$$\int_\Omega b(x) \left(\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} - \frac{f(t_1 u)}{t_1 u} \right) u^2 dx = 0.$$

Since $t_1 u, \; \tilde{t_1} u \in \mathcal{N}^+$ by Lemma 1.8 we have $supp\{t_1 u\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$ and $supp\{\tilde{t_1} u\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$. It follows there exists $\varepsilon > 0$ and $x_0 \in \Omega$ such that $B_\varepsilon(x_0) \subset \Omega \setminus \overline{\Omega_0}$ and

$$0 = \int_\Omega b(x) \left(\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} - \frac{f(t_1 u)}{t_1 u} \right) u^2 dx \geq \int_{B_\varepsilon(x_0)} b(x) \left(\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} - \frac{f(t_1 u)}{t_1 u} \right) u^2 dx > 0,$$

because $f(s)/(s)$ is increasing by (f_3), $b(x) > 0$ for $x \in B_\varepsilon(x_0)$, leading to an absurd. We conclude that there exists only one t_1 such that $t_1 u \in \mathcal{N}^+$. □

Lemma 1.10. *The function*

$$\mathcal{A} := \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \frac{u}{\|u\|} \in L^- \right\} \to (0, +\infty)$$
$$u \mapsto t(u)$$

is continuous.

Proof. Let $\frac{u}{\|u\|} \in L^-$, we shall prove that $T(u) := \int_\Omega (|\nabla u|^2 - \lambda u^2) dx < 0$ is continuous. We have that $T^{-1}\{(-\infty, 0)\}$ is open in $H_0^1(\Omega)$ and $u \mapsto t(u)$ is defined in an open subset of $H_0^1(\Omega)$. To prove continuity, we will use the Implicit Function Theorem. Let $g : \mathbb{R}^+ \times H_0^1(\Omega) \to \mathbb{R}$ of class C^1 defined by $g(t,u) = t\|u\|^2 - \lambda t \int_\Omega u^2 dx + \int_\Omega b(x) f(tu) u dx$. Consider (t_0, u_0) such that $g(t_0, u_0) = 0$ and $u_0 > 0$. For $t_0 u_0 \in \mathcal{N}^+$, we have that

$$t_0 \|u_0\|^2 - \lambda t_0 \int_\Omega u_0^2 dx = -\int_\Omega b(x) f(t_0 u_0) u_0 dx$$

$$\Longleftrightarrow t_0^2[\|u_0\|^2 - \lambda \int_\Omega u_0^2 dx] = -\int_\Omega b(x) f(t_0 u_0) t_0 u_0 dx$$

$$\Longleftrightarrow t_0^2[\|u_0\|^2 - \lambda \int_\Omega u_0^2 dx] = -\int_\Omega b(x) \frac{f(t_0 u_0)}{t_0 u_0} t_0^2 u_0^2 dx$$

$$\Longleftrightarrow \|u_0\|^2 - \lambda \int_\Omega u_0^2 dx = \int_\Omega b(x) \frac{f(t_0 u_0)}{t_0 u_0} u_0^2 dx.$$

Differentiating the function g with respect to t and using the hypothesis (f_3), we have that

$$\begin{aligned}
\frac{\partial g(t_0, u_0)}{\partial t} &= \|u_0\|^2 - \lambda \int_\Omega a(x) u_0^2 dx + \int_\Omega b(x) f'(t_0 u_0) u_0^2 dx \\
&= -\int_\Omega b(x) \frac{f(t_0 u_0)}{t_0 u_0} u_0^2 dx + \int_\Omega b(x) f'(t_0 u_0) u_0^2 dx \\
&= \int_\Omega b(x) \left[f'(t_0 u_0) u_0^2 - \frac{f(t_0 u_0)}{t_0 u_0} u_0 \right] dx > 0.
\end{aligned}$$

By the Implicit Function Theorem, the function $\Psi : \mathcal{A} \to \mathbb{R}^+$ defined by $t = t(u)$ is of class C^1 in a neighborhood V of u_0 and $g(t, u) = g(t(u), u) = 0$ in V. □

Proof of Theorem 2. Since $\mathcal{N} = \mathcal{N}^+ \cup \{0\}$ is bounded by Lemma 1.5, there is $C > 0$ such that $\|u\| \leq C$ for all $u \in \mathcal{N}$. Using equalities (1.7) and Remark 1.5

$$\begin{aligned}
|I(u)| &= \left| \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2) dx + \int_\Omega b(x) F(u) dx \right| \\
&= \left| -\frac{1}{2} \int_\Omega b(x) f(u) u dx + \int_\Omega b(x) F(u) dx \right| \\
&= \left| \frac{1}{2} \int_\Omega b(x) [2F(u) - f(u)u] dx \right| \\
&\leq \frac{1}{2} \int_\Omega |b(x)| |2F(u) - f(u)u| dx \\
&\leq \frac{\|b\|_\infty}{2} \int_\Omega (2|F(u)| + |f(u)||u|) dx \\
&\leq C \int_\Omega (\frac{\varepsilon}{2}|u| + \frac{M}{q}|u|^q + \varepsilon|u| + M|u|^q) dx \\
&\leq C \int_\Omega |u|^2 dx + C \int_\Omega |u|^q dx \\
&\leq C\|u\|_2^2 + C\|u\|_q^q \\
&\leq C\|\nabla u\|_2^2 + C\|\nabla u\|_2^q \\
&\leq C.
\end{aligned}$$

Thus, I is bounded in $\mathcal{N}$. We claim that $\inf_{u \in \mathcal{N}} I(u) < 0$. Indeed, let ϕ_1 be the first eigenfunction of $-\Delta$ in Ω, associated of the first eigenvalue $\lambda_1(\Omega)$. Then $\phi_1 \in L^-$ and by Lemma 2.9 there exists $t_1 > 0$ such that $t_1\phi_1 \in \mathcal{N} \setminus \{0\} = \mathcal{N}^+$. By Lemma 1.4, $I(t_1\phi_1) \leq 0$, on the other hand, using the Remark 1.1 and that $t_1\phi_1 > 0$ in all domain Ω,

$$\begin{aligned}
I(t_1\phi_1) &= I(t_1\phi_1) - \frac{1}{2} J(t_1\phi_1) \\
&= \frac{1}{2} \int_\Omega b(x)(2F(t_1\phi_1) - f(t_1\phi_1)t_1\phi_1) dx \\
&= \frac{1}{2} \int_{\Omega_+} b(x)(2F(t_1\phi_1) - f(t_1\phi_1)t_1\phi_1) dx < 0.
\end{aligned}$$

Thus,

$$\inf_{u\in\mathcal{N}^+} I(u) := -m < 0. \tag{1.9}$$

Let (u_n) be a minimizing sequence in $\mathcal{N}^+$. Since $\mathcal{N}^+$ is bounded by Lemma 1.5, then (u_n) is bounded and, up to a subsequence, we have $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$. By (1.9) and by equalities (1.14) and (1.15)

$$\begin{aligned}
0 > -m = \inf_{u\in\mathcal{N}^+} I(u) = \lim_{n\to\infty} I(u_n) &= \lim_{n\to\infty}\left\{\frac{1}{2}\int_\Omega b(x)[2F(u_n) - f(u_n)u_n]dx\right\}\\
&= \lim_{n\to\infty}\int_\Omega b(x)F(u_n)dx - \lim_{n\to\infty}\frac{1}{2}\int_\Omega b(x)f(u_n)u_n dx\\
&= \int_\Omega b(x)F(u_0)dx - \frac{1}{2}\int_\Omega b(x)f(u_0)u_0 dx\\
&= \frac{1}{2}\int_\Omega b(x)[2F(u_0) - f(u_0)u_0]dx.
\end{aligned}$$

This implies that $supp\{u_0\}\cap(\Omega\setminus\overline{\Omega_0}) \neq \emptyset$, then $u_0 \not\equiv 0$ and

$$\begin{aligned}
\int_\Omega b(x)\left[f'(u_0) - \frac{f(u_0)}{u_0}\right]u_0^2 dx &= \int_{\Omega\setminus\overline{\Omega_0}} b(x)\left[f'(u_0) - \frac{f(u_0)}{u_0}\right]u_0^2 dx\\
&= \int_{\Omega_+} b(x)\left[f'(u_0) - \frac{f(u_0)}{u_0}\right]u_0^2 dx > 0,
\end{aligned}$$

concluding that $u_0 \in \mathcal{N}^+$. It follows that

$$I(u_0) = \frac{1}{2}\int_\Omega b(x)[2F(u_0) - f(u_0)u_0]dx = \lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^+} I(u) < 0.$$

Thus, u_0 is a non-trivial critical point of I in $\mathcal{N}^+$ and by Lemma 1.6, $I'(u_0) = 0$. Without loss of generality, we may consider u_0 positive. Indeed, since F is an even function, then

$$\begin{aligned}
\inf_{u\in\mathcal{N}^+} I(u) &= I(u_0) = I(u_0^+ - u_0^-)\\
&= \frac{1}{2}\int_\Omega |\nabla(u_0^+ - u_0^-)|^2 dx - \frac{\lambda}{2}\int_\Omega (u_0^+ - u_0^-)^2 dx + \int_\Omega b(x)F(u_0^+ - u_0^-)dx\\
&= \frac{1}{2}\int_\Omega (|\nabla u_0^+|^2 + |\nabla u_0^-|^2)dx - \frac{\lambda}{2}\int_\Omega [(u_0^+)^2 + (u_0^-)^2] + \int_\Omega b(x)F(u_0^+ - u_0^-)dx\\
&= \frac{1}{2}\int_{\{u_0\geq 0\}} (|\nabla u_0^+|^2 - \lambda(u_0^+)^2)dx + \int_{\{u_0\geq 0\}} b(x)F(u_0^+)dx\\
&+ \frac{1}{2}\int_{\{u_0<0\}} (|\nabla(-u_0^-)|^2 - \lambda(-u_0^-)^2)dx + \int_{\{u_0<0\}} b(x)F(-u_0^-)dx\\
&= \frac{1}{2}\int_{\{u_0\geq 0\}} (|\nabla u_0^+|^2 - \lambda(u_0^+)^2)dx + \int_{\{u_0\geq 0\}} b(x)F(u_0^+)dx\\
&+ \frac{1}{2}\int_{\{u_0<0\}} (|\nabla(u_0^-)|^2 - \lambda(u_0^-)^2)dx + \int_{\{u_0<0\}} b(x)F(u_0^-)dx\\
&= \frac{1}{2}\int_{\{u_0\geq 0\}} (|\nabla u_0^+|^2 - \lambda(u_0^+)^2)dx + \int_{\{u_0\geq 0\}} b(x)F(u_0^+)dx\\
&+ \frac{1}{2}\int_{\{u_0<0\}} (|\nabla u_0^-|^2 - \lambda(u_0^-)^2)dx + \int_{\{u_0<0\}} b(x)F(u_0^-)dx\\
&= \int_\Omega (|\nabla|u_0||^2 - \lambda|u_0|^2)dx + \int_\Omega b(x)F(|u_0|)dx\\
&= I(|u_0|).
\end{aligned}$$

Hence, $u_0 \geq 0$. Moreover, assuming $b \in C^{0,\alpha}(\Omega)$, it follows by Hopf lemma that $u_0 > 0$ (see the end of the proof of Theorem 3 for details).

Suppose there exist two positive classical solutions u_1 and u_2 of (P), with $u_1 \neq u_2$ then

$$-\Delta u_1 - \lambda u_1 + b(x)f(u_1) = 0 \text{ in } \Omega \text{ and } u_1 = 0 \text{ in } \partial\Omega, \tag{1.10}$$

$$-\Delta u_2 - \lambda u_2 + b(x)f(u_2) = 0 \text{ in } \Omega \text{ and } u_2 = 0 \text{ in } \partial\Omega. \tag{1.11}$$

Dividing (1.10) by u_1 and (1.11) by u_2 we obtain

$$\frac{-\Delta u_1}{u_1} = \lambda - b(x)\frac{f(u_1)}{u_1} \tag{1.12}$$

and

$$\frac{-\Delta u_2}{u_2} = \lambda - b(x)\frac{f(u_2)}{u_2}. \tag{1.13}$$

Subtracting equation (1.12) of equation (1.13), it follows that

$$\frac{-\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} = b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right). \tag{1.14}$$

Multiplying the equation (1.14) by $(u_1^2 - u_2^2)$ and integrating over Ω

$$\int_\Omega (u_1^2 - u_2^2)\left(\frac{-\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right)dx = \int_\Omega b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx. \tag{1.15}$$

From the proof of uniqueness of solution in [3]

$$\begin{aligned}
\int_\Omega (u_1^2 - u_2^2)\left(\frac{-\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right)dx &= \int_\Omega -u_1\Delta u_1 dx + \int_\Omega u_2^2\frac{\Delta u_1}{u_1}dx \\
&\quad + \int_\Omega u_1^2\frac{\Delta u_2}{u_2}dx - \int_\Omega u_2\Delta u_2 dx \\
&= \int_\Omega \nabla u_1 \cdot \nabla u_1 dx - \int_\Omega \nabla\left(\frac{u_2^2}{u_1}\right)\cdot\nabla u_1 dx \\
&\quad - \int_\Omega \nabla\left(\frac{u_1^2}{u_2}\right)\cdot\nabla u_2 dx + \int_\Omega \nabla u_2\cdot\nabla u_2 dx \\
&= \int_\Omega |\nabla u_1|^2 dx - \int_\Omega \left(2\frac{u_2}{u_1}\nabla u_2 - \frac{u_2^2}{u_1^2}\nabla u_1\right)\cdot\nabla u_1 dx \\
&\quad - \int_\Omega \left(2\frac{u_1}{u_2}\nabla u_1 - \frac{u_1^2}{u_2^2}\nabla u_2\right)\cdot\nabla u_2 dx + \int_\Omega |\nabla u_2|^2 dx \\
&= \int_\Omega \left\{\left|\nabla u_1 - \frac{u_1}{u_2}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1}\nabla u_1\right|^2\right\}dx \geq 0.
\end{aligned}$$

Substituting in (1.15), it follows that

$$\int_\Omega b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx \geq 0. \tag{1.16}$$

As we assume $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$, by Lemma 1.2 it holds $\overline{L^-} \cap B^0 = \emptyset$, and since $u_1,\ u_2 \in \mathcal{N}^+$, then $\frac{u_1}{\|u_1\|},\ \frac{u_2}{\|u_2\|} \in L^-$ by Lemma 1.9. Moreover, by Lemma 1.8 it holds $supp\{u_1\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$, $supp\{u_2\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$. Furthermore, since $f(s)/s$ is increasing,

we have the following possibilities:
(i) if $u_1 > u_2$, then

$$\int_{\Omega\setminus\Omega_0} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx < 0,$$

which is a contradiction with (1.16).
(ii) if $u_1 < u_2$, then, once again

$$\int_{\Omega\setminus\Omega_0} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx < 0,$$

which is a contradiction with (1.16).
(iii) If there are subsets of $A \cup B = \Omega \setminus \overline{\Omega_0}$, $A \neq \emptyset$ and $B \neq \emptyset$ open in $\mathbb{R}^N$, such that $u_1 - u_2 > 0$ in A, $u_2 - u_1 > 0$ in B, by (1.15), we have

$$\begin{aligned}
0 &\leq \int_\Omega b(x)\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1^2 - u_2^2)dx = \int_{\Omega\setminus\overline{\Omega_0}} b(x)\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1^2 - u_2^2)dx \\
&= \int_A b(x)\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1^2 - u_2^2)dx + \int_B b(x)\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1^2 - u_2^2)dx \\
&< 0,
\end{aligned}$$

because in A, $\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) < 0$, $(u_1^2 - u_2^2) > 0$ and $b(x) > 0$ and in B, $\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) > 0$, $(u_1^2 - u_2^2) < 0$ and $b(x) > 0$. Therefore, we conclude that $u_1 \equiv u_2$. □

1.4 Sign-changing solution

In this section we will also assume $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$.

Lemma 1.11. *If (u_n) is a $(PS)_c$ sequence for I restricted to $\mathcal{N}^+$, with $c \leq 0$, then, up to a subsequence, (u_n) converges to u in $H_0^1(\Omega)$.*

Proof. Let (u_n) be a $(PS)_c$ sequence for I restricted to $\mathcal{N}^+$, which is bounded by Lemma 1.5. Then $I'(u_n) \to 0$, by Lemma 1.6 and Lemma 1.7, we have, $I'(u_n)\varphi \to I'(u_0)\varphi = 0, \quad \forall \varphi \in C_0^\infty(\Omega)$.

Notice that

$$\begin{aligned}
\langle I'(u_n) - I'(u_0), u_n - u_0\rangle &= I'(u_n).(u_n - u_0) - I'(u_0).(u_n - u_0) \\
&= \int_\Omega \nabla u_n \nabla(u_n - u_0)dx - \lambda\int_\Omega u_n(u_n - u_0)dx \\
&+ \int_\Omega b(x)f(u_n)(u_n - u_0)dx - \int_\Omega \nabla u_0 \nabla(u_n - u_0)dx \\
&+ \lambda\int_\Omega u_0(u_n - u_0)dx - \int_\Omega b(x)f(u_0)(u_n - u_0)dx \\
&= \|u_n - u_0\|^2 - \lambda\int_\Omega (u_n - u_0)^2 dx \\
&- \int_\Omega b(x)[f(u_n) - f(u_0)](u_n - u_0)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_n - u_0\|^2 =& \langle I'(u_n) - I'(u_0), u_n - u_0\rangle + \lambda\int_\Omega (u_n - u_0)^2 dx \\
&+ \int_\Omega b(x)[f(u_n) - f(u_0)](u_n - u_0)dx, \qquad (1.1)
\end{aligned}$$

and applying the limit as n goes to infinity in (1.1) it results that, up to a subsequence, $u_n \to u_0$ in $H_0^1(\Omega)$, because u_n is bounded and $(PS)_c$ sequence, using Sobolev embedding and Lebesgue Dominated Theorem. Thus, the functional I satisfies the $(PS)_c$ condition. □

The next result is based on Lemma 5.2 in [15], and suits our settings.

Lemma 1.12. *Let u_0 be a positive solution of the problem* (P) *and $v^j : \Omega \to \mathbb{R},\ j \in \mathbb{N}$, of functions, a sequence satisfying $\|v^j - u_0\|_{C^1(\overline{\Omega})} \to 0$. Then there exists $j_0 \in \mathbb{N}$ such that $v^j(x) > 0,\ \forall x \in \Omega,\ \forall\ j \geq j_0$.*

Proof. Indeed, we have that $\partial\Omega$ is C^1 and $u_0 = 0$ on $\partial\Omega$, then every differentiable curve $\gamma : [-1, 1] \to \partial\Omega,\ \gamma(0) = x_0 \in \partial\Omega$, we obtain $u_0(\gamma(t)) = 0$, therefore

$$\frac{d}{dt}(u_0(\gamma(t))) = \nabla u_0(\gamma(t))\gamma'(t) = 0,$$

and if $t = 0$ we have $\nabla u_0(\gamma(0))\gamma'(0) = 0$. Replacing the value of $\gamma(0) = x_0$ we have that $\nabla u_0(x_0)\gamma'(0) = 0$, in other words, $\nabla u_0(x_0)$ is perpendicular to the zero level at the point x_0. This guarantees that the normal exterior to $\partial\Omega$ at the point x_0 is parallel to $\nabla u(x_0)$ and since $u_0(x) > 0$ in Ω so we can write the outer unitary normal at x_0 as

$$\nu_{x_0} = \frac{\nabla u_0(x_0)}{\|\nabla u_0(x_0)\|}.$$

Given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, if $x \in N_{\delta_0} := \{x \in \Omega : dist(x, \partial\Omega) \leq \delta_0\}$, then $\left|\frac{f(u(x))}{u(x)}\right| < \varepsilon$. Indeed, suppose by contradiction that this is not true. Then there exists $\varepsilon_0 > 0$ such that for any $\delta_n = \frac{1}{n} > 0$, there exists x_n satisfying $dist(x_n, \partial\Omega) < \delta_n$ and $\left|\frac{f(u(x_n))}{u(x_n)}\right| > \varepsilon_0$.

Since $dist(x_n, \partial\Omega) \to 0$, and $\overline{\Omega}$ is a compact set, there is $x_0 \in \partial\Omega$ such that $x_n \to x_0$. The functions f and u_0 are continuous, hence $u_0(x_n) \to u(x_0) = 0$ and $f(u_0(x_n)) \to f(u(x_0)) = 0$, because $x_0 \in \partial\Omega$, but $\lim_{n\to\infty} \left|\frac{f(u(x_n))}{u(x_n)}\right| \geq \varepsilon_0$, which contradicts (f_1).

It follows from the hypothesis (f_1), that $f(u_0(x)) = o(|u_0(x)|)u_0(x)$ for all $x \in N_{\delta_0}$ and as u_0 is a positive solution to the problem (P), then

$$\begin{aligned}
-\Delta u_0(x) &= \lambda u_0(x) - b(x)f(u_0(x)) \\
&= \lambda u_0(x) - b(x)o(|u_0|)u_0(x) \\
&\geq \lambda u_0(x) - b(x)\varepsilon u_0(x) \\
&\geq \lambda u_0(x) - \|b\|_\infty \varepsilon u_0(x) \\
&= (\lambda - \|b\|_\infty \varepsilon)u_0(x).
\end{aligned}$$

Taking $\varepsilon < \frac{\lambda}{2\|b\|_\infty}$, then

$$-\Delta u_0(x) = \lambda u_0(x) - o(|u_0(x)|) > 0, \tag{1.2}$$

for each $x \in N_{\delta_0}$. In addition, $u_0(x) > u_0(x_0) = 0$, for all $x \in int(N_{\delta_0})$, thus $\inf_{N_{\delta_0}} u(x) = 0$. Note that N_{δ_0} is regular because the set Ω is regular and u_0 is continuous in $\overline{N_{\delta_0}}$. Then by Hopf's lemma $\frac{\partial u_0}{\partial \nu_{x_0}}(x_0) > 0$ for all $x_0 \in u_0^{-1}\{0\} \cap \partial N_{\delta_0}$, then it holds $\frac{\partial u_0}{\partial \nu_{x_0}}(x_0) > 0$ for $x \in u_0^{-1}\{0\} \cap \partial\Omega$. Since $\frac{\partial u_0}{\partial \nu_x}(x)$ is continuous for all $x \in \partial\Omega$ and $\partial\Omega$ is compact, then $\frac{\partial u_0}{\partial \nu_x}(x) \geq \delta > 0$ for all $x \in \partial\Omega$. In fact,

$$\frac{\partial u_0}{\partial \nu_x}(x) = \left\langle \nabla u_0(x), \frac{\nabla u_0(x)}{\|\nabla u_0(x)\|} \right\rangle = \|\nabla u_0(x)\| \geq \delta > 0,\ \forall\ x \in \partial\Omega. \tag{1.3}$$

If $y \in \Omega$ and $y - x_0 = \alpha\nu_{x_0}$ with $\alpha > 0$, then

$$\langle \nabla u_0(x_0), y - x_0 \rangle = \left\langle \nabla u_0(x_0), \alpha \frac{\nabla u_0(x_0)}{\|\nabla u_0(x_0)\|} \right\rangle = \|\nabla u_0(x_0)\|\alpha > 0.$$

Let $t(x_0)$ be such that if $0 < t < t(x_0)$ and $y = x_0 + t\nu_{x_0}$, then by the continuity of the $\|\nabla u_0(\cdot)\|$, we have

$$\|\nabla u_0(y)\| \geq \frac{1}{2} \min_{x_0 \in \partial\Omega} \|\nabla u_0(x_0)\| = \frac{1}{2}\delta, \quad \forall\ 0 < t < t(x_0). \tag{1.4}$$

Take $y \in \Omega$ such that $y = x + t\nu_x$ with $x \in \partial\Omega$ and $0 < t < t(x)$, for (1.4) and by the continuity of the $\|\nabla u_0(\cdot)\|$ we have

$$\|\nabla u_0(y)\| \geq \frac{\delta}{2},\ \forall\ 0 < t < t(x).$$

Consider the open ball $B_{t(x)}(x)$ such that $\partial\Omega \subset \bigcup_{x \in \partial\Omega} B_{t(x)}(x)$ and by the compactness of $\partial\Omega$ it follows $\partial\Omega \subset \bigcup_{k=1}^{n} B_{t(x_k)}(x_k)$, in other words, $\partial\Omega$ has a finite subcover. Let $y \in \bigcup_{k=1}^{n} B(x_k) \cap \Omega$, and $x \in \partial\Omega$ such that $y - x$ is perpendicular to $\partial\Omega$, therefore, we can write, $y - x = t\nu_x$. So, if $y \in \bigcup_{k=1}^{n} B_{t(x_k)}(x_k) \cap \Omega \subset \bigcup_{x_0 \in \partial\Omega} B_{t(x_0)}(x_0) \cap \Omega$, then $u_0(y) > 0$. Now, let $K = \Omega \setminus \bigcup_{k=1}^{n} B_{t(x_k)}(x_k)$ be closed and bounded, therefore compact, it follows that exists $\delta_2 > 0$ such that

$$u_0(y) > \delta_2 > 0 \quad \forall y \in K. \tag{1.5}$$

Moreover, from the compactness of K and using the norm of supremum we have

$$\|v^j - u_0\|_{L^\infty(\overline{\Omega})} \longrightarrow 0.$$

Therefore

$$|u_0(y) - v^j(y)| < \frac{\delta_2}{2} \quad \forall y \in K \text{ and } \forall j \geq j_0,$$

thus, by the triangular inequality $u_0(y) - \frac{\delta_2}{2} < v^j(y)$ and by (1.5) we have

$$\delta_2 - \frac{\delta_2}{2} < v^j(y), \quad \forall\ y \in K \text{ and } \forall\ j \geq j_0.$$

Then, $v^j(y) > 0$ for all $y \in K$ and for all $j \geq j_0$. On the other hand, for all $y \in \bigcup_{k=1}^{n} B_{t(x_k)}(x_k) \cap \Omega$, using again Taylor's formula and the fact that $\|v^j - u_0\|_{C^{1,\alpha}(\overline{\Omega})} \to 0$, if $j \to +\infty$, with $u|_{\partial\Omega}$ and denoting $o_1(j)$, where $o_j(1) \to 0$ when $j \to 0$, we have for $x \in \partial\Omega$ such that $y - x$ is perpendicular to $\partial\Omega$,

$$\begin{aligned}
v^j(y) &= v^j(x) + \nabla v^j(x)\cdot(y-x) + o(\|y-x\|)\\
&= \left\langle \nabla v^j(x), t\frac{\nabla u_0(x)}{\|\nabla u_0(x)\|}\right\rangle + o(t) + o_1(j)\\
&= \left\langle \nabla v^j(x) - \nabla u_0(x) + \nabla u_0(x), t\frac{\nabla u_0(x)}{\|\nabla u_0(x)\|}\right\rangle + o(t) + o_1(j)\\
&= \left\langle \nabla v^j(x) - \nabla u_0(x), t\frac{\nabla u_0(x)}{\|\nabla u_0(x)\|}\right\rangle + t\|\nabla u_0(x)\| + o(t) + o_1(j)\\
&= \frac{t}{\|\nabla u_0(x)\|}\langle \nabla v^j(x) - \nabla u_0(x), \nabla u_0(x)\rangle + t\|\nabla u_0(x)\| + o(t) + o_1(j)\\
&\geq -\frac{t}{\|\nabla u_0(x)\|}\|\nabla v^j - \nabla u_0\|_\infty \|\nabla u_0(x)\| + \|\nabla u_0(x)\|t + o(t) + o_1(j)\\
&= -\|\nabla v^j - \nabla u_0\|_\infty t + \|\nabla u_0(x)\|t + o(t) + o_1(j)\\
&= (-\varepsilon+\delta)\frac{t}{2} > 0, \quad \forall j \geq j_0,
\end{aligned}$$

because $\|\nabla v^j - \nabla u_0\|_{L^\infty(\overline{\Omega})} < \varepsilon$ for all $n \geq n_0$, ε sufficiently small and (1.3). This concludes that $v^j(y) > 0$ for all y in Ω and for all $j \geq j_0$, which completes the proof. □

Let u_0 be the critical point of I obtained by Theorem 2 and the critical level $-m$ defined in (1.9).

Lemma 1.13. *There exists $\rho > 0$ and $\delta > 0$ such that $\rho < 2\|u_0\|$,*

$$I(u) \geq \delta - m$$

for $u \in \partial B_\rho(u_0) \cap \mathcal{N}$.

Proof. First, let us recall that the Nehari manifold $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^0 = \{J^{-1}(0)\}$ is a closed subset in $H_0^1(\Omega)$. Furthermore, $I : \mathcal{N} \to \mathbb{R}$ is continuous and bounded from below, by the proof of Theorem 2 with $0 > I(u) \geq -m$.
Now, suppose by contradiction that for every fixed ρ with $0 < \rho < 2\|u_0\|$ there is a sequence $(u_n) \subset \mathcal{N} \cap \partial B_\rho(u_0)$ such that $I(u_n) \to -m = \inf_{u\in\mathcal{N}^+} I(u)$, as $n \to \infty$. Define $|\rho_j| = \frac{1}{j}$, so the sequence $(u_n^j) \subset \mathcal{N} \cap \partial B_{\rho_j}(u_0)$ satisfies $I(u_n^j) \to -m$, as $n \to \infty$. We can apply Ekeland's Variational Principle to $I|_{\mathcal{N}}$, where $\mathcal{N}$ is a closed metric space. Therefore, by Corollary 3 of [12], there is a sequence, for each fixed $j > 0$, $(v_n^j) \subset \mathcal{N} \cap \partial B_{\rho_j}(u_0)$ such that, if $n \to +\infty$

a) $I|_{\mathcal{N}}(v_n^j) \to -m$;

b) $\|v_n^j - u_n^j\| \to \rho_j$;

c) $\|I'|_{\mathcal{N}}(v_n^j)\| \to 0$.

This means that (v_n^j) is a (PS) sequence of $I|_{\mathcal{N}}$ the functional restricted to $\mathcal{N}^+$. Since $-m < 0$, by Lemma 1.7 (v_n^j) has a subsequence (PS) for I such that $I'(v_n^j) \to 0$ and by Lemma 1.11, up to a subsequence, we have that $v_n^j \to v^j$ if $n \to +\infty$. It follows from the continuity of I and the uniqueness of the limit that

$$I(v^j) = -m, \; I'(v^j) = 0, \; v^j \in \mathcal{N} \cap B_{\rho_j}(u_0)$$

and $\|v^j - u_0\|_{H_0^1(\Omega)} \to 0$. Taking $w^j = v^j - u_0$ and using regularity theory for elliptic operators, as in Section 1.3, we have that $\|v^j - u_0\|_{C^{1,\alpha}(\overline{\Omega})} \to 0$.

Since $\|v^j - u_0\| = \rho_j \to 0$, with $0 < \rho_j < 2\|u_0\|$ and $v^j > 0$ for j large enough, by Lemma 1.12, we have (v^j) is a sequence of positive critical points for I that converge to u_0 in the norm of $H_0^1(\Omega)$, which contradicts the uniqueness of the positive solution of I given by Theorem 2. □

Note that by the previous Lemma 1.13, we obtained the first geometry of the Mountain Pass Theorem around the minimum u_0. From now on, we will translate the functional I so that it behaves like the Mountain Pass Theorem on the Nehari manifold.

Consider the translated functional $\tilde{I} : H_0^1(\Omega) \to \mathbb{R}$,

$$\tilde{I}(u) := I(u) + m = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx + \int_\Omega b(x)F(u)dx + m.$$

Theorem 7. *Assume $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ and f satisfies $(f_1) - (f_4)$. Let $u_0 \geq 0$ and $-u_0 \leq 0$ be local minima of I on $\mathcal{N}^+$, then*

(i) $\tilde{I}(u_0) = 0 = \tilde{I}(-u_0)$;

(ii) there exists $0 < \rho < 2\|u_0\|$ and $\delta > 0$ such that $\tilde{I}(u) \geq \delta > 0$ for any $u \in \partial B_\rho(u_0) \cap \mathcal{N}$.

Moreover, $\tilde{I}$ satisfies $(PS)_c$ condition with

$$0 < c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \tilde{I}(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], \mathcal{N}) : \gamma(0) = u_0, \gamma(1) = -u_0\}$. Then there exists a non-trivial solution u^* of problem (P) satisfying $I(u^*) = c^* > -m$, where $c^* = c - m$.

Proof. (i) $\tilde{I}(u_0) = I(u_0) + m = -m + m = 0$ and $\tilde{I}(-u_0) = I(-u_0) + m = I(u_0) + m = 0$. (ii) By Lemma 1.13 there exists $\rho > 0$, $\rho < \|u_0 - (-u_0)\| = 2\|u_0\|$, and $\delta > 0$ such that

$$I(u) \geq \delta - m$$

for $u \in \partial B_\rho(u_0) \cap \mathcal{N}$. By the definition of the functional $\tilde{I}$ we have

$$\tilde{I}(u) = I(u) + m \geq \delta - m + m = \delta > 0$$

for $u \in \partial B_\rho(u_0) \cap \mathcal{N}$, and item (ii) is verified.

Therefore, $\tilde{I}$ satisfies the geometry of the Mountain Pass Theorem, and so the same is true for I. Let us evoque Ghoussoub's Theorem [[14], Theorem 3.2]. Note that the Nehari manifold is a Finsler variety because it is a closed submanifold of class C^1, with $\mathcal{T}_u\mathcal{N}$ carying the norm induced by the inclusion $\mathcal{T}_u\mathcal{N} \subset \mathcal{T}_u H_0^1(\Omega) \cong H_0^1(\Omega)$ by [[26], Chapter II, Section 3.7]. We also have that the set $\mathcal{F} = \Gamma$ is a homotopically stable family. In fact, making $X = \mathcal{N}$ which is a complete metric space, then $B = \{-u_0, u_0\}$ is a closed subset in $\mathcal{N}$. Since $\gamma(0) = u_0$ and $\gamma(1) = -u_0$, we have that any element $\gamma([0,1])$ in Γ contains B. Furthermore, for all $A = \gamma([0,1]) \in \Gamma$ and $\eta : [0,1] \times \mathcal{N} \to \mathcal{N}$ continuous, satisfying $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times \mathcal{N}) \cup ([0,1] \times B)$ then $\gamma \circ \eta(1) = -u_0 \in B$. Moreover, by item (ii),

$$0 \geq c^* = c - m > \{I(\gamma(0)),\ I(\gamma(1))\},$$

and thus satisfies the hypothesis (F_0) of Ghoussoub's Theorem, and hence there is a $(PS)_{c^*}$ sequence (u_n) for I restricted to $\mathcal{N}^+$. By Lemma 1.7, (u_n) is a $(PS)_{c^*}$ sequence for the functional I in $H_0^1(\Omega)$ and by Lemma 1.11, up to a subsequence, $u_n \to u^* \in \mathcal{N}$, giving

$$I(u^*) = c^* \text{ and } I'(u^*) = 0.$$

Therefore, u^* is a critical point of the functional I on $\mathcal{N}$ and $-m < c^* \leq 0$. Since the Nehari manifold $\mathcal{N}$ is a natural constraint, then u^* is a solution of problem (P) in $H_0^1(\Omega)$. □

Note that $u^* \in \mathcal{N}$ may be the trivial solution. The next theorem gives a sufficient condition for solution u^* not to be null.

Proof of Theorem 3: First, we want to show that $I(u^*) = c^* < 0$ which gives u^* is not trivial. Let us consider the first positive eigenfunction, normalized in $H_0^1(\Omega)$ and denoted by ϕ_1 associated with the first eigenvalue $\lambda_1(\Omega)$ of the problem $-\Delta$ in Ω, consider also a normalized eigenfunction ϕ_2 associated with the second eigenvalue $\lambda_2(\Omega)$, ϕ_1^0 the first positive eigenfunction (normalized to $H_0^1(\Omega_0)$) associated with the first eigenvalue $\lambda_1(\Omega_0)$ and ϕ_2^0 a normalized eigenfunction associated with the second eigenvalue $\lambda_2(\Omega_0)$ of problem $-\Delta$ in Ω_0, each with Dirichlet boundary condition. Note that the supports of $\phi_i^0,\ i = 1, 2$ are subsets of Ω_0.

We claim that

$$\int_\Omega \phi_1\phi_2 dx = 0 \tag{1.6}$$

and

$$\int_{\Omega_0} \phi_1^0\phi_2^0 dx = 0. \tag{1.7}$$

Indeed, it follows from the spectral theory that, $\int_\Omega \nabla\phi_1 \cdot \nabla\phi_2 dx = 0$ and $\int_{\Omega_0} \nabla\phi_1^0 \cdot \nabla\phi_2^0 dx = 0$. Since the eigenfunctions are regular functions, then by the Divergence Theorem, we have

$$0 = \int_\Omega \nabla\phi_1\nabla\phi_2 dx = -\int_\Omega \Delta\phi_1\phi_2 dx$$

which implies

$$\int_\Omega \lambda_1\phi_1\phi_2 dx = 0.$$

Since $\lambda_1 \neq 0$, we have

$$\int_\Omega \phi_1\phi_2 dx = 0.$$

Similarly, using the Divergence Theorem, it holds

$$\int_{\Omega_0} \phi_1^0\phi_2^0 dx = 0.$$

In order to construct a convenient path in Γ not passing through zero, define $w \in H_0^1(\Omega)$ by $w := t_1(\phi_1 + \varepsilon\phi_1^0) + t_2(\phi_2 + \varepsilon\phi_2^0)$ with constants $t_1, t_2 > 0$ and for some $\varepsilon > 0$, to be chosen sufficiently small. Using equalities (1.6), (1.7) and also the hypothesis (f_4) we obtain

$$
\begin{aligned}
I(w) &= \frac{1}{2}\int_\Omega(|\nabla w|^2-\lambda w^2)dx+\int_\Omega b(x)F(w)dx\\
&= \frac{1}{2}\int_\Omega[|\nabla(t_1(\phi_1+\varepsilon\phi_1^0)+t_2(\phi_2+\varepsilon\phi_2^0))|^2-\lambda(t_1(\phi_1+\varepsilon\phi_1^0)+t_2(\phi_2+\varepsilon\phi_2^0))^2]dx\\
&+\int_\Omega b(x)F(w)dx\\
&= \frac{t_1^2}{2}\int_\Omega[\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\phi_1+\varepsilon\phi_1^0)-\lambda(\phi_1+\varepsilon\phi_1^0)(\phi_1+\varepsilon\phi_1^0)]dx\\
&+\frac{t_1t_2}{2}\int_\Omega[\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\phi_2+\varepsilon\phi_2^0)-\lambda(\phi_1+\varepsilon\phi_1^0)(\phi_2+\varepsilon\phi_2^0)]dx\\
&+\frac{t_2t_1}{2}\int_\Omega\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\phi_1+\varepsilon\phi_1^0)-\lambda(\phi_2+\varepsilon\phi_2^0)(\phi_1+\varepsilon\phi_1^0)]dx\\
&+\frac{t_2^2}{2}\int_\Omega[\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\phi_2+\varepsilon\phi_2^0)-\lambda(\phi_2+\varepsilon\phi_2^0)(\phi_2+\varepsilon\phi_2^0)]dx\\
&+\int_\Omega b(x)F(w)dx\\
&= \frac{t_1^2}{2}\int_\Omega[\nabla(\phi_1+\varepsilon\phi_1^0)\nabla\phi_1-\lambda(\phi_1+\varepsilon\phi_1^0)\phi_1]dx+\frac{t_1t_2}{2}\int_\Omega[\nabla(\phi_1+\varepsilon\phi_1^0)\nabla\phi_2\\
&-\lambda(\phi_1+\varepsilon\phi_1^0)\phi_2]dx+\frac{t_2t_1}{2}\int_\Omega[\nabla(\phi_2+\varepsilon\phi_2^0)\nabla\phi_1-\lambda(\phi_2+\varepsilon\phi_2^0)\phi_1]dx\\
&+\frac{t_2^2}{2}\int_\Omega[\nabla(\phi_2+\varepsilon\phi_2^0)\nabla\phi_2-\lambda(\phi_2+\varepsilon\phi_2^0)\phi_2]dx+\frac{t_1t_1}{2}\int_\Omega[\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\varepsilon\phi_1^0)\\
&-\lambda(\phi_1+\varepsilon\phi_1^0)(\varepsilon\phi_1^0)]dx+\frac{t_1t_2}{2}\int_\Omega[\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\varepsilon\phi_2^0)-\lambda(\phi_1+\varepsilon\phi_1^0)(\varepsilon\phi_2^0)]dx\\
&+\frac{t_2t_1}{2}\int_\Omega[\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\varepsilon\phi_1^0)-\lambda(\phi_2+\varepsilon\phi_2^0)(\varepsilon\phi_1^0)]dx+\frac{t_2t_2}{2}\int_\Omega[\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\varepsilon\phi_2^0)\\
&-\lambda(\phi_2+\varepsilon\phi_2^0)(\varepsilon\phi_2^0)]dx+\int_\Omega b(x)F(w)dx\\
&= \frac{t_1^2}{2}\int_\Omega(|\nabla\phi_1|^2-\lambda\phi_1^2)dx+\frac{t_1^2}{2}\varepsilon\int_\Omega(\nabla\phi_1^0\nabla\phi_1-\lambda\phi_1^0\phi_1)dx\\
&+\frac{t_1t_2}{2}\varepsilon\int_\Omega(\nabla\phi_1^0\nabla\phi_2-\lambda\phi_1^0\phi_2)dx+\frac{t_2t_1}{2}\varepsilon\int_\Omega(\nabla\phi_2^0\nabla\phi_1\\
&-\lambda\phi_2^0\phi_1)dx+\frac{t_2^2}{2}\int_\Omega(|\nabla\phi_2|^2-\lambda\phi_2^2)dx+\frac{t_2^2}{2}\varepsilon\int_\Omega(\nabla\phi_2^0\nabla\phi_2-\lambda\phi_2^0\phi_2)dx\\
&+\frac{t_1^2}{2}\varepsilon\int_\Omega[\nabla\phi_1\nabla\phi_1^0-\lambda\phi_1\phi_1^0]dx+\frac{t_1^2}{2}\varepsilon^2\int_\Omega[|\nabla\phi_1^0|^2-\lambda(\phi_1^0)^2]dx\\
&+\frac{t_1t_2}{2}\varepsilon\int_\Omega[\nabla\phi_1\nabla\phi_2^0-\lambda\phi_1\phi_2^0]dx+\frac{t_2t_1}{2}\varepsilon\int_\Omega[\nabla\phi_2\nabla\phi_1^0\\
&-\lambda\phi_2\phi_1^0]dx+\frac{t_2^2}{2}\varepsilon\int_\Omega[\nabla\phi_2\nabla\phi_2^0-\lambda\phi_2\phi_2^0]dx\\
&+\frac{t_2}{2}\varepsilon^2\int_\Omega[|\nabla\phi_2^0|^2-\lambda(\phi_2^0)^2]dx+\int_\Omega b(x)F(w)dx\\
&\le \frac{t_1^2}{2}(\lambda_1(\Omega)-\lambda)\int_\Omega\phi_1^2dx+\frac{t_2^2}{2}(\lambda_2(\Omega)-\lambda)\int_\Omega\phi_2^2dx+\frac{t_1^2}{2}\varepsilon^2(\lambda_1(\Omega_0)-\lambda)\int_\Omega(\phi_1^0)^2dx\\
&+\frac{t_2^2}{2}\varepsilon^2(\lambda_2(\Omega_0)-\lambda)\int_\Omega(\phi_2^0)^2dx+\varepsilon\sum_{i=1}^2\sum_{j=1}^2 t_it_j\left\{\int_\Omega(\nabla\phi_i\nabla\phi_j^0-\lambda\phi_i\phi_j^0)dx\right\}\\
&+\int_\Omega b(x)F(w)dx.
\end{aligned}
$$

Note that,

$$
\begin{aligned}
\|w\|^2 &= \int_\Omega |\nabla(t_1(\phi_1+\varepsilon\phi_1^0)+t_2(\phi_2+\varepsilon\phi_2^0))|^2dx\\
&= t_1^2\int_\Omega|\nabla\phi_1|^2dx + t_1^2\varepsilon^2\int_\Omega|\nabla\phi_1^0|^2dx + t_2^2\int_\Omega|\nabla\phi_2|^2dx + t_2^2\varepsilon^2\int_\Omega|\nabla\phi_2^0|^2dx\\
&+ \varepsilon\sum_{i=1}^2\sum_{j=1}^2 t_it_j\int_\Omega\nabla\phi_i\nabla\phi_j^0dx\\
&\le t_1^2+t_2^2+t_1^2\varepsilon^2+t_2^2\varepsilon^2+\varepsilon(t_1^2+2t_1t_2+t_2^2)\|\nabla\phi_i\|_2^2\|\nabla\phi_j\|_2^2\\
&= t_1^2+t_2^2+t_1^2\varepsilon^2+t_2^2\varepsilon^2+\varepsilon(t_1^2+2t_1t_2+t_2^2)\\
&\le t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2)+2\varepsilon(t_1^2+t_2^2)\le t_1^2+t_2^2+3\varepsilon(t_1^2+t_2^2). \qquad (1.8)
\end{aligned}
$$

Furthermore, using (1.8) and Remark 1.5, considering the same real number $\varepsilon>0$ in the definition of the function w, we have

$$
\begin{aligned}
\int_\Omega b(x)F(w)dx &\le \int_\Omega b(x)\left(\frac{\varepsilon}{2}|w|^2+\frac{C_2}{q}|w|^q\right)dx\\
&\le \|b\|_\infty\frac{\varepsilon}{2}\int_\Omega|w|^2dx+\|b\|_\infty\frac{C_2}{q}\int_\Omega|w|^qdx\\
&\le C\left[\varepsilon\|w\|^2+C\|w\|^q\right]\\
&\le C\left[\varepsilon(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2))+(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}}\right].
\end{aligned}
$$

Using Holder's inequality for $p=q=2$ and that $\phi_i,\ \phi_j^0$ are normalized eigenfunctions we have

$$
\begin{aligned}
\varepsilon\sum_{i=1}^2\sum_{j=1}^2 t_it_j\left\{\int_\Omega(\nabla\phi_i\nabla\phi_j^0-\lambda\phi_i\phi_j^0)dx\right\} &\le \varepsilon\sum_{i=1}^2\sum_{j=1}^2 t_it_j\left\{\int_\Omega|\nabla\phi_i\nabla\phi_j^0-\lambda\phi_i\phi_j^0|dx\right\}\\
&\le \varepsilon\sum_{i=1}^2\sum_{j=1}^2 t_it_j\left\{\int_\Omega(|\nabla\phi_i\nabla\phi_j^0|+\lambda|\phi_i\phi_j^0|)dx\right\}\\
&\le \varepsilon(t_1^2+2t_1t_2+t_2^2)(\|\nabla\phi_1\|_2\|\nabla\phi_j^0\|_2+\lambda\|\phi_i\|_2\|\phi_j\|_2)\\
&\le 2\varepsilon(t_1^2+t_2^2)(1+C\lambda)\\
&= C\varepsilon(t_1^2+t_2^2).
\end{aligned}
$$

Taking, $0<\varepsilon<1$ and $0<t_1^2+t_2^2<1$, since $\frac{q}{2}>1$, its follows

$$
\begin{aligned}
I(w) \leq{}& \frac{(t_1^2+t_2^2)}{2} C \max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} + \frac{(t_1^2+t_2^2)}{2}\varepsilon^2 C \max\{\lambda_1(\Omega_0)-\lambda, \lambda_2(\Omega_0)-\lambda\} \\
&+ \varepsilon C(t_1^2+t_2^2) + C\Big[\varepsilon\Big(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2)\Big) + \Big(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2)\Big)^{\frac{q}{2}}\Big] \\
\leq{}& \frac{(t_1^2+t_2^2)}{2} C \max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} + \frac{(t_1^2+t_2^2)}{2}\varepsilon^2 C \max\{\lambda_1(\Omega_0)-\lambda, \lambda_2(\Omega_0)-\lambda\} \\
&+ C\varepsilon(t_1^2+t_2^2) + C\varepsilon(t_1^2+t_2^2) + C\varepsilon^2(t_1^2+t_2^2) + C2^{\frac{q}{2}-1}(t_1^2+t_2^2)^{\frac{q}{2}} + C2^{\frac{q}{2}-1}(\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}} \\
\leq{}& \frac{(t_1^2+t_2^2)}{2} C \max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} + \frac{(t_1^2+t_2^2)}{2}\varepsilon^2 C \max\{\lambda_1(\Omega_0)-\lambda, \lambda_2(\Omega_0)-\lambda\} \\
&+ C\varepsilon(t_1^2+t_2^2) + C\varepsilon^2(t_1^2+t_2^2) + C(t_1^2+t_2^2)^{\frac{q}{2}} + C(\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}} \\
\leq{}& \frac{(t_1^2+t_2^2)}{2} C \max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} + O(\varepsilon(t_1^2+t_2^2)).
\end{aligned}
$$

Without loss of generality, we can assume $\max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} = \lambda_1(\Omega)-\lambda < 0$, thus

$$
\begin{aligned}
I(w) &\leq \frac{(t_1^2+t_2^2)}{2} C(\lambda_1(\Omega)-\lambda) + C\varepsilon(t_1^2+t_2^2) \\
&= C(t_1^2+t_2^2)\left[\frac{\lambda_1(\Omega)-\lambda}{2} + C\varepsilon\right].
\end{aligned}
$$

Taking $0 < \varepsilon \leq \frac{\lambda-\lambda_1(\Omega)}{4}$, then

$$
I(w) \leq C(t_1^2+t_2^2)\left(\frac{(\lambda_1(\Omega)-\lambda)}{4}\right) := -\delta_1 < 0.
$$

Now, let $w_1 := t_1(\phi_1+\varepsilon\phi_1^0)$, $w_2 := t_2(\phi_2+\varepsilon\phi_2^0)$ and $w_\theta := \cos(\theta)w_1 + \sin(\theta)w_2$, such that

$$
\begin{aligned}
w_{\frac{\pi}{4}} &= \cos\Big(\frac{\pi}{4}\Big)(t_1(\phi_1+\varepsilon\phi_1^0)) + \sin\Big(\frac{\pi}{4}\Big)(t_2(\phi_2+\varepsilon\phi_2^0)) \\
&= \frac{\sqrt{2}}{2}\left[t_1(\phi_1+\varepsilon\phi_1^0) + t_2(\phi_2+\varepsilon\phi_2^0)\right] \\
&= \frac{\sqrt{2}}{2} w
\end{aligned}
$$

with $\|w_{\frac{\pi}{4}}\| = \left\|\frac{\sqrt{2}}{2}w\right\| = \frac{\sqrt{2}}{2}\|w\|$, and for all $\theta \in [0,\pi]$,

$$
\begin{aligned}
I(w_\theta) &\leq C\frac{t_1^2}{2}\cos^2(\theta)(\lambda_1(\Omega)-\lambda) + C\frac{t_2^2}{2}\sin^2(\theta)(\lambda_2(\Omega)-\lambda) \\
&+ C\varepsilon^2\frac{t_1^2}{2}\cos^2(\theta)(\lambda_1(\Omega_0)-\lambda) + C\varepsilon^2\frac{t_2^2}{2}\sin^2(\theta)(\lambda_2(\Omega_0)-\lambda) \\
&+ C(\varepsilon\|w_\theta\|^2 + \|w_\theta\|^q) \\
&\leq C(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)\{\max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} \\
&+ O(\varepsilon(t_1^2+t_2^2)) \\
&\leq C(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)\Bigg[\max\{\lambda_1(\Omega)-\lambda, \lambda_2(\Omega)-\lambda\} \\
&+ \frac{C(\varepsilon(t_1^2+t_2^2))}{\cos^2(\theta)t_1^2+\sin^2(\theta)t_2^2}\Bigg],
\end{aligned}
$$

and because $(t_1^2+t_2^2)/(\cos^2(\theta)t_1^2+\sin^2(\theta)t_2^2)$ is bounded by positive constants from below and above, uniformly for $\theta\in[0,\pi]$, then similarly to the calculations for $I(w)$, we obtain that there exists $\delta_2>0$ such that

$$I(w_\theta)<-\delta_2<0,$$

for all $\theta\in[0,\pi]$. Finally, define the following curve in $H_0^1(\Omega)$ given by

$$\gamma(s):=\begin{cases}[(1-3s)u_0+3s(w_1)], & s\in[0,1/3]\\ w_{\theta(s)}, \quad s\in[1/3,2/3] & \text{and} \quad \theta(s)=3(s-1/3)\pi\\ [3(1-s)(-w_1)+3(s-2/3)(-u_0)], & s\in[2/3,1].\end{cases}$$

Now, let us show that for $0\le s\le 1$ there exists $t\gamma(s)$ such that $\tilde{\gamma}(s):=t_{\gamma(s)}\gamma\in\mathcal{N}^+$. Note that $\int_\Omega(|\nabla\gamma(s)|^2-\lambda\gamma(s)^2)dx<0$ when $s\in[0,1/3]$, $s\in[1/3,2/3]$ and $s\in[2/3,1]$. In fact, for $s\in[0,1/3]$, we have

$$\begin{aligned}\int_\Omega\Big(|\nabla\gamma(s)|^2 &- \lambda\gamma(s)^2\Big)dx=\int_\Omega\Big(|\nabla((1-3s)u_0+3s(w_1))|^2-\lambda((1-3s)u_0+3s(w_1))^2\Big)dx\\ &= (1-3s)^2\int_\Omega\Big(|\nabla u_0|^2-\lambda u_0^2\Big)dx+2(1-3s)(3s)\int_\Omega\Big(\nabla u_0\nabla w_1-\lambda u_0w_1\Big)dx\\ &+ (3s)^2\int_\Omega(|\nabla w_1|^2-\lambda(w_1)^2\Big)dx.\end{aligned}$$

Notice that

$$\begin{aligned}\int_\Omega\Big(|\nabla w_1|^2-\lambda w_1^2\Big)dx &= \int_\Omega\Big(|\nabla(t_1(\phi_1+\varepsilon\phi_1^0))|^2-\lambda(t_1(\phi_1+\varepsilon\phi_1^0))^2\Big)dx\\ &= t_1^2\int_\Omega\Big(|\nabla\phi_1|^2-\lambda\phi_1^2\Big)dx+2t_1^2\varepsilon\int_\Omega\Big(\nabla\phi_1\nabla\phi_1^0-\lambda\phi_1\phi_1^0\Big)dx\\ &+ t_1^2\varepsilon^2\int_\Omega\Big(|\nabla\phi_1^0|^2-\varepsilon(\phi_1^0)^2\Big)dx\\ &\le Ct_1^2(\lambda_1(\Omega)-\lambda)+Ct_1^2\varepsilon^2(\lambda_1(\Omega_0)-\lambda)+Ct_1^2\varepsilon\\ &\le Ct_1^2(\lambda_1(\Omega)-\lambda)+O(\varepsilon t_1^2)<0. \end{aligned}\tag{1.9}$$

On the other hand, since u_0 is a positive solution to the problem (P), $b(x)\ge 0$, f is continuous and $w_1>0$, because, $\phi_1,\phi_1^0>0$, using the weak formulation, we have

$$\int_\Omega\Big(\nabla u_0\nabla w_1-\lambda u_0w_1\Big)dx=-\int_\Omega b(x)f(u_0)w_1dx<0.\tag{1.10}$$

and finally,

$$\int_\Omega\Big(|\nabla u_0|^2-\lambda u_0^2\Big)dx=-\int_\Omega b(x)f(u_0)u_0dx<0.\tag{1.11}$$

Therefore, from inequalities (1.9), (1.10) and (1.11), for $s\in[0,1/3]$,

$$\int_\Omega\Big(|\nabla\gamma(s)|^2-\lambda\gamma(s)^2\Big)dx<0.$$

Let $s \in [1/3, 2/3]$, we have

$$\begin{aligned}
\int_\Omega \Big(|\nabla\gamma(s)|^2 - \lambda\gamma(s)^2\Big)dx &= \int_\Omega \Big(|\nabla w_\theta|^2 - \lambda(w_\theta)^2\Big)dx \\
&= \int_\Omega \Big(|\nabla(\cos(\theta)w_1 + \sin(\theta)w_2)|^2 - \lambda(|\nabla(\cos(\theta)w_1 + \sin(\theta)w_2)\Big)dx \\
&= \cos^2(\theta)\int_\Omega \Big(|\nabla w_1|^2 - \lambda w_1^2\Big)dx + 2\cos(\theta)\sin(\theta)\int_\Omega \Big(\nabla w_1\nabla w_2 - \lambda w_1 w_2\Big)dx \\
&+ \sin^2(\theta)\int_\Omega \Big(|\nabla w_2|^2 - \lambda w_2^2\Big)dx
\end{aligned}$$

Using the Hölder inequality

$$\begin{aligned}
\int_\Omega \Big(\nabla w_1\nabla w_2 - \lambda w_1 w_2\Big)dx &= \int_\Omega \Big(\nabla(t_1(\phi_1 + \varepsilon\phi_1^0))\nabla(t_2(\phi_2 + \varepsilon\phi_2^0)) - \lambda(t_1(\phi_1 + \varepsilon\phi_1^0))(\phi_2 + \varepsilon\phi_2^0))\Big)dx \\
&= t_1t_2\int_\Omega (\nabla\phi_1\nabla\phi_2 - \phi_1\phi_2)dx + t_1t_2\varepsilon\int_\Omega (\nabla\phi_1\nabla\phi_1^0 - \lambda\phi_1\phi_1^0)dx \\
&+ t_1t_2\varepsilon\int_\Omega (\nabla\phi_1^0\nabla\phi_2 - \lambda\phi_1^0\phi_2)dx + t_1t_2\varepsilon^2\int_\Omega (\nabla\phi_1^0\phi_2^0 - \lambda\phi_1^0\phi_2^0)dx \\
&\leq t_1t_2\varepsilon(\|\nabla\phi_1\|_2\|\nabla\phi_2^0\|_2 - \lambda\|\phi_1\|_2\|\phi_2^0\|_2) \\
&+ t_1t_2\varepsilon(\|\nabla\phi_1\|_2\|\nabla\phi_2^0\|_2 - \lambda\|\phi_1\|_2\|\phi_2^0\|_2) = Ct_1t_2\varepsilon \qquad (1.12)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \Big(|\nabla w_2|^2 \;-\; \lambda w_2^2\Big)dx &= \int_\Omega \Big(|\nabla(t_2(\phi_2 + \varepsilon\phi_2^0))|^2 - \lambda(t_2(\phi_2 + \varepsilon\phi_2^0))^2\Big)dx \\
&= t_2^2\int_\Omega \Big(|\nabla\phi_2|^2 - \lambda\phi_2^2\Big)dx + 2t_2^2\varepsilon\int_\Omega \Big(\nabla\phi_2\nabla\phi_2^0 - \lambda\phi_2\phi_2^0\Big)dx \\
&+ t_2^2\varepsilon^2\int_\Omega \Big(|\nabla\phi_2^0|^2 - \varepsilon(\phi_2^0)^2\Big)dx \\
&\leq Ct_2^2(\lambda_2(\Omega) - \lambda) + Ct_2^2\varepsilon^2\lambda_2(\Omega_0) - \lambda) + Ct_2^2\varepsilon \\
&\leq Ct_2^2(\lambda_2(\Omega) - \lambda) + O(\varepsilon t_2^2) < 0. \qquad (1.13)
\end{aligned}$$

Using (1.9), (1.12) and (1.13) we obtain

$$\begin{aligned}
\int_\Omega \Big(|\nabla\gamma(s)|^2 - \lambda\gamma(s)^2\Big)dx &= \cos^2(\theta)Ct_1^2(\lambda_1(\Omega) - \lambda) + O(\varepsilon t_1^2) \\
&+ 2\cos^2(\theta)\sin(\theta)C\varepsilon t_1t_2 + \sin^2(\theta)Ct_2^2(\lambda_2(\Omega) - \lambda) + O(\varepsilon t_2^2) \\
&\leq C(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)\max\{\lambda_1(\Omega) - \lambda, \lambda_2(\Omega) - \lambda\} \\
&+ O(\varepsilon(t_1^2 + t_2^2)) \\
&\leq C(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)\Bigg\{\max\{\lambda_1(\Omega) - \lambda, \lambda_2(\Omega) - \lambda\} \\
&+ \frac{C(\varepsilon(t_1^2 + t_2^2))}{\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2}\Bigg\} < 0,
\end{aligned}$$

because $(t_1^2 + t_2^2)/(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)$ is bounded by positive constants from below and above, uniformly for $\theta \in [0, \pi]$ and t_1 and t_2 positive, and $\varepsilon > 0$ is sufficiently small.

Let $s \in [2/3, 1]$, using the equalities in (1.9), (1.10) and (1.11), we have

$$\begin{aligned}
\int_\Omega (|\nabla \gamma(s)|^2 &- \lambda(\gamma(s))^2) dx = \int_\Omega \Big(|\nabla (3(1-s)(-w_1) + 3(s-2/3)(-u_0))|^2 \\
&- \lambda (3(1-s)(-w_1) + 3(s-2/3)(-u_0))^2 \Big) dx \\
&= (3(1-s)^2 \int_\Omega \Big(|\nabla w_1|^2 - \lambda w_1^2 \Big) dx + (3(s-2/3))^2 \int_\Omega \Big(|\nabla u_0|^2 - \lambda u_0^2 \Big) dx \\
&+ 6(1-s)3(s-\frac{2}{3}) \int_\Omega \Big(\nabla w_1 \nabla u_0 - \lambda w_1 u_0 \Big) dx \\
&< 0.
\end{aligned}$$

Recalling Lemma 1.10, we have that $t_{\gamma(s)}$ is continuous and then $\tilde{\gamma}(s) \in \mathcal{N}^+$ for all $s \in [0,1]$. From Lemma 1.4 we conclude that $I(\tilde{\gamma}(s)) < 0$ for all $s \in [0,1]$. Thus, $I(\tilde{\gamma}(s)) \leq \max_{0 \leq s \leq 1} I(\tilde{\gamma}(s)) < 0$ for $s \in [0,1]$. By the definition of the min-max level c, it follows that $I(u^*) = c^* < \max_{0 \leq s \leq 1} I(\tilde{\gamma}(s)) < 0$.

Finally, suppose by contradiction that u^* is non-trivial and non-negative. Then the set $\tilde{\Omega} \subset \Omega$ in which $u^* = 0$ is bounded, and the set of boundary points $\partial\tilde{\Omega} \subset \Omega$ is bounded. Let $x_0 \in \partial\tilde{\Omega}$ be such that $u^*(x_0) = 0$. Furthermore, since $u^* \in C^1$ (see section 1.1), then $\partial\tilde{\Omega}$ is regular and compact.

Given $\delta > 0$ sufficiently small, there exists $\delta_1 > 0$ such that, if $x \in N_{\delta_1} := \{x \in \Omega \backslash \tilde{\Omega} : dist(x, \partial\tilde{\Omega}) < \delta_1\}$, then $|u^*(x)| < \delta$. It follows from hypothesis (f_1) that $f(u^*(x)) = o(|u^*(x)|)u^*(x)$ for all $x \in N_{\delta_1}$, and since $u^* \geq 0$ by assumption, then $-\Delta u^*(x) = \lambda u^*(x) - o(|u^*(x)|) \geq 0$, for all $x \in N_{\delta_1}$. Moreover $u^*(x) > 0$ for all $int(N_{\delta_1})$, thus $\inf_{N_{\delta_1}} u^*(x) = 0$.

Note that N_{δ_1} is regular because the subset $\partial\tilde{\Omega}$ is regular and u^* is continuous in $\overline{N}_{\delta_1}$. Then, by Hopf lemma $\frac{\partial u^*}{\partial \nu_x}(x) > 0$, for all x such that $u^*(x) \in \partial\tilde{\Omega}$, and ν_x is the exterior normal vector to $\partial\tilde{\Omega}$ at x, namely $Du^*(x) \neq 0$, which is impossible in an interior minimum point of u^*. Thus, $u^* > 0$ which is impossible by the uniqueness of the positive solution. The same result we obtain when u^* is non-positive. Therefore, u^* changes sign, which concludes the proof. $\square$

Chapter 2

Asymptotically Linear Problem

In this chapter, we are interested in investigating the existence of a positive solution and a sign-changing solution for the problem (P) with f being asymptotically linear. We know about few works in the literature on the logistic problem with this nonlinearity behavior. In this regard, we are able to prove that such a problem has a nontrivial solution and a sign-changing solution in $H_0^1(\Omega)$. Following the ideas from the work by Brown and Zhang [4], the search for a solution began initially as we studied the properties of the Nehari manifold associated with the functional. We split the Nehari manifold into disjoint subsets and observed that under certain conditions, it is possible to show that $\mathcal{N}^0$ is a null set, $\mathcal{N}^-$ is an empty set, and $\mathcal{N}^+$ is a non-empty and bounded set. Having done that, we minimized the functional I over $\mathcal{N}^+$ and obtained a critical point. Furthermore, we show that the critical point is nontrivial, meaning that the solution to the problem (P) is positive and classical. By employing techniques similar to those in the renowned article by Brezis and Oswald [3], we were able to establish the uniqueness of the solution. Upon obtaining the existence of a solution, we questioned whether it was possible to demonstrate the existence of a sign-changing solution for this problem. Faced with this inquiry, we relied primarily on the work of Fernandes and Maia [13]. Finally, through the Mountain Pass Theorem constrained to the Nehari manifold, we obtained a nontrivial critical point. By once again employing the ideas from [13], we showed that the critical point is a sign-changing solution.

Here we consider the problem

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P'}$$

where $\Omega \subset \mathbb{R}^N$ is bounded open with $\partial\Omega$ smooth, λ is a positive real parameter, $a, b : \overline{\Omega} \to \mathbb{R}$ are functions in $L^\infty(\Omega)$ such that $a(x) \geq \underline{a} > 0$ a.e. in Ω and $b(x)$ is non-negative and $b(x) = 0$ in a connected subset $\Omega_0 \subset\subset \Omega$ with positive Lebesgue measure and smooth boundary, in other words, $\partial\Omega_0 \in C^2$. The f function of class $C^1(\mathbb{R})$ is odd and satisfies the conditions $(f_1), (f_3), (f_4)$, $(f_2)'$ and $(f_5)'$ mentioned in the introduction.

Example 2.1. *The function* $f(s) = \frac{s^3}{1+s^2}$ *for* $s > 0$ *and* $l_\infty = 1$ *satisfies the hypotheses above.*

Indeed,

(f_1) $\lim_{s\to 0} \frac{f(s)}{s} = \lim_{s\to 0} \frac{s^2}{1+s^2} = 0.$

$(f_2)'$ $\lim_{s\to \infty} \frac{f(s)}{s} = \lim_{s\to \infty} \frac{s^2}{1+s^2} = 1.$

(f_3) $\frac{f(s)}{s} - f'(s) = \frac{s^2}{1+s^2} - \frac{s^4+3s^2}{(1+s^2)^2} = \frac{s^2(1+s^2)-s^4-3s^2}{(1+s^2)^2} = -\frac{2s^2}{(1+s^2)^2} < 0$ implies that $\frac{f(s)}{s} - f'(s) < 0$ for $s > 0$.

(f_4) Note that

$$f'(s) = \frac{3s^2(1+s^2) - s^3(2s)}{(1+s^2)^2} = \frac{3s^2+3s^2-2s^4}{(1+s^2)^2} = \frac{3s^2+s^4}{(1+s^2)^2}$$

and by L'Hôpital theorem

$$\lim_{s\to\infty} f'(s) = \lim_{s\to\infty} \frac{6s+4s^3}{4s+4s^3} = \lim_{s\to\infty} \frac{6+12s^2}{4+12s^2} = 1.$$

Then, given $\varepsilon > 0$, there is R_ε such that for all $s > R_\varepsilon$, we have

$$|f'(s)| < 1 + \varepsilon.$$

Note that fixed $0 < q-2 < 2^* - 2$, there is R_2 such that $1+\varepsilon < |s|^{q-2}$. Taking $R = \max\{R_\varepsilon, R_2\}$ we have for all $s > R$

$$|f'(s)| < |s|^{q-2}. \tag{2.1}$$

Since f is $C^1(\mathbb{R})$, we have for $0 \le s \le R$ that for $C > 0$

$$|f'(s)| \le C. \tag{2.2}$$

By (2.1) and (2.2), we conclude that

$$|f'(s)| < C + |s|^{q-2}.$$

$(f_5)'$ For $F(s) = \int_0^s \frac{t^3}{1+t^2} dt = \frac{1}{2}(s^2 - ln(s^2+1))$. We set $G(s) = f(s)s - 2F(s)$, that is,

$$\begin{aligned} G(s) &= \frac{s^4}{1+s^2} - s^2 + ln(1+s^2) \\ &= \frac{s^4 - s^2 + ln(1+s^2) - s^4 + s^2 ln(1+s)}{1+s^2} \\ &= \frac{ln(1+s^2)(1+s^2) - s^2}{1+s^2} \\ &= ln(1+s^2) - \frac{s^2}{1+s^2}. \end{aligned}$$

We have that $G(0) = 0$ and for $s > 0$

$$\begin{aligned} G'(s) &= \frac{2s}{1+s^2} - \left[\frac{2s(1+s^2) - s^2 2s}{(1+s^2)^2}\right] \\ &= \frac{2s}{1+s^2} - \left[\frac{2s}{(1+s^2)^2}\right] \\ &= \frac{2s}{1+s^2}\left[1 - \frac{1}{1+s^2}\right] \\ &= \frac{2s}{1+s^2}\left[\frac{s^2}{1+s^2}\right] \\ &= \frac{2s^3}{(1+s^2)^2} > 0. \end{aligned}$$

Then, $G(s) > 0$ para $s > 0$. In addition,

$$\begin{aligned}\lim_{s\to\infty} G(s) &= \lim_{s\to\infty} ln(1+s^2) - \frac{s^2}{1+s^2} \\ &= \lim_{s\to\infty} ln(1+s^2) - \lim_{s\to\infty} \frac{s^2}{1+s^2} \\ &= \lim_{s\to\infty} ln(1+s^2) - 1 \\ &= +\infty.\end{aligned}$$

Example 2.2. *Example of a function that satisfies condition* (b_1). *Let* $b : B_2(0) \subset \mathbb{R}^N \to \mathbb{R}$ *be given by*

$$b(x) = \begin{cases} e^{\frac{1}{0,5^2-|x|^2}}, & |x| > 0,5, \\ 0, & |x| \leq 0,5. \end{cases}$$

Consider $\lambda_1(\Omega) < \lambda$ *with* $\lambda_1(\Omega)$ *close to* λ. *We set*

$$a(x) = \frac{\lambda_1(\Omega)}{\lambda}\left[\frac{1+2|x|^2}{2+|x|^2}\right].$$

Taking $l_\infty > \lambda_1(\Omega)$. *Then*

$$\begin{aligned} g(x) &:= \frac{\lambda a(x) - \lambda_1(\Omega)}{l_\infty} \\ &= \frac{\lambda_1(\Omega)\left(\frac{1+2|x|^2}{2+|x|^2}\right) - \lambda_1(\Omega)}{l_\infty} = \frac{\lambda_1(\Omega)}{l_\infty}\left(\frac{1+2|x|^2}{2+|x|^2} - 1\right) \\ &= \frac{\lambda_1(\Omega)}{l_\infty}\left(\frac{|x|^2-1}{2+|x|^2}\right).\end{aligned}$$

The $b(x)$ *function is non-negative, the* $a(x)$ *is positive, the* $g(x)$ *function changes sign, and* $b(x) > g(x)$.

Remark 2.1. *By* (f_1), *we have that given* $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$$|f(s)| \leq \varepsilon|s|, \tag{2.3}$$

for all $|s| \leq \delta$. *By* (f_4), *it follows that for* $|s| > R > 1$,

$$|f(s)| \leq \int_0^s |f'(t)|dt \leq \int_0^s a_1(1+|t|^{q-2})dt = a_1(|s| + M|s|^{q-1}). \tag{2.4}$$

Since $|s| > 1$, *then* $|1| < |s|^{q-1}$, *and*

$$|f(s)| \leq M_1|s|^{q-1}, \quad \forall\ |s| > R > 1.$$

On the other hand, for $\delta < |s| < R$, *and since* f *is continuous, então*

$$\left|\frac{f(s)}{s^{q-1}}\right| < M_2, \quad M_2 > 0.$$

Thus, for $\delta < |s| < R$,

$$|f(s)| \leq M_2|s|^{q-1}. \tag{2.5}$$

Adding up the inequalities (2.3), (2.4) *and* (2.5), *we have*

$$|f(s)| \leq \varepsilon|s| + M|s|^{q-1}, \tag{2.6}$$

which implies that

$$|F(s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{M}{q}|s|^q \tag{2.7}$$

for all $s > 0$.

Remark 2.2. *Note that of the hypothesis* $(f_2)'$ *and L'Hôpital's theorem, we have*

$$\lim_{s\to\infty} \frac{F(s)}{s^2} = \frac{l_\infty}{2}. \tag{2.8}$$

In fact by $(f_2)'$, *we have that* f *is increasing, then by L'Hôpital's theorem, we have*

$$\lim_{s\to\infty} \frac{F(s)}{s^2} = \lim_{s\to\infty} \frac{f(s)}{2s} = \frac{l_\infty}{2}.$$

Remark 2.3. *By* $(f_2)'$ *and* (f_3), *we have that if* $s > 0$ *and* $\frac{f(s)}{s}$ *is increasing, then* $\frac{f(s)}{s} < l_\infty$.

We will look for a solution in the space $H_0^1(\Omega)$ with the standard standard

$$\|u\|^2 = \int_\Omega |\nabla u|^2 dx.$$

We consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ associated with the problem (P') is given by

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)F(u)dx, \tag{2.9}$$

of class C^2 with derivative given by

$$I'(u)v = \int_\Omega (\nabla u \nabla v - \lambda a(x)uv)dx + \int_\Omega b(x)f(u)vdx,$$

for all $u, v \in H_0^1(\Omega)$. In addition, we set

$$J(u) := I'(u)u = \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)f(u)udx,$$

and we set Nehati manifold like set

$$\mathcal{N} = \{u \in H_0^1(\Omega) : J(u) = I'(u)u = 0\}.$$

Note that, $u \in \mathcal{N}$ if and only if

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x)f(u)u\ dx. \tag{2.10}$$

There is a fibrering map associated with I defined by

$$\phi_u(t) := I(tu) = \frac{t^2}{2}\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)F(tu)dx, \tag{2.11}$$

and its derivative in the variable t is given by

$$\phi_u'(t) = t\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)f(tu)u\ dx. \tag{2.12}$$

The following lemma relates the Nehari manifold to the first and second derivatives of the fibrering map.

Lemma 2.1. *If $u \in \mathcal{N}$, then $\phi_u'(1) = 0$. In addition, i) if $\phi_u''(1) > 0$, then $u \in \mathcal{N}^+$; ii) if $\phi_u''(1) < 0$, then $u \in \mathcal{N}^-$; iii) if $\phi_u''(1) = 0$, then $u \in \mathcal{N}^0$.*

Proof. For the first part, consider $u \in \mathcal{N}$. By equalities (2.10) and (2.11), we have

$$\phi_u'(1) = -\int_\Omega b(x)f(u)udx + \int_\Omega b(x)f(u)udx = 0.$$

The second part follows from the definition of the sets above. Indeed, if $\phi_u''(1) > 0$, then

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)f'(u)u^2dx > 0,$$

in other words, $u \in \mathcal{N}^+$, and this proves item (i). The other items follow in a similar way. □

We define the set in $\mathcal{N}$.

$$\begin{aligned}
\mathcal{N}^+ &= \left\{u \in \mathcal{N}; \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)f'(u)u^2\ dx > 0\right\}\\
&= \left\{u \in \mathcal{N}; -\int_\Omega b(x)f(u)u\ dx + \int_\Omega b(x)f'(u)u^2\ dx > 0\right\}\\
&= \left\{u \in \mathcal{N}; \int_\Omega b(x)(f'(u)u^2 - f(u)u)dx > 0\right\}\\
&= \left\{u \in \mathcal{N}; \int_\Omega b(x)\left(f'(u) - \frac{f(u)}{u}\right)u^2\ dx > 0\right\}.
\end{aligned}$$

Similarly, the sets

$$\begin{aligned}
\mathcal{N}^0 &= \left\{u \in \mathcal{N}; \int_\Omega b(x)\left(f'(u) - \frac{f(u)}{u}\right)u^2\ dx = 0\right\},\\
\mathcal{N}^- &= \left\{u \in \mathcal{N}; \int_\Omega b(x)\left(f'(u) - \frac{f(u)}{u}\right)u^2\ dx < 0\right\}.
\end{aligned}$$

Moreover, we also define the following sets

$$\begin{aligned}
L^+ &= \left\{u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx > 0\right\},\\
L^0 &= \left\{u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = 0\right\},\\
L^- &= \left\{u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx < 0\right\},
\end{aligned}$$

and with respect to the nonlinear term, we define

$$\begin{aligned}
B^+ &= \left\{u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega b(x)f(u)u\ dx > 0\right\},\\
B^0 &= \left\{u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega b(x)f(u)u\ dx = 0\right\},\\
B^- &= \left\{u \in H_0^1(\Omega); \|u\| = 1, \int_\Omega b(x)f(u)u\ dx < 0\right\}.
\end{aligned}$$

Remark 2.4. *The assumption* (f_3) *and* $b(x) \geq 0$ *ensure that* $\mathcal{N}^- = \emptyset$. *Additionally, if* $\lambda_1(a) < \lambda < \lambda_1^0(a)$, *we have* $L^- \neq \emptyset$. *In fact, taking* ϕ_1 *as the first positive eigenfunction associated with the first eigenvalue* $\lambda_1(a)$ *of the problem* (P_1). *Then*

$$\int_\Omega (|\nabla \phi_1|^2 - \lambda a(x)\phi_1^2)dx = \int_\Omega (\lambda_1(a) - \lambda)a(x)\phi_1^2 dx < 0.$$

And also, $L^+ \neq \emptyset$, *as taking* ϕ_1^0 *as the first positive eigenfunction associated with the first eigenvalue* $\lambda_1^0(a)$ *of the problem* (P_1) *restricted to* Ω_0, *define*

$$\varphi(x) = \begin{cases} \phi_1^0(x), & x \in \Omega_0 \\ 0, & x \in \Omega \backslash \overline{\Omega_0}. \end{cases}$$

We have that $\|\varphi\| = \|\phi_1^0\| = 1$. *Thus,*

$$\begin{aligned} \int_\Omega (|\nabla \varphi|^2 - \lambda a(x)\varphi^2)dx &= \int_{\Omega_0} (|\nabla \varphi|^2 - \lambda a(x)\varphi^2)dx + \int_{\Omega\backslash\overline{\Omega_0}} (|\nabla \varphi|^2 - \lambda a(x)\varphi^2)dx \\ &= \int_{\Omega_0} (|\nabla \phi_1^0|^2 - \lambda a(x)(\phi_1^0)^2)dx \\ &= \int_{\Omega_0} (\lambda_1^0(a) - \lambda)a(x)(\phi_1^0)^2 dx > 0, \end{aligned}$$

and then, $\varphi \in L^+$.

Remark 2.5. *If* $0 < \lambda < \lambda_1(a)$, *then* $\mathcal{N}^+ = \emptyset$ *since, as we will see in Section 2.2,* $\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)\,dx$ *will be a norm, and by the hypotheses* (f_1), $\frac{f(s)}{s}$ *increasing for* $s > 0$, *and* $b(x) \geq 0$, *we have that the equality* (1.7) *will not occur. On the other hand, if* $\lambda_1(a) < \lambda < \lambda_1^0(a)$, *then Lemma 2.9 will ensure that* $\mathcal{N}^+ \neq \emptyset$.

We consider the subsets $\Omega_+ = \{x \in \Omega : b(x) > 0\}$. Note that if b is continuous, then $\Omega_0 = \overline{\Omega_0}$ is a closed subset of $\mathbb{R}^N$.

Lemma 2.2. *If* $0 < \lambda < \lambda_1^0(a)$ *then* $\overline{L^-} \cap B^0 = \emptyset$.

Proof. Suppose the contrary, i.e., that there is $u \in \overline{L^-} \cap B^0$, then $\|u\| = 1$ and

$$0 = \int_\Omega b(x)f(u)udx = \int_{\Omega_+} b(x)f(u)udx.$$

Note that $f(s)s \geq 0$ and $b(x) > 0$ in Ω_+, then

$$\int_\Omega b(x)f(u)u = 0 \text{ implies that } b(x)f(u(x))u(x) = 0 \;\text{ q.t.p in } \Omega^+.$$

Consider $x \in \Omega_+$, that is, $b(x) > 0$, then $f(u(x))u(x) = 0$ *a.e.* in Ω_+, and by hypotheses (f_1) and (f_3) we have that $u(x) = 0$ *a.e.* in Ω_+. Thus, $supp\{u\} \subset \overline{\Omega_0}$. In addition, since $u \in L^-$ and $supp\{u\} \subset \overline{\Omega_0}$, if $0 < \lambda < \lambda_1^0(a)$, then

$$\begin{aligned} 0 \leq \int_{\Omega_0} (\lambda_1^0(a) - \lambda)a(x)u^2 dx &\leq \int_{\Omega_0} (|\nabla u|^2 - \lambda a(x)u^2)dx \\ &= \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx \\ &\leq 0. \end{aligned}$$

Since $a(x) > 0$, this implies $u \equiv 0$, which contradicts $\|u\| = 1$. It follows that $\overline{L^-} \cap B^0 = \emptyset$. □

Lemma 2.3. *Suppose that* $0 < \lambda < \lambda_1^0(a)$*, then* $\mathcal{N}^0 = \{0\}$.

Proof. Suppose by contradiction there is $u_0 \in \mathcal{N}^0 \backslash \{0\}$, i.e., u_0 satisfies

$$\int_\Omega b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx = 0.$$

Thus

$$\int_{\Omega_0} b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 + \int_{\Omega \backslash \Omega_0} b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx = 0,$$

which implies that

$$\int_{\Omega \backslash \Omega_0} b(x) \left(f'(u_0) - \frac{f(u_0)}{u_0} \right) u_0^2 dx = 0.$$

If $supp\{u_0\} \cap \Omega \backslash \overline{\Omega_0} \neq \emptyset$, by (f_3), we have a contradiction. Thus $supp\{u_0\} \subset \overline{\Omega_0}$. It follows that

$$\int_\Omega b(x) f(u_0) u_0 dx = \int_{\Omega_0} b(x) f(u_0) u_0 dx = 0,$$

and so, $\frac{u_0}{\|u_0\|} \in B^0$. In addition,

$$0 = -\int_\Omega b(x) f(u_0) u_0 dx = \int_\Omega (|\nabla u_0|^2 - \lambda a(x) u_0^2) dx,$$

implies that

$$\frac{\int_{\Omega_0} |\nabla u_0|^2 dx}{\int_{\Omega_0} a(x) u_0^2 dx} = \lambda.$$

Thus,

$$\lambda_1^0(a) \leq \frac{\int_{\Omega_0} |\nabla u_0|^2 dx}{\int_{\Omega_0} a(x) u_0^2 dx} = \lambda,$$

which is a contradiction. □

Lemma 2.4. *For all* $u \in \mathcal{N}$*, we have* $I(u) \leq 0$.

Proof. Let $u \in \mathcal{N}$, then by (2.10) and by hypothesis (f_5), we have

$$\begin{aligned} I(u) &= \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda a(x) u^2) dx + \int_\Omega b(x) F(u) dx \\ &= -\frac{1}{2} \int_\Omega b(x) f(u) u dx + \int_\Omega b(x) F(u) dx \\ &= \frac{1}{2} \int_\Omega b(x)[2F(u) - f(u)u] dx \leq 0. \end{aligned}$$

Moreover, $I(u) = 0$ if and only if $u = 0$ or $\text{supp}\{u\} \subset \overline{\Omega_0}$. □

Up to now, we assume (b_1), which can be seen in more detail through the following observation:

Remark 2.6. *Let $\lambda_1(a) < \lambda < \lambda_1^0(a)$, $\psi_1 > 0$ be the first eigenfunction in $H_0^1(\Omega)$ associated with the first eigenvalue $\lambda_1(a)$ of the problem (P_1), and $\varphi_1 > 0$ be the first eigenfunction in $H_0^1(\Omega)$ associated with the first eigenvalue $\lambda_1(\Omega)$ of the problem $-\Delta$ in Ω, with Dirichlet boundary conditions denoted by (P_2). Since ψ_1 is a weak solution of the problem (P_1) and φ_1 is a weak solution of the problem (P_2), then*

$$\int_\Omega \nabla\psi_1 \nabla v dx = \lambda_1(a) \int_\Omega a(x)\psi_1 v dx, \quad \forall v \in H_0^1(\Omega) \tag{2.13}$$

and

$$\int_\Omega \nabla\varphi_1 \nabla v dx = \lambda_1(\Omega) \int_\Omega \varphi_1 v dx, \quad \forall w \in H_0^1(\Omega). \tag{2.14}$$

Taking $v = \varphi_1$ in the equation (2.13) and $w = \psi_1$ in the equation (2.14), we have that

$$\int_\Omega \nabla\psi_1 \nabla\varphi_1 dx = \lambda_1(a) \int_\Omega a(x)\psi_1\varphi_1 dx$$

and

$$\int_\Omega \nabla\varphi_1 \nabla\psi_1 dx = \lambda_1(\Omega) \int_\Omega \varphi_1\psi_1 dx.$$

Thus,

$$\lambda_1(a) \int_\Omega a(x)\psi_1\varphi_1 dx = \lambda_1(\Omega) \int_\Omega \varphi_1\psi_1 dx$$

or equivalently

$$\int_\Omega (\lambda_1(a)a(x) - \lambda_1(\Omega))\psi_1\varphi_1 dx = 0.$$

Thus, we must have that $(\lambda_1(a)a(x) - \lambda_1(\Omega))$ changes sign, because $\psi_1\varphi_1 > 0$. In addition, since $a(x) > 0$ a.e in Ω and $\lambda_1(a) < \lambda$, then $\lambda a(x) - \lambda_1(\Omega) > \lambda_1(a)a(x) - \lambda_1(\Omega)$, from this it follows that $\lambda a(x) - \lambda_1(\Omega)$ can take on negative values, which is in line with (b_1).

Note that if $\Omega_1 = \{x \in \Omega : \lambda a(x) - \lambda_1(\Omega) < 0\}$, then we can have $\Omega_0 \subset \Omega_1$, and a hypothesis (b_1) will be satisfied.

Lemma 2.5. *Suppose that $0 < \lambda < \lambda_1^0(a)$ and f satisfies $(f_1), (f_2)' - (f_5)'$ and b satisfies (b_1). Then $\mathcal{N}^+$ is a bounded.*

Proof. Suppose otherwise, then there exists a sequence $(u_n) \in \mathcal{N}^+$ such that $\|u_n\| \to +\infty$. Consider the sequence $v_n := \frac{u_n}{\|u_n\|}$. Up to a subsequence, (v_n) is bounded in $H_0^1(\Omega)$, $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ in $L^p(\Omega)$ for $2 \le p < 2^*$ e $v_n(x) \to v(x)$ a.e. in Ω. Since $\frac{f(s)}{s}$ is increasing for $s > 0$, by hypothesis (f_3), $\frac{f(s)}{s}$ is even and satisfies hypothesis (f_1), we obtain

$$b(x)\frac{f(\|u_n\|v_n(x))}{\|u_n\|v_n(x)}v_n^2(x) \ge 0.$$

Note initially that $u_n \in \mathcal{N}^+$, then $\Omega_n := supp\{u_n\} \cap (\Omega \backslash \overline{\Omega_0}) \ne \emptyset$ and $|\Omega_n| > 0$, then, $supp\{v_n\} \cap (\Omega \backslash \overline{\Omega_0}) \ne \emptyset$. Thus, for all $n \in \mathbb{N}$,

$$\int_{\Omega_n} b(x)f(u_n)u_n(x)dx > 0 \Longleftrightarrow \int_\Omega b(x)\frac{f(\|u_n\|v_n)}{\|u_n\|v_n}v_n^2 dx \ge 0.$$

Thus, dividing $J(u_n) = 0$ por $\|u_n\|^2$, we have

$$
\begin{aligned}
0 &= \int_\Omega |\nabla u_n|^2 dx - \lambda \int_\Omega a(x) u_n^2 dx + \int_\Omega b(x) f(u_n) u_n dx \\
&= \int_\Omega |\nabla v_n|^2(x) dx - \lambda \int_\Omega a(x) v_n^2 dx + \int_\Omega b(x) \frac{f(u_n)}{u_n} v_n^2 dx \\
&= \int_\Omega |\nabla v_n|^2 dx - \lambda \int_\Omega a(x) v_n^2 dx + \int_\Omega b(x) \frac{f(\|u_n\| v_n)}{\|u_n\| v_n} v_n^2 dx \\
&> \int_\Omega |\nabla v_n|^2 dx - \lambda \int_\Omega a(x) v_n^2 dx. \qquad (2.15)
\end{aligned}
$$

Thus,

$$
0 > 1 - \lambda \int_\Omega a(x) v_n^2 dx. \qquad (2.16)
$$

Taking the limit as $n \to \infty$ in equation (2.16) and using the compact embedding of Sobolev $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$
0 > 1 - \lambda \int_\Omega a(x) v^2 dx.
$$

If $v = 0$, we obtain a contradiction, therefore, $v \neq 0$. Thus, we have two possibilities: $supp\{v\} \subset \overline{\Omega_0}$ or $supp\{v\} \cap (\Omega \backslash \overline{\Omega_0}) \neq \emptyset$ has positive measure. If $supp\{v\} \subset \overline{\Omega_0}$, and since $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, it follows from the inequality (2.15)

$$
\begin{aligned}
0 &\geq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 dx - \lambda \int_\Omega a(x) v^2 dx \\
&\geq \int_{\Omega_0} |\nabla v|^2 dx - \lambda \int_{\Omega_0} a(x) v^2 dx \\
&\geq (\lambda_1^0(a) - \lambda) \int_{\Omega_0} a(x) v^2 dx \\
&> 0,
\end{aligned}
$$

which is an absurd. We conclude that $supp\{v\} \cap (\Omega \backslash \Omega_0)$ has positive measure. Let $\tilde{\Omega} := \{x \in \Omega \backslash \overline{\Omega_0} : v(x) \neq 0\}$ and $|\tilde{\Omega}| > 0$. Since $\lim_{n\to\infty} v_n(x) = v(x)$ we have $u_n(x) = \|u_n\| v_n(x)$ and $\lim_{n\to\infty} u_n(x) = \pm\infty$ almost every $x \in \tilde{\Omega}$. Take $r_n(x) = \|u_n\| v_n(x)$, then $r_n(x) \to \pm\infty$. Thus, using Remark 2.2

$$
\lim_{n\to\infty} \frac{F(r_n(x))}{r_n^2(x)} = \frac{l_\infty}{2}, \text{ almost every } \; x \in \tilde{\Omega}.
$$

Following from this and the Fatou's lemma,

$$
\liminf_{n\to\infty} \int_\Omega b(x) \frac{F(r_n(x))}{r_n^2(x)} v_n^2(x) dx \geq \int_{\tilde{\Omega}} \liminf_{n\to\infty} b(x) \frac{F(r_n(x))}{r_n^2(x)} v_n^2(x) dx = \int_{\tilde{\Omega}} b(x) \frac{l_\infty}{2} v^2(x) dx > 0 \qquad (2.17)
$$

On the other hand, since $u_n \in \mathcal{N}^+$ by Lemma 2.4, we have

$$\begin{aligned}
0 \geq \frac{I(u_n)}{\|u_n\|^2} &= \frac{I(\|u_n\|v_n)}{\|u_n\|^2} \\
&= \frac{\|u_n\|^2}{2\|u_n\|^2}\int_\Omega (|\nabla v_n|^2 - \lambda a(x)v_n^2)dx + \frac{1}{\|u_n\|^2}\int_\Omega b(x)F(\|u_n\|v_n)dx \\
&= \frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda a(x)v_n^2)dx + \int_\Omega b(x)\frac{F(\|u_n\|v_n)}{\|u_n\|^2}dx \\
&= \frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda a(x)v_n^2)dx + \int_\Omega b(x)\frac{F(\|u_n\|v_n(x))}{\|u_n\|^2}\frac{v_n^2(x)}{v_n^2(x)}dx \\
&= \frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda a(x)v_n^2)dx + \int_\Omega b(x)\frac{F(\|u_n\|v_n(x))}{(\|u_n\|v_n(x))^2}v_n^2(x)dx.
\end{aligned}$$

Now, using (2.17), $v_n \rightharpoonup v \neq 0$, by the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, given the hypothesis $a \in L^\infty(\Omega)$, Lebesgue's Dominated Convergence Theorem, Fatou's lemma and (b_1), we obtain

$$\begin{aligned}
0 \geq \frac{I(u_n)}{\|u_n\|^2} &= \liminf_{n\to\infty}\left\{\frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda a(x)v_n^2)dx + \int_\Omega b(x)\frac{F(r_n(x))}{r_n^2(x)}v_n^2(x)dx\right\} \\
&\geq \frac{1}{2}\int_\Omega |\nabla v|^2dx - \frac{\lambda}{2}\int_\Omega a(x)v^2dx + \int_\Omega b(x)\liminf_{n\to\infty}\frac{F(r_n(x))}{r_n^2(x)}v_n^2(x)dx \\
&\geq \frac{\lambda_1(\Omega)}{2}\int_\Omega v^2dx - \frac{\lambda}{2}\int_\Omega a(x)v^2dx + \int_\Omega b(x)\frac{l_\infty}{2}v^2(x)dx \\
&= \frac{1}{2}\left\{\int_\Omega (\lambda_1(\Omega) - \lambda a(x) + b(x)l_\infty)v^2dx\right\} \\
&> 0,
\end{aligned}$$

resulting in an absurd. Therefore, $\mathcal{N}^+$ is a bounded. □

Lemma 2.6. *Suppose that u_0 is a critical point of I restricted to $\mathcal{N}$ such that $u_0 \notin \mathcal{N}^0$, then $I'(u_0) = 0$ in $H^{-1}(\Omega)$.*

Proof. If u_0 is a critical point for I restricted to $\mathcal{N}$, then u_0 is a minimizer of $I(u)$ subject to the constraint $J(u) = 0$. Hence, by the Theorem Lagrange Multiplier, there exists $\mu \in \mathbb{R}$ such that $I'(u_0) = \mu J'(u_0)$. Thus,

$$\langle I'(u_0), u_0\rangle = \mu\langle J'(u_0), u_0\rangle. \tag{2.18}$$

Replacing $J(u_0) = 0$ into (2.18), and using (1.7), we have

$$\begin{aligned}
\langle J'(u_0),u_0\rangle &= 2\int_\Omega (|\nabla u_0|^2 - \lambda u_0^2)dx + \int_\Omega b(x)f'(u_0)u_0^2dx + \int_\Omega b(x)f(u_0)u_0dx \\
&= \int_\Omega b(x)\left(f'(u_0) - \frac{f(u_0)}{u_0}\right)u_0^2dx.
\end{aligned}$$

Since $u_0 \notin \mathcal{N}^0$, it follows that $\langle J'(u_0), u_0\rangle \neq 0$ and applying (2.18) we obtain $\mu = 0$, that is $I'(u_0) = 0$. □

Proof of the Theorem 4. Assuming by contradiction that there exists a positive solution u to problem (P'), then, multiplying the first equation of problem (P') by ϕ_1^0, the first positive eigenfunction in $H_0^1(\Omega)$ associated with the first eigenvalue $\lambda_1^0(\Omega)$ of the problem (P_1) restricted to Ω_0, we obtain

$$-\Delta u\phi_1^0 = \lambda a(x)u\phi_1^0 - b(x)f(u)\phi_1^0.$$

By integrating by parts in this open set with a smooth boundary $\partial\Omega_0$, we obtain

$$\begin{aligned} 0 &= \int_{\Omega_0} \nabla u \nabla \phi_1^0 dx - \lambda \int_{\Omega_0} a(x)u\phi_1^0 dx + \int_{\partial\Omega_0} u\frac{\partial \phi_1^0}{\partial \eta} dx + \int_{\Omega_0} b(x)f(u)\phi_1^0 dx \\ &= -(\lambda - \lambda_1^0(a)) \int_{\Omega_0} a(x)u\phi_1^0 dx + \int_{\partial\Omega_0} u\frac{\partial \phi_1^0}{\partial \eta} dx + \int_{\Omega_0} b(x)f(u)\phi_1^0 dx, \end{aligned}$$

where η is the outward unit normal vector on $\partial\Omega_0$. On the other hand, $\int_{\Omega_0} b(x)f(u)\phi_1^0\,dx = 0$ and

$$\int_{\partial\Omega_0} u\frac{\partial \phi_1^0}{\partial \eta} dx < 0 \quad \text{and} \quad (\lambda - \lambda_1(\Omega_0)) \int_{\Omega_0} a(x)u\phi_1^0 dx > 0,$$

and this yields a contradiction. □

Note that, given a sequence $(u_n) \in \mathcal{N}^+$, we will now present some convergences that will be used throughout this chapter. By Lemma 2.5, assuming hypothesis $(f_2)'$, u_n is bounded, and then, up to a subsequence, we have $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \to u_0$ in $L^p(\Omega)$, and $u_n(x) \to u_0(x)$ almost everywhere in Ω. As in Chapter 1, it holds

$$\lim_{n\to\infty} \int_\Omega b(x)F(u_n)dx = \int_\Omega b(x)F(u_0)dx, \tag{2.19}$$

$$\lim_{n\to\infty} \int_\Omega b(x)f(u_n)u_n\,dx = \int_\Omega b(x)f(u_0)u_0\,dx, \tag{2.20}$$

and

$$\lim_{n\to\infty} \int_\Omega b(x)f'(u_n)u_n^2 dx = \int_\Omega b(x)f'(u_0)u_0^2 dx. \tag{2.21}$$

The next lemma is analogous to the one found in [6], here proven with the necessary adaptations.

Lemma 2.7. *Assume $0 < \lambda < \lambda_1^0(a)$ and (b_1). Every sequence (u_n) of Palais-Smale, $(PS)_c$, with $c \le 0$ of the functional I restricted to $\mathcal{N}^+$ has a subsequence $(PS)_c$ of I in $H_0^1(\Omega)$.*

Proof. Let $(u_n) \subset \mathcal{N}^+$ be a Palais-Smale, $(PS)_c$, sequence of the functional I restricted to $\mathcal{N}^+$. Then, by Lemma 2.5, (u_n) is bounded in $H_0^1(\Omega)$. Using the Lagrange multipliers theorem for the derivative of the functional I constrained to $\mathcal{N}^+$, we can write

$$\nabla I(u_n) = \nabla_{\mathcal{N}^+} I(u_n) + \mu_n \nabla J(u_n). \tag{2.22}$$

If $c < 0$, since $\mathcal{N}^+$ is bounded in $H_0^1(\Omega)$, then, up to a subsequence, $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ for $2 \le p < 2^*$. By Lemma 2.4 we have $I(u_n) < 0$, then

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{2}\int_\Omega a(x)u_n^2 dx + \int_\Omega b(x)F(u_n)dx < 0.$$

If $u_n \rightharpoonup u = 0$ in $H_0^1(\Omega)$, then $u_n \to 0$ in $L^p(\Omega)$, $2 \le p < 2^*$. Through weak convergence, Sobolev embedding, and Lebesgue's Dominated Convergence Theorem,

$$0 \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \le 0.$$

Thus,

$$\lim_{n\to\infty} \|u_n\| = 0.$$

Since I is continuous, then

$$c = \lim_{n\to\infty} I(u_n) = I(u) = 0$$

which is a contradiction. Therefore, $u \neq 0$.

By hypothesis (f_4), for any $v \in H_0^1(\Omega)$ it holds

$$\left| \int_\Omega [f'(u_n)u_n + f(u_n)]v dx \right| \;\; \leq \;\; C\|v\|,$$

where C is a positive constant.
Thus,

$$\begin{aligned} |\langle \nabla J(u_n), v\rangle| &= |2\langle u_n, v\rangle - \int_\Omega [f'(u_n)u_n + f(u_n)]v| \\ &\leq C\|v\|, \quad \forall v \in H_0^1(\Omega). \end{aligned}$$

This shows that $(\nabla J(u_n))$ is bounded in $H_0^1(\Omega)$. Since $|\langle \nabla J(u_n), u_n\rangle| \leq \|\nabla J(u_n)\|\|u_n\| < C$, passing to a subsequence, we have that $|\langle \nabla J(u_n), u_n\rangle| \to \rho \geq 0$. Let us show that $\rho > 0$. If $\text{supp}\{u\} \cap (\Omega \backslash \overline{\Omega_0}) \neq \emptyset$ and has positive measure, then by (2.21) and by (f_3)

$$\begin{aligned} \rho &= \lim_{n\to\infty} |\langle \nabla J(u_n), u_n\rangle| \\ &= \lim_{n\to\infty} \int_\Omega b(x)[f'(u_n)u_n^2 - f(u_n)u_n]dx \\ &= \int_\Omega b(x)[f'(u)u^2 - f(u)u]dx > 0. \end{aligned}$$

Taking the inner product of (2.22) com $u_n \in \mathcal{N}^+$, we have

$$\begin{aligned} 0 = \langle I'(u_n), u_n\rangle &= \langle \nabla_{\mathcal{N}^+} I(u_n), u_n\rangle + \mu_n\langle \nabla J(u_n), u_n\rangle \\ &= o_n(1) + \mu_n\langle \nabla J(u_n), u_n\rangle. \end{aligned}$$

It follows that $\mu_n \to 0$ since $(\nabla J(u_n))$ is bounded in $H_0^1(\Omega)$ and $|J'(u_n)u_n| \to \rho > 0$. Therefore, taking the limit in (2.22)

$$\nabla I(u_n) = \nabla_{\mathcal{N}^+} I(u_n) + o_n(1),$$

which implies $I'(u_n) \to 0$.

If $supp\{u\} \subset \overline{\Omega_0}$, then

$$0 \geq \liminf_{n\to\infty} I(u_n) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega a(x)u^2 dx \geq (\lambda_1^0(a) - \lambda)\int_{\Omega_0} a(x)u^2 dx > 0,$$

which is a absurd.

On the other hand, if $c = 0$, como $(u_n) \subset \mathcal{N}$ is bounded, we have that $u_n \rightharpoonup u$, by (2.19) and (2.20)

$$\begin{aligned} 0 = c &= \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) - \frac{1}{2}I'(u_n)u_n \\ &= \lim_{n\to\infty} \int_\Omega b(x)(F(u_n) - \frac{1}{2}f(u_n)u_n)dx \\ &= \int_\Omega b(x)(F(u) - \frac{1}{2}f(u)u)dx. \end{aligned}$$

It follows that $u = 0$ or $\text{supp}\{u\} \subset \overline{\Omega_0}$. If $\text{supp}\{u\} \subset \overline{\Omega_0}$, then by the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ and by Lebesgue's Dominated Convergence Theorem

$$\begin{aligned} 0 &= \liminf_{n\to\infty} I(u_n) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega a(x)u^2 dx \\ &\geq (\lambda_1^0(a) - \lambda)\int_{\Omega_0} a(x)u^2 dx. \end{aligned}$$

Since $a(x) > 0$, we have that $u = 0$, that is, $u_n \rightharpoonup 0$ when $n \to \infty$. By the compact embedding and Lebesgue's Dominated Convergence Theorem

$$0 = \lim_{n\to\infty} I(u_n) \geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \geq \int_\Omega |\nabla u|^2 \geq 0$$

implies $\|u_n\| \to 0$ when $n \to \infty$. Since I is a functional of class C^1, then

$$I(u_n) \to I(0) = 0 \text{ and } I'(u_n) \to I'(0) = 0$$

when $n \to \infty$. This completes the proof of the lemma. □

2.1 Regularity of the solution

In this section, we will assume that $a, b \in C^{0,\alpha}(\Omega)$. Consider the equation

$$-\Delta u = \lambda a(x)u - b(x)f(u) = k(x)(1 + |u|) = g(x,u)$$

where

$$k(x) = \frac{\lambda a(x)u - b(x)f(u)}{1 + |u|}.$$

Let us show that $k \in L^{\frac{N}{2}}_{loc}(\Omega)$. Note that

$$\begin{aligned} |k(x)| &= \left|\frac{\lambda a(x)u - b(x)f(u)}{1+|u|}\right| \\ &\leq \frac{\lambda\|a\|_\infty|u| + \|b\|_\infty|f(u)|}{1+|u|} \\ &\leq \frac{\lambda\|a\|_\infty|u|}{1+|u|} + \frac{\|b\|_\infty(\varepsilon|u| + M|u|^{q-1})}{1+|u|} \\ &\leq \lambda\|a\|_\infty + \frac{\|b\|_\infty\varepsilon|u|}{1+|u|} + \frac{\|b\|_\infty M|u|^{q-2}|u|}{1+|u|} \\ &\leq \lambda\|a\|_\infty + \|b\|_\infty\varepsilon + \|b\|_\infty M|u|^{q-2} =: c_1 + c_2|u|^{q-2}, \end{aligned}$$

with c_1, c_2 positive constants. We observe that

$$\begin{aligned} \int_\Omega |k(x)|^{\frac{N}{2}} dx &\leq \int_\Omega (c_1 + c_2|u|^{q-2})^{\frac{N}{2}} dx \\ &= c_1|\Omega| + c_2\int_\Omega |u|^{(q-2)\frac{N}{2}} dx < \infty, \end{aligned}$$

since for $2 < q < 2^*$, we have to

$$0 < q - 2 < 2^* - 2 \implies 0 < (q-2)\frac{N}{2} < (2^* - 2)\frac{N}{2} \implies$$

$$(2^*-2)\frac{N}{2}=\left(\frac{2N}{N-2}-2\right)\frac{N}{2}=\left(\frac{2N-2N+4}{N-2}\right)\frac{N}{2}=\left(\frac{4}{N-2}\right)\frac{N}{2}=2^*.$$

Thus, $k\in L^{\frac{N}{2}}_{(loc)}(\Omega)$. Therefore, by the Brezis - Kato Theorem [26], $u\in L^s(\Omega)$, for any $1\leq s<+\infty$. Thus, $-\Delta u=\lambda a(x)u-b(x)f(u)\in L^s(\Omega)$, because,

$$\begin{aligned}\int_\Omega|\lambda a(x)u-b(x)f(u)|^s dx &\leq \int_\Omega(\lambda\|a\|_\infty|u|+\|b\|_\infty|f(u)|)^s dx\\ &\leq \int_\Omega(\lambda\|a\|_\infty|u|+\|b\|_\infty(\varepsilon|u|+M|u|^{q-1})]^s dx\\ &\leq 2^{s-1}\lambda^s\|a\|^s_\infty\int_\Omega|u|^s dx+2^{s-1}(\|b\|_\infty\varepsilon)^s\int_\Omega(|u|+|u|^{q-1})^s dx\\ &= C\|u\|^s_s+C|\Omega|+C\|u\|^{(q-1)s}<\infty.\end{aligned}$$

Since $q-1>1$, $(q-1)\leq(q-1)s<+\infty$ and $u\in L^s(\Omega)$, taking $\overline{s}:=(q-1)s$, with $1<\overline{s}<+\infty$, we have that

$$\int_\Omega|\lambda a(x)u-b(x)f(u)|^{\overline{s}}dx<\infty.$$

Therefore, $u\in W^{2,\overline{s}}(\Omega)$ for all $1<s<\infty$, and by Sobolev's immersion, $W^{2,s}(\Omega)\hookrightarrow C^{1,\alpha}(\overline{\Omega})$, $u\in C^{1,\alpha}(\overline{\Omega})$. It follows from this and from the assumptions that $f\in C^1(\Omega)$ and $a,b\in C^{0,\alpha}(\Omega)$ that $\lambda a(x)u-b(x)f(u)\in C^{0,\alpha}(\Omega)$. Finally, by Schauder's theorem $u\in C^{2,\alpha}(\Omega)$. And with this we show that $u\in C^{2,\alpha}(\Omega)\cap C(\overline{\Omega})$ is a classic solution of the problem (P).

2.2 The case $0<\lambda<\lambda_1(a)$

Let's see that in this case, the norm $\|u\|_\lambda=\int_\Omega\nabla u\nabla v\,dx-\lambda\int_\Omega a(x)uv\,dx$ is equivalent to the norm $\|u\|=\int_\Omega|\nabla u|^2\,dx$ of $H^1_0(\Omega)$. In fact, consider the eigenvalue problem (P_1) and define the application $H^1_0(\Omega)\times H^1_0(\Omega)$ by

$$B_\lambda(u,v)=\int_\Omega\nabla u\nabla v dx-\lambda\int_\Omega a(x)uvdx.$$

Note that B_λ is a bilinear form that satisfies

i) $B_\lambda(u,v)=B_\lambda(v,u)$;

ii) $|B_\lambda(u,v)|\leq\|u\|\|v\|$;

iii) $B_\lambda(u,u)\geq\|u\|^2$.

In fact, (i) it follows

$$\begin{aligned}B_\lambda(u,v) &= \int_\Omega\nabla u\nabla v dx-\lambda\int_\Omega a(x)uvdx\\ &= \int_\Omega\nabla v\nabla u dx-\lambda\int_\Omega a(x)vudx\\ &= B_\lambda(v,u).\end{aligned}$$

ii) By Holder's inequality and Sobolev embeddings, we have

$$\begin{aligned} |B_\lambda(u,v)| &= \left| \int_\Omega \nabla u \nabla v dx - \lambda \int_\Omega a(x) uv dx \right| \\ &\leq \int_\Omega |\nabla u \cdot \nabla v| dx + \lambda \int_\Omega |a(x)||uv| dx \\ &\leq \|u\|\|v\| + \lambda \|a\|_\infty C \|u\|\|v\| \\ &= C_\lambda \|u\|\|v\|. \end{aligned}$$

iii) Since $\lambda_1(a) \int_\Omega a(x) u^2 dx \leq \int_\Omega |\nabla u|^2 dx$, then

$$\begin{aligned} B_\lambda(u,u) &= \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega a(x) u^2 dx \\ &\geq \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{\lambda_1(a)} \int_\Omega |\nabla u^2| dx \\ &= \left(1 - \frac{\lambda}{\lambda_1(a)}\right) \|u\|^2 \\ &\geq 0. \end{aligned}$$

Thus, $B_\lambda(u,v)$ defines the norm

$$\|u\|_\lambda^2 := \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega a(x)|u|^2 dx \leq \int_\Omega |\nabla u|^2 dx = \|u\|^2,$$

that is, $\|u\|_\lambda \leq \|u\|$. On the other hand,

$$\begin{aligned} \|u\|_\lambda^2 &= \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega a(x)|u|^2 dx \\ &\geq \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{\lambda_1(a)} \int_\Omega |\nabla u|^2 dx \\ &= \left(1 - \frac{\lambda}{\lambda_1(a)}\right) \int_\Omega |\nabla u|^2 dx \\ &= \left(\frac{\lambda_1(a) - \lambda}{\lambda_1(a)}\right) \|u\|^2, \end{aligned}$$

that is, if $\lambda < \lambda_1(a)$, then

$$\|u\|_\lambda \geq C\|u\|,$$

where C is a positive constant. Therefore, $\|\cdot\|_\lambda$ and $\|\cdot\|$ are equivalent.

In addition,

$$I(u) = \frac{1}{2}\|u\|_\lambda^2 + \int_\Omega b(x) F(u) dx \geq 0.$$

Thus, I is bounded from below by 0. On the other hand,

$$\begin{aligned} I(u) &= \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda a(x) u^2) dx + \int_\Omega b(x) F(u) dx \\ &\geq \frac{1}{2}\|u\|_\lambda \\ &\geq \frac{C}{2}\|u\|^2 \to \infty, \end{aligned}$$

if $\|u\| \to \infty$. Therefore, I is coercive, and by the minimization theorem, there exists $u_0 \in H_0^1(\Omega)$ such that

$$I(u_0) = \inf_{u \in H_0^1(\Omega)} I(u) \geq 0.$$

Since $I(0) = 0$ then

$$I(u_0) = \frac{1}{2}\int_\Omega (|\nabla u_0|^2 - \lambda a(x)u_0^2)dx + \int_\Omega b(x)F(u_0)dx = 0,$$

and $u_0 = 0$ is a minimum point of I in $H_0^1(\Omega)$. Suppose there exists $u_1 \in H_0^1(\Omega)$ as a positive solution of (P), then $u_1 \in \mathcal{N}^+$, however, by the Remark 2.4

$$0 < \int_\Omega (|\nabla u_1|^2 - \lambda a(x)u_1^2)dx = -\int_\Omega b(x)f(u_1)u_1 \leq 0,$$

which is a absurd. Therefore, for the case $0 < \lambda < \lambda_1(a)$, the problem (P') has no positive solution.

2.3 O caso $\lambda_1(a) < \lambda < \lambda_1^0(a)$

The condition $\lambda_1(a) < \lambda < \lambda_1^0(a)$ implies that $\overline{L^-} \cap B^0 = \emptyset$ by Lemma 2.2. In addition, Lemma 2.3 ensures that $\mathcal{N}^0 = \{0\}$.

Lemma 2.8. *Assume $\lambda_1(a) < \lambda < \lambda_1^0(a)$. If $u \in \mathcal{N}^+$, then $supp\{u\} \cap (\Omega \setminus \overline{\Omega_0}) \neq \emptyset$.*

Proof. Let $u \in \mathcal{N}^+$, that is,

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)f(u)udx = 0, \tag{2.1}$$

and suppose by contradiction that $supp\{u\} \cap (\Omega \setminus \overline{\Omega_0}) = \emptyset$, thus $supp\{u\} \subset \overline{\Omega_0}$. Then,

$$\int_\Omega b(x)f(u)udx = 0 \tag{2.2}$$

which implies that (2.1)

$$\frac{\int_{\Omega_0} |\nabla u|^2 dx}{\int_{\Omega_0} a(x)u^2 dx} = \lambda,$$

that is,

$$\lambda_1^0(a) \leq \frac{\int_{\Omega_0} |\nabla u|^2 dx}{\int_{\Omega_0} a(x)u^2 dx} = \lambda$$

which is a contradiction with the hypothesis of the lemma. □

In the setting that $f(s)$ is asymptotically linear, as s goes to infinity, not all functions u in $H_0^1(\Omega)\backslash\{0\}$ are projectable on the Nehari manifold. In order to obtain a subset of functions in $H_0^1(\Omega)$ which are projectable we define the set

$$\mathcal{E} = \{u \in H_0^1(\Omega)\backslash\{0\} : \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)l_\infty u^2 dx > 0\}.$$

Note that $\mathcal{E} \neq \emptyset$, as taking ϕ_1^0 positive eigenfunction associated with the first eigenvalue $\lambda_1^0(a)$ of problem (P_1) restricted to Ω_0, define

$$\varphi(x)=\begin{cases}\phi_1^0(x), & x\in\Omega_0\\ 0, & x\in\Omega\backslash\overline{\Omega_0}.\end{cases}$$

Then

$$\int_\Omega(|\nabla\varphi|^2-\lambda a(x)(\varphi)^2)dx \quad + \quad \int_\Omega b(x)l_\infty(\varphi)^2=\int_{\Omega_0}(\lambda_1^0(a)-\lambda)a(x)(\phi_1^0)^2dx>0.$$

Thus, $\varphi\in\mathcal{E}$.

Remark 2.7. *$L^-\cap\mathcal{E}\neq\emptyset$.*

In fact, let ψ_1 be the first eigenfunction of the eigenvalue problem (P_1), then

$$\int_\Omega(|\nabla\psi_1|^2-\lambda a(x)(\psi_1)^2)dx=\int_\Omega(\lambda_1(a)-\lambda)a(x)(\psi_1)^2dx<0.$$

On the other hand, since $\psi_1\in H_0^1(\Omega)$ and using (b_1), we have that

$$\begin{aligned}\int_\Omega|\nabla\psi_1|^2dx \quad &- \quad \lambda\int_\Omega a(x)(\psi_1)^2dx+\int_\Omega b(x)l_\infty(\psi_1)^2dx\\ &\geq \quad \lambda_1(\Omega)\int_\Omega(\psi_1)^2dx-\lambda\int_\Omega a(x)(\psi_1)^2dx+\int_\Omega b(x)l_\infty(\psi_1)^2dx\\ &= \quad \int_\Omega[\lambda_1(\Omega)-\lambda a(x)+b(x)l_\infty](\psi_1)^2dx\\ &> \quad 0.\end{aligned}$$

Therefore, $\psi_1\in L^-\cap\mathcal{E}$.

Lemma 2.9. *Assume $\lambda_1(a)<\lambda<\lambda_1^0(a)$ and f satisfies $(f_1),(f_2)'-(f_4)$. Then, there is a unique real number $t=t(u)>0$ such that $tu\in\mathcal{N}^+$ if and only if $\frac{u}{\|u\|}\in L^-\cap\mathcal{E}$.*

Proof. Suppose that $\frac{u}{\|u\|}\in L^-\cap\mathcal{E}$, we will show that there is t such that $tu\in\mathcal{N}^+$. Let $v=\frac{u}{\|u\|}$, that is, $\int_\Omega(|\nabla v|^2-\lambda a(x)v^2)dx<0$. If $supp\{u\}\subset\overline{\Omega_0}$, then $supp\{v\}\subset\overline{\Omega_0}$, thus $\int_\Omega b(x)f(v)vdx=0$, implies that $v=\frac{u}{\|u\|}\in\overline{L^-}\cap B^0$, which is an absurd by Lemma 2.2. Thus, $supp\{u\}\cap(\Omega\setminus\overline{\Omega_0})\neq\emptyset$ and has positive measure, then we can take $t\to0^+$ and by hypothesis (f_1), we have that given $\varepsilon>0$, there is δ such that if $0<t<\delta$, then

$$\left|b(x)\frac{f(tu)}{tu}u^2\right|\leq\|b\|_\infty\left|\frac{f(tu)}{tu}\right|u^2\leq\varepsilon\|b\|_\infty u^2 \ \in L^1(\Omega).$$

By the Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{t\to0^+}\int_\Omega b(x)\frac{f(tu)}{tu}u^2\ dx=0. \tag{2.3}$$

Since $\frac{u}{\|u\|}\in L^-$

$$\begin{aligned}\lim_{t\to0^+}\frac{\phi_u'(t)}{t}&=\lim_{t\to0^+}\int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx+\lim_{t\to0^+}\int_\Omega b(x)\frac{f(tu)}{tu}u^2dx\\ &=\int_\Omega(|\nabla u|^2-\lambda a(x)u^2)dx<0.\end{aligned} \tag{2.4}$$

On the other hand, since $u \in \mathcal{E}$ an using Fatou's Lemma, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{\phi_u'(t)}{t} &\geq \liminf_{t\to\infty} \left\{ \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)\frac{f(tu)}{tu}u^2 dx \right\} \\ &> \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x) l_\infty u^2 dx \\ &> 0. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), it follows that there exists t such that $tu \in \mathcal{N}^+$.
Conversely, if there exists $t > 0$ such that $tu \in \mathcal{N}^+$, we have by Lemma 2.8 that $supp\{tu\} \cap (\Omega\backslash\overline{\Omega_0}) \neq \emptyset$ and by equation (2.10),

$$t^2 \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x) f(tu)(tu)dx.$$

Thus,

$$t^2 \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x)\frac{f(tu)}{tu} t^2 u^2 dx$$

which implies that $\frac{u}{\|u\|} \in L^-$.

Furthermore, it follows from $(f_2)'$ that

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx > -\int_\Omega b(x) l_\infty u^2 dx$$

and, therefore,

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x) l_\infty u^2 dx > 0$$

which shows that $u \in \mathcal{E}$.

Finally, we will show that the projection onto the Nehari manifold is unique. Suppose that there are $0 < t_1 < \tilde{t_1}$ such that $t_1 u,\ \tilde{t_1} u \in \mathcal{N}^+$. From this, it follows that

$$t_1^2 \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x) f(t_1 u) t_1 u dx, \tag{2.6}$$

$$\tilde{t_1}^2 \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x) f(\tilde{t_1} u) \tilde{t_1} u dx. \tag{2.7}$$

Dividing the equation (2.6) by t_1^2 and a equation (2.7) by $\tilde{t_1}^2$ we have that

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x)\frac{f(t_1 u)}{t_1 u} u^2 dx, \tag{2.8}$$

$$\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx = -\int_\Omega b(x)\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} u^2 dx. \tag{2.9}$$

Subtracting the equation (2.8) from the equation (2.9) results in

$$\int_\Omega b(x)\left(\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} - \frac{f(t_1 u)}{t_1 u}\right) u^2 dx = 0.$$

Since $t_1 u, \tilde{t_1} u \in \mathcal{N}$, by Lemma 2.8, $\text{supp}\{t_1 u\} \cap (\Omega\backslash\overline{\Omega_0}) \neq \emptyset$ and $\text{supp}\{\tilde{t_1} u\} \cap (\Omega\backslash\overline{\Omega_0}) \neq \emptyset$

From this, it follows that there exist $\varepsilon > 0$ and $x_0 \in \Omega$ such that $B_\varepsilon(x_0) \subset \Omega \setminus \Omega_0$ and

$$\int_\Omega b(x)\left(\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} - \frac{f(t_1 u)}{t_1 u}\right) u^2 dx \geq \int_{B_\varepsilon(x_0)} b(x)\left(\frac{f(\tilde{t_1} u)}{\tilde{t_1} u} - \frac{f(t_1 u)}{t_1 u}\right) u^2 dx > 0,$$

but this is an absurd because $f(s)/(s)$ is increasing by (f_3), $b(x) > 0$ for $x \in B_\varepsilon(x_0)$. Therefore, we conclude that there exists a unique t_1 such that $t_1 u \in \mathcal{N}^+$. □

Lemma 2.10. *The function*

$$\mathcal{A} := \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \frac{u}{\|u\|} \in L^- \right\} \to (0, +\infty)$$
$$u \mapsto t(u)$$

is continuous.

Proof. Let $\frac{u}{\|u\|} \in L^-$, that is, $T(u) := \int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx < 0$ and T is continuous. We have that $T^{-1}\{(-\infty, 0)\}$ is open in $H_0^1(\Omega)$ and $u \mapsto t(u)$ is defined in an open subset of $H_0^1(\Omega)$. To prove continuity, we will use the Implicit Function Theorem. Let $g : \mathbb{R}^+ \times H_0^1(\Omega) \to \mathbb{R}$ of class C^1 defined for $g(t,u) = t\|u\|^2 - \lambda t \int_\Omega a(x)u^2 dx + \int_\Omega b(x)f(tu)udx$. Consider (t_0, u_0) such that $g(t_0, u_0) = 0$ and $u_0 > 0$. For $t_0 u_0 \in \mathcal{N}^+$, we have that

$$t_0\|u_0\|^2 - \lambda t_0 \int_\Omega a(x)u_0^2 dx = -\int_\Omega b(x)f(t_0u_0)u_0 dx$$

$$\iff t_0^2[\|u_0\|^2 - \lambda \int_\Omega a(x)u_0^2 dx] = -\int_\Omega b(x)f(t_0u_0)t_0u_0 dx$$
$$\iff t_0^2[\|u_0\|^2 - \lambda \int_\Omega a(x)u_0^2 dx] = -\int_\Omega b(x)\frac{f(t_0u_0)}{t_0u_0}t_0^2u_0^2 dx$$
$$\iff \|u_0\|^2 - \lambda \int_\Omega a(x)u_0^2 dx = \int_\Omega b(x)\frac{f(t_0u_0)}{t_0u_0}u_0^2 dx.$$

Differentiating the function g with respect to t and using the hypothesis (f_3), we have that

$$\begin{aligned}
\frac{\partial g(t_0,u_0)}{\partial t} &= \|u_0\|^2 - \lambda \int_\Omega a(x)u_0^2 dx + \int_\Omega b(x)f'(t_0u_0)u_0^2 dx \\
&= -\int_\Omega b(x)\frac{f(t_0u_0)}{t_0u_0}u_0^2 dx + \int_\Omega b(x)f'(t_0u_0)u_0^2 dx \\
&= \int_\Omega b(x)\left[f'(t_0u_0)u_0^2 - \frac{f(t_0u_0)}{t_0u_0}u_0\right] dx > 0.
\end{aligned}$$

By the Implicit Function Theorem, the function $\Psi : \mathcal{A} \to \mathbb{R}^+$ defined for $t = t(u)$ is of class C^1 in a neighborhood V of u_0 and $g(t,u) = g(t(u), u) = 0$ in V. □

Proof of the Theorem 5. Since $\mathcal{N} = \mathcal{N}^+ \cup \{0\}$ is bounded by Lemma 2.5, there is $C > 0$ such that $\|u\| \leq C$ for all $u \in \mathcal{N}$. Making calculations analogous to those found in the proof of Theorem 2, we have I is bounded from below in $\mathcal{N}^+$. We observe that $\inf_{u\in\mathcal{N}^+} I(u) < 0$. In fact, let ϕ_1 be the first eigenfunction associated with the first eigenvalue $\lambda_1(a)$ of problem (P_1), then $\phi_1 \in L^-$ and by Lemma 2.8 existe $t > 0$ tal que $t\phi_1 \in \mathcal{N} \setminus \{0\} = \mathcal{N}^+$. By Lemma 2.4, $I(t\phi_1) \leq 0$, on the other hand, using the hypothesis (f_5) and that $t\phi_1 \neq 0$

$$\begin{aligned}
I(t\phi_1) &= I(t\phi_1) - \frac{1}{2}J(t\phi_1) \\
&= \frac{1}{2}\int_\Omega b(x)(2F(t\phi_1) - f(t\phi_1)t\phi_1)dx \\
&= \frac{1}{2}\int_{\Omega_+} b(x)(2F(t\phi_1) - f(t\phi_1)t\phi_1)dx < 0.
\end{aligned}$$

Thus,

$$\inf_{u\in\mathcal{N}^+} I(u) < 0. \tag{2.10}$$

Therefore, there is $m > 0$ such that

$$\inf_{u \in \mathcal{N}^+} I(u) = -m. \tag{2.11}$$

Let (u_n) be a minimizing sequence in $\mathcal{N}^+$. By Lemma 2.5, $\mathcal{N}^+$ is bounded, then (u_n) is bounded, and up to a subsequence we have $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$. It follows from 2.11 and equalities (2.19) and (2.20) that

$$\begin{aligned} 0 > \inf_{u \in \mathcal{N}^+} I(u) = \lim_{n\to\infty} I(u_n) &= \lim_{n\to\infty} \left\{ \frac{1}{2} \int_\Omega b(x)[2F(u_n) - f(u_n)u_n]dx \right\} \\ &= \lim_{n\to\infty} \int_\Omega b(x)F(u_n)dx - \lim_{n\to\infty} \frac{1}{2} \int_\Omega b(x)f(u_n)u_n dx \\ &= \int_\Omega b(x)F(u_0)dx - \frac{1}{2} \int_\Omega b(x)f(u_0)u_0 dx \\ &= \frac{1}{2} \int_\Omega b(x)[2F(u_0) - f(u_0)u_0]dx. \end{aligned}$$

This implies that $supp\{u_0\} \cap (\Omega \setminus \Omega_0) \neq \emptyset$, thus $u_0 \not\equiv 0$ and then

$$\begin{aligned} \int_\Omega b(x) \left[f'(u_0) - \frac{f(u_0)}{u_0} \right] u_0^2 dx &= \int_{\Omega\setminus\Omega_0} b(x) \left[f'(u_0) - \frac{f(u_0)}{u_0} \right] u_0^2 dx \\ &= \int_{\Omega_+} b(x) \left[f'(u_0) - \frac{f(u_0)}{u_0} \right] u_0^2 dx > 0. \end{aligned}$$

Thus, $u_0 \in \mathcal{N}^+$ it follows that

$$\begin{aligned} I(u_0) &= \frac{1}{2} \int_\Omega b(x)[2F(u_0) - f(u_0)u_0]dx \\ &= \lim_{n\to\infty} I(u_n) = \inf_{u \in \mathcal{N}^+} I(u) < 0. \end{aligned}$$

Therefore, u_0 is a non-trivial critical point of I in $\mathcal{N}^+$. We will now verify that u_0 is positive. In fact, since f is odd, then F is an even function. Hence

$$
\begin{aligned}
\inf_{u\in\mathcal{N}^+} I(u) &= I(u_0) = I(u_0^+ - u_0^-)\\
&= \frac{1}{2}\int_\Omega |\nabla(u_0^+ - u_0^-)|^2 dx - \frac{\lambda}{2}\int_\Omega a(x)(u_0^+ - u_0^-)^2 dx + \int_\Omega b(x)F(u_0^+ - u_0^-)dx\\
&= \frac{1}{2}\int_\Omega (|\nabla u_0^+|^2 + |\nabla u_0^-|^2)dx - \frac{\lambda}{2}\int_\Omega a(x)[(u_0^+)^2 + (u_0^-)^2] + \int_\Omega b(x)F(u_0^+ - u_0^-)dx\\
&= \frac{1}{2}\int_{\{u_0\geq 0\}} (|\nabla u_0^+|^2 - \lambda a(x)(u_0^+)^2)dx + \int_{\{u_0\geq 0\}} b(x)F(u_0^+)dx\\
&+ \frac{1}{2}\int_{\{u_0< 0\}} (|\nabla(-u_0^-)|^2 - \lambda a(x)(-u_0^-)^2)dx + \int_{\{u_0< 0\}} b(x)F(-u_0^-)dx\\
&= \frac{1}{2}\int_{\{u_0\geq 0\}} (|\nabla u_0^+|^2 - \lambda a(x)(u_0^+)^2)dx + \int_{\{u_0\geq 0\}} b(x)F(u_0^+)dx\\
&+ \frac{1}{2}\int_{\{u_0< 0\}} (|\nabla(u_0^-)|^2 - \lambda a(x)(u_0^-)^2)dx + \int_{\{u_0< 0\}} b(x)F(u_0^-)dx\\
&= \frac{1}{2}\int_{\{u_0\geq 0\}} (|\nabla u_0^+|^2 - \lambda a(x)(u_0^+)^2)dx + \int_{\{u_0\geq 0\}} b(x)F(u_0^+)dx\\
&+ \frac{1}{2}\int_{\{u_0< 0\}} (|\nabla u_0^-|^2 - \lambda a(x)(u_0^-)^2)dx + \int_{\{u_0< 0\}} b(x)F(u_0^-)dx\\
&= \int_\Omega (|\nabla |u_0||^2 - \lambda a(x)|u_0|^2)dx + \int_\Omega b(x)F(|u_0|)dx\\
&= I(|u_0|),
\end{aligned}
$$

and then, $u_0 \geq 0$. Moreover assuming $a, b \in C^{0,\alpha}(\Omega)$, it follows by Hopf lemma that $u_0 > 0$ (see the end of the proof of Theorem 6 for details).

Claim 2.1. *If $u \in H_0^1(\Omega)$ is a positive solution of problem (P') then $\lambda < \lambda_1^{a,b}$. In fact, let $f(s) := l_\infty s - g(s)$, with $g(s) > 0$ a decreasing function. Let $\psi_1 > 0$ the first eigenfuction associated with $\lambda_1^{a,b}$. Then,*

$$\int_\Omega (\nabla\psi_1\nabla u + b(x)l_\infty\psi_1 u)dx = \lambda_1^{a,b}\int_\Omega a(x)\psi_1 u dx \tag{2.12}$$

and

$$\int_\Omega (\nabla u\nabla\psi_1 + b(x)l_\infty u\psi_1)dx = \int_\Omega \lambda a(x)u\psi_1 dx + \int_\Omega b(x)g(u)dx.$$

Thus,

$$\lambda_1^{a,b}\int_\Omega a(x)\psi_1 u dx = \int_\Omega \lambda a(x)u\psi_1 dx + \int_\Omega b(x)g(u)dx$$

implies

$$(\lambda_1^{a,b} - \lambda)\int_\Omega a(x)\psi_1 u dx = \int_\Omega b(x)g(u)dx > 0.$$

Therefore, $\lambda_1^{a,b} > \lambda$.

Claim 2.2. $\lambda_1^{a,b} < \lambda_1^0(a)$.

Indeed, let φ_1^0 the first eigenfunction associated with $\lambda_1^0(a)$, taking φ_1^0 as the test function in the equation (2.12), we have

$$\int_{\Omega_0} (\nabla\psi_1\nabla\varphi_1^0 + b(x)l_\infty\psi_1\varphi_1^0)dx = \lambda_1^{a,b}\int_{\Omega_0} a(x)\psi_1\varphi_1^0 dx.$$

Since $b = 0$ *in* Ω_0*, then*

$$\int_{\Omega_0} \nabla\psi_1 \nabla\varphi_1^0 dx = \lambda_1^{a,b} \int_{\Omega_0} a(x)\psi_1\varphi_1^0 dx. \tag{2.13}$$

Note that φ_1^0 *is solution to the problem* (P_1) *restricted to* Ω_0*, that is,* φ_1^0 *satisfies*

$$\Delta\varphi_1^0 = \lambda_1^0(a)a(x)\varphi_1^0 \text{ in } \Omega_0$$

and $\varphi_1^0 = 0$ *in* $\partial\Omega_0$*. Multiplying this equation by* ψ_1 *and integrating in* Ω_0*, we obtain*

$$\int_{\Omega_0} -\psi_1\Delta\varphi_1^0 dx = \lambda_1^0(a) \int_{\Omega_0} a(x)\varphi_1^0\psi_1 dx. \tag{2.14}$$

By the Divergence Theorem for $\vec{F} = \psi_1\nabla\varphi_1^0$

$$\int_{\Omega_0} div(\psi_1\nabla\varphi_1^0)dx = \int_{\partial\Omega_0} dx\frac{\partial\varphi_1^0}{\partial\eta}dS$$

implies

$$\int_{\Omega_0} \nabla\psi_1\nabla\varphi_1^0 dx + \int_{\Omega_0} \psi_1\nabla\varphi_1^0 dx = \int_{\partial\Omega_0} \psi_1\frac{\partial\varphi_1^0}{\partial\eta}dS.$$

Thus,

$$\int_{\Omega_0} \nabla\psi_1\nabla\varphi_1^0 dx - \int_{\partial\Omega_0} \psi_1\frac{\partial\varphi_1^0}{\partial\eta}dS = -\int_{\Omega_0} \psi_1\Delta\varphi_1^0 dx.$$

Substituting in (2.14)*, we have*

$$\int_{\Omega_0} \nabla\psi_1\nabla\varphi_1^0 dx - \int_{\partial\Omega_0} \psi_1\frac{\partial\varphi_1^0}{\partial\eta}dS = \lambda_1^0(a)\int_{\Omega_0} a(x)\psi_1\varphi_1^0 dx.$$

By (2.13)*, we obtain*

$$\lambda_1^{a,b}\int_{\Omega_0} a(x)\psi_1\varphi_1^0 dx - \int_{\partial\Omega_0} \psi_1\frac{\partial\varphi_1^0}{\partial\eta}dS = \lambda_1^0(a)\int_{\Omega_0} a(x)\psi_1\varphi_1^0 dx$$

and

$$(\lambda_1^{a,b} - \lambda_1^0(a))\int_{\Omega_0} a(x)\psi_1\varphi_1^0 dx = \int_{\partial\Omega_0} \psi_1\frac{\partial\varphi_1^0}{\partial\eta}dS < 0$$

Therefore, $\lambda_1^{a,b} < \lambda_1^0(a)$*.*

To obtain the uniqueness, suppose there are two positive classical solutions u_1 and u_2 of (P'), with $u_1 \neq u_2$ then

$$-\Delta u_1 - \lambda a(x)u_1 + b(x)f(u_1) = 0 \text{ in } \Omega \text{ and } u_1 = 0 \text{ on } \partial\Omega \tag{2.15}$$

$$-\Delta u_2 - \lambda a(x)u_2 + b(x)f(u_2) = 0 \text{ in } \Omega \text{ and } u_2 = 0 \text{ on } \partial\Omega. \tag{2.16}$$

Dividing the equation (2.15) for u_1 and (2.16) for u_2 we have

$$\frac{-\Delta u_1}{u_1} = \lambda a(x) - b(x)\frac{f(u_1)}{u_1} \tag{2.17}$$

and

$$\frac{-\Delta u_2}{u_2} = \lambda a(x) - b(x)\frac{f(u_2)}{u_2}. \tag{2.18}$$

Subtracting equation (2.17) from equation (1.13) results in

$$\frac{-\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} = b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right). \tag{2.19}$$

Multiplying the equation (2.19) by $(u_1^2 - u_2^2)$ and integrating over Ω

$$\int_\Omega (u_1^2 - u_2^2)\left(\frac{-\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right) dx = \int_\Omega b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx. \tag{2.20}$$

By proving the uniqueness of the solution in [3]

$$\begin{aligned}\int_\Omega (u_1^2 - u_2^2)\left(\frac{-\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2}\right) dx &= \int_\Omega -u_1\Delta u_1 dx + \int_\Omega u_2^2\frac{\Delta u_1}{u_1}dx \\ &+ \int_\Omega u_1^2\frac{\Delta u_2}{u_2}dx - \int_\Omega u_2\Delta u_2 dx \\ &= \int_\Omega \nabla u_1\cdot\nabla u_1 dx - \int_\Omega \nabla\left(\frac{u_2^2}{u_1}\right)\cdot\nabla u_1 dx \\ &- \int_\Omega \nabla\left(\frac{u_1^2}{u_2}\right)\cdot\nabla u_2 dx + \int_\Omega \nabla u_2\cdot\nabla u_2 dx \\ &= \int_\Omega |\nabla u_1|^2 dx - \int_\Omega \left(2\frac{u_2}{u_1}\nabla u_2 - \frac{u_2^2}{u_1^2}\nabla u_1\right)\cdot\nabla u_1 dx \\ &- \int_\Omega \left(2\frac{u_1}{u_2}\nabla u_1 - \frac{u_2^2}{u_2^2}\nabla u_2\right)\cdot\nabla u_2 dx + \int_\Omega |\nabla u_2|^2 dx \\ &= \int_\Omega \left\{\left|\nabla u_1 - \frac{u_1}{u_2}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1}\nabla u_1\right|^2\right\} dx \geq 0.\end{aligned}$$

Hence and from (2.20), it follows that

$$\int_\Omega b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx \geq 0. \tag{2.21}$$

Since we have $\overline{L^-}\cap B^0 = \emptyset$ and $u_1,\ u_2 \in \mathcal{N}^+$, then $\frac{u_1}{\|u_1\|},\ \frac{u_2}{\|u_2\|} \in L^-$. By Lemma 2.8 we have $supp\{u_1\}\cap(\Omega\setminus\overline{\Omega_0}) \neq \emptyset,\ supp\{u_2\}\cap(\Omega\setminus\overline{\Omega_0}) \neq \emptyset$. In addition, since $\frac{f(s)}{s}$ is increasing, we have the following cases

(i) if $u_1 > u_2$ then

$$\int_{\Omega\setminus\Omega_0} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx < 0,$$

which is a contradiction with (2.21).

(ii) if $u_1 < u_2$ then, one more time,

$$\int_{\Omega\setminus\Omega_0} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx < 0,$$

which is a contradiction with (2.21).

(iii) there are subsets of $A\cup B = \Omega\setminus\overline{\Omega_0}$, $A \neq \emptyset$ and $B \neq \emptyset$ open in $\mathbb{R}^N$, such that $u_1 - u_2 > 0$ in A, $u_2 - u_1 > 0$ in B and by (1.15), we have

$$
\begin{aligned}
0 &\leq \int_{\Omega} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx \\
&= \int_{\Omega\setminus\overline{\Omega_0}} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx \\
&= \int_{A} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx + \int_{B} b(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_1^2 - u_2^2)dx \\
&< 0,
\end{aligned}
$$

because in A, $\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) < 0$, $(u_1^2 - u_2^2) > 0$ and $b(x) > 0$ and in B, $\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) > 0$, $(u_1^2 - u_2^2) < 0$ e $b(x) > 0$.

Therefore, by the cases (i) and (ii) the case (iii) it can't occur. We conclude that $u_1 \equiv u_2$.

□

2.4 Sign-changing solution

In this section, let us also assume $\lambda_1(a) < \lambda < \lambda_1^0(a)$.

Theorem 8. *If (u_n) is a $(PS)_c$ sequence for I restricted to $\mathcal{N}^+$, with $c \leq 0$, then up to a subsequence (u_n) converges to u in $H_0^1(\Omega)$.*

Proof. Let (u_n) be a $(PS)_c$ sequence for I restricted to $\mathcal{N}^+$, which is a bounded set by Lemma 2.5. Then $I'(u_n) \to 0$, by Lemma 2.6 and Lemma 2.7, isto é,

$$I'(u_n)\varphi \to I'(u_0)\varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

Note that

$$
\begin{aligned}
\langle I'(u_n) - I'(u_0), u_n - u_0\rangle &= I'(u_n).(u_n - u_0) - I'(u_0).(u_n - u_0) \\
&= \int_\Omega \nabla u_n \nabla(u_n - u_0)dx - \lambda\int_\Omega a(x)u_n(u_n - u_0)dx \\
&+ \int_\Omega b(x)f(u_n)(u_n - u_0)dx - \int_\Omega \nabla u_0\nabla(u_n - u_0)dx \\
&+ \lambda\int_\Omega a(x)u_0(u_n - u_0)dx - \int_\Omega b(x)f(u_0)(u_n - u_0)dx \\
&= \|u_n - u_0\|^2 - \lambda\int_\Omega a(x)(u_n - u_0)^2dx \\
&+ \int_\Omega b(x)[f(u_n) - f(u_0)](u_n - u_0)dx. \qquad (2.1)
\end{aligned}
$$

In this way,

$$
\begin{aligned}
\|u_n - u_0\|^2 =& \langle I'(u_n) - I'(u_0), u_n - u_0\rangle + \lambda\int_\Omega a(x)(u_n - u_0)^2dx \\
&- \int_\Omega b(x)[f(u_n) - f(u_0)](u_n - u_0)dx,
\end{aligned}
$$

applying the limit when n goes to infinity in (2.1), we have up to a subsequence $u_n \to u_0$, in $H_0^1(\Omega)$, because, u_n be a $(PS)_c$, sequence and bounded, by the compact embedding of Sobolev and the Lebesgue Dominated Convergence Theorem, the functional I satisfies the Palais-Smale condition at the level c. □

The next result is based on Lemma 5.2 in [15], and suits our settings.

Lemma 2.11. *Let u_0 be a positive solution of the problem* (P') *and $v^j : \Omega \to \mathbb{R},\ j \in \mathbb{N}$, of functions, a sequence satisfying $\|v^j - u_0\|_{C^1(\overline{\Omega})} \to 0$. Then there are $j_0 \in \mathbb{N}$ such that $v^j(x) > 0,\ \forall x \in \Omega,\ \forall\ j \geq j_0$.*

Proof. In fact, we have that Ω é C^1 and $u_0 = 0$ in $\partial\Omega$, then for every differentiable curve $\gamma : [-1, 1] \to \partial\Omega,\ \gamma(0) = x_0 \in \partial\Omega$, we have $u_0(\gamma(t)) = 0$, thus

$$\frac{d}{dt}(u_0(\gamma(t))) = \nabla u_0(\gamma(t))\gamma'(t) = 0,$$

and if $t = 0$ we have that $\nabla u_0(\gamma(0))\gamma'(0) = 0$. Replacing the value of $\gamma(0)$ we have that $\nabla u_0(x_0)\gamma'(0) = 0$, that is $\nabla u_0(x_0)$ is perpendicular to the zero level at the point x_0. This ensures that the normal exterior to the $\partial\Omega$ on point x_0 is parallel to the $\nabla u(x_0)$ and so we can write

$$\nu_{x_0} = \frac{\nabla u_0(x_0)}{\|\nabla u_0(x_0)\|}.$$

Given $\varepsilon > 0$, we claim that there exists $\delta_0 > 0$ such that, if $x \in N_{\delta_0} := \{x \in \Omega : dist(x, \partial\Omega) \leq \delta_0\}$, then $\left|\frac{f(u(x))}{u(x)}\right| < \varepsilon$. Indeed, suppose by contradiction that this is not true. Then there exists $\varepsilon_0 > 0$ such that for any $\delta_n = \frac{1}{n} > 0$, there exists x_n satisfying $dist(x_n, \partial\Omega) < \delta_n$ and $\left|\frac{f(u(x_n))}{u(x_n)}\right| > \varepsilon_0$.

Since $dist(x_n, \partial\Omega) \to 0$, and $\overline{\Omega}$ is a compact set, there is $x_0 \in \partial\Omega$ such that $x_n \to x_0$. The functions f and u_0 are continuous, hence $u_0(x_n) \to u(x_0) = 0$ and $f(u_0(x_n)) \to f(u(x_0)) = 0$, because $x_0 \in \partial\Omega$, but $\lim_{n\to\infty}\left|\frac{f(u(x_n))}{u(x_n)}\right| \geq \varepsilon_0$, which contradicts (f_1).

It follows from the hypothesis (f_1), that $f(u_0(x)) = o(|u_0(x)|)u_0(x)$ for all $x \in N_{\delta_0}$ and as u_0 being a positive solution to the problem (P'), then

$$\begin{aligned}
-\Delta u_0(x) &= \lambda a(x)u_0(x) - b(x)f(u_0(x)) \\
&= \lambda a(x)u_0(x) - b(x)o(|u_0|)u_0(x) \\
&\geq \lambda a(x)u_0(x) - b(x)\varepsilon u_0(x) \\
&\geq \lambda a(x)u_0(x) - \|b\|_\infty \varepsilon u_0(x) \\
&= (\lambda a(x) - \|b\|_\infty \varepsilon)u_0(x).
\end{aligned}$$

Taking $\varepsilon < \frac{\lambda}{\|b\|_\infty}$, then

$$-\Delta u = \lambda a(x)u - o(|u|) > 0, \tag{2.2}$$

since $a(x) \geq \underline{a} > 0$. In addition, $u_0(x) > u_0(x_0) = 0$, for all $x \in int(N_{\delta_0})$, thus $\inf_{N_{\delta_0}} u(x) = 0$. Note that N_{δ_0} is regular because the set Ω is regular and u_0 is continuous in $\overline{N_{\delta_0}}$. Then by Hopf's lemma $\frac{\partial u_0}{\partial \nu_{x_0}}(x_0) > 0$ for all $x_0 \in u_0^{-1}\{0\} \cap \partial N_{\delta_0}$, ν_{x_0} is the exterior normal vector ∂N_{δ_0} in x_0, then it holds $\frac{\partial u_0}{\partial \nu_{x_0}}(x_0) > 0$ for $x \in u_0^{-1}\{0\} \cap \partial\Omega$.

Since $\frac{\partial u_0}{\partial \nu_x}(x)$ is continuous for all $x \in \partial\Omega$ and $\partial\Omega$ is compact, then $\frac{\partial u_0}{\partial \nu_x}(x) \geq \delta > 0$ for all $x \in \partial\Omega$. In fact,

$$\frac{\partial u_0}{\partial \nu_x}(x) = \left\langle \nabla u_0(x), \frac{\nabla u_0(x)}{\|\nabla u_0(x)\|} \right\rangle = \|\nabla u_0(x)\| \geq \delta > 0,\ \forall\ x \in \partial\Omega. \tag{2.3}$$

If $y \in \Omega$ and $y - x_0 = \alpha\nu_{x_0}$ with $\alpha > 0$, then

$$\langle \nabla u_0(x_0), y - x_0 \rangle = \left\langle \nabla u_0(x_0), \alpha\frac{\nabla u_0(x_0)}{\|\nabla u_0(x_0)\|} \right\rangle = \|\nabla u_0(x_0)\|\alpha > 0.$$

Let $t(x_0)$ be such that if $0 < t < t(x_0)$ and $y = x_0 + t\nu_{x_0}$, so for continuity of $\|\nabla u_0(\cdot)\|$, we have

$$\|\nabla u_0(y)\| \geq \frac{1}{2} \min_{x_0 \in \partial\Omega} \|\nabla u_0(x_0)\| = \frac{1}{2}\delta, \quad \forall\, 0 < t < t(x_0). \tag{2.4}$$

Taking $y \in \Omega$ such that $y = x + t\nu_x$ with $x \in \partial\Omega$ and $0 < t < t(x)$, by (2.4) and by the continuity $\|\nabla u_0(\cdot)\|$ we have

$$\|\nabla u_0(y)\| \geq \frac{\delta}{2},\ \forall\, 0 < t < t(x).$$

Consider the open ball $B_{t(x)}(x)$ such that $\partial\Omega \subset \bigcup_{x \in \partial\Omega} B_{t(x)}(x)$ and by the compactness of $\partial\Omega$ it follows $\partial\Omega \subset \bigcup_{k=1}^{n} B_{t(x_k)}(x_k)$, in other words, $\partial\Omega$ has a finite subcover. Let $y \in \bigcup_{k=1}^{n} B(x_k) \cap \Omega$, and $x \in \partial\Omega$ such that $y-x$ is perpendicular to $\partial\Omega$, therefore, we can write, $y - x = t\nu_x$. So, if $y \in \bigcup_{k=1}^{n} B_{t(x_k)}(x_k) \cap \Omega \subset \bigcup_{x_0 \in \partial\Omega} B_{t(x_0)}(x_0) \cap \Omega$, then $u_0(y) > 0$. Now, let $K = \Omega \setminus \bigcup_{k=1}^{n} B_{t(x_k)}(x_k)$ be closed and bounded, therefore compact, it follows there exists $\delta_2 > 0$ such that

$$u_0(y) > \delta_2 > 0 \quad \forall y \in K. \tag{2.5}$$

Moreover, from the compactness of K and using the norm of supremum we have

$$\|v^j - u_0\|_{L^\infty(\overline{\Omega})} \longrightarrow 0$$

therefore

$$|u_0(y) - v^j(y)| < \frac{\delta_2}{2} \quad \forall y \in K \text{ e } \forall j \geq j_0,$$

thus, by the triangular inequality $u_0(y) - \frac{\delta_2}{2} < v^j(y)$ e por (2.5) we have

$$\delta_2 - \frac{\delta_2}{2} < v^j(y), \quad \forall\, y \in K \text{ and } \forall\, j \geq j_0.$$

Then, $v^j(y) > 0$ for all $y \in K$ and for all $j \geq j_0$. On the other hand, for all $y \in \bigcup_{k=1}^{n} B_{t(x_k)}(x_k) \cap \Omega$, using again Taylor's formula and the fact that $\|v^j - u_0\|_{C^{1,\alpha}(\overline{\Omega})} \to 0$, if $j \to +\infty$, with $u|_{\partial\Omega}$ and denoting $o_1(j)$, where $o_j(1) \to 0$ when $j \to 0$ we have for $x \in \partial\Omega$ such that $y - x$ is perpendicular to $\partial\Omega$

$$\begin{aligned}
v^j(y) &= v^j(x) + \nabla v^j(x) \cdot (y - x) + o(\|y - x\|) \\
&= \left\langle \nabla v^j(x), t\frac{\nabla u_0(x)}{\|\nabla u_0(x)\|} \right\rangle + o(t) + o_1(j) \\
&= \left\langle \nabla v^j(x) - \nabla u_0(x) + \nabla u_0(x), t\frac{\nabla u_0(x)}{\|\nabla u_0(x)\|} \right\rangle + o(t) + o_1(j) \\
&= \left\langle \nabla v^j(x) - \nabla u_0(x), t\frac{\nabla u_0(x)}{\|\nabla u_0(x)\|} \right\rangle + t\|\nabla u_0(x)\| + o(t) + o_1(j) \\
&= \frac{t}{\|\nabla u_0(x)\|}\langle \nabla v^j(x) - \nabla u_0(x), \nabla u_0(x)\rangle + t\|\nabla u_0(x)\| + o(t) + o_1(j) \\
&\geq -\frac{t}{\|\nabla u_0(x)\|}\|\nabla v^j - \nabla u_0\|_\infty \|\nabla u_0(x)\| + \|\nabla u_0(x)\|t + o(t) + o_1(j) \\
&= -\|\nabla v^j - \nabla u_0\|_\infty t + \|\nabla u_0(x)\|t + o(t) + o_1(j) \\
&= (-\varepsilon + \delta)\frac{t}{2} > 0, \quad \forall j \geq j_0,
\end{aligned}$$

because $\|\nabla v^j - \nabla u_0\|_{L^\infty(\overline{\Omega})} < \varepsilon$ for all $n \geq n_0$, ε sufficiently small and (2.3). This concludes that $v^j(y) > 0$ for all y in Ω and for all $j \geq j_0$, which completes the proof. □

Lemma 2.12. *Assume that b satisfies (b_1). There exists $\rho > 0$ and $\delta > 0$ such that $\rho < 2\|u_0\|$,*

$$I(u) \geq \delta - m$$

for $u \in \partial B_\rho(u_0) \cap \mathcal{N}$.

Proof. First, let us recall that the Nehari manifold $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^0 = \{J^{-1}(0)\}$ is a closed subset in $H_0^1(\Omega)$. Furthermore, $I : \mathcal{N} \to \mathbb{R}$ is continuous and bounded from below, by the proof of Theorem 5 with $0 > I(u) \geq -m$.
Now, suppose by contradiction that for every fixed ρ with $0 < \rho < 2\|u_0\|$ there is a sequence $(u_n) \subset \mathcal{N} \cap \partial B_\rho(u_0)$ such that $I(u_n) \to -m = \inf_{u \in \mathcal{N}^+} I(u)$, as $n \to \infty$. Define $|\rho_j| = \frac{1}{j}$, so the sequence $(u_n^j) \subset \mathcal{N} \cap \partial B_{\rho_j}(u_0)$ satisfies $I(u_n^j) \to -m$, as $n \to \infty$. We can apply Ekeland's Variational Principle to $I|_{\mathcal{N}}$, where $\mathcal{N}$ is a closed metric space. Therefore, by Corollary 3 of [12], there is a sequence, for each fixed $j > 0$, $(v_n^j) \subset \mathcal{N} \cap \partial B_{\rho_j}(u_0)$ such that, if $n \to +\infty$

a) $I|_{\mathcal{N}}(v_n^j) \to -m$;

b) $\|v_n^j - u_n^j\| \to \rho_j$;

c) $\|I'|_{\mathcal{N}}(v_n^j)\| \to 0$.

This means that (v_n^j) is a (PS) sequence of $I|_{\mathcal{N}}$ the functional restricted to $\mathcal{N}^+$. Since $-m < 0$, by Lemma 2.7 (v_n^j) has a subsequence (PS) for I such that $I'(v_n^j) \to 0$ and by Theorem 8, up to a subsequence, we have that $v_n^j \to v^j$ if $n \to +\infty$. It follows from the continuity of I and the uniqueness of the limit that

$$I(v^j) = -m, \; I'(v^j) = 0, \; v^j \in \mathcal{N} \cap B_{\rho_j}(u_0)$$

and $\|v^j - u_0\|_{H_0^1(\Omega)} \to 0$. Taking $w^j = v^j - u_0$ and using regularity theory for elliptic operators, as in Section 2.3, we have that $\|v^j - u_0\|_{C^{1,\alpha}(\overline{\Omega})} \to 0$.

Since $\|v^j - u_0\| = \rho_j \to 0$, with $0 < \rho_j < 2\|u_0\|$ and $v^j > 0$ for j large enough, by Lemma 1.12, we have (v^j) we have that (v^j) is a sequence of positive critical points for I that converge to u_0 in the norm of $H_0^1(\Omega)$, which contradicts the uniqueness of the positive solution of I given by Theorem 5. □

Consider the translated functional $\tilde{I} : H_0^1(\Omega) \to \mathbb{R}$

$$\begin{aligned}\tilde{I}(u) &:= I(u) + m \\ &= \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda a(x)u^2)dx + \int_\Omega b(x)F(u)dx + m.\end{aligned}$$

Theorem 9. *Assume $\lambda_1(a) < \lambda < \lambda_1^0(a)$, b satisfies (b_1) and f satisfies $(f_1), (f_3), (f_4)$ and $(f_2)', (f_5)'$. Let $u_0 \geq 0$ and $-u_0 \leq 0$ be local minima of I on $\mathcal{N}^+$, then*

(i) $\tilde{I}(u_0) = 0$;

(ii) there exists $0 < \rho < 2\|u_0\|$ and $\delta > 0$ such that $\tilde{I}(u) \geq \delta > 0$ for any $u \in \partial B_\rho(u_0) \cap \mathcal{N}$;

(iii) $\tilde{I}(-u_0) = 0$.

Moreover, $\tilde{I}$ satisfies $(PS)_c$ condition with

$$0 < c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \tilde{I}(\gamma(t))$$

where $\Gamma := \{\gamma \in C([0,1], \mathcal{N}) : \gamma(0) = u_0, \gamma(1) = -u_0\}$. Then there exists a non-trivial solution u^* of problem (P) satisfying $I(u^*) = c^* > -m$, where $c^* = c - m$.

Proof. (i) $\tilde{I}(u_0) = I(u_0) + m = -m + m = 0$.
(ii) By Lemma 1.13 there are $\rho > 0$ and $\delta > 0$ such that

$$I(u) \geq \delta - m$$

for $u \in \partial B_\rho(u_0) \cap \mathcal{N}$.By the definition of the functional $\tilde{I}$ we have

$$\tilde{I}(u) = I(u) + m \geq \delta - m + m = \delta > 0$$

for $u \in \partial B_\rho(u_0) \cap \mathcal{N}$, and then we show the item (ii).
(iii) $\tilde{I}(-u_0) = I(-u_0) + m = -m + m = 0$ and $-u_0 \notin B_\rho(u_0)$, thus $\rho < \|u_0 - (-u_0)\| = 2\|u_0\|$.

Therefore, $\tilde{I}$ satisfies the geometry of the Mountain Pass Theorem, and so the same is true for I. Let us use Ghoussoub's Theorem [[14], Theorem 3.2]. Note that the Nehari manifold is a Finsler variety because it is a closed submanifold of class C^1, with $\mathcal{T}_u\mathcal{N}$ carying the norm induced by the inclusion $\mathcal{T}_u\mathcal{N} \subset \mathcal{T}_u H_0^1(\Omega) \cong H_0^1(\Omega)$ by [[26], Chapter II, Section 3.7]. We also have that the set $\mathcal{F} = \Gamma$ is a homotopically stable family. In fact, making $X = \mathcal{N}$ which is a complete metric space, then $B = \{-u_0, u_0\}$ is a closed subset in $\mathcal{N}$. Since $\gamma(0) = u_0$ and $\gamma(1) = -u_0$, we have that any element $\gamma([0,1])$ in Γ contains B. Furthermore, for all $A = \gamma([0,1]) \in \Gamma$ and $\eta : [0,1] \times \mathcal{N} \to \mathcal{N}$ continuous, satisfying $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times \mathcal{N}) \cup ([0,1] \times B)$ implies $\gamma \circ \eta(1) = -u_0 \in B$. Moreover, by item (ii),

$$0 \geq c^* = c - m > \{I(\gamma(0)),\ I(\gamma(1))\} = -m,$$

and thus satisfies hypothesis (F_0) of Ghoussoub's Theorem, then there exists a sequence (u_n) in $\mathcal{N}$ which is $(PS)_{c^*}$ restricted to $\mathcal{N}$. By Lemma 1.7 the (u_n) is a sequence $(PS)_{c^*}$ for the functional I in $H_0^1(\Omega)$ eand by Theorem 8 the functional satisfies the condition $(PS)_{c^*}$. Therefore, up to a subsequence, $u_n \to u^* \in \mathcal{N}$ such that

$$I(u^*) = c^* \text{ and } \ I'(u^*) = 0.$$

Hence, u^* is a critical point of the functional I restricted to $\mathcal{N}$, and $-m < c^* \leq 0$. As the Nehari manifold $\mathcal{N}$ is a natural constraint, u^* is a solution in $H_0^1(\Omega)$ for the problem (P'). □

Note that $u^* \in \mathcal{N}$ could be the trivial solution. In what follows, we will present a sufficient condition for $u^* \neq 0$.

Proof of the Theorem 6. We want to show that $I(u^*) = c^* < 0$, which implies that u^* is not trivial. Let us consider the first positive eigenfunction, normalized in $H_0^1(\Omega)$, and denoted by ϕ_1 associated with the first eigenvalue $\lambda_1(a)$ of the problem (P_1), the (normalized) eigenfunction ϕ_2 associated with the second eigenvalue $\lambda_2(a)$ of (P_1), ϕ_1^0 the first positive (normalized) eigenfunction in $H_0^1(\Omega_0)$ associated with the first eigenvalue $\lambda_1^0(a)$ of the problem (P_1) restricted to the Ω_0, and the same for ϕ_2^0. Note that the supports of ϕ_i^0, $i = 1, 2$ are subsets of Ω_0.

$$\int_\Omega a(x)\phi_1\phi_2 dx = 0 \tag{2.6}$$

and

$$\int_{\Omega_0} a(x)\phi_1^0\phi_2^0 dx = 0. \tag{2.7}$$

In fact, it follows from spectral theory that, $\int_\Omega \nabla\phi_1 \cdot \nabla\phi_2 dx = 0$ and $\int_{\Omega_0} \nabla\phi_1^0 \cdot \nabla\phi_2^0 dx = 0$. As eigenfunctions are regular functions, and Ω and Ω_0 are regular domains, by the Divergence Theorem, we have that

$$0 = \int_\Omega \nabla\phi_1\nabla\phi_2 dx = -\int_\Omega \Delta\phi_1\phi_2 dx = \int_\Omega \lambda_1(a)a(x)\phi_1\phi_2 dx.$$

Follows from the fact $\lambda_1(a) \neq 0$, that

$$\int_\Omega a(x)\phi_1\phi_2 dx = 0.$$

Similarly, using the divergence theorem, it follows that

$$\int_{\Omega_0} a(x)\phi_1^0\phi_2^0 dx = 0.$$

In order to construct a convenient path in Γ not passing through zero, define $w \in H_0^1(\Omega)$ by $w := t_1(\phi_1 + \varepsilon\phi_1^0) + t_2(\phi_2 + \varepsilon\phi_2^0)$ with constants $t_1, t_2 > 0$ and for some $\varepsilon > 0$, to be chosen sufficiently small. Using equalities (2.6), (2.7) and also the hypothesis (f_4) we obtain

$$
\begin{aligned}
I(w) &= \frac{1}{2}\int_\Omega (|\nabla w|^2 - \lambda a(x) w^2)dx + \int_\Omega b(x)F(w)dx \\
&= \frac{1}{2}\int_\Omega [|\nabla(t_1(\phi_1+\varepsilon\phi_1^0) + t_2(\phi_2+\varepsilon\phi_2^0))|^2 - \lambda a(x)(t_1(\phi_1+\varepsilon\phi_1^0) + t_2(\phi_2+\varepsilon\phi_2^0))^2]dx \\
&+ \int_\Omega b(x)F(w)dx \\
&= \frac{t_1^2}{2}\int_\Omega [\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\phi_1+\varepsilon\phi_1^0) - \lambda a(x)(\phi_1+\varepsilon\phi_1^0)(\phi_1+\varepsilon\phi_1^0)]dx \\
&+ \frac{t_1t_2}{2}\int_\Omega [\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\phi_2+\varepsilon\phi_2^0) - \lambda a(x)(\phi_1+\varepsilon\phi_1^0)(\phi_2+\varepsilon\phi_2^0)]dx \\
&+ \frac{t_2t_1}{2}\int_\Omega [\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\phi_1+\varepsilon\phi_1^0) - \lambda a(x)(\phi_2+\varepsilon\phi_2^0)(\phi_1+\varepsilon\phi_1^0)]dx \\
&+ \frac{t_2^2}{2}\int_\Omega [\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\phi_2+\varepsilon\phi_2^0) - \lambda a(x)(\phi_2+\varepsilon\phi_2^0)(\phi_2+\varepsilon\phi_2^0)]dx \\
&+ \int_\Omega b(x)F(w)dx \\
&= \frac{t_1^2}{2}\int_\Omega [\nabla(\phi_1+\varepsilon\phi_1^0)\nabla\phi_1 - \lambda a(x)(\phi_1+\varepsilon\phi_1^0)\phi_1]dx + \frac{t_1t_2}{2}\int_\Omega [\nabla(\phi_1+\varepsilon\phi_1^0)\nabla\phi_2 \\
&- \lambda a(x)(\phi_1+\varepsilon\phi_1^0)\phi_2]dx + \frac{t_2t_1}{2}\int_\Omega [\nabla(\phi_2+\varepsilon\phi_2^0)\nabla\phi_1 - \lambda a(x)(\phi_2+\varepsilon\phi_2^0)\phi_1]dx \\
&+ \frac{t_2^2}{2}\int_\Omega [\nabla(\phi_2+\varepsilon\phi_2^0)\nabla\phi_2 - \lambda a(x)(\phi_2+\varepsilon\phi_2^0)\phi_2]dx + \frac{t_1t_1}{2}\int_\Omega [\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\varepsilon\phi_1^0) \\
&- \lambda a(x)(\phi_1+\varepsilon\phi_1^0)(\varepsilon\phi_1^0)]dx + \frac{t_1t_2}{2}\int_\Omega [\nabla(\phi_1+\varepsilon\phi_1^0)\nabla(\varepsilon\phi_2^0) - \lambda a(x)(\phi_1+\varepsilon\phi_1^0)(\varepsilon\phi_2^0)]dx \\
&+ \frac{t_2t_1}{2}\int_\Omega [\nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\varepsilon\phi_1^0) - \lambda a(x)(\phi_2+\varepsilon\phi_2^0)(\varepsilon\phi_1^0)]dx + \frac{t_2t_2}{2}\int_\Omega \nabla(\phi_2+\varepsilon\phi_2^0)\nabla(\varepsilon\phi_2^0) \\
&- \lambda\int_\Omega a(x)(\phi_2+\varepsilon\phi_2^0)(\varepsilon\phi_2^0)dx + \int_\Omega b(x)F(w)dx \\
&= \frac{t_1^2}{2}\int_\Omega (|\nabla\phi_1|^2 - \lambda a(x)\phi_1^2)dx + \frac{t_1^2}{2}\varepsilon\int_\Omega (\nabla\phi_1^0\nabla\phi_1 - \lambda a(x)\phi_1^0\phi_1)dx \\
&+ \frac{t_1t_2}{2}\int_\Omega (\nabla\phi_1\nabla\phi_2 - \lambda a(x)\phi_1\phi_2)dx + \frac{t_1t_2}{2}\varepsilon\int_\Omega (\nabla\phi_1^0\nabla\phi_2 - \lambda a(x)\phi_1^0\phi_2)dx \\
&+ \frac{t_2t_1}{2}\int_\Omega (\nabla\phi_2\nabla\phi_1 - \lambda a(x)\phi_2\phi_1)dx + \frac{t_2t_1}{2}\varepsilon\int_\Omega (\nabla\phi_2^0\nabla\phi_1 - \lambda a(x)\phi_2^0\phi_1)dx \\
&+ \frac{t_2^2}{2}\int_\Omega (|\nabla\phi_2|^2 - \lambda a(x)\phi_2^2)dx + \frac{t_2^2}{2}\varepsilon\int_\Omega (\nabla\phi_2^0\nabla\phi_2 - \lambda a(x)\phi_2^0\phi_2)dx \\
&+ \frac{t_1^2}{2}\varepsilon\int_\Omega [\nabla\phi_1\nabla\phi_1^0 - \lambda a(x)\phi_1\phi_1^0]dx + \frac{t_1^2}{2}\varepsilon^2\int_\Omega [|\nabla\phi_1^0|^2 - \lambda a(x)(\phi_1^0)^2]dx \\
&+ \frac{t_1t_2}{2}\varepsilon\int_\Omega [\nabla\phi_1\nabla\phi_2^0 - \lambda a(x)\phi_1\phi_2^0]dx + \frac{t_1t_2}{2}\varepsilon^2\int_\Omega [\nabla\phi_1^0\nabla\phi_2^0 - \lambda a(x)\phi_1^0\phi_2^0]dx \\
&+ \frac{t_2t_1}{2}\varepsilon\int_\Omega [\nabla\phi_2\nabla\phi_1^0 - \lambda a(x)\phi_2\phi_1^0]dx + \frac{t_2t_1}{2}\varepsilon^2\int_\Omega [\nabla\phi_2^0\nabla\phi_1^0 - \lambda a(x)\phi_2^0\phi_1^0]dx \\
&+ \frac{t_2^2}{2}\varepsilon\int_\Omega [\nabla\phi_2\nabla\phi_2^0 - \lambda a(x)\phi_2\phi_2^0]dx + \frac{t_2}{2}\varepsilon^2\int_\Omega [|\nabla\phi_2^0|^2 - \lambda a(x)(\phi_2^0)^2]dx + \int_\Omega b(x)F(w)dx.
\end{aligned}
$$

Thus, by (2.6) and (2.7),

$$
\begin{aligned}
I(w) \leq & \frac{t_1^2}{2}(\lambda_1(a)-\lambda)\int_\Omega a(x)\phi_1^2 dx + \frac{t_2^2}{2}(\lambda_2(a)-\lambda)\int_\Omega a(x)\phi_2^2 dx \\
&+ \frac{t_1^2}{2}\varepsilon^2(\lambda_1^0(a)-\lambda)\int_\Omega a(x)(\phi_1^0)^2 dx \\
&+ \frac{t_2^2}{2}\varepsilon^2(\lambda_2^0(a)-\lambda)\int_\Omega a(x)(\phi_2^0)^2 dx + \varepsilon\sum_{i=1}^{2}\sum_{j=1}^{2} t_i t_j \left\{\int_\Omega (\nabla\phi_i\nabla\phi_j^0 - \lambda a(x)\phi_i\phi_j^0)dx\right\} \\
&+ \int_\Omega b(x)F(w)dx.
\end{aligned}
$$

Note that,

$$
\begin{aligned}
\|w\| &= \left(\int_\Omega |\nabla(t_1(\phi_1+\varepsilon\phi_1^0)+t_2(\phi_2+\varepsilon\phi_2^0))|^2 dx\right)^{\frac{1}{2}} \\
&= \left(t_1^2\int_\Omega |\nabla\phi_1|^2 dx + t_1^2\varepsilon^2\int_\Omega |\nabla\phi_1^0|^2 dx + t_2^2\int_\Omega |\nabla\phi_2|^2 dx + t_2^2\varepsilon^2\int_\Omega |\nabla\phi_2^0|^2 dx\right. \\
&\left. + \varepsilon\sum_{i=1}^{2}\sum_{j=1}^{2} t_i t_j \int_\Omega \nabla\phi_i\nabla\phi_j^0 dx\right)^{\frac{1}{2}} \\
&\leq (t_1^2+t_2^2+t_1^2\varepsilon^2+t_2^2\varepsilon^2+\varepsilon(t_1^2+2t_1t_2+t_2^2)\|\nabla\phi_i\|_2^2\|\nabla\phi_j\|_2^2)^{\frac{1}{2}} \\
&= (t_1^2+t_2^2+t_1^2\varepsilon^2+t_2^2\varepsilon^2+\varepsilon(t_1^2+2t_1t_2+t_2^2) \\
&\leq (t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2)+2\varepsilon(t_1^2+t_2^2))^{\frac{1}{2}} \leq \sqrt{t_1^2+t_2^2+3\varepsilon^2(t_1^2+t_2^2)}. \qquad (2.8)
\end{aligned}
$$

Furthermore, using (2.8) and Remark 2.1, considering the same real number $\varepsilon > 0$ in the definition of the function w, we have

$$
\begin{aligned}
\int_\Omega b(x)F(w)dx &\leq \int_\Omega b(x)\left(\frac{\varepsilon}{2}|w|^2 + \frac{C_2}{q}|w|^q\right)dx \\
&\leq \|b\|_\infty \frac{\varepsilon}{2}\int_\Omega |w|^2 dx + \|b\|_\infty \frac{C_2}{q}\int_\Omega |w|^q dx \\
&\leq C\left[\varepsilon\|w\|^2 + C\|w\|^q\right] \\
&\leq C\left[\varepsilon(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2)) + (t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}}\right].
\end{aligned}
$$

Using Holder's inequality for $p = q = 2$ and that ϕ_i, ϕ_j^0 are normalized eigenfunctions we have

$$
\begin{aligned}
\varepsilon\sum_{i=1}^{2}\sum_{j=1}^{2} t_i t_j \left\{\int_\Omega (\nabla\phi_i\nabla\phi_j^0 - \lambda a(x)\phi_i\phi_j^0)dx\right\} &\leq \varepsilon\sum_{i=1}^{2}\sum_{j=1}^{2} t_i t_j \left\{\int_\Omega |\nabla\phi_i\nabla\phi_j^0 - \lambda a(x)\phi_i\phi_j^0| dx\right\} \\
&\leq \varepsilon\sum_{i=1}^{2}\sum_{j=1}^{2} t_i t_j \left\{\int_\Omega (|\nabla\phi_i\nabla\phi_j^0| + \lambda|a(x)\phi_i\phi_j^0|)dx\right\} \\
&\leq \varepsilon(t_1^2+2t_1t_2+t_2^2)(\|\nabla\phi_1\|_2\|\nabla\phi_j^0\|_2 \\
&+ \lambda\|a\|_\infty\|\phi_i\|_2\|\phi_j\|_2) \\
&\leq 2\varepsilon(t_1^2+t_2^2)(1+C\lambda\|a\|_\infty) \\
&= C\varepsilon(t_1^2+t_2^2).
\end{aligned}
$$

Taking, $0<\varepsilon<1$ and $0<t_1^2+t_2^2<1$, since $\frac{q}{2}>1$, its follows

$$
\begin{aligned}
I(w) \leq & \frac{(t_1^2+t_2^2)}{2}C\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}+\frac{(t_1^2+t_2^2)}{2}\varepsilon^2C\max\{\lambda_1^0(a)-\lambda,\lambda_2^0(a)-\lambda\}\\
&+\varepsilon C(t_1^2+t_2^2)+C[\varepsilon(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2))+(t_1^2+t_2^2+\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}}]\\
\leq & \frac{(t_1^2+t_2^2)}{2}C\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}+\frac{(t_1^2+t_2^2)}{2}\varepsilon^2C\max\{\lambda_1^0(a)-\lambda,\lambda_2^0(a)-\lambda\}\\
&+C\varepsilon(t_1^2+t_2^2)+C\varepsilon(t_1^2+t_2^2)+C\varepsilon^2(t_1^2+t_2^2)+C2^{\frac{q}{2}-1}(t_1^2+t_2^2)^{\frac{q}{2}}+C2^{\frac{q}{2}-1}(\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}}\\
\leq & \frac{(t_1^2+t_2^2)}{2}C\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}+\frac{(t_1^2+t_2^2)}{2}\varepsilon^2C\max\{\lambda_1^0(a)-\lambda,\lambda_2^0(a)-\lambda\}\\
&+C\varepsilon(t_1^2+t_2^2)+C\varepsilon^2(t_1^2+t_2^2)+C(t_1^2+t_2^2)^{\frac{q}{2}}+C(\varepsilon(t_1^2+t_2^2))^{\frac{q}{2}}\\
\leq & \frac{(t_1^2+t_2^2)}{2}C\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}+O(\varepsilon(t_1^2+t_2^2)),
\end{aligned}
$$

where $C=C(\|a\|_\infty)$. Without loss of generality, we can assume $\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}=\lambda_1(a)-\lambda<0$, thus

$$
\begin{aligned}
I(w) &\leq \frac{(t_1^2+t_2^2)}{2}C(\lambda_1(a)-\lambda)+C\varepsilon(t_1^2+t_2^2)\\
&= C(t_1^2+t_2^2)\left[\frac{\lambda_1(a)-\lambda}{2}+C\varepsilon\right].
\end{aligned}
$$

Taking $0<\varepsilon\leq\frac{\lambda-\lambda_1(a)}{4}$ then

$$
I(w)\leq C(t_1^2+t_2^2)\left(\frac{(\lambda_1(a)-\lambda)}{4}\right):=-\delta_1<0.
$$

Now, let $w_1:=t_1(\phi_1+\varepsilon\phi_1^0)$, $w_2:=t_2(\phi_2+\varepsilon\phi_2^0)$ and $w_\theta:=\cos(\theta)w_1+\sin(\theta)w_2$, such that

$$
\begin{aligned}
w_{\frac{\pi}{4}} &= \cos\left(\frac{\pi}{4}\right)(t_1(\phi_1+\varepsilon\phi_1^0))+\sin\left(\frac{\pi}{4}\right)(t_2(\phi_2+\varepsilon\phi_2^0))\\
&= \frac{\sqrt{2}}{2}\left[t_1(\phi_1+\varepsilon\phi_1^0)+t_2(\phi_2+\varepsilon\phi_2^0)\right]\\
&= \frac{\sqrt{2}}{2}w
\end{aligned}
$$

with $\|w_{\frac{\pi}{4}}\|=\left\|\frac{\sqrt{2}}{2}w\right\|=\frac{\sqrt{2}}{2}\|w\|$, and for all $\theta\in[0,\pi]$,

$$
\begin{aligned}
I(w_\theta) &\leq C\frac{t_1^2}{2}\cos^2(\theta)(\lambda_1(a)-\lambda)+C\frac{t_2^2}{2}\sin^2(\theta)(\lambda_2(a)-\lambda)+Ct_1^2\varepsilon\cos^2(\theta)\\
&+ C\sin(\theta)\cos(\theta)t_1t_2\varepsilon+Ct_2^2\varepsilon\sin^2(\theta)+C\varepsilon^2\frac{t_1^2}{2}\cos^2(\theta)(\lambda_1^0(a)-\lambda)+\\
&+ C\varepsilon^2\frac{t_2^2}{2}\sin^2(\theta)(\lambda_2^0(a)-\lambda)+C(\varepsilon\|w_\theta\|^2+\|w_\theta\|^q)\\
&\leq C(\cos^2(\theta)t_1^2+\sin^2(\theta)t_2^2)\{\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}\\
&+ O(\varepsilon(t_1^2+t_2^2))\\
&\leq C\left(\cos^2(\theta)t_1^2+\sin^2(\theta)t_2^2\right)\Bigg[\max\{\lambda_1(a)-\lambda,\lambda_2(a)-\lambda\}\\
&+ \frac{C(\varepsilon(t_1^2+t_2^2))}{\cos^2(\theta)t_1^2+\sin^2(\theta)t_2^2}\Bigg].
\end{aligned}
$$

where $(t_1^2+t_2^2)/(\cos^2(\theta)t_1^2+\sin^2(\theta)t_2^2)$ is bounded by positive constants from below and above, uniformly for $\theta\in[0,\pi]$, then similarly to the calculations for $I(w)$, we obtain that there exists $\delta_2>0$ such that

$$I(w_\theta)<-\delta_2<0,$$

for all $\theta\in[0,\pi]$. Finally, define the following curve in $H_0^1(\Omega)$ given by

$$\gamma(s):=\begin{cases}[(1-3s)u_0+3s(w_1)], & s\in[0,1/3]\\ w_{\theta(s)}, \quad s\in[1/3,2/3] \quad \text{e} \quad \theta(s)=3(s-1/3)\pi\\ [3(1-s)(-w_1)+3(s-2/3)(-u_0)], & s\in[2/3,1].\end{cases}$$

Now, let us show that for $0\le s\le 1$ there exists $t\gamma(s)$ such that $\tilde{\gamma}(s):=t_{\gamma(s)}\gamma\in\mathcal{N}^+$. Note that $\int_\Omega(|\nabla\gamma(s)|^2-\lambda\gamma(s)^2)dx<0$ when $s\in[0,1/3]$, $s\in[1/3,2/3]$ and $s\in[2/3,1]$. In fact, for $s\in[0,1/3]$, we have

$$\begin{aligned}\int_\Omega\Big(|\nabla\gamma(s)|^2-\lambda a(x)\gamma(s)^2\Big)dx &= \int_\Omega|\nabla((1-3s)u_0+3s(w_1))|^2dx\\ &- \lambda\int_\Omega a(x)((1-3s)u_0+3s(w_1))^2dx\\ &= (1-3s)^2\int_\Omega\Big(|\nabla u_0|^2-\lambda a(x)u_0^2\Big)dx\\ &+ 2(1-3s)(3s)\int_\Omega\Big(\nabla u_0\nabla w_1-\lambda a(x)u_0w_1\Big)dx\\ &+ (3s)^2\int_\Omega(|\nabla w_1|^2-\lambda a(x)(w_1)^2)dx.\end{aligned}$$

Notice that

$$\begin{aligned}\int_\Omega\Big(|\nabla w_1|^2-\lambda a(x)w_1^2\Big)dx &= \int_\Omega\Big(|\nabla(t_1(\phi_1+\varepsilon\phi_1^0))|^2-\lambda a(x)(t_1(\phi_1+\varepsilon\phi_1^0))^2\Big)dx\\ &= t_1^2\int_\Omega\Big(|\nabla\phi_1|^2-\lambda a(x)\phi_1^2\Big)dx\\ &+ 2t_1^2\varepsilon\int_\Omega\Big(\nabla\phi_1\nabla\phi_1^0-\lambda a(x)\phi_1\phi_1^0\Big)dx \qquad (2.9)\\ &+ t_1^2\varepsilon^2\int_\Omega\Big(|\nabla\phi_1^0|^2-\lambda a(x)(\phi_1^0)^2\Big)dx\\ &\le C\|a\|_\infty t_1^2(\lambda_1(a)-\lambda)+C\|a\|_\infty t_1^2\varepsilon^2(\lambda_1^0(a)-\lambda)\\ &+ Ct_1^2\varepsilon \qquad (2.10)\\ &\le Ct_1^2(\lambda_1(a)-\lambda)+O(\varepsilon t_1^2)<0. \qquad (2.11)\end{aligned}$$

On the other hand, since u_0 is a positive solution to the problem (P), $b(x)\ge 0$, f is continuous and $w_1>0$, because, $\phi_1,\phi_1^0>0$, using the weak formulation, we have

$$\int_\Omega\Big(\nabla u_0\nabla w_1-\lambda a(x)u_0w_1\Big)dx=-\int_\Omega b(x)f(u_0)w_1dx<0. \qquad (2.12)$$

In addition,

$$\int_\Omega\Big(|\nabla u_0|^2-\lambda a(x)u_0^2\Big)dx=-\int_\Omega b(x)f(u_0)u_0dx<0. \qquad (2.13)$$

Finally, by (2.9), (2.12) and (2.13), for $s\in[0,1/3]$,

$$\int_\Omega\Big(|\nabla\gamma(s)|^2-\lambda a(x)\gamma(s)^2\Big)dx<0.$$

Let $s \in [1/3, 2/3]$, we have

$$\begin{aligned}
\int_\Omega \Big(|\nabla\gamma(s)|^2 - \lambda a(x)\gamma(s)^2\Big)dx &= \int_\Omega \Big(|\nabla w_\theta|^2 - \lambda a(x)(w_\theta)^2\Big)dx \\
&= \int_\Omega \Big(|\nabla(\cos(\theta)w_1 + \sin(\theta)w_2)|^2 - \lambda a(x)((\cos(\theta)w_1 + \sin(\theta)w_2)\Big)dx \\
&= \cos^2(\theta)\int_\Omega \Big(|\nabla w_1|^2 - \lambda a(x)w_1^2\Big)dx \\
&+ 2\cos(\theta)\sin(\theta)\int_\Omega \Big(\nabla w_1 \nabla w_2 - \lambda a(x)w_1 w_2\Big)dx \\
&+ \sin^2(\theta)\int_\Omega \Big(|\nabla w_2|^2 - \lambda a(x)w_2^2\Big)dx
\end{aligned}$$

Using the Hölder inequality

$$\begin{aligned}
\int_\Omega \Big(\nabla w_1\nabla w_2 - \lambda a(x)w_1w_2\Big)dx &= \int_\Omega \nabla(t_1(\phi_1+\varepsilon\phi_1^0))\nabla(t_2(\phi_2+\varepsilon\phi_2^0))dx \\
&- \lambda\int_\Omega a(x)(t_1(\phi_1+\varepsilon\phi_1^0))t_2((\phi_2+\varepsilon\phi_2^0))dx \\
&= t_1t_2\int_\Omega (\nabla\phi_1\nabla\phi_2 - \lambda a(x)\phi_1\phi_2)dx + t_1t_2\varepsilon\int_\Omega(\nabla\phi_1\nabla\phi_1^0 - \lambda a(x)\phi_1\phi_1^0)dx \\
&+ t_1t_2\varepsilon\int_\Omega(\nabla\phi_1^0\nabla\phi_2 - \lambda a(x)\phi_1^0\phi_2)dx + t_1t_2\varepsilon^2\int_\Omega(\nabla\phi_1^0\phi_2^0 - \lambda a(x)\phi_1^0\phi_2^0)dx \\
&\le t_1t_2\varepsilon(\|\nabla\phi_1\|_2\|\nabla\phi_2^0\|_2 + \lambda\|a\|_\infty\|\phi_1\|_2\|\phi_2^0\|_2) \\
&+ t_1t_2\varepsilon(\|\nabla\phi_1\|_2\|\nabla\phi_2^0\|_2 + \lambda\|a\|_\infty\|\phi_1\|_2\|\phi_2^0\|_2) = Ct_1t_2\varepsilon \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \Big(|\nabla w_2|^2 - \lambda a(x)w_2^2\Big)dx &= \int_\Omega\Big(|\nabla(t_2(\phi_2+\varepsilon\phi_2^0))|^2 - \lambda a(x)(t_2(\phi_2+\varepsilon\phi_2^0))^2\Big)dx \\
&= t_2^2\int_\Omega\Big(|\nabla\phi_2|^2 - \lambda a(x)\phi_2^2\Big)dx + 2t_2^2\varepsilon\int_\Omega\Big(\nabla\phi_2\nabla\phi_2^0 - \lambda a(x)\phi_2\phi_2^0\Big)dx \\
&+ t_2^2\varepsilon^2\int_\Omega\Big(|\nabla\phi_2^0|^2 - \lambda a(x)(\phi_2^0)^2\Big)dx \\
&\le C\|a\|_\infty t_2^2(\lambda_2(a)-\lambda) + C\|a\|_\infty t_2^2\varepsilon^2(\lambda_2^0(a)-\lambda) + Ct_2^2\varepsilon \\
&\le Ct_2^2(\lambda_2(a)-\lambda) + O(\varepsilon t_2^2) < 0. \qquad (2.15)
\end{aligned}$$

Using (2.9), (2.14) and (2.15) we obtain

$$\begin{aligned}
\int_\Omega\Big(|\nabla\gamma(s)|^2 - \lambda a(x)\gamma(s)^2\Big)dx &\le \cos^2(\theta)Ct_1^2(\lambda_1(a)-\lambda) + O(\varepsilon t_1^2) \\
&+ 2\cos(\theta)\sin(\theta)C\varepsilon t_1t_2 + \sin^2(\theta)Ct_2^2(\lambda_2(a)-\lambda) + O(\varepsilon t_2^2) \\
&\le C(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)\max\{\lambda_1(a)-\lambda, \lambda_2(a)-\lambda\} \\
&+ O(\varepsilon t_1^2) + O(\varepsilon t_2^2) + 2\cos^2(\theta)\sin(\theta)C\varepsilon t_1t_2 < 0,
\end{aligned}$$

because $(t_1^2+t_2^2)/(\cos^2(\theta)t_1^2 + \sin^2(\theta)t_2^2)$ is bounded by positive constants from below and above, uniformly for $\theta \in [0,\pi]$ and t_1 and t_2 positive, and $\varepsilon > 0$ is sufficiently small.

Let $s \in [2/3, 1]$, using the equalities in (1.9), (1.10) and (1.11), we have

$$\begin{aligned}
\int_\Omega (|\nabla\gamma(s)|^2 &- \lambda a(x)(\gamma(s))^2)dx = \int_\Omega |\nabla(3(1-s)(-w_1) + 3(s-2/3)(-u_0))|^2 dx \\
&- \lambda \int_\Omega a(x)(3(1-s)(-w_1) + 3(s-2/3)(-u_0))^2 dx \\
&= (3(1-s))^2 \int_\Omega \Big(|\nabla w_1|^2 - \lambda a(x) w_1^2\Big) dx + (3(s-2/3))^2 \int_\Omega \Big(|\nabla u_0|^2 - \lambda a(x) u_0^2)\Big) dx \\
&+ 6(1-s)3(s-\frac{2}{3}) \int_\Omega \Big(\nabla w_1 \nabla u_0 - \lambda a(x) w_1 u_0\Big) dx \\
&< 0.
\end{aligned}$$

Recalling Lemma 2.10, we have that $t_{\gamma(s)}$ is continuous and then $\tilde{\gamma}(s) \in \mathcal{N}^+$ for all $s \in [0, 1]$. From Lemma 2.4 and by (2.10) we conclude that $I(\tilde{\gamma}(s)) < 0$ for all $s \in [0, 1]$. Thus, $I(\tilde{\gamma}(s)) \leq \max_{0 \leq s \leq 1} I(\tilde{\gamma}(s)) < 0$ for $s \in [0, 1]$. By the definition of the min-max level c, it follows that $I(u^*) = c^* < \max_{0 \leq s \leq 1} I(\tilde{\gamma}(s)) < 0$.

Finally, suppose by contradiction that u^* is non-trivial and non-negative. Then the set $\tilde{\Omega} \subset \Omega$ in which $u^* = 0$ is bounded, and the set of boundary points $\partial\tilde{\Omega} \subset \Omega$ is bounded. Let $x_0 \in \partial\tilde{\Omega}$ be such that $u^*(x_0) = 0$. Furthermore, since $u^* \in C^1$ (see section 2.1), then $\partial\tilde{\Omega}$ is regular and compact.

Given $\delta > 0$ sufficiently small, there exists $\delta_1 > 0$ such that, if $x \in N_{\delta_1} := \{x \in \Omega\backslash\tilde{\Omega} : dist(x, \partial\tilde{\Omega}) < \delta_1\}$, then $|u^*(x)| < \delta$. It follows from hypothesis (f_1) that $f(u^*(x)) = o(|u^*(x)|)u^*(x)$ for all $x \in N_{\delta_1}$, and since $u^* \geq 0$ by assumption, then $-\Delta u^*(x) = \lambda a(x) u^*(x) - o(|u^*(x)|) \geq 0$, for all $x \in N_{\delta_1}$. Moreover $u^*(x) > 0$ for all $int(N_{\delta_1})$, thus $\inf_{N_{\delta_1}} u^*(x) = 0$.

Note that N_{δ_1} is regular because the subset $\partial\tilde{\Omega}$ is regular and u^* is continuous in $\overline{N}_{\delta_1}$. Then, by Hopf lemma $\frac{\partial u^*}{\partial \nu_x}(x) > 0$, for all x such that $u^*(x) \in \partial\tilde{\Omega}$, and ν_x is the exterior normal vector to $\partial\tilde{\Omega}$ at x, namely $Du^*(x) \neq 0$, which is impossible in an interior minimum point of u^*. Thus, $u^* > 0$ which is impossible by the uniqueness of the positive solution. The same result we obtain when u^* is non-positive. Therefore, u^* changes sign and the proof is complete.

□

Appendix A

Spectral Theory in Bounded Domains

In this section, we want to develop a study of the eigenvalue problem with indefinite weight in a limited domain. We see that it is possible to guarantee the existence of a sequence of positive and negative eigenvalues and other properties. The existence of the first eigenvalue with its respective first eigenfunction was fundamental in the first chapter, since we used it to show that some sets were non-empty. In addition, other properties of the eigenvalue problem with weight were used in Chapter 2. This section is based on [8] and [16].

Consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \mu m(x) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P_3}$$

where $m(x) \in L^r(\Omega)$, with $r > \frac{N}{2}$, and changes sign.

Let

$$a(u,v) = \int_\Omega \nabla u \nabla v dx$$

and consider

$$\begin{cases} a(u,v) = \mu \int_\Omega m(x) uv dx \quad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{PA}$$

We know that

i) $a(u,v) = a(v,u)$,

ii) $|a(u,v)| \leq \|u\|\|v\|$,

iii) $a(u,u) \geq C\|u\|^2$.

Notice that fixed $u \in H_0^1(\Omega)$, the functional

$$v \to \int_\Omega muv dx$$

is linear in $H_0^1(\Omega)$. By Riesz's representation theorem, there exists an element in $H_0^1(\Omega)$, denoted by Tu, such that

$$(Tu, v)_a = \int_\Omega muv dx.$$

We have the functional above is symmetrical and bounded. Indeed

$$
\begin{aligned}
\|Tu\|_a^2 &= (Tu, Tu)_a \\
&= \int_\Omega muTu dx \\
&\leq \left(\int_\Omega |m|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int_\Omega (|u||Tu|)^{\frac{N}{N-2}} dx\right)^{\frac{N-2}{N}} \\
&\leq \|m\|_{L^{\frac{N}{2}}} \left(\int_\Omega |u|^{\frac{N}{N-2}} |Tu|^{\frac{N}{N-2}} dx\right)^{\frac{N-2}{N}} \\
&\leq \|m\|_{\frac{N}{2}} \left(\left(\int_\Omega |u|^{\frac{2N}{N-2}} dx\right)^{\frac{1}{2}} \left(\int_\Omega |Tu|^{\frac{2N}{N-2}} dx\right)^{\frac{1}{2}}\right)^{\frac{N-2}{N}} \\
&= \|m\|_{\frac{N}{2}} \left(\int_\Omega |u|^{2*} dx\right)^{\frac{1}{2*}} \left(\int_\Omega |Tu|^{2*} dx\right)^{\frac{1}{2*}} \\
&= \|m\|_{\frac{N}{2}} \|u\|_{2*} \|Tu\|_{2*} \\
&\leq C_1 \|m\|_{\frac{N}{2}} \|u\| \|Tu\|.
\end{aligned}
$$

Let us show that T is compact. Let (u_n) be a bounded sequence at $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive, then up to a subsequence, we have $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Therefore,

$$
\begin{aligned}
\|Tu_n - Tu\|^2 &\leq (Tu_n - Tu, Tu_n - Tu) \\
&= \int_\Omega m(u_n - u)(Tu_n - Tu) dx \\
&\leq \left(\int_\Omega |m|^r\right)^{\frac{1}{r}} \left(\int_\Omega |u_n - u|^{r'} |Tu_n - Tu|^{r'} dx\right)^{\frac{1}{r'}} \\
&= \|m\|_{L^r} \left(\left(\int_\Omega |u_n - u|^{r'\theta'} dx\right)^{\frac{1}{\theta'}} \left(\int_\Omega |Tu_n - Tu|^{r'\theta} dx\right)^{\frac{1}{\theta}}\right)^{\frac{1}{r'}} \\
&= \|m\|_{L^r} \left(\left(\int_\Omega |u_n - u|^{r' \frac{2^*}{2^* - r'}}\right)^{\frac{2^* - r'}{2^*}} \left(\int_\Omega |Tu_n - Tu|^{r' \frac{2^*}{r'}} dx\right)^{\frac{r'}{2^*}}\right)^{\frac{1}{r'}} \\
&= \|m\|_{L^r} \left(\int_\Omega |u_n - u|^{r' \frac{2^*}{2^* - r'}}\right)^{\frac{2^* - r'}{2^*} \frac{1}{r'}} \left(\int_\Omega |Tu_n - Tu|^{2^*} dx\right)^{\frac{1}{2^*}} \\
&= \|m\|_{L^r} \left(\int_\Omega |u_n - u|^s\right)^{\frac{1}{s}} \left(\int_\Omega |Tu_n - Tu|^{2^*} dx\right)^{\frac{1}{2^*}} \\
&\leq C_1 \|m\|_{L^r} \left(\int_\Omega |u_n - u|^s\right)^{\frac{1}{s}} \|Tu_n - Tu\|.
\end{aligned}
$$

Since $u_n \to u$ in $L^s(\Omega)$, then $Tu_n \to Tu$ in $H_0^1(\Omega)$, so T is compact. Thus, we have the following characterizations:

Lemma A.1. *If*

$$\lambda_1 = \sup\{(Tx, x) : \|x\| = 1\} > 0,$$

then there is $\varphi_1 \in H_0^1(\Omega)$, *with* $\|\varphi_1\| = 1$, *such that*

$$(T\varphi_1, \varphi_1) = \lambda_1, \quad T\varphi_1 = \lambda_1 \varphi_1.$$

Lemma A.2. *If*

$$\lambda_1 = \inf\{(Tx, x) : \|x\| = 1\} < 0,$$

then there is $\varphi_{-1} \in H_0^1(\Omega)$, *with* $\|\varphi_{-1}\| = 1$, *such that*

$$(T\varphi_{-1}, \varphi_{-1}) = \lambda_{-1}, \quad T\varphi_{-1} = \lambda_{-1}\varphi_{-1}.$$

Lemma A.3. *For each* n *positive,*

$$\lambda_n = \sup_{F_n} \inf\{(Tx, x) : \|x\| = 1, x \in F_n\},$$

where the supreme is taken over all subspaces F_n *of* H *with dimension* n. *Similarly, we present the description of* λ_{-n},

$$\lambda_{-n} = \inf_{F_n} \sup\{(Tx, x) : \|x\| = 1, x \in F_n\}.$$

Remark A.1. *By Lemma A.3, we can write*

$$\lambda_n = \sup_{F_n} \inf\{(Tx, x) : \|x\| = 1, x \in F_n\}$$

and

$$\lambda_{-n} = \inf_{F_n} \sup\{(Tx, x) : \|x\| = 1, x \in F_n\}.$$

Theorem A.1. *Let* $m, \tilde{m} : \overline{\Omega} \to \mathbb{R}$ *be functions in* $L^r(\Omega)$, *with* $r > \frac{N}{2}$, *such that* $m(x) \leq \tilde{m}(x)$ *for* $x \in \Omega$. *Suppose that for a given* n, $n = \pm 1, \pm 2, \ldots$, *the eigenvalues* $\mu_n(m)$ *and* $\mu_n(\tilde{m})$ *exist. Then,*

$$\mu_n(m) \geq \mu_n(\tilde{m}).$$

Proof. Let $u \in F_n$ be such that $\|u\| = 1$. Since $m(x) \leq \tilde{m}(x)$, we have

$$\int_\Omega mu^2 dx \leq \int_\Omega \tilde{m}u^2 dx,$$

which implies that

$$\frac{1}{\mu_n(m)} = \sup_{F_n} \inf \int_\Omega mu^2 dx \leq \sup_{F_n} \inf \int_\Omega \tilde{m}u^2 dx = \frac{1}{\mu_n(\tilde{m})}.$$

□

Let $\mu_n(m)$ be a continuous function of m in the norm of $L^{\frac{N}{2}}(\Omega)$. In other words, if $m_j \in L^r(\Omega)$ converges in the norm of $L^{\frac{N}{2}}(\Omega)$, to $m \in L^r(\Omega)$, then

$$\mu_n(m_j) \to \mu_n(m).$$

Proof. Since $m_j \in L^r(\Omega)$ converges in the norm of $L^{\frac{N}{2}}(\Omega)$, to $m \in L^r(\Omega)$ we have that $m_j(x)$ converges to $m(x)$ almost always in Ω and there exists $g \in L^{\frac{N}{2}}(\Omega)$ such that

$$|m_j(x)| \leq g(x).$$

Thus, for $\|u\| = 1$, $m_j(x)u^2$ converges to $m(x)u^2$ almost always in Ω. In addition,

$$|m_j(x)u^2| \leq g(x)u^2,$$

and by Holder's Inequality,

$$\int_\Omega |g(x)u^2|dx \le \left(\int_\Omega |g(x)|^{\frac{N}{2}}dx\right)^{\frac{2}{N}} \left(\int_\Omega u^{2^*}dx\right)^{\frac{2}{2^*}} < \infty,$$

because, $g \in L^{\frac{N}{2}}(\Omega)$ and $H_0^1(\Omega)$ is continuously immersed in $L^{2^*}(\Omega)$. Therefore, by Lebesgue's Dominated Convergence Theorem,

$$\int_\Omega m_j u^2 dx \to \int_\Omega m u^2 dx$$

and consequently

$$\mu_n(m_j) \to \mu_n(m).$$

□

The eigenvalue problem (PA) has a double sequence of eigenvalues

$$... \le \mu_{-2} < \mu_{-1} < 0 < \mu_1 < \mu_2 \le ...,$$

whose variational characterization is

$$\begin{cases} \frac{1}{\mu_n(m)} = \sup_{F_n} \inf\{\int_\Omega m u^2 dx : \|u\| = 1, u \in F_n\} \\ \\ \frac{1}{\mu_{-n}(m)} = \sup_{F_n} \inf\{\int_\Omega m u^2 dx : \|u\| = 1, u \in F_n\}, \end{cases} \tag{A.1}$$

where F_n varies over all n-dimensional subspaces of $H_0^1(\Omega)$. The corresponding eigenfunctions are such that

$$\begin{cases} a(\varphi_n, v) = \mu_n \int_\Omega m\varphi_n v, \quad \forall v \in H_0^1(\Omega) \\ \\ a(\varphi_n, \varphi_n) = 1 \\ \\ \frac{1}{\mu_n} = \int_\Omega m\varphi_n^2 dx \end{cases} \tag{A.2}$$

Proof. By the observation 2.1 and by the definition of the T operator, we obtain the variational characterization. Since

$$a(u, v) = \mu \int_\Omega m(x)uvdx \quad \forall v \in H_0^1(\Omega)$$

take $u = \varphi_n$, we have

$$a(\varphi_n, v) = \mu \int_\Omega m(x)\varphi_n v dx, \quad \forall v \in H_0^1(\Omega).$$

Now taking $\varphi_n = u$, we obtain

$$a(\varphi_n, \varphi_n) = \mu_n \int_\Omega m\varphi_n^2 dx = \mu_n(T\varphi_n, \varphi_n) = \frac{\mu_n}{\mu_n} = 1.$$

□

The next result can be seen in detail in [2].

Let H be a separable Hilbert space and let T be a compact self-adjoint operator. Then there exists a Hilbert basis composed of eigenvectors of T.

Appendix B

The Classical Min-Max Principle

Now we will enunciate the homotopically stable family definition and a theorem known as Ghoussoub's theorem that can be found in the book [14], Theorem 3.2. Let be B a closed subset of X. A class $\mathcal{F}$ of compacts subset of X is a homotopically - stable family with boundary B provided

(a) any subset in $\mathcal{F}$ contains B;

(b) for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all (t,x) in $(\{0\} \times X) \cup ([0,1] \times B)$ we have that $\eta(1 \times A) \in \mathcal{F}$.

Theorem B.1. *Let φ be a C^1- functional on a complete connected C^1- Finsler manifold (without boundary) and consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B. Let $c = c(\varphi, \mathcal{F}) = \inf\limits_{A \in \mathcal{F}} \max_{x \in A} \varphi(x)$ and suppose that*
(F_0) $\sup \varphi(B) < c$.
Then, for any sequence of sets $(A_n)_n$ in $\mathcal{F}$ such that $\lim\limits_{n} \sup\limits_{A_n} \varphi = c$, there is a sequence $(x_n)_n$ in X such that

(i) $\lim\limits_{n} \varphi(x_n) = c$

(ii) $\lim\limits_{n} \|d\varphi(x_n)\| = 0$

(iii) $\lim\limits_{n} dist(x_n, A_n) = 0.$

Moreover, if $d\varphi$ is uniformly continuous, then x_n can be chosen to be in A_n for each n. See the proof in Chapter 3 of [14].

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