Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

# Splittings of profinite groups and its applications

por

Mattheus Pereira da Silva Aguiar

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Mattheus Pereira da Silva Aguiar †

sob orientação do

Prof. Dr. Pavel Zalesski

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos necessários para obtenção do título de Doutor em Matemática.

†O autor contou com apoio financeiro CAPES durante a realização deste trabalho.

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## Doutor em Matemática.

Área de Concentração: Álgebra

Examinada e aprovada por:

**Prof. Dr. Pavel Zalesski (Mat – UnB)**
**Orientador**

**Prof. Dr. John William MacQuarrie (MAT – UFMG)**

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**Prof. Dr. Theo Allan Darn Zapata (MAT – UnB)**

**Brasília, 14 de Julho de 2023**

† O autor contou com apoio financeiro da CAPES durante a realização deste trabalho.

*For me, the subject is of secondary importance: I want to convey what is alive between me and the subject.*

**Claude Monet**

# Agradecimentos

Esta tese finaliza a minha formação acadêmica. Por ter traçado uma trajetória peculiar na matemática, desenvolvo aqui um relato do meu percurso, mencionando as pessoas que foram determinantes nesse processo. Durante a leitura dos agradecimentos, sugiro que o leitor escute o primeiro movimento do Concerto para Violino e Orquestra em Ré Menor op. 47 de Jean Sibelius, executado pela violinista sul-coreana Soyoung Yoon.

Agradeço a Deus, que me proporcionou vida, saúde, energia, oportunidades e sabedoria para enfrentar o diversos desafios que a vida me apresentou.

As minhas primeiras motivadoras encontrei em casa, minha mãe Cláudia Pereira e minha irmã Thayná Pereira. Elas sempre estiveram comigo nas minhas conquistas, vibrando junto e me motivando a seguir em frente. Foram também fundamentais nos meus fracassos, ao ajudar a me reerguer e mostrar que era possível construir um novo caminho, dessa vez de sucesso. Minha eterna gratidão e amor a vocês, obrigado por tudo que fizeram e fazem por mim.

Também agradeço aos meus avós, Adelina da Silva Pereira e José Walter da Silva por todo o apoio e amor que me dedicam.

Tive o primeiro contato com a escola antes de completar um ano de idade, na Escola Infatil Mini-Ninho. Localizada em um casarão histórico no centro de Diamantina (MG), foi cenário de parte considerável da minha infância. Tenho as melhores lembranças desse período, em que fui acolhido com um amor único, indescritível e que fez parte da minha formação escolar e pessoal. A diretora, Climene Rocha e uma das minhas primeiras professoras, Tereza Cristina Rocha, acompanharam toda a minha caminhada, me ajudaram a superar desafios nas mais diversas situações e ainda são extremamente importantes e queridas por mim.

Após minha família enfrentar dificuldades financeiras, tive que me mudar para uma escola pública municipal, experiência essa que ficou marcada de forma traumática na minha infância. A falta de empatia, ressentimento por parte dos professores e alunos, falta de infraestrutura, tudo isso contribuiu para que minha experiência fosse infeliz. Muito

me alegrou quando soube que poderia voltar ao Mini-Ninho com uma bolsa integral. Sou eternamente grato por isso.

Ao final do Ensino Fundamental I, na antiga 3ª série (hoje 4º ano), mudei para a Escola Estadual Professora Júlia Kubitscheck, em que enfrentei novos desafios. A escola era excelente, apesar de alguns professores não possuírem grande empatia. Algo que me marcou nesse momento foram as fichas de fatos. A professora passava uma ficha e ditava multiplicações e divisões que deveriam ser feitas em um curto intervalo de tempo e registradas. Posteriormente, passávamos o nosso cartão para o colega ao lado e recebíamos o dele. Cada cartão deveria ser corrigido e colorido de azul se a operação estivesse correta e de vermelho caso contrário. Ficava triste e sem entender o porquê coloria tantas fichas de azul e a minha voltava repleta de quadrinhos vermelhos. Sempre fui bastante competitivo e queria mudar essa situação, mostrar a mim mesmo que era capaz de realizar aquela atividade que parecia ser tão fácil para os demais, mas que se apresentava difícil para mim. Minha mãe foi crucial no processo de reversão dessa tendência. Treinava comigo todos os dias, até que eu ficasse bom. Terminei a 4ª série no Júlia com uma das notas mais altas em matemática da turma.

Ingressei então no Colégio Tiradentes da PMMG, através de sorteio. Eram 5 vagas para 55 candidatos e eu fui o terceiro sorteado. Segui nesse Colégio até a conclusão do 3º ano e foi lá que descobri a minha paixão pela matemática. Mesmo não sendo professores de matemática, Francisco Tadeu e Maria Rosália Carneiro marcaram minha trajetória. Ao ser valorizado em cada uma das minhas pequenas conquistas, ganhei forças para enfrentar desafios cada vez maiores e hoje finalizo esse trabalho para obter a mais alta titulação acadêmica.

Dentre todos os professores de matemática que tive durante a minha formação básica, Marcela Sena foi a que mais se destacou. Me ensinou a superar meus limites, buscar novos conhecimentos e foi fundamental na conquista das minhas 4 medalhas na OBMEP. Como parte da premiação fui agraciado por 4 anos com o Programa de Iniciação Científica Júnior (PIC-Jr), onde Marciene Torres e Murilo Hendrik me mostraram novas formas de escrever e pensar matemática. Até hoje me lembro vividamente da palestra do Prof. Dr. Anderson Porto no meu primeiro ano do PIC-Jr. O ano era 2009, eu tinha 12 anos e o tema era geometrias não-euclidianas. Foi fantástico descobrir que existia um tal espaço hiperbólico em que a soma dos ângulos internos de um triângulo era menor que

180º. Obrigado Anderson, pela sua empolgação e por ter me apresentado a matemática de alto nível. Tendo visto em mim potencial, ele me convidou, antes mesmo que eu terminasse o ensino médio, para cursar as disciplinas de Equações Diferenciais Ordinárias e Cálculo Numérico. Assim que ingressei no Bacharelado em Ciência e Tecnologia da UFVJM pude contar com a sua orientação: incansável, dedicada e atenta aos detalhes; características essas provavelmente herdadas do grande Pavel Zalesski, como pude constatar posteriormente.

Finalizei o Bacharelado em Ciência e Tecnologia no tempo regulamentar, 3 anos. Agradeço em especial Maria Cecíllia Alecrim, minha amiga desde as fatídicas fichas de fatos do Júlia Kubitscheck até hoje. Obrigado por sempre me instilar ânimo e me fazer mostrar o melhor de mim em todas as situações críticas. Também agradeço a Álvaro Santos, meu parceiro no ensino médio e nos primeiros anos de faculdade. Aprendi muito com você e sou eternamente grato por isso.

O próximo passo natural seria seguir para a Engenharia Mecânica na mesma instituição de ensino. Mas a Escola de Álgebra, organizada pela UFMG e atipicamente sediada em Diamantina, mudaria o rumo desse processo.

Desde que entrei na graduação, não me enxergava atuando como engenheiro. Apesar de algumas matérias técnicas serem bastante interessantes, como soldagem, não me sentia realizado com o exercício da profissão. Sempre gostei de pensar; quanto mais abstrato o pensamento, melhor.

O melhor matemático que eu conhecia à época era aquele que havia me fascinado com espaços hiperbólicos. Me lembro que, ainda no PIC-Jr, perguntei à Marciene o que o Anderson estudava. Ela riu e disse que nunca tinha tido coragem de perguntar, porque provavelmente era algo bastante complexo. Eu nunca tive medo de coisas complexas; em verdade, elas me fascinam. Então fui eu mesmo perguntar.

Posso afirmar com clareza que não foi só uma pergunta. Foram 5 anos de perguntas. O leitor pode indagar como isso foi possível e a explicação é simples: sempre que descobria uma estrutura matemática ou uma propriedade, Anderson me apresentava outra. E eu era desafiado a entender esse novo conceito. Esse ciclo se repetiu de forma indefinida.

Descobri que o Anderson estudava ações de grupos em grafos, a Teoria de Bass-Serre. Posteriormente descobri que existe também uma versão profinita dessas ações. E um dos desenvolvedores da versão profinita morava no Brasil e tinha orientado o Anderson no doutorado. Surgiu então a mística do Professor Pavel Zalesski.

Passei a estudar com o Anderson a teoria abstrata de grafos. Tínhamos um grupo de estudos com mais três alunas da engenharia (química, geológica e alimentos). Apesar de nos encontrarmos com frequência, o progresso acabou se tornando lento. Ao final da graduação, incrementei os estudos com o Prof. Dr. Douglas Santiago, o qual me ajudou com os Teoremas das funções Implícita e Inversa (note que nesse momento eu não havia cursado sequer um curso de análise na reta. Aprendi tudo o que precisava por conta própria, lendo o livro de Análise do Elon Lages). Ao mesmo tempo, estudei Espaços Métricos e Topologia Geral com o Prof. Dr. Leonardo Gomes. Ao final, chegamos a discutir propriedades elementares de grupos fundamentais de espaços topológicos.

Quando ocorreu a XXIV Escola de Álgebra (2016), tive contato com matemáticos de renome internacional, inclusive o medalhista fields Efim Zelmanov. Infelizmente Pavel Zalesski não estava presente, então ainda não pude conhecê-lo. Estava em Cambridge fazendo mais um de seus não-enumeráveis pós-doutorados. Pude assistir a várias palestras sobre grupos profinitos e percebi que aquilo sim me fascinava. Não era a engenharia que me deslumbrava, mas sim pensar sobre aqueles objetos de definição tão complexa que era necessário o título de mestre para compreendê-los de fato.

Cabe pontuar que queria ter apresentado um pôster naquela Escola de Álgebra. O Professor Anderson me explicou que era um evento de altíssimo nível e que o mínimo aceitável seria apresentar a demonstração do Serre para o Teorema de Nielsen-Schreier (subgrupos de grupos livres também são livres). Eu não fazia ideia à época do que era um grupo livre. Confesso que tentei, mas era demais para mim no momento. Esse Teorema teve três aparições memoráveis na minha jornada: a primeira foi quando apresentei-o na II Semana Acadêmica de Matemática da UFVJM (2017), meu último evento da graduação; a segunda foi quando demonstrei a versão profinita utilizando a teoria profinita de Bass-Serre, no III Workshop in Groups and Algebras da UFMG (2019). Cabe ressaltar que essa demonstração é o que mais me orgulha na dissertação de mestrado, por ter sido original; e a terceira, quando já como professor, tive um aluno, o Cadete Intendente Ian, da Aeronáutica, apresentando a versão clássica do Teorema na XXVI Escola de Álgebra (2023).

Foi também na XXIV Escola de Álgebra que conheci Daniela Oliveira e José Alves. Como no conto "El avión de la bella durmiente" de Gabriel García Márquez, não chegamos

a trocar sequer uma palavra naquele momento. Eu somente admirava aquelas pessoas incríveis, que falavam sobre matemática com uma naturalidade ímpar e traçavam planos infalíveis. Queria me tornar um dia como elas.

Acordei do sonho que foi a Escola de Álgebra e decidi que era o momento de agir. Mesmo sem as disciplinas necessárias, somente com o estudo individual e a tutoria dos professores já citados, candidatei-me e fui aprovado para um mestrado em matemática na Universidade Federal de Minas Gerais. Antes de sair da UFVJM, pedi ao Anderson algum material sobre grupos profinitos, porque queria estudar esse tema no mestrado. Ele disse que tinha acabado de receber um livro do Luis Ribes, lançado naquele mesmo ano com o título "Profinite graphs and groups". Pronto, era exatamente o que eu queria estudar. O Professor John MacQuarrie, que seria meu orientador de mestrado, me chamou em seu gabinete para escolhermos o tema da dissertação logo na primeira semana. Com sua calma natural, ele abriu o seu valiosíssimo armário (há o valor de um carro popular só em livros de matemática) e me apresentou um livro novíssimo, que nunca tinha sido aberto. Para a minha surpresa, era o livro do Ribes! Ele então perguntou se eu me interessava por aquele livro. E o sim foi enfático e radiante. Então ele concluiu a reunião como uma expressão: foi mais fácil do que eu imaginava. Começamos na semana que vem da página 1.

E durante um ano e meio navegamos nas águas turvas do livro do Ribes, que na verdade é um compilado de artigos das mais diversas épocas. O próprio Ribes definiu, em 1979, juntamente com Gildenhuys, o conceito de grafo profinito. Após enfrentar duros desafios no mestrado, estava apto para defender. Estava tomando café com uns colegas e meu orientador me chama na porta da sala. Eis que encontro na minha frente o Professor Pavel Zalesski! Finalmente foi-me dada a oportunidade de personificar o indivíduo que só existia nas histórias mais longínquas. Anderson já havia me dito que: Pavel só se dirige a você depois de ter qualificado no mestrado. Dito e feito!

Agradeço enormemente ao Prof. Dr. John MacQuarrie, meu orientador de mestrado, por ter desenvolvido minha maturidade matemática de forma p-ádica. O título de mestre foi obtido com muito esforço, mas sem o senhor ele não seria possível. Agradeço também à Prof. Drª. Ana Cristina Vieira, que me apoiou em vários momentos cruciais da minha trajetória e sempre acreditou que eu seria capaz de superar qualquer desafio que se impusesse em meu caminho.

Faço um adendo que, nesse episódio do café, os colegas que me acompanhavam eram os esperados pelo leitor: Dr. José Alves e Drª Daniela Oliveira. Eles tornaram-se amigos de longa data e me apoiaram durante todo o processo, doloroso, de formação matemática. Sou eternamente grato a eles e a todos os outros que se somaram a esse time, como o Dr. Lucas Reis, Dr. Moacir Aloísio e Dr. Mateus Figueira.

Antes mesmo da defesa de mestrado, fui aprovado no doutorado da UnB. E sim, o leitor acertou! O meu orientador seria ninguém menos que o Prof. Dr. Pavel Zalesski. Defendi a dissertação com sucesso e estava finalmente apto a estudar com o Pavel. Nunca tinha visitado Brasília. Me mudei em agosto de 2019 sem conhecer ninguém. Mas como bem aprendi com minha mãe, isso era uma oportunidade para fazer novas amizades e conhecer novas culturas. Agradeço Dr. Hércules Carvalho, Dr. Lucas Britto, Drª. Renata Alves, Drª. Sara Rodrigues, Drª. Nathália Gonçalves e Dr. João Pedro Papalardo pela amizade, companheirismo e pelas infindáveis horas de estudo. Também agradeço os Professores Dr. Igor Lima, Dr. Raimundo Bastos e Dr. Theo Zapata pelas orientações, repreeensões e direcionamentos ao longo de todo esse caminho.

Pego um buraco de minhoca para me deslocar no espaço-tempo e agradecer também aos Cadetes Intendentes da Aeronáutica, Turmas Ártermis e Athos. Vocês foram meus primeiros alunos e muito me orgulha poder ensinar matemática a mentes tão brilhantes. Em especial, da Turma Ártemis, cito as Cad Júlia, Cad Maria Gabriela, Cad Mylena, Cad Carolina, Cad Giroto, Cad Máximo, Cad Barcellos e os Cad Bramucci e Cad Bruno em nome dos quais cumprimento todos os demais. Da Turma Athos, cito o Cad Braga César, as Cad Shelsea, Cad Júlia Brandão, Cad Ana Laura e Cad Júlia Sousa, em nome dos quais cumprimento todos os demais.

Agradeço também aos meus colegas da Turma de Oficiais da Aeronáutica, Xavante. Cito os Tenentes Cormanich, Willian, Rampazo e a Tenente Suzuki. Obrigado por todo o apoio e por terem me tornado mais confiante e capaz de superar desafios antes intransponíveis. Agradeço também às Professoras Drª. Marina e Drª. Vitória pela amizade e companheirismo e ao Prof. Dr. Marcus pelas numerosas horas de conversa e aconselhamento, tanto matemático quanto de vida.

Voltando pela outra extremidade do buraco de minhoca, retorno ao momento da chegada em Brasília em agosto de 2019. Cheguei na UnB e a primeira coisa que fiz foi procurar a sala do Pavel. Estava lá, no mezanino, o local mais nobre do departamento.

Mas fechada e vazia. Enviei um e-mail perguntando se ele iria na UnB naquele dia. A resposta foi frustrante: estou no México, só chego na semana que vem.

Desalentado, fui para a biblioteca da universidade e passei a folhear o livro do Ribes. Eis que me deparo com uma pergunta em aberto passível de ser compreendida, a Open Question 6.7.1 (nas últimas páginas da primeira parte do livro). Passei a esboçar umas ideias. Afinal, se o Pavel demoraria uma semana, quando ele chegasse já queria mostrar que havia pensado em algo.

A semana que parecia infindável finalmente se passou e pude encontrar com meu novo orientador. Similarmente à primeira reunião do mestrado, essa também foi curta, mas apresentou novos desafios. De pronto já fui questionado sobre o que eu sabia. E a resposta também foi incisiva: os seis primeiros capítulos do livro do Ribes, exceto o 4. Ele então murmurou: ok, então podemos começar a pesquisar. Tenho um problema para você. Foi aí que eu o surpreendi e disse que já estava pensando em um problema (sim, a Open Question 6.7.1). Apresentei a questão e o Pavel disse: mas isso é muito fácil, você define os grupos de $(\mathcal{G}, \Gamma)$ como completamento do caso abstrato. Me mantive convicto. Expliquei que essa estratégia só funcionava se o grafo $\Gamma$ em questão fosse finito, que era a construção do Ribes (posteriormente descobri que o Pavel era coautor desse artigo!). A Open Question no entanto se referia ao caso em que $\Gamma$ era infinito. Ele então se deu conta da sutileza do problema e exclamou, primeiro com seus fulminates olhos azuis e depois com palavras: esse problema é interessante, vamos começar por ele!

Menos de seis meses depois já tínhamos uma resposta afirmativa para a Open Question 6.7.1 do Ribes (cf. Theorem 1). Os outros anos renderam trabalhos ainda melhores, que serão vastamente explorados nessa tese. Trabalhar com Pavel é ao mesmo tempo enriquecedor e frustrante. Chega a ser impossível conceber alguém que tenha tanto domínio sobre construções livres quanto ele. Muito me honra ter tido a oportunidade de fazer um doutorado com esse pesquisador ímpar, de conhecimento muito além do finito, profinito.

Ao meu orientador Prof. Dr. Pavel Zalesski, obrigado por ter me introduzido como protagonista no mundo da pesquisa científica. Posso afirmar com toda a certeza que aquele desejo inenarrável de conhecê-lo e trabalhar com o senhor não foi só cumprido, mas superado. Confesso que não foi fácil e que o senhor demandou de mim muito mais do que eu me acreditava capaz. Mas foi superando barreiras que hoje apresento nessa tese

de doutorado o melhor de tudo o que produzi até então.

Aos membros da banca, Prof. Dr. John William MacQuarrie (MAT – UFMG) e Prof. Dr. Slobodan Tanushevski (IME – UFF), Prof. Dr. Theo Allan Darn Zapata (MAT – UnB), por aceitarem prontamente o convite para avaliação deste trabalho e pelas valiosas sugestões.

À Universidade de Brasília, pela infraestrutura e recursos oferecidos para a realização deste trabalho.

À CAPES pelo apoio financeiro.

Peço desculpas àqueles que injusta e involuntariamente tenham sido omitidos.

# Dedicatória

*Aos jovens brasileiros e brasileiras que sonham um dia desvendar os mistérios mais profundos da abstração matemática dedico como fonte de inspiração esta tese de doutorado.*

# Papers

(A) M. P. S. Aguiar and P.A. Zalesskii, *The profinite completion of the fundamental group of infinite graphs of groups*, *Isr. J. Math.*, **250**, (2022), 429-462.

(B) M. P. S. Aguiar and P.A. Zalesskii, *Generalized Stallings' decomposition theorems for pro-p groups*, *Ann. Sc. norm. super. Pisa - Cl. sci.* (to appear).

# Resumo

## Decomposições de grupos profinitos e suas aplicações


Nessa tese estudamos um dos principais objetos da teoria combinatória de grupos profinitos: decomposições de grupos profinitos como extensões HNN ou produtos livres amalgamados.

Respondemos três problemas em aberto propostos por Luis Ribes em seu livro de 2017 "Profinite Graphs and Groups" (veja Open Questions 6.7.1, 15.11.10 e 15.11.11 de [31]). Esses resultados generalizam os teoremas principais de [7] e [45]. Também generalizamos a versão pro-$p$ do célebre Teorema da decomposição de Stallings para decomposições sobre grupos pro-$p$ infinitos. Essa construção estende consideravelmente o resultado de Weigel-Zalesski de 2017 e não possui correspondente no caso abstrato. Por fim, mostramos que a acessibilidade generalizada de grupos pro-$p$ finitamente gerados é fechada para comensurabilidade.

Produtos amalgamados profinitos e extensões HNN profinitas podem ser considerados casos particulares de um grupo fundamental de grafo de grupos, o qual denotaremos por $\Pi_1(\mathcal{G},\Gamma)$. Dessa forma, se dado grupo profinito $G$ possui uma decomposição $G = \Pi_1(\mathcal{G},\Gamma)$ para algum grafo profinito de grupos $(\mathcal{G},\Gamma)$, obtemos não só propriedades do grupo $G$ mas também de grafo de grupos $(\mathcal{G},\Gamma)$.

Na primeira parte, dado um grupo abstrato $G$ que se decompõe como o grupo fundamental de um grafo infinito de grupos, construímos um grafo profinito de grupos $(\overline{\mathcal{G}},\overline{\Gamma})$ tal que $\Gamma$ mergulha em $\overline{\Gamma}$ e o completamento profinito de $G$ se decompõe como $\widehat{G} = \Pi_1(\overline{\mathcal{G}},\overline{\Gamma})$. Isso responde um Problema em Aberto de Ribes. Com essa construção em mãos, respondemos dois outros Problemas em Aberto de Ribes. O primeiro está relacionado com o fecho de normalizadores e generaliza o teorema principal de um artigo escrito por Ribes e Zalesski (cf. [34]). O segundo está relacionado com a separabilidade por conjugação de subgrupo de grupos virtualmente livres, generalizando o resultado principal de um artigo escrito por Chagas e Zalesski (cf. [7]). Nossa estratégia para resolver os problemas supracitados foi descrever o grupo fundamental profinito de um grafo de grupos na linguagem de caminhos. Essa nova definição se comporta bem quando da aplicação de limites inversos, o que facilita a interrelação entre as configurações abstrata e profinita da Teoria de Bass-Serre.

Continuamos nossa jornada investigando o célebre Teorema da Decomposição de Stallings. Este estabelece que a decomposição de um subgrupo $H$ de índice finito de um grupo finitamente gerado $G$ como um produto livre amalgamado ou uma extensão HNN sobre um grupo finito implica o mesmo para $G$. A versão pro-$p$ desse resultado foi obtida por Weigel e Zalesski (cf. [45]) em 2017. Nós mostramos que, na categoria de grupos pro-$p$, os teoremas de decomposição valem além de cisões sobre grupos finitos. Se $G$ é um grupo pro-$p$ finitamente gerado que possui um subgrupo normal aberto $H$ que se decompões como $H = \Pi_1(\mathcal{H}, \Delta)$, e supomos que classes de conjugação de grupos de vértices são $G$-invariantes, então $G$ também se decompõe como $G = \Pi_1(\mathcal{G}, \Gamma)$ (cf. Teorema 11). Se $H$ é um produto pro-$p$ livre não trivial obtemos, como um caso particular, o Teorema de Weigel-Zalesski supracitado. A principal ferramenta por trás da demonstração é nosso Teorema da Limitação, que estabelece um limitante para $E(\Gamma)$, a saber $|E(\Gamma)| \leqslant |E(\Delta)|$.

Acrescentamos ao nosso Teorema da Limitação o seguinte resultado: se $G$ é um grupo pro-$p$ finitamente gerado que possui um subgrupo normal aberto $H$ agindo sobre uma árvore pro-$p$ $T$, com $\{H_v \mid v \in V(T)\}$ sendo $G$-invariante, então $G$ se decompõe como $G = \Pi_1(\mathcal{G}, \Gamma)$. Com esses resultados em mãos, obtemos uma poderosa aplicação: a acessibilidade generalizada de grupos pro-$p$ finitamente gerados é fechada para comensurabilidade. Finalizamos a tese mostrando que nosso Teorema 9 também vale para o exemplo de grupo pro-$p$ inacessível dado por Wilkes.

**Palavras-chave:** Teoria Combinatória de Grupos; grupos profinitos; grafos infinitos de grupos; decomposição de Stallings.

# Abstract

In this thesis we study one of the main objects in profinite combinatorial group theory: splittings of profinite groups as HNN-extensions or amalgamated free products.

We answer three Open Questions proposed by Luis Ribes in his 2017 book "Profinite Graphs and Groups" (see Open Questions 6.7.1, 15.11.10, and 15.11.11 of [31]). These results generalize the main Theorems of [7] and [34]. We also generalize the pro-$p$ version of the celebrated Stallings' decomposition theorem to splittings over infinite pro-$p$ groups. This extends by far the Weigel-Zalesski result from 2017 and it does not have any abstract analogs. Finally we prove that generalized accessibility of finitely generated pro-$p$ groups is closed for commensurability.

Profinite amalgamated products and profinite HNN-extensions can be considered as particular cases of profinite fundamental groups of graphs of groups, which we denote by $\Pi_1(\mathcal{G}, \Gamma)$. Hence, if a profinite group $G$ has a splitting $G = \Pi_1(\mathcal{G}, \Gamma)$ for some profinite graph of groups $(\mathcal{G}, \Gamma)$, we obtain not only properties of the group $G$ but also properties of the graph of groups $(\mathcal{G}, \Gamma)$.

In the first part, given an abstract group $G$ that splits as the fundamental group of an infinite graph of groups, we construct a profinite graph of groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ such that $\Gamma$ embeds in $\overline{\Gamma}$ and the profinite completion of $G$ splits as $\Pi_1(\overline{\mathcal{G}}, \overline{\Gamma})$. This answers an Open Question of Ribes. With this construction in hand, we answer two more Open Questions of Ribes. The first concerns the closure of normalizers, which generalizes the main Theorem of a paper by Ribes and Zalesski (cf. [34]). The second is related to subgroup conjugacy separability of virtually free groups, generalizing the main Theorem of a paper by Chagas and Zalesski (cf. [7]). Our strategy for solving the problems above is to describe the profinite fundamental group of a graph of groups in the language of paths. Since it behaves very well via inverse limits, it facilitates the interrelation between the abstract and the profinite settings.

We continue our journey by investigating the Stallings' decomposition Theorem. It states that the splitting of a finite index subgroup $H$ of a finitely generated group $G$ as an amalgamated free product or an HNN-extension over a finite group implies the same

for $G$. The pro-$p$ version of this result was obtained by Weigel and Zalesskii (see [45]) in 2017. We proved that, in the category of pro-$p$ groups, splitting theorems hold beyond splittings over finite groups. In fact, if $G$ is a finitely generated pro-$p$ group having an open normal subgroup $H$ that splits as $H = \Pi_1(\mathcal{H}, \Delta)$, and we suppose conjugacy classes of vertex groups are $G$-invariant then $G$ also splits as $G = \Pi_1(\mathcal{G}, \Gamma)$ (see Theorem 11). If $H$ is a non-trivial free pro-$p$ product we obtain, as a particular case, the aforementioned Weigel-Zalesski Theorem. The main tool behind the proof is our Limitation Theorem, which establishes a bound for $E(\Gamma)$, namely $|E(\Gamma)| \leqslant |E(\Delta)|$.

We attach to our Limitation Theorem the following result: if $G$ is a finitely generated pro-$p$ group having an open normal subgroup $H$ acting on a pro-$p$ tree $T$, with $\{H_v \mid v \in V(T)\}$ being $G$-invariant, then $G$ splits as $G = \Pi_1(\mathcal{G}, \Gamma)$. With these results in hand, we provide a powerful application: generalized accessibility of finitely generated pro-$p$ groups is closed for commensurability. We finish the thesis by showing that our Theorem 9 holds even for Wilkes' example of a pro-$p$ inaccessible group.

**Key-words:** Combinatorial group theory; profinite groups; infinite graphs of groups; Stallings' decomposition.

# INTRODUCTION

Combinatorial group theory may be characterized as the theory of groups that are given by a presentation. This notion was introduced by Walther von Dyck in 1882 and it was improved in the subsequent years by the extensive use of free constructions (a term introduced by Remeslennikov et al. in a survey), namely free products with amalgamation and Higman-Neumann-Neumann extensions, or simply HNN-extensions. Free products with trivial amalgamation, also known as free products, were introduced by Otto Schreier in 1927 and HNN-extensions were introduced by Graham Higman, Bernhard H. Neumann, and Hanna Neumann in 1949. Free constructions represent just a part of a bigger structure, with free products with amalgamation being the disconnected case and HNN-extensions the connected one. Stallings proposed the concept of a bipolar structure, but the interpretation that lasted was given by Jean Pierre Serre in his Structure Theorem: a fundamental group of a graph of groups is either a free product with amalgamation or an HNN-extension.

The theory of profinite groups was inaugurated in 1964 by J.-P. Serre in his book Cohomologie Galoisienne. Later, in 1978, Ribes and Gidenhuys published a paper ([17]) where they introduced the concept of a profinite graph. The natural continuation was the development of a profinite Bass-Serre theory, whose foundations were laid in the '80s. This new technology was essential due to the difficulty of obtaining basic results on amalgamated free products or HNN-extensions by existing methods. An element of a profinite group can not be written as a word of generators and so the classical methods of the Combinatorial Group Theory are absent in the class of profinite groups.

The content of the doctoral thesis of Pavel Zalesski established the starting point of the profinite Bass-Serre theory. The thesis appeared in 1988 and consisted of three papers ([51],[53], [52]), the first two in collaboration with his Ph.D. supervisor Oleg Mel'nikov. There, Zalesski establishes the theory of covering spaces of profinite graphs (Definition 1.3.1) and defines the fundamental group of a connected profinite graph (Definition 1.3.3). He also studies the profinite fundamental group of a profinite graph of groups $\Pi_1(\mathcal{G}, \Gamma)$ (Definition 1.6.7) and the action of this group $\Pi_1(\mathcal{G}, \Gamma)$ on its standard profinite tree $S(\mathcal{G}, \Gamma)$ (Definition 1.6.16). Finally, he investigates subgroups of fundamental groups of graphs of profinite groups. The considerations are restricted to graphs $(\mathcal{G}, \Gamma)$ of profinite groups over finite graphs $\Gamma$, although at the end of [51] there is a suggestive comment that we transcribe here:

> **"In a number of cases, $\Pi_1(\mathcal{G}, \Gamma)$ can be represented as an inverse limit of fundamental groups of finite graphs of groups; for example, for free products over spaces".**

This phenomenon is more general than expected at the time by Zalesski. We give, 34 years later, a complete characterization of $\Pi_1(\mathcal{G}, \Gamma)$ as an inverse limit of fundamental groups of finite graphs of groups in Proposition 2.1.4. We go beyond and establish the fundamental group of a graph of groups with a base point in Section 2.1, more specifically in Theorem 2.2.10, making the concept closer to the classical topological definition and facilitating the use of projective limits.

Subsequent papers by Luis Ribes, Oleg Mel'nikov and Pavel Zalesski consolidated the profinite Bass-Serre theory and established its independence. It is important to mention that some of the classical properties of the abstract case are missing in the profinite version. For example, it is quite simple to show that every finite graph has a maximal subtree (see Proposition 1.1.24). However, this is not always true in the profinite case (see Example 2.1.7). It is a crucial aspect, since one of the forms of defining $\pi_1(\mathcal{G}, \Gamma)$ in the abstract case makes use of the maximal subtree of $\Gamma$. Hence we cannot define $\Pi_1(\mathcal{G}, \Gamma)$ in terms of maximal subtrees.

A book that organizes the most important results of the Ribes-Zalesski-Mel'nikov theory was published by Ribes in 2017 (see [31]). The first two parts of the book address the pro-$\mathcal{C}$ version of the classical Bass-Serre theory. There Ribes defines a profinite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$, its pro-$\mathcal{C}$ fundamental group $\Pi_1(\mathcal{G}, \Gamma)$, and its standard pro-$\mathcal{C}$ tree $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ on which $\Pi_1(\mathcal{G}, \Gamma)$ naturally acts. In fact, the action of the fundamental group on the standard tree obtains $\Gamma$ itself as the quotient graph, i.e. $\Pi_1(\mathcal{G}, \Gamma)\backslash S^{\mathcal{C}}(\mathcal{G}, \Gamma) = \Gamma$.

The third part of the book is dedicated to applications in abstract groups, where the methods are based on an interplay between Bass-Serre and Ribes-Zalesski-Mel'nikov's theories. More concretely, if $G$ is the abstract fundamental group $G = \pi_1(\mathcal{G}, \Gamma)$ of a finite graph of groups $(\mathcal{G}, \Gamma)$ then the profinite completion $\widehat{G}$ is the fundamental group $\widehat{G} = \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ over the same graph with fibers being the profinite completions. However, it works only when the underlying graph $\Gamma$ is finite, and sometimes, for example when $G$ is infinitely generated free-by-finite, the underlying graph $\Gamma$ is infinite.

This motivated Ribes to ask whether there is a reasonable way to define a graph of groups $(\mathcal{G}', \overline{\Gamma})$ over a profinite graph $\overline{\Gamma}$ containing $\Gamma$ such that $\Pi_1(\mathcal{G}', \overline{\Gamma}) = \widehat{\pi_1(\mathcal{G}, \Gamma)}$ when $\Gamma$ is infinite and the vertex groups are finite. Another question of Ribes in the same direction is whether the action of $\Pi_1(\mathcal{G}', \overline{\Gamma})$ on $S(\mathcal{G}', \overline{\Gamma})$ (note that $\Pi_1(\mathcal{G}', \overline{\Gamma})\backslash S(\mathcal{G}', \overline{\Gamma}) = \overline{\Gamma}$) extends naturally the action of $\pi_1(\mathcal{G}, \Gamma)$ on $S^{abs}(\mathcal{G}, \Gamma)$, where $S^{abs}(\mathcal{G}, \Gamma)$ is the abstract standard tree of $(\mathcal{G}, \Gamma)$. In fact, $\pi_1(\mathcal{G}, \Gamma)\backslash S^{abs}(\mathcal{G}, \Gamma) = \Gamma$ (cf. [31, Open Question 6.7.1] for the precise formulation).

Our first result answers these questions; in fact, we state it more generally, namely for the pro-$\mathcal{C}$ completion, where $\mathcal{C}$ is the class of finite groups closed for subgroups and extensions. Throughout this thesis, all theorems and corollaries numbered without section indication will be original results.

**Theorem 1.** *Let $(\mathcal{G}, \Gamma)$ be an abstract reduced graph of finite groups over an abstract graph $\Gamma$ such that $\pi_1(\mathcal{G}, \Gamma)$ is residually $\mathcal{C}$. Then there exists a profinite graph $\overline{\Gamma}$ that contains $\Gamma$ as a dense subgraph and a profinite graph of finite groups $(\overline{\mathcal{G}}, \overline{\Gamma})$, with $\overline{\mathcal{G}}(m) = \mathcal{G}(m)$, whenever $m \in \Gamma$, such that*

*(a) The pro-$\mathcal{C}$ fundamental group $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ is the pro-$\mathcal{C}$ completion of $\pi_1(\mathcal{G}, \Gamma)$;*

*(b) The standard tree $S^{abs}(\mathcal{G}, \Gamma)$ for $G$ is densely embedded in the standard pro-$\mathcal{C}$ tree $S^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ in such a way that the action of $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ on $S^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ extends the natural action of $\pi_1(\mathcal{G}, \Gamma)$ on $S^{abs}(\mathcal{G}, \Gamma)$.*

This theorem allows us to answer Open Question 15.11.10 of [31] on the density of the normalizer. Let $G$ be a virtually free group and H a finitely generated subgroup of

$G$. If we consider the profinite topology of G, then the closure of the normalizer of $H$ in $G$ is equal to the normalizer of the closure of $H$ in the profinite completion of $G$.

**Theorem 2.** *Let $G$ be a virtually free abstract group and $H$ a finitely generated subgroup of $G$. Then*

$$\overline{N_G(H)} = N_{\widehat{G}}(\overline{H}).$$

Theorem 2 generalizes the main result of [34], where it was proved for the finitely generated case. In fact, we prove a more general result, namely for the pro-$\mathcal{C}$ completion of a free-by-$\mathcal{C}$ group assuming that $H$ is closed in the pro-$\mathcal{C}$ topology (see Theorem 2.4.2).

Let $G = \pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a graph of groups, and suppose $G$ is residually $\mathcal{C}$. Let $\overline{\mathcal{G}}(m)$ be the closure of $\mathcal{G}(m)$ in $G_{\widehat{\mathcal{C}}}$. We prove a generalization of Theorem 1 by showing that we can define a profinite graph $\overline{\Gamma}$ that contains $\Gamma$ as a dense subgraph and a profinite graph of groups $(\overline{\mathcal{G}}, \overline{\Gamma})$, where the groups $\mathcal{G}(m)$ are not necessarily finite.

**Theorem 3.** *Let $(\mathcal{G}, \Gamma)$ be a reduced graph of groups and $G = \pi_1(\mathcal{G}, \Gamma)$ be its fundamental group. Assume that $G$ is residually $\mathcal{C}$. Then,*

*(a) There exists a profinite graph of pro-$\mathcal{C}$ groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ such that $\Gamma$ is densely embedded in $\overline{\Gamma}$;*

*(b) for each $m \in \Gamma$, the vertex group is $\overline{\mathcal{G}}(m)$;*

*(c) The fundamental pro-$\mathcal{C}$ group $\Pi = \Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ of $(\overline{\mathcal{G}}, \overline{\Gamma})$ is the pro-$\mathcal{C}$ completion of $G$, so that all the vertex groups of $(\overline{\mathcal{G}}, \overline{\Gamma})$ embed in $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ ( i.e. $(\overline{\mathcal{G}}, \overline{\Gamma})$ is injective in the terminology of [31]).*

*(d) If in addition we assume that $\mathcal{G}(m)$ is closed in the pro-$\mathcal{C}$ topology of $G$ for every $m \in \Gamma$, the standard tree $S^{abs} = S(\mathcal{G}, \Gamma)$ of the graph of groups $(\mathcal{G}, \Gamma)$ embeds densely in the standard pro-$\mathcal{C}$-tree $S = S^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ of the profinite graph of profinite groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ in such a way that the action of $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ on $S^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ extends the natural action of $\pi_1(\mathcal{G}, \Gamma)$ on $S^{abs}(\mathcal{G}, \Gamma)$.*

In order to prove this theorem we give a new description of the profinite fundamental group of a profinite graph of profinite groups; namely, we transport to the profinite context the definition of the fundamental group of a graph of groups in the language of paths. This gives the possibility of working with morphisms of profinite graphs of groups and to use their projective limits.

We apply the above results to establish subgroup conjugacy separability of virtually free groups, answering the Ribes' Open Question 15.11.11 (cf. [31]). A group $G$ is said to be subgroup conjugacy separable if whenever $H_1$ and $H_2$ are finitely generated subgroups of $G$ then $H_1$ and $H_2$ are conjugate in $G$ if and only if their images in every finite quotient are conjugate.

**Theorem 4.** *Let $G$ be a virtually free group. Then $G$ is subgroup conjugacy separable.*

In fact, we prove the subgroup conjugacy $\mathcal{C}$-separability of a residually $\mathcal{C}$ free-by-$\mathcal{C}$ abstract group $G$. This generalizes the main result of [7] where it is proved for finitely generated free-by-$\mathcal{C}$ abstract groups.

In 1965, J-P. Serre showed that a torsion-free virtually free pro-$p$ group must be free (cf. [39]). This motivated him to ask the question of whether the same statement holds also in the discrete context. His question was answered positively some years later. In several papers (cf. [40], [42], [44]), J.R. Stallings and R.G. Swan showed that free groups are precisely the groups of cohomological dimension 1. At the same time J-P. Serre himself showed that in a torsion-free group $G$, the cohomological dimension of a subgroup of finite index coincides with the cohomological dimension of $G$ (cf. [38]).

One of the major tools for obtaining this type of result - the theory of ends - provided deep results also in the presence of torsion. The first result to be mentioned is 'Stallings' decomposition theorem' (cf. [43]).

**Theorem** (J.R. Stallings)**.** *Let $G$ be a finitely generated group containing a subgroup $H$ of finite index which splits as a non-trivial free amalgamated product or HNN-extension over a finite group. Then $G$ also splits either as a free product with amalgamation or as an HNN-extension over a finite group.*

The pro-$p$ version of the above theorem was proved by Thomas Weigel and Pavel Zalesski in [45] generalizing the result of W.N. Herfort and Pavel Zalesski in [23], where it was proved for virtually free pro-$p$ groups. Chapter 3 of this thesis will be devoted to showing that, in the category of pro-$p$ groups, splitting theorems hold beyond splitting over finite groups. More precisely, the result holds for splittings over a general pro-$p$ group $K$ provided that the factors and the the base group are *indecomposable* over any conjugate of any subgroup of $K$, i.e. do not split as a free amalgamated pro-$p$ product or pro-$p$ HNN-extension. Note that, in the pro-$p$ case, an amalgamated free pro-$p$ product or HNN-extension might be not *proper* (see Subsections 2.4 and 2.5), i.e. the free factors or the base group do not embed in general in the free amalgamated product or in the HNN-extension. In this thesis, every amalgamated free pro-$p$ product and every HNN-extension will be proper.

**Theorem 5.** *Let $H = H_1 \amalg_K H_2$ be a free amalgamated pro-p product of finitely generated pro-p groups $H_1, H_2$ that are indecomposable over any conjugate of any subgroup of $K$. Let $G$ be a pro-p group having $H$ as an open normal subgroup. Then $G$ splits as a free amalgamated pro-p product $G = G_1 \amalg_L G_2$ such that $G_i \cap H$ are contained in some conjugate of $H_i$, $i = 1, 2$ and $L \cap H$ is contained in some conjugate of $K$.*

Following [37, Section 6.1] we say that a pro-$p$ group $G$ has the FA property if, for any pro-$p$ tree $T$ on which $G$ acts, $T^G \neq \varnothing$, i.e. if $G$ acts on a pro-$p$ tree $T$ then it has a global fixed point. Of course, if $H_1, H_2$ do not split as a free amalgamated pro-$p$ product or HNN-extension at all then Theorem 5 holds independently of $K$.

The class of FA pro-$p$ groups is quite large and includes many important examples. All Fab pro-$p$ groups, i.e., pro-$p$ groups whose open subgroups have finite abelianization are FA pro-$p$ groups. Note that Fab pro-$p$ groups include all just-infinite pro-$p$ groups and play a very important role in class field theory (in particular have importance to the Fontaine-Mazur Conjecture, cf. [5]), $p$-adic representation theory [24] and include for example all open pro-$p$ subgroups of $SL_n(\mathbb{Z}_p)$. The pro-$p$ completion of the Grigorchuk, Gupta-Sidki groups, the Nottingham pro-$p$ group and other branch groups are FA pro-$p$ groups as well. Splittings as amalgamated free products of Fab analytic pro-$p$ groups occur naturally in the study of generalized RAAG pro-$p$ groups [29, Subsection 5.5] where it is also proved that an amalgamated free pro-$p$ product of uniformly powerful pro-$p$ groups is always proper. Thus Theorem 5 applies to these splittings of generalized RAAG pro-$p$ groups.

In fact, if $H_1, H_2$ are FA, we even do not need a hypothesis of normality on $H$ for odd $p$.

**Corollary 6.** *Let $p > 2$ and $H = H_1 \amalg_K H_2$ be a free amalgamated pro-p product of finitely generated FA pro-p groups $H_1, H_2$. Let $G$ be a pro-p group having $H$ as an open subgroup.*

*Then $G$ splits as a free amalgamated pro-$p$ product $G = G_1 \amalg_L G_2$ such that $G_i \cap H$ are contained in some conjugate of $H_i$, $i = 1, 2$ and $L \cap H$ is contained in some conjugate of $K$.*

For an HNN-extension the corresponding statement admits two types of splittings.

**Theorem 7.** *Let $H = HNN(H_1, K, t)$ be a pro-$p$ HNN-extension of a finitely generated pro-$p$ group $H_1$ that is indecomposable over any conjugate of any subgroup of $K$. Let $G$ be a pro-$p$ group having $H$ as an open normal subgroup. Then $G$ splits as a free amalgamated pro-$p$ product $G = G_1 \amalg_L G_2$ or HNN-extension $G = (G_1, L, t)$ such that $G_i \cap H$, $i = 1, 2$ is contained in some conjugate of $H_1$, and $L \cap H$ is contained in some conjugate of $K$.*

If $H_1$ is FA then for $p > 2$ we can drop the hypothesis of normality on $H$.

**Corollary 8.** *Let $p > 2$ and $H = HNN(H_1, K, t)$ be a pro-$p$ HNN-extension of a finitely generated FA pro-$p$ group $H_1$. Let $G$ be a pro-$p$ group having $H$ as an open subgroup. Then $G$ splits as a pro-$p$ HNN-extension $G = (G_1, L, t)$ such that $G_1 \cap H$ is contained in some conjugate of $H_1$, and $L \cap H$ is contained in some conjugate of $K$.*

Of course, in general, the factors of an amalgamated free pro-$p$ product $H = H_1 \amalg_K H_2$ or the base group of a pro-$p$ HNN-extension $H = HNN(H_1, K, t)$ can split further, so to extend our results to a more general context we need to have some pro-$p$ version of a JSJ-decomposition, i.e. $H$ should be the fundamental pro-$p$ group of a graph of pro-$p$ groups whose vertex groups do not split further over edge groups. Thus we need to use the Ribes-Zalesski-Mel'nikov theory in the pro-$p$ case.

**Theorem 9.** *Let $G$ be a finitely generated pro-$p$ group having an open normal subgroup $H$ acting on a pro-$p$ tree $T$. Suppose $\{H_v \mid v \in V(T)\}$ is $G$-invariant. Then $G$ is the fundamental group of a profinite graph of pro-$p$ groups such that each vertex group intersected with $H$ stabilizes a vertex of $T$. In particular, $G$ splits as a non-trivial free amalgamated pro-$p$ product or a pro-$p$ HNN-extension.*

If the stabilizers $H_v$ are FA, then the $G$-invariancy $\{H_v \mid v \in V(T)\}$ is automatic; moreover, if the $H_v$ are Fab, then we can drop the normality assumption on $H$.

**Corollary 10.** *Let $G$ be a finitely generated pro-$p$ group having an open subgroup $H$ acting on a pro-$p$ tree $T$ such that each stabilizer $H_v$ is Fab. Then $G$ is the fundamental* *group of a profinite graph of pro-p groups such that each vertex group intersected with $H$ stabilizes a vertex of $T$. In particular, $G$ splits as a non-trivial free amalgamated pro-p product or a pro-p HNN-extension.*

Note also that Theorem 9 does not require necessarily the existence of JSJ decomposition or even accessibility (see Section 2 for definition); in other words, we do not require that $H\backslash T$ is finite. Indeed, G. Wilkes [46] constructed an example of a finitely generated inaccessible pro-$p$ group $G$ (that acts on a pro-$p$ tree with infinite $G\backslash T$), but our theorem holds for his example as well (see Section 3.4).

However, if we assume accessibility, we can say more.

**Theorem 11.** *Let $G$ be a finitely generated pro-p group having an open normal subgroup $H$ that splits as the fundamental pro-p group of a finite graph of finitely generated pro-p groups $(\mathcal{H}, \Delta)$. Suppose conjugacy classes of vertex groups are $G$-invariant. Then $G$ is the fundamental group of a reduced finite graph of pro-p $(\mathcal{G}, \Gamma)$ groups whose vertex and edge groups intersected with $H$ are subgroups of the vertex and edge groups of $H$ respectively. Moreover, $|E(\Gamma)| \leqslant |E(\Delta)|$.*

Once more, if the vertex groups $\mathcal{H}(v)$ are Fab, then we can omit $G$-invariancy and normality hypotheses.

**Corollary 12.** *Let $G$ be a finitely generated pro-p group having an open subgroup $H$ that splits as a finite graph of finitely generated pro-p groups $(\mathcal{H}, \Delta)$. Suppose the vertex groups of $(\mathcal{H}, \Delta)$ are Fab. Then $G$ is the fundamental group of a reduced finite graph of pro-p groups $(\mathcal{G}, \Gamma)$ such that its vertex and edge groups intersected with $H$ are subgroups of vertex and edge groups of $H$ respectively. Moreover, $|E(\Gamma)| \leqslant |E(\Delta)|$.*

Theorem 11 is a generalization of the pro-$p$ version of Stallings' decomposition theorem proved in [45], namely if in Theorem 11 we suppose that $H$ is a non-trivial free pro-$p$ product, we obtain as a particular case the pro-$p$ version of [45, Theorem 1.1]. Theorem 1.4 of [49] gives an example of a situation when Theorem 11 is applicable, namely if all vertex groups are Poincaré duality of dimension $n$ ($PD^n$ pro-$p$ groups) and the edge groups have cohomological dimension $\leqslant n-1$.

The proofs of Theorem 11 and Corollaries 6, 8, and 12 are more subtle and require the following theorem that is of independent interest. Note that for an open subgroup

$H$ of the fundamental pro-$p$ group $G = \Pi_1(\mathcal{G}, \Gamma, v)$ of a finite graph of pro-$p$ groups, the pro-$p$ version of the Bass-Serre theorem for subgroups works, i.e. $H = \Pi_1(\mathcal{H}, H\backslash S(G))$ in the standard manner (see Proposition 1.6.23).

**Theorem 13** (Limitation Theorem). *Let $G = \Pi_1(\mathcal{G}, \Gamma, v)$ be the fundamental pro-p group of a finite reduced graph of pro-p groups. Let $H$ be an open normal subgroup of $G$ and $H = \Pi_1(\mathcal{H}, \Delta, v')$ be a decomposition as the fundamental pro-p group of a reduced graph of pro-p groups $(\mathcal{H}, \Delta, v')$ obtained from $(\mathcal{H}, H\backslash S(G))$ via an reduction process. Then $|E(\Delta)| \geqslant |E(\Gamma)|$. Moreover, for $p > 2$ the inequality is strict unless $\Gamma = \Delta$.*

Recall that two pro-$p$ groups $G_1, G_2$ are *commensurable* if there exist $H_1$ open in $G_1$ and $H_2$ open in $G_2$ such that $H_1 \cong H_2$. Theorem 13 allows us to prove that the accessibility of a pro-$p$ group with respect to a family $\mathcal{F}$ of pro-$p$ groups is preserved by commensurability. For accessible abstract groups such a result can be deduced from the Stallings splitting theorem; we are not aware of such a result for accessible groups with respect to a family of infinite groups in the abstract situation.

**Theorem 14.** *Let $\mathcal{F}$ be a family of pro-p groups closed for commensurability. Let $G$ be a finitely generated pro-p group and $H$ an open subgroup of $G$. Then $G$ is $\mathcal{F}$-accessible if and only if $H$ is $\mathcal{F}$-accessible.*

Note that the hypothesis of non-splitting in Theorems 5, 7 and 9 are essential. The pro-5 completion of the triangle group $G = \langle x, y \mid x^5, y^5, (xy)^5 \rangle$, for example, contains the pro-5 completion $\widehat{S}$ of a surface group $S$ as a subgroup of index 5. The group $\widehat{S}$ is a free pro-5 product of free pro-5 groups with cyclic amalgamation, but $G$ does not split as a non-trivial amalgamated free pro-5 product. Indeed, if it did, i.e. if $G = G_1 \amalg_H G_2$ then all torsion elements $x, y$, and $xy$ have to belong to some free factor up to conjugation, but then they belong to the normal closure of the same free factor, say $G_1^G$; it means that $G_1^G = G$ which is impossible, since $G/G_1^G \cong G_2/H^{G_2} \neq 1$.

This thesis is organized as follows. Chapter 1 is dedicated to recalling the elements of the Bass-Serre theory for abstract and profinite graphs that will be used throughout the text and establish the notation to be used in subsequent sections. The content of the next two chapters is completely original. In Chapter 2 we introduce a new definition of the fundamental group of a graph of pro-$\mathcal{C}$-groups at base point and prove that it is equivalent to the one that appears in [31]. The second section is the technical heart of the chapter and contains the proof of Theorem 3. We construct explicitly the graph of profinite groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ starting with an infinite graph of groups $(\mathcal{G}, \Gamma)$. With this construction in hand, Theorem 1 follows immediately. In Section 3 we prove Theorem 2 on the closure of normalizers. The techniques used here are largely based on Bass-Serre and Ribes-Zalesski-Mel'nikov's theories and their close interrelation. We finish the chapter with Section 4, which contains the proof of Theorem 4.

The third Chapter is organized as follows. Section 1 contains the proof of the Limitation Theorem and its applications. Section 2 starts with the proof of Theorem 9. Then with the Limitation Theorem in hand, we prove Theorem 11. Theorems 5 and 7 then follow immediately, but their corollaries require some work. Section 3 deals with finitely generated pro-$p$ accessible groups, where we prove Theorem 14. In the last section, we show that our Theorem 9 also works for Wilkes' example of a finitely generated inaccessible pro-$p$ group.

Brasília, 2023

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# CONTENTS

# CHAPTER 1

# BASS-SERRE AND RIBES-ZALESSKI-MEL'NIKOV'S THEORIES

This chapter offers an overview of Bass-Serre and Ribes-Zalesski-Mel'nikov's theories. Bass-Serre's theory is based on the books of Serre (cf. [37]) and Dunwoody (cf. [11]); Ribes-Zalesski-Mel'nikov's theory is based on the book of Ribes (cf. [31]). Many of the details and examples provided here derive from Ph.D. course notes given by Pavel Zalesski between 2000-2010.

## 1.1 Abstract and profinite graphs

Graphs are our main object of study throughout this work since they interact very nicely with groups. From the beginning, we will deviate slightly from classical Bass-Serre theory in order to make it compatible with the profinite version. We start with the definition of an abstract graph that will be used freely without further mention:

**Definition 1.1.1** (Abstract graph)**.** An abstract graph, or simply a graph, is a set $\Gamma$ constituted by two disjoint subsets, $V(\Gamma) \neq \varnothing$ and $E(\Gamma) = \Gamma - V(\Gamma)$, where the first is called the vertices set and the second the edges set, together with two maps $d_0, d_1$ :

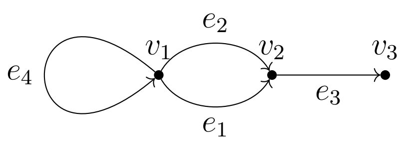
$\Gamma \to V(\Gamma)$ called incidence maps. These maps have the property that $d_i|_{V(\Gamma)} = \mathrm{id}_{V(\Gamma)}$, for $i = 0, 1$.

We have the following example

**Example 1.1.2.** Let $\Gamma = V(\Gamma) \cup E(\Gamma)$ be the following graph where

- $V(\Gamma) = \{v_1, v_2, v_3\}$
- $E(\Gamma) = \{e_1, e_2, e_3, e_4\}$

Then we can represent $\Gamma$ by




This abstract graph has 3 vertices, and 4 edges. In addition, $e_1, e_2, e_3, e_4$ have extremities $\{v_1, v_1\}$; $\{v_1, v_2\}$; $\{v_1, v_2\}$ and $\{v_2, v_3\}$ respectively. Note that $d_0(e_1) = v_1 = d_0(e_2)$, $d_1(e_1) = v_2 = d_1(e_1)$, $d_0(e_3) = v_2$, $d_1(e_3) = v_3$, $d_0(e_4) = v_1 = d_1(e_4)$, and $d_0(v_i) = v_i = d_1(v_i)$ for all $v_i \in V(\Gamma)$.

Now the reader must have some intuition about these objects and it is worth highlighting some details that are hidden in the definition. Firstly, the vertex set is never empty, but the edge set can be empty. In this case, if $\Gamma$ has more than one vertex, they will be isolated from each other. Subsequently, it is enough to evaluate the incidence maps on the edges, since they are the identity on the vertices, so they provide nothing new. Finally, the initial and final vertices of an edge $e$ ($d_0(e)$ and $d_1(e)$, respectively) can be the same $v \in V(\Gamma)$. In this case, we say that $e$ is a loop with basis $v$ or simply that $e$ is a loop if the basis is clear. The edge $e_4$ in Example 1.1.2 is a loop with basis $v_1$.

Two vertices are said to be connected if both are extremities of the same edge and two edges are said to be parallel if they have the same starting and ending points. The edges $e_1$ and $e_2$ in Example 1.1.2 are parallel and the vertices $v_2$ and $v_3$ are connected due to the edge $e_3$.

It is possible to add topology to these structures, giving rise to profinite graphs. First we recall the concept of inverse limit.

Let $(I, \preceq)$ be a directed partially oriented set (i.e., $\preceq$ is an order relation in $I$ and if $i, j \in I$, there exists some $k \in I$ such that $i, j \preceq k$).

**Definition 1.1.3** (Inverse system)**.** An inverse system of topological spaces consists of a family $(X_i)_{i\in I}$, and a collection of continuous mappings $\varphi_{ij} : X_i \to X_j$ whenever $i \succeq j$ such that the diagrams

$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & X_k \\ & X_j \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

commute, whenever $i \preceq j \preceq k$.

If $Y$ is another topological space, we can define continuous maps $\psi_i : Y \to X_i$ for every $i \in I$. They are said to be compatible if $\varphi_{ij}\psi_i = \psi_j$, whenever $i \succeq j$.

**Definition 1.1.4** (Inverse limit)**.** We say that a topological space $X$, together with a set of compatible maps $\varphi_i : X \to X_i$ (called projections) is the inverse limit of the inverse system $\{X_i, \varphi_{ij}, I\}$ if the following universal property is satisfied

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[dr, "\psi_i"'] & X \arrow[d, "\varphi_i"] \\ & X_i \end{tikzcd}$$

whenever $Y$ is a topological space and $\psi_i : Y \to X_i$ is a set of compatible continuous mappings, then there is a unique continuous mapping $\psi : Y \to X$ such that $\varphi_i\psi = \psi_i$ for all $i \in I$.

Although this universal property is useful in many cases, [30, Proposition 1.1.1] provides another form of seeing $X = \varprojlim_{i\in I} X_i$: as the subset of $\prod_{i\in I} X_i$ (endowed with the product topology) consisting of those tuples $(m_i)$ with $\varphi_{ij}(m_i) = m_j$, whenever $i \succeq j$.

**Definition 1.1.5** (Profinite space)**.** Using the previous notation, $X$ is said to be a profinite space if $X = \varprojlim_{i\in I} X_i$ where each $X_i$ if finite and endowed with the discrete topology.

By [30, Proposition 1.1.12], a topological space $X$ is profinite if, and only if, it is Hausdorff, compact and totally disconnected. It is clear that every finite space endowed with the discrete topology is also profinite.

The first reference to profinite (or boolean) graphs appears in [17]. Later the concept was developed by Luis Ribes, Oleg Melnikov, and Pavel Zalesski in order to obtain a profinite version of the celebrated Bass-Serre theory. Roughly speaking, a profinite graph is a profinite space with a graph structure.

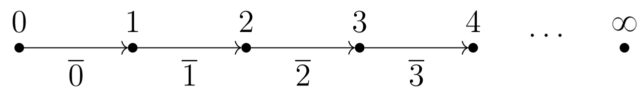
We provide the formal definition that appears in [31]:

**Definition 1.1.6** (Profinite graph, [31], Section 2.1, page 29)**.** A profinite graph is a profinite space $\Gamma$ with a distinguished closed nonempty subset $V(\Gamma)$ called the vertex set, $E(\Gamma) = \Gamma - V(\Gamma)$ the edge set and two continuous maps $d_0, d_1 : \Gamma \to V(\Gamma)$ whose restrictions to $V(\Gamma)$ are the identity map $id_{V(\Gamma)}$. We refer to $d_0$ and $d_1$ as the incidence maps of the profinite graph $\Gamma$.

As usual in profinite constructions, finite abstract graphs with the discrete topology are profinite graphs. But we provide another example to help the intuition.

**Example 1.1.7** ([31], Example 2.1.1(b))**.** Let $N = \{0, 1, 2, \cdots\}$ and $\overline{N} = \{\overline{n} \mid n \in N\}$ be two copies of the set of the natural numbers (each one with the discrete topology). Define $\Gamma = N \cup \overline{N} \cup \{\infty\}$ to be the one-point compactification of the space $N \cup \overline{N}$. Hence $\Gamma$ is a profinite space because it is compact, Hausdorff and totally disconnected. We can introduce a profinite graph structure into $\Gamma$ by setting:

- $V(\Gamma) = N \cup \{\infty\}$;
- $E(\Gamma) = \overline{N}$;
- $d_0(\overline{n}) = n$ for $\overline{n} \in E(\Gamma)$ and $d_0(n) = n$ for $n \in V(\Gamma)$;
- $d_1(\overline{n}) = n + 1$ for $\overline{n} \in E(\Gamma)$ and $d_1(n) = n$ for $n \in V(\Gamma)$.




In this case the subset of edges $E(\Gamma) = \overline{N} = \{\overline{n} \mid n \in N\}$ is open, but not closed in $\Gamma$. Indeed, we are taking the discrete topology, so $\overline{N}$ is open, but it is not compact and therefore not closed.

**Definition 1.1.8** (Abstract subgraph)**.** A nonempty subset $\Delta$ of an abstract graph $\Gamma$ is called an abstract subgraph, or simply a subgraph, of $\Gamma$ if whenever $m \in \Gamma$, then $d_j(m) \in \Gamma$ $(j = 0, 1)$.

When working with topological spaces, additional conditions on a subset of a profinite graph are needed to constitute a profinite subgraph. It has to be a closed subset in order to be profinite. We state this in the following definition:

**Definition 1.1.9** (Profinite subgraph)**.** A nonempty closed subset $\Delta$ of a profinite graph $\Gamma$ is called a profinite subgraph of $\Gamma$ if it admits a subgraph structure, i.e. whenever $m \in \Delta$, then $d_j(m) \in \Delta$ $(j = 0, 1)$.

Given a profinite graph $\Gamma$, any finite subgraph $\Delta$ of $\Gamma$ is a profinite subgraph. In example 1.1.7, since the open neighbourhoods of $\{\infty\}$ are all the sets $\Gamma - C$, where $C$ is a compact subspace of $N \cup \widetilde{N}$, they contain all but finitely many elements of $\Gamma$. Hence they are closed, because their complement is open. Therefore, if such a neighbourhood admits a subgraph structure, it is a profinite subgraph of $\Gamma$.

Maps between graphs work nicely, but they have some particularities as follows

**Definition 1.1.10** (Morphism of abstract graphs)**.** A morphism of abstract graphs

$$\alpha : \Gamma \to \Delta$$

is a map that commutes with $d_0$ and $d_1$, i.e., $d_j(\alpha(m)) = \alpha(d_j(m))$, for all $m \in \Gamma$ and $j = 0, 1$ and $\alpha(e) \in E(\Delta)$ for every $e \in E(\Gamma)$.

If $\alpha$ is a surjective morphism (respectively injective, bijective), we say that $\alpha$ is an epimorphism (respectively, monomorphism, isomorphism). An isomorphism $\alpha : \Gamma \to \Gamma$ of the graph $\Gamma$ to itself is called an automorphism.

**Definition 1.1.11.** Let $\Delta \subseteq \Gamma$, where $\Gamma, \Delta$ are abstract graphs and $a \in \Gamma$. Define

$$q : \Gamma \to \Gamma$$

by $q(x) = x$ for every $x \in \Gamma - \Delta$ and $q(x) = a$ for all $x \in \Delta$. The relation $x \equiv y \iff q(x) = q(y)$ is an equivalence relation on $\Gamma$.

**Definition 1.1.12** (Quotient graph)**.** We define an abstract graph $\Gamma/\Delta = q(\Gamma)$ such that

- $V(\Gamma/\Delta) = q(V(\Gamma))$;
- $E(\Gamma/\Delta) = \Gamma/\Delta - V(\Gamma/\Delta)$;
- $d_0, d_1 : \Gamma/\Delta \to V(\Gamma/\Delta)$ defined by $d_j(q(m)) = q(d_j(m))$, $(j = 0, 1; m \in \Gamma/\Delta)$.

The graph $\Gamma/\Delta$ is called the quotient abstract graph.

The major changes in relation to the abstract context start to appear at this point, where the $q$-morphism allows mapping edges to vertices. This works very well with inverse limit constructions that are not needed in the abstract case.

**Definition 1.1.13** ($q$-morphism of profinite graphs)**.** Let $\Gamma$ and $\Delta$ be profinite graphs. A $q$-morphism or a quasi-morphism of profinite graphs $\alpha : \Gamma \to \Delta$ is a continuous map such that $d_j(\alpha(m)) = \alpha(d_j(m))$, for all $m \in \Gamma$ and $j = 0, 1$. If in addition $\alpha(e) \in E(\Delta)$ for every $e \in E(\Gamma)$, we say that $\alpha$ is a morphism.

The equality $d_j(\alpha(m)) = \alpha(d_j(m))$, $(j = 0, 1; m \in \Gamma)$ implies that a $q$-morphism of profinite graphs maps vertices to vertices. Indeed, take $m \in V(\Gamma)$. Then $d_j(\alpha(m)) \in V(\Delta)$ because $d_j : \Delta \to V(\Delta)$. Since $d_j(\alpha(m)) = \alpha(d_j(m))$, this implies that $\alpha(d_j(m)) = \alpha(m) \in V(\Gamma)$, as desired. On the other hand, $q$-morphisms can map edges to vertices, unlike the abstract case.

We also have a profinite version of a quotient graph.

**Example 1.1.14** ([31], Example 2.1.2)**.** Let $\Gamma$ be a profinite graph and $\Delta$ a profinite subgraph of $\Gamma$. We can define a natural continuous map $\alpha : \Gamma \to \Gamma/\Delta$, where $\Gamma/\Delta$ is endowed with the quotient topology, and a profinite graph structure on the space $\Gamma/\Delta$ inherited from $\Gamma$ as follows:

- $V(\Gamma/\Delta) = \alpha(V(\Gamma))$;
- $d_j(\alpha(m)) = \alpha(d_j(m))$ $(j = 0, 1)$;

for all $m \in \Gamma$ (note that we only need to define the vertex set and the incidence maps on the edges, because they are trivial on the vertices). Then $\alpha$ is a q-morphism of profinite graphs and we call $\Gamma/\Delta$ a quotient graph of $\Gamma$. We can say that $\Gamma/\Delta$ is obtained from $\Gamma$ by collapsing $\Delta$ to a point. Observe that $\alpha$ maps any edge of $\Gamma$ which is in $\Delta$ to a vertex of $\Gamma/\Delta$.

Next, we enlarge the notion of vertex connectivity

**Definition 1.1.15.** A path $p$ in an abstract graph $\Gamma$ is a finite sequence

$$p = v_0, e_1^{\varepsilon_1}, v_1, \cdots, e_n^{\varepsilon_n}, v_n,$$

where $n \geqslant 0$, $v_0, v_1, \cdots, v_n \in V(\Gamma)$, $e_1, e_2, \cdots, e_n \in E(\Gamma)$, $\varepsilon_i = \pm 1$ $(i = 1, \cdots, n)$, and $d_0(e_i^{\varepsilon_i}) = v_{i-1}$, $d_1(e_i^{\varepsilon_i}) = v_i$, for $i = 1, \cdots, n$.

Its normally abbreviated by $e_1^{\varepsilon_1}, \cdots, e_1^{\varepsilon_1}$, where $n \geqslant 0$.

We say that $p = p(v_0, v_n)$ is a path between $v_0$ and $v_n$ of length $n$. Two vertices $u, v \in \Gamma$ are connected if there is a path between $u$ and $v$. The fact that $u$ is connected to $v$ defines an equivalence relation on $\Gamma$ and the equivalence class is called the connected component of $\Gamma$. We can observe similarities with the topological notion of connectedness. We will be exploring it in the profinite case.

**Definition 1.1.16.** An abstract graph $\Gamma$ is connected if given two vertices $u, v \in \Gamma$, there is a path between $u$ and $v$.

A connected graph has just one connected component, itself.

**Definition 1.1.17.** Let $p = v_0, e_1^{\varepsilon_1}, v_1, \cdots, e_n^{\varepsilon_n}, v_n$ be a path in $\Gamma$. We say that $p$ is reduced if, for all $i = 1, \cdots, n$, we have: $e_i = e_{i+1} \Rightarrow \varepsilon_i = \varepsilon_{i+1}$.

In the case that $p$ is not reduced, there is $e_i \in E(\Gamma)$ such that $e_i = -e_{i+1}$ and $\varepsilon_i = \varepsilon_{i+1}$, so we have

$$\begin{aligned} p &= v_0, e_1^{\varepsilon_1}, v_1, \cdots, e_{i-1}^{\varepsilon_{i-1}}, v_{i-1}, e_i^{\varepsilon_i}, v_i, e_{i+1}^{\varepsilon_{i+1}}, v_{i+1}, e_{i+2}^{\varepsilon_{i+2}}, v_{i+2}, \cdots, e_n^{\varepsilon_n}, v_n \\ &= v_0, e_1^{\varepsilon_1}, v_1, \cdots, e_{i-1}^{\varepsilon_{i-1}}, v_{i-1}, e_i^{\varepsilon_i}, v_i, e_i^{-\varepsilon_i}, v_{i-1}, e_{i+2}^{\varepsilon_{i+2}}, v_{i+2}, \cdots, e_n^{\varepsilon_n}, v_n \\ &= v_0, e_1^{\varepsilon_1}, v_1, \cdots, e_{i-1}^{\varepsilon_{i-1}}, v_{i-1}, e_{i+2}^{\varepsilon_{i+2}}, v_{i+2}, \cdots, e_n^{\varepsilon_n}, v_n \end{aligned}$$

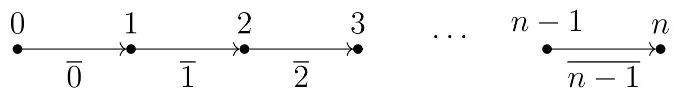
This operation is called the **simple reduction** of $p$. By successive simple reductions, we can transform $p$ into a reduced path, called the reduced form of $p$. The simple reduction is an equivalence relation, being the reduced form the representative of the class. The reduced form of a path over an abstract graph is unique.

Another change in the profinite case is the one of connectivity. As we will show, there exist profinite graphs that are connected, but have vertices with no edge beginning or ending at them.

**Definition 1.1.18** (Connected profinite graph)**.** A profinite graph $\Gamma$ is said to be connected if whenever $\varphi : \Gamma \to A$ is a $q$-morphism of profinite graphs onto a finite graph, then $A$ is connected as an abstract graph.

The following example shows a connected profinite graph which is not connected as an abstract graph and has a vertex with no edge beginning or ending at it.

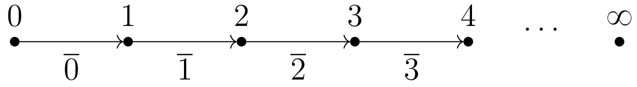
**Example 1.1.19** ([31], Example 2.1.8)**.** The graph $\Gamma$ of Example 1.1.7 is a connected profinite graph; to see this consider the connected finite graphs $\Gamma_n$




with vertices $V(\Gamma_n) = \{0, 1, 2, 3, \cdots, n\}$ and edges $E(\Gamma_n) = \{\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{n-1}\}$ such that $d_0(\overline{\imath}) = i$, $d_1(\overline{\imath}) = i+1$ $(i = 0, 1, \cdots, n-1)$ and $d_j(i) = i$ for $i \in V(\Gamma)$, $i = 0, 1, \cdots, n; j = 0, 1$. If $n \leqslant m$, define $\varphi_{m,n} : \Gamma_m \to \Gamma_n$ to be the map of graphs that sends the segment $[0, n]$ identically to $[0, n]$, and the segment $[n, m]$ to the vertex $n$. Then $\{\Gamma_n, \varphi_{m,n}\}$ is an inverse system of finite graphs, and

$$\Gamma = \varprojlim_{n \in N} \Gamma_n,$$

where $\infty = (n)_{n \in N}$ (note here the importance of the q-morphism definition. We are allowed to send the edge $[n, m]$ to the vertex $n$, what would not be possible with the abstract definition of a morphism). Our graph $\Gamma$ becomes the following:




Hence $\Gamma$ is a connected profinite graph, because any morphism of $G$ to a finite graph can be factored through a finite connected graph (cf. [31, Proposition 2.1.5]), so the finite graph has to be connected. We observe that there is no edge $e$ of $\Gamma$ which has $\{\infty\}$ as one of its vertices; and so $\Gamma$ is not connected as an abstract graph.

The next proposition shows that if $E(\Gamma)$ is closed, we cannot have a vertex like $\{\infty\}$ in a connected profinite graph, such that there is no edge starting or ending at it. In [31, Proposition 2.1.4], it is shown that every profinite graph can be written as an inverse limit of its finite quotient graphs.

**Proposition 1.1.20** ([31], Proposition 2.1.6 (c))**.** *Let $\Gamma$ be a connected profinite graph. If $|\Gamma| > 1$, then $\Gamma$ has at least one edge. Furthermore, if the set of edges $E(\Gamma)$ of $\Gamma$ is closed in $\Gamma$, then for any vertex $v \in V(\Gamma)$, there exists and edge $e \in E(\Gamma)$ such that either $v = d_0(e)$ or $v = d_1(e)$.*

*Proof.* We give an idea of the proof. By [31, Proposition 2.1.4], as $E(\Gamma)$ is closed, it can be written as an inverse limit

$$E(\Gamma) = \varprojlim_{i \in I} E(\Gamma_i).$$

Put $v_i = \varphi_i(v)$, where $\varphi_i : \Gamma \to \Gamma_i$ is the projection. Since $\Gamma_i$ is a connected finite graph, $S_i = d_0^{-1}(v_i) \cup d_1^{-1}(v_i) \cap E(\Gamma) \neq \varnothing$. Moreover, $\varphi_{ij}(S_i) \subseteq S_j$. Hence the collection $(S_i)_{i \in I}$ is an inverse system of nonempty finite sets. By [30, Proposition 1.1.4], $\varprojlim_{i \in I} S_i \neq \varnothing$. Let $e \in \varprojlim_{i \in I} S_i$. Then $e$ is an edge of $\Gamma$ with either $d_0(e) = v$ or $d_1(e) = v$. □

We finish this section with the concept of trees. They are going to be essential objects in this thesis. The reason they are so special will be uncovered later.

**Definition 1.1.21** (Circuit)**.** A circuit on an abstract graph is a reduced path of a vertex $v$ to itself, with length at least 1.

An abstract graph without circuits is a forest. A connected forest is called a tree. It is possible to show that if $T$ is a tree and $v, w \in V(T)$ there is only one reduced path $p(v, w)$ connecting $v$ to $w$. Indeed, if there were two paths $p_1(v, w)$ and $p_2(v, w)$, it would be possible to define a circuit $p_3(v, v) = p_1(v, w)[p_2(v, w)]^{-1}$. If

$$p(v, w) = v, e_1^{\varepsilon_1}, v_1, \cdots, e_n^{\varepsilon_n}, w$$

is the reduced path between $v$ and $w$, we define

$$l(p(v, w)) = d(v, w) = n.$$

We say that the distance between $v$ and $w$ is $n$. The path $[v, w]$ (unique reduced path) is called a geodesic.

**Definition 1.1.22.** Given a vertex $v \in V(\Gamma)$, we will define $\mathrm{star}(v) = d_0^{-1}(v) \cup d_1^{-1}(v) \cap E(\Gamma)$, i.e., the set of all edges that have their beginning or end at $v$. We define the degree of $v$ as $|\mathrm{star}(v)|$.

For the next result, we are going to use the following classical Lemma

**Lemma 1.1.23** (Zorn's Lemma)**.** *Let $(A, \leq)$ be a partially ordered set in which each totally ordered subset has supremum. Then $A$ has a maximal element.*

**Theorem 1.1.24.** *Every connected abstract graph $\Gamma$ has a maximal subgraph $T$ that is a tree and $V(\Gamma) = V(T)$ (also called a spanning subtree of $\Gamma$).*

*Proof.* We first claim that there exists a maximal subtree $T$ of $\Gamma$. In fact, the set $\mathcal{S}$ of all subgraphs of $\Gamma$ that are trees forms a partially ordered set by inclusion . Note also that $\mathcal{S} \neq \varnothing$, because $\{v\} \in \mathcal{S}$, where $v \in \Gamma$ is arbitrary.

Let $T_1 \subseteq T_2 \subseteq \cdots$ be a chain of subtrees of $\Gamma$. Then $T = \bigcup_{i=1}^{\infty} \in \mathcal{S}$ is also a subtree of $\Gamma$. It follows from Zorn's Lemma (cf. Lemma 1.1.23) that $\mathcal{S}$ has a maximal element $T$ and so $\Gamma$ has a maximal subtree $T$, proving the claim.

Finally, we claim that $V(\Gamma) = V(T)$. Suppose by contradiction that $V(T) \neq V(\Gamma)$, i.e., there exists $v \in V(\Gamma) - V(T)$. Consider $w \in T$ and let $p(v, w)$ be a reduced path from $v$ to $w$. It exists because $\Gamma$ is connected. Then there exists an edge $e$ in $p(v, w)$ that has extremities $a \in T$ and $u \in \Gamma - T$. Therefore, $T \cup \{e, u\}$ is a subtree of $\Gamma$ containing $T$, which is maximal. Therefore, $V(\Gamma) = V(T)$, as desired. □

As we are going to see later, this does not hold for profinite graphs (cf. Example 2.1.7).

## 1.2 Groups acting on graphs

We are investigating in this section how groups interact with graphs and what properties can be inferred by these actions.

**Definition 1.2.1.** We say that a group $G$ acts on a set $X$, or that $X$ is a $G$-set, if there exists a homomorphism $\varphi : G \to \mathrm{Sym}(X)$

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ & & \\ (g,x) & \longmapsto & gx \end{array}$$

The action is called transitive if it possesses only a single orbit, i.e., for every pair of elements $x, y \in X$, there is a group element $g$ such that $gx = y$.

**Definition 1.2.2** (Pseudovariety[1] of finite groups)**.** A nonempty class of finite groups $\mathcal{C}$ is a pseudovariety if it is closed under taking subgroups, quotients and finite direct products. A pseudovariety of finite groups $\mathcal{C}$ is said to be extension-closed if whenever

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

is an exact sequence of finite groups with $K, H \in \mathcal{C}$, then $G \in \mathcal{C}$.

**Definition 1.2.3** (Pro-$\mathcal{C}$ group)**.** Let $\mathcal{C}$ be a nonempty pseudovariety of finite groups. Define a pro-$\mathcal{C}$ group $G$ as an inverse limit

$$G = \varprojlim_{i \in I} G_i$$

of a surjective inverse system $\{G_i, \varphi_{ij}, I\}$ of groups $G_i$ in $\mathcal{C}$, where each group $G_i$ is endowed with the discrete topology.

We think of such a pro-$\mathcal{C}$ group $G$ as a topological group, whose topology is inherited from the product topology on $\prod_{i \in I} G_i$. If $\mathcal{C}$ is the pseudovariety of all finite groups, we call a pro-$\mathcal{C}$ group $G$ a profinite group. If it is the pseudovariety of all finite $p$-groups, $G$ is said to be a pro-$p$ group. Observe that every pro-$\mathcal{C}$ group is also a profinite group.

From now on we shall use the following convenient notations. Let $G$ be a topological group and $H$ a subgroup of $G$. Then $H \leqslant_o G$, $H \leqslant_c G$, $H \lhd_o G$, $H \lhd_c G$, $H \leqslant_f G$ and $H \lhd_f G$ will indicate, respectively: $H$ is an open subgroup; $H$ is a closed subgroup; $H$ is an open normal subgroup; $H$ is a closed normal subgroup; $H$ is a subgroup of finite index; $H$ is a normal subgroup of finite index.

We define the action of a profinite group on a profinite graph and the quotient graph by an action of a profinite group. These concepts will be useful to understand Galois coverings in the next section.

[1] In [30] it this is referred as a variety of profinite groups

**Definition 1.2.4.** Let $G$ be a profinite group and let $\Gamma$ be a profinite graph. We say that the profinite group $G$ acts on the profinite graph $\Gamma$ on the left, or that $\Gamma$ is a $G$-graph, if

(i) $G$ acts continuously on the topological space $\Gamma$ on the left, i.e., there is a continuous map $G \times \Gamma \to \Gamma$, denoted by $(g, m) \mapsto gm$, $g \in G$, $m \in \Gamma$, such that

$$(gh)m = g(hm) \text{ and } 1m = m,$$

for all $g, h \in G$, $m \in \Gamma$, where 1 is the identity element of $G$; and

(ii) $d_j(gm) = gd_j(m)$, for all $g \in G$, $m \in \Gamma$, $j = 0, 1$.

**Definition 1.2.5** ($G$-map of graphs)**.** Let $G$ be a profinite group that acts continuously on two profinite graphs $\Gamma$ and $\Gamma'$. A q-morphism of graphs $\varphi : \Gamma \to \Gamma'$ is called a $G$-map of graphs if

$$\varphi(gm) = g\varphi(m)$$

for all $m \in \Gamma$, $g \in G$.

**Definition 1.2.6** (Stabilizer)**.** Assume that a profinite group acts on a profinite graph $\Gamma$ and let $m \in \Gamma$. Define $G_m = \{g \in G \mid gm = m\}$ to be the stabilizer, or $G$-stabilizer of the element $m$.

It follows from the continuity of the action and the compactness of $G$ that $G_m$ is a closed subgroup of $G$. We have that

$$G_m \leqslant G_{d_j(m)}$$

for every $m \in \Gamma$, $j = 0, 1$, because $G_{d_j(m)} = \{g \in G \mid gd_j(m) = d_j(m)\}$ and as $gd_j(m) = d_j(gm)$, $G_{d_j(m)} = \{g \in G \mid d_j(gm) = d_j(m)\}$.

**Definition 1.2.7** (Free action)**.** If the stabiliser $G_m$ of every element $m \in \Gamma$ is trivial, i.e., $G_m = 1$, we say that $G$ acts freely on $\Gamma$.

**Definition 1.2.8** ($G$-orbit)**.** If $m \in \Gamma$, the $G$-orbit of $m$ is the closed subset $Gm = \{gm \mid g \in G\}$.

The next definition is very important for Galois coverings, which will be presented in the next chapter:

**Definition 1.2.9** (Quotient graph under the action of a profinite group)**.** Let $G$ be a profinite group that acts on a profinite graph $\Gamma$. In particular, $G$ acts on $V(\Gamma)$ (also a profinite space, because it is a closed subset of $\Gamma$) and $E(\Gamma)$. The space

$$G\backslash\Gamma = \{Gm \mid m \in \Gamma\}$$

of $G$-orbits with the quotient topology is a profinite space which admits a natural graph structure as follows:

- $V(G\backslash\Gamma) = G\backslash V(\Gamma)$;
- $d_j(Gm) = Gd_j(m)$ $(j = 0, 1)$.

The profinite graph $G\backslash\Gamma$ is called the quotient graph of $\Gamma$ under the action of $G$. The corresponding quotient map

$$\varphi : \Gamma \to G\backslash\Gamma$$

is an epimorphism of profinite graphs given by $m \mapsto Gm$ $(m \in \Gamma, g \in G)$.

**Remark 1.2.10.** The map $\varphi$ sends edges to edges (it is a morphism). Indeed, given $e \in E(\Gamma)$ the element $\varphi(e) = Ge$ belongs to $G\backslash E(\Gamma) = G\backslash(\Gamma - V(\Gamma)) = G\backslash\Gamma - G\backslash V(\Gamma) = G\backslash\Gamma - V(G\backslash\Gamma) = E(G\backslash\Gamma)$.

**Remark 1.2.11.** If $N \lhd_c G$, there is an induced action of $G/N$ on $N\backslash\Gamma$ defined by

$$(gN)(Nm) = N(gm)$$

for $g \in G$, $m \in \Gamma$.

**Proposition 1.2.12** ([31], Proposition 2.2.1)**.** *Let a profinite group $G$ act on a profinite graph $\Gamma$.*

*(a) Let $\mathcal{N}$ be a collection of closed normal subgroups of $G$ filtered from below (i.e., the intersection of any two groups in $\mathcal{N}$ contains a group in $\mathcal{N}$) and assume that*

$$G = \varprojlim_{N \in \mathcal{N}} G/N.$$

*Then the collection of graphs* $\{N\backslash\Gamma \mid N \in \mathcal{N}\}$ *is an inverse system in a natural way and*

$$\Gamma = \varprojlim_{N\in\mathcal{N}} N\backslash\Gamma.$$

*(b) Let* $N \lhd_c G$. *For* $m \in \Gamma$, *denote by* $m'$ *the image of* $m$ *in* $N\backslash\Gamma$. *Consider the natural action of* $G/N$ *on* $N\backslash\Gamma$ *defined above. Then* $(G/N)_{m'}$ *is the image of* $G_m$ *under the natural epimorphism* $G \to G/N$. *In particular, if* $G_m \leqslant N$, *for all* $m \in \Gamma$, *then* $G/N$ *acts freely on* $N\backslash\Gamma$.

Let $\Gamma$ be a profinite graph. If $\{\Gamma_i, \varphi_{ij}, I\}$ is an inverse system of profinite $G$-graphs and $G$-maps over the directed poset $I$, then

$$\Gamma = \varprojlim_{i\in I} \Gamma_i$$

is in a natural way a profinite $G$-graph, defined by $V(\Gamma) = \varprojlim_{i\in I} V(\Gamma_i)$ and the incidence maps as the compositions.

**Proposition 1.2.13** ([31], Proposition 2.2.2)**.** *Let a profinite group* $G$ *act on a profinite graph* $\Gamma$.

*(a) Then there exists a decomposition*

$$\Gamma = \varprojlim_{i\in I} \Gamma_i$$

*of* $\Gamma$ *as the inverse limit of a system of finite quotient* $G$*-graphs* $\Gamma_i$ *and* $G$*-maps* $\varphi_{ij} : \Gamma_i \to \Gamma_j$ $(i \succeq j)$ *over a directed poset* $(I, \preceq)$.

*(b) If* $G$ *is finite and acts freely on* $\Gamma$, *then the decomposition of part* (*a*) *can be chosen so that* $G$ *acts freely on each* $\Gamma_i$.

In the category of topological rings, a profinite ring $\Lambda$ is an inverse limit of an inverse system $\{\Lambda_i, \varphi_{ij}\}$ of finite discrete rings. We always assume that rings have an identity element, denoted usually by 1 and that homomorphisms of rings send identity elements to identity elements. Consider a commutative profinite ring $R$ and a profinite group $G$. Define the complete group algebra

$$R\,[[G]] = \varprojlim_{U\in\mathcal{U}} \left[R(G/U)\right],$$

of the ordinary group algebras $[R(G/U)]$, where $\mathcal{U}$ is the collection of all open normal subgroups of $G$. One can express $R[[G]]$ as an inverse limit of finite rings $[(R/I)(G/U)]$, where $I$ and $U$ range over all the open ideals of $R$ and the open normal subgroups of $G$, respectively.

**Definition 1.2.14** (pro-$\mathcal{C}$-tree)**.** Let $\Gamma$ be a profinite graph. Define $E^*(\Gamma) = \Gamma/V(\Gamma)$ to be the quotient space of $\Gamma$ (viewed as a profinite space) modulo the subspace of vertices $V(\Gamma)$. Let $R$ be a profinite ring and consider the free profinite $R$-modules $R[[(E^*(\Gamma), *)]]$ and $R[[V(\Gamma)]]$ on the pointed profinite space $(E^*(\Gamma), *)$ and on the profinite space $V(\Gamma)$, respectively. Denote by $C(\Gamma, R)$ the chain complex

$$0 \longrightarrow R[[(E^*(\Gamma), *)]] \xrightarrow{d} R[[V(\Gamma)]] \xrightarrow{\varepsilon} R \longrightarrow 0$$

of free profinite $R$-modules and continuous $R$-homomorphisms $d$ and $\varepsilon$ determined by $\varepsilon(v) = 1$, for every $v \in V(\Gamma)$, $d(\overline{e}) = d_1(e) - d_0(e)$, where $\overline{e}$ is the image of an edge $e \in E(\Gamma)$ in the quotient space $E^*(\Gamma)$, and $d(*) = 0$. Obviously $Im(d) \subseteq Ker(\varepsilon)$ and one defines the homology groups of $\Gamma$ as the homology groups of the chain complex $C(\Gamma, R)$ in the usual way:

$$H_0(\Gamma, R) = Ker(\varepsilon)/Im(d) \text{ and } H_1(\Gamma, R) = Ker(d).$$

One says that $\Gamma$ is a pro-$\mathcal{C}$ tree if the sequence $C(\Gamma, Z_{\hat{\mathcal{C}}})$ is exact.

By [31, Proposition 2.3.2] a profinite graph is connected if and only if $H_0(\Gamma, R) = 0$, independently of the choice of the profinite ring $R$. Therefore a profinite graph $\Gamma$ is a pro-$\mathcal{C}$ tree if and only if it is connected and $H_1(\Gamma, Z_{\hat{\mathcal{C}}}) = 0$. If $v$ and $w$ are elements of a tree (respectively $\mathcal{C}$-tree) $T$, one denotes by $[v, w]$ the smallest subtree (consider the intersection of all paths from $v$ to $w$ in $T$) (respectively $\mathcal{C}$-tree) of $T$ containing $v$ and $w$.

We collect some results that will be important in the development of the text

**Theorem 1.2.15** (Theorem 3.9 of [32])**.** *Suppose that a finite $p$-group $G$ acts on a pro-$p$ tree $T$. Then $G = G_v$ for some vertex $v \in V(T)$.*

**Theorem 1.2.16** (Theorem 4.1.2 of [31] or Theorem 2.6 of [51])**.** *Let a pro-$\mathcal{C}$ group $G$ act freely on a pro-$\mathcal{C}$ tree $T$. Then $G$ is a projective group.*

We note that, in the abstract case, a group that acts freely on a tree is itself free. There is also a graph on which a given group $G$ acts naturally. It is called the Cayley graph of $G$ and it was introduced in the very first paper about profinite graphs by Gildenhuys and Ribes (cf. [17]). define it as follows

**Definition 1.2.17** (Cayley graph)**.** Let $G$ be a group and $X$ a subset of $G$. Put $\widetilde{X} = X \cup \{1\}$. We can define the Cayley graph $\Gamma(G, X)$ of the group $G$ with respect to the subset $X$ as follows:

- $\Gamma(G, X) = G \times \widetilde{X}$
- $V(\Gamma(G, X)) = \{(g, 1) \mid g \in G\}$
- $d_j : \Gamma(G, X) = G \times \tilde{X} \to V(\Gamma(G, X))$
  - $d_0(g, x) = g$
  - $d_1(g, x) = gx$

For the profinite case, we ask $G$ to be a profinite group, $X$ to be a closed subset of $G$ and $G \times \widetilde{X}$ to have the product topology with $j = 0, 1$. Note that, in this case, $G \times \widetilde{X}$ is a profinite space, because $G$ is a pro-$\mathcal{C}$ group (and in particular a profinite space), $X$ is a closed subspace of a profinite space, and so it compact and hence profinite. Hence, $G \times \widetilde{X}$ with the product topology is also profinite.

We can identify the space $V(\Gamma(G, X))$ with $G$ through the homeomorphism $(g, 1) \mapsto g$ $(g \in G)$.

In the profinite case, the incidence maps $d_0$ and $d_1$ are continuous (one is just a projection and the other is a left multiplication) and they are the identity map when restricted to $V(\Gamma(G, X)) = \{(g, 1) \mid g \in G\} = G$. Therefore $\Gamma(G, X) = G \times \tilde{X}$ is a profinite graph. Note that the space of edges is $E(\Gamma(G, X)) = \Gamma(G, X) - V(\Gamma(G, X)) = G \times (X - \{1\})$:

$$g \bullet \xrightarrow{(g,x)} \bullet\ gx$$

where $x \in X - \{1\}$. It is already open, because $V(\Gamma(G, X))$ is closed in $\Gamma(G, X)$. It is closed if, and only if 1 is an isolated point of $\tilde{X}$. Indeed, note that if $1 \notin X$, then $V(\Gamma(G, X)) = G$ and $E(\Gamma(G, X)) = G \times X$, and in this case $E(\Gamma(G, X))$ is clopen. On the other hand, if $1 \in X$ we have that $\tilde{X} = X$. If in addition 1 is an isolated point of $X$ (for example, if $X$ is finite), then $X - \{1\}$ is also a closed subspace and we have

$\Gamma(G,X) = \Gamma(G, X - \{1\})$. Note that the Cayley graph $\Gamma(G,X)$ does not contain loops since the elements of the form $(g,1)$ are vertices by definition.

Let $\varphi : G \to H$ be a continuous homomorphism of profinite groups and let $X$ be a closed subset of $G$, and so compact. Put $Y = \varphi(X)$, which is a compact subset of $H$ by the continuity of $\varphi$ and therefore closed. So we can define the Cayley graph of $G$ with respect to the subset $X$, denoted by $\Gamma(G,X)$ and of $H$ with respect to the subset $\varphi(X)$, denoted by $\Gamma(H,\varphi(X))$. Then $\varphi$ induces a q-morphism of the corresponding Cayley graphs

$$\tilde{\varphi} : \Gamma(G,X) \to \Gamma(H,Y).$$

In particular, if $U$ is an open normal subgroup of $G$ and $X_U = \varphi_U(X)$, where $\varphi_U : G \to G/U$ is the canonical epimorphism, then $\varphi_U$ induces a corresponding epimorphism of Cayley graphs $\widetilde{\varphi_U} : \Gamma(G,X) \to \Gamma(G/U, X_U)$. So we can construct with the morphisms $\widetilde{\varphi_U} : \Gamma(G,X) \to \Gamma(G/U, X_U)$ and $\widetilde{\varphi_{U_{ij}}} : \Gamma(G/U_1, X_{U_1}) \to \Gamma(G/U_2, X_{U_2})$ an inverse system such that

$$\Gamma(G,X) = \varprojlim_{U \triangleleft_o G} \Gamma(G/U, X_U)$$

is a decomposition of $\Gamma(G,X)$ as an inverse limit of finite Cayley graphs.

We give as example the Cayley graph of $\mathbb{Z}_p$

**Example 1.2.18** ([31], Example 2.1.1(c))**.** Let $p$ be a prime number and let $\mathbb{Z}_p$ be the additive group of the ring of $p$-adic integers. According to the definition above, the Cayley graph of $\mathbb{Z}_p$ is

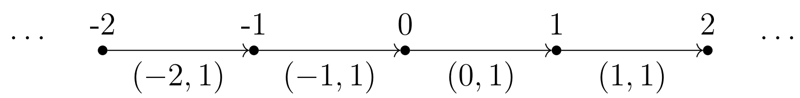
$$\Gamma = \Gamma(\mathbb{Z}_p, \{1\}) = \mathbb{Z}_p \times \{1\}$$

with set of vertices $V(\Gamma) \cong \mathbb{Z}_p$ and incidence maps $d_0(g,\{1\}) = g$, $d_1(g,\{1\}) = g+1$, $g \in \mathbb{Z}_p$. In this case, both $V(\Gamma)$ and $E(\Gamma)$ are closed, and so profinite spaces.

The subgroup of integers $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ and the topology of $\mathbb{Z}$ induced by the topology of $\mathbb{Z}_p$ is the discrete topology. Let

$$\Gamma(\mathbb{Z}, \{1\}) = \left\{ g \in V(\Gamma) \,\middle|\, g \in \mathbb{Z} \right\} \cup \left\{ (g, \{1\}) \,\middle|\, g \in \mathbb{Z} \right\}.$$

Then $\Gamma(\mathbb{Z}, \{1\})$ is an abstract discrete graph



which is dense in the profinite graph $\Gamma = \Gamma(\mathbb{Z}_p, \{1\})$. More generally, let $\beta$ be a fixed element of $\mathbb{Z}_p$ and define

$$\Gamma(\mathbb{Z}+\beta, \{1\}) = \left\{g \in V(\Gamma) \,\middle|\, g \in \mathbb{Z}+\beta\right\} \cup \left\{(g, \{1\}) \,\middle|\, g \in \mathbb{Z}+\beta\right\}.$$

Then $\Gamma(\mathbb{Z}+\beta, \{1\})$ is an abstract discrete graph

$$\begin{tikzcd} \cdots & \overset{\beta-2}{\bullet} \arrow[r, "(\beta-2,1)"'] & \overset{\beta-1}{\bullet} \arrow[r, "(\beta-1,1)"'] & \overset{\beta}{\bullet} \arrow[r, "(\beta,1)"'] & \overset{\beta+1}{\bullet} \arrow[r, "(\beta+1,1)"'] & \overset{\beta+2}{\bullet} & \cdots \end{tikzcd}$$

which is also dense in the profinite graph $\Gamma = \Gamma(\mathbb{Z}_p, \{1\})$. Note that $\Gamma = \Gamma(\mathbb{Z}_p, \{1\})$ is a disjoint union of uncountably many abstract discrete graphs of the form $\Gamma(\mathbb{Z}+\beta, \{1\})$:

$$\Gamma = \Gamma(\mathbb{Z}_p, \{1\}) = \bigcup_{\lambda \in \Lambda} \Gamma(\mathbb{Z}+\beta_\lambda, \{1\})$$

## 1.3 The fundamental group of a graph

This section is based on a paper by Pavel Zalesski (cf. [52]).

**Definition 1.3.1** (Galois covering of profinite graphs)**.** Let $G$ be a pro-$\mathcal{C}$ group that acts freely on a profinite graph $\Gamma$. The natural epimorphism of profinite graphs $\zeta : \Gamma \to \Delta = G\backslash\Gamma$ of $\Gamma$ onto the quotient graph by the action of $G$, $\Delta = G\backslash\Gamma$ is called a Galois $\mathcal{C}$-covering of the profinite graph $\Delta$. The associated pro-$\mathcal{C}$ group $G$ is called the group associated with the Galois covering $\zeta$ and we denote it by $G = G(\Gamma|\Delta)$. If $\Gamma$ is finite, one says that the Galois covering $\zeta$ is finite. The Galois covering is said to be connected if $\Gamma$ is connected.

**Definition 1.3.2** (Universal Galois $\mathcal{C}$-covering)**.** Let $\Gamma$ be a connected profinite graph. A universal Galois $\mathcal{C}$-covering $\tilde{\zeta} : \tilde{\Gamma} \to \Gamma$ is a connected Galois $\mathcal{C}$-covering that respects the following universal property: given any q-morphism $\beta : \Gamma \to \Delta$ to a connected profinite graph $\Delta$, any connected Galois $\mathcal{C}$-covering $\xi : \Sigma \to \Delta$, and any vertices $m \in \widetilde{\Gamma}$, $s \in \Sigma$ such that $\beta\zeta(m) = \xi(s)$, there exists a q-morphism of profinite graphs $\alpha : \tilde{\Gamma} \to \Sigma$, such that $\beta\zeta = \xi\alpha$ and $\alpha(m) = s$.

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r, dotted, "\alpha"] \arrow[d, "\tilde{\zeta}"'] & \Sigma \arrow[d, "\xi"] \\ \Gamma \arrow[r, "\beta"'] & \Delta \end{tikzcd} \tag{1.1}$$

We say that $\alpha$ lifts $\beta$, or that $\alpha$ is a lifting (q-morphism) of $\beta$.

Once $m \in \widetilde{\Gamma}$ and $s \in \Sigma$ with $\beta\zeta(m) = \xi(s)$ are given, the lifting q-morphism $\alpha$ is unique. Indeed, suppose the lifting is not unique. We have then $\alpha_1$ and $\alpha_2$ such that

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r, dotted, bend left, "\alpha_1"] \arrow[r, dotted, bend right, "\alpha_2"'] \arrow[d, "\tilde{\zeta}"'] & \Sigma \arrow[d, "\xi"] \\ \Gamma \arrow[r, "\beta"'] & \Delta \end{tikzcd} \tag{1.2}$$

and $\beta\zeta = \xi\alpha_1$,$\beta\zeta = \xi\alpha_2$. Hence $\xi\alpha_1 = \xi\alpha_2$ and we have a connected Galois covering $\xi$, a connected profinite graph $\widetilde{\Gamma}$ and $\alpha_1, \alpha_2 : \widetilde{\Gamma} \to \Sigma$ q-morphisms of profinite graphs with $\xi\alpha_1 = \xi\alpha_2$ and $\alpha_1(m) = \alpha_2(m)$ (commutativity of the diagram), for some $m \in \widetilde{\Gamma}$. By uniqueness of the diagram, since $\alpha_1$ and $\alpha_2$ agree on one point, by [31, Lemma 3.1.7] they must be equal. Hence $\alpha_1 = \alpha_2$ as desired.

Note also that if the map $\beta$ is surjective, so is $\alpha$ by [31, Corollary 3.1.6]. Furthermore, it follows from [31, Proposition 3.1.3] that it is sufficient to check the universal property above for finite Galois $\mathcal{C}$-coverings $\xi : \Sigma \to \Delta$.

**Definition 1.3.3** (cf. Theorem 3.7.1 of [31] or cf. in the original Theorem 2.8 of [34])**.** If $\tilde{\zeta}$ is the *universal* Galois $\mathcal{C}$-covering of a connected profinite graph, then we define $G = \pi_1^{\mathcal{C}}(\Gamma)$ as the pro-$\mathcal{C}$ fundamental group of $\Gamma$.

**Definition 1.3.4** ($\mathcal{C}$-simply connected profinite graph)**.** We say that a connected profinite graph $\Gamma$ is $\mathcal{C}$-simply connected if $\pi_1^{\mathcal{C}}(\Gamma) = 1$.

Let $\zeta : \widetilde{\Gamma} \to \Gamma$ be a Galois covering of a profinite graph $\Gamma$ and let $G = G(\widetilde{\Gamma}|\Gamma)$ be the associated profinite group. Then we may think of $G$ as a closed subgroup of $\mathrm{Aut}\,(\widetilde{\Gamma})$ as follows. Each element $g \in G$ determines a continuous automorphism

$$\begin{aligned} \nu_g : \widetilde{\Gamma} &\to \widetilde{\Gamma} \\ m &\mapsto gm \end{aligned}$$

Moreover, the map

$$\begin{aligned} \nu : G &\to \mathrm{Aut}\,(\widetilde{\Gamma}) \\ g &\mapsto \nu_g \end{aligned}$$

is a homomorphism, and it is injective because $G$ acts freely on $\Gamma$.

Now endow $\mathrm{Aut}\,(\widetilde{\Gamma})$ with the compact-open topology, i.e., the one generated by a sub-base of open sets of the form

$$B(K,U) = \{f \in \mathrm{Aut}\,(\widetilde{\Gamma}) \mid f(K) \subseteq U\}$$

where $K$ ranges over the compact subsets of $\widetilde{\Gamma}$ and $U$ ranges over the open subsets of $\widetilde{\Gamma}$.

We claim that the monomorphism $\nu : G \to \mathrm{Aut}\,(\widetilde{\Gamma})$ defined above is continuous. If $K, U \subset \widetilde{\Gamma}$, the first compact and the second open subset of $\widetilde{\Gamma}$, we show that $\nu^{-1}(B(K,U))$ is open in $G$. To see this, it is enough to find an open neighborhood $W$ of $g \in G$ contained in $\nu^{-1}(B(K,U))$. Since $gK \subseteq U$, for each $x \in K$, there exist open neighborhoods $W_x$ of $g$ in $G$ and $V_x$ of $x$ in $\widetilde{\Gamma}$ such that $W_xV_x \subseteq U$. Since $K$ is compact, there exist finitely many points $x_1, x_2, \cdots, x_n$ in $K$ so that $\bigcup_{i=1}^{n} V_{x_i} = K$. Then

$$W = \bigcap_{i=1}^{n} W_{x_i}$$

is the desired neighborhood of $g$. This proves the claim.

Finally, since $G$ is compact, the above implies that $\nu$ maps $G$ homeomorphically onto $\mathrm{Im}\,(\nu)$, if the compact-open topology on $\mathrm{Aut}\,(\widetilde{\Gamma})$ is Hausdorff. In fact, take distinct elements $f_1, f_2 \in \mathrm{Aut}\,(\widetilde{\Gamma})$. Then there exists an $m \in \mathrm{Aut}\,(\widetilde{\Gamma})$ with $f_1(m) \neq f_2(m)$. Let $U_i$ be a neighborhood of $f_i(m)$, $(i = 1,2)$ such that $U_1 \cap U_2 = \varnothing$. Then $f_i \in B(\{m\}, U_i)$, $(i = 1,2)$ and $B(\{m\}, U_1) \cap B(\{m\}, U_2) = \varnothing$, proving the assertion.

These ideas can seem quite abstract since it is not clear how to calculate the fundamental profinite group of a given profinite graph. The next results provide a way to obtain it for finite graphs.

**Construction 1.3.5** ([31], Construction 3.5.1)**.** Let $\Gamma$ be a finite connected graph and let $T$ be a subtree of $\Gamma$ with $V(T) = V(\Gamma)$ ($T$ need not equal $\Gamma$). Denote by $X = \Gamma/T$ the corresponding quotient space with canonical map

$$\omega : \Gamma \to X = \Gamma/T.$$

Consider the element $* = \omega(T)$ as a distinguished point of $X$. Let $F = F_{\mathcal{C}}(X, *)$ be the free pro-$\mathcal{C}$ group on the pointed profinite space $(X, *)$ and consider $(X, *)$ a subspace of

$F_{\mathcal{C}}(X,*)$ in the natural way. Define a profinite graph $\Upsilon_{\mathcal{C}}(\Gamma,T)$ as follows:

- $\Upsilon_{\mathcal{C}}(\Gamma,T) = F_{\mathcal{C}}(X,*) \times \Gamma$;
- $V(\Upsilon_{\mathcal{C}}(\Gamma,T)) = F_{\mathcal{C}}(X,*) \times V(\Gamma)$;
- $d_0(r,m) = (r, d_0(m))$;
- $d_1(r,m) = (r\omega(m), d_1(m))$,

$(r \in F,\ m \in \Gamma)$. Next define an action of $F$ on the graph $\Upsilon_{\mathcal{C}}(\Gamma,T)$ by

$$r'(r,m) = (r'r, m)$$

$(r, r' \in F, m \in \Gamma)$. Clearly, this is a free action (because $F$ is a free group) and $F\backslash\Upsilon_{\mathcal{C}}(\Gamma,T) = \Gamma$. Therefore the natural epimorphism

$$\upsilon : \Upsilon_{\mathcal{C}}(\Gamma,T) \to \Gamma$$

that sends $(r,m)$ to $m$ $(r \in F, m \in \Gamma)$ is a Galois $\mathcal{C}$-covering.

This Galois covering $\upsilon : \Upsilon_{\mathcal{C}}(\Gamma,T) \to \Gamma$ is connected (cf. [31, Lemma 3.5.2]). In fact, as constructed, $\upsilon$ is Universal.

**Theorem 1.3.6** ([31], Theorem 3.5.3)**.** *Let $\Gamma$ be a finite connected graph and let $T$ be a maximal subtree of $\Gamma$ ($T$ is a spanning $\mathcal{C}$-simply connected profinite subgraph of $\Gamma$). Then one has the following properties:*

*(a) The Galois $\mathcal{C}$-covering $\upsilon : \Upsilon_{\mathcal{C}}(\Gamma,T) \to \Gamma$ of Construction 1.3.5 is universal.*

*(b) Let $(X,*) = (\Gamma/T,*)$; then*

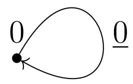
$$\pi_1^{\mathcal{C}}(\Gamma) = F_{\mathcal{C}}(X,*)$$

*is a free pro-$\mathcal{C}$ group of finite rank $|\Gamma| - |T|$;*

*(c) The universal Galois $\mathcal{C}$-covering $\upsilon : \Upsilon_{\mathcal{C}}(\Gamma,T) \to \Gamma$ is independent of the maximal subtree $T$ chosen.*

Next, we provide some examples of fundamental groups of profinite graphs using the Construction 1.3.5:

**Example 1.3.7** ([31], Exercise 3.3.4(a))**.** Let $\mathcal{C}$ be a pseudovariety of finite groups and $L(0)$ be the loop




Take a maximal subtree $T = \{0\}$ of $L(0)$ and $\omega : L(0) \to L(0)/T = L(0)$. The universal Galois covering of $L(0)$, by Construction 1.3.5 and Theorem 1.3.6, is

$$\begin{aligned} \Upsilon_{\mathcal{C}}(L(0), T) &= F_{\mathcal{C}}(L(0)/T, \omega(T)) \times L(0) \\ &= F_{\mathcal{C}}(L(0), \{0\}) \times L(0), \end{aligned}$$

Also by Theorem 1.3.6, $rank(F) = |L(0)| - |\{0\}| = 2 - 1 = 1$, thus $F = Z_{\hat{\mathcal{C}}}$ (where $F = F_{\mathcal{C}}(L(0), \{0\}))$ and

$$\begin{aligned} F_{\mathcal{C}}(L(0), \{0\}) \times L(0) &= Z_{\hat{\mathcal{C}}} \times \{0\} \\ &= \Gamma(Z_{\hat{\mathcal{C}}}, \{1\}). \end{aligned}$$

Thereafter $\Upsilon_{\mathcal{C}}(L(0), T) = \Gamma(Z_{\hat{\mathcal{C}}}, 1)$ is defined by

$$\begin{aligned} V(\Upsilon_{\mathcal{C}}(L(0), \{0\})) &= V(\Gamma(Z_{\hat{\mathcal{C}}}, 1)) \\ &= \{(g, 1) \mid g \in Z_{\hat{\mathcal{C}}}\} = Z_{\hat{\mathcal{C}}}; \end{aligned}$$

- $d_0(r, m) = (r, d_0(m))$;
- $d_1(r, m) = (r\omega(m), d_1(m))$;

$(r \in F,\ m \in L(0))$, with an action of $F = Z_{\hat{\mathcal{C}}}$ on the graph

$$\Upsilon_{\mathcal{C}}(L(0), \{0\}) = \Gamma(Z_{\hat{\mathcal{C}}}, \{1\})$$

by $r'(r, m) = (r'r, m)$, $(r, r' \in F, m \in L(0))$. Therefore,

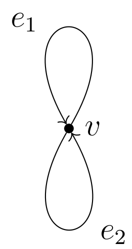
$$Z_{\hat{\mathcal{C}}} \backslash \Gamma(Z_{\hat{\mathcal{C}}}, \{1\}) = L(0)$$

and

$$\pi_1^{\mathcal{C}}(L(0)) \cong Z_{\hat{\mathcal{C}}}.$$

We construct a Galois profinite covering (i.e. the associated group $G = G(\Gamma|\Delta)$ is a profinite group) of the graph $\Gamma$ of Example **??**.

**Example 1.3.8** (A covering space of a profinite graph)**.** Let $\Gamma$ be the following graph:




Since $\Gamma$ is finite, it is automatically profinite. We can construct a Galois profinite covering $p : \Upsilon_{\mathcal{C}}(\Gamma, T) \to \Gamma$ for $\Gamma$ as follows. First take a maximal subtree $T = \{v\}$ of $\Gamma$ and $\omega : \Gamma \to X = \Gamma/T$. The Universal Galois covering of $\Gamma$, by Construction 1.3.5 and Theorem 1.3.6, is

$$\begin{aligned} \Upsilon_{\mathcal{C}}(\Gamma, T) &= F_{\mathcal{C}}(X, \omega(T)) \times \Gamma \\ &= F_{\mathcal{C}}(X, \{v\}) \times \Gamma, \end{aligned}$$

Also by Theorem 1.3.6, $rank(F) = |\Gamma| - |\{v\}| = 3 - 1 = 2$, thus $F = (F_2)_{\hat{\mathcal{C}}}$ (where $F = F_{\mathcal{C}}(X, \{v\})$) and

$$\begin{aligned} F_{\mathcal{C}}(X, \{v\}) \times \Gamma &= (F_2)_{\hat{\mathcal{C}}} \times \Gamma \\ &= \Gamma((F_2)_{\hat{\mathcal{C}}}, \{e_1, e_2\}). \end{aligned}$$

Thereafter $\Upsilon_{\mathcal{C}}(\Gamma, T) = \Gamma((F_2)_{\hat{\mathcal{C}}}, \{e_1, e_2\})$ is defined by

$$V(\Upsilon_{\mathcal{C}}(\Gamma, \{v\})) = V(\Gamma((F_2)_{\hat{\mathcal{C}}}, \{e_1, e_2\})).$$

- $d_0(r, m) = (r, d_0(m))$;
- $d_1(r, m) = (r\omega(m), d_1(m))$;

($r \in F$, $m \in \Gamma$), with an action of $F$ on the graph

$$\Upsilon_{\mathcal{C}}(\Gamma, \{v\}) = \Gamma((F_2)_{\hat{\mathcal{C}}}, \{e_1, e_2\})$$

by $r'(r, m) = (r'r, m)$, $(r, r' \in F, m \in \Gamma)$. Therefore,

$$(F_2)_{\hat{\mathcal{C}}} \backslash \Gamma((F_2)_{\hat{\mathcal{C}}}, \{e_1, e_2\}) = \Gamma$$

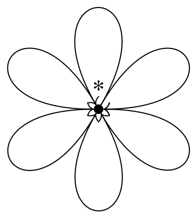
and

$$\pi_1^{\mathcal{C}}(\Gamma) \cong (F_2)_{\hat{\mathcal{C}}}.$$

In fact, these examples constitute part of a broader idea that we present in the next Proposition. First we need an example.

**Example 1.3.9** ([31], Example 3.1.1(b))**.** Let $(X, *)$ be a pointed profinite space (i.e., $X$ is a profinite space with a distinguished point $*$). Define a profinite graph $B = B(X, *)$ by $B = X$, $V(B) = \{*\}$ and $d_i(x) = \{*\}$, $(x \in X)$ for $i = 0, 1$. This graph $B(X, *)$ is named the bouquet of loops associated to $(X, *)$.

For example, if $X$ has 7 points, $B(X, *)$ is the graph




The next Proposition highlights that, as in the abstract case, the most important thing is the number of edges outside the maximal subtree. They determine the rank of the free pro-$\mathcal{C}$ group $\pi_1^{\mathcal{C}}(\Gamma)$.

**Proposition 1.3.10** (Proposition 3.8.1,[31])**.** *The Cayley graph $\Gamma(F_{\mathcal{C}}(X, *), X)$ of a free pro-$\mathcal{C}$ group on a pointed profinite space $(X, *)$ with respect to $X$ is $\mathcal{C}$-simply connected. In fact, $\Gamma(F_{\mathcal{C}}(X, *), X)$ is the universal Galois $\mathcal{C}$-covering space of the bouquet of loops $B = B(X, *)$ and $\pi_1^{\mathcal{C}}(B) = F_{\mathcal{C}}(X, \ast)$.*

*Proof.* Consider the bouquet of loops $B = B(X, *)$ associated with $(X, \ast)$. In this case, the subgraph $T = \{*\}$ is a spanning $\mathcal{C}$-simply connected profinite subgraph of $B$ and

$\Upsilon_{\mathcal{C}}(B,T)$ coincides with the Cayley graph $\Gamma(F_{\mathcal{C}}(X,*),X)$. By construction, $\Gamma(F_{\mathcal{C}}(X,\ast),X)$ is the universal Galois $\mathcal{C}$-covering graph of $B$, with $\pi_1^{\mathcal{C}}(B)=\Gamma(F_{\mathcal{C}}(X,*))$. It implies that $\Gamma(F_{\mathcal{C}}(X,*))$ is $\mathcal{C}$-simply connected. $\square$

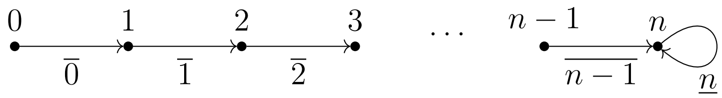
Hence, if a free pro-$\mathcal{C}$ group $G$ acts on a pro-$\mathcal{C}$ tree the quotient graph can be a bouquet of loops. The next result describes the action when we have a pro-$\mathcal{C}$ tree.

**Theorem 1.3.11** (Theorem 4.1.1 of [31] and Corollary 3.6 of [32])**.** *Let a pro-$\mathcal{C}$ group $G$ act on a pro-$\mathcal{C}$ tree $T$. We define $\widetilde{G}$ as the subgroup of $G$ generated by the vertex stabilizers, i.e.,*

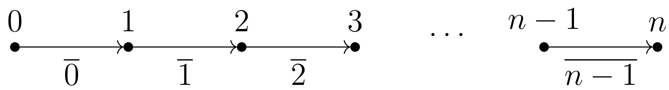
$$\widetilde{G}=\overline{\langle G_m \mid m\in T\rangle},$$

*where $G_m$ is the $G$-stabilizer of $m$. Then $\widetilde{G}\backslash T$ is a tree and $G/\widetilde{G}$ is a projective pro-$\mathcal{C}$ group. In particular, for the pro-$p$ case, $G/\widetilde{G}$ is free pro-$p$.*

**Example 1.3.12** ([31], Exercise 3.3.4(b))**.** Let $\mathcal{C}$ be a pseudovariety of finite groups and $n$ be a natural number. Consider the finite graph $L(n)$




Take a maximal subtree $T$ of $L(n)$,




and $\omega: L(n)\to L(n)/T=L(0)$, $\omega(T)=\{0\}$. The universal Galois covering of $L(0)$, by Construction 1.3.5 and Theorem 1.3.6, is

$$\begin{aligned}\Upsilon_{\mathcal{C}}(L(n),L(0)) &= \Upsilon_{\mathcal{C}}(L(0),\{0\})\\ &= F_{\mathcal{C}}(L(0),\{0\})\times L(n),\end{aligned}$$

By Example 1.3.7, $F_{\mathcal{C}}(L(0),\{0\})\cong Z_{\hat{\mathcal{C}}}$. Therefore,

$$\Upsilon_{\mathcal{C}}(L(n),L(0))=\Gamma(Z_{\hat{\mathcal{C}}},\{1\}),$$

$$Z_{\hat{\mathcal{C}}}\backslash\Gamma(Z_{\hat{\mathcal{C}}},\{1\})=L(0)$$

and

$$\pi_1^{\mathcal{C}}(L(n)) \cong \pi_1^{\mathcal{C}}(L(0)) \cong Z_{\hat{\mathcal{C}}}.$$

Suppose now that $\Gamma$ is infinite. How to determine its fundamental group, since Theorem 1.3.6 and Construction 1.3.5 do not hold in this case? The finite case restriction is needed because we need a maximal profinite subtree. Although every abstract graph has a maximal subtree, not every profinite graph has a profinite subtree, as we show in Example 2.1.7. In order to calculate the fundamental profinite group of $\Gamma$ in Example 2.2.5, in Section 2.2 we construct $\pi_1(\Gamma, v)$ as an inverse limit $\pi_1(\Gamma, v) = \varprojlim \widehat{\pi_1(\Gamma_i, v_i)}$. This is an original result of this thesis.

## 1.4 Graphs of groups

**Definition 1.4.1** (Pseudo-sheaf of (profinite) groups, Section 5.1 of [31])**.** Let $T$ be a (profinite space) set. A pseudo-sheaf of (profinite) groups over $T$ is a triple $(\mathcal{G}, \pi, T)$, where $\mathcal{G}$ is a (profinite space) set and $\pi : \mathcal{G} \to T$ is a (continuous) surjection satisfying the following conditions:

(a) For every $t \in T$, the fiber $\mathcal{G}(t) = \pi^{-1}(t)$ over $t$ is a (profinite) group (whose topology is induced by the topology of $\mathcal{G}$ as the subspace topology);

and only for the profinite case,

(b) If we define

$$\mathcal{G}^2 = \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \pi(g) = \pi(h)\},$$

then the map $\mu : \mathcal{G}^2 \to \mathcal{G}$ given by $\mu_{\mathcal{G}}(g, h) = gh^{-1}$ is continuous.

**Definition 1.4.2.** A morphism $\underline{\alpha} = (\alpha, \alpha') : (\mathcal{G}, \pi, T) \to (\mathcal{G}', \pi', T')$ of sheaves of (profinite) groups consists of a pair of (continuous) maps $\alpha : \mathcal{G} \to \mathcal{G}'$ and $\alpha' : T \to T'$ such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \mathcal{G}' \arrow[d, "\pi'"] \\ T \arrow[r, "\alpha'"'] & T' \end{tikzcd}$$

commutes and the restriction of $\alpha$ to $\mathcal{G}(t)$ is a homomorphism from $\mathcal{G}(t)$ into $\mathcal{G}'(\alpha'(t))$, for each $t \in T$.

If $\alpha$ and $\alpha'$ are injective, the morphism $\underline{\alpha}$ is said to be a monomorphism and the image of a monomorphism $\mathcal{G} \to \mathcal{G}'$ is called a subsheaf of the pseudo-sheaf $\mathcal{G}'$.

**Example 1.4.3.** If $(\mathcal{G}, \pi, T)$ is a pseudo-sheaf and $T'$ is a (closed) subspace of $T$, then the triple

$$(\pi^{-1}(T'), \pi|_{\pi^{-1}(T')}, T')$$

is a subsheaf of $(\mathcal{G}, \pi, T)$.

**Example 1.4.4** (Example 5.1.1 of [31])**.** Let $T = \{1, 2, \cdots, n\}$ be a finite discrete space with $n$ points and let $G_1, G_2, \cdots, G_n$ be pro-$\mathcal{C}$ groups. Define the space $\mathcal{G} = \dot{\bigcup}_{i=1}^{n} G_i$ to have the disjoint topology. Let $\pi : \mathcal{G} \to T$ be the map that sends $G_i$ to $i$, $(i = 1, \cdots, n)$. Then $(\mathcal{G}, \pi, T)$ is in a natural way a pseudo-sheaf over the space $T$ with $\mathcal{G}(i) = G_i$ $(i = 1, \cdots, n)$.

**Definition 1.4.5** (Fiber homomorphism)**.** Let $(\mathcal{G}, \pi, T)$ be a pseudo-sheaf of pro-$\mathcal{C}$ groups and let $H$ be a pro-$\mathcal{C}$ group. We can think of $H$ as the fiber of a pseudo-sheaf over a singleton space. Hence we have the fiber homomorphism

$$\alpha : \mathcal{G} \to H$$

where $\alpha$ is a continuous pseudo-sheaf morphism, i.e., the restriction to each fiber $\mathcal{G}(t)$ is a homomorphism.

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & H \arrow[d, "\pi'"] \\ T \arrow[r, "\alpha'"'] & \{1\} \end{tikzcd}$$

**Definition 1.4.6** ((Profinite) graph of (profinite) groups)**.** Let $\Gamma$ be a connected (profinite) graph with incidence maps $d_0, d_1 : \Gamma \to V(\Gamma)$. A (profinite) graph of groups over $\Gamma$ is a pseudo-sheaf $(\mathcal{G}, \pi, \Gamma)$ of (profinite) groups over $\Gamma$ together with two morphisms of sheaves $(\partial_i, d_i) : (\mathcal{G}, \pi, \Gamma) \to (\mathcal{G}_V, \pi, V(\Gamma))$, where $(\mathcal{G}_V, \pi, V(\Gamma))$ is a restriction pseudo-sheaf of $(\mathcal{G}, \pi, \Gamma)$ and the restriction of $\partial_i$ to $\mathcal{G}_v$ is the identity map $id_{\mathcal{G}_V}$, $i = 0, 1$; in addition, we assume that the restriction of $\partial_i$ to each fiber $\mathcal{G}(m)$ is an injection.

**Definition 1.4.7.** A morphism of graphs of groups $\underline{\nu} = (\nu, \nu') : (\mathcal{G}, \Gamma) \longrightarrow (\mathcal{H}, \Delta)$ is a morphism of sheaves such that $\nu\partial_i = \partial_i\nu$.

Let $I$ be a partially ordered set, $\{(\mathcal{G}_i, \pi_i, \Gamma_i), \nu_{ij}\}$ an inverse system of finite graphs of $\mathcal{C}$-groups. Then $(\mathcal{G}, \pi, \Gamma) = \varprojlim_{i\in I}(\mathcal{G}_i, \pi_i, \Gamma_i)$ is a profinite graph of pro-$\mathcal{C}$ groups.

We finish the section with the definition of the graph of pro-$\mathcal{C}$ groups $(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$ associated with a profinite graph of pro-$\mathcal{C}$-groups $(\mathcal{G}, \Gamma)$ that appears in [51, Subsection 2.5] and [31, page 185].

**Definition 1.4.8.** Let $(\mathcal{G}, \Gamma)$ be a profinite graph of pro-$\mathcal{C}$ groups. Let $\widetilde{\Gamma}$ be the universal $\mathcal{C}$-covering of the profinite graph $\Gamma$ and $\zeta : \widetilde{\Gamma} \to \Gamma$ be the covering map. Consider the pull-back

$$\begin{tikzcd} \widetilde{\mathcal{G}} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\zeta}"'] & \widetilde{\Gamma} \arrow[d, "\zeta"] \\ \mathcal{G} \arrow[r, "\pi"'] & \Gamma \end{tikzcd} \tag{1.3}$$

of the maps $\pi : \mathcal{G} \to \Gamma$ and $\zeta : \widetilde{\Gamma} \to \Gamma$. This means that $\widetilde{\mathcal{G}}$ is defined by

$$\widetilde{\mathcal{G}} = \{(x, \tilde{m}) \in \mathcal{G} \times \widetilde{\Gamma} \mid \pi(x) = \zeta(\tilde{m}), x \in \mathcal{G}, \tilde{m} \in \widetilde{\Gamma}\} \subseteq \mathcal{G} \times \widetilde{\Gamma},$$

where $\tilde{\pi} : \widetilde{\mathcal{G}} \to \widetilde{\Gamma}$ and $\tilde{\zeta} : \widetilde{\mathcal{G}} \to \mathcal{G}$ be the restrictions to $\widetilde{\mathcal{G}}$ of the canonical projections from $\mathcal{G} \times \widetilde{\Gamma}$ to $\widetilde{\Gamma}$ and $\mathcal{G}$ respectively. For a fixed $\tilde{m} \in \widetilde{\Gamma}$, define

$$\widetilde{\mathcal{G}}(\tilde{m}) = \tilde{\pi}^{-1}(\tilde{m}) = \mathcal{G}(\zeta(\tilde{m})) \times \{\tilde{m}\},$$

which is a group with respect to the operation $(x, \tilde{m})(x', \tilde{m}) = (xx', \tilde{m})$, $x, x' \in \mathcal{G}(\zeta(\tilde{m}))$ and clearly $\widetilde{\mathcal{G}}(\tilde{m}) \cong \mathcal{G}(\zeta(\tilde{m}))$.

Then $(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$ is a profinite graph of the above defined groups $\widetilde{\mathcal{G}}(\tilde{m})$ over the universal covering graph $\widetilde{\Gamma}$ of $\Gamma$ with boundary maps $\partial_i(g) = (\partial_i(\tilde{\zeta}(g)), d_i(\tilde{m}))$ for $g \in \widetilde{\mathcal{G}}(\tilde{m})$.

## 1.5 Free constructions

Two essential particular cases of the fundamental group of a finite graph of pro-$\mathcal{C}$ groups are amalgamated free pro-$\mathcal{C}$ products and pro-$\mathcal{C}$ HNN-extensions. We start with the abstract notions since the pro-$\mathcal{C}$ version is a completion of the abstract with the appropriate topology. We follow Robinson's book (cf. [35]) for the abstract setting and Ribes and Zalesski's (cf. [30]) for the profinite one. We start with free groups and free products.

**Definition 1.5.1.** Let $X$ be a subset of a group $F$. Then $F$ is a free group with basis $X$, provided the following holds: if $\varphi$ is any function from the set $X$ into a group $H$, then there exists a unique extension of $\varphi$ to a homomorphism $\varphi*$ from $F$ into $H$.

$$\begin{tikzcd} F \arrow[dr, dotted, "\varphi*"] & \\ X \arrow[u, hook, "\imath"] \arrow[r, "\varphi"'] & H \end{tikzcd}$$

It is possible to show that $X$ generates $G$, so one calls generators the elements of $X$. Define $X^{\pm 1} = X \cup X^{-1}$, where $X^{-1}$ is just an isomorphic copy of $X$. We call letters the elements of $X^{\pm 1}$. A word is a finite sequence of letters, $w = (a_1, a_2, \cdots, a_n)$, $n \geqslant 0$, all $a_i \in X^{\pm 1}$. We say that a word is reduced if it does not contain 1 nor $aa^{-1}$, $a \in X^{\pm 1}$. There is an intuitive reduction process, so we define an equivalence relation between two words if they have the same reduced form.

The elements of a free group are equivalence classes of words. Note that each equivalence class contains just one reduced word and the reduced form is unique. If $X$ is a finite set, the number of its elements is called the rank of $F$, denoted by $r_F$.

**Definition 1.5.2** (Free product)**.** (cf. Definition 2.1.1 of [35]) Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of groups. The free product of $G_\lambda$'s, denoted by $*_{\lambda\in\Lambda} G_\lambda$ is a group $G$ and a family of homomorphisms $f_\lambda : G_\lambda \to G$ such that for every group $H$ and for every family of homomorphisms $g_\lambda : G_\lambda \to H$ there exists a unique homomorphism $\varphi : G \to H$ such that $g_\lambda = \varphi \circ f_\lambda$, for all $\lambda \in \Lambda$, i.e., the following diagram is commutative:

$$\begin{tikzcd} G = *_{\lambda\in\Lambda} G_\lambda \arrow[dr, dotted, "\varphi"] & \\ G_\lambda \arrow[u, hook, "f_\lambda"] \arrow[r, "g_\lambda"'] & H \end{tikzcd}$$

The elements of a free product are equivalence classes of words to which each letter belongs to some $G_\lambda$, and where two letters are equivalent if they are consecutive in a word and belong to the same $G_\lambda$. As for the free groups, note that each equivalence class contains just one reduced word and the reduced form is unique. This reduced form is called 'the normal form'.

Given a presentation $G_\lambda = \langle X_\lambda \mid R_\lambda \rangle$, it is possible to show that

$$G = *_{\lambda\in\Lambda} G_\lambda = \langle \dot{\bigcup_{\lambda\in\Lambda}} X_\lambda \mid \dot{\bigcup_{\lambda\in\Lambda}} R_\lambda \rangle.$$

### 1.5.1 Free products with amalgamation

**Definition 1.5.3** (Free product with amalgamation). Let $H, G_1, G_2$ be groups and let $f_1 : H \hookrightarrow G_1$, $f_2 : H \hookrightarrow G_2$ be monomorphisms. The free product of $G_1$ and $G_2$ with $H$ amalgamated is a group $G$ with homomorphisms $\varphi_1 : G_1 \to G$ and $\varphi_2 : G_2 \to G$ such that the following diagram is commutative

$$\begin{tikzcd} H \arrow[r, hook, "f_1"] \arrow[d, hook, "f_2"'] & G_1 \arrow[d, "\varphi_1"] \\ G_2 \arrow[r, "\varphi_2"'] & G \end{tikzcd}$$

and for every group $K$ and homomorphisms $\psi_1 : G_1 \to K$,$\psi_2 : G_2 \to K$, such that $\psi_1 \circ f_1 = \psi_2 \circ f_2$, there exists a unique homomorphism $\psi : G \to K$ such that $\psi \circ \varphi_1 = \psi_1$ and $\psi \circ \varphi_2 = \psi_2$. In terms of a diagram, we have the following pushout

$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "\varphi_1"'] \arrow[ddr, bend left, "\psi_1"] & \\ G_2 \arrow[r, "\varphi_2"] \arrow[drr, bend right, "\psi_2"'] & G \arrow[dr, dotted, "\psi"'] & \\ & & K \end{tikzcd}$$

We denote the free product with amalgamation by $G = G_1 *_H G_2$. Here we use Robinson's notation for homomorphisms of groups: $f_1(H) = H^{f_1}$. The groups $G_1, G_2, H$ can be seen as subgroups of $G$ (this is not always true for the profinite case, as we will see). If $R = \{h^{f_1}(h^{f_2})^{-1} \mid h \in H\}$, we have that

$$G = G_1 *_H G_2 = \frac{G_1 * G_2}{R}.$$

Also, if $G_\lambda = \langle X_\lambda \mid R_\lambda \rangle$, $\lambda \in \{1, 2\}$, then

$$G = G_1 *_H G_2 = \langle X_1 \cup X_2 \mid R_1, R_2, h^{f_1}(h^{f_2})^{-1}, h \in H \rangle.$$

**Example 1.5.4.** Let $A = \langle a \rangle$ and $B = \langle a \rangle$ be cyclic groups of orders 4 and 6 respectively. The free product $A * B$ has a presentation $\langle a, b \mid a^4, b^6 \rangle$. Since $a^2$ and $b^3$ both have order 2, the subgroups $\langle a^2 \rangle$ and $\langle b^3 \rangle$ are isomorphic. We may therefore form the free product with amalgamation $G$ determined by the isomorphism $\varphi : \langle a^2 \rangle \to \langle b^3 \rangle$. This emerges by identifying $a^2$ and $b^3$. Therefore $G$ has presentation

$$G = C_4 *_{C_2} C_6 = \langle a, b \mid a^4, a^2 = b^3 \rangle.$$

**Definition 1.5.5** ([30], Section 9.2)**.** Let $G_1$ and $G_2$ be pro-$\mathcal{C}$ groups and let $f_i : H \to G_i$ $(i = 1, 2)$ be continuous monomorphisms of pro-$p$ groups. An amalgamated free pro-$\mathcal{C}$ product of $G_1$ and $G_2$ with amalgamated subgroup $H$ is defined to be a pushout of $f_i$ $(i = 1, 2)$

$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "\varphi_1"] \\ G_2 \arrow[r, "\varphi_2"'] & G \end{tikzcd}$$

in the category of pro-$\mathcal{C}$ groups, i.e., a pro-$\mathcal{C}$ group $G$ together with continuous homomorphisms $\varphi_i : G_i \to G$ $(i = 1, 2)$ satisfying the following universal property: for any pair of continuous homomorphisms $\psi_i : G_i \to K$ $(i = 1, 2)$ into a pro-$\mathcal{C}$ group $K$ with $\psi_1 f_1 = \psi_2 f_2$, there exists a unique continuous homomorphism $\psi : G \to K$ such that the following diagram is commutative:

$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "\varphi_1"'] \arrow[ddr, bend left, "\psi_1"] & \\ G_2 \arrow[r, "\varphi_2"] \arrow[drr, bend right, "\psi_2"'] & G \arrow[dr, dotted, "\psi"'] & \\ & & K \end{tikzcd}$$

This amalgamated free pro-$\mathcal{C}$ product, also referred to as free pro-$\mathcal{C}$ product with amalgamation, is denoted by $G = G_1 \amalg_H G_2$.

Following the abstract notion, we can consider $H$ as a common subgroup of $G_1$ and $G_2$ and think of $f_1$ and $f_2$ as inclusions. However, unlike the abstract case where the canonical homomorphisms

$$\varphi_i^{abs} : G_i \to G_1 \star_H G_2$$

$(i = 1, 2)$ are always monomorphisms (cf. Theorem I.1 in [37]), the corresponding maps in the category of pro-$\mathcal{C}$ groups

$$\varphi_i : G_i \to G_1 \amalg_H G_2$$

$(i = 1, 2)$ are not always injective, as the following example shows

**Example 1.5.6** (Example 9.2.9 of [30], non-proper free amalgamated pro-$p$ product)**.** Let $H$ be an abelian finitely generated pro-$p$ group of order $p^n$, where $1 \leqslant n \leqslant \infty$. Put $K = H \times H$ and let $T$ be a procyclic group of order $p^n$. We shall use additive notation for $T$ and multiplicative notation for $H$. Define two actions of $T$ on $K$ as follows

$$t(g, h) = (gh^t, h) \text{ and } t(g, h) = (g, g^t h) \ (t \in T, \text{ and } g, h \in H).$$

We refer to these actions as the first and the second action, respectively. These two actions are continuous. Define $G_1 = K \rtimes_\varphi T$ and $G_2 = K \rtimes_\psi T$ to be semidirect products using the first and the second actions respectively. Consider the amalgamated free pro-$p$ product $G = G_1 \coprod_K G_2$ of $G_1$ and $G_2$ amalgamating $K$. We show that $G$ is not proper.

Suppose by contraction that it is proper. Let $H_1$ be a normal subgroup of index $p$ in $H$. We observe that $K_1 = H_1 \times H_1$ is normal in $G_1$ and $G_2$. Hence $K_1$ is normal in $G$ and $G/K_1$ can be written as the following pro-$p$ product $G/K_1 = G_1/k_1 \coprod_{K/K_1} G_2/K_1$. We claim that $K/K_1 = H/H_1 \times H/H_1$ does not contain nontrivial proper subgroups which are normal in both $G_1/K_1$ and $G_2/K_1$. Indeed, assume that $\Delta$ is a nontrivial subgroup of $K/K_1$ which satisfies the previous assumption. Let $1 \neq (g, h) \in \Delta$, where $g, h \in H/H_1$. Then either $g$ or $h$ is nontrivial, say $g \neq 1$. Hence, $h = g^t$ for some $1 \leqslant t \leqslant p$. So, using the action of $T$ on $H/H_1 \times H/H_1$ determined by $\psi$, one has $(-t)(g, h) = (g, g^{-t}h) = (g, 1)$. Now using the action of $T$ on $H/H_1 \times H/H_1$ determined by $\psi$ again, one has $1 \cdot (g, 1) = (g, g)$. Thus we get that $(g, 1)$ and $(1, g) = (g^{-1}, 1)(g, g)$ belong to $\Delta$. Thus $\Delta = K/K_1$. This proves the claim.

It follows that $K/K_1$ is a finite minimal normal subgroup of $G/K_1$. However, this is impossible since $K/K_1$ is noncyclic and contains a central element of $G/K_1$ according to [30, Lemma 9.2.8]. This contradiction proves that $G$ is not proper.

This motivates the next definition:

**Definition 1.5.7.** An amalgamated free pro-$\mathcal{C}$ product $G = G_1 \amalg_H G_2$ will be called proper if the canonical homomorphisms $\varphi_i$ $(i = 1, 2)$ are monomorphisms. In that case we shall identify $G_1$, $G_2$ and $H$ with their images in $G$, when no possible confusion arises.

Throughout this thesis, all free pro-$\mathcal{C}$ products with amalgamation will be proper.

### 1.5.2 HNN-extensions

As with the previous free constructions, HNN-extensions are defined through a universal property, as follows:

**Definition 1.5.8** (Universal property of HNN-extensions)**.** Let $G$ be a group with isomorphic subgroups $A, B$ with $\varphi : A \to B$ being the isomorphism. Then an HNN- extension of $G$ with respect to $A$ and $B$ is a group $G^*$, a homomorphism $f : G \to G^*$ and an element $t \in G^*$ satisfying $f(a)^t = f(a^\varphi)$, for all $a \in A$. For every group $K$ with an element $k \in K$ and a homomorphism $h : G \to K$ satisfying $g(a)^k = g(a^\varphi)$, $\forall a \in A$, then there exists a unique homomorphism $\psi : G^* \to K$, with $\psi(G^*) = K$ and such that $\psi \circ f = h$.

By construction, we give a presentation of $G^*$ that will be simpler to use than the definition. We also provide a topological example to help intuition. Then we go to applications in finite group theory.

**Definition 1.5.9** (Presentation of an HNN-extension)**.** Let $G$ be a group and let $A$ and $B$ be subgroups of $G$ with $\varphi : A \to B$ an isomorphism. The HNN-extension of $G$ relative to $A, B$ and $\varphi$ is the group

$$G^* = \langle G, t \mid R_G,\, a^t = a^\varphi \rangle.$$

The group $G$ is called the base of $G^*$, $t$ is called the stable letter and $A, B$ are the associated groups.

This construction was introduced by Graham Higman, Bernhard Neumann, and Hanna Neumann in 1949. The next example shows how it appears in algebraic topology.

**Example 1.5.10.** Let $X$ be a topological arcwise connected space and suppose $U$ and $V$ are both arcwise connected subspaces of $X$. Let $h : U \to V$ be a homeomorphism.

Choose a base point $u \in U$ for the fundamental groups of $U$ and $X$ and $h(u) = v \in V$ as a basepoint for $\pi_1(V)$. Let $I$ be the unit interval and let $C = U \times I$. Identify $U \times \{0\}$ with $U$ and $U \times \{1\}$ with $V$ by the homeomorphism $h$. Let $Z$ be the resulting space (what we have done is to attach a handle to $X$).

The Seifert-van Kampen Theorem (see [20, Theorem 1.20]) can be used to show that

$$\pi_1(X)^* = \pi_1(Z) = \langle \pi_1(X), t \mid \pi_1(U)^t = \pi_1(V) \rangle$$

We provide now examples in group theory

**Example 1.5.11** (Dihedral group)**.** Let $D_{2n} = \langle r, s \mid r^n, s^2, r^s = r^{-1} \rangle$ be the dihedral group with $2n$ elements. Since $\langle r^{\frac{n}{2}} \rangle$ and $\langle s \rangle$ are both cyclic groups of order two, we can take an isomorphism $\varphi : \langle s \rangle \to \langle r^{\frac{n}{2}} \rangle$. Hence

$$D_{2n}^* = \langle r, s, t \mid r^n, s^2, r^s = r^{-1}, s^t = s^{\varphi} \rangle.$$

We observe that this group is infinite since $t$ has infinite order. We note that the torsion of $D_{2n}$ appears in $D_{2n}^*$.

**Example 1.5.12** (Quaternions)**.** Let $Q_8 = \langle r, s \mid r^4, r^2 = s^2, r^s = r^{-1} \rangle$ be the quaternions group. Since $\langle rs \rangle$ and $\langle r \rangle$ are both cyclic groups of order four, we can take an isomorphism $\varphi : \langle rs \rangle \to \langle r \rangle$. Hence

$$Q_8^* = \langle r, s, t \mid r^4, r^2 = s^2, r^s = r^{-1}, (rs)^t = r^{\varphi} \rangle.$$

We observe that this group is also infinite since $t$ has infinite order. Also reduced sequences containing powers of $t$ must be of infinite order by Britton's Lemma.

**Definition 1.5.13** ([30], Section 9.4)**.** Let $H$ be a profinite group and let $f : A \to B$ be a continuous isomorphism between closed subgroups $A$, $B$, and $H$. A pro-$\mathcal{C}$ HNN-extension of $H$ with associated groups $A$, $B$ consists of a pro-$\mathcal{C}$ group $G = HNN(H, A, f)$, an element $t \in G$ called the stable letter, and a continuous homomorphism $\varphi : H \to G$ with $t(\varphi(a))t^{-1} = \varphi f(a)$ and satisfying the following universal property: for any pro-$\mathcal{C}$ group $K$, any $k \in K$ and any continuous homomorphism $\psi : H \to K$ satisfying $k(\psi(a))k^{-1} = \psi f(a)$ for all $a \in A$, there is a continuous homomorphism $\omega : G \to K$ with $\omega(t) = k$ such that the diagram

$$\begin{tikzcd} G \arrow[dr, dotted, "\omega"] & \\ H \arrow[u, "\varphi"] \arrow[r, "\psi"'] & K \end{tikzcd}$$

is commutative.

In contrast with the abstract situation, the canonical homomorphism $\varphi : H \to G = HNN(H, A, f)$ is not always a monomorphism. When $\varphi$ is a monomorphism, we shall call $G = HNN(H, A, f)$ a proper pro-$\mathcal{C}$ HNN-extension. Throughout this thesis, all pro-$\mathcal{C}$ HNN-extensions will be proper.

## 1.6 The fundamental group of a graph of groups

This section is dedicated to the fundamental group of a graph of groups. It is the main technical object of this thesis and it appears as a generalization of free constructions presented in the previous section.

**Definition 1.6.1** ($G$-transversal)**.** Let $G$ be a profinite group that acts on a connected profinite graph $\Gamma$, and let $\varphi : \Gamma \to \Delta = G\backslash\Gamma$ be the canonical quotient map. A $G$-transversal or a transversal of $\varphi$ is a closed subset $J$ of $\Gamma$ such that $\varphi_{|J} : J \to \Delta$ is a homeomorphism. Associated with this transversal there is a continuous $G$-section or section of $\varphi$, $j : \Delta \to \Gamma$, i.e., a continuous mapping such that $\varphi j = id_\Delta$ and $j(\Delta) = J$.

Note that, in general, $J$ is not a graph.

**Definition 1.6.2** (0-transversal)**.** We say that a transversal $J$ is a 0-transversal if $d_0(m) \in J$, for each $m \in J$; in this case we refer to $j$ as a 0-section.

Note that if $j$ is a 0-section, then $jd_0 = d_0 j$.

**Definition 1.6.3** (Profinite graph of pro-$\mathcal{C}$ groups, [31], Definition 6.1.1)**.** Let $\Gamma$ be a connected profinite graph with incidence maps $d_0, d_1 : \Gamma \to V(\Gamma)$. We can define a profinite graph of pro-$\mathcal{C}$ groups over $\Gamma$ as a pseudo-sheaf $(\mathcal{G}, \pi, \Gamma)$ of pro-$\mathcal{C}$ groups together with two morphisms of sheaves

$$(\partial_0, d_0), (\partial_1, d_1) : (\mathcal{G}, \pi, \Gamma) \to (\mathcal{G}_V, \pi|_{\pi^{-1}(V(\Gamma))}, V(\Gamma)),$$

where the restriction of $\partial_i$ to $\mathcal{G}_V$ is the identity map $id_{\mathcal{G}_V}$ $(i = 0, 1)$; in addition, we assume that the restriction of $\partial_i$ to each fiber $\mathcal{G}(m)$ is an injection $(m \in \Gamma)$, $(i = 0, 1)$.

Note that $\mathcal{G}_V$ denotes the restriction subsheaf of $\mathcal{G}$ to the space $V(\Gamma)$, which we term the 'vertex subsheaf of $\mathcal{G}$'. It is indeed a pseudo-sheaf because the vertex set $V(\Gamma)$ is closed in $\Gamma$, so it is also a profinite space and the vertex pseudo-sheaf is well defined (cf. Example 1.4.3). The vertex groups of a graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \pi, \Gamma)$ are the groups $\mathcal{G}(v)$ with $v \in V(\Gamma)$, and the edge groups are the groups $\mathcal{G}(e)$, with $e \in E(\Gamma)$.

Let $(\mathcal{G}, \pi, \Gamma)$ be a graph of pro-$\mathcal{C}$ groups over $\Gamma$, and let

$$\zeta : \tilde{\Gamma} \to \Gamma$$

be a universal Galois $\mathcal{C}$-covering of the profinite graph $\Gamma$. Choose a continuous 0-section $j$ of $\zeta$ (it exists because the fundamental group $\pi_1^{\mathcal{C}}(\Gamma)$ acts freely on $\tilde{\Gamma}$, cf. [30, Lemma 5.6.5]), and denote by $J = j(\Gamma)$ the corresponding 0-transversal. Associated with $j$ there is a continuous function

$$\chi : \Gamma \to \pi_1^{\mathcal{C}}(\Gamma)$$

defined by $\chi(m)$ being the unique element of $\pi_1^{\mathcal{C}}(\Gamma)$ such that $\chi(m)(jd_1(m)) = d_1 j(m)$.

**Definition 1.6.4** ($J$-specialization)**.** Given a pro-$\mathcal{C}$ group $H$, define a $J$-specialization of the graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \pi, \Gamma)$ in $H$ to consist of a pair $(\beta, \beta')$, where

$$\beta : (\mathcal{G}, \pi, \Gamma) \to H$$

is a morphism from the pseudo-sheaf $(\mathcal{G}, \pi, \Gamma)$ to $H$, and where

$$\beta' : \pi_1^{\mathcal{C}}(\Gamma) \to H$$

is a continuous homomorphism satisfying the following conditions:

$$\beta(x) = \beta\partial_0(x) = (\beta'\chi(m))(\beta\partial_1(x))(\beta'\chi(m))^{-1} \tag{1.4}$$

for all $x \in \mathcal{G}$, where $m = \pi(x)$. Note that the definition of the map $\chi$ depends uniquely on the 0-section $j$ and the corresponding $G$-transversal $J$. The following diagram shows the compositions:

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\beta"] \arrow[dd, "\partial_i"'] \arrow[ddr, bend left, "\pi"] & H \\
& \pi_1^{\mathcal{C}}(\Gamma) \arrow[u, "\beta'"'] \\
\mathcal{G}_V \arrow[uur, bend left, "\beta|_{\mathcal{G}_V}"] & \Gamma \arrow[u, "\chi"']
\end{tikzcd} \tag{1.5}$$

**Example 1.6.5** ([31], Example 6.1.2(a))**.** Assume that the graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \pi, \Gamma)$ has trivial edge groups, i.e., $\mathcal{G}(e) = 1$, for every edge $e \in \Gamma$. Then we may think of a $J$-specialization $(\beta, \beta')$ of $(\mathcal{G}, \pi, \Gamma)$ in a pro-$\mathcal{C}$ group $H$ as simply a morphism $\beta$ from the pseudo-sheaf $(\mathcal{G}, \pi, \Gamma)$ to $H$, since conditions (1.4) are automatic in this case.

Indeed, (1.4) gives us that:

$$\beta(\mathcal{G}(m)) = \beta\partial_0(\mathcal{G}(m)) = (\beta'\chi(m))(\beta\partial_1(\mathcal{G}(m)))(\beta'\chi(m))^{-1}$$

So, if $m \in E(\Gamma)$, we have that:

$$\begin{aligned} \beta(\mathcal{G}(e)) &= \beta\partial_0(\mathcal{G}(e)) \\ \beta(1) &= \beta\partial_0(1). \end{aligned}$$

By diagram 1.7,

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\beta"] \arrow[d, "\partial_0"'] & H \\
\mathcal{G}_V \arrow[ur, "\beta|_{\mathcal{G}_V}"'] &
\end{tikzcd}$$

so this diagram commutes for every $e \in E(\Gamma)$. If $v \in V(\Gamma)$, $\partial_0(\mathcal{G}(v)) = id_{\mathcal{G}(V)} = \mathcal{G}(v)$, then

$$\begin{aligned} \beta(\mathcal{G}(v)) &= \beta\partial_0(\mathcal{G}(v)) \\ \beta(\mathcal{G}(v)) &= \beta(\mathcal{G}(v)). \end{aligned}$$

and we have nothing to show. Thus, for $\partial_1$ and $e \in E(\Gamma)$, we have:

$$\begin{aligned} \beta(\mathcal{G}(e)) &= (\beta'\chi\pi(\mathcal{G}(e)))(\beta\partial_1(\mathcal{G}(e)))(\beta'\chi\pi(\mathcal{G}(e)))^{-1} \\ \beta(1) &= (\beta'\chi\pi(1))(\beta\partial_1(1))(\beta'\chi\pi(1))^{-1} \end{aligned}$$

By diagram 1.7,

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\beta"] \arrow[dd, "\partial_i"'] \arrow[ddr, bend left, "\pi"] & H \\
& \pi_1^{\mathcal{C}}(\Gamma) \arrow[u, "\beta'"'] \\
\mathcal{G}_V \arrow[uur, bend left, "\beta|_{\mathcal{G}_V}"] & \Gamma \arrow[u, "\chi"']
\end{tikzcd} \tag{1.6}$$

and the condition of Equation (1.4) is automatically satisfied.

**Example 1.6.6** ([31], Example 6.1.2(b))**.** If $\Gamma$ is $\mathcal{C}$-simply connected, then $\pi_1^{\mathcal{C}}(\Gamma) = 1$ and $\tilde{\Gamma} = \Gamma = J$. Then we can refer to a 'specialization' rather than '$J$-specialization': it is just a morphism $\beta : \mathcal{G} \to H$ such that

$$\beta(x) = \beta\partial_0(x) = \beta\partial_1(x)$$

for all $x \in \mathcal{G}$. It happens for example if $\Gamma$ is a finite tree or, more generally, an inverse limit of finite trees.

Now we are able to define the fundamental group of a graph of pro-$\mathcal{C}$ groups using a universal property. In the next chapter, we will present a more intuitive definition, using loops and a base point.

**Definition 1.6.7** (The fundamental group of a graph of pro-$\mathcal{C}$ groups)**.** Choose a continuous 0-section $j$ of the universal Galois $\mathcal{C}$-covering $\zeta : \widetilde{\Gamma} \to \Gamma$ and denote by $J = j(\Gamma)$ the corresponding 0-transversal. We define a fundamental pro-$\mathcal{C}$ group of the graph of groups $(\mathcal{G}, \pi, \Gamma)$ with respect to the 0-transversal $J$ to be a pro-$\mathcal{C}$ group $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$, together with a $J$-specialisation $(\nu, \nu')$ of $(\mathcal{G}, \pi, \Gamma)$ in $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ satisfying the following universal property:

$$\begin{tikzcd}
& \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) \arrow[dd, dotted, "\delta"'] & \\
\mathcal{G} \arrow[ur, "\nu"] \arrow[dr, "\beta"'] & & \pi_1^{\mathcal{C}}(\Gamma) \arrow[ul, "\nu'"'] \arrow[dl, "\beta'"] \\
& H &
\end{tikzcd}$$

whenever $H$ is a pro-$\mathcal{C}$ group and $(\beta, \beta')$ a $J$-specialisation of $(\mathcal{G}, \pi, \Gamma)$ in $H$, there exists a unique continuous homomorphism

$$\delta : \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) \to H$$

such that $\delta\nu = \beta$ and $\delta\nu' = \beta'$. We refer to $(\nu, \nu')$ as a universal $J$-specialisation of $(\mathcal{G}, \pi, \Gamma)$.

Note that in the pro-$\mathcal{C}$ case the vertex groups do not necessarily embed in the fundamental group (namely, $\nu$ restricted on fibers might not be a monomorphism); when it is the case, the profinite graph of pro-$\mathcal{C}$ groups is called *injective*.

**Remark 1.6.8.** Observe that a profinite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ can be replaced with a natural graph of quotient groups $(\overline{\mathcal{G}}, \Gamma)$ by replacing $\mathcal{G}(m)$ by its image $\nu(\mathcal{G}(m))$ in $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ for every $m \in \Gamma$; then $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \Gamma)$ and $(\overline{\mathcal{G}}, \Gamma)$ is injective (see [31, Section 6.4] for details). This means that we do not lose generality restricting our attention to injective profinite graphs of pro-$\mathcal{C}$ groups.

By [31, Theorem 6.2.4] the definition of $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ does not depend on the choice of the transversal $J$. Proposition 6.2.1 of [31] shows that, in fact,

$$\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \frac{\left(W \amalg \pi_1^{\mathcal{C}}(\Gamma)\right)}{N},$$

where $\amalg$ denotes the free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups, and where $N$ is the topological closure of the normal subgroup of the group $W \amalg \pi_1^{\mathcal{C}}(\Gamma)$ generated by the set

$$\{\partial_0(x)^{-1}(\chi\pi(x))\partial_1(x)(\chi\pi(x))^{-1} \mid x \in \mathcal{G}\}.$$

The next example shows that an amalgamated free pro-$\mathcal{C}$ product appears as a particular case of the fundamental pro-$\mathcal{C}$ group of a profinite graph of pro-$\mathcal{C}$ groups.

**Example 1.6.9** ([31], Example 6.2.3(d)). Let $G_1$, $G_2$ and $H$ be pro-$\mathcal{C}$ groups and consider the following graph of groups:

$$\overset{G_1}{\bullet} \xrightarrow[H]{\qquad} \overset{G_2}{\bullet}$$

In this case

$$\mathcal{G} = G_1 \,\dot{\bigcup}\, G_2 \,\dot{\bigcup}\, H,$$

endowed with the disjoint union topology. Since our graph $\Gamma$ is a tree, it is $\mathcal{C}$-simply connected and $\pi_1^{\mathcal{C}}(\Gamma) = 1$ (cf. Example 1.6.5). Hence the composition $\chi\pi(x)$ is the zero map for all $x \in \mathcal{G}$ and our J-specialization becomes the following

$$\begin{tikzcd} G_1 \cup G_2 \arrow[dr, bend right, "\pi"'] & \{1\} \\ & \Gamma \arrow[u, "\chi"'] \end{tikzcd} \tag{1.7}$$

Therefore

$$\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \frac{G_1 \coprod G_2}{N},$$

where N is generated by

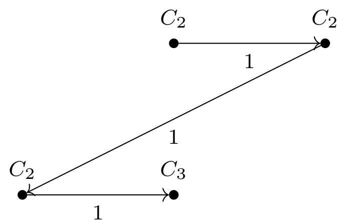
$$\{\partial_0(x) = \partial_1(x) \mid x \in \mathcal{G}\}.$$

We can conclude that the fundamental pro-$\mathcal{C}$ group will be

$$G = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = G_1 \amalg_H G_2.$$

In a more concrete way, using the results of the previous section, we have the following

**Example 1.6.10.** Consider the graph of groups $(\mathcal{G}, \Gamma)$:




Since $C_2 * C_2 \cong D_\infty$ and $C_2 * C_3 \cong PSL_2(Z)$, its abstract fundamental group is

$$\pi_1(\mathcal{G}, \Gamma) = D_\infty * PSL_2(Z),$$

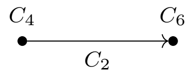
since

On the other hand, its profinite fundamental group is

$$\Pi_1(\mathcal{G}, \Gamma) = D \coprod \widehat{PSL_2(\mathbb{Z})}.$$

where $D$ is the infinite dihedral pro-2 group, $D \cong \mathbb{Z}_2 \rtimes C_2$.

**Example 1.6.11.** Consider the graph of groups $(\mathcal{G}, \Gamma)$:




It is known that $SL_2(\mathbb{Z}) = \langle A, B \rangle$, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

We note that $A^2 = -I = B^3$. Since $\langle A^2 \rangle \lhd SL_2(\mathbb{Z})$, because it is a central subgroup, we have that $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\langle A^2 \rangle$. Therefore, $SL_2(\mathbb{Z}) = \langle A, B \mid A^4, A^2 = B^3 \rangle$. By Example 1.5.4,

$$SL_2(\mathbb{Z}) \cong C_4 \ast_{C_2} C_6.$$

Hence we have the fundamental group $\pi_1(\mathcal{G}, \Gamma) = SL_2(\mathbb{Z})$.

On the other hand, its profinite fundamental group is

$$\Pi_1(\mathcal{G}, \Gamma) = \widehat{SL_2(\mathbb{Z})}.$$

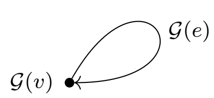
We observe that $\widehat{SL_2(\mathbb{Z})} \neq SL_2(\widehat{\mathbb{Z}})$. In fact, if we consider the natural continuous epimorphism

$$\varphi : \widehat{SL_2(\mathbb{Z})} \to SL_2(\widehat{\mathbb{Z}}),$$

Oleg Mel'nikov proved that Ker $\varphi$ is a free profinite group of countably infinite rank (cf. [28]), hence the map $\varphi$ cannot be injective.

The profinite HNN-extension is also a particular case of the profinite fundamental group of a profinite graph of profinite groups. In fact, obtaining the fundamental group of a graph of groups is an iterated process of computing free amalgamated products and HNN-extensions.

**Example 1.6.12** ([31], Example 6.2.3(e))**.** Let $\mathcal{G}(v)$ and $\mathcal{G}(e)$ be profinite groups and consider the following graph of groups:



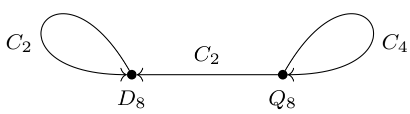
Then its profinite fundamental group will be the profinite HNN-extension

$$G = \mathrm{HNN}(\mathcal{G}(v), \partial_0(\mathcal{G}(e)), t, f)$$

of $\mathcal{G}(v)$, where $f : \partial_0(\mathcal{G}(e)) \to \partial_1(\mathcal{G}(e))$ is the isomorphism defined by $\partial_0(x) \mapsto \partial_1(x)$, for all $x \in \mathcal{G}(e)$ and $t$ is the stable letter related to $e \in E(\Gamma)$.

The next example combines everything we have developed in this section:

**Example 1.6.13.** Consider the graph of groups $(\mathcal{G}, \Gamma)$




We can calculate the fundamental group by parts. Example 1.5.11 provides that the fundamental group of the left loop is $D_8^* = \langle r, s, t \mid r^4, s^2, r^s = r^{-1}, s^t = s^{\varphi}\rangle$. On the other hand, Example 1.5.12 tells us that the fundamental group of the right loop is $Q_8^* = \langle r, s, t \mid r^4, r^2 = s^2, r^s = r^{-1}, (rs)^t = r^{\varphi}\rangle$. We can identify these two subgroups via an isomorphism $\varphi : \langle r^2\rangle_{Q_8} \to \langle r^2\rangle_{D_8}$, so we have a free amalgamated product. Therefore the fundamental group of $(\mathcal{G}, \Gamma)$ is

$$\pi_1(\mathcal{G}, \Gamma) = D_8^* *_{C_2} Q_8^*.$$

On the other hand, its pro-2 fundamental group is

$$\Pi_1(\mathcal{G}, \Gamma) = \widehat{D_8^*} \coprod_{\mathbb{Z}_2} \widehat{Q_8^*}.$$

where $\widehat{H}$ is the pro-2 completion of a given group $H$, $\coprod$ is the free pro-2 product, and $\mathbb{Z}_2$ is the 2-adic procyclic group.

We present the definitions of the standard tree and standard pro-$\mathcal{C}$ tree on which the fundamental group naturally acts following [37] and [31] respectively; we shall need them in the following sections.

**Definition 1.6.14** (Standard tree)**.** Let $G = \pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of the graph of abstract groups $(\mathcal{G}, \Gamma)$. There is an abstract standard graph $S^{abs} = S^{abs}(\mathcal{G}, \Gamma)$

which is in fact a tree (cf. [37], Sec. I.5.3). Let $T$ be a maximal subtree of $\Gamma$. We define $S^{abs}$ by

$$V(S^{abs}) = \bigcup_{v\in V(\Gamma)} G/\mathcal{G}(v) \text{ and } E(S^{abs}) = \bigcup_{e\in E(\Gamma)} G/\mathcal{G}(e)$$

and its incidence maps

$$d_0(g\mathcal{G}(e)) = g\mathcal{G}(d_0(e)),\, d_1(g\mathcal{G}(e)) = gt_e\mathcal{G}(d_1(e))$$

$(g \in G, e \in E(\Gamma))$ and $t_e = 1, \forall e \in E(T)$.

Let $(\mathcal{G}, \Gamma)$ be a profinite graph of pro-$\mathcal{C}$ groups. When $\Gamma$ is finite, the definition of the standard pro-$\mathcal{C}$ tree is much simpler and we will use it in Chapter 3.

**Definition 1.6.15** (Standard (universal) pro-$\mathcal{C}$ tree (cf. [31] Example 6.3.1))**.** Associated with the finite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ there is a corresponding standard pro-$\mathcal{C}$ tree (or universal covering graph) $S = S(G) = \bigcup\kern-0.6em\cdot_{m\in\Gamma} G/\Pi(m)$ (cf. [51, Proposition 3.8]). The vertices of $S$ are cosets of the form $g\Pi(v)$, with $v \in V(\Gamma)$ and $g \in G$; its edges are the cosets of the form $g\Pi(e)$, with $e \in E(\Gamma)$; choosing a maximal subtree $D$ of $\Gamma$, the incidence maps of $S$ are given by the formulas:

$$d_0(g\Pi(e)) = g\Pi(d_0(e)); \quad d_1(g\Pi(e)) = gt_e\Pi(d_1(e)) \;\; (e \in E(\Gamma), t_e = 1 \text{ if } e \in D).$$

There is a natural continuous action of $G$ on $S$ given by

$$g(g'\Pi(m)) = gg'\Pi(m),$$

where $g, g' \in G$, $m \in \Gamma$. Clearly $G\backslash S = \Gamma$. There is a standard connected transversal $s : \Gamma \to S$, given by $m \mapsto \Pi(m)$. Note that $s_{|D}$ is an isomorphism of graphs and the elements $t_e$ satisfy the equality $d_1(s(e)) = t_e s(d_1(e))$. Using the map $s$, we shall identify $\Pi(m)$ with the stabilizer $G_{s(m)}$ for $m \in \Gamma$:

$$\Pi(e) = G_{s(e)} = G_{d_0(s(e))} \cap G_{d_1(s(e))} = \Pi(d_0(e)) \cap t_e\Pi(d_1(e))t_e^{-1} \tag{1.8}$$

with $t_e = 1$ if $e \in D$. Remark also that since $\Gamma$ is finite, $E(S)$ is compact.

If the profinite graph $\Gamma$ is infinite and does not have a maximal profinite subtree, we need the general construction below, the $\mathcal{C}$-standard graph associated with a graph of pro-$\mathcal{C}$ groups, as follows:

**Definition 1.6.16** ($\mathcal{C}$-standard graph of a graph of pro-$\mathcal{C}$ groups)**.** Let $(\mathcal{G}, \pi, \Gamma)$ be a graph of pro-$\mathcal{C}$ groups over a connected profinite graph $\Gamma$, $j : \Gamma \to \widetilde{\Gamma}$ be a continuous 0-section of the universal Galois $\mathcal{C}$-covering $\zeta : \widetilde{\Gamma} \to \Gamma$ of $\Gamma$, and let $J = j(\Gamma)$ be the corresponding 0-transversal. Let $(\gamma, \gamma')$ be a $J$-specialisation of $(\mathcal{G}, \pi, \Gamma)$ in the fundamental pro-$\mathcal{C}$ group of $(\mathcal{G}, \pi, \Gamma)$, $\Pi = \Pi_1^{\mathcal{C}}(\mathcal{G}, \pi, \Gamma)$. Then we can define a profinite graph $S = S^{\mathcal{C}}(\mathcal{G}, \Gamma) = S^{\mathcal{C}}(\mathcal{G}, \Gamma, \Pi)$ which is canonically associated to the graph of groups $(\mathcal{G}, \pi, \Gamma)$ and $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$.

For $m \in \Gamma$, define $\Pi(m) = \gamma(\mathcal{G}(m))$. As a topological space, $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is defined to be the quotient space of $\Gamma \times \Pi$ modulo the equivalence relation $\sim$ given by

$$(m, h) \sim (m', h') \text{ if } m = m', h^{-1}h' \in \Pi(m) \ (m, m' \in \Gamma, h, h' \in \Pi).$$

So, as a set, $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is the disjoint union

$$S^{\mathcal{C}}(\mathcal{G}, \Gamma) = \bigcup_{m \in \Gamma} \Pi/\Pi(m).$$

Denote by $\alpha : \Gamma \times \Pi \to S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ the quotient map. The projection $p' : \Gamma \times \Pi \to \Gamma$ induces a continuous epimorphism $p : S^{\mathcal{C}}(\mathcal{G}, \Gamma) \to \Gamma$, such that $p^{-1}(m) = \Pi/\Pi(m)$ and $p' = p\alpha$.

To make $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ into a profinite graph we define the subspace of vertices of $S^{\mathcal{C}}(\mathcal{G}, \Gamma, H)$ by $V(S^{\mathcal{C}}(\mathcal{G}, \Gamma, H)) = p^{-1}(V(\Gamma))$ and the incidence maps by

$$d_0(hH(m)) = hH(d_0(m))$$

$$d_1(hH(m)) = h(\gamma'\chi(m))H(d_1(m)),$$

$(h \in H, m \in \Gamma)$. The definition of $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is independent, up to isomorphism, of the choice of the 0-section $j$ (cf. [31], Theorem 6.3.3)

There is a natural continuous action of $\Pi = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ on the graph $S^{\mathcal{C}}(\mathcal{G}, \Gamma)$ given by

$$g(g'\Pi(m)) = gg'\Pi(m),$$

$(g, g' \in \Pi, m \in \Gamma)$.

The $\mathcal{C}$-standard graph is in fact a $\mathcal{C}$-tree when $\mathcal{C}$ is a pseudovariety of finite groups which is extension closed (cf. [31], Corollary 6.3.6).

By this definition, there are numerous ways to produce graph of groups with the same profinite fundamental group, through fictitious edges, i.e., the edge group is isomorphic to its extremity vertex group. This induces the identity map on the fundamental group, so we can remove it from the graph of groups without losing any important data, as follows:

**Definition 1.6.17** (Reduced graph of groups)**.** A profinite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ is said to be reduced if for every edge $e$, which is not a loop, neither $\partial_1 : \mathcal{G}(e) \to \mathcal{G}(d_1(e))$ nor $\partial_0 : \mathcal{G}(e) \to \mathcal{G}(d_0(e))$ is an isomorphism; we say that an edge $e$ is fictitious if it is not a loop and one of the edge maps $\partial_i$ is an isomorphism.

**Remark 1.6.18.** Any finite graph of groups can be transformed into a reduced finite graph of groups by collapsing fictitious edges using the following procedure. If $e$ is a fictitious edge, we can remove $\{e\}$ from the edge set of $\Gamma$, and identify $d_0(e)$ and $d_1(e)$ to a new vertex $y$. Let $\Gamma'$ be the finite graph given by $V(\Gamma') = y \cup V(\Gamma)\backslash\{d_0(e), d_1(e)\}$ and $E(\Gamma') = E(\Gamma)\backslash\{e\}$, and let $(\mathcal{G}', \Gamma')$ denote the finite graph of groups based on $\Gamma'$ given by $\mathcal{G}'(y) = \mathcal{G}(d_1(e))$ if $\partial_0(e)$ is an isomorphism, and $\mathcal{G}'(y) = \mathcal{G}(d_0(e))$ if $\partial_0(e)$ is not an isomorphism. This procedure can be continued until there are no fictitious edges. The resulting finite graph of groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ is reduced.

**Remark 1.6.19.** As mentioned before, the reduction procedure described above does not change the fundamental group (as a group given by a presentation), i.e. choosing a maximal subtree to contain the collapsing edge, the morphism $(\mathcal{G}, \Gamma) \to (\mathcal{G}', \Gamma')$ induces the identity map on the fundamental group with the presentation given by eliminating redundant relations associated with fictitious edges that are just collapsed by reduction.

The reduction procedure cannot be applied, however, if $\Gamma$ is infinite profinite since the removal of an edge results in a non-compact object.

The reduction procedure allows us to refine the main result of [23] as follows:

**Theorem 1.6.20.** *Let $G$ be a finitely generated pro-$p$ group with a free open subgroup $F$. Then $G$ is the pro-$p$ fundamental group of a reduced finite graph of finite $p$-groups $(\mathcal{G}, \Gamma)$ with orders of vertex groups bounded by $[G : F]$. Moreover, if $G = \Pi_1(\mathcal{G}', \Gamma')$ is another splitting as a reduced finite graph of finite $p$-groups then $|\Gamma| = |\Gamma'|, |V(\Gamma)| = |V(\Gamma')|, |E(\Gamma)| = |E(\Gamma')|$.*

*Proof.* By [23, Theorem 1.1] $G$ is the pro-$p$ fundamental group of a finite graph of finite $p$-groups $(\mathcal{G}, \Gamma)$ with orders of vertex groups bounded by $[G : F]$ and applying the reduction procedure we get the first statement. Maximal finite subgroups of $G$ are exactly the vertex groups of $(\mathcal{G}, \Gamma)$ and $(\mathcal{G}', \Gamma')$ up to conjugation by Theorem 1.6.22, so $|V(\Gamma)| = |V(\Gamma')|$. Now by [9, Proposition 3.4] $G/\widetilde{G} = \pi_1(\Gamma) = \pi_1(\Gamma')$ is a free pro-$p$ groups of rank $|E(\Gamma)| - |V(\Gamma)| + 1 = |E(\Gamma')| - |V(\Gamma')| + 1$ implying $|E(\Gamma)| = |E(\Gamma')|$ and $|\Gamma| = |\Gamma'|$. The proof is complete.

□

**Definition 1.6.21** (FA group, cf. Section 6.1 of [37])**.** Let $G$ be a pro-$p$ group acting on a pro-$p$ tree $T$. The set $T^G$ of fixed points of $G$ in $T$ is a subgraph of $T$. We say that $G$ is FA (or has property FA) if $T^G \neq \varnothing$ for any tree $T$ on which $G$ acts.

The following lemma will be very useful in the abstract and profinite cases.

**Theorem 1.6.22** (cf. Theorem 7.1.2 of [31] and Theorem I.15 of [37])**.** .

*(a) Let $K$ be a finite subgroup of the fundamental group $\pi = \pi_1(\mathcal{G}, \Gamma)$ of a graph of groups $(\mathcal{G}, \Gamma)$. Then*

$$K \leqslant g\pi(v)g^{-1},$$

*for some $v \in V(\Gamma)$ and $g \in \pi_1(\mathcal{G}, \Gamma)$.*

*(b) Let $K$ be a finite subgroup of the fundamental pro-$\mathcal{C}$ group $\Pi = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ of an injective graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$. Then*

$$K \leqslant g\Pi(v)g^{-1},$$

*for some $v \in V(\Gamma)$ and $g \in \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$.*

It is worth mentioning that item (a) of this Proposition is equivalent to showing that every finite group $K$ is FA (cf. Definition 1.6.21). Note that $K$ acts on the abstract standard tree $S^{abs}(\mathcal{G}, \Gamma)$. In fact, $K$ fits all the conditions of [37, Theorem I.15]. It cannot be an amalgam, since amalgams are infinite groups; for the same reason, it cannot have quotients isomorphic to $\mathbb{Z}$. Finally, $K$ is clearly finitely generated.

We shall use in the last chapter the following results from [53], [9] and [31]. Note that they only hold in the pro-$p$ case. The first states that for open subgroups of the fundamental pro-$p$ group of the finite graph of pro-$p$ groups the subgroup theorem of the Bass-Serre theory works. The second states that geodesics are fixed by the action on a tree. The third states that a finitely generated pro-$p$ group acting on a pro-$p$ tree splits over an edge stabilizer.

**Proposition 1.6.23.** *([53, Corollary 4.5 combined with 5.4] or [31, Theorem 6.6.1]) Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the pro-p fundamental group of a finite graph of pro-p groups and $H$ an open subgroup of $G$. Let $s : H\backslash S(G) \to S(G)$ be a connected transversal. Then $H = \Pi_1(\mathcal{H}, H\backslash S(G))$ with $\mathcal{H}(m) = H_{s(m)}$ for each $m \in H\backslash S(G)$.*

**Proposition 1.6.24.** *(Corollary 4.1.6 of [31]) Suppose that a pro-p group $G$ acts on a pro-p tree $T$, and let $v$ and $w$ be two different vertices of $T$. Then the set of edges $E([v, w])$ of the chain $[v, w]$ is nonempty, and $G_v \cap G_w \leqslant G_e$ for every $e \in E([v, w])$.*

**Theorem 1.6.25.** *([9, Theorem 4.2]) Let $G$ be a finitely generated pro-p group acting on a pro-p tree $T$ without global fixed points. Then $G$ splits non-trivially as a free amalgamated pro-p product or pro-p HNN-extension over some stabilizer of an edge of $T$.*

# CHAPTER 2

# THE PROFINITE FUNDAMENTAL GROUP OF AN INFINITE GRAPH OF GROUPS

## 2.1 Decomposition of an injective profinite graph of groups

The entirety of this chapter is a novelty (published in the Israel Journal of Mathematics, 2022).

We show in this section that an injective profinite graph of profinite groups $(\mathcal{G},\Gamma)$ decomposes as an inverse limit $(\mathcal{G},\Gamma)=\varprojlim_{i\in I}(\mathcal{G}_i,\Gamma_i)$ of finite graphs of finite groups. In fact, this characterizes injective graphs of profinite groups.

**Lemma 2.1.1.** *Let $(\mathcal{G},\Gamma)$ be a profinite graph of pro-$\mathcal{C}$-groups. Suppose there exists a decomposition $(\mathcal{G},\Gamma)=\varprojlim_{i\in I}(\mathcal{G}_i,\Gamma_i)$ as a surjective inverse limit of finite graphs of finite $\mathcal{C}$-groups. Let $\underline{\alpha}=(\alpha,\alpha'):(\mathcal{G},\Gamma)\longrightarrow(\mathcal{H},\Delta)$ be a surjective morphism to a finite graph of finite $\mathcal{C}$-groups and $\eta:(\mathcal{G},\Gamma)\longrightarrow H$ be a fiber homomorphism (cf. Definition 1.4.5) to a finite $\mathcal{C}$-group $H$. Then $\underline{\alpha},\eta$ factor via some $(\mathcal{G}_k,\Gamma_k)$, i.e. there exist a morphism $\underline{\alpha}_k:(\mathcal{G}_k,\Gamma_k)\longrightarrow(\mathcal{H},\Delta)$ and a fiber homomorphism $\eta_k:(\mathcal{G}_k,\Gamma_k)\longrightarrow H$ such that $\underline{\alpha}=\underline{\alpha}_k\underline{\pi}_k$ and $\eta=\eta_k\underline{\pi}_k$, where $\underline{\pi}_i=(\pi_i,\pi_i'):(\mathcal{G},\Gamma)\longrightarrow(\mathcal{G}_i,\Gamma_i)$ is the natural projection.*

The statement of the lemma is based on the following commutative diagrams, where the induced homomorphisms are highlighted by dotted lines.

$$\begin{tikzcd}
(\mathcal{G},\Gamma) \arrow[rr, "\underline{\alpha}"] \arrow[dr, "\underline{\pi}_k"'] & & (\mathcal{H},\Delta) \\
& (\mathcal{G}_k,\Gamma_k) \arrow[ur, dotted, "\underline{\alpha}_k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{G},\Gamma) \arrow[rr, "\eta"] \arrow[dr, "\underline{\pi}_k"'] & & H \\
& (\mathcal{G}_k,\Gamma_k) \arrow[ur, dotted, "\eta_k"'] &
\end{tikzcd}$$

*Proof.* We adapt the proof of [31, Lemma 2.1.5]).

Let $S$ be the equivalence relation on $\mathcal{G}$ whose equivalence classes are the clopen sets $\alpha^{-1}(h), h \in \mathcal{H}$ and $R$ the equivalence relation on $\Gamma$ whose equivalence classes are the clopen sets $\alpha'^{-1}(m), m \in \Delta$;

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\alpha"] \arrow[d] & \mathcal{G}/S = \mathcal{H} \arrow[d] \\
\Gamma \arrow[r, "\alpha'"'] & \Gamma/R = \Delta
\end{tikzcd}$$

then $(\mathcal{G}/S, \Gamma/R) = (\mathcal{H},\Delta)$ and $\underline{\alpha} = (\alpha,\alpha')$ is the natural projection $(\mathcal{G},\Gamma) \longrightarrow (\mathcal{G}/S, \Gamma/R)$. Similarly, for $i \in I$, let $S_i$ be the equivalence relation on $(\mathcal{G},\Gamma)$ whose equivalence classes are the clopen sets $\pi_i^{-1}(h), h \in \mathcal{G}_i$ and $R_i$ the equivalence relation on $(\mathcal{G},\Gamma)$ whose equivalence classes are the clopen sets $(\pi_i')^{-1}(m), m \in \Gamma_i$, so that $\underline{\pi}_i = (\pi_i,\pi_i') : (\mathcal{G},\Gamma) \longrightarrow (\mathcal{G}/S_i, \Gamma/R_i)$ is the natural projection.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\pi_i"] \arrow[d] & \mathcal{G}/S_i = \mathcal{G}_i \arrow[d] \\
\Gamma \arrow[r, "\pi_i'"'] & \Gamma/R_i = \Gamma_i
\end{tikzcd}$$

Since

$$(\mathcal{G},\Gamma) = \varprojlim_{i\in I}(\mathcal{G}_i,\Gamma_i),$$

the intersections $\bigcap_{i\in I} S_i$, $\bigcap_{i\in I} R_i$ are the diagonal subsets of $\mathcal{G}\times\mathcal{G}$ and $\Gamma\times\Gamma$, respectively. Note that $S, S_i$ are clopen subsets of $\mathcal{G}\times\mathcal{G}$ and $R, R_i$ are clopen subsets of $\Gamma\times\Gamma$.

In order to seal the argument, we use two crucial properties: compactness of $\mathcal{G}\times\mathcal{G}$ and $\Gamma\times\Gamma$, and $I$ being a poset. It follows from the compactness of $\mathcal{G}\times\mathcal{G}$ and $\Gamma\times\Gamma$ that there exists a finite subset $J$ of $I$ such that $\bigcap_{j\in J} S_j \subseteq S$ and $\bigcap_{j\in J} R_j \subseteq R$. Since the poset $I$ is directed, there exists a $k \in I$ such that $S_k \subseteq \bigcap_{j\in J} S_j \subseteq S$ and $R_k \subseteq \bigcap_{j\in J} R_j \subseteq R$. This means that there exists a morphism of graphs of groups $\underline{\alpha}_k : (\mathcal{G}_k,\Gamma_k) \longrightarrow (\mathcal{H},\Delta)$ such that $\underline{\alpha} = \underline{\alpha}_k\underline{\pi}_k$.

The statement about a fiber homomorphism $\eta_k : (\mathcal{G}_k,\Gamma_k) \longrightarrow H$ follows from the first since we can consider $H$ as a graph of groups $(H, \{v\})$ with the underlying graph being one vertex $v$. $\square$

The next lemma deals with a double inverse system structure. It will be useful to recover the data from intricate arguments that need more than one reduction to finite situations. For example, in the proof of the main result of this section, starting from a graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$, we obtain a reduction to a profinite graph of $\mathcal{C}$-groups $(\mathcal{G}_U, \Gamma)$, such that $(\mathcal{G}, \Gamma) = \varprojlim_{U \lhd_o G}(\mathcal{G}_U, \Gamma)$. However, we need more, the graph must be finite as well. The next lemma allows us to prove our main result for $\varprojlim_{U \lhd_o G}(\mathcal{G}_U, \Gamma)$ instead of proving it for $(\mathcal{G}, \Gamma)$, since the data can be recovered via inverse limits.

**Lemma 2.1.2.** *Let $(\mathcal{G}, \Gamma)$ be a profinite graph of pro-$\mathcal{C}$-groups. Suppose there exists a decomposition $(\mathcal{G}, \Gamma) = \varprojlim_{U \in \mathcal{U}}(\mathcal{G}_U, \Gamma)$ as an inverse limit of profinite graph of finite quotient $\mathcal{C}$-groups over the same graph $\Gamma$. Assume further that for each $U \in \mathcal{U}$ the graph of groups $(\mathcal{G}_U, \Gamma)$ decomposes as a surjective inverse limit $(\mathcal{G}_U, \Gamma) = \varprojlim_{i \in I_U}(\mathcal{G}_{i,U}, \Gamma_i)$. Then $(\mathcal{G}_{i,U}, \Gamma_i)$ form naturally an inverse system such that $(\mathcal{G}, \Gamma) = \varprojlim_{U \in \mathcal{U}, i \in I_U}(\mathcal{G}_{i,U}, \Gamma_i)$.*

*Proof.* We follow the proof of [31, Proposition 3.1.3] making all the appropriate changes. Denote by $\varphi_{i,U} : (\mathcal{G}_U, \Gamma) \longrightarrow (\mathcal{G}_{i,U}, \Gamma_i)$ the canonical projection. Define the indexing set $I = \bigcup_{U \in \mathcal{U}} I_U$. We relabel the elements of $I_U$: an element $i \in I_U$ will be denoted from now on by $(i, U)$. If $(i, U), (j, V) \in I$, we say $(i, U) \leqslant (j, V)$ if $U \leqslant V$ and there exists a morphism of graph of groups $\alpha : (\mathcal{G}_{j,V}, \Gamma_j) \longrightarrow (\mathcal{G}_{i,U}, \Gamma_i)$ such that the diagram

$$\begin{tikzcd}
 & (\mathcal{G}_V, \Gamma) \arrow[r, "\varphi_{j,V}"] \arrow[d] & (G_{j,V}, \Gamma_j) \arrow[d, dotted, "\alpha"] \\
(\mathcal{G}, \Gamma) \arrow[ur] \arrow[dr] & & \\
 & (\mathcal{G}_U, \Gamma) \arrow[r, "\varphi_{j,U}"'] & (G_{i,U}, \Gamma_i)
\end{tikzcd}$$

commutes. Observe that $\alpha$ is unique, if it exists, because $\varphi_{j,V}$ is surjective. Hence $(I, \leqslant)$ is a partially ordered set. We also observe that the restriction of $\leqslant$ to $I_U$ coincides with the partial order of $I_U$ $(U \in \mathcal{U})$. We claim that this ordering makes $(I, \leqslant)$ into a directed poset.

To see this consider $(j, V), (i, U) \in I$. Let $L \geqslant V, U$. Then by Lemma 2.1.1 there exists $(k, L) \in I_L$ and morphisms of graphs of groups $\underline{\alpha}_i : (\mathcal{G}_{k,L}, \Gamma_k) \longrightarrow (\mathcal{G}_{i,U}, \Gamma_i)$,

$\underline{\alpha}_j : (\mathcal{G}_{k,L}, \Gamma_k) \longrightarrow (\mathcal{G}_{j,V}, \Gamma_j)$ such that the diagram

$$\begin{tikzcd}
 & (\mathcal{G}_V, \Gamma) \arrow[r, "\varphi_{j,V}"] & (G_{j,V}, \Gamma_j) \\
(\mathcal{G}_L, \Gamma) \arrow[ur] \arrow[r] \arrow[dr] & (\mathcal{G}_{Lk}, \Gamma_k) \arrow[ur, "\alpha_j"'] \arrow[dr, "\alpha_i"] & \\
 & (\mathcal{G}_U, \Gamma) \arrow[r, "\varphi_{j,U}"'] & (G_{i,U}, \Gamma_i)
\end{tikzcd}$$

commutes.

All maps in this diagram are surjective; it follows that $\alpha_i, \alpha_j$ are unique. This shows that $(I, \leqslant)$ is directed and it follows that $(\mathcal{G}, \Gamma) = \varprojlim_{U \in \mathcal{U}, i \in I_U} (\mathcal{G}_{i,U}, \Gamma_i)$. $\square$

**Theorem 2.1.3.** *Let $\mathcal{C}$ be an extension closed pseudovariety of finite groups. The sheaf of pro-$\mathcal{C}$ groups $(\mathcal{G}, \pi, \Gamma)$ decomposes as a surjective inverse limit $(\mathcal{G}, \pi, \Gamma) = \varprojlim_{i \in I} (\mathcal{G}_i, \pi_i, \Gamma_i)$ of finite sheaves of groups in $\mathcal{C}$.*

*Proof.* Let $G = \coprod_{\Gamma}^{\mathcal{C}} \mathcal{G}$ be the free pro-$\mathcal{C}$ product of the sheaf $(\mathcal{G}, \pi, \Gamma)$. By a result of Melnikov (cf. [31, Theorem 5.3.4]), $G = \varprojlim_{i \in I} G_i$ where each $G_i$ is a free pro-$\mathcal{C}$ product of finite groups, and $\Gamma = \varprojlim_{i \in I} \Gamma_i$, where each $\Gamma_i$ is a finite graph. Hence we can write $G_i = \coprod H_{i,t}$, so $G = \varprojlim \coprod H_{i,t}$. So we construct an associated sheaf of finite groups $(\mathcal{G}_i, \pi_i, \Gamma_i)$ as follows:

$$\mathcal{G}_i = \{(m_i, h_i) \in \Gamma_i \times G_i \mid h_i \in H_{i,t}\}$$

where $\pi : \mathcal{G}_i \to \Gamma_i$ is the restriction of the natural projection $\Gamma_i \times G_i \to \Gamma_i$. It is clear that $G_i = \coprod_{\Gamma_i}^{\mathcal{C}} \mathcal{G}_i$.

Now construct the canonical fiber homomorphism $\omega : \mathcal{G} \to G$ defined by $\omega(m, g) = g$, $(g \in G_i, m \in \Gamma)$. We define the map $\omega_i : \mathcal{G}_i \to G_i$ by identifying the identity elements of the fibers. For $i \geq j$, we have the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\varphi_i"] & G_i \arrow[r, "\varphi_{ij}"] & G_j \\
\mathcal{G} \arrow[u, "\omega"] \arrow[r, "\nu_i"'] & \mathcal{G}_i \arrow[u, "\omega_i"'] \arrow[r, "\nu_{ij}"'] & \mathcal{G}_j \arrow[u, "\omega_j"']
\end{tikzcd} \tag{2.1}$$

where $\nu_{ij} : \mathcal{G}_i \to \mathcal{G}_j$ is defined as follows. The map $\omega_i$ becomes injective on $\mathcal{G}_i - (\Gamma_i \times \{1\})$. Hence,

$$\nu_{ij} = \omega_j^{-1} \circ \varphi_{ij} \circ \omega_i.$$

For the identity elements, since we have the canonical commutative diagram of the sheaves,

$$\begin{tikzcd} \mathcal{G}_i \arrow[r, "\nu_{ij}"] \arrow[d, "\pi_i"'] & \mathcal{G}_j \arrow[d, "\pi_j"] \\ \Gamma_i \arrow[r, "\nu'_{ij}"'] & \Gamma_j \end{tikzcd}$$

define

$$\nu_{ij} = \pi_j^{-1} \circ \nu'_{ij} \circ \pi_i.$$

Let $\underline{\nu}_{ij} = (\nu_{ij}, \nu'_{ij})$ be the pseudo-sheaf map. Hence $((\mathcal{G}, \Gamma), \underline{\nu}_{ij}, I)$ form an inverse system. By construction, we have that

$$\mathcal{G} = \varprojlim \mathcal{G}_i,$$

as desired.

□

We are finally ready to prove the main result in this section. It will be crucial to define the profinite fundamental group of a graph of groups with a base point.

**Proposition 2.1.4.** *Let $(\mathcal{G}, \Gamma)$ be an injective profinite graph of pro-$\mathcal{C}$-groups. Then $(\mathcal{G}, \Gamma)$ decomposes as an inverse limit $(\mathcal{G}, \Gamma) = \varprojlim_{i \in I} (\mathcal{G}_i, \Gamma_i)$ of finite graphs of $\mathcal{C}$-groups.*

*Proof.* Let $G = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ be the fundamental group of $(\mathcal{G}, \Gamma)$ with respect to a universal specialization $(\nu, \nu') : (\mathcal{G}, \Gamma) \longrightarrow \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$. Let $\mathcal{U}$ be the collection of all open normal subgroups of $G$.

$$\begin{tikzcd} & \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) & \\ \mathcal{G} \arrow[ur, "\nu"] & & \pi_1^{\mathcal{C}}(\Gamma) \arrow[ul, "\nu'"'] \end{tikzcd}$$

Let $U \in \mathcal{U}$. Since $\nu(\mathcal{G}(m))U/U$ is finite, it is closed in $G/U$. Now, by [31, Lemma 5.2.1(d)], noticing that $\Gamma$ is a profinite space, one has that $\mathcal{G}_U = \bigcup_{m \in \Gamma} \nu(\mathcal{G}(m))U/U$ is closed in $G/U$. Therefore the set $\mathcal{G}_U = \{(g, m) \in G/U \times \Gamma \mid m \in \Gamma, g \in \nu(\mathcal{G}(m))U/U\}$ is a sheaf. For simplicity we identify from now on the fiber $\mathcal{G}_U(m) = \nu(\mathcal{G}(m))U/U \times \{m\}$ with the subgroup $\nu(\mathcal{G}(m))U/U$ in $G/U$.

Define then a profinite graph of finite groups, $(\mathcal{G}_U, \Gamma)$ putting

$$\partial_j : \mathcal{G}_U(m) \longrightarrow \mathcal{G}_U(d_j(m))$$

by $\partial_j(\nu(g)U) = \nu\partial_j(g)U$, for $g \in \mathcal{G}(m)$, $j = 0, 1$. To see that it is well-defined let $g, h \in \mathcal{G}(m)$ such that $\nu(g)U = \nu(h)U$ and let $\chi : \Gamma \longrightarrow \pi_1^{\mathcal{C}}(\Gamma)$ be the map from Definition 1.6.4. Then

$$\partial_0(\nu(g)U) = \nu\partial_0(g)U = \nu(g)U = \nu(h)U = \nu(\partial_0(h))U = \partial_0(\nu(h)U)$$

and

$$\partial_1(\nu(g)U) = \nu\partial_1(g)U = (\nu'\chi(m))^{-1}\nu(g)\nu'\chi(m)U =$$

$$= (\nu'\chi(m))^{-1}\nu(h)\nu'\chi(m)U = \nu\partial_1(h)U = \partial_1(\nu(h)U).$$

Put $G_U = \Pi_1^{\mathcal{C}}(\mathcal{G}_U, \Gamma)$ and define $\tilde{U} = \langle U \cap \mathcal{G}(v)^g \mid v \in V(\Gamma), g \in G\rangle$. Note that $\mathcal{G}(v)$ embeds in $G$, since $(\mathcal{G}, \Gamma)$ is injective. Hence $\mathcal{G}(v)^g$ lies inside $G = \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$. It is not difficult to see, using a similar argument to the proof of [31, Corollary 5.5.9], that $G_U = G/\tilde{U}$ and so $G = \varprojlim_{U \triangleleft_o G} G_U$. Thus by Lemma 2.1.2 it suffices to show the proposition for $(\mathcal{G}_U, \Gamma)$.

Let $(\nu_U, \nu'_U) : (\mathcal{G}_U, \Gamma) \longrightarrow G_U$ be the universal specialization. The projection $G/U \times \Gamma \to G/U$ restricts to a continuous map $\rho : \mathcal{G}_U \to G/U$ that sends $\mathcal{G}_U(m)$ identically to the subgroup $\nu(\mathcal{G}(m))U/U$ of $G/U$. So $\rho$ induces a unique continuous homomorphism $f_U : G_U \to G/U$ such that $f_U\nu_U = \rho$. Hence $f_U\nu_U$ is injective on $\mathcal{G}_U(m)$, for each $m \in \Gamma$.

By Theorem 2.1.3, the sheaf $(\mathcal{G}_U, \pi_U, \Gamma)$ decomposes as a surjective inverse limit

$$(\mathcal{G}_U, \pi_U, \Gamma) = \varprojlim_{i \in I} (\mathcal{G}_i, \pi_i, \Gamma_i)$$

of finite sheaves $(\mathcal{G}_i, \pi_i, \Gamma_i)$. Let $\underline{\varphi}_i = (\varphi_i, \varphi'_i) : (\mathcal{G}_U, \Gamma) \longrightarrow (\mathcal{G}_i, \Gamma_i)$ be the natural projection.

$$\begin{tikzcd} \mathcal{G}_U \arrow[r, "\varphi_i"] \arrow[d] & \mathcal{G}_i \arrow[d] \\ \Gamma \arrow[r, "\varphi'_i"'] & \Gamma_i \end{tikzcd}$$

We need to turn this inverse limit decomposition into a decomposition of graph of groups, i.e to define the structure of graphs of finite groups on $(\mathcal{G}_i, \Gamma_i)$ and show that we have an inverse system of graphs of groups.

There exists $i_0 \in I$ such that the continuous maps $f_U\nu_U, f_U\nu'_U\chi$ factor via $(\mathcal{G}_{i_0}, \Gamma_{i_0})$ (cf. Lemma 2.1.1), i.e. there exist a fiber homomorphism $\beta_{i_0} : \mathcal{G}_{i_0} \longrightarrow G/U$ and a continuous map $\beta'_{i_0} : \Gamma_{i_0} \longrightarrow G/U$ such that $f_U\nu_U = \beta_{i_0}\varphi_{i_0}$ and $f_U\nu'_U\chi = \beta'_{i_0}\varphi'_{i_0}$.

For $i \geqslant i_0$ define a fiber homomorphism $\beta_i : \mathcal{G}_i \longrightarrow G/U$ and a continuous map $\beta'_i : \Gamma_i \longrightarrow G/U$ by $\beta_i = \beta_{i_0}\varphi_{i,i_0}$ , $\beta'_i = \beta'_{i_0}\varphi'_{i,i_0}$, where

$$\varphi_{i,i_0} : (\mathcal{G}_i, \Gamma_i) \longrightarrow (\mathcal{G}_{i_0}, \Gamma_{i_0}), \varphi'_{i,i_0} : \Gamma_i \longrightarrow \Gamma_{i_0}$$

These morphisms of the corresponding inverse systems are represented in the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}_i \arrow[r, "\varphi_{i,i_0}"] \arrow[d] & \mathcal{G}_{i_0} \arrow[d] \\ \Gamma_i \arrow[r, "\varphi'_{i,i_0}"'] & \Gamma_{i_0} \end{tikzcd}$$

Then

$$f_U\nu_U = \beta_i\varphi_i, f_U\nu'_U\chi = \beta'_i\varphi'_i \qquad (*)$$

for every $i \geqslant i_0$.

Without loss of generality, we assume from now on that all $i \geqslant i_0$. In particular, $\beta_i$ is injective on $\mathcal{G}_i(m)$ for all $m \in \Gamma_i$ and for each $i \in I$. It follows that the natural morphisms $\varphi_i : \mathcal{G}_U \longrightarrow \mathcal{G}_i$ are injective on fibers. Thus we have the following commutative diagram:

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi'_i"] \arrow[d, "\chi"'] & \Gamma_i \arrow[dd, "\beta'_i"] \\ \pi_1^{\mathcal{C}}(\Gamma) \arrow[d, "\nu'_U"'] & \\ \Pi_1^{\mathcal{C}}(\mathcal{G}_U, \Gamma) \arrow[r, "f_U"] & G/U \\ \mathcal{G}_U \arrow[u, "\nu_U"] \arrow[r, "\varphi_i"'] & \mathcal{G}_i \arrow[u, "\beta_i"'] \end{tikzcd}$$

Define a graph of groups structure on $(\mathcal{G}_i, \Gamma_i)$ by considering $\partial_j\varphi_i(g) = \varphi_i\partial_j(g)$, $g \in \mathcal{G}_i(m)$ on $(\mathcal{G}_i, \Gamma_i)$, $(j = 0, 1)$. We show that $\partial_j$ is well-defined on $(\mathcal{G}_i, \Gamma_i)$, $(j = 0, 1)$.

Choose $g \in \mathcal{G}_U(e)$, $h \in \mathcal{G}(e')$ such that $\varphi_i'(e) = \varphi_i'(e')$ and $\varphi_i(g) = \varphi_i(h)$. Since $f_U\nu_U$ is injective on $\mathcal{G}_U(m)$ and $\beta_i\varphi_i(x) = f_U\nu_U(x)$ for every $x \in \mathcal{G}_U$ by $(*)$, it suffices to show that $f_U\nu_U\partial_j(g) = f_U\nu_U\partial_j(h)$ for $j = 0, 1$. Since $\nu_U(x) = \nu_U\partial_0(x)$ for any $x \in \mathcal{G}_U$ we have

$$f_U\nu_U\partial_0(g) = f_U\nu_U(g) = \beta_i\varphi_i(g) = \beta_i\varphi_i(h) = f_U\nu_U(h) = f_U\nu_U\partial_0(h); \qquad (**)$$

so $\partial_0$ is well-defined. Now we need to show that $f_U\nu_U\partial_1(g) = f_U\nu_U\partial_1(h)$. But

$$f_U\nu_U\partial_1(g) = f_U((\nu_U'\chi(e))^{-1}\nu_U(g)\nu_U'\chi(e)) = f_U(\nu_U'\chi(e)^{-1})f_U(\nu_U(g))f_U\nu_U'\chi(e) \overset{(*)}{=}$$

$$\overset{(*)}{=} (\beta_i'\varphi_i'(e))^{-1}f_U\nu_U(g)\beta_i'\varphi_i'(e) \overset{(**)}{=} (\beta_i'\varphi_i'(e))^{-1}f_U\nu_U(h)\beta_i'\varphi_i'(e) =$$

$$= (\beta_i'\varphi_i'(e'))^{-1}f_U\nu_U(h)\beta_i'\varphi_i'(e') \overset{(*)}{=} f_U(\nu_U'\chi(e'))^{-1}\nu_U(h)\nu_U'\chi(e')) = f_U\nu_U\partial_1(h)$$

as desired.

It remains to observe that $\partial_i$ are injective on $(\mathcal{G}_i, \Gamma_i)$ since $f_U\nu_U$ is injective on $\mathcal{G}_U(m)$ for all $m \in \Gamma$. Thus $(\mathcal{G}_U, \Gamma) = \varprojlim_i(\mathcal{G}_i, \Gamma_i)$ is a decomposition as an inverse limit of finite graphs of finite $\mathcal{C}$-groups as required. $\square$

**Remark 2.1.5.** The hypothesis of injectivity of $(\mathcal{G}, \Gamma)$ If $\mathcal{C}$ consists of all finite groups, then $(\mathcal{G}, \Gamma)$ is automatically injective. Indeed, in this case $\Pi_1^{abs}(\mathcal{G}_i, \Gamma_i)$ is virtually free and so $\Pi_1(\mathcal{G}_i, \Gamma_i) = \widehat{\Pi_1^{abs}(\mathcal{G}_i, \Gamma_i)}$; thus $(\mathcal{G}_i, \Gamma_i)$ is injective for each $i \in I$. It follows that $(\mathcal{G}, \Gamma) = \varprojlim_i(\mathcal{G}_i, \Gamma_i)$ is injective.

Combining Proposition 2.1.4 and Remark 2.1.5 we can state the following

**Corollary 2.1.6.** *Let $(\mathcal{G}, \Gamma)$ be a profinite graph of profinite groups. Then $(\mathcal{G}, \Gamma)$ decomposes as an inverse limit $(\mathcal{G}, \Gamma) = \varprojlim_{i\in I}(\mathcal{G}_i, \Gamma_i)$ of finite graphs of finite groups if and only if $(\mathcal{G}, \Gamma)$ is injective.*

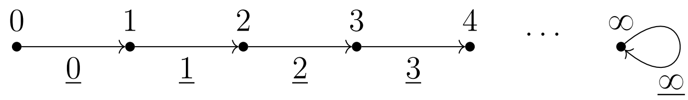
We finish the section with a very interesting example. It is well known that every connected abstract graph has a spanning subtree (cf. Theorem 1.1.24), but this is not always true for profinite graphs. Indeed, the next example shows a connected profinite graph with no spanning $\mathcal{C}$-simply connected profinite subgraph (cf. Definition 1.3.4). Hence, it is not possible to define the fundamental group of a graph of groups via maximal trees, as done in Section 1.3.3 through Construction 1.3.5 and Theorem 1.3.6. In the next chapter, we construct an infinite profinite graph of groups that has a maximal subtree.

**Example 2.1.7** ([31], Example 3.4.1)**.** Let $N = \{0, 1, 2, \cdots\}$ be the set of natural numbers with the discrete topology and let $\overline{N} = N \cup \{\infty\}$ be the one-point compactification of $N$. Define a profinite graph $\Gamma = \overline{N} \times \{0, 1\}$ with space of vertices and edges

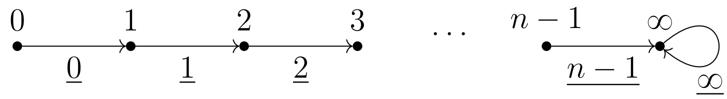
- $V(\Gamma) = \{i = (i, 0) \mid i \in \overline{N}\}$;
- $E(\Gamma) = \{\underline{i} = (i, 1) \mid i \in \overline{N}\}$;
- $d_0(\underline{i}) = i$ for $\tilde{n} \in E(\Gamma)$ and $d_0(\underline{i}) = i$ for $\underline{i} \in V(\Gamma)$;
- $d_1(\underline{i}) = i + 1$ for $\underline{i} \in E(\Gamma)$ and $d_1(i) = i$ for $i \in V(\Gamma)$.

where $\infty + 1 = \infty$.

So we can represent $\Gamma$ as follows:

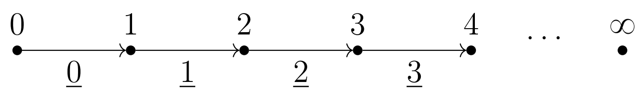



Observe that $V(\Gamma)$ and $E(\Gamma)$ are disjoint and they are both profinite spaces, because they are both clopen. Note that $\Gamma$ is the inverse limit of the following finite connected graphs $\Gamma(n)$ $(n \geqslant 0)$




where the canonical map $\Gamma(n + 1) \to \Gamma(n)$ sends $\underline{i}$ to $\underline{i}$ identically, if $i \leqslant n - 1$, and it sends $\underline{n}$ and $\underline{\infty}$ to $\underline{\infty}$. Hence $\Gamma$ is a connected profinite graph. We claim that any infinite connected profinite subgraph $\Gamma'$ of $\Gamma$ coincides with $\Gamma$; first note that the profinite graph $\Delta$ defined by

- $V(\Delta) = \{i = (i, 0) \mid i \in \overline{N}\}$;
- $E(\Delta) = \{\underline{i} = (i, 1) \mid i \in N\}$;
- $d_0(\underline{i}) = i$ for $\tilde{n} \in E(\Delta)$ and $d_0(\underline{i}) = i$ for $\underline{i} \in V(\Delta)$;
- $d_1(\underline{i}) = i + 1$ for $\underline{i} \in E(\Delta)$ and $d_1(i) = i$ for $i \in V(\Delta)$.



is not a spanning profinite subgraph of $\Gamma$, because it is not a closed subset of $\Gamma$. Indeed, suppose by contradiction that $\Delta$ is closed in $\Gamma$. Since $V(\Delta) = V(\Gamma)$ and it is a closed subset of $\Gamma = V(\Gamma) \cup E(\Gamma)$, we only have to verify that $E(\Delta)$ is closed in $E(\Gamma)$. But $E(\Gamma)$ is closed in $\Gamma$ (because it is compact), so it is profinite. Thus, $E(\Delta)$ must be compact, absurd. Therefore, $\Delta$ is not a profinite subgraph of $\Gamma$.

To prove the claim, since $\Gamma'$ is connected and contains all the vertices of $\Gamma$, it must contain all the edges of the form $\underline{i}$ $(i = 0, 1, \cdots)$; therefore since $\Gamma'$ is compact, it also contains $\underline{\infty}$; this proves the claim. On the other hand, if $\mathcal{C}$ is a pseudovariety of finite groups, we see that $\pi_1^{\mathcal{C}}(\Gamma) \cong Z_{\hat{\mathcal{C}}}$ (cf. Example 1.3.12). Hence $\Gamma$ does not contain any spanning $\mathcal{C}$-simply connected profinite subgraph.

## 2.2 The fundamental pro-$\mathcal{C}$ group of $(\mathcal{G}, \Gamma)$ with a base point

We concluded in the previous section that given an injective profinite graph $(\mathcal{G}, \Gamma)$ of pro-$\mathcal{C}$ groups, we can decompose it as an inverse limit $(\mathcal{G}, \Gamma) = \varprojlim_{i\in I}(\mathcal{G}_i, \Gamma_i)$, where each $(\mathcal{G}_i, \Gamma_i)$ is a finite graph of finite $\mathcal{C}$-groups. The goal of this section is to define the pro-$\mathcal{C}$ fundamental group of $(\mathcal{G}, \Gamma)$ with a base point via an inverse limit of completions of its finite abstract analogs, i.e.,

$$\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v) = \varprojlim_{i\in I}(\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_{\hat{\mathcal{C}}},$$

where $\pi_1(\mathcal{G}_i, \Gamma_i, v_i)$ is the usual abstract fundamental group of $(\mathcal{G}_i, \Gamma_i)$ with a base point $v \in V(\Gamma_i)$.

This is a new concept in the Ribes-Zalesski theory and has the advantage of behaving better with respect to inverse limits and morphisms of graphs of groups. We highlight that it is not possible to use the classical definition of the fundamental group of graph of groups with respect to a maximal subtree, because the image of a maximal subtree under an epimorphism of graphs is not a maximal subtree in general. We start with the classical definition of the fundamental group of a graph of groups with a base point in the abstract case (cf. [37, Sect. I.5.1]).

**Definition 2.2.1** (The group $F(\mathcal{G}, \Gamma)$ [37, Sect. I.5.1])**.** Let $(\mathcal{G}, \Gamma)$ be an abstract graph of groups. The path group $F(\mathcal{G},\Gamma)$ is defined by $F(\mathcal{G},\Gamma) = W_1/N$, where $W_1 = \big(*_{v\in V(\Gamma)}\mathcal{G}(v)\big) * F(E(\Gamma))$, where $F(E(\Gamma))$ denotes the free group with basis $E(\Gamma)$ and $N$ is a normal subgroup of $W_1$ generated by the set $\{\partial_0(x)^{-1}e\partial_1(x)e^{-1} \mid x \in \mathcal{G}(e), e \in E(\Gamma)\}$.

**Definition 2.2.2** (Words of $F(\mathcal{G},\Gamma)$ [37, Sec. I.5.1, Definition 9])**.** Let $c : v_0, e_0, \cdots, e_n, v_n$, be a path in $\Gamma$ with length $n = l(c)$ such that $v_j \in V(\Gamma), e_j \in E(\Gamma)$, $j = 0, \cdots, n$. A word of type $c$ in $F(\mathcal{G},\Gamma)$ is a pair $(c,\mu)$ where $\mu = (g_0, \cdots, g_n)$ is a sequence of elements $g_j \in \mathcal{G}(v_j)$. The element $|c,\mu| : g_0, e_0, g_1, e_1, \cdots, e_n, g_n$ of $F(\mathcal{G},\Gamma)$ is said to be associated with the word $(c,\mu)$.

**Definition 2.2.3** (The fundamental group of $(\mathcal{G},\Gamma)$ [37, Sect. I.5.1, Definition 9(a)])**.** Let $v$ be a vertex of $\Gamma$. We define $\pi_1(\mathcal{G},\Gamma,v)$ as the set of elements of $F(\mathcal{G},\Gamma)$ of the form $|c,\mu|$, where $c$ is a path whose extremities both equal $v$.

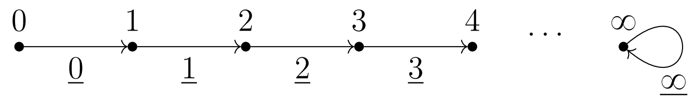
One sees immediately that $\pi_1(\mathcal{G},\Gamma,v)$ is a subgroup of $F(\mathcal{G},\Gamma)$, called the fundamental group of $(\mathcal{G},\Gamma)$ at $v$. In particular, if $\mathcal{G}$ consists of trivial groups only then $\pi_1(\mathcal{G},\Gamma,v)$ becomes the usual fundamental group of the graph $\Gamma$ and denoted by $\pi_1(\Gamma,v)$. It can be viewed of course as a subgroup that consists of set of elements of $F(\mathcal{G},\Gamma)$ of the form $|c,\mu| : g_0, e_0, g_1, e_1, \cdots, e_n, g_n$, where $c$ is a path whose extremities both equal $v$ and $g_0 = 1 = g_1 = \ldots = g_n$. This way $G = \pi_1(\mathcal{G},\Gamma,v)$ is a semidirect product $\pi_1(\mathcal{G},\Gamma,v) = \langle \mathcal{G}(v) \mid v \in V(\Gamma)\rangle^G \rtimes \pi_1(\Gamma,v)$.

Now we are able to implement the first step of our construction: the notion of $\pi_1^{\mathcal{C}}(\Gamma,v)$. Compare this setting to Section 1.3.3, where we defined the fundamental group of a profinite graph, $\pi_1^{\mathcal{C}}(\Gamma)$ as the group associated with the universal Galois covering of $\Gamma$.

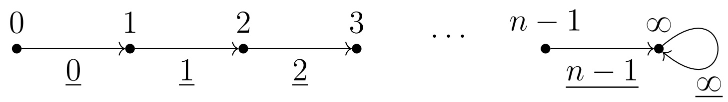
**Definition 2.2.4.** If $\Gamma$ is a connected finite graph, its pro-$\mathcal{C}$ fundamental group $\pi_1^{\mathcal{C}}(\Gamma,v)$ can be defined as the pro-$\mathcal{C}$ completion $\pi_1(\Gamma,v)_{\hat{\mathcal{C}}}$ of $\pi_1(\Gamma,v)$. If $\Gamma$ is a connected profinite graph and $\Gamma = \varprojlim \Gamma_i$ its decomposition as an inverse limit of finite graphs $\Gamma_i$, then $\pi_1^{\mathcal{C}}(\Gamma,v)$ can be defined as the inverse limit $\pi_1^{\mathcal{C}}(\Gamma,v) = \varprojlim \pi_1^{\mathcal{C}}(\Gamma_i,v_i)$, where $v_i$ is the image of $v$ in $\Gamma_i$ (see [31, Proposition 3.3.2 (b)]).

With this Construction in hand, we obtain the fundamental group $\pi_1^{\mathcal{C}}(\Gamma,v)$, where $\Gamma$ is the infinite profinite graph of Example 2.1.7 that does not have a maximal subtree.

**Example 2.2.5.** Our graph $\Gamma$ is as follows




It can be written as the inverse limit of the following finite connected graphs $\Gamma(n)$ $(n \geqslant 0)$




We write then

$$\Gamma = \varprojlim_{n \in N} \Gamma(n).$$

As we obtained in Example 1.3.12, the fundamental group of $\Gamma(n)$ is

$$\pi_1^{\mathcal{C}}(\Gamma(n)) = Z_{\hat{\mathcal{C}}}$$

for every $n \in N$. Choose $\{0\}$ as the base point of $\Gamma$. It projects to itself on each $\Gamma(n)$. Then, by Definition 2.2.4,

$$\begin{aligned} \pi_1^{\mathcal{C}}(\Gamma, \{0\}) &= \varprojlim \pi_1^{\mathcal{C}}(\Gamma(n), \{0\}) \\ &= \varprojlim Z_{\hat{\mathcal{C}}} \\ &= Z_{\hat{\mathcal{C}}}. \end{aligned}$$

Hence we obtained the fundamental group $\pi_1^{\mathcal{C}}(\Gamma) = Z_{\hat{\mathcal{C}}}$ of an infinite connected profinite graph that does not have a maximal subtree (cf. Example 2.1.7). It is a novelty and was not possible before, using the classical setting present in Section 1.3.3.

We shall use a similar approach to define the fundamental pro-$\mathcal{C}$ group $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v)$ with a base point $v$. However, this next step is more subtle and requires the following Proposition by Hyman Bass:

**Proposition 2.2.6** ([3, Proposition 2.4])**.** *A morphism of graphs of groups*

$$\underline{\alpha} = (\alpha, \alpha') : (\mathcal{G}, \Gamma) \to (\mathcal{G}', \Gamma')$$

*induces a morphism of fundamental groups*

$$\beta : \pi_1(\mathcal{G}, \Gamma, v) \to \pi_1(\mathcal{G}', \Gamma', v')$$

*defined by* $\beta(|c, \mu|) = |\alpha'(c), \alpha(\mu)|$*, where* $\alpha(\mu) = (\alpha(g_0), \alpha(g_1), \cdots, \alpha(g_n))$*,* $g_i \in \mathcal{G}(v_i)$ *and* $\beta(v) = \alpha'(v) = v'$*.*

Since we are always interested to recover data via inverse limits, these induced morphisms allow us to obtain induced inverse limits, as follows:

**Proposition 2.2.7.** *An inverse limit* $(\mathcal{G}, \Gamma) = \varprojlim_{i \in I}(\mathcal{G}_i, \Gamma_i)$ *of finite abstract graphs of finite groups (that are in* $\mathcal{C}$*) induces an inverse limit* $\varprojlim_{i \in I}(\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_{\hat{\mathcal{C}}}$ *of the pro-*$\mathcal{C}$ *completions of fundamental abstract groups* $\pi_1(\mathcal{G}_i, \Gamma_i, v_i)$*.*

*Proof.* Let $\{(\mathcal{G}_i, \Gamma_i), \underline{\alpha}_{ij}, I\}$ be the corresponding inverse system of finite abstract graphs of groups. By Proposition 2.2.6, the morphism $\underline{\alpha}_{ij} : (\mathcal{G}_i, \Gamma_i) \to (\mathcal{G}_j, \Gamma_j)$ induces a morphism of fundamental groups $\beta_{ij} : \pi_1(\mathcal{G}_i, \Gamma_i, v_i) \to \pi_1(\mathcal{G}_j, \Gamma_j, v_j)$ defined by $\beta(|c, \mu|) = |\alpha'(c), \alpha(\mu)|$, where $\alpha(\mu) = (\alpha(g_0), \alpha(g_1), \cdots, \alpha(g_n))$, $g_j \in G(v_j)$ and $\beta(v_i) = \alpha'(v_i) = v_i'$.

Therefore, we can construct an inverse system $\{(\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_{\hat{\mathcal{C}}}, \beta_{ij}, I\}$. Its inverse limit is $\varprojlim_{i \in I}(\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_{\hat{\mathcal{C}}}$, as desired. □

We are ready to define the pro-$\mathcal{C}$ fundamental group $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v)$ of a profinite graph of pro-$\mathcal{C}$ groups with base vertex $v$ of $\Gamma$. Note that it works only for an injective profinite graph of pro-$\mathcal{C}$-groups (see Remark 2.1.5). On the other hand, given a graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ that is not injective, it can become injective by replacing $\mathcal{G}(m)$ with its image $\nu(\mathcal{G}(m))$ in $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ for every $m \in \Gamma$ (cf. Remark 1.6.8). This means that we do not lose generality restricting our attention to injective profinite graphs of pro-$\mathcal{C}$ groups.

**Definition 2.2.8.** Let $(\mathcal{G}, \Gamma)$ be an injective profinite graph of pro-$\mathcal{C}$ groups and $(\mathcal{G}, \Gamma) = \varprojlim_{i \in I}(\mathcal{G}_i, \Gamma_i)$ be the decomposition as the inverse limit of finite graphs of finite $\mathcal{C}$-groups (See Proposition 2.1.4). Let $v$ be a vertex of $\Gamma$. The group $\varprojlim_{i \in I}(\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_{\hat{\mathcal{C}}}$ from Proposition 2.2.7 will be called the pro-$\mathcal{C}$ fundamental group of the graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ at the point $v$ and denoted by $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v)$.

The next proposition shows that the definition of $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v)$ does not depend on the decomposition of $(\mathcal{G}, \Gamma)$ as an inverse limit of finite graphs of finite groups. By [31, Proposition 6.5.1], we have that $\Pi_1^{\mathcal{C}}(\mathcal{G}_i, \Gamma_i, v_i) = (\pi_1(\mathcal{G}_i, \Gamma_i, v_i))_{\hat{\mathcal{C}}}$.

**Proposition 2.2.9.** *Let $(\mathcal{G},\Gamma)$ be an injective profinite graph of pro-$\mathcal{C}$-groups and $(\mathcal{G},\Gamma)=\varprojlim_{i\in I}(\mathcal{G}_i,\Gamma_i)$ be a decomposition as an inverse limit of finite graphs of finite $\mathcal{C}$-groups. Then*

$$\Pi_1^{\mathcal{C}}(\mathcal{G},\Gamma,v)=\varprojlim_{i\in I}\Pi_1^{\mathcal{C}}(\mathcal{G}_i,\Gamma_i,v_i).$$

*Proof.* Let $\underline{\eta}:(\mathcal{G},\Gamma)\longrightarrow(\mathcal{H},\Delta)$ be an epimorphism to a finite graph of $\mathcal{C}$-groups $(\mathcal{H},\Delta)$. Choose $v\in V(\Gamma)$ and let $v_0$ be its image in $\Delta$. It suffices to show that the natural epimorphism $\Pi_1^{\mathcal{C}}(\mathcal{G},\Gamma,v)\longrightarrow\Pi_1^{\mathcal{C}}(\mathcal{H},\Delta,v_0)$ factors through some $\Pi_1^{\mathcal{C}}(\mathcal{G}_i,\Gamma_i,v_i)$.

Since $(\mathcal{H},\Delta)$ is finite, by Lemma 2.1.1 $\underline{\eta}$ factors through some $\underline{\eta}_i:(\mathcal{G}_i,\Gamma_i)\longrightarrow(\mathcal{H},\Delta)$, i.e. $\underline{\eta}=\underline{\eta}_i\underline{\pi}_i$, where $\underline{\pi}_i:(\mathcal{G},\Gamma)\longrightarrow(\mathcal{G}_i,\Gamma_i)$ is the natural projection.

$$\begin{tikzcd}
(\mathcal{G},\Gamma) \arrow[rr, "\underline{\eta}"] \arrow[dr, "\underline{\pi}_i"'] & & (\mathcal{H},\Delta) \\
& (\mathcal{G}_i,\Gamma_i) \arrow[ur, dotted, "\underline{\eta}_i"'] &
\end{tikzcd}$$

Let $v_i$ be the image of $v$ in $\Gamma_i$. By Proposition 2.2.6 $\underline{\eta}_i$ induces the natural homomorphism $\pi_1(\mathcal{G}_i,\Gamma_i,v_i)\longrightarrow\pi_1(\mathcal{H},\Delta,v_0)$ that in turn induces the homomorphism of pro-$\mathcal{C}$ completions

$$\Pi_1^{\mathcal{C}}(\mathcal{G}_i,\Gamma_i,v_i)=(\pi_1(\mathcal{G}_i,\Gamma_i,v_i))_{\hat{\mathcal{C}}}\longrightarrow(\pi_1(\mathcal{H},\Delta,v_0))_{\hat{\mathcal{C}}}=\Pi_1^{\mathcal{C}}(\mathcal{H},\Delta,v_0).$$

Thus we have a diagram

$$\begin{tikzcd}
\Pi_1^{\mathcal{C}}(\mathcal{G},\Gamma,v) \arrow[d] \arrow[dr] & \\
\Pi_1^{\mathcal{C}}(\mathcal{G}_i,\Gamma_i,v_i) \arrow[r] & \Pi_1^{\mathcal{C}}(\mathcal{H},\Delta,v_0)
\end{tikzcd}$$

that commutes on vertex groups and underlying graphs. Hence this diagram commutes and the proof is finished.

□

We shall show now that for injective $(\mathcal{G},\Gamma)$ this definition and Definition 1.6.7 are equivalent.

**Theorem 2.2.10.** *Let $(\mathcal{G},\Gamma)$ be an injective profinite graph of pro-$\mathcal{C}$ groups. Then*

$$\Pi_1^{\mathcal{C}}(\mathcal{G},\Gamma,v)\cong\Pi_1^{\mathcal{C}}(\mathcal{G},\Gamma).$$

*Proof.* Let $\widetilde{\Gamma}$ be the universal $\mathcal{C}$-covering of the profinite graph $\Gamma$ and $\zeta : \widetilde{\Gamma} \to \Gamma$ be the covering map.

Consider a graph of groups $(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$ from Definition 1.4.8. Choose a base point $v \in \Gamma$. The action of $\pi_1^{\mathcal{C}}(\Gamma, v)$ on $\widetilde{\Gamma}$ induces the action on $(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$ that in turn induces the action of $\pi_1^{\mathcal{C}}(\Gamma, v)$ on $\Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$, i.e. we have a semidirect product $\Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}, \widetilde{\Gamma}) \rtimes \pi_1(\Gamma, v)$.

Choose $\tilde{v} \in \widetilde{\Gamma}$ such that $\zeta(\tilde{v}) = v$. Let $(\mathcal{G}, \Gamma) = \varprojlim_i (\mathcal{G}_i, \Gamma_i)$ be a decomposition of $\Gamma$ as the inverse limit of finite quotient graphs of groups $(\mathcal{G}_i, \Gamma_i)$ (cf. Proposition 2.1.4). By Proposition 3.3.2 [31] the decomposition $\Gamma = \varprojlim_i \Gamma_i$ induces an inverse system of pairs of compatible morphisms $\zeta_i : \widetilde{\Gamma}_i \to \Gamma_i$, $f_{ij} : \pi_1(\Gamma_i) \to \pi_1(\Gamma_j)$ such that $\pi_1^{\mathcal{C}}(\Gamma) = \varprojlim_i \pi_1^{\mathcal{C}}(\Gamma_i, v_i)$, $\zeta = \varprojlim_i \zeta_i$, where $v_i$ is the image of $v$ in $\Gamma_i$. This defines a decomposition as the inverse limit $(\widetilde{\mathcal{G}}, \widetilde{\Gamma}) = \varprojlim_i (\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i)$ (as $(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i)$ is defined as a pull-back of $\mathcal{G}_i \longrightarrow \Gamma_i$ and $\zeta_i : \widetilde{\Gamma}_i \longrightarrow \Gamma$, see Definition 1.4.8).

Let $\tilde{v}_i$ be the image of $\tilde{v}$ in $\widetilde{\Gamma}_i$ and $c : v_i = v_0, e_0, \cdots, e_n, v_n = v_i$, be a circuit of length $n$ such that $v_j \in V(\Gamma), e_j \in E(\Gamma)$, $j = 0, \cdots, n$ that we regard also as an element of $\pi_1(\Gamma_i, v_i)$. Let $\widetilde{\Gamma}_i^{abs}$ be the connected component of $\widetilde{\Gamma}_i$ (regarded as an abstract graph) containing $\tilde{v}_i$. Then $\widetilde{\Gamma}_i^{abs}$ is the usual universal cover of $\Gamma_i$ (see [31, Proposition 8.2.4]). It follows that the circuit $c$ lifts to the unique path $\tilde{c} : \tilde{v}_i = \tilde{v}_0, \tilde{e}_0, \ldots, \tilde{e}_n, \tilde{v}_n = c\tilde{v}_i$ from $\tilde{v}_i$ to $c\tilde{v}_i$ in $\widetilde{\Gamma}_i^{abs}$.

Denote by $(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs})$ the graph of groups obtained by the restriction of $(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i)$ to $\widetilde{\Gamma}_i^{abs}$ and let $\pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, \tilde{v}_i)$ be its fundamental group. Then $\pi_1(\Gamma_i, v_i)$ acts naturally on $(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs})$ that induces the action of $\pi_1(\Gamma_i, v_i)$ on $\pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, \tilde{v}_i)$ so that we can consider the semidirect product $\pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, \tilde{v}_i) \rtimes \pi_1(\Gamma_i, v_i)$.

An element $|c, \mu| : g_0, e_0, g_1, e_1, \cdots, e_n, g_n$ in $\pi_1(\mathcal{G}_i, \Gamma_i, v_i)$, where $\mu = (g_0, \cdots, g_n)$ is a sequence of elements $g_j \in \mathcal{G}(v_j)$, lifts to a unique element $(\tilde{\mu}, c) \in \pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, \tilde{v}_i) \rtimes \pi_1(\Gamma_i, v_i)$, where $\tilde{\mu} = \tilde{g}_0 \tilde{g}_1 \cdots \tilde{g}_n \in \pi_1(\widetilde{\mathcal{G}}, \widetilde{\Gamma}, \tilde{v}_i)$ with $\tilde{g}_j = \tilde{\zeta}^{-1}_{|\widetilde{\mathcal{G}}(\tilde{v}_j)}(g_j)$.

Hence we can define a map $\psi_i : \pi_1(\mathcal{G}_i, \Gamma_i, v_i) \to \pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, \tilde{v}_i) \rtimes \pi_1(\Gamma_i, v_i)$ by $\psi_i(|c, \mu|) = (\tilde{\mu}, c)$. To see that $\psi_i$ is a homomorphism, let $|c, \mu| = |c_1, \mu_1||c_2, \mu_2|$ and let $\tilde{c}_1$, $\tilde{c}_2$ be liftings of $c_1$, $c_2$ respectively. Then $\psi_i(|c, \mu|) = (\tilde{\mu}, c) = (\tilde{\mu}_1(c_1 \tilde{\mu}_2 c_1^{-1}), c_1 c_2)$ as needed.

Thus $\psi_i$ is a homomorphism that is clearly bijective, i.e. $\psi_i$ is an isomorphism. Moreover, for $i > j$ the commutative diagram

$$
\begin{tikzcd}
\widetilde{\Gamma}_j \arrow[r] \arrow[d, "\zeta_j"'] & \widetilde{\Gamma}_i \arrow[d, "\zeta_i"] \\
\Gamma_j \arrow[r] & \Gamma_i
\end{tikzcd}
$$

induces the commutative diagram

$$
\begin{tikzcd}
\pi_1(\mathcal{G}_j, \Gamma_j, v_j) \arrow[r] \arrow[d, "\psi_j"'] & \pi_1(\mathcal{G}_i, \Gamma_i, v_i) \arrow[d, "\psi_i"] \\
\pi_1(\widetilde{\mathcal{G}}_j, \widetilde{\Gamma}_j^{abs}, v_j) \rtimes \pi_1(\Gamma_j, v_j) \arrow[r] & \pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, v_i) \rtimes \pi_1(\Gamma_i, v_i)
\end{tikzcd}
$$

It is straightforward to check that the pro-$\mathcal{C}$ completion of $\pi_1(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i^{abs}, \tilde{v}_i) \rtimes \pi_1(\Gamma_i, v_i)$ gives $\Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i, \tilde{v}) \rtimes \pi_1^{\mathcal{C}}(\Gamma_i, v_i)$ as defined above. This shows that $\psi_i$ induces the isomorphism

$$\hat{\psi}_i : \Pi_1^{\mathcal{C}}(\mathcal{G}_i, \Gamma_i, v_i) \to \Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i, \tilde{v}_i) \rtimes \pi_1^{\mathcal{C}}(\Gamma_i, v_i)$$

and we have the following commutative diagram

$$
\begin{tikzcd}
\Pi_1^{\mathcal{C}}(\mathcal{G}_j, \Gamma_j, v_j) \arrow[r] \arrow[d, "\hat{\psi}_j"'] & \Pi_1^{\mathcal{C}}(\mathcal{G}_i, \Gamma_i, v_i) \arrow[d, "\hat{\psi}_i"] \\
\Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}_j, \widetilde{\Gamma}_j, \tilde{v}_j) \rtimes \pi_1^{\mathcal{C}}(\Gamma_j, v_j) \arrow[r] & \Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}_i, \widetilde{\Gamma}_i, \tilde{v}_i) \rtimes \pi_1^{\mathcal{C}}(\Gamma_i, v_i)
\end{tikzcd}
$$

Then $\psi = \varprojlim_i \hat{\psi}_i$ is an isomorphism $\psi : \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v) \longrightarrow \Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}, \widetilde{\Gamma}, \tilde{v}) \rtimes \pi_1^{\mathcal{C}}(\Gamma, v)$. Note that $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) \cong \Pi_1^{\mathcal{C}}(\widetilde{\mathcal{G}}, \widetilde{\Gamma}) \rtimes \pi_1^{\mathcal{C}}(\Gamma, v)$ by [31, Proposition 6.5.1]).

Finally, we take a closer look on the map $\beta : \mathcal{G} \to \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v)$. If we go back to the finite case for a moment, since $\Gamma_i$ is finite, we follow Example 1.4.4, so

$$\mathcal{G}_i = \bigcup_{v_j \in V(\Gamma)} \mathcal{G}_i(v_j)$$

we can define a map

$$\beta_i : \mathcal{G}_i \to \pi_1(\mathcal{G}_i, \Gamma_i, v_i)$$

by

$$\beta_i(\mathcal{G}_i(v_j)) = |v_j, \mu|,$$

where $v_j \in V(\Gamma)$. This map is clearly injective.

On the other hand, define

$$\nu_i : \mathcal{G}_i \to \pi_1(\mathcal{G}_i, \Gamma_i, v_i)$$

by the natural inclusion

$$\nu_i(\mathcal{G}_i(v_j)) = (\mathcal{G}_i(v_j)),$$

where $v_j \in V(\Gamma)$. This map is also clearly injective. The universal property of $\pi_1(\mathcal{G}_i, \Gamma_i)$ provides us the following commutative diagram

$$\begin{tikzcd}
& \pi_1(\mathcal{G}_i, \Gamma_i) \arrow[dd, dotted, "\delta_i"'] & \\
\mathcal{G}_i \arrow[ur, "\nu_i"] \arrow[dr, "\beta_i"'] & & \pi_1(\Gamma_i, v_i) \arrow[ul, "\nu_i'"'] \arrow[dl, "\beta_i'"] \\
& \pi_1(\mathcal{G}_i, \Gamma_i, v_i) &
\end{tikzcd}$$

Now define

$$\delta_i : \pi_1(\mathcal{G}_i, \Gamma_i) \to \pi_1(\mathcal{G}_i, \Gamma_i, v_i)$$

by

$$\delta_i(\mathcal{G}_i(v_j)) = |v_j, \mu|.$$

Its kernel is trivial, since the only vertex group that maps to the identity on $\pi_1(\mathcal{G}_i, \Gamma_i, v_i)$ is the trivial group. Hence $\delta_i : \pi_1(\mathcal{G}_i, \Gamma_i) \to \pi_1(\mathcal{G}_i, \Gamma_i, v_i)$ is an isomorphism. Define now $\beta = \varprojlim_i \beta_i$, $\nu = \varprojlim_i \nu_i$ and $\delta = \varprojlim_i \delta_i$. The universal property of $\Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ provides us the following commutative diagram

$$\begin{tikzcd}
& \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) = \Pi_1^{\mathcal{C}}(\tilde{\mathcal{G}}, \tilde{\Gamma}) \rtimes \pi_1^{\mathcal{C}}(\Gamma, v) \arrow[dd, dotted, "\delta"'] & \\
\mathcal{G} \arrow[ur, "\nu"] \arrow[dr, "\beta"'] & & \pi_1^{\mathcal{C}}(\Gamma, v) \arrow[ul, "\nu'"'] \arrow[dl, "\beta'"] \\
& \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v) = \Pi_1^{\mathcal{C}}(\tilde{\mathcal{G}}, \tilde{\Gamma}, \tilde{v}) \rtimes \pi_1^{\mathcal{C}}(\Gamma, v) &
\end{tikzcd}$$

The maps $\nu'$ and $\beta'$ are the inclusions of $\pi_1^{\mathcal{C}}(\Gamma, v)$ in the semidirect products, so they are canonically defined. On the other hand, $\nu : \mathcal{G} \to \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma)$ is the universal specialization map, so it is also uniquely defined.

Therefore $\varprojlim_i \delta_i = \delta : \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma) \to \Pi_1^{\mathcal{C}}(\mathcal{G}, \Gamma, v)$ is an isomorphism, as desired. □

## 2.3 The pro-$\mathcal{C}$ completion of the fundamental group $\pi_1(\mathcal{G}, \Gamma, v)$

The Ribes-Zalesski-Mel'nikov theory can be used very effectively in the study of certain abstract groups. One sees a group $G$ as the fundamental group of a graph of groups and then the profinite completion $\widehat{G}$ is the fundamental group $\widehat{G} = \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ of the same graph of the profinite completions of edge and vertex groups. Through this view, it is possible to apply geometric techniques to obtain algebraic results. Until now, one could use this approach only for finitely generated groups, assuming that the graph $\Gamma$ in the graph of groups $(\mathcal{G}, \Gamma)$ is finite. The main reason for this assumption is that $\Gamma$ is consequently a profinite graph and so $(\widehat{\mathcal{G}}, \Gamma)$ is automatically a profinite graph of pro-$\mathcal{C}$-groups, where each $\widehat{\mathcal{G}}(m)$ is the pro-$\mathcal{C}$ completion of $\mathcal{G}$. One also has a natural way of associating $\pi_1(\mathcal{G}, \Gamma)$ with $\Pi = \Pi_1(\widehat{\mathcal{G}}, \Gamma)$ and $S^{abs} = S^{abs}(\mathcal{G}, \Gamma)$ with $S = S(\widehat{\mathcal{G}}, \Gamma)$. With the aim to apply this technique to abstract groups which are not necessarily finitely generated, we construct a profinite graph of pro-$\mathcal{C}$ groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ where an infinite abstract graph $\Gamma$ in densely embedded in $\overline{\Gamma}$ and the above properties are preserved, the content of Theorem 3. We assume that $(\mathcal{G}, \Gamma)$ is reduced, i.e. whenever $e$ is an edge of $\Gamma$ which is not a loop, then $\partial_i(\mathcal{G}(e))$ is a proper subgroup of $\mathcal{G}(d_i(e))$ $(i = 0, 1)$ (cf. Remark 1.6.18). This of course does not affect the generality of the argument since for any graph of groups we can collapse fictitious edges successfully (i.e. the edges that are not loops such that $\partial_i(\mathcal{G}(e)) = \mathcal{G}(d_i(e))$ for $i = 0, 1$) to arrive at the reduced graph of groups.

We finish this section with the proof of Theorem 1 which answers Ribes Open Question 6.7.1 of [31].

Parts $(a)$, $(b)$ and $(c)$ of Theorem 3 are the subject of the following

**Theorem 2.3.1.** *Let $(\mathcal{G}, \Gamma)$ be a reduced graph of groups and $G = \pi_1(\mathcal{G}, \Gamma, v)$ its fundamental group. Assume that $G$ is residually $\mathcal{C}$ and denote by $\overline{\mathcal{G}}(m)$ the closure of $\mathcal{G}(m)$ in $G_{\hat{\mathcal{C}}}$. Then* *(a)* *There exists an injective profinite graph of pro-$\mathcal{C}$ groups $(\overline{\mathcal{G}},\overline{\Gamma})$ such that $\Gamma$ is densely embedded in $\overline{\Gamma}$;*

*(b)* *for each $m \in \Gamma$ its vertex group is $\overline{\mathcal{G}}(m)$;*

*(c)* *The fundamental pro-$\mathcal{C}$ group $\Pi = \Pi_1^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma})$ of $(\overline{\mathcal{G}},\overline{\Gamma})$ is the pro-$\mathcal{C}$ completion of $G$ and so $(\overline{\mathcal{G}},\overline{\Gamma})$ is injective.*

*(d)* *The graph of groups $(\overline{\mathcal{G}},\overline{\Gamma})$ decomposes as a surjective inverse limit $(\overline{\mathcal{G}},\overline{\Gamma}) = \varprojlim(\mathcal{G}_U,\Gamma_U)$ of finite graphs of finite $\mathcal{C}$-groups and $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma},v) = \varprojlim(\pi_1(\mathcal{G}_U,\Gamma_U,v_U))_{\hat{\mathcal{C}}}$.*

*Proof.* Let $\mathcal{U}$ be the collection of all open normal subgroups of $G$ in the pro-$\mathcal{C}$ topology of $G$. For $m \in \Gamma$, $U \in \mathcal{U}$, define $\mathcal{G}_U(m) = \mathcal{G}(m)U/U$. As $\mathcal{G}_U(m) \leqslant G/U$, one concludes that each $\mathcal{G}_U(m) \in \mathcal{C}$.

Define the profinite space $\mathcal{G}_U = \bigcup_{m\in\Gamma}\mathcal{G}_U(m)$ and $R_U$ to be the following equivalence relation in $\Gamma$: given $v,w \in V(\Gamma)$, $v \sim_{R_U} w$ if $\mathcal{G}_U(v) = \mathcal{G}_U(w)$ and given $e,e' \in E(\Gamma)$, $e \sim_{R_U} e'$ if $d_0(e) \sim_{R_U} d_0(e')$ and $d_1(e) \sim_{R_U} d_1(e')$. Hence, the quotient graph $\Gamma_U$ defined by $\Gamma_U = \Gamma/R_U$ is finite.

Define then a finite graph of finite groups, $(\mathcal{G}_U,\Gamma_U)$ by putting $\mathcal{G}_U(\bar{m}) = \mathcal{G}_U(m)$ where $\bar{m}$ is the equivalence class of $m$ and defining $\partial_i(gU/U) = \partial_i(g)U/U$. To see that the maps $\partial_i$ are well-defined, one follows precisely the argument of the third paragraph of the proof of Proposition 2.1.4 ignoring the topology.

Put $G_U = \pi_1(\mathcal{G}_U,\Gamma_U)$ and, for an open subgroup $Y \leqslant_o U$, $\underline{\alpha}_{YU} = (\alpha_{YU},\alpha'_{YU})$ : $(\mathcal{G}_Y,\Gamma_Y) \to (\mathcal{G}_U,\Gamma_U)$ to be the epimorphism of graphs of groups defined by $\alpha(gY) = gU$ and $\alpha'(mR_Y) = mR_U$, for $m \in \Gamma$ and $g \in \mathcal{G}(m)$. These are the canonical quotient maps, so we have the following commutative diagram:

$$\begin{tikzcd}
 & (\mathcal{G}_U,\Gamma_U) \arrow[dr, "\underline{\alpha}_{UW}"] & \\
(\mathcal{G}_Y,\Gamma_Y) \arrow[ur, "\underline{\alpha}_{YU}"] \arrow[dr, "\underline{\alpha}_{YZ}"'] & & (\mathcal{G}_W,\Gamma_W) \\
 & (\mathcal{G}_Z,\Gamma_Z) \arrow[ur, "\underline{\alpha}_{ZW}"'] &
\end{tikzcd}$$

Indeed, $(\alpha_{UW}\circ\alpha_{YU})(gY) = \alpha_{UW}(\alpha_{YU}(gY)) = \alpha_{UW}(gU) = gW$ and $(\alpha_{ZW}\circ\alpha_{YZ})(gY) = \alpha_{ZW}(\alpha_{YZ}(gY)) = \alpha_{ZW}(gZ) = gW$. For the graph maps,

$$(\alpha'_{UW}\circ\alpha'_{YU})(mR_Y) = \alpha'_{UW}(\alpha'_{YU}(mR_Y)) = \alpha'_{UW}(mR_U) = mR_W$$

and

$$(\alpha'_{ZW} \circ \alpha'_{YZ})(mR_Y) = \alpha'_{ZW}(\alpha'_{YZ}(mR_Y)) = \alpha'_{ZW}(mR_Z) = mR_W.$$

Hence

$$\underline{\alpha}_{UW} \circ \underline{\alpha}_{YU} = \underline{\alpha}_{ZW} \circ \underline{\alpha}_{YZ} \tag{2.2}$$

for every $m \in \Gamma$ and $g \in \mathcal{G}(m)$.

Choose a vertex $v$ in $\Gamma$ and denote by $v_U$ its image in $\Gamma_U$. By Proposition 2.2.6 this diagram induces the diagram of fundamental groups

$$\begin{tikzcd}
 & \pi_1(\mathcal{G}_U, \Gamma_U, v_U) \arrow[dr, "\beta_{UW}"] & \\
\pi_1(\mathcal{G}_Y, \Gamma_Y, v_Y) \arrow[ur, "\beta_{YU}"] \arrow[dr, "\beta_{YZ}"'] & & \pi_1(\mathcal{G}_W, \Gamma_W, v_W) \\
 & \pi_1(\mathcal{G}_Z, \Gamma_Z, v_Z) \arrow[ur, "\beta_{ZW}"'] &
\end{tikzcd}$$

Given $|c, \mu| \in \pi_1(\mathcal{G}_Y, \Gamma_Y)$, we have

$$\beta_{UW}(\beta_{YU}(|c, \mu|)) = \beta_{UW}(|\alpha'_{YU}(c), \alpha_{YU}(\mu)|) = |(\alpha'_{UW} \circ \alpha'_{YU})(c), (\alpha_{UW} \circ \alpha_{YU})(\mu)|$$

and

$$\beta_{ZW}(\beta_{YZ}(|c, \mu|)) = \beta_{ZW}(|\alpha'_{YZ}(c), \alpha_{YZ}(\mu)|) = |(\alpha'_{YZ} \circ \alpha'_{YZ})(c), (\alpha_{ZW} \circ \alpha_{YZ})(\mu)|,$$

which implies, by Equation (2.2), that

$$\beta_{UW} \circ \beta_{YU} = \beta_{ZW} \circ \beta_{YZ} \tag{2.3}$$

for every $|c, \mu| \in \pi_1(\mathcal{G}_Y, \Gamma_Y)$. Therefore the diagram commutes.

Thus we have an inverse system of graphs of groups $\{(\mathcal{G}_U, \Gamma_U), \underline{\alpha}_{YU}\}$. Put $(\overline{\mathcal{G}}, \overline{\Gamma}) = \varprojlim(\mathcal{G}_U, \Gamma_U)$. By Proposition 2.2.9, we have

$$\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma}, v) = \varprojlim(\pi_1(\mathcal{G}_U, \Gamma_U, v_U))_{\hat{\mathcal{C}}}. \tag{2.4}$$

Note that by equation (2.4), $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma},v) = G_{\hat{\mathcal{C}}}$ and so $(\overline{\mathcal{G}},\overline{\Gamma})$ is an injective profinite graph of pro-$\mathcal{C}$ groups, since $\overline{\mathcal{G}}(m)$ are naturally subgroups of $G_{\hat{\mathcal{C}}}$.

To finish the proof we show that the natural map $\alpha' : \Gamma \longrightarrow \overline{\Gamma}$ is an injection. Since $(\mathcal{G},\Gamma)$ is reduced $\mathcal{G}(v)$ and $\mathcal{G}(w)$ are distinct subgroups in $G$ whenever vertices $v$ and $w$ of $\Gamma$ are distinct. Therefore, as $G$ is residually $\mathcal{C}$, there exists $U$ such that $v \not\sim_{R_U} w$. This shows that $\alpha'_{|V(\Gamma)}$ is injective. Let $T$ be a maximal subtree of $\Gamma$. Since $T$ has a unique edge connecting two vertices, $\alpha'_T$ is also injective. But the image of the natural map $\chi : \Gamma \longrightarrow \pi_1(\Gamma)$, where $\chi(m)$ is the unique element of $\pi_1(\Gamma)$ such that $\chi(m)(jd_1(m)) = d_1 j(m)$, and that sends $T$ to 1 can be viewed as a basis of $\pi_1(\Gamma)$. Since $\pi_1(\Gamma) \leqslant G$ is residually $\mathcal{C}$ it follows that $\alpha'_{|E(\Gamma)}$ is injective as well. □

Define $\Gamma = \varprojlim_{i\in I} \Gamma_i$. As each $\Gamma_i$ is a profinite $G$-graph, we have that $V(\Gamma_i) = V(G\backslash\overline{\Gamma_i}) = G\backslash V(\overline{\Gamma_i})$, by the definition of the vertex set of a quotient graph by the action of $G$. Therefore, $V(\Gamma) = \varprojlim_{i\in I} V(\Gamma_i) = \varprojlim_{i\in I} G\backslash V(\overline{\Gamma_i}) = G\backslash(\varprojlim_{i\in I} V(\overline{\Gamma_i})) = G\backslash V(\overline{\Gamma})$, where $\overline{\Gamma} = \varprojlim_{i\in I} \overline{\Gamma_i}$.

**Corollary 2.3.2.** *We maintain the hypothesis of Theorem 2.3.1. Suppose in addition that $G$ contains a free normal subgroup $\Phi$ such that $G/\Phi \in \mathcal{C}$. Then there exists an open normal subgroup in the pro-$\mathcal{C}$ topology $V \leqslant \Phi$ such that for every open normal subgroup $U \leqslant V$ of $G$ the fundamental group $\pi_1(\mathcal{G}_U,\Gamma_U,v_U)$ that appear in (d) is residually $\mathcal{C}$. In particular, $(\mathcal{G}_U,\Gamma_U)$ is injective.*

*Proof.* Let $(\overline{\mathcal{G}},\overline{\Gamma}) = \varprojlim(\mathcal{G}_U,\Gamma_U)$, $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma},v) = \varprojlim(\pi_1(\mathcal{G}_U,\Gamma_U,v_U))_{\hat{\mathcal{C}}}$ be the decompositions of Theorem 2.3.1 (d), and choosing all $U \leqslant \Phi$. Then the natural epimorphism $\widehat{G} \longrightarrow G/\Phi$ factors via $(\pi_1(\mathcal{G}_V,\Gamma_V,v_V))_{\hat{\mathcal{C}}}$ for some $V$ and therefore via every $(\pi_1(\mathcal{G}_U,\Gamma_U,v_U))_{\hat{\mathcal{C}}}$ for $U \leqslant V$, i.e. we have a homomorphism $\psi_U : (\pi_1(\mathcal{G}_U,\Gamma_U,v_U))_{\hat{\mathcal{C}}} \longrightarrow G/\Phi$ which is injective on vertex groups. Then the kernel of its restriction on $\pi_1(\mathcal{G}_U,\Gamma_U,v_U)$ is free and so $\pi_1(\mathcal{G}_U,\Gamma_U,v_U)$ is free-by-$\mathcal{C}$ group. Thus $\pi_1(\mathcal{G}_U,\Gamma_U,v_U)$ is residually $\mathcal{C}$. □

The following theorem covers the remaining part $(d)$ of Theorem 3.

**Theorem 2.3.3.** *We continue with the hypotheses and notation of Theorem 2.3.1. Furthermore, we assume that $\mathcal{G}(m)$ is closed in the pro-$\mathcal{C}$ topology of $G$, for every $m \in \Gamma$. Then the standard tree $S^{abs} = S(\mathcal{G},\Gamma)$ of the graph of groups $(\mathcal{G},\Gamma)$ is embedded densely in the standard $\mathcal{C}$-tree $S = S^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma})$ of the profinite graph of profinite groups $(\overline{\mathcal{G}},\overline{\Gamma})$ and the action of $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma})$ on $S^{\mathcal{C}}(\overline{\mathcal{G}},\overline{\Gamma})$ extends the natural action of $\pi_1(\mathcal{G},\Gamma)$ on $S^{abs}(\mathcal{G},\Gamma)$.*

*Proof.* Let $\widetilde{\Gamma}, \widetilde{\widetilde{\Gamma}}$ be the abstract universal covering and $\mathcal{C}$-universal covering of the abstract graph $\Gamma$ and profinite graph $\overline{\Gamma}$ respectively. Let $\zeta : \widetilde{\Gamma} \to \Gamma$, $\overline{\zeta} : \widetilde{\widetilde{\Gamma}} \to \overline{\Gamma}$ be the covering and $\mathcal{C}$-covering maps. Consider a profinite graph of pro-$\mathcal{C}$-groups $(\widetilde{\widetilde{\mathcal{G}}}, \widetilde{\widetilde{\Gamma}})$ from Definition 1.4.8 and its abstract version $(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$.

If one denotes $\widetilde{G} = \pi_1(\widetilde{\mathcal{G}}, \widetilde{\Gamma}, \tilde{v_0})$, we have the standard tree $\widetilde{S^{abs}}(\widetilde{\mathcal{G}}, \widetilde{\Gamma})$, defined by $\widetilde{S^{abs}} = \bigcup_{\tilde{m}\in\widetilde{\Gamma}} \widetilde{G}/\widetilde{\mathcal{G}}(\tilde{m})$, $V(\widetilde{S^{abs}}) = \bigcup_{\tilde{v}\in V(\widetilde{\Gamma})} \widetilde{G}/\widetilde{\mathcal{G}}(\tilde{v})$ and incidence maps $d_0(g\widetilde{\mathcal{G}}(\tilde{e})) = g\widetilde{\mathcal{G}}(d_0(\tilde{e}))$ and $d_1(g\widetilde{\mathcal{G}}(\tilde{e})) = g\widetilde{\mathcal{G}}(d_1(\tilde{e}))$, $(g \in \widetilde{G}, \tilde{v} \in V(\widetilde{\Gamma}), \tilde{e} \in E(\widetilde{\Gamma}))$.

In a similar manner, denote $\widetilde{\Pi} = \Pi_1^{\mathcal{C}}(\widetilde{\widetilde{\mathcal{G}}}, \widetilde{\widetilde{\Gamma}}, \tilde{v}_0)$ and take its standard $\mathcal{C}$-tree, that is defined by $\widetilde{S} = \widetilde{S}(\widetilde{\widetilde{\mathcal{G}}}, \widetilde{\widetilde{\Gamma}}) = \bigcup_{\tilde{\tilde{m}}\in\tilde{\tilde{\Gamma}}} \widetilde{\Pi}/\widetilde{\Pi}(\tilde{\tilde{m}})$, $V(\widetilde{S}) = \bigcup_{\tilde{\tilde{v}}\in V(\tilde{\tilde{\Gamma}})} \widetilde{\Pi}/\widetilde{\Pi}(\tilde{\tilde{v}})$ and incidence maps $d_0(g\widetilde{\Pi}(\tilde{\tilde{e}})) = g\widetilde{\Pi}(d_0(\tilde{\tilde{e}}))$ and $d_1(g\widetilde{\Pi}(\tilde{\tilde{e}})) = g(\widetilde{\Pi}(d_1(\tilde{\tilde{e}}))$, $(g \in \widetilde{\Pi}, \tilde{\tilde{v}} \in V(\widetilde{\widetilde{\Gamma}}), \tilde{\tilde{e}} \in E(\widetilde{\widetilde{\Gamma}}))$.

By Theorem 2.3.1, $\Gamma$ is densely embedded in $\overline{\Gamma}$ and $\Pi = G_{\hat{\mathcal{C}}}$ , so there are natural inclusions $\imath_1 : \Gamma \to \overline{\Gamma}$ and $\imath_2 : G \to \Pi$. One defines a morphism of graphs

$$\tilde{\varphi} : \widetilde{S^{abs}}(\widetilde{\mathcal{G}}, \widetilde{\Gamma}) \to \widetilde{S}(\widetilde{\widetilde{\mathcal{G}}}, \widetilde{\widetilde{\Gamma}})$$

putting $g\widetilde{\mathcal{G}}(\tilde{v}) \mapsto g\widetilde{\Pi}(\tilde{v})$, $g\widetilde{\mathcal{G}}(\tilde{e}) \mapsto g\widetilde{\Pi}(\tilde{e})$, $(g \in \widetilde{G}, \tilde{v} \in V(\widetilde{\Gamma}), \tilde{e} \in E(\widetilde{\Gamma}))$. As $\mathcal{G}(m)$ $(m \in \Gamma)$ is closed in the pro-$\mathcal{C}$ topology of $G$ for every $m \in \Gamma$, $\widetilde{\mathcal{G}}(m)$ $(\tilde{m} \in \widetilde{\Gamma})$ is also closed in $\widetilde{G}$ for every $\tilde{m} \in \widetilde{\Gamma}$ and the morphism $\tilde{\varphi}$ is injective.

Now by [31, Theorem 6.3.3] there is a canonical isomorphism $\widetilde{S} \cong S$ and similarly, $\widetilde{S^{abs}} \cong S^{abs}$, from which we deduce that the $G$-tree $S^{abs} = S(\mathcal{G}, \Gamma)$ embeds densely in the standard pro-$\mathcal{C}$ $\Pi$-tree $S = S^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma})$ as needed. □

Now one uses Theorem 3 (Theorems 2.3.1 and 2.3.3) to prove Theorem 1:

*Proof of Theorem 1.* As $\pi_1(\mathcal{G}, \Gamma)$ is residually $\mathcal{C}$, we can apply Theorem 2.3.1 to conclude part $(a)$. On the other hand, since each $\mathcal{G}(m)$ is a finite group in $\mathcal{C}$, it is closed in the pro-$\mathcal{C}$ topology of $\pi_1(\mathcal{G}, \Gamma)$, so that the hypotheses of Theorem 2.3.3 also hold. This concludes the proof of $(b)$. □

For the next Theorem, we need the following lemmas. Their proofs can be found in [31, Lemma 8.1.1], page 238, and [31, Lemma 8.2.1], page 241.

**Lemma 2.3.4** (cf. Lemma 1.1 of [34] or Lemma 8.1.1 of [31]). *Let $G$ be an abstract group that is residually $\mathcal{C}$. Assume that $G$ acts freely on an abstract tree $T$ and endow $G$ with its pro-$\mathcal{C}$ topology. Let $K$ be a closed subgroup of $G$ in this topology and let $\Delta$ be a* *finite subgraph of the quotient graph $K\backslash T$. Then there exists an open subgroup $V$ of $G$ containing $K$ such that the natural map of graphs*

$$\tau_V : K\backslash T \to V\backslash T$$

*is injective on $\Delta$.*

Consider now the following situation. Let $H$ be an abstract group that is embedded as a dense subgroup in an infinite pro-$\mathcal{C}$ group $\widetilde{H}$. Assume that $T^{abs}$ is an abstract tree that is embedded as a dense subgraph of a pro-$\mathcal{C}$ tree $T$. We assume further that $\widetilde{H}$ acts continuously on the pro-$\mathcal{C}$ tree $T$ in such a way that $T^{abs}$ is $H$-invariant and such that $H\backslash T^{abs}$ is a finite graph, and suppose that the $H$-stabilizer of each vertex is finite.

**Lemma 2.3.5** (cf. Lemma 1.4 of [34] or Lemma 8.2.1 of [31])**.** *Assume in addition that the natural epimorphism of graphs*

$$H\backslash T^{abs} \to \widetilde{H}\backslash T$$

*is an isomorphism. Then there exists a unique minimal $H$-invariant subtree $D^{abs}$ of $T^{abs}$ and its closure $D = \overline{D^{abs}}$ in $T$ is the unique minimal $\widetilde{H}$-invariant $\mathcal{C}$-subtree of $T$; moreover, $D^{abs} = T^{abs} \cap D$ and $H\backslash D^{abs} = \widetilde{H}\backslash D$ is finite.*

Now we use Theorem 1 to prove the main technical result on the interrelation of $S$ and $S^{abs}$ for a virtually free group. In fact, we adapt the proof of [34, Proposition 1.6] to the infinitely generated case.

**Proposition 2.3.6.** *Let $G = \pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a reduced graph of finite groups having a free subgroup $\Phi$ such that $G/\Phi \in \mathcal{C}$. Let $H = \langle h_1, \ldots, h_r\rangle$ be an infinite finitely generated subgroup of $G$ closed in the pro-$\mathcal{C}$ topology of $G$, and let $\overline{H}$ be its closure in the pro-$\mathcal{C}$ group $G_{\hat{\mathcal{C}}}$. Then the standard tree $S^{abs}$ has a unique minimal $H$-invariant subtree $D^{abs}$, and its closure $D$ in the standard pro-$\mathcal{C}$ tree $S$ is the unique minimal $\overline{H}$-invariant pro-$\mathcal{C}$ subtree of $S$; furthermore $S^{abs} \cap D = D^{abs}$, $\overline{D^{abs}} = D$ and $H\backslash D^{abs} = \overline{H}\backslash D$ is finite.*

*Proof.* Choose a vertex $v_0$ of $\Gamma$, and denote by $\tilde{v}_0$ the vertex $\tilde{v}_0 = 1\Pi^{abs}(v_0) = 1\Pi(v_0)$ in

$S^{abs} \subseteq S$. Define a subgraph $D^{abs}$ of $S^{abs}$ as follows

$$D^{abs} = \bigcup_{i=1}^{r} H[\tilde{v}_0, h_i\tilde{v}_0].$$

Put $L = \bigcup_{i=1}^{r}[\tilde{v}_0, h_i\tilde{v}_0]$; this is obviously a finite connected graph. Then $D^{abs} = HL$. Since $L \cap h_iL \neq \varnothing$ $(i = 1, \ldots, r)$, we have that $D^{abs}$ is a connected subgraph of the tree $S^{abs}$, and so $D^{abs}$ is a tree; clearly it is $H$-invariant. Hence its closure

$$D = \overline{D^{abs}} = \bigcup_{i=1}^{r} \overline{H}[\tilde{v}_0, h_i\tilde{v}_0]$$

in $S$ is a pro-$\mathcal{C}$ subtree of $S$; clearly it is $\overline{H}$-invariant.

Since $H$ is infinite and each $\mathcal{G}(m)$ is finite, our result will follow by Lemma 2.3.5 after we show that the epimorphism of graphs $H\backslash D^{abs} \longrightarrow \overline{H}\backslash D$ is in fact an isomorphism. To see this we distinguish two cases.

*Case 1.* Assume that $H \leqslant \Phi$.

Since the $G$-stabilizers of the elements of $S^{abs}$ are finite groups, $\Phi$ acts freely on $S^{abs}$. By Lemma 2.3.4, there exists an open subgroup $V$ of $\Phi$ (and so of $G$) containing $H$ such that the map of graphs

$$H\backslash D^{abs} \longrightarrow H\backslash S^{abs} \longrightarrow V\backslash S^{abs}$$

is injective. Next observe that for every $m \in \Gamma$, we have the following equality of double cosets

$$V\backslash G/\mathcal{G}(m) = \overline{V}\backslash G_{\hat{\mathcal{C}}}/\mathcal{G}(m)$$

because $V$ has finite index in $G$; hence, one deduces that $V\backslash S^{abs}$ is a subgraph of $\overline{V}\backslash S$ from Theorem 1. Therefore, from the commutative diagram

$$\begin{array}{ccccc} H\backslash D^{abs} & \longrightarrow & H\backslash S^{abs} & \longrightarrow & V\backslash S^{abs} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{H}\backslash D & \longrightarrow & \overline{H}\backslash S & \longrightarrow & \overline{V}\backslash S \end{array}$$

we deduce that the left vertical map is injective finishing this case.

*Case 2.* General case.

Define $K = \Phi \cap H$. Note that $K$ is closed in $\Phi$ and that $K\backslash D^{abs}$ is finite (because $K$ has finite index in $H$). So Lemma 2.3.4 can be used again. Mimicking the argument in Case 1 shows that $K\backslash D^{abs} = \overline{K}\backslash D$. What this says is that if $t, t' \in D^{abs}$, and $\overline{K}t = \overline{K}t'$, then $Kt = Kt'$.

Now since $K$ has a finite index in $H$, we have finite unions $H = \bigcup Kx_i$ and $\overline{H} = \bigcup \overline{K}x_i$ (for some representatives $x_i \in H$ of the cosets $K\backslash H$). Let $t, t' \in D^{abs}$, and assume that $\overline{H}t = \overline{H}t'$. We want to show that then $Ht = Ht'$. By hypothesis, we have $\bigcup \overline{K}x_i t = \bigcup \overline{K}x_i t'$. So for each $i$, there are some $i'$ and $i''$ such that $\overline{K}x_i t = \overline{K}x_{i'}t'$ and $\overline{K}x_i t' = \overline{K}x_{i''}t$; hence by Case 1 $Kx_i t = Kx_{i'}t'$ and $Kx_i t' = Kx_{i''}t$. Therefore, $\bigcup Kx_i t = \bigcup Kx_i t'$, i.e., $Ht = Ht'$. □

## 2.4 Closure of normalizers

Let $G$ be an abstract group that is residually $\mathcal{C}$ and let $H$ be a finitely generated closed (in the pro-$\mathcal{C}$ topology of $G$) subgroup of $G$. In this section we study the relationship between the normalizer $N_G(H) = \{x \in R \mid x^{-1}Hx\}$ of $H$ in $G$ and the normalizer $N_{G_{\hat{\mathcal{C}}}}(\overline{H})$ of $\overline{H}$ in $G_{\hat{\mathcal{C}}}$. When $G$ contains an open free abstract subgroup, we show (Theorem 2) that $\overline{N_G(H)} = N_{G_{\hat{\mathcal{C}}}}(\overline{H})$. This answers Ribes Open Question 15.11.10 of [31] and generalizes the main result of [34], where the theorem was proved for finitely generated groups. In particular, we show that $\overline{N_G(H)} = N_{\hat{G}}(\overline{H})$ when $G$ is virtually free and $H$ is a finitely generated subgroup of $G$ (Corollary 2.4.3).

Ribes and Zalesski have proved the following in [34, Theorem 2.6]:

**Theorem 2.4.1.** *Let $G$ be a finitely generated free-by-$\mathcal{C}$ abstract group. Let $H$ be a finitely generated subgroup of $G$ which is closed in the pro-$\mathcal{C}$ topology of $G$. Then $\overline{N_G(H)} = N_{G_{\hat{\mathcal{C}}}}(\overline{H})$, where the closure $\overline{N_G(H)}$ is taken in $G_{\hat{\mathcal{C}}}$.*

We state Theorem 2 in a more general form, namely for the pro-$\mathcal{C}$ case.

**Theorem 2.4.2.** *Let $G$ be a group having normal free subgroup $\Phi$ such that $G/\Phi \in \mathcal{C}$. Let $H$ be a finitely generated subgroup of $G$. Then $\overline{N_G(H)} = N_{G_{\hat{\mathcal{C}}}}(\overline{H})$, where the closure $\overline{N_G(H)}$ is taken in $G_{\hat{\mathcal{C}}}$.*

*Proof.* Obviously $\overline{N_G(H)} \leqslant N_{G_{\hat{\mathcal{C}}}}(\overline{H})$. We need to prove the opposite containment.

We continue with the notations of Sections 2 and 3. According to a result of Scott (cf. [36]), $G$ is the abstract fundamental group $\pi_1(\mathcal{G}, \Gamma, v)$ of a graph of groups $(\mathcal{G}, \Gamma)$

over a graph $\Gamma$ such that each $\mathcal{G}(m)$ ($m \in \Gamma$) is a finite group. A finite subgroup of $G$ is isomorphic to a subgroup of $G/\Phi$, and so $\mathcal{G}(m)$ is in $\mathcal{C}$. By Theorem 2.3.1, there exists a profinite graph of pro-$\mathcal{C}$ groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ such that $\Pi_1(\overline{\mathcal{G}}, \overline{\Gamma}) = \widehat{G}$. It decomposes as a surjective inverse limit $(\overline{\mathcal{G}}, \overline{\Gamma}) = \varprojlim(\mathcal{G}_U, \Gamma_U)$ of finite graphs of finite $\mathcal{C}$-groups and $\Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma}, v) = \varprojlim(\pi_1(\mathcal{G}_U, \Gamma_U, v_U))_{\hat{\mathcal{C}}}$. Let $\pi_U : \Pi_1^{\mathcal{C}}(\overline{\mathcal{G}}, \overline{\Gamma}, v) \longrightarrow (\pi_1(\mathcal{G}_U, \Gamma_U, v_U))_{\hat{\mathcal{C}}}$ be the natural projection. Since $\pi_U$ is induced by the morphism $(\overline{\mathcal{G}}, \overline{\Gamma}) \longrightarrow (\mathcal{G}_U, \Gamma_U)$, one has $\pi_U(\pi_1(\mathcal{G}, \Gamma)) \leqslant \pi_1(\mathcal{G}_U, \Gamma_U)$. Put $G_U = \pi_1(\mathcal{G}_U, \Gamma_U, v_U)$ and $H_U = Cl(\pi_U(H))$, where $Cl$ means the closure in the pro-$\mathcal{C}$ topology of $G_U$, and note that by Corollary 2.3.2, the group $G_U$ is residually $\mathcal{C}$. Then by Theorem 2.4.1 $\overline{N_{G_U}(H_U)} = N_{(G_U)_{\hat{\mathcal{C}}}}(\overline{H_U})$. Since $\overline{H}_U = \overline{\pi_U(H)}$, $N_{G_{\hat{\mathcal{C}}}}(\overline{H}) = \varprojlim_U N_{(G_U)_{\hat{\mathcal{C}}}}(\overline{H_U})$ and $\overline{N_G(H)} = \varprojlim_U \overline{N_{G_U}(H_U)}$ one deduces that $\overline{N_G(H)} = N_{G_{\hat{\mathcal{C}}}}(\overline{H})$. □

In [21], Marshall Hall proved that a finitely generated subgroup $H$ of a free abstract group $\Phi$ is closed in the profinite topology of $\Phi$. It follows easily that a finitely generated subgroup of a virtually free (or free-by-finite) abstract group $G$ is automatically closed in the profinite topology of $G$. Therefore one has Theorem 2.4.2 as a Corollary.

**Corollary 2.4.3.** *Let $G$ be a virtually free abstract group and let $H$ be a finitely generated subgroup of $G$. Then*

$$\overline{N_G(H)} = N_{\hat{G}}(\overline{H}).$$

We shall finish the section proving the same equality for centralizers of cyclic subgroups, generalizing [31, Corollaries 13.10, 13.1.12]. However, first, we need the following lemma:

**Lemma 2.4.4.** *Let $G$ be a group having a normal free subgroup $\Phi$ of $G$ such that $G/\Phi \in \mathcal{C}$. Let $H$ be a cyclic non-trivial subgroup of $\Phi$. Then*

$$C_G(H) = C_G(Cl(H)),$$

*where $Cl(H)$ denotes the closure of $H$ in the pro-$\mathcal{C}$ topology of $G$.*

*Proof.* As $\Phi$ is closed in $G$, $Cl(H)$ is also the closure of $H$ in the pro-$\mathcal{C}$ topology of $\Phi$. By [33, Proposition 3.4], $Cl(H)$ is cyclic and contains $H$ as a subgroup of finite index. Say $Cl(H) = \langle x \rangle$ and $H = \langle x^n \rangle$. Now, if $a \in G$ and $a^{-1}x^n a = x^n$, then both $a^{-1}xa$ and $x$

are $n$-th roots of $x^n$. Since in a free abstract group $n$-th roots are unique, we deduce that $a^{-1}xa = x$ and the result follows. □

**Proposition 2.4.5.** *Let $G$ be a free-by-$\mathcal{C}$ abstract group. If $H$ is an infinite cyclic subgroup of $G$, then*

$$C_{G_{\hat{\mathcal{C}}}}(\overline{H}) = \overline{C_G(H)}.$$

*Proof.* Consider the natural homomorphism

$$\varphi : N_G(H) \to Aut(H) \cong \mathbb{Z}/2\mathbb{Z}.$$

Then $Ker(\varphi) = C_G(H)$.

Assume first that $H$ is closed. Note that

$$\overline{C_G(H)} \leqslant C_{G_{\hat{\mathcal{C}}}} \leqslant N_{G_{\hat{\mathcal{C}}}}(\overline{H}) = \overline{N_G(H)}$$

(for the last equality we use Theorem 2). Since the index of $\overline{C_G(H)}$ in $\overline{N_G(H)}$ is at most 2, the result follows easily: suppose $C_{G_{\hat{\mathcal{C}}}}(\overline{H}) = \overline{N_G(H)}$ and let $g \in N_G(H)$; then $g \in C_{G_{\hat{\mathcal{C}}}}(\overline{H})$, and so $g \in C_G(H)$, i.e., $C_G(H) = N_G(H)$. Hence $\overline{C_G(H)} = C_{G_{\hat{\mathcal{C}}}}(\overline{H})$.

Assume now that $H$ is a cyclic subgroup of $\Phi$, not necessarily closed. By Lemma 2.4.4, $C_G(H) = C_G(Cl(H))$. Therefore using the result above for the closed subgroup $Cl(H)$,

$$\overline{C_G(H)} = \overline{C_G(Cl(H))} = C_{G_{\hat{\mathcal{C}}}}(\overline{Cl(H)}) = C_{G_{\hat{\mathcal{C}}}}(\overline{H}),$$

since $\overline{H} = \overline{Cl(H)}$. □

## 2.5 Subgroup conjugacy separability

A group $G$ is said to be subgroup conjugacy $\mathcal{C}$-separable if whenever $H_1$ and $H_2$ are finitely generated closed subgroups of $G$ (in its pro-$\mathcal{C}$ topology), then $H_1$ and $H_2$ are conjugate in $G$ if and only if their images in every quotient $G/N \in \mathcal{C}$ are conjugate, or equivalently for residually $\mathcal{C}$ groups, if and only if their closures in $G_{\hat{\mathcal{C}}}$ are conjugate. In this section, we prove the subgroup conjugacy $\mathcal{C}$-separability of a free-by-$\mathcal{C}$ group $G$ (Theorem 4). This answers Open Question 15.11.11 of [31] and generalizes the main result of [7] where it is proved for finitely generated free-by-$\mathcal{C}$ groups. In particular, if

$G$ is virtually free then $G$ is subgroup conjugacy separable (Corollary 2.4.3). One uses again the technique of groups acting on trees and the interrelation between abstract and profinite graphs and groups.

Thus, for the rest of the section fix a group $G$ having a free subgroup $\Phi$ with $G/\Phi \in \mathcal{C}$. According to Scott [36], $G$ splits as the fundamental group of a graph of groups $(\mathcal{G}, \Gamma)$ over a graph $\Gamma$, i.e., $G = \pi_1(\mathcal{G}, \Gamma)$, such that $\mathcal{G}(m) \in \mathcal{C}$ for every $m \in \Gamma$. In fact, we may assume that $(\mathcal{G}, \Gamma)$ is reduced, i.e. whenever $e$ is an edge of $\Gamma$ which is not a loop, then the order of the finite group $\mathcal{G}(e)$ is strictly smaller than the order of $\mathcal{G}(d_i(e))$ $(i = 0, 1)$ (cf. Remark 1.6.18). Then $G_{\hat{\mathcal{C}}}$ is the pro-$\mathcal{C}$ fundamental group of the profinite graph of groups $(\overline{\mathcal{G}}, \overline{\Gamma})$ constructed in Theorem 2.3.1.

**Lemma 2.5.1.** $(\overline{\mathcal{G}}, \overline{\Gamma})$ *is reduced, i.e.* $\mathcal{G}(e) \neq \mathcal{G}(d_i(e))$ *for all* $e \in \overline{\Gamma}$, $i = 1, 2$.

*Proof.* As in the proof of Theorem 3, let $R_U$ be the following equivalence relation in $\Gamma$ defined by $v, w \in V(\Gamma)$, $v \sim_{R_U} w$ if $\mathcal{G}_U(v) = \mathcal{G}_U(w)$ and given $e, e' \in E(\Gamma)$, $e \sim_{R_U} e'$ if $d_0(e) \sim_{R_U} d_0(e')$ and $d_1(e) \sim_{R_U} d_1(e')$, where all $U$ can be taken inside $\Phi$ and $\mathcal{G}_U(m) = \mathcal{G}(m)U/U$. Hence, the quotient graph $\Gamma_U$ defined by $\Gamma_U = \Gamma/R_U$ is finite and we have the finite graph of groups $(\mathcal{G}_U, \Gamma_U)$ such that $\mathcal{G}_U(\bar{m}) = \mathcal{G}(m)$ where $\bar{m}$ is the equivalence class of $m$ and $\partial_i(gU/U) = \partial_i(g)U/U$.

Therefore, if for a given $e \in E(\Gamma)$ which is not a loop neither $\partial_1 : \mathcal{G}(e) \to \mathcal{G}(d_1(e))$ nor $\partial_0 : \mathcal{G}(e) \to \mathcal{G}(d_0(e))$ is an isomorphism, then neither $\partial_1 : \mathcal{G}_U(e) \to \mathcal{G}_U(d_1(e))$ nor $\partial_0 : \mathcal{G}_U(e) \to \mathcal{G}_U(d_0(e))$ can be an isomorphism, so each finite graph of groups $(\mathcal{G}_U, \Gamma_U)$ is reduced. Moreover, for a normal subgroup $V \leqslant U$ of finite index in $G$ the morphism $\eta_{VU} : (\mathcal{G}_V, \Gamma_V) \longrightarrow (\mathcal{G}_U, \Gamma_U)$ restricted to each vertex group $\mathcal{G}(v)$ is an isomorphism. Hence the inverse limit preserves the property of being reduced and the proof is finished. □

**Theorem 2.5.2.** *Let $G$ be a free-by-$\mathcal{C}$ group. Then $G$ is subgroup conjugacy $\mathcal{C}$-separable.*

*Proof.* We continue with the notation of the section. Let $H_1$ and $H_2$ be finitely generated closed subgroups of $G$. Since $G$ is residually $\mathcal{C}$, it suffices to prove that if $\gamma \in G_{\hat{\mathcal{C}}}$ and $\overline{H_1} = \gamma \overline{H_2} \gamma^{-1}$, then there exists some $g \in G$ such that $H_1 = gH_2g^{-1}$.

As usual, we denote by $S^{abs}$ and $S$ the abstract standard tree of $(\mathcal{G}, \Gamma)$ and the standard pro-$\mathcal{C}$-tree of $(\overline{\mathcal{G}}, \overline{\Gamma})$. We divide the proof in two cases, when $H_1$ is infinite and when $H_1$ is finite.

Case 1. $H_1$ is infinite (hence so is $H_2$). By Proposition 2.3.6, $S^{abs}$ has a unique minimal $H_i$-invariant subtree $D_i^{abs}$, and $D_i = \overline{D_i^{abs}}$ is the unique minimal $\overline{H_i}$-invariant pro-$\mathcal{C}$ tree of $S$ ($i = 1, 2$). Then $\gamma D_2$ is a minimal $\overline{H_1}$-invariant pro-$\mathcal{C}$ tree of $S$, and hence $D_1 = \gamma D_2$.

By Theorem 2.3.3 and Proposition 2.3.6 one has that $D_i^{abs}$ is a connected component of $D_i$ considered as an abstract graph and any other component of $D_i$ has the form $\beta D_i^{abs}$, for some $\beta \in \overline{H_i}$ ($i = 1, 2$). Therefore $\gamma D_2^{abs}$ is an abstract connected component of $D_1$. It follows that there exists some $\tilde{h}_1 \in \overline{H_1}$ such that

$$\tilde{h}_1 \gamma D_2^{abs} = D_1^{abs}.$$

Since $H_2$ is infinite and the $G$-stabilizer of any $m \in S^{abs}$ is finite, the tree $D_2^{abs}$ must contain at least one edge; say $e \in E(D_2^{abs})$. Then $\tilde{h}_1\gamma e \in D_1^{abs} \subseteq S^{abs}$. Since $G \backslash S^{abs} = \Gamma \subseteq \overline{\Gamma}$ by Theorem 1, there exists some $g_1 \in G$ such that $g_1 e = \tilde{h}_1 \gamma e$. Hence $g_1^{-1}\tilde{h}_1\gamma$ is in the $G_{\hat{\mathcal{C}}}$-stabilizer of $e$, which in fact coincides with the $G$-stabilizer of $e$ since it is finite. Therefore $g_1^{-1}\tilde{h}_1\gamma \in G$, and so $\tilde{h}_1\gamma = g \in G$. Finally, taking into account that $H_1$ and $H_2$ are closed, we have

$$H_2 = G \cap \overline{H_2} = G \cap \overline{H_1}^{\gamma} = G \cap \overline{H_1}^{(\tilde{h}_1)^{-1}g} = G \cap \overline{H_1^g} = H_1^g,$$

as desired.

Case 2. $H_1$ is finite (hence so is $H_2$). Since $\overline{\Phi}$ is open in $G_{\hat{\mathcal{C}}}$, $G_{\hat{\mathcal{C}}} = G\overline{\Phi}$. Hence $\gamma = g'\gamma'$, where $g' \in G$ and $\gamma' \in \overline{\Phi}$. Then $H_2 = H_1^{\gamma} = (H_1^{g'})^{\gamma'}$. So, replacing $H_1$ with $H_1^{g'}$ and $\gamma$ with $\gamma'$, we may assume that $\gamma \in \overline{\Phi}$. Hence $\overline{\Phi}H_1 = \overline{\Phi}H_2$, and so $\Phi H_1 = \Phi H_2$, because $\Phi$ is closed in $G$. Since in fact $\Phi H_i$ is open in $G$, one has $(\Phi H_i)_{\hat{\mathcal{C}}} \leqslant G_{\hat{\mathcal{C}}}$ ($i = 1, 2$). Thus from now on we may assume that

$$G = \Phi H_1 = \Phi H_2 = \Phi \rtimes H_1 = \Phi \rtimes H_2.$$

This implies that $H_1$ and $H_2$ are maximal finite subgroups of $G$, and so they are $G$-stabilizers of some vertices $S^{abs} \subseteq S$, say $v_1$ and $v_2$, respectively (cf. Proposition 1.6.22).

Recall that by Lemma 2.5.1, $(\overline{\mathcal{G}}, \overline{\Gamma})$ is reduced. Then, if $v$ is one of the vertices of $\tilde{e}$ and $\overline{G}_v = \overline{G}_{\tilde{e}}$, it must be because the image of $\tilde{e}$ in $\overline{\Gamma}$ is a loop.

Since $H_1$ stabilises $\gamma v_2$ and $v_1$, it must stabilise every element of the chain $[\gamma v_2, v_1]$ in $S$ (see Proposition 1.6.24); therefore, since $H_1$ is maximal, it is the $\overline{G}$-stabilizer of each element of the chain $[\gamma v_2, v_1]$. In particular, the endpoints of any edge of this chain have the same $\overline{G}$-stabilizers. By the comment above, this means that the projection on $\overline{\Gamma}$ of any edge of $[\gamma v_2, v_1]$ must be a loop. Since the image of $[\gamma v_2, v_1]$ in $\overline{\Gamma}$ is a connected subgraph of the graph $\overline{\Gamma}$, one deduces that the image of $[\gamma v_2, v_1]$ in $\overline{\Gamma}$ has a unique vertex. Therefore $\gamma v_2$ and $v_1$ are in the same $\overline{G}$-orbit. Hence $\overline{G}v_2 = \overline{G}\gamma v_2 = \overline{G}v_1$.

Since $G\backslash S^{abs} = \Gamma$ is densely embedded in $\overline{\Gamma} = \overline{G}\backslash S$, one deduces that $Gv_2 = Gv_1$. Say $gv_2 = v_1$, where $g \in G$. Then $H_1 = G_{v_1} = gG_{v_2}g^{-1} = gH_2g^{-1}$. $\square$

By the M. Hall theorem every finitely generated subgroup of a virtually free group is closed in the profinite topology. Thus we have our Theorem 4 as a Corollary:

**Corollary 2.5.3.** *Let $G$ be a virtually free group. Then $G$ is subgroup conjugacy separable.*

# CHAPTER 3

# GENERALIZED STALLINGS' DECOMPOSITION THEOREMS FOR PRO-$p$ GROUPS

The entirety of this chapter is a novelty (published in the Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 2023).

The celebrated Stallings' decomposition theorem states that the splitting of a finite index subgroup $H$ of a finitely generated group $G$ as an amalgamated free product or an HNN-extension over a finite group implies the same for $G$. We generalize the pro-$p$ version of it proved by Weigel and Zalesskii in [45] to splittings over infinite pro-$p$ groups. This generalization does not have any abstract analogs. We also prove that generalized accessibility of finitely generated pro-$p$ groups is closed for commensurability.

## 3.1 The Limitation Theorem for virtually free pro-$p$ groups

We prove a special case of the main technical result of this chapter, namely Theorem 13. We will provide the additional elements to prove Theorem 13 in the next section. Unless explicitly stated otherwise, we maintain the notation of the previous chapters.

**Theorem 3.1.1.** *Let $G = \Pi_1(\mathcal{G}, \Gamma, v)$ be the fundamental pro-p group of a finite reduced graph of finite p-groups. Let $H$ be an open subgroup of $G$ and $H = \Pi_1(\mathcal{H}, \Delta, v')$ be a decomposition as the fundamental pro-p group of a reduced finite graph of finite p-groups. Then $|E(\Delta)| \geqslant |E(\Gamma)|$.*

*Proof.* Using induction on the index $[G : H]$ we may assume that $[G : H] = p$. Consider the action of $G$ on its standard pro-$p$ tree $S(G)$. Then $G/H$ acts naturally on the quotient graph $H\backslash S(G)$, i.e.

$$\begin{array}{rcl} \rho : G/H \times H\backslash S(G) & \rightarrow & H\backslash S(G) \\ (gH, Hg'\Pi(m)) & \mapsto & (Hgg'\Pi(m)), \end{array}$$

since $H$ is a normal subgroup of $G$.

Denote by $V_1$ the set of fixed vertices by this action and by $V_2$ the moving ones. By Proposition 1.6.23, $H = \Pi_1(\mathcal{H}, H\backslash S(G))$. Moreover, $\mathcal{H}(w)$ is a $G$-conjugate of some vertex group $\mathcal{G}(v) \leqslant H$ for each $w \in V_2$; indeed, if $w = g\mathcal{G}(v) \in V_2$ for some $g \in G$, since $w = gH\Pi(v)$ by definition, then $\mathcal{G}(v) \subset H\Pi(v)$ and consequently $\mathcal{G}(v) \leqslant H$, so $G$ acts non-trivially on $G/H\mathcal{G}(v) = G/H$ implying that $\mathcal{H}(w) = \mathcal{G}(w)$ is conjugate to $\mathcal{G}(v)$. If $(\mathcal{H}, H\backslash S(G))$ is not reduced, we can apply the procedure described in Remark 1.6.18 to obtain the reduced graph of finite $p$-groups $(\mathcal{H}, \Delta)$. Since $G$ is virtually free pro-$p$ one can use the Theorem 1.6.20 to deduce that it suffices to prove the statement for $(\mathcal{H}, \Delta)$.

Identifying $V_1$ with its bijective image in $\Gamma$ we have that for each $v \in V_1$ the vertex group $\mathcal{H}(v) = \mathcal{G}(v) \cap H$ is of index $p$ in $\mathcal{G}(v)$.

If $V_1 = \varnothing$ then by the previous paragraph all the edge and vertex groups of $(\mathcal{H}, H\backslash S(G))$ are conjugates of some edge and vertex groups of $(\mathcal{G}, \Gamma)$. It follows that $(\mathcal{H}, H\backslash S(G))$ is reduced, since $(\mathcal{G}, \Gamma)$ is by hypothesis. We have that $|H\backslash E(S(G))| = p|E(\Gamma)|$ and the result follows in this case.

Assume that $V_1$ is non-empty. Denote by $\Gamma(V_i)$ the spanned graph of $V_i$, $i = 1, 2$ and $E_{12}$ the edges that connect vertices of $V_1$ to vertices of $V_2$. If $(\mathcal{H}, H\backslash S(G))$ is not reduced, then the fictitious edges can be only the moved ones that are in $E(\Gamma(V_1)) \cup E_{12}$. Moreover, only one such edge from its $G/H$-orbit can be collapsed. Indeed, suppose $e_{12} \in E_{12}$ with extremities $v_1 \in V_1, v_2 \in V_2$ is fictitious and so can be collapsed into the new vertex $\mathbf{v_2}$. This can happen because $\mathcal{H}(v_1)$ has index $p$ in $\mathcal{G}(v_1)$ and so can be equal to the edge group, i.e. $\mathcal{H}(v_1) = \mathcal{H}(e_{12})$ is proper in $\mathcal{H}(v_2) = \mathcal{G}(v_2)$. This $v_1$ is identified after the collapse of $e_{12}$ with $v_2$ (and the rest of the vertices of $Ge_{12}$ are in $\Gamma(V_2)$) and so, after collapsing, $\mathcal{H}(gv_2) \neq \mathcal{H}(ge_{12}) \neq \mathcal{H}(v_2)$. On the other hand, if $e \in E(\Gamma(V_1))$ then, after collapsing it, all the other edges from its orbit become loops. Here are the pictures for the case $p = 2$, where $g \in G/H$.

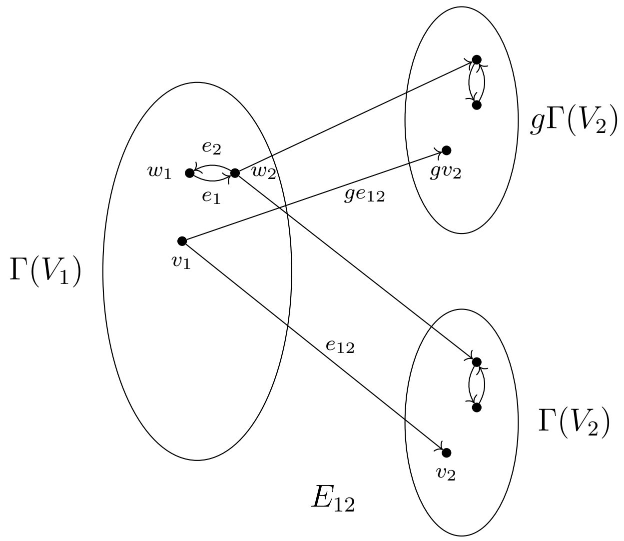



Figure 3.1: Graph of groups $(\mathcal{H}, S(G)/H)$

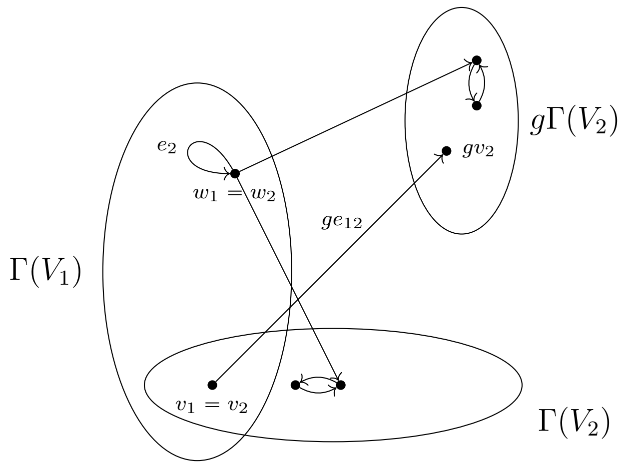



Figure 3.2: Reduced graph of groups $(\mathcal{H}, \Delta)$ assuming that $e_1$ and $e_{12}$ are collapsed

Thus we can deduce that $E(\Delta) \geqslant |E(\Gamma(V_1))| + p|E(\Gamma(V_2))| + (p-1)|E_{12}| \geqslant |E(\Gamma(V_1))| + |E(\Gamma(V_2))| + |E_{12}| \geqslant E(\Gamma)$. This finishes the induction and concludes the theorem. $\square$

**Remark 3.1.2.** It follows from the first 3 lines of the proof of Theorem 3.1.1 that $|\Delta| \leqslant [G : H]|\Gamma|$, $|E(\Delta)| \leqslant [G : H]|E(\Gamma)|$ and $|V(\Delta)| \leqslant [G : H]|V(\Gamma)|$.

**Remark 3.1.3.** The proof also shows that for $p > 2$ one has $|E(\Gamma)| < |E(\Delta)|$ unless $V_2 = \varnothing$ and no edge from $\Gamma(V_1)$ is moved, and therefore $\Gamma = \Delta$. For $p = 2$ the equality $|E(\Delta)| = |E(\Gamma)|$ can happen either in the case $\Delta = \Gamma$ or if for every edge $e \in E_{12}$ one has $[\mathcal{G}(d_0(e)) : \mathcal{G}(e)] = 2$ and $E(\Gamma_2) = \varnothing$.

## 3.2 Finitely generated pro-$p$ groups acting virtually on pro-$p$ trees

In this section, we prove the main results stated in the introduction and deduce several consequences. The proof of Theorem 9 follows the proof of [9, Lemma 4.1] whose original idea appears in the proof of the main result of [45].

**Theorem 9.** *Let $G$ be a finitely generated pro-$p$ group having an open normal subgroup $H$ acting on a pro-$p$ tree $T$. Suppose $\{H_v \mid v \in V(T)\}$ is $G$-invariant. Then $G$ is the fundamental group of a profinite graph $(\mathcal{G}, \Gamma)$ of pro-$p$ groups such that each $H_v$ is conjugate into a vertex group of $G$ and each vertex group of $G$ intersected with $H$ stabilizes a vertex of $T$. In particular, $G$ splits as a non-trivial free amalgamated pro-$p$ product or a pro-$p$ HNN-extension.*

*Proof.* Let $\mathcal{U}$ be the collection of open normal subgroups $U$ of $G$ contained in $H$. Denote by $\widetilde{U}$ the topological closure of $U$ generated by the $U$-stabilizers of the vertices of $T$, i.e.,

$$\widetilde{U} = \overline{\langle U \cap H_v \mid v \in V(T)\rangle}.$$

Then $\widetilde{U}$ is a closed normal subgroup of $G$, since $U$ is normal and $\{H_v \mid v \in V(T)\}$ is stable under conjugation of $G$.

Note that $\widetilde{U}\backslash T$ is a pro-$p$ tree (cf. Proposition 1.3.11) and $H/\widetilde{U}$ acts on $\widetilde{U}\backslash T$ with $U/\widetilde{U}$ acting freely. Therefore $G/\widetilde{U}$ contains the open normal subgroup $U/\widetilde{U}$ which is finitely generated and free pro-$p$ (cf. Theorem 1.2.16). By Theorem 1.6.20, $G/\widetilde{U}$ is isomorphic to the pro-$p$ fundamental group $\Pi_1(\mathcal{G}_U, \Gamma_U, v_U)$ of a finite graph of finite $p$-groups.

Although neither the finite graph $\Gamma_U$ nor the finite graph of finite $p$-groups $\mathcal{G}_U$ is uniquely determined by $U$ (resp. $\widetilde{U}$), the index $U$ in the notation shall express that both these objects are depending on $U$. Using the procedure described at Remark 1.6.18 we have a morphism $\eta : (\mathcal{G}_U, \Gamma_U) \to (\overline{\mathcal{G}}_U, \overline{\Gamma}_U)$ to a reduced graph of groups.

For $V \subseteq U$ both open and normal in $G$, the decomposition $G/\widetilde{V} = \Pi_1(\overline{\mathcal{G}}_V, \overline{\Gamma}_V, \overline{v}_V)$ gives rise to a natural decomposition of $G/\widetilde{U}$ as the fundamental group

$$G/\widetilde{U} = \Pi_1(\mathcal{G}_{V,U}, \overline{\Gamma}_V, \overline{v}_V)$$

of a finite graph of finite $p$-groups $(\mathcal{G}_{V,U}, \overline{\Gamma}_V)$, where the vertex and edge groups are $\mathcal{G}_{V,U}(x) = \overline{\mathcal{G}}_V(x)\tilde{U}/\tilde{U}$, $x \in \overline{\Gamma}_V$. Thus we have a morphism $\underline{\nu}_{V,U} : (\overline{\mathcal{G}}_V, \overline{\Gamma}_V) \to (\mathcal{G}_{V,U}, \overline{\Gamma}_V)$ of graphs of groups

$$\begin{array}{ccc} \overline{\mathcal{G}}_V & \xrightarrow{\nu_{V,U}} & \mathcal{G}_{V,U} \\ \big\downarrow & & \big\downarrow \\ \overline{\Gamma}_V & \xrightarrow[\nu'_{V,U}]{} & \overline{\Gamma}_V \end{array}$$

By Proposition 2.2.7, this morphism $\underline{\nu}_{V,U}$ of graphs of groups induces a homomorphism of fundamental groups, so we define $\underline{\nu}_{V,U}$ in such a manner that the induced homomorphism on the pro-$p$ fundamental groups coincides with the canonical projection $\varphi_{V,U} : G/\widetilde{V} \to G/\widetilde{U}$. For this, choose a reduction morphism $\eta_U : (\mathcal{G}_{V,U}, \overline{\Gamma}_V) \to (\overline{\mathcal{G}}_{V,U}, \overline{\Gamma}_U)$ to a finite reduced graph of groups $(\overline{\mathcal{G}}_{V,U}, \overline{\Gamma}_U)$ (it is not unique); it induces the identity map on the fundamental group $G/\widetilde{U}$ (see Remark 1.6.19) and so $\eta_U\underline{\nu}_{V,U}$ induces the homomorphism $\Pi_1(\overline{\mathcal{G}}_V, \overline{\Gamma}_V, \overline{v}_V) \to \Pi_1(\overline{\mathcal{G}}_{V,U}, \overline{\Gamma}_U, \overline{v}_U)$ on the pro-$p$ fundamental groups that coincides with the canonical projection $\varphi_{UV} : G/\widetilde{V} \to G/\widetilde{U}$.

Using the aforementioned reduction, we have that $G/\widetilde{U} = \Pi_1(\overline{\mathcal{G}}_U, \overline{\Gamma}_U, \overline{v}_U)$. Then the number of isomorphism classes of finite reduced graphs of finite $p$-groups $(\mathcal{G}'_U, \Gamma')$ which are based on $\Gamma'$ satisfying $G/\widetilde{U} \simeq \Pi_1(\mathcal{G}'_U, \Gamma', v_0)$ is finite (cf. [45, Corollary 3.3]).

Let $\Omega_U$ be a set containing a copy of every such isomorphism class. Since $G$ is finitely generated, we may choose $V_i$, $i \in \mathbb{N}$, to be a decreasing chain of open normal subgroups of $G$ with $V_0 = U$ and $\bigcap_i V_i = \{1\}$. For $X \subseteq \Omega_{V_i}$ define $T(X)$ to be the set of all reduced graphs of groups in $\Omega_{V_{i-1}}$ that can be obtained from graphs of groups of $X$ by the procedure of reduction explained above (note that $T$ is not a map).

Define

$$\Omega_1 = T(\Omega_{V_1}), \Omega_2 = T(T(\Omega_{V_2})), \cdots, \Omega_i = T^i(\Omega_{V_i})$$

and note that $\Omega_i$ is a non-empty subset of $\Omega_U$ for every $i \in \mathbb{N}$. Clearly $\Omega_{i+1} \subseteq \Omega_i$ and since $\Omega_U$ is finite there is an $i_1 \in \mathbb{N}$ such that $\Omega_j = \Omega_{i_1}$ for all $j > i_1$ and we denote this $\Omega_{i_1}$ by $\Sigma_U$. Then $T(\Sigma_{V_i}) = \Sigma_{V_{i-1}}$ and so we can construct an infinite sequence of graphs of groups $(\mathcal{G}_{V_j}, \Gamma_j) \in \Omega_{V_j}$ such that $(\mathcal{G}_{V_{j-1}}, \Gamma_{j-1}) \in T(\mathcal{G}_{V_j}, \Gamma_j)$ for all $j$. This means that $(\mathcal{G}_{V_j}, \Gamma_{V_j})$ can be reduced to $(\mathcal{G}_{V_{j-1}}, \Gamma_{j-1})$, i.e. this sequence $\{(\mathcal{G}_{V_j}, \Gamma_j)\}$ is an inverse system of reduced finite graphs of groups satisfying the required conditions. Therefore $(\mathcal{G}, \Gamma) = \varprojlim(\mathcal{G}_{V_j}, \Gamma_j)$ is a reduced profinite graph of finitely generated pro-$p$ groups satisfying $G \simeq \Pi_1(\mathcal{G}, \Gamma, v)$. Since the stabilizer of any vertex of $\tilde{U}\backslash T$ in $H/\tilde{V}_j$ is finite and so by Theorem 1.6.22 it is conjugated into a vertex group of $(\mathcal{G}_{V_j}, \Gamma_j)$. Therefore for any vertex in $T$ its vertex stabilizer is conjugate into a vertex of $\Pi_1(\mathcal{G}, \Gamma, v)$.

Moreover, denoting by $x_V$ the image of $x \in \Gamma$ in $\Gamma_V$ we have $\mathcal{G}(x) = \varprojlim \mathcal{G}_{V_j}(x_{V_j})$ if $x$ is either a vertex or an edge of $\Gamma$. Since $\mathcal{G}_{V_j}(x) \cap H/\widetilde{V_j}$ fixes a vertex in $\widetilde{V_j}\backslash T$ for each $V_j$, and the set of fixed vertices of $\mathcal{G}_{V_j}(x) \cap H/\widetilde{V_j}$ is compact, the inverse limit argument implies that $\mathcal{G}(x) \cap H$ fixes a vertex of $T$.

A finitely generated pro-$p$ group that acts on a pro-$p$ tree splits as an amalgamated free pro-$p$ product or pro-$p$ HNN-extension over the stabilizer of an edge (cf. Theorem 1.6.25). Using the fact that the fundamental pro-$p$ group of a graph of pro-$p$ groups acts on its standard pro-$p$ tree (cf. Definition 1.6.16) we can deduce that $G$ splits as non-trivial free amalgamated pro-$p$ product or pro-$p$ HNN-extension. This finishes the proof of the theorem. □

**Corollary 3.2.1.** *In the hypotheses of Theorem 9 one has* $|E(\Gamma)| \leqslant |E(H\backslash T)|$. *Moreover, if* $p > 2$, $|H\backslash T| < \infty$ *and* $\Gamma \neq H\backslash T$, *then the inequality is strict.*

*Proof.* It makes sense to prove the statement assuming that $H\backslash T$ is finite. Let $\Gamma_{V_j}$ be as in the proof of Theorem 9. By Theorem 3.1.1 combined with Theorem 1.6.20, $|E(\Gamma_{V_j})| \leqslant |E(H\backslash T)|$ for each $j$. Hence $|E(\Gamma)| \leqslant |E(H\backslash T)|$ as required. Moreover, if $\Gamma \neq H\backslash T$ then $|E(\Gamma_{V_j})| < |E(H\backslash T)|$ for each $j$ by Remark 3.1.3 and hence $|E(\Gamma)| < |E(H\backslash T)|$. □

One of the obstacles to obtaining the main structure result in the pro-$p$ version of Bass-Serre theory is that a maximal subtree of a profinite graph $\Gamma$ does not always exist. The next corollary shows that, for the finitely generated case, this difficulty can be surpassed.

**Corollary 3.2.2.** *In the hypotheses of Theorem 9, the graph $\Gamma$ possesses a closed maximal pro-$p$ subtree.*

*Proof.* By [37, Section 2.3, Corollary 2] the inverse image of a maximal subtree under a collapse is a maximal subtree. Hence we can choose maximal subtrees $D_j$ of $\Gamma_{V_j}$ from the proof of Theorem 9 such that they form the inverse subsystem. Then $D = \varprojlim D_j$ is a pro-$p$ tree with $V(D) = V(\Gamma)$. $\square$

**Definition 3.2.3.** Following [37, Section 6.1] we say that a pro-$p$ group $G$ has the FA property if, for any pro-$p$ tree $T$ on which $G$ acts, $T^G \neq \varnothing$, i.e. if $G$ acts on a pro-$p$ tree $T$ then it has a global fixed point.

By Theorem 1.6.25, a pro-$p$ group $G$ is FA if it does not split as an amalgamated free pro-$p$ product or pro-$p$ HNN-extension. In this sense, it is used in the introduction.

**Definition 3.2.4** (Fab pro-$p$ groups)**.** We say that a pro-$p$ group $G$ is Fab if every open subgroup of $G$ has finite abelianization.

**Lemma 3.2.5.** *Let $G$ be a finitely generated Fab pro-$p$ group. Then $G$ is FA (cf. Definition 1.6.21), i.e. if $G$ acts on a pro-$p$ tree $T$ then it has a global fixed point.*

*Proof.* Suppose $G$ act on a a pro-$p$ tree $T$. We shall use the first paragraph of the proof of Theorem 9. According to it for any open normal subgroup $U$ of $G$, $U/\widetilde{U}$ is free pro-$p$, and since $G$ is Fab this quotient has to be trivial. Therefore $G/\widetilde{U}$ is finite and so by Theorem 1.2.15 the set of fixed points $(\widetilde{U}\backslash T)^{G/\widetilde{U}}$ is non-empty. Then $T^G = \varprojlim_U (\widetilde{U}\backslash T)^{G/\widetilde{U}}$ is non-empty and the lemma is proved. $\square$

**Corollary 10.** Let $G$ be a finitely generated pro-$p$ group having an open subgroup $H$ acting on a pro-$p$ tree $T$ such that each stabilizer $H_v$ is Fab. Then $G$ is the fundamental group of a profinite graph of pro-$p$ groups such that each vertex group intersected with $H$ stabilizes a vertex of $T$. In particular, $G$ splits as a non-trivial free amalgamated pro-$p$ product or a pro-$p$ HNN-extension.

*Proof.* Let $N = H_G$ be the normal core of $H$ in $G$. Since $H_v$ is Fab and $N_v$ is open in $H_v$, $N_v$ is $Fab$ and hence so is $N_v^g$. Then by Lemma 3.2.5 $N_v^g$ must fix a vertex of $T$. Hence, the hypotheses of Theorem 9 are satisfied for $N$ and the result follows. □

We are ready to prove Theorem 13. It will be crucial to deal with the generalized version of Stallings' decomposition theorem and with accessibility in finitely generated pro-$p$ groups.

**Theorem 13.** [Limitation Theorem] *Let $G = \Pi_1(\mathcal{G}, \Gamma, v)$ be the fundamental pro-p group of a finite reduced graph of pro-p groups. Let $H$ be an open normal subgroup of $G$ and $H = \Pi_1(\mathcal{H}, \Delta, v')$ be a decomposition as the fundamental pro-p group of a reduced graph of pro-p groups $(\mathcal{H}, \Delta, v')$ obtained by a reduction process from $(\mathcal{H}, H\backslash S(G))$. Then $|E(\Delta)| \geqslant |E(\Gamma)|$. Moreover, for $p > 2$ the inequality is strict unless $\Gamma = \Delta$.*

*Proof.* We consider the action of $H$ on the standard pro-$p$ tree $S(G)$ of $G$ and observe that since $H$ is normal in $G$, the set $\{H_v \mid v \in S(G)\}$ is $G$-invariant. Hence the result follows from Corollary 3.2.1. □

An action of a pro-$p$ group on a pro-$p$ tree $T$ is called $k$-acylindrical if the stabilizer of any geodesic in $T$ of length greater than $k$ is trivial. We say that a profinite graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$ is $k$-acylindrical if the action of the fundamental group $\Pi_1(\mathcal{G}, \Gamma)$ on its standard pro-$p$ tree is $k$-acylindrical.

Hence, we obtain as a particular case of our Limitation Theorem (cf. Theorem 13), the pro-$p$ version of Sela's Theorem proved by Castellano and Zalesski (cf. [6])

**Theorem 3.2.6.** *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a finite reduced $k$-acylindrical graph of pro-p groups. Then $|E(\Gamma)| \leqslant d(G)(4k+1) - 1$ and $|V(\Gamma)| \leqslant 4kd(G)$.*

It provides a bound for $|E(\Gamma)|$ when $(\mathcal{G}, \Gamma)$ is a finite reduced $k$-acylindrical graph of pro-$p$ groups. However, our Limitation Theorem (cf. Theorem 13) is much stronger and provides a bound for $|E(\Gamma)|$ when $(\mathcal{G}, \Gamma)$ is any finite reduced graph of pro-$p$ groups.

**Theorem 11.** *Let $G$ be a finitely generated pro-p group having an open normal subgroup $H$ that splits as the fundamental pro-p group of a finite graph of finitely generated pro-p groups $(\mathcal{H}, \Delta)$. Suppose the set of conjugacy classes of all vertex groups of $H$ is $G$-invariant. Then $G$ is the fundamental group of a reduced finite graph $(\mathcal{G}, \Gamma)$ of pro-p* *groups such that the vertex groups of $H$ are conjugated into vertex groups of $G$ and the vertex and edge groups of $G$ intersected with $H$ are conjugated to subgroups of vertex and edge groups of $H$ respectively. Moreover, $|E(\Gamma)| \leqslant |E(\Delta)|$.*

*Proof.* Consider the action of $H$ on its standard pro-$p$ tree $S(H)$. Since the set of the conjugacy classes of vertex groups of $H$ is $G$-invariant and the vertex stabilizers for this action are exactly the conjugates of the vertex groups, we can apply Theorem 9 to obtain a splitting of $G$ as the fundamental group of a profinite graph of pro-$p$ groups $G = \Pi_1(\mathcal{G}, \Gamma, v)$, where the vertex groups of $H$ are conjugate into vertex groups of $G$ and the vertex groups of $G$ intersected with $H$ are subgroups of vertex groups of $H$. Moreover, since $\Delta$ is finite, by Corollary 3.2.1 $|E(\Gamma)| \leqslant |E(\Delta)|$.

We are left with the statement about edge stabilizers. We use here the decomposition $(\mathcal{G}, \Gamma) = \varprojlim(\mathcal{G}_{V_j}, \Gamma_j)$ as a reduced profinite graph of finitely generated pro-$p$ groups satisfying $G \simeq \Pi_1(\mathcal{G}, \Gamma, v)$ from the proof of Theorem 9. Since $\Gamma$ is finite we may assume that $\Gamma = \Gamma_j$ for all $j$. Since $(\mathcal{G}_{V_j}, \Gamma_j)$ is reduced, the vertex stabilizers of $\widetilde{V}_j\backslash S(H)$ in $H/\widetilde{V}_j$ are exactly the maximal finite subgroups of the $H/\widetilde{V}_j$ (cf. Theorem 1.6.22). This implies, in particular, that, for $V_{j+1} \leqslant_o V_j$, the maximal finite subgroups of $H/\widetilde{V}_{j+1}$ map onto maximal finite subgroups of $H/\widetilde{V}_j$. It induces a bijection of the conjugacy classes of the maximal finite subgroups of $H/\widetilde{V}_{j+1}$ and $H/\widetilde{V}_j$. Then, if $e$ is an edge of $\Gamma$, starting from some $j$, one has $\mathcal{G}_{V_j}(e) = \mathcal{G}'_{V_j}(v) \cap \mathcal{G}_{V_j}(w)^g$, where $v, w$ are the extremity vertices of $e$ and are maximal finite subgroups of $G/\widetilde{V}_j$.

Let $H_{V_j} = \Pi_1(\mathcal{H}_j, \Delta_j, v'_j)$ be the splitting of $H_{V_j}$ as an open subgroup of $\Pi_1(\mathcal{G}_{V_j}, \Gamma_j, v_j)$ (see Proposition 1.6.23). Then $H_{V_j} \cap \mathcal{G}_{V_j}(e) \leqslant \mathcal{H}_j(e')^h$ for some $e' \in E(\Delta_j)$, $h \in H_{V_j}$. It follows that $H_{V_j} \cap \mathcal{G}_{V_j}(e)$ is contained in the intersection of at least two distinct maximal finite subgroups of $H_{V_j}$ (some vertex stabilizers of $\widetilde{H}_{V_j}\backslash S(H)$). Hence $\mathcal{G}(e) \cap H$ is contained in the intersection of at least two distinct vertex stabilizers of $T$ and so fixes an edge of $T$. This finishes the proof. $\square$

With Theorem 11 in hand, we prove Corollary 12, Theorem 5, Corollary 6 and Corollary 8 respectively.

**Corollary 12.** *Let $G$ be a finitely generated pro-p group having an open subgroup $H$ that splits as a finite graph of finitely generated pro-p groups $(\mathcal{H}, \Delta)$. Suppose the vertex groups of $(\mathcal{H}, \Delta)$ are Fab. Then $G$ is the fundamental group of a reduced finite graph of* *pro-p groups* $(\mathcal{G},\Gamma)$ *such that its vertex and edge groups intersected with* $H$ *are subgroups of vertex and edge groups of* $H$ *respectively. Moreover,* $|E(\Gamma)| \leqslant |E(\Delta)|$.

*Proof.* Let $N = H_G$ be the normal core of $H$ in $G$. Remember that $H_G = \bigcap_{g\in G} g^{-1}Hg$ is the largest normal subgroup of $G$ that is contained in $H$. By Proposition 1.6.23, $N$ is the fundamental group $\Pi_1(\mathcal{N},\Omega)$ of a finite graph of finite $p$-groups whose vertex groups are open in the conjugates of vertex groups of $(\mathcal{H},\Delta)$. Since $\mathcal{H}(v)$, $v \in V(\Delta)$ is Fab and $N \cap \mathcal{H}(v)^g$ is open in $H_v^g$, the group $N \cap \mathcal{H}(v)^g$ is $Fab$. Then Lemma 3.2.5 shows that $N_v^g$ must fix a vertex of the standard pro-$p$ tree $S(H)$ and so the set of the conjugacy classes of all vertex groups of $N$ is $G$-invariant. Hence the result follows from Theorem 11. □

**Theorem 5.** *Let* $H = H_1 \amalg_K H_2$ *be a free amalgamated pro-p product of finitely generated pro-p groups* $H_1, H_2$ *that are indecomposable over any conjugate of any subgroup of* $K$*. Let* $G$ *be a pro-p group having* $H$ *as an open normal subgroup. Then* $G$ *splits as a free amalgamated pro-p product* $G = G_1 \amalg_L G_2$ *such that* $H_i$ *is conjugate into some* $G_j$ $(i,j = 1,2)$ *and* $G_i \cap H$ *is contained in some conjugate of* $H_i$*,* $i = 1,2$ *as well as* $L \cap H$ *is contained in some conjugate of* $K$*.*

*Proof.* Since $H_1, H_2$ are indecomposable over any conjugate of any subgroup of $K$, by Theorem 1.6.25 any conjugate of $H_1$ and $H_2$ in $G$ must fix a vertex of the standard pro-$p$ tree $S(H)$. By Theorem 11, $G$ is the fundamental group of graph of groups $\Pi_1(\mathcal{G},\Gamma)$ with one edge only (cf. Example 1.6.9) and $H_1, H_2$ are conjugate into the vertex groups. Consider the action of $G$ on its standard pro-$p$ tree and recall that $\tilde{H}$ means the subgroup generated by the vertex stabilizers. However, $H/\tilde{H}$ is trivial in this case (as $H_1, H_2$ stabilize some vertices) and since $\tilde{H} \leqslant \tilde{G}$, $G/\tilde{G}$ is finite. Hence $G$ can not be an HNN-extension, as in the case of an HNN-extension $G/\tilde{G} = \mathbb{Z}_p$. Thus $\Gamma$ is one edge with two vertices and so $G = G_1 \amalg_L G_2$, where $G_1, G_2$ are vertex groups and $L$ is the edge group.

□

**Corollary 6.** *Let* $p > 2$ *and* $H = H_1 \amalg_K H_2$ *be a free amalgamated pro-p product of finitely generated FA pro-p groups* $H_1, H_2$*. Let* $G$ *be a pro-p group having* $H$ *as an open subgroup. Then* $G$ *splits as a free amalgamated pro-p product* $G = G_1 \amalg_L G_2$ *such that* $G_i \cap H$ *is contained in some conjugate of* $H_i$*,* $i = 1,2$ *and* $L \cap H$ *is contained in some conjugate of* $K$*.*

*Proof.* We use induction on $[G : H]$. The base of induction $[G : H] = p$ follows from Theorem 5 as $H$ is normal in $G$ in this case. Suppose $[G : H] > p$ and $H < N \leqslant G$ with $[N : H] = p$. Then, by Theorem 5, $N = N_1 \amalg_M N_2$ such that $H_i, i = 1, 2$ are conjugate into $N_j, j = 1, 2$ and $N_j \cap H$ are conjugate into $H_i, i = 1, 2$. To apply the induction step, we just need to show that $N_1$ and $N_2$ are FA.

Assume for definiteness that $H_1$ is conjugated into $(N_1 \cap H)$. Then we can suppose w.l.o.g $H_1 \leqslant N_1 \cap H$. Since $H_1$ is not conjugated into $H_2$ we deduce that $N_1 \cap H$ is conjugate into $H_1$ and hence is equal to $H_1$ (a pro-$p$ group can not be conjugated to its proper subgroup). Then $H_1$ has at most index $p$ in $N_1$ and so $N_1$ can not split, because otherwise $H_1$ would split by Proposition 1.6.23 and this splitting is non-trivial by Theorem 13. Thus $N_1$ is FA by Theorem 1.6.25.

If $N_2 \cap H$ is conjugate into $H_2$ then by the same argument one deduces that $N_2$ is FA.

We claim that $N_2 \cap H$ is not conjugated into $H_1$. Suppose it is, so $H$ is contained in the normal closure $G_1^G$ of $G_1$ in $G$ and, by Proposition 1.6.23, $G_1^G$ splits as the fundamental group of the finite graph of pro-$p$ groups $(\mathcal{G}_1, \Delta)$ that we may assume to be reduced (see Remark 1.6.19). Moreover, by Theorem 13, $E(\Delta) = 1$ only if $p = 2$. Since the $p = 2$ case is excluded by the hypothesis, the proof is complete. □

Theorem 7 follows by direct application of Theorem 9 combined with Theorem 1.6.25 and Corollary 3.2.1 (cf. Example 1.6.12).

**Corollary 8.** *Let $p > 2$ and $H = HNN(H_1, K, t)$ be a pro-p HNN-extension of a finitely generated FA pro-p group $H_1$. Let $G$ be a pro-p group having $H$ as an open subgroup. Then $G$ splits as a pro-p HNN-extension $G = (G_1, L, t)$ such that $G_1 \cap H$ is contained in some conjugate of $H_1$, and $L \cap H$ is contained in some conjugate of $K$.*

*Proof.* We use induction on $[G : H]$. The base of induction $[G : H] = p$ follows from Theorem 7 as $H$ is normal in $G$ in this case. Suppose $[G : H] > p$ and $H < N \leqslant G$ with $[N : H] = p$. Then, by Theorem 5, either $N = N_1 \amalg_M N_2$ or $N = HNN(N_1, M, t)$ with $H_1$ conjugate into $N_1$ and $N_1 \cap H$ conjugate into $H_1$. But for $p > 2$ the first case $N = N_1 \amalg_M N_2$ does not occur, since viewing $N$ as the fundamental group of the graph of groups $(\mathcal{N}, \Gamma)$ with $\Gamma$ being an edge with two distinct vertices and viewing $H$ as the fundamental group of the graph of groups $(\mathcal{H}, \Delta)$ with $\Delta$ being an edge with one vertex we have $\Gamma \neq \Delta$ and so, by Theorem 13, $E(\Delta)$ must have more than one edge, a contradiction.

Thus $N = HNN(N_1, M, t)$ with $H_1$ conjugate into $N_1$ and $N_1 \cap H$ conjugate into $H_1$ implying that $H_1$ and $N_1 \cap H$ are conjugate so, w.l.o.g, we may assume that with $N_1 \cap H = H_1$. Then $H_1$ has index at most $p$ in $N_1$. Then $N_1$ can not split, because then $H_1$ would split non-trivially by Proposition 1.6.23 and Theorem 13, contradicting the hypothesis. So $N_1$ is FA by Theorem 1.6.25.

Since $N_1$ is FA we can apply the induction step to deduce the result.

□

## 3.3 Generalized accessible pro-$p$ groups

Abstract accessibility was studied in a series of papers by M.J. Dunwoody (cf. [12] [13],[14],[15]), where he proved that every finitely presented group is accessible, but not every finitely generated group over an arbitrary family of groups. In fact, he presented an example of a finitely generated inaccessible group. Generalized accessible groups were studied by Bestvina and Feighn ([4]). The pro-$p$ version of accessibility was introduced by G. Wilkes in [46]. Chatzidakis and Zalesski generalized this definition as follows:

**Definition 3.3.1** (Generalized accessible pro-p group, cf. Definition 5.1 of [9])**.** Let $\mathcal{F}$ be a family of pro-$p$ groups. We say that a pro-$p$ group $H$ is $\mathcal{F}$-accessible if any splitting of $H$ as the fundamental group of a reduced finite graph $(\mathcal{G}, \Gamma)$ of pro-$p$ groups such that the edge groups are in $\mathcal{F}$ has a bound on $\Gamma$.

Now we prove Theorem 14. Recall that two pro-$p$ groups $G_1, G_2$ are *commensurable* if there exist $H_1$ open in $G_1$ and $H_2$ open in $G_2$ such that $H_1 \cong H_2$. Theorem 13 allows us to prove that the accessibility of a pro-$p$ group with respect to a family $\mathcal{F}$ of pro-$p$ groups is preserved by commensurability. For accessible abstract groups such a result can be deduced from the Stallings splitting theorem; we are not aware of such a result for accessible groups with respect to a family of infinite groups in the abstract situation.

**Theorem 14.** Let $\mathcal{F}$ be a family of pro-$p$ groups closed for commensurability. Let $G$ be a finitely generated pro-$p$ group and $H$ an open subgroup of $G$. Then $G$ is $\mathcal{F}$-accessible if and only if $H$ is $\mathcal{F}$-accessible.

*Proof.* Using the obvious induction on $[G : H]$ we may assume that $[G : H] = p$ and so $H$ is normal in $G$.

Suppose $H$ is $\mathcal{F}$-accessible and $G$ is not. Then for any $n \in \mathcal{N}$ there exists a finite reduced graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$ such that $G = \Pi_1(\mathcal{G}, \Gamma, v)$ with edge groups in $\mathcal{F}$ and $|E(\Gamma)| > n$. It follows from the proof of Theorem 9 that there exists an open normal subgroup $U$ of $G$ contained in $H$ such that $G/\widetilde{U} = \Pi_1(\mathcal{G}_U, \Gamma)$ is the fundamental group of a reduced quotient graph of finite $p$-groups of $(\mathcal{G}_U, \Gamma)$ over the same underlying graph $\Gamma$. Then, by Theorem 13, $H/\tilde{U} = \Pi_1(\mathcal{H}_U, \Delta_U, v_U)$ is the fundamental group of a finite reduced graph of pro-$p$ groups with $|E(\Delta_U)| > n$. It follows that $|E(\Delta_V)| > n$ for each open normal $V$ contained in $U$. By the proof of Theorem 9, the set $\{(\mathcal{H}_V, \Delta_V) \mid V \leqslant_o U\}$ contains a subset that form a surjective inverse system $\{(\mathcal{H}_{V_j}, \Delta_{V_j})\}$ with $(\mathcal{H}, \Delta) = \varprojlim(\mathcal{H}_{V_j}, \Delta_{V_j})$ being the reduced graph of pro-$p$ groups such that $H = \Pi_1(\mathcal{H}, \Delta)$. Moreover, it is proved in Theorem 11 that edge groups of $(\mathcal{H}, \Delta)$ are virtually $\mathcal{F}$. Therefore, $|E(\Delta)| > n$ for an arbitrary chosen $n \in \mathcal{N}$ contradicting $\mathcal{F}$-acessibility of $H$.

Suppose now $G$ is $\mathcal{F}$-accessible with accessibility number $m$ and $H$ is not. Then for any $n \in \mathcal{N}$ there exists a finite reduced graph of pro-$p$ groups $(\mathcal{H}, \Delta)$ such that $H = \Pi_1(\mathcal{H}, \Delta)$ with edge groups in $\mathcal{F}$ and $|E(\Delta)| > n$. Again it follows from the proof of Theorem 9 that there exists an open normal subgroup $U$ of $G$ contained in $H$ such that $H_U = (\mathcal{H}_U, \Delta, v)$ is the fundamental group of a reduced quotient graph of finite $p$-groups with the same underlying graph $\Delta$. On the other hand, the graph of groups $(\overline{\mathcal{G}}_U, \overline{\Gamma}_U)$ with $\widetilde{G}_U = \Pi_1(\overline{\mathcal{G}}_U, \overline{\Gamma}_U, \overline{v}_U)$ constructed in the proof of Theorem 9 must have at most $m$ edges and therefore by Theorem 1.6.20 and Remark 3.1.2, $\Delta$ has at most $m[G : H]$ edges. This contradiction completes the proof of the theorem. $\square$

## 3.4 Adaptation of Wilkes' example

In this section we show that our Theorem 9 also works for the inaccessible finitely generated group $J$ presented by Wilkes in [46, Section 4.2]. This means that any pro-$p$ group containing $J$ as an open subgroup splits as the fundamental group of a profinite graph of pro-$p$ groups, and in fact as free amalgamated product over a finite group.

**Example 3.4.1.** First define the map $\mu_n : \{0, ..., p^{n+1} - 1\} \longrightarrow \{0, ..., p^n - 1\}$ by sending an integer to its remainder modulo $p^n$. Define $H_n = \mathbb{F}_p[\{0, ..., p^n - 1\}]$ to be the $\mathbb{F}_p$-vector space with basis $\{h_0, ..., h_{p^n-1}\}$. There are inclusions $H_n \subseteq H_{n+1}$ given by inclusions of bases, and retractions $\eta_n : H_{n+1} \longrightarrow H_n$ defined by $h_k \to h_{\mu_n(k)}$. Note also that there is a natural action of $\mathbb{Z}/p^n\mathbb{Z}$ on $H_n$ given by cyclic permutation of the basis elements, and that these actions are compatible with the retractions $\eta_i$. The inverse limit of the $H_n$ along these retractions is the completed group ring $H_\infty = \mathbb{F}_p[[\mathbb{Z}_p]]$ with multiplication ignored. The continuous action of $\mathbb{Z}_p$ on the given basis of $H_\infty$ allows to form a sort of a pro-$p$ wreath product $H_\omega = \mathbb{F}_p[[\mathbb{Z}_p]] \rtimes \mathbb{Z}_p = \varprojlim(H_n \rtimes \mathbb{Z}/p^n)$ which is a pro-$p$ group into which $H_\infty$ embeds.

Next set $K_n = \mathbb{F}_p \times H_n = \langle k_n \rangle \times H_n$. Set $G_1 = K_1 \times \mathbb{F}_p$. For $n > 1$, let $G_n$ be a finite $p$-group with presentation $G_n = \langle k_{n-1}, k_n, h_0, \ldots, h_{p^n-1} \mid k_i^p = h_i^p = 1, h_i \leftrightarrow h_j, k_{n-1} \leftrightarrow h_i$ for all $i \neq p^{n-1}$, $k_n = [k_{n-1}, h_{p^{n-1}}]$ central$\rangle$ where $\leftrightarrow$ denotes the relation 'commutes with'.

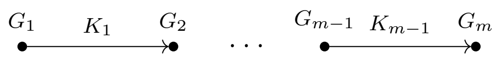
The choice of generator names describes maps $H_n \longrightarrow G_n$, $K_{n-1} \longrightarrow G_n$, and $K_n \longrightarrow G_n$. One may easily see that all three of these maps are injections. Define a retraction map

$$\rho_n : G_n \longrightarrow K_{n-1}$$

by killing $k_n$ and by sending $h_k \to h_{\mu_{n-1}(k)}$. Note that $\rho_n$ is compatible with $\eta_n : H_n \to H_{n-1}$ that is, there is a commuting diagram

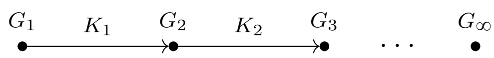
$$\begin{tikzcd}
K_{n-1} \arrow[r, hook, shift left] & G_n \arrow[l, shift left, "\rho_n"] \\
H_{n-1} \arrow[u, hook] \arrow[r, hook, shift left] & H_n \arrow[u, hook] \arrow[l, shift left, "\eta_n"]
\end{tikzcd}$$

Define $\Pi_1(\mathcal{G}_m, \Gamma_m, v_m)$ to be the pro-$p$ fundamental group of the following graph of groups:




Note that the retraction $\rho_n : G_n \longrightarrow K_{n-1}$ induces the retraction $P_{m+1} \longrightarrow P_m$ represented by the collapse the last right edge of the picture.

Then $P = \varprojlim_{m \in \mathcal{N}} \Pi_1(\mathcal{G}_m, \Gamma_m, v_m)$ is the fundamental group of the following profinite graph of pro-$p$ groups



where the vertex at infinity is a one point compactification of the edge set of the graph and so does not have an incident edge to it; thus the edge set is not compact. The vertex group $G_\infty$ of the vertex at infinity is $G_\infty = K_\infty = \varprojlim_{i\in\mathcal{N}} K_i = H_\infty$. Let $J = P \amalg_{H_\infty} H_\omega$. Then $J$ is the fundamental group of the following profinite graph of groups

$$\overset{G_1}{\bullet} \xrightarrow{\;K_1\;} \overset{G_2}{\bullet} \quad \cdots \quad \overset{H_\infty}{\bullet} \xrightarrow{\;H_\infty\;} \overset{H_\omega}{\bullet}$$

By [46, Section 4.3], this graph of pro-$p$ groups is injective and by [46, Section 4.4] $J = \langle G_1, H_\omega\rangle$. Since $G_1$ is finite and $H_\omega$ is 2-generated, $J$ is finitely generated (in fact for $p = 2$ the group $J$ is 3-generated). Collapsing the right edge we shall get the reduced graph of pro-$p$ groups since no vertex group equals to an edge group of an incident edge. Note that the latter graph of groups has a unique vertex $\infty$ whose vertex group is infinite and isomorphic to $\mathbb{F}_p \wr \mathbb{Z}_p$ which does not split over a finite $p$-group, so satisfies the hypotheses of Theorem 9. This means that any pro-$p$ group $G$ containing $J$ as an open subgroup, splits as the fundamental group of a profinite graph of pro-$p$ groups $(\mathcal{G}, \Gamma)$. Note however, that since $J$ is generated by vertex groups arguing as in the proof of Theorem 5 one can deduce that $G/\tilde{G}$ is finite and so is trivial (cf. [9, Proposition 3.4]. By 1.3.11 $\Gamma$ must be a pro-$p$ tree and so $G$ splits as a free amalgamated pro-$p$ product by Theorem 1.6.25.

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