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Instituto de Ciências Exatas
Departamento de Matemática

Ricci-Bourguignon Flows and Warped Products

por

Valter Borges

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Valter Borges †

sob orientação da

Profa. Dra. Keti Tenenblat

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos necessários para obtenção do título de Doutor em Matemática.

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Valter Borges Sampaio Júnior

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Profa. Dra. Keti Tenenblat– Orientadora (MAT-UnB)

Prof. Dr. João Paulo dos Santos (MAT-UnB)

Prof. Dr. Detang Zhou (UFF)

Prof. Dr. Enani de Sousa Ribeiro Júnior (UFC)

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"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

Carl Friedrich Gauss

Resumo

Esta tese tem três propósitos principais. Primeiro, investigamos soluções do fluxo de Ricci que preservam a estrutura de produto torcido. Neste caso, nós mostramos que a fibra é uma variedade de Einstein e a equação do Fluxo de Ricci na variedade produto é equivalente a um sistema de equações de evolução na base. Em seguida, consideramos soluções do fluxo Ricci que preservam a estrutura do produto torcido e são definidos para todo tempo negativo, as chamadas soluções anciãs. Nós provamos a não-existência de tais soluções quando o produto torcido tem base compacta e a constante de Einstein de sua fibra é não-positiva.

Em segundo lugar, estudamos quase solitons de Ricci em produtos torcidos. Mostramos que um quase soliton de Ricci gradiente em um produto torcido, $(B^n \times_h F^m, g, f, \lambda)$, cuja função potencial f depende da fibra, ou é um sóliton de Ricci ou λ não é constante e o produto torcido, a base, e a fibra são variedades de Einstein que admitem campos de vetores conformes. Assumindo a completude, uma classificação é fornecida para os quase solitons Ricci em produtos torcidos, cujas funções potenciais dependem da fibra. Uma propriedade importante da função potencial é provada, mais precisamente, sua decomposição em termos de funções que dependem ou da base ou da fibra. No caso de um sóliton de Ricci completo, provamos que a função potencial depende apenas da base.

O terceiro tema ocupa-se do fluxo de Ricci-Bourguignon em uma variedade compacta de dimensão 3. Esta é uma família de equações de evolução a 1 parâmetro, ρ, e nós mostramos que a importante estimativa de Hamilton-Ivey é verdadeira quando esse parâmetro está no intervalo $(-1/2, 1/4)$. Como consequência dessa desigualdade, mostramos que as soluções anciãs, em variedades tridimensionais compactas, possuem curvatura seccional não negativa, quando sua curvatura escalar é uniformemente limitada no tempo e $\rho \in (-1/2, 1/4)$.

Palavras Chave: Fluxo de Ricci, Fluxo de Ricci-Bourguignon, Desigualdade de Hamilton-Ivey, Soliton de Ricci, Almost Soliton de Ricci, Variedade de Einstein, Produto Warped, Campos Conformes.

Abstract

This thesis has three main purposes. First we investigate solutions of the Ricci flow which preserve the warped product structure. In this case, we show that the fiber is an Einstein manifold and the Ricci Flow equation on the product manifold is equivalent to a system of evolution equations on the base. We then turn to the solutions of the Ricci Flow that preserve the warped product structure and are defined for all negative time, the so called ancient solutions. We prove the nonexistence of such solutions if the base is compact and the Einstein constant of its fiber is non positive.

Secondly we study Ricci Almost Soliton on warped products. It is shown that a gradient Ricci almost soliton on a warped product, $(B^n \times_h F^m, g, f, \lambda)$ whose potential function f depends on the fiber, is either a Ricci soliton or λ is not constant and the warped product, the base and the fiber are Einstein manifolds, which admit conformal vector fields. Assuming completeness, a classification is provided for the Ricci almost solitons on warped products, whose potential functions depend on the fiber. An important decomposition property of the potential function in terms of functions which depend either on the base or on the fiber is proven. In the case of a complete Ricci soliton, the potential function depends only on the base.

The third theme is concerned with the Ricci-Bourguignon Flow on a compact 3-dimensional manifold. This is a family of evolution equations on a parameter ρ and we show that the important Hamilton-Ivey estimate holds when ρ lies in $(-1/2, 1/4)$. As a consequence of this inequality, we show that ancient solutions on compact three dimensional manifolds with scalar curvature uniformly bounded on time, has positive sectional curvature, provided $\rho \in (-1/2, 1/4)$.

Keywords: Ricci Flow, Ricci-Bourguignon Flow, Hamilton-Ivey Inequality, Ricci Soliton, Ricci Almost Soliton, Einstein Manifold, Warped Product, Conformal Fields.

Contents

Introduction

A family of Riemannian metrics $g(t)$, $t \in [0, T)$, on a manifold M^n is called a *Ricci Flow* if the following evolution equation is satisfied:

$$\frac{\partial}{\partial t} g(t) = -2Ric(t), \tag{1}$$

where $Ric(t)$ is the Ricci tensor of the metric $g(t)$.

Since it was introduced, the Ricci flow has drown the attention of many mathematicians, mainly because it has shown to be a powerful technique in order to solve many important problems in Differential Geometry, such as the Poincaré Conjecture [44]. Its first appearance was in a paper written by Richard Hamilton [29], where compact three manifolds carrying metrics of positive Ricci curvature were classified by using the Ricci flow. One of the main reasons, why Ricci flow was successfully implemented in dimension three, is because in this dimension Ricci flow preserves positivity of Ricci curvature. Even in this dimension, if one removes the assumption of positivity on the Ricci tensor, the problem of controlling the Ricci flow becomes much harder. One approach that may help understanding the general case, where the lack of positivity of the Ricci tensor is present, is to put more symmetry into the problem. One can assume, for instance, the manifold to be rotationally symmetric, a symmetric space, a Lie group and so on. A first problem to be analyzed is if the flow preserves such a symmetry. Another problem to be concerned with is to understand the singular solutions, if any, inside the chosen class of symmetry. An important notion in the process of understanding such singular solution is that of ancient solutions. A solution of the Ricci Flow is called an *ancient solution* if it is defined on a set $t \in (-\infty, T)$, $T \in \mathbb{R}$.

In this thesis, we investigate solutions of the Ricci flow which preserves warped products. We characterize such solutions and then we get necessary conditions for this property to be preserved, since the Ricci flow does not preserve warped product structure in general. More precisely we have:

Theorem A *Let* $(M^{n+m} = B^n \times_{h_0} F^m, g_0)$ *be a warped product of* (B^n, g_B^0) *and* (F^m, g_F^0) *with non constant warping function* $h_0 : B \to (0, \infty)$ *and* $g_0 = g_B^0 + h_0^2 g_F^0$. *Let* $(M^{n+m}, g(t))$, $t \in [0, \varepsilon)$, $\varepsilon \in (0, \infty]$, *be a Ricci flow such that* $g(0) = g_0$. *The flow* $(M^{n+m}, g(t))$ *preserves the warped product structure of* (M^{n+m}, g_0) *if, and only if,* (F^m, g_F^0) *is an Einstein manifold and there exists a family of smooth functions* $u(t) : B \to \mathbb{R}$ *such that*

$$g(t) = g_B(t) + e^{2u(t)} g_F^0 \tag{2}$$

$$\frac{\partial}{\partial t} g_B(t) = -2Ric(g_B(t)) + 2m\nabla_{g_B(t)}\nabla_{g_B(t)} u(t) + 2m du(t) \otimes du(t), \tag{3}$$

$$\frac{\partial}{\partial t} u(t) = \Delta_{g_B(t)} u(t) + m|\nabla_{g_B(t)} u(t)|^2 - \frac{R_F^0}{m} e^{-2u(t)}, \tag{4}$$

where R_F^0 *is the constant scalar curvature of the fiber.*

Aiming the understanding of singularities of the Ricci flow which preserves the warped product property, using the Scalar Maximum Principle for evolving metrics, we prove that there are no ancient solutions that are warped product along the time t, its fiber has non positive scalar curvature and its base is compact.

Theorem B *Let* $(M^{n+m} = B^n \times F^m, g(t))$ *be an ancient Ricci Flow that is warped product along the time* t *and that has compact base. Then* $(F^m, g_F(t))$ *is Einstein for each time* t *with positive Einstein constant.*

A *self-similar* solution of the Ricci Flow is a solution of the Ricci flow whose evolution is performed by means of scaling and diffeomorphisms. Applying the last theorem to self-similar solutions we prove that neither shrinking nor steady Ricci Flow that preserves warped product can exist when its base is compact and its fiber has non positive scalar curvature (see Corollary 5), a result proved in [23] using an Elliptic Maximum Principle.

Self-similar solutions of the Ricci flow have a static formulation (i.e., described by an equation independent of time). They are called *Ricci solitons*, and are defined as Riemannian manifolds (M^n, g) for which there are a vector field X and a constant λ satisfying the equation

$$Ric + \frac{1}{2}\mathfrak{L}_X g = \lambda g.$$

When the vector field X is the gradient of a function $f : M \to \mathbb{R}$, then the Ricci soliton is called *gradient Ricci soliton* and we use the notation (M, g, f, λ). *Ricci almost solitons* were introduced by Pigola and collaborators [46] by allowing the constant λ to become a function. The second goal of this thesis is to understand semi-Riemannian warped product manifolds admitting a structure of Ricci almost solitons. As a starting result, we have an important decomposition property of the potential function in terms of functions which depend either on the base or on the fiber. Such fact allows us to break down the fundamental equation of a Ricci almost soliton into equations on the base and on the fiber.

The result is based on the following proposition:

Proposition C *Let $(B^n \times_h F^m, g, f, \lambda)$ be a Ricci almost soliton defined on a semi-Riemannian warped product manifold, where the base (B^n, g_B) or the fiber (F^m, g_F) are either Riemannian or semi-Riemannian manifolds, $h : B \to \mathbb{R}$ is a positive smooth function and $g = g_B + h^2 g_F$. Then the potential function f can be decomposed as*

$$f = \beta + h\varphi, \tag{5}$$

where $\beta : B \to \mathbb{R}$ and $\varphi : F \to \mathbb{R}$ are smooth functions.

This decomposition has some consequences. Among them is the one concerning complete warped product Ricci solitons with non constant warping function. Before we state the result, it is worth remarking that in most of the examples of Ricci solitons built on warped products the assumption in which the warping function does not depend on the fiber is carried out. The result then says that in the presence of completeness of the base, this is satisfied automatically.

Corollary D *Let $(B \times_h F, g, f, \lambda)$ be a Ricci soliton on a complete non trivial semi-Riemannian warped product. Then f does not depend on the fiber.*

Another consequence of the decomposition stated above is that when the potential function depends on the fiber a rigidity result is obtained, in the sense that when the function λ is not constant, the manifold is necessarily an Einstein manifold. More concisely:

Theorem E *If $(B^n \times_h F^m, g, f, \lambda)$ is a non-trivial warped product Ricci almost soliton with f non constant on F, then either λ is not constant and $(B^n \times_h F^m, g)$ is an Einstein manifold or $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci soliton.*

Assuming completeness, the above theorem together with the previous corollary tells us that a Ricci almost soliton on a warped product manifold with non constant warping function and potential function depending on its fiber must be an Einstein manifold. Now the fundamental equation of the Ricci almost soliton forces the gradient of the potential function to become a conformal field, from where a classification is provided for such spaces (see Theorem 15).

The third theme to be studied in this thesis is the Hamilton-Ivey estimate in the context of the *Ricci-Bourguignon Flow*, which is a family of Riemannian metrics $g(t)$, $t \in [0, T)$, on a manifold M^n, satisfying the following perturbation of the Ricci Flow

$$\frac{\partial}{\partial t} g(t) = -2(Ric(t) - \rho R(t) g(t)), \tag{6}$$

where $Ric(t)$ and $R(t)$ are the Ricci tensor and the scalar curvature, respectively, of the metric $g(t)$ and ρ is a real parameter. To better explain the results, in what follows we will comment on previous results both for the Ricci Flow and for the Ricci-Bourguignon Flow. The Hamilton-Ivey estimate was first proved independently by Ivey [31] and Hamilton [28] for the Ricci Flow, that is, when $\rho = 0$. Some consequences of these estimates were given by both authors. In [14] Catino, Cremaschi and coworkers generalized several properties of the Ricci flow to the case where $\rho \neq 0$. See also Cremaschi's thesis [17] for more on that, including a list of open problems. Regarding the existence of solutions in the compact case, for general dimension n, they found out that it is true once the constant ρ lies in the range $(-\infty, 1/(2(n-1)))$, a result also proved in [41]. For dimension 3 we then have that a solution exists when $\rho \in (-\infty, 1/4)$ for any initial metric. Since the Ricci flow is a particular case of this family of flows studied in [18] for $\rho = 0$, a natural question is whether the Hamilton-Ivey estimate is true or not for the Ricci-Bourguinon flow corresponding to some $\rho \neq 0$. It turns out that Catino and coworkers, in the same work, showed that such an estimate is true for solutions of the Ricci-Bourguinon flow on compact manifolds, provided $\rho \in [0, 1/6)$. Our main goal concerning the Ricci-Bourguignon flow in this thesis is to investigate this important inequality for all ρ in which a solution is guaranteed, i.e., for $\rho \in (-\infty, 1/4)$. When ρ is positive and below 1/4 we proved that solutions to the Ricci-Bourguignon in dimension 3 satisfy the Hamilton-Ivey estimate, which extends the result earlier mentioned. We then have,

Theorem F *Let M^3 be a compact three manifold, $\rho \in [0, 1/4)$ and g_0 be a Riemannian metric on M satisfying the normalized assumption $\min_{p\in M} \nu_0(p) \geq -1$, where ν_0 is the smallest sectional curvature of g_0. If $g(t)$, $t \in [0, T)$, is the solution of the Ricci-Bourguignon Flow corresponding to ρ satisfying $g(0) = g_0$, then the scalar curvature $R(t)$ of $g(t)$ satisfies*

$$R \geq -\nu(\log(-\nu) + \log(1 + 2(1 - 4\rho)t) - 3), \tag{7}$$

at any point (p, t) where the smallest sectional curvature $\nu(p, t)$ of $g_p(t)$ is negative.

Turning to the negative case, we proved that the result does hold, provided the parameter ρ is above $-1/2$. In this case the estimate takes a shape that is slightly different from that in the positive case, since one needs to deform the eigenvalues of the curvature operator, in order to be able to apply the Tensor Maximum Principle, proved for the Laplace operator in [27] and extended to other differential operators later in [14]. That explains the constants in the statement below.

Theorem G *Let M^3 be a compact three manifold, $\rho \in (-1/2, 0]$, $\theta_1 = 4\theta_2 = 1/2(2\rho^2 - 2\rho + 1)$ and let g_0 be a Riemannian metric on M satisfying the normalized assumption $\min_{p\in M} \nu_0(p) \geq -1$, where ν_0 is the smallest sectional curvature of g_0. If $g(t)$, $t \in [0, T)$, is the solution of the Ricci-Bourguignon Flow corresponding to ρ satisfying $g(0) = g_0$, then the scalar curvature $R(t)$ of $g(t)$ satisfies*

$$R \geq -\nu(\theta_1 \log(-\nu) + \theta_2 \log(1 + 2(1 + 2\rho)t) - 3), \tag{8}$$

at any point (p, t) where the smallest sectional curvature $\nu(p, t)$ of $g_p(t)$ is negative.

Among the most important consequences of the Hamilton-Ivey estimate is the one where ancient solutions of the Ricci flow are proven to have nonnegative sectional curvature [28]. This geometric consequence was obtained in [14] for singularity models of the Ricci-Bourguignon Flow when $\rho \in [0, 1/4)$. It follows from Theorem G that it is true for ancient solutions of the Ricci-Bourguignon Flow with $\rho \in (-1/4, 1/2)$, as we state below.

Theorem H *Let $(M^3, g(t))$, $t \in (-\infty, T)$, be a compact ancient solution of the Ricci-Bourguignon flow with uniformly bounded scalar curvature. Assume that $\rho \in (-1/2, 1/4)$. Then $g(t)$ has nonnegative sectional curvature, for as long as it exists.*

Brasília, November 28, 2018

Valter Borges

Chapter 1

Preliminaries

The aim of this chapter is to present the preliminaries we need in order to state and prove the main results of this thesis.

In the first section, Section 1.1, we introduce the Ricci-Bourguignon Flow. We give a list of the results that will be used in Chapter 4. We outline the important Uhlenbeck's trick for the equation of the curvature operator, used for the first time by Hamilton in [29], and then we present features of the three dimensional case. We end this section introducing the concept of almost Ricci soliton.

In Section 1.2, we describe Riemannian and semi-Riemannian warped products. Such manifolds will be considered both in Chapter 2 and Chapter 3.

In Section 1.3, we introduce the important notion of conformal vector fields and then we concentrate in the gradient ones. We present a local characterization which says that Riemannian manifolds carrying such vector fields are a warped product of a line and an $(n-1)$-dimensional hypersurface in a neighborhood of a regular point. This section ends with the classification of Einstein manifolds carrying conformal vector fields.

1.1 Ricci-Bourguignon Flow

In 1981 Bourguignon [7] introduced the problem of studying an evolutionary version of the Einstein equation. He suggested the investigation of a one parameter family of evolutionary equations and indicated some directions to be followed. It turns out that when the parameter is zero, then the evolution corresponds to the Ricci flow, that has been intensively studied in the last years. The Ricci flow plays an important role in proving important results in Differential Geometry. The celebrated Poincaré Conjecture, for example, was proved in [44] by Perelman, following Hamilton's program using Ricci Flow. In his seminal paper, Hamilton [29] showed the power of this technique by classifying closed 3 manifolds with positive Ricci curvature. Since then, Ricci flow has been intensively investigated.

The family of flows was later called the Ricci-Bourguignon flow, and some results known for the Ricci flow were extended to the Ricci-Bourguignon Flow by Catino, Cremaschi and collaborators [14]. It is worthwhile to note that some of these results are sensitive to the range of the parameter that occurs in its definition (see Definition (1)).

1.1.1 Definition and Existence

Let $g(t)$, $t \in [0, T)$, be a family of Riemannian metrics on an $n-$dimensional manifold M^n. For each $t \in [0, T)$, let $\nabla(t)$, $Rm(t)$, $Ric(t)$ and $R(t)$ be the Levi-Civita connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively, associated to the metric $g(t)$. In what follows we will omit the t whenever it is clear from the context.

Definition 1 We say that a 1-parameter family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$ on a manifold M^n is a *solution of the Ricci-Bourguignon flow* if it satisfies the equation

$$\frac{\partial}{\partial t} g = -2(Ric - \rho R g), \tag{1.1}$$

where ρ is a given constant. We say that a Ricci Bourguignon Flow $\{g(t)\}_{t\in[0,T)}$ on M starts at g_0 if $g(0) = g_0$.

The tensor on the right hand side of (1.1) has special interest for certain values of ρ. For instance, if $\rho = 1/(2(n-1))$ the corresponding tensor is the Schouten tensor, an important tensor in conformal geometry. When $\rho = 0$ it is the Ricci tensor and the flow gives origin to the *Ricci flow*.

A general existence result concerning the Ricci-Bourguignon Flow on compact manifolds is the following:

Theorem 1 ([14], [41]) *Consider a compact Riemannian manifold (M^n, g_0) of dimension n. If $\rho < 1/(2(n-1))$, then there exists a unique Ricci-Bourguignon Flow starting at g_0.*

This theorem was first proved by Hamilton [29] in the case where $\rho = 0$. We observe that the proof of the theorem above highlights the nonexistence of Ricci-Bourguignon Flows for generic initial metrics on compact manifolds, when $\rho > 1/(2(n-1))$. It is still an open problem to determine whether (1.1) admits a solution for general initial metrics in the case $\rho = 1/(2(n-1))$, as it was observed in [14].

1.1.2 Evolution Equation of the Curvature Operator

For later use we first consider the general case of a smooth family of metrics $\{g(t)\}_{t\in(-\varepsilon,\varepsilon)}$ satisfying the more general evolution equation

$$\frac{\partial}{\partial t} g = \sigma, \tag{1.2}$$

where σ is a symmetric two tensor depending smoothly on t. Our goal here is to collect the variation formulas for the curvature tensors of $g(t)$. The proof of the proposition below can be found either in [5], page 62, or in [18], Chapter 3. We use local coordinates to state the formulas.

Proposition 1 *If a family of metrics $\{g(t)\}_{t\in(-\varepsilon,\varepsilon)}$ evolves via (1.2), then:*

1. *the inverse $g^{-1}(t)$ of the metric $g(t)$ evolves by the equation*

$$\frac{\partial}{\partial t} g^{ij} = -g^{ai} g^{jb} \sigma_{ab}; \tag{1.3}$$

2. *the curvature operator $Rm(t)$ of the metric $g(t)$ evolves by the equation*

$$\begin{aligned}\frac{\partial}{\partial t} R^l_{ijk} = \frac{1}{2} g^{lr} \big[&\nabla_i \nabla_j \sigma_{kr} + \nabla_i \nabla_k \sigma_{jr} - \nabla_i \nabla_r \sigma_{jk} - \nabla_j \nabla_i \sigma_{kr} \\ &- \nabla_j \nabla_k \sigma_{ir} + \nabla_j \nabla_r \sigma_{ik} \big];\end{aligned} \tag{1.4}$$

3. *the Ricci tensor $Ric(t)$ of the metric $g(t)$ evolves by the equation*

$$\frac{\partial}{\partial t} R_{ij} = \frac{1}{2} g^{ab} (\nabla_a \nabla_i \sigma_{jb} + \nabla_a \nabla_j \sigma_{ib} - \nabla_a \nabla_b \sigma_{ij} - \nabla_i \nabla_j \sigma_{ab}); \tag{1.5}$$

4. *the scalar curvature $R(t)$ of the metric $g(t)$ evolves by the equation*

$$\frac{\partial}{\partial t}R = \Delta tr(\sigma) - div(div\ \sigma) - \langle \sigma, Ric \rangle\,, \tag{1.6}$$

where $tr(\sigma)$ is the trace of $\sigma(t)$ with respect to $g(t)$ and div, Δ and $\langle , \rangle$ are taken with respect to $g(t)$.

As an application of the evolution equation (1.3), Proposition 1, we have the following result, which will be important in Section 2.2.

Proposition 2 *Let $(M^n, g(t), u(t))$, $t \in [0, T)$, be a family satisfying the coupled system of evolution equations*

$$\frac{\partial}{\partial t}g(t) = -2Ric(t) + 2m\nabla_{g(t)}\nabla_{g(t)}u(t) + 2mdu(t) \otimes du(t), \tag{1.7}$$

$$\frac{\partial}{\partial t}u(t) = \Delta_{g(t)}u(t) + m|\nabla_{g(t)}u(t)|^2 - \frac{r}{m}e^{-2u(t)}, \tag{1.8}$$

where $m \in \mathbb{N}$ and $r \in \mathbb{R}$. Then

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\nabla_{g(t)}u|^2 = -2|\nabla_{g(t)}\nabla_{g(t)}u|^2 + m\left\langle \nabla_{g(t)}u, \nabla_{g(t)}(|\nabla_{g(t)}u|^2)\right\rangle \tag{1.9}$$
$$-2m|\nabla_{g(t)}u|^4 + \frac{4r}{m}e^{-2u}|\nabla_{g(t)}u|^2.$$

Proof: Fix a normal coordinate system $(x^1, \ldots, x^n)$ around a point in M. We will first prove that:

1. The following identities hold

$$\nabla_i\Delta u = \Delta\nabla_i u - R_{ia}\nabla^a u, \tag{1.10}$$

$$\Delta|\nabla u|^2 = 2\langle \nabla u, \Delta\nabla u\rangle + 2|\nabla\nabla u|^2. \tag{1.11}$$

2. The inverse $g^{-1}(t)$ of the metric $g(t)$ evolves by the equation

$$\frac{\partial}{\partial t}g^{ij} = 2g^{ai}g^{jb}(R_{ab} - m\nabla_a\nabla_b u - m\nabla_a u\nabla_b u). \tag{1.12}$$

3. The evolution equation of ∇u is

$$\frac{\partial}{\partial t}\nabla u = \left(\Delta\nabla^j u + R_a^j\nabla^a u + 2\left(\frac{r}{m}e^{-2u} - m|\nabla u|^2\right)\nabla^j u\right)\partial_j. \tag{1.13}$$

Let us prove (1.10) and (1.11):

$$\begin{aligned}
\partial_i(\Delta u) =& \partial_i(g^{ab}\nabla_a\nabla_b u) \\
=& \underbrace{\partial_i(g^{ab})}_{=0}\nabla_a\nabla_b u + g^{ab}\underbrace{\partial_i\nabla_a\nabla_b u}_{\nabla_i\nabla_a\nabla_b u} \\
=& g^{ab}\nabla_i\nabla_a\nabla_b u \\
=& g^{ab}\nabla_a\nabla_b\nabla_i u - R_{ia}\nabla^a u \\
=& \Delta\nabla_i u - R_{ia}\nabla^a u,
\end{aligned}$$

where from the third to the fourth line we used the Ricci identity, $\nabla_i\nabla_a\nabla_b u = \nabla_a\nabla_b\nabla_i u + \nabla^c u R_{iacb}$. On the other hand,

$$\begin{aligned}
\Delta|\nabla u|^2 =& g^{ab}\nabla_a\nabla_b(g^{cd}\nabla_c u\nabla_d u) \\
=& g^{ab}g^{cd}\nabla_a(\nabla_b\nabla_c u\nabla_d u + \nabla_c u\nabla_b\nabla_d u) \\
=& g^{ab}g^{cd}(\nabla_a\nabla_b\nabla_c u\nabla_d u + \nabla_b\nabla_c u\nabla_a\nabla_d u + \nabla_a\nabla_c u\nabla_b\nabla_d u + \nabla_c u\nabla_a\nabla_b\nabla_d u) \\
=& g^{cd}\Delta\nabla_c u\nabla_d u + g^{cd}\nabla_c u\Delta\nabla_d u + g^{ab}g^{cd}\nabla_b\nabla_c u\nabla_a\nabla_d u + g^{ab}g^{cd}\nabla_a\nabla_c u\nabla_b\nabla_d u \\
=& \langle\Delta\nabla u, \nabla u\rangle + \langle\Delta\nabla u, \nabla u\rangle + |\nabla\nabla u|^2 + |\nabla\nabla u|^2 \\
=& 2\langle\Delta\nabla u, \nabla u\rangle + 2|\nabla\nabla u|^2.
\end{aligned}$$

If we choose σ as

$$\sigma_{ij} = -2R_{ij} + 2m\nabla_i\nabla_j u + 2m\nabla_i u\nabla_j u, \tag{1.14}$$

then item (1.12) follows immediately from (1.3) of Proposition 2.

Now let us prove (1.13):

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^j u =& \frac{\partial}{\partial t}\left(g^{ij}\partial_i u\right) = \frac{\partial}{\partial t}\left(g^{ij}\right)\partial_i u + g^{ij}\frac{\partial}{\partial t}\left(\partial_i u\right) = \frac{\partial}{\partial t}\left(g^{ij}\right)\partial_i u + g^{ij}\partial_i\left(\frac{\partial}{\partial t}u\right) \\
=& 2g^{ai}g^{jb}\partial_i u\left(R_{ab} - m\nabla_a\nabla_b u - m\nabla_a u\nabla_b u\right) + g^{ij}\partial_i\left(\Delta u + m|\nabla u|^2 - \frac{r}{m}e^{-2u}\right)
\end{aligned}$$

$$
\begin{aligned}
=&2g^{jb}\Big(R_{ab}\nabla^a u - m\nabla_a\nabla_b u\nabla^a u - m\nabla_a u\nabla_b u\nabla^a u\Big)\\
&+ g^{ij}\Big(\nabla_i\Delta u + m\nabla_i(|\nabla u|^2) + 2\frac{r}{m}e^{-2u}\nabla_i u\Big)\\
=&2g^{jb}\Big(R_{ab}\nabla^a u - m\nabla_b\nabla_{\nabla u}u - m|\nabla u|^2\nabla_b u\Big)\\
&+ g^{ij}\Big(\Delta\nabla_i u - R_{ia}\nabla^a u + 2m\nabla_i\nabla_{\nabla u}u + 2\frac{r}{m}e^{-2u}\nabla_i u\Big)\\
=&2R^j_a\nabla^a u - 2m|\nabla u|^2\nabla^j u + \Delta\nabla^j u - R^j_a\nabla^a u + 2\frac{r}{m}e^{-2u}\nabla^j u\\
=&\Delta\nabla^j u + R^j_a\nabla^a u + 2\left(\frac{r}{m}e^{-2u} - m|\nabla u|^2\right)\nabla^j u.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 =&\frac{\partial}{\partial t}|\nabla u|^2 - \Delta|\nabla u|^2 = \frac{\partial}{\partial t}(\nabla^j u\nabla_j u) - 2\langle\nabla u, \Delta\nabla u\rangle - 2|\nabla\nabla u|^2\\
=&\frac{\partial}{\partial t}(\nabla^j u)\nabla_j u + \nabla^j u\nabla_j\left(\frac{\partial}{\partial t}u\right) - 2\langle\nabla u, \Delta\nabla u\rangle - 2|\nabla\nabla u|^2\\
=&\left[\Delta\nabla^j u + R^j_a\nabla^a u + 2\left(\frac{r}{m}e^{-2u} - m|\nabla u|^2\right)\nabla^j u\right]\nabla_j u\\
&+ \nabla^j u\nabla_j\left[\Delta u + m|\nabla u|^2 - \frac{r}{m}e^{-2u}\right] - 2\langle\nabla u, \Delta\nabla u\rangle - 2|\nabla\nabla u|^2\\
=&\langle\nabla u, \Delta\nabla u\rangle + Ric(\nabla u, \nabla u) + 2\left(\frac{r}{m}e^{-2u} - m|\nabla u|^2\right)|\nabla u|^2\\
&+ \nabla^j u\nabla_j\Delta u + m\nabla^j u\nabla_j|\nabla u|^2 + \frac{2r}{m}e^{-2u}\nabla^j u\nabla_j u\\
&- 2\langle\nabla u, \Delta\nabla u\rangle - 2|\nabla\nabla u|^2\\
=&-\langle\nabla u, \Delta\nabla u\rangle - 2|\nabla\nabla u|^2 + 2\left(\frac{2r}{m}e^{-2u} - m|\nabla u|^2\right)|\nabla u|^2\\
&+ Ric(\nabla u, \nabla u) + m\left\langle\nabla u, \nabla|\nabla u|^2\right\rangle\\
&+ \nabla^j u\Delta\nabla_j u - \nabla^j u R^a_j\nabla_a u\\
=&m\left\langle\nabla u, \nabla|\nabla u|^2\right\rangle - 2|\nabla\nabla u|^2 + 2\left(\frac{2r}{m}e^{-2u} - m|\nabla u|^2\right)|\nabla u|^2\\
&+ Ric(\nabla u, \nabla u) - \langle\nabla u, \Delta\nabla u\rangle\\
&+ \langle\nabla u, \Delta\nabla u\rangle - Ric(\nabla u, \nabla u)\\
=&-2|\nabla\nabla u|^2 + m\left\langle\nabla u, \nabla|\nabla u|^2\right\rangle + 2\left(\frac{2r}{m}e^{-2u} - m|\nabla u|^2\right)|\nabla u|^2.
\end{aligned}
$$

■

In [14], the authors proved that under the Ricci-Bourguignon flow, that is when $\sigma = -2(Ric - \rho Rg)$, the Riemann curvature tensor evolves according to a complicated reaction-diffusion equation. More precisely, (1.4) gives rise to:

Proposition 3 ([14]) *If $g(t)$, $t \in [0, T)$, is a Ricci-Bourguignon Flow, then its curvature tensor $Rm(t)$ evolves accordingly to the reaction-diffusion equation*

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &- g^{ab}(R_{ajkl}R_{bi} + R_{iakl}R_{bj} + R_{ijal}R_{bk} + R_{ijka}R_{bl}) \\ &- \rho(\nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik}) + 2\rho R R_{ijkl}, \end{aligned} \tag{1.15}$$

where $B_{ijkl} = g^{ab}g^{cd}R_{iajc}R_{kbld}$ and the time dependent Laplacian $\Delta = \Delta_t$ is defined by

$$\Delta_t R(t)_{ijkl} = g(t)^{ab}\nabla(t)_{\frac{\partial}{\partial x_a}}\nabla(t)_{\frac{\partial}{\partial x_b}} R(t)_{ijkl}.$$

The proof is a straightforward calculation that uses the Bianchi identities several times to go from (1.4) to (1.15). For more details, see [29] or [18] for the case $\rho = 0$ and [14] or [17] for the general case.

In what follows we describe the Uhlembeck's trick, which consists in using a suitable evolving bundle isometry for rewriting (1.15), in order to obtain an equation simpler than the previous one. It was first implemented by Hamilton in [29]. Later Catino and coworkers generalized it for the Ricci-Bouguignon Flow in [14]

Let $g(t)$, $t \in [0, T)$, be a Ricci-Bourguignon Flow on a manifold M^n. Consider a family of bundle isomorphisms $\varphi(t) : TM \to TM$, $t \in [0, T)$, satisfying

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = Ric^\sharp \circ \varphi(t) - \rho R\varphi(t), \\ \varphi(0) = Id_{TM}, \end{cases} \tag{1.16}$$

where $Ric^\sharp : TM \to TM$ is the self adjoint operator, depending on t, defined by

$$g(t)(Ric^\sharp(X), Y) = Ric(t)(X, Y),$$

for all $X,\ Y \in TM$. The existence of solutions for equation (1.16) follows from the fact that, for each $p \in M$ it gives rise to an ODE system in the vector space T_pM, whose vector field is smooth and then the general existence result applies [18].

An important fact is that the family of metrics $\varphi(t)^*g(t)$ does not depend on t (see [18], page 181) and then, $g_0 = \varphi(t)^*g(t)$, for all $t \in [0, T)$. This means that

$$\varphi(t) : (TM, g_0) \to (TM, g(t))$$

is a family of isometries. Next consider the pulled back Levi-Civita connections $D(t)$ defined by

$$D(t)_X Y = \varphi^*(\nabla_X(\varphi^* Y)),$$

where we have omitted the t of $\varphi(t)$ and $\nabla(t)$ in the right hand side for the sake of simplicity. Notice that for each $t \in [0, T)$, since φ is a bundle isometry, the connection $D(t)$ is compatible with g_0. One can represent the Euclidean bundle with a family of connections $(TM, g_0, D(t))$ by using an abstract notation, namely $(V, l, D(t))$, for emphasizing the fact that the connection used is not the Levi-Civita connection. We still denote by $D(t)$ the natural extension of the previously defined connection to the tensor bundles associated to V.

The importance of the pull back by $\varphi(t)$ of the Riemannian curvature operator, $P(t) = \varphi^* Rm(t) : \wedge^2 V \to \wedge^2 V$, is due to the proposition below.

Proposition 4 ([14]) *The operators $Rm(t)$ and $P(t)$ have the same eigenvalues.*

Before stating the next proposition, we would like to note that given a Euclidean bundle with a family of metric connections $(V, l, D(t))$, there is a natural way of defining the Laplacian of a tensor. For instance, if $Q \in End_{SA}(\wedge^2 V)$, the *bundle of self adjoint endomorphisms of* $\wedge^2 V$, then $\Delta_D Q : \wedge^2 V \to \wedge^2 V$ is defined as

$$\Delta_D Q = l^{ab} D(t)_{\frac{\partial}{\partial x_a}} D(t)_{\frac{\partial}{\partial x_b}} Q.$$

Remark 1 (see [18], page 184) Consider a Lie Algebra $\mathfrak{g}$ with Lie bracket $[,]$ and inner product $\langle , \rangle$. Let $\{e^\alpha\}$ be a basis of $\mathfrak{g}$ and $C_\gamma^{\alpha\beta}$ be the constant structures, defined by

$$[e^\alpha, e^\beta] = C_\gamma^{\alpha\beta} e^\gamma.$$

If L is a bilinear form in $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, and $\{e_\alpha^*\}$ is the algebraic dual basis of $\{e^\alpha\}$, one can see L as an element of $\mathfrak{g} \otimes_S \mathfrak{g}$ whose components are given by $L_{\alpha\beta} = L(e_\alpha^*, e_\beta^*)$.

We define the **Lie algebra squared of** L, $L^{\#} \in \mathfrak{g} \otimes_S \mathfrak{g}$, as the bilinear form whose coordinates are

$$(L^{\#})_{\alpha\beta} = C_{\alpha}^{\gamma\delta} C_{\beta}^{\varepsilon\zeta} L_{\gamma\epsilon} L_{\delta\zeta}.$$

Using the inner product we can now regard $L^{\#}$ as a symmetric operator on L.

We end this remark by saying how the construction above applies to the setting of the Ricci-Bourguignon Flow. Since g_0 and the bracket of vector fields extend in a canonical way to $\wedge^2 T_pM$ they provide a Lie algebra naturally isomorphic to $SO(n)$. On the other hand, Rm can be seen as a bilinear form on $\wedge^2 T_pM$, from which we can define $Rm^{\#}$.

Now we can state the main consequence of the Uhlembeck's trick.

Proposition 5 ([14]) *The family of operators $P(t) : \wedge^2 V \to \wedge^2 V$, $t \in [0, T)$, evolves accordingly to the equation*

$$\frac{\partial}{\partial t} P = \mathcal{L}P + 2P^2 + 2P^{\#} - 4\rho tr_{g_0}(P)P, \tag{1.17}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L}Q = \Delta_D Q - 2\rho\varphi^*(\nabla\nabla tr_l(Q)) \odot l, \tag{1.18}$$

and $\odot$ is the Kulkarni-Nomizu product.

We end this section with the following important proposition used in the applications of the maximum principle.

Proposition 6 ([14]) *If $\rho < 1/(2(n-1))$, then the differential operator (1.18) is uniformly elliptic.*

1.1.3 Maximum Principles

One of the most important tools in investigating partial differential equations is the Maximum Principle. In this section we are going to state some versions of this principle to be used in the forthcoming chapters. We start with the Scalar Maximum Principle and then we state an important generalization of it to tensor equations, formulated first by Hamilton [29].

Theorem 2 ([18, 19]) *Let $(M^k, g(t))$, $t \in [0,T)$, be a family of compact Riemannian manifolds smooth with respect to t. Let $u : M \times [0,T) \to \mathbb{R}$ be a function satisfying*

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta_{g(t)}\right) u(t) \leq \left\langle X(t), \nabla_{g(t)} u(t) \right\rangle + f(u(t),t), \\ u(0) \leq c, \end{cases} \tag{1.19}$$

where, $X(t)$ is a smooth one parameter family of vector fields and $f : \mathbb{R} \times [0,T) \to \mathbb{R}$ is continuous in the first factor and locally Lipschitz in the second factor. Consider the solution $\varphi : [0, \tilde{T}) \to \mathbb{R}$ of the initial value problem

$$\begin{cases} \dfrac{d\varphi}{dt}(t) = f(\varphi(t), t), \\ \varphi(0) = c. \end{cases} \tag{1.20}$$

Then $u(p,t) \leq \varphi(t)$, for all $(p,t) \in M \times [0, \min\{T, \tilde{T}\})$.

Theorem 2 is also true if we reverse all the inequalities. Then, if in (1.19) we have equality, it follows that

Corollary 1 ([18, 19]) *Let $(M^k, g(t))$, $t \in [0,T)$, be a family of compact Riemannian manifolds smooth with respect to t. Let $u : M \times [0,T) \to \mathbb{R}$ be a solution of*

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta_{g(t)}\right) u(t) = \left\langle X(t), \nabla_{g(t)} u(t) \right\rangle + f(u(t),t), \\ c_1 \leq u(0) \leq c_2, \end{cases} \tag{1.21}$$

where, $X(t)$ is a smooth one parameter family of vector fields and $f : \mathbb{R} \times [0,T) \to \mathbb{R}$ is locally Lipschitz in the first factor and continuous in the second factor. Fix $j \in \{1,2\}$ and consider the solution $\varphi_j : [0, \tilde{T}_j) \to \mathbb{R}$ of the initial value problem

$$\begin{cases} \dfrac{d\varphi_j}{dt}(t) = f(\varphi_j(t), t), \\ \varphi_j(0) = c_j. \end{cases} \tag{1.22}$$

Then $\varphi_1(t) \leq u(p,t) \leq \varphi_2(t)$, for all $(p,t) \in M \times [0, \min\{T, \tilde{T}_1, \tilde{T}_2\})$.

Now we will state the Vectorial Maximum Principle for Time-Dependent Sets. Let $(E, h, D(t))$ be a vector bundle over $(M, g(t))$, where h is a bundle metric and $D(t)$ is a family of linear connections compatible with h, for each $t \in [0,T)$.

Theorem 3 ([14]) *Let $u : [0,T) \to \Gamma(E)$ be a smooth solution of the parabolic equation*

$$\frac{\partial}{\partial t}u = \mathcal{L}u + F(u,t) \tag{1.23}$$

and $F : E \times [0,T) \to E$ a continuous map, locally lipschtiz in the E factor and $F(v,t) \in E_p$ for every $p \in M$, $v \in E_p$ and $t \in [0,T)$. For every $t \in [0,T)$, let $K(t) \subset E$ be a closed subbundle, invariant under parallel translation with respect to $D(t)$, convex in the fibers and such that the space time track

$$\mathcal{T} = \{(v,t) \in E \times \mathbb{R} : v \in K(t),\ t \in [0,T)\} \tag{1.24}$$

is closed in $E \times [0,T)$. Suppose that for every $t_0 \in [0,T)$, $K(t_0)$ is preserved by the associated ODE

$$\frac{\partial}{\partial t}Q = F(Q,t) \tag{1.25}$$

i.e., any solution $Q(t)$ of the ODE (1.25) that starts in $K(t_0)_p$ remains in $K(t)_p$, as long as it exists. If $u(0)$ is contained in $K(0)$, then $u(p,t) \in K(t)_P$, for every $p \in M$, $t \in [0,T)$.

We will end this section with the following proposition, useful in the proofs when checking invariance under parallel transport of sets.

Proposition 7 ([19], Lemma 10.11) *Let $(\mathcal{W},h)$ be a metric fiber bundle of rank $r \in \mathbb{N}$ over a manifold (M^n,g) and let $\mathcal{V} = End_{SA}(\mathcal{W})$ be the bundle of self adjoint endomorphisms of $\mathcal{W}$. Suppose $G : \Gamma \to \mathbb{R}$ is a function, where*

$$\Gamma = \{(x_1,\ldots,x_r) \in \mathbb{R}^r : x_1 \geq \ldots \geq x_r\}. \tag{1.26}$$

Given $c \in \mathbb{R}$, let

$$\mathcal{L}_c = \{u \in \mathcal{V} : G(\lambda_1(u),\ldots,\lambda_r(u)) \leq c\}, \tag{1.27}$$

where $\lambda_1(u) \geq \ldots \geq \lambda_r(u)$ are the eigenvalues of $u \in \mathcal{V}_c$. The subset $\mathcal{L}_c \subset \mathcal{V}$ is invariant by parallel translations.

1.1.4 Ricci-Bourguiginon Flow in Dimension 3

One characteristic present when working with Ricci Flow is to detect quantities preserved by the flow, as long as the flow exists. For instance, the positivity of the curvature operator is preserved in any dimension [20], and it was used in many papers. In dimension 3, besides those excellent properties satisfied by this flow in any dimension, the Ricci flow preserves the positivity of the Ricci tensor. To figure out properties of the curvature operator that are preserved by the flow, we must look at its associated ODE, introduced by Hamilton in [27]:

$$\frac{\partial}{\partial t}Q = 2Q^2 + 2Q^{\#} - 4\rho tr_{g_0}(Q)Q. \tag{1.28}$$

In dimension three, a family of operators $Q_p(t) \in S^2(\wedge^2 T_pM)$ is a solution to (1.28) if, and only if, its eigenvalues λ, μ and ν satisfy the system (see [27]),

$$\begin{cases} \lambda' = 2\lambda^2 + 2\mu\nu - 4\rho\lambda(\lambda+\mu+\nu), \\ \mu' = 2\mu^2 + 2\lambda\nu - 4\rho\mu(\lambda+\mu+\nu), \\ \nu' = 2\nu^2 + 2\lambda\mu - 4\rho\nu(\lambda+\mu+\nu). \end{cases} \tag{1.29}$$

Since we are interested in working with dimension 3, from now on, we will use system (1.29). As a first application of the system, we have that the inequality $\lambda \geq \mu \geq \nu$ is preserved in time by the system (1.29), as observed in [14] and in [28]. More precisely:

Proposition 8 *Let $\lambda(t)$, $\mu(t)$ and $\nu(t)$, $t \in [0,T)$, be a solution of (1.29) satisfying $\lambda(0) \geq \mu(0) \geq \nu(0)$. Then $\lambda(t) \geq \mu(t) \geq \nu(t)$, for all $t \in [0,T)$. Furthermore, if $\lambda(t_0) = \mu(t_0)$ (resp. $\mu(t_0) = \nu(t_0)$ or $\lambda(t_0) = \nu(t_0)$) for a $t_0 \in [0,T)$, then $\lambda \equiv \mu$ (resp. $\mu \equiv \nu$ or $\lambda \equiv \nu$).*

Proof: Using (1.29) one can see that

$$\begin{aligned}(\lambda-\mu)' &= 2(\lambda^2-\mu^2) + 2(\mu\nu - \lambda\nu) - 4\rho(\lambda+\mu+\nu)(\lambda-\mu) \\ &= 2(\lambda-\mu)\Big[\lambda+\mu-\nu-2\rho(\lambda+\mu+\nu)\Big] \quad .\end{aligned} \tag{1.30}$$

Therefore, if $(\lambda-\mu)(t_0) = 0$, then (1.30) guarantees that $(\lambda-\mu)'(t_0) = 0$. By the theorem of Existence and Uniqueness of solutions to an ODE, $(\lambda-\mu)(t)$ must be the solution identically zero, which gives $\lambda(t) = \mu(t)$, for all $t \in [0, T)$. It implies that if $\lambda(0) > \mu(0)$, then $\lambda(t) > \mu(t)$, for all $t \in [0, T)$. The other cases are proven analogously. ■

On the other hand, the trace of $Q_p(t)$, $tr(Q_p(t)) = \lambda + \mu + \nu$, satisfies the following differential inequality, important in the proof of Lemma 4 and Lemma 5, in Chapter 2.

Proposition 9 *If $Q_p(t)$ is a solution of (1.28), then its trace satisfies*

$$(tr(Q_p(t)))' \geq \frac{4}{3}(1-3\rho)tr(Q_p(t))^2. \tag{1.31}$$

Proof: Using (1.29) we get

$$\begin{aligned}(trQ_p(t))' =& (\lambda+\mu+\nu)' \\ =& \underbrace{\lambda^2+\mu^2+\nu^2+2\lambda\mu+2\lambda\nu+2\mu\nu}_{(\lambda+\mu+\nu)^2} \\ &+ \lambda^2+\mu^2+\nu^2-4\rho(\lambda+\mu+\nu)^2 \\ =& \lambda^2+\mu^2+\nu^2+(1-4\rho)(\lambda+\mu+\nu)^2. \end{aligned} \tag{1.32}$$

On the other hand, using the fact that the norm of the sum $\| \|_\Sigma$ and the Euclidean norm $\| \|_E$ on $\mathbb{R}^3$ satisfy

$$\| \|_E^2 \geq \frac{1}{3}\| \|_\Sigma^2,$$

we conclude that

$$\lambda^2+\mu^2+\nu^2 \geq \frac{1}{3}(\lambda+\mu+\nu)^2. \tag{1.33}$$

Now, it follows from (1.32) and (1.33) that

$$\begin{aligned}(trQ_p(t))' &\geq \frac{1}{3}(\lambda+\mu+\nu)^2+(1-4\rho)(\lambda+\mu+\nu)^2. \\ &= \frac{4}{3}\left(1-3\rho\right)(\lambda+\mu+\nu)^2.\end{aligned}$$

■

We end this section by stating the Tensor Maximum Principle applied to the Ricci-Bourquignon Flow concerning equation (1.17).

Theorem 4 *Let $g(t)$, $t \in [0,T)$, be a Ricci-Bourguignon Flow on a compact Riemannian manifold (M^n, g_0) so that $g(0) = g_0$. For every $t \in [0,T)$, let $K(t) \subset End_{SA}(\wedge^2 TM)$ be a closed subbundle, invariant under parallel translation with respect to $D(t)$, convex in the fibers and such that the space time track*

$$\mathcal{T} = \{(P,t) \in End_{SA}(\wedge^2 TM) \times \mathbb{R} : P \in K(t),\ t \in [0,T)\} \tag{1.34}$$

is closed in $End_{SA}(\wedge^2 TM) \times [0,T)$. Suppose that for every $t_0 \in [0,T)$, $K(t_0)$ is preserved by the associated ODE. If $Rm(p,0) \in K_p(0)$ for every $p \in M$, then $Rm(p,t) \in K_p(t)$, for every $(p,t) \in M \times [0,T)$.

1.1.5 Ricci Almost Solitons

The concept of Ricci almost soliton was introduced in [46], generalizing the notion of Ricci soliton.

Definition 2 A *Ricci almost soliton* (M, g, X, λ) is a Riemannian or semi-Riemannian manifold (M, g) with a vector field X and a smooth function $\lambda : M \to \mathbb{R}$ satisfying the following fundamental equation

$$Ric + \frac{1}{2}\mathfrak{L}_X g = \lambda g.$$

If the vector field X is the gradient field of some function $f : M \to \mathbb{R}$, then the soliton is called a *gradient Ricci almost soliton*, or just *Ricci almost soliton*. In this case, it is denoted by (M, g, f, λ) and the fundamental equation becomes

$$Ric + \nabla\nabla f = \lambda g, \tag{1.35}$$

where f is called the *potential function* and $\nabla\nabla f$ is the Hessian of f with respect to the metric g.

Definition 3 We say that a Ricci almost soliton is *shrinking, steady, expanding* or *undefined* if the function λ is positive, null, negative or changes sign, respectively. If λ is constant, then the Ricci almost soliton reduces to what is called just of *Ricci soliton.*

The importance of the Ricci solitons is due to their relation with the Ricci flow. In fact, they are stationary solutions of the Ricci flow, which were introduced by Hamilton [29]. If the function λ is not constant, then Ricci almost solitons evolve under the Ricci flow changing only by conformal diffeomorphisms (see [49] and [26] page 4). Another relation with geometric flows is obtained by choosing specific functions for λ, for which the corresponding Ricci almost solitons are self similar solutions of the Ricci-Bourguignon flow [8]. On the other hand, Ricci almost solitons can be viewed as a generalization of Einstein manifolds [5], as one can easily see by considering a constant function on an Einstein manifold.

1.2 Semi-Riemannian Warped Product Manifolds

The content included in this section is based on the classical book [43], written by O'Neil, page 204. Most of the results appeared first in the paper [6], written by Bishop and O'Neil.

1.2.1 Definition and Properties

Let (B^n, g_B) and (F^m, g_F) be semi-Riemannian manifolds of dimensions n and m, respectively and denote by $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ the canonical projections.

Definition 4 (Warped Product) The *warped product between* (B^n, g_B) *and* (F^m, g_F) with *warping function* $h : B \to (0, \infty)$ is the product manifold $B \times F$ endowed with the metric g, defined by

$$g = \pi^* g_B + (h \circ \pi)^2 \sigma^* g_F. \tag{1.36}$$

In this case we denote $B \times_h F$ and $g = g_B + h^2 g_F$.

Note that $B \times_h F$ is a semi-Riemannian manifold of dimension $n + m$.

Warped products were introduced by Bishop and O'Neil in [6], where they gave examples of manifolds of negative curvature. One of the most important points in assuming that a manifold is a warped product is that it is possible to reduce geometric quantities (such as curvatures), to similar quantities on the base and on the fiber, depending also on the warping functions and its derivatives. In the case of the curvatures, this is expressed by identities, sometimes called O'Neil formulas for a warped product, that can be found in the main reference [43], page 204. We will only state those which will be used in this thesis. For the Ricci curvature of the warped product one has

Proposition 10 (Ricci Curvature) *Let $B^n \times_h F^m$ be a semi-Riemannian warped product. Then its Ricci tensor is given by*

$$\begin{cases} Ric(X,Y) = Ric_B(X,Y) - mh^{-1}\nabla_B\nabla_B h(X,Y), \\ Ric(X,U) = 0, \\ Ric(U,V) = Ric_F(U,V) - [h\Delta_B h + (m-1)|\nabla_B h|^2]g_F(U,V), \end{cases} \tag{1.37}$$

where X, Y are vector fields lifted from B and U, V are vector fields lifted from F.

Warped product Einstein manifolds have been studied largely (see [34] and references therein) where one of the main goals is to obtain new examples of Einstein manifolds having this property. Using Proposition 10, the Einstein condition on a warped product reduces to a system of equations on the base and on the fiber, as we can see in the next proposition.

Proposition 11 ([34]) *A semi-Riemannian warped product, $B^n \times_h F^m$, is an Einstein space with Einstein constant $a \in \mathbb{R}$ if, and only if, there is a constant $c \in \mathbb{R}$ so that*

$$\begin{cases} Ric_B - mh^{-1}\nabla_B\nabla_B h = a(m+n-1)g_B, \\ h\Delta_B h + (m-1)|\nabla_B h|^2 + a(m+n-1)h^2 = c(m-1), \\ Ric_F = c(m-1)g_F. \end{cases} \tag{1.38}$$

If the base B is a connected interval $I \subset \mathbb{R}$, then Proposition 11 takes a simpler form, which we state below for future references.

Corollary 2 *A semi-Riemannian warped product of the form $I \times_h F^m$, where $I \subset \mathbb{R}$, is an Einstein space if, and only if, (F^m, g_F) is an Einstein space and the function h satisfies*

$$h'' \pm ah = 0 \qquad \text{and} \qquad \pm (h')^2 + ah^2 = c, \tag{1.39}$$

where a is the Einstein constant of $I \times_h F^m$ and c is the Einstein constant of F.

Let $f : B \times F \to \mathbb{R}$ be a smooth function. If f depends only on the base, its gradient agrees with its gradient viewed as a function only on B (see [43], page 204.). This implies that the hessian of f with respect to B coincides with the hessian of f with respect to $B \times_h F$. In the case where f depends both on the base and on the fiber, using the expressions for the Levi-Civita connection of $B \times_h F$ (see [43], page 204.), we have the following:

Proposition 12 (Hessian of a Function) *Let $B^n \times_h F^m$ be a semi-Riemannian warped product. Then the Hessian of a function $f : B^n \times F^m \to \mathbb{R}$ is given by*

$$\begin{cases} \nabla\nabla f(X,Y) = \nabla_B \nabla_B f(X,Y), \\ \nabla\nabla f(X,U) = X(U(f)) - h^{-1} X(h) U(f), \\ \nabla\nabla f(U,V) = \nabla_F \nabla_F f(U,V) + h(\nabla_B h) f g_F(U,V), \end{cases} \tag{1.40}$$

where X, Y are vector fields lifted from B and U, V are vector fields lifted from F.

Concerning completeness we have the following criterion for building a complete warped product in the Riemannian setting, that can also be found in [43] (see page 209).

Proposition 13 ([6]) *A Riemannian warped product $B \times_h F$ is complete if, and only if, B and F are complete.*

When the signature of a semi-Riemannian warped product is not zero, the situation is not so simple. In fact, Been and Busemann showed that $(\mathbb{R} \times \mathbb{R}, dx^2 - e^{2x} dy^2)$ is not a complete semi-Riemannian manifold. In fact, they showed that there are light like geodesics that cannot be extended to $\mathbb{R}$, see [43] (page 209). Their example shows that there is no result similar to Proposition 13 for indefinite signature.

For our purposes we have the following result that guarantees the non completeness of the semi-Riemannian warped product, whenever the gradient of the warping function is a parallel vector field on the base. For more results on completeness of semi-Riemannian manifolds see [12]

Proposition 14 *Let $B \times_h F$ be a non trivial warped product of the Riemanian or semi-Riemannian manifolds (B^n, g_B) and (F^n, g_F). If $\nabla_B h$ is a parallel vector field on B, then $B \times_h F$ is not complete.*

Proof: Suppose by contradiction that $B \times_h F$ is complete. Consider $p_0 \in B$ and $v_0 \in T_{p_0}B$ such that $dh_{p_0}v_0 \neq 0$. Let γ be the geodesic such that $\gamma(0) = p_0$ and $\gamma'(0) = v_0$. Since $\nabla_B h$ is parallel, it follows that

$$\begin{aligned}(h \circ \gamma)''(t) &= \gamma'(\gamma'(h)) = \gamma'(\gamma'(h)) - \nabla_{B_{\gamma'}}\gamma'(h) \\ &= \nabla_B \nabla_B h(\gamma', \gamma') = 0.\end{aligned}$$

Therefore, there exist constants $a_0, b_0 \in \mathbb{R}$, so that

$$(h \circ \gamma)(t) = a_0 t + b_0.$$

Observe that

$$a_0 = (h \circ \gamma)'(0) = dh_{p_0}v_0 \neq 0.$$

By assumption γ is defined on $\mathbb{R}$, hence we may consider $t_0 = -b_0/a_0 \in \mathbb{R}$. However, $h(\gamma(-b_0/a_0)) = 0$, which contradicts the fact that $h \neq 0$. ■

1.3 Conformal Fields on Semi-Riemannian Manifolds

The study of conformal vector fields started long ago with the work of Brinkmann [9]. In this work, he gave a local characterization of manifolds admitting such fields. Later on, Obata [42], Kanay [32], Yano [50] and Kerbrat [33] studied global aspects of such manifolds obtaining a classification for some specific cases. The goal of this section is to collect part of these results that are going to be used in this thesis.

1.3.1 Definition and Examples

Let (M^n, g) be a semi-Riemannian manifold of dimension $n \geq 2$. The main notion to be studied in this section is introduced in the definition below.

Definition 5 A vector field X on (M^n, g) is called a *conformal field* if there exists a smooth function $\varphi : M \to \mathbb{R}$ such that

$$\mathfrak{L}_X g = 2\varphi g. \tag{1.41}$$

If the field X is the gradient of a function $\phi : M \to \mathbb{R}$, it is called a *gradient conformal field* and equation (1.41) becomes

$$\nabla\nabla\phi = \varphi g. \tag{1.42}$$

Here is the geometric interpretation of equation (1.41). Consider a point $p \in M$ and let $\Phi : U \times (-\varepsilon, \varepsilon) \subset M \times \mathbb{R} \to M$ be the local flow of X. We know that Φ satisfies $\Phi(p, 0) = p$ and

$$\frac{\partial \Phi}{\partial t}(p, t) = X(\Phi(p, t)), \tag{1.43}$$

for all $(p, t) \in U \times (-\varepsilon, \varepsilon)$. By definition of Lie derivative, we know that

$$(\mathfrak{L}_X g)(X, Y) = \left(\frac{\partial}{\partial t}\bigg|_{t=t_0} \Phi_t^* g \right)(X, Y), \tag{1.44}$$

where $\Phi_t^* g$ is the pull-back of g by $\Phi_t : U \to M$, defined by $\Phi_t(p) = \Phi(t, p)$. From (1.41) and (1.44) it follows that

$$\frac{\partial}{\partial t}\Phi_t^* g = 2(\phi \circ \Phi_t)\Phi_t^* g.$$

Integrating the ODE above one obtains that, for each $t \in (-\varepsilon, \varepsilon)$, there exists a function $\psi_t : U \to M$ for which

$$\Phi_t^* g = \psi_t g. \tag{1.45}$$

In other words, the diffeomorphisms Φ_t induced by the local flow of a field satisfying (1.41) act by conformal transformations, which justifies its denomination.

Now we will see in Proposition 16 that if an Einstein manifold admits a conformal vector field satisfying (1.41), then it also admits a gradient conformal field. This result can be found in [38]. Before stating this result we need the following proposition, also stated in [38].

Proposition 15 ([38]) *Let (M^n, g) be a semi-Riemannian manifold of dimension $n \geq 2$ and X a conformal field satisfying (1.41). Then*

$$\mathfrak{L}_X Ric = -(n-2)\nabla\nabla\varphi - \Delta\varphi g. \tag{1.46}$$

Proof: Consider $g(t) = \Phi_t^* g$, where Φ_t is the flow of the conformal field X, as in (1.43), and $\phi(t) = \phi \circ \Phi_t$. With this notation we have

$$\frac{\partial}{\partial t} g(t) = 2\phi(t) g(t). \tag{1.47}$$

We also note that

$$\mathfrak{L}_X Ric = \frac{\partial}{\partial t} Ric(g(t)), \tag{1.48}$$

where $Ric(g(t)) = \Phi_t^* Ric(g)$, by the diffeomorphism invariance of the Ricci tensor. Using formula (1.5), it follows that the right hand side of (1.48) is given, in local coordinates, by

$$\begin{aligned} \frac{\partial}{\partial t} R_{jk} &= g^{ab}(\nabla_a \nabla_j \phi g_{kb} + \nabla_a \nabla_k \phi g_{jb} - \nabla_a \nabla_b \phi g_{jk} - \nabla_j \nabla_k \phi g_{ab}) \\ &= \nabla_k \nabla_j \phi + \nabla_j \nabla_k \phi - \Delta\phi g_{jk} - n\nabla_j \nabla_k \phi \\ &= -(n-2)\nabla_j \nabla_k \phi - \Delta\phi g_{jk}. \end{aligned} \tag{1.49}$$

Now, (1.46) follows by considering (1.48) and (1.49) together. ■

Proposition 16 ([38]) *Let (M^n, g) be a semi-Riemannian Einstein manifold of dimension $n \geq 3$ and X a conformal field satisfying (1.41) for $\varphi : M \to \mathbb{R}$. Then $\nabla\varphi$ is also a conformal field satisfying*

$$\nabla\nabla\varphi + \frac{a}{n-1}\varphi g = 0, \tag{1.50}$$

where a is the unnormalized Einstein constant of (M^n, g).

Proof: Since (M^n, g) is Einstein, it follows that $Ric = ag$. Taking the Lie derivative in both sides of this equation one obtains

$$2a\varphi g = a\mathfrak{L}_X g = \mathfrak{L}_X Ric = -(n-2)\nabla\nabla\varphi - \Delta\varphi g, \tag{1.51}$$

where we have used equations (1.41) and (1.46). Taking the trace in equation (1.51) we get $\Delta\varphi = \dfrac{n}{n-1}a\varphi$. Substituting this expression into (1.51), we obtain (1.50), as desired. ■

When the conformal field is a gradient vector field of the form (1.42) and the manifold is Einstein, it follows from the Bochner formula (see Proposition 17 below) that φ is, up to the not normalized Einstein constant, equals to ϕ. To prove this result we will state below the referred version of the Bochner formula. For a proof in the Riemannian case, see Lemma 2.1 of [45]. We observe that the same proof is valid for any signature.

Proposition 17 ([45]) *Let (M, g) be a Riemannian or semi-Riemannian manifold and let $\varphi : M \to \mathbb{R}$ be a smooth function. Then*

$$div(\nabla\nabla\varphi)(X) = Ric(\nabla\varphi, X) + X(\Delta\varphi), \tag{1.52}$$

for all $X \in \mathfrak{X}(M)$.

With this version of Bochner formula, we can provide a simple proof of the proposition below when $n \geq 2$. For another proof when $n \geq 3$ see ([38]).

Proposition 18 *Let (M^n, g) be an Einstein manifold with dimension $n \geq 2$ and normalized Einstein constant a. If $\phi : M \to \mathbb{R}$ is a smooth function such that $\nabla\phi$ is a conformal vector field satisfying (1.42) for some smooth function $\varphi : M \to \mathbb{R}$, then there is a constant $b \in \mathbb{R}$ such that $\varphi = -a\phi - b$.*

Proof: It is easy to see that $\Delta\phi = n\varphi$ and that $div(\nabla\nabla\phi)(X) = X(\varphi)$, for all $X \in \mathfrak{X}(M)$. Using Bochner formula, we have

$$(n-1)X(\varphi + a\phi) = 0.$$

Since X is an arbitrary field and $n \geq 2$, it follows that there is a constant b satisfying the assertion. ■

From now on we will focus on conformal vector fields satisfying the equation

$$\nabla\nabla\varphi + (c\varphi + b)g = 0, \tag{1.53}$$

where $b,\ c \in \mathbb{R}$. Note that if $c \neq 0$ we can assume that $b = 0$ replacing φ by $\varphi - b/c$. Equation (1.53) has been largely studied since the 1920's, starting with Brinkman's work [9] on conformal transformations between semi-Riemannian Einstein manifolds.

In what follows we present examples of semi-Riemannian manifolds that admit non-constant solutions for equation (1.53). We are following the notation used in [43].

Example 1 [Semi-Euclidean Space] Let $\mathbb{R}^n_\varepsilon$ be the linear space $\mathbb{R}^n$ with the semi-Riemannian metric of index ξ

$$\langle v, w\rangle_\varepsilon = \sum_{j=1}^{n} \varepsilon_j v_j w_j.$$

If φ is a non constant solution of (1.53), then a straightforward calculation shows that c must be zero and that, for all $b \in \mathbb{R}$, a generic solution to (1.53) in $\mathbb{R}^n_\varepsilon$ is given by

$$\varphi(x_1, \ldots, x_n) \quad = \quad -(b/2)\textstyle\sum_{j=1}^{n} \varepsilon_j x_j^2 + \langle A_\varepsilon, x\rangle_\varepsilon + A_{n+1} \tag{1.54}$$

where $A_\varepsilon = (\varepsilon_1 A_1, \ldots, \varepsilon_n A_n) \in \mathbb{R}^n_\varepsilon$ and $A_{n+1} \in \mathbb{R}$.

Example 2 [Pseudospheres] The pseudosphere [43], with dimension n and index ε, is defined as

$$\mathbb{S}^n_\varepsilon(1/\sqrt{c}) = \{x \in \mathbb{R}^{n+1}_\varepsilon; \langle x, x\rangle_\varepsilon = 1/c\}, \qquad \text{where } c > 0.$$

It is connected if, and only if, $0 \leq \varepsilon \leq n-1$, and simply connected if, and only if, $0 \leq \varepsilon \leq n-2$. Furthermore, each connected component of $\mathbb{S}^n_\varepsilon(1/\sqrt{c})$ is a complete semi-Riemannian manifold of dimension n, index ε and constant curvature c. It is not difficult to see that the functions in Example 1 with $A_{n+2} = 0$ in the expression (1.54) i.e., $\varphi_{A_\varepsilon}(x) = \langle A_\varepsilon, x\rangle_\varepsilon$, provide all the functions satisfying (1.53) for $c > 0$. Note that

$\varphi_{A_\varepsilon}(x) = \langle A_\varepsilon, x\rangle_\varepsilon$ is the height function with respect to A_ε on the pseudosphere. .

Example 3 [Pseudohyperbolic Spaces] Similarly to the example above, the pseudohyperbolic space [43], with dimension n and index ε, is defined as

$$\mathbb{H}^n_\varepsilon(1/\sqrt{-c}) = \{x \in \mathbb{R}^{n+1}_{\varepsilon+1}; \langle x, x\rangle_{\varepsilon+1} = 1/c\}, \qquad \text{where } c < 0.$$

It is connected if, and only if, $2 \leq \varepsilon \leq n$ and simply connected if and only if $1 \leq \varepsilon \leq n-2$. Furthermore each connected component of $\mathbb{H}^n_\varepsilon(1/\sqrt{-c})$ is a complete semi-Riemannian manifold of dimension n, index ε and constant curvature c. As in the previous example, the functions in Example 1 with $A_{n+2} = 0$ in the expression (1.54) i.e., $\varphi_{A_{\varepsilon+1}}(x) = \langle A_{\varepsilon+1}, x\rangle_{\varepsilon+1}$, provide all the functions satisfying (1.53), for $c < 0$. Note that $\varphi_{A_{\varepsilon+1}}(x) = \langle A_{\varepsilon+1}, x\rangle_{\varepsilon+1}$ is the height function with respect to $A_{\varepsilon+1}$ on the pseudohyperbolic space.

Example 4 [Warped Products] Let $\pm I \times_h N^{n-1}$ be a warped product semi-Riemannian manifold, where $I \subset \mathbb{R}$ is a connected interval and N^{n-1} is an arbitrary semi-Riemannian manifold. Then a simple calculation shows that the function

$$\varphi(s, p) = \int_{s_0}^{s} h(t)dt$$

solves equation (1.53), when h satisfies

$$h'' \pm ch = 0. \tag{1.55}$$

For our purposes, it is important to know if a height function has zeros or not. This is because height functions can occur as warping functions, as we will see in Theorem 10, and warping functions do not admit zeros. We end this subsection with a proposition which reveals the hyperquadrics that admit such functions.

Proposition 19 *Let $\varphi_A : \mathbb{R}^{n+1}_\varepsilon \to \mathbb{R}$ be the height function with respect to $A \in \mathbb{R}^{n+1}_\varepsilon$, $A \neq 0$ and $n \geq 2$. Then φ_A has no zeros on $\mathbb{S}^n_\varepsilon(1/\sqrt{c})$ (resp. $\mathbb{H}^n_\varepsilon(1/\sqrt{-c})$ if, and only if, $\varepsilon = n$ (resp. $\varepsilon = 1$) and A is a space like (resp. time like) or light like vector.*

Proof: We first prove the proposition in the case of the sphere. Since we are considering $\mathbb{S}^n_\varepsilon(1/\sqrt{c}) \neq \emptyset$, we can assume $0 \leq \varepsilon \leq n$, i.e., $\varepsilon \neq n+1$. Moreover, φ_A is a linear function, hence $\mathbb{R}^{n+1} = \mathrm{Ker}(\varphi_A) \oplus \mathrm{Im}(\varphi_A)$. where $\mathrm{Ker}(\varphi_A) = (A)^{\perp} \subset \mathbb{R}^{n+1}$. Since $A \neq 0$, it follows that $\dim\{\mathrm{Ker}(\varphi_A)\} = n \geq 2$ and $\dim\{\mathrm{Im}(\varphi_A)\} = 1$. In what follows, we will analyze each case according to A being a time like, space like or light like vector. We will consider an appropriate orthonormal basis in each case, $\{e_1, \ldots, e_\varepsilon, e_{\varepsilon+1}, \ldots, e_{n+1}\}$ for $\mathbb{R}^{n+1}_\varepsilon$ such that $e_1, ..., e_\varepsilon$ are time like and $e_{\varepsilon+1}, \ldots, e_{n+1}$ are space like.

Suppose that A is time like. In this case, $1 \leq \varepsilon \leq n$ and we choose the basis such that $e_\varepsilon = A/\sqrt{| < A, A >_\varepsilon |}$. Therefore, e_{n+1} and e_ε are orthogonal hence, $(1/\sqrt{c})e_{n+1} \in A^{\perp} \cap \mathbb{S}^n_\varepsilon(1/\sqrt{c})$, i.e., φ_A has zeros on the sphere.

Suppose that A is space like. We consider the basis on $\mathbb{R}^{n+1}_\varepsilon$, such that $e_{\varepsilon+1} = A_\varepsilon/|A_\varepsilon|$. If $0 \leq \varepsilon \leq n-1$, then $e_{\varepsilon+1}$ and e_{n+1} are orthogonal and hence $(1/\sqrt{c})e_{n+1} \in A^{\perp} \cap \mathbb{S}^n_\varepsilon(1/\sqrt{c})$. If $\varepsilon = n$ then $A^{\perp}$ is negative definite since it is generated by $\{e_1, \ldots, e_n\}$. Therefore $A^{\perp} \cap \mathbb{S}^n_n(1/\sqrt{c}) = \emptyset$, i.e., φ_A has no zeros on the sphere.

Suppose that A is light like, then $1 \leq \epsilon \leq n$ and it is not so difficult to see that there exist orthogonal vectors $V_1, V_2 \in \mathbb{R}^{n+1}$ such that $V_1 \neq 0$ is time like, $V_2 \neq 0$ is space like and $A = V_1 + V_2$. We consider the basis so that $e_\varepsilon = V_1/|\sqrt{< V_1, V_1 >_\varepsilon}|$ and $e_{\varepsilon+1} = V_2/|V_2|$. If $\varepsilon \leq n-1$, then $(1/\sqrt{c})e_{\varepsilon+2} \in (A^{\perp} \cap \mathbb{S}^n_\varepsilon(1/\sqrt{c})$. Therefore, φ_A has no zeros on the sphere if, and only if, $\varepsilon = n$.

This completes the proof for the case of the sphere. Considering suitable changes, the proof for the hyperbolic space is similar. ■

1.3.2 Special Coordinate System Around a Regular Point

In the proof of Theorem 14 (see Chapter 3), it will be important to work on the set of regular points of a solution of (1.53). Such an argument works since as we will see below, the referred set is a dense subset of the manifold, in the case where it is complete. To support what was just said, one has the following result due to Kerbrat [33]. The proof can also be found in Kuenel's paper [39].

Proposition 20 ([33]) *Let $\varphi : M \to \mathbb{R}$ be a solution of the equation (1.53). Then the critical points of φ are isolated.*

The local classification below is due to Brinkmann [9] and can also be found in the excellent survey [36]. It is of fundamental inportance for the global classification of complete semi-Riemannian manifolds admitting solutions to equation (1.53).

Proposition 21 ([9]) *Let (M, g) be a pseudo-Riemannian-manifold. The following are equivalent:*

1. *There is a non constant solution φ of*

$$\nabla\nabla\varphi - (\Delta\varphi/n)g = 0,$$

in a neighborhood of a point $p \in M$ such that $g(\nabla\varphi, \nabla\varphi) \neq 0$.

2. *There is a neighborhood U of $p \in M$, a smooth function $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $\varphi'(t) \neq 0$, for all $t \in (-\varepsilon, \varepsilon)$ and a pseudo-Riemannian manifold (N, g_N) such that (U, g) is isometric to the warped product*

$$((-\varepsilon, \varepsilon) \times_{\varphi'} N, \pm dt^2 + (\varphi')^2 g_N),$$

where $sgn(g(\varphi', \varphi')) = \pm 1$.

1.3.3 Classification Results

To state the classification that we are interested in, since one is working with semi-Riemannian manifolds, it is necessary to treat separately the cases when the non-constant field is light like in an open set. For this reason we introduce the definition below.

Definition 6 We say that a vector field X is *improper* if there is an open set where X is light like. If there is no such an open set then X is called a *proper* vector field.

Theorems 5-7 will be of fundamental importance in the proof of our classification results in Chapter 3. They provide the classification results of complete semi-Riemannian Einstein manifolds admitting a non-constant solution of equation (1.53), and consequently for Einstein manifolds admitting general conformal fields. This fact follows from Proposition 1.3. These theorems assert the uniqueness of the examples given in Subsection 1.3.1, provided the conformal gradient vector field is proper. The improper case was analized by Brinkman [9] showing, among other things, that $\nabla\varphi$ must be parallel. Since then, manifolds carrying parallel improper vector fields are called *Brinkman spaces.*

Theorem 5 ([33]) *A complete semi-Riemannian manifold, (M^n, g), with $n \geq 2$, admits a non constant solution of the equation $\nabla\nabla\varphi + bg = 0$ $b \neq 0$ if, and only if, it is isometric to the semi-Euclidean space $\mathbb{R}^n_\varepsilon$.*

This result is a particular case of a theorem proved by Kerbrat [33], where the author classifies spaces carrying vector fields satisfying more general equations.

Theorem 6 *A complete semi-Riemannian Einstein manifold, (M^n, g), with $n \geq 2$, admits a non constant solution φ of the equation $\nabla\nabla\varphi = 0$ if, and only if, it is isometric to*

1. *$\mathbb{R} \times N^{n-1}$, where (N, g_N) is a complete semi-Riemannian Einstein manifold, if $\nabla\varphi$ is a proper vector field (see [33]);*

2. *a Brinkman space, if $\nabla\varphi$ is an improper vector field and $n \geq 3$ (see [9]).*

Theorem 7 *A complete semi-Riemannian Einstein manifold, (M^n, g), with $n \geq 2$ and index ε, admits a non constant solution of the equation $\nabla\nabla\varphi + c\varphi g = 0$ with $c \neq 0$ if, and only if, it is isometric to*

1. *$\mathbb{S}^n_\varepsilon(1/\sqrt{c})$, when $0 \leq \varepsilon \leq n-2$; the covering of $\mathbb{S}^n_{n-1}(1/\sqrt{c})$ when $\varepsilon = n-1$ and the upper part of $\mathbb{S}^n_n(1/\sqrt{c})$ when $\varepsilon = n$ if $c > 0$ and φ has some critical point (see [42] for the case $\varepsilon = 0$ and [33] for the cases $\varepsilon \neq 0$);*

2. *$\mathbb{H}^n_\varepsilon(1/\sqrt{|c|})$, when $2 \leq \varepsilon \leq n-1$; the covering of $\mathbb{H}^n_1(1/\sqrt{|c|})$ when $\varepsilon = 1$ and the upper part of $\mathbb{H}^n_0(1/\sqrt{|c|})$ when $\varepsilon = 0$, if $c < 0$ and φ has some critical point (see [32] for the case $\varepsilon = 0$ and [33] for the cases $\varepsilon \neq 0$);*

3. *$(\mathbb{R} \times N^{n-1}, \pm dt^2 + \cosh^2(\sqrt{|c|}\,t) g_N)$, where (N^{n-1}, g_N) is a semi-Riemannian Einstein manifold, if φ has no critical points (see [33]);*

4. *$(\mathbb{R} \times N^{n-1}, \pm dt^2 \pm e^{2\sqrt{|c|}\,t} g_N)$, where (N^{n-1}, g_N) is a Riemannian Einstein manifold, if φ has no critical points (see [33]).*

Chapter 2

Ricci Flow Preserving Warped Product

In this chapter we will study in which cases the Ricci flow preserves warped product. To be more precise, we will study when a solution for the Ricci flow whose initial metric is a warped product is still a warped product for positive times. This is the content of Section 2.1, where we present a structural theorem, Theorem 8, which gives necessary and sufficient conditions for the Ricci flow to preserve the warped product structure. In Section 2.2, using Parabolic Maximum Principles, we provide non existence results concerning ancient solutions for Ricci Flows preserving warped products with compact base.

2.1 Definition and Structural Theorem

The notion that we introduce in this section was motivated by the study of Ricci solitons on warped product manifolds. Since a Ricci soliton generates a solution to the Ricci flow which is self-similar (i.e., evolves by diffeomorphisms and change of scales) it is natural to ask whether the warped product property is also preserved in time if the initial metric is a Ricci soliton on a warped product manifold. It turns out that the answer is positive (see the proof of Corollary 5), and the corresponding self-similar solution is built from two families of metrics, one coming from the base manifold and the other (constant in time) coming from the fiber and a family of warping functions.

Having this in mind, we introduce below a class of Ricci flow encoding the behavior described above. It says that the Ricci flow evolves as a warped product manifold.

Definition 7 Let $(M^{n+m} = B^n \times_{h_0} F^m, g_0)$ be a warped product of (B^n, g_B^0) and (F^m, g_F^0) with non constant warping function $h_0 : B \to (0, \infty)$ and $g_0 = g_B^0 + h_0^2 g_F^0$. Let $(M^{n+m}, g(t))$, $t \in [0, \varepsilon)$, $\varepsilon \in (0, \infty]$, be a Ricci flow such that $g(0) = g_0$. We say that the Ricci Flow *preserves the warped product structure* if there exist smooth families

1. $\{g_B(t); t \in [0, T)\}$, of metrics in B;

2. $\{h(t); t \in [0, T)\}$, of non constant functions in B;

3. $\{g_F(t); t \in [0, T)\}$, of metrics in F

such that for each $t \in [0, T)$

$$g(t) = \pi^* g_B(t) + (\pi^* h)^2 \sigma^* g_F(t), \tag{2.1}$$

where $\pi : M^{n+m} \to B^n$ and $\sigma : M^{n+m} \to F^m$ are the canonical projections into the base and the fiber, respectively.

In the product case $(B^n \times F^m, g_0 = g_B^0 + g_F^0)$, the situation is well understood. To see this, consider $g_B(t)$ and $g_F(t)$, Ricci Flows on B and F, respectively, with $g_B(0) = g_B^0$ and $g_F(0) = g_F^0$. It follows that $g(t) = g_B(t) + g_F(t)$ is a Ricci Flow on $B \times F$ with $g(0) = g_0$. Conversely, a Ricci Flow $g(t)$ on the product manifold $B^n \times F^m$ starting at the product metric $g_0 = g_B^0 + g_F^0$ is equal to $g_B(t) + g_F(t)$, if one has uniqueness of solution. It is worth to say that uniqueness is guaranteed if, for instance, the manifold is compact, see Hamilton [29]. In the non compact case, see [48] for a condition that leads to uniqueness. Therefore, it seems reasonable to separate the case when the warping function is constant as a trivial case.

Our goal in the next result is to answer the question of when the Ricci flow preserves the warped product structure in the sense of Definition 7.

Theorem 8 *Let $(M^{n+m} = B^n \times_{h_0} F^m, g_0)$ be a warped product of (B^n, g_B^0) and (F^m, g_F^0) with non constant warping function $h_0 : B \to (0, \infty)$ and $g_0 = g_B^0 + h_0^2 g_F^0$. Let $(M^{n+m}, g(t))$, $t \in [0, \varepsilon)$, $\varepsilon \in (0, \infty]$, be a Ricci flow such that $g(0) = g_0$. The* *flow* $(M^{n+m}, g(t))$ *preserves the warped product structure of* (M^{n+m}, g_0) *if, and only if,* (F^m, g_F^0) *is an Einstein manifold and there exists a family of smooth functions* $u(t) : B \to \mathbb{R}$ *such that*

$$g(t) = g_B(t) + e^{2u(t)} g_F^0 \tag{2.2}$$

$$\frac{\partial}{\partial t} g_B(t) = -2Ric(g_B(t)) + 2m\nabla_{g_B(t)}\nabla_{g_B(t)} u(t) + 2mdu(t) \otimes du(t), \tag{2.3}$$

$$\frac{\partial}{\partial t} u(t) = \Delta_{g_B(t)} u(t) + m|\nabla_{g_B(t)} u(t)|^2 - \frac{R_F^0}{m} e^{-2u(t)}, \tag{2.4}$$

where R_F^0 *is the constant scalar curvature of the fiber.*

In order to prove Theorem 8 we need the following lemma that separates variables on a product manifold. It will also be important in Chapter 3, for the proof of Theorem 10.

Lemma 1 *Let* $B^n \times F^m$ *be a semi-Riemannian manifold and let* $h : B^n \to \mathbb{R}$ *and* $\varphi : F^m \to \mathbb{R}$ *be non constant differentiable functions. Let* $\mu_1,\ \rho_1 : D \subset B \to \mathbb{R}$ *and* $\mu_2,\ \rho_2 : G \subset F \to \mathbb{R}$ *be differentiable functions, where* $D \times G$ *is connected. Then*

$$h(p)\mu_2(q) + \varphi(q)\mu_1(p) = \rho_1(p) + \rho_2(q), \quad \forall (p, q) \in D \times G. \tag{2.5}$$

if, and only if, there are constants $b,\ \tilde{b},\ c,\ \tilde{c} \in \mathbb{R}$ *such that*

$$\begin{cases} \mu_1 = ch + \tilde{c}, \\ \rho_1 = -bh + \tilde{b}, \\ \mu_2 = -c\varphi - b, \\ \rho_2 = \tilde{c}\varphi - \tilde{b}, \end{cases} \tag{2.6}$$

for all $p \in D$ *and* $q \in G$.

Proof.: Assume that the relation (2.5) holds. Since h and φ are not constant, we consider $(p_0, q_0) \in D \times G$ such that p_0 and q_0 are regular points of the functions h and φ, respectively. Then there exists a vector field X_1 on a connected neighborhood $D_1 \subset D$ of p_0 and a vector field U_1 on a connected neighborhood $G_1 \subset G$ of q_0 such that

$$X_1(h)(p) \neq 0, \quad U_1(\varphi)(q) \neq 0, \quad \forall p \in D_1,\ q \in G_1.$$

Consider $X_1, X_2, ..., X_n$ and $U_1, U_2, ..., U_m$ orthogonal frames locally defined in (neighborhoods that we still denote by) D_1 and G_1, respectively. Applying the vector fields X_k, $k = 1, ..., n$ and U_α, $\alpha = 1, ..., m$ to the relation (2.5) we get that

$$X_k(h)U_\alpha(\mu_2) = -X_k(\mu_1)U_\alpha(\varphi), \qquad \forall k,\ \alpha. \tag{2.7}$$

In particular, we have

$$\frac{X_1(\mu_1)}{X_1(h)} = -\frac{U_1(\mu_2)}{U_1(\varphi)} = c, \qquad \text{in } D_1 \text{ and } G_1,$$

for some constant $c \in \mathbb{R}$. Hence

$$X_1(\mu_1) = cX_1(h) \quad \text{in } D_1 \quad \text{and} \quad U_1(\mu_2) = -cU_1(\varphi) \quad \text{in } G_1. \tag{2.8}$$

We want to show that this expression holds for all X_i and U_α. Fix $p_1 \in D_1$ and consider $X_i(h)(p_1)$ for $i \geq 2$. If $X_i(h)(p_1) \neq 0$, shrinking D_2 if necessary, we can assume that $X_i(h) \neq 0$ in D_1. Then it follows from (2.7) and (2.8) that in D_1

$$\frac{X_i(\mu_1)}{X_i(h)} = -\frac{U_1(\mu_2)}{U_1(\varphi)} = c.$$

Therefore,

$$X_i(\mu_1) = cX_i(h) \qquad \text{in } D_1.$$

If $X_i(h)(p_1) = 0$, then it follows from (2.7) that $U_1(\varphi)X(\mu_1)(p_1) = 0$ and therefore $X_i(h)(p_1) = cX_i(\mu_1)(p_1)$. We conclude that for all i and α we have

$$X_i(\mu_1 - ch) = 0 \qquad \text{in } D_1.$$

Similarly, we get

$$U_\alpha(\mu_2 + c\varphi) = 0 \qquad \text{in } G_1.$$

From the last two expressions we conclude that there exist constants $\tilde{c}, b \in \mathbb{R}$ such that

$$\mu_1 - ch = \tilde{c}, \quad \text{in } D_1 \qquad \mu_2 + c\varphi = -b, \quad \text{in } G_1.$$

It follows from (2.5) that

$$\rho_1 + bh = \tilde{c}\varphi - \rho_2 = \tilde{b}.$$

Therefore, we obtained (2.6) in $D_1 \times G_1$.

If there is $p_1 \in D \backslash D_1$, using (2.5) in p_1 and (2.6) in $q \in G_1$ we have

$$\varphi(q)(-ch(p_1) + \mu_1(p_1) - \tilde{c}) = \rho_1(p_1) + bh(p_1) - \tilde{b},$$

for all $q \in G_1$. Applying X_1 on the above identity and how φ is not constant on G_1 it follows that (2.6) holds on $D \times G_1$. Analogously if there is $q_1 \in G \backslash G_1$, we can use (2.5) in q_1, (2.6) in $p \in D_1$ and the non constancy of h on D_1 to prove (2.6) on whole $D \times G$. ■

Proof of Theorem 8: Since the flow $g(t)$ preserves the warped product structure, there are smooth families $g_B(t)$, $g_F(t)$ and $h(t)$ satisfying condition (2.1). Considering frames $\{E_j\} \subset \mathfrak{X}(B)$ and $\{U_\alpha\} \subset \mathfrak{X}(F)$ lifted from B and F, respectively, the Ricci flow equation becomes equivalent to

$$\frac{\partial}{\partial t} g(t)_{jk} = -\, 2Ric(g(t))_{jk}, \tag{2.9}$$

$$\frac{\partial}{\partial t} g(t)_{j\alpha} = -\, 2Ric(g(t))_{j\alpha}, \tag{2.10}$$

$$\frac{\partial}{\partial t} g(t)_{\alpha\beta} = -\, 2Ric(g(t))_{\alpha\beta}, \tag{2.11}$$

where $j, k \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, \ldots, m\}$. We first observe that both sides of (2.10) are identically zero. In fact, using the second equality of (1.37) one gets that the right hand side of (2.10) is identically zero. To see that the left hand side is also identically zero we will use the fact that the canonical projections $\pi : B^m \times F^n \to B^n$ and $\sigma : B^m \times F^n \to F^n$ are constant with respect to t, and then, for any $j \in \{1, \ldots, n\}$ and $\alpha \in \{1, \ldots, m\}$:

$$\begin{aligned}
\frac{\partial}{\partial t} g(t)_{j\alpha} &= \left(\frac{\partial}{\partial t} g(t)\right)(E_j, U_\alpha) \\
&= \left(\frac{\partial}{\partial t}\left[\pi^* g_B(t) + (\pi^* h(t))^2 \sigma^* g_B(t)\right]\right)(E_j, U_\alpha) \\
&= \left(\pi^*\left[\frac{\partial}{\partial t} g_B(t)\right] + \frac{\partial}{\partial t}(\pi^* h(t))^2 \sigma^* g_B(t) + (\pi^* h(t))^2 \sigma^*\left[\frac{\partial}{\partial t} g_B(t)\right]\right)(E_j, U_\alpha)
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\partial}{\partial t} g_B(t)\right] (d\pi(E_j), \underbrace{d\pi(U_\alpha))}_{=0} + \frac{\partial}{\partial t}(\pi^* h(t))^2 [g_B(t)] (\underbrace{d\sigma(E_j)}_{=0}, d\sigma(U_\alpha)) \\
&+ (\pi^* h(t))^2 \left[\frac{\partial}{\partial t} g_B(t)\right] (\underbrace{d\sigma(E_j)}_{=0}, d\sigma(U_\alpha)) \\
&= 0,
\end{aligned}
$$

as we claimed. Therefore, the Ricci flow equation preserves the warped product structure if, and only if (2.9) and (2.11) are satisfied. Using Proposition 10 and the fact that the canonical projections do not depend on t, these equations are equivalent to

$$
\frac{\partial}{\partial t} g_B(t)_{jk} = -2Ric(g_B(t))_{jk} + 2m h(t)^{-1} \nabla_{g_B(t)} \nabla_{g_B(t)} h(t)_{jk}, \tag{2.12}
$$

$$
\begin{aligned}
h(t)^2 \frac{\partial}{\partial t} g_F(t)_{\alpha\beta} =& 2\left[h(t)\Delta_{g_B(t)} h(t) + (m-1)|\nabla_{g_B(t)} h(t)|^2 - h(t)\frac{\partial}{\partial t} h(t)\right] g_F(t)_{\alpha\beta} \\
&- 2Ric(g_F(t))_{\alpha\beta},
\end{aligned} \tag{2.13}
$$

where $j, k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, m\}$.

Equation (2.12) is a differential equation purely on the base. However, equation (2.13) involves the base and the fiber. We will separate the variables in this equation in order to obtain equations on the base and on the fiber that are equivalent to it. Fix $t_0 \in [0, \varepsilon)$ and assume that $\{U_\alpha\}$, $\alpha \in \{1, \dots, m\}$, is an orthonormal frame in a connected open set $G \subset F$, with respect to $g_F(t_0)$.

Take $\alpha = \beta$ in (2.13) and define smooth functions $\rho_1(t_0) : B \to R$ and $\rho_2(t_0, \alpha), \mu_2(t_0, \alpha) : G \subset F \to R$ as

$$
\rho_1(t_0) = 2[h(t)\Delta_{g_B(t)} h(t) + (m-1)|\nabla_{g_B(t)} h(t)|^2 - h(t)\frac{\partial}{\partial t} h(t)]. \tag{2.14}
$$

$$
\mu_2(t_0, \alpha) = \frac{\partial}{\partial t} g_F(t)_{\alpha\alpha}, \tag{2.15}
$$

$$
\rho_2(t_0, \alpha) = -2Ric(g_F(t))_{\alpha\alpha}, \tag{2.16}
$$

With these considerations, equation (2.13) can be rewritten as

$$
h(t_0)^2 \mu_2(t_0, \alpha) = \rho_1(t_0) + \rho_2(t_0, \alpha), \tag{2.17}
$$

for all points in $B \times G$. By the Lemma 1 we conclude the existence of constants $a(t_0)$ and $b(t_0)$ satisfying

$$\begin{cases} \rho_1(t_0) = -a(t_0,\alpha)h(t_0)^2 + b(t_0,\alpha), \\ \mu_2(t_0,\alpha) = -a(t_0,\alpha), \\ \rho_2(t_0,\alpha) = -b(t_0,\alpha). \end{cases} \tag{2.18}$$

By the first equation of (2.18) we can see that $a(t_0,\alpha)$ and $b(t_0,\alpha)$ do not depend on α, so we will call them just by $a(t_0)$ and $b(t_0)$, respectively. By using (2.14), (2.15), (2.16) and (2.18) we arrive at

$$\begin{cases} 2[h(t)\Delta_{g_B(t)}h(t) + (m-1)|\nabla_{g_B(t)}h(t)|^2 - h(t)\frac{\partial}{\partial t}h(t)] = -a(t_0,\alpha)h(t_0)^2 + b(t_0,\alpha), \\ \frac{\partial}{\partial t}g_F(t)_{\alpha\alpha} = -a(t_0) = -a(t_0)\delta_{\alpha\alpha}, \\ -2Ric(g_F(t))_{\alpha\alpha} = -b(t_0) = -b(t_0)\delta_{\alpha\alpha}. \end{cases} \tag{2.19}$$

Take $\alpha \neq \beta$ in equation (2.13). Since $h(t_0)$ is not constant, we conclude that

$$\begin{cases} \frac{\partial}{\partial t}g_F(t)_{\alpha\beta} = 0 = -a(t_0)\delta_{\alpha\beta}, \\ -2Ric(g_F(t))_{\alpha\beta} = 0 = -b(t_0)\delta_{\alpha\beta}. \end{cases} \tag{2.20}$$

Since $t_0 \in [0,\varepsilon)$ is arbitrary, (2.19) and (2.20) are enough to conclude that

$$\begin{cases} \frac{\partial}{\partial t}h(t) = \Delta_{g_B(t)}h(t) + (m-1)h(t)^{-1}|\nabla_{g_B(t)}h(t)|^2 + \frac{1}{2}a(t)h(t) - \frac{1}{2}b(t)h(t)^{-1}, \\ \frac{\partial}{\partial t}g_F(t) = -a(t)g_F(t), \\ Ric(g_F(t)) = \frac{b(t)}{2}g_F(t), \end{cases} \tag{2.21}$$

for all $t \in [0,\varepsilon)$, where the first equation occurs on B and the last two equations hold for all points in $G \subset F$. Since G was chosen to define the orthonormal frame and we can do this on a neighborhood of each point of F, we conclude that system (2.21) holds, for the same functions $a(t)$ and $b(t)$, on the whole manifold $B \times F$. In particular $(F, g_F(t))$ is an Einstein manifold, for each $t \in [0,\varepsilon)$. Notice that the last two equations in (2.21) imply that $a(t)$ and $b(t)$ are smooth functions in the variable t. Using the last two equations and the scalar invariance of the Ricci tensor we have

$$g_F(t) = \exp\Big(- \int_0^t a(s)ds \Big) g_F^0 \tag{2.22}$$

and

$$b(t) = \frac{2R_0^F}{m} \exp\Big(\int_0^t a(s)ds \Big). \tag{2.23}$$

To see this, notice that from the second equation of (2.21), given a non vanishing vector field $U \in \mathfrak{X}(F)$, we have

$$\frac{\partial}{\partial t} \ln(g_F(t)(U,U)) = -a(t),$$

what implies (2.22). By the invariance of the Ricci tensor we get

$$\frac{b(t)}{2} \exp\Big(- \int_0^t a(s)ds \Big) g_F^0 = \frac{R_F^0}{m} g_F^0,$$

what gives (2.23). Define $u(t) = \ln(h(t)) - \frac{1}{2}\int_0^t a(s)ds$ and observe that

$$\begin{aligned} g(t) &= g_B(t) + h(t)^2 g_F(t) \\ &= g_B(t) + \exp\big(2(\ln(h(t)) - \tfrac{1}{2}\textstyle\int_0^t a(s)ds)\big) g_F^0 \\ &= g_B(t) + e^{2u(t)} g_F^0. \end{aligned}$$

Using the identities

$$\begin{cases} \frac{\partial}{\partial t} h(t) = (\frac{\partial}{\partial t} u(t) + \frac{1}{2} a(t)) h(t) \\ \Delta_{g_B(t)} h(t) = (\Delta_{g_B(t)} u(t) + |\nabla_{g_B(t)} u(t)|^2) h(t) \\ |\nabla_{g_B(t)} h(t)|^2 = |\nabla_{g_B(t)} u(t)|^2 h(t)^2 \end{cases}$$

and the first equation of (2.21) we have that $h(t)$ satisfies

$$\begin{aligned} (\tfrac{\partial}{\partial t} u(t) + \tfrac{1}{2} a(t)) h(t) &= (\Delta_{g_B(t)} u(t) + |\nabla_{g_B(t)} u(t)|^2) h(t) + (m-1) h(t) |\nabla_{g_B(t)} u(t)|^2 \\ &\quad + \tfrac{1}{2} a(t) h(t) - \tfrac{1}{2} b(t) h(t)^{-1}, \end{aligned}$$

and since $b(t)h^{-2} = \frac{2R_F^0}{m} e^{-2u(t)}$, it follows that

$$\frac{\partial}{\partial t} u(t) = \Delta_{g_B(t)} u(t) + m |\nabla_{g_B(t)} u(t)|^2 - \frac{2R_F^0}{m} e^{-2u(t)}.$$

The converse is a direct computation. ■

Assume we have uniqueness of the Ricci Flow when $g(0) = g_B^0 + h_0^2 g_F^0$ is a warped product metric on the manifold $B \times F$. Let $g(t)$, $t \in [0, T)$, be the unique solution of the Ricci Flow with this initial condition. Theorem 8 tells us that if the fiber (F^n, g_F^0) is not an Einstein manifold, the flow $g(t)$ does not preserve warped product structure, in the sense of Definition 7.

Corollary 3 *Let $(M^{n+m} = B^n \times F^m, g(t))$, $t \in [0, \varepsilon)$, $\varepsilon \in (0, \infty]$, be a Ricci flow with $g(0) = g_B^0 + h_0^2 g_F^0$, where $h_0 : B \to (0, \infty)$ is a non constant smooth function. If $(F^m, g_F(0))$ is not an Einstein manifold then the flow does not preserve the warped product structure.*

Note the resemblance with the case of Ricci solitons on warped products, where a necessary condition for its existence is that the fiber is an Einstein manifold. So we can see this as a generalization of this fact, proved in [47].

Given two solutions of the Ricci flow $(B^n, g_B(t))$ and $(F^m, g_F(t))$, it is easy to see that its product $(B^n \times F^m, g(t) = g_B(t) + g_F(t))$ is also a solution of the Ricci flow. It is natural to ask whether this result still true for the warped product $(B^n \times F^m, g(t) = g_B(t) + h(t)^2 g_F(t))$, where $h(t) : B \to (0, \infty)$ is a family of positive functions. It follows from Theorem 8 that when the metric $g_B(t)$ is complete, for each $t \in [0, T)$, then there is no such a warping function. In other words,

Corollary 4 *Let $(B^n, g_B(t))$, $t \in [0, T)$, be a Ricci Flow so that $g_B(t)$ is complete for each t and $(F^m, g_F(t))$, $t \in [0, T)$, be any family of metrics. There is no family of non constant functions $h(t) : B \to (0, \infty)$, $t \in [0, T)$, on B so that the warped product $(B^n \times F^m, g(t) = g_B(t) + h(t)^2 g_B(t))$ is a Ricci Flow.*

Proof: Suppose by contradiction that there is a family $h(t) : B \to (0, \infty)$, $t \in [0, T)$, so that $g(t) = g_B(t) + h(t)^2 g_F(t)$ is a Ricci Flow. Since by hypothesis $g(t)$ is a Ricci Flow that preserves warped product, it satisfies equation (2.3), and using the fact that $g_B(t)$ is a Ricci Flow on B, we conclude that $\nabla_B \nabla_B h(t) = 0$. Let $t_0 \in [0, T)$ so that $\nabla_{g_B(t_0)} h(t_0) \neq 0$ and consider a geodesic $\gamma : \mathbb{R} \to B$ with respect to $(B^n, g_B(t_0))$ so that $\gamma'(0) = \nabla_{g_B(t_0)} h(t_0)$. It follows that $(h(t_0) \circ \gamma)''(s) = 0$, for all $s \in \mathbb{R}$. Then there are constants a and b, depending of γ, so that $(h(t_0) \circ \gamma)(s) = as + b$. Notice that $\gamma(s)$ is defined in $\mathbb{R}$. Since $a = (h(t_0) \circ \gamma)'(0) = |\nabla_{g_B(t_0)} h(t_0)|^2 \neq 0$, one can take $s_0 = -b/a$ in the domain of γ to obtain $h(t_0)(\gamma(s_0)) = (h(t_0) \circ \gamma)(s_0) = 0$. But this is a contradiction, since $h(t)$ is positive for each t. ■

2.2 Nonexistence of Ancient Solutions of the Ricci Flow that Preserves Warped Product

Ancient solutions occur naturally in the process of understanding singularities that occur in finite time.

Definition 8 We say that a solution $(M^{n+m} = B^n \times F^m, g(t))$ of the Ricci Flow is an *ancient solution* if it is defined for each $t \in (-\infty, t_0)$, where $t_0 \in (-\infty, \infty]$.

Similar to Definition 7, we introduce the ancient solutions that have the property of being a warped product.

Definition 9 An ancient solution $(M^{n+m} = B^n \times F^m, g(t))$, $t \in (-\infty, T)$, of the Ricci Flow is *warped product along the time* t if there are smooth families

1. $\{g_B(t); t \in (-\infty, T)\}$, of metrics on B;
2. $\{h(t); t \in (-\infty, T)\}$, of non constant functions on B;
3. $\{g_F(t); t \in (-\infty, T)\}$, of metrics on F,

such that for each $t \in (-\infty, T)$

$$g(t) = \pi^* g_B(t) + (\pi^* h)^2 \sigma^* g_F(t), \tag{2.24}$$

where $\pi : M^{m+n} \to B^n$ and $\sigma : M^{m+n} \to F^n$ are the canonical projections into the base and the fiber, respectively.

Shrinking and expanding Ricci solitons are self-similar solutions of the Ricci Flow and it turns out that they are ancient solutions. In [22], the authors used Elliptic Maximum Principle to prove that both shrinking and steady warped product Ricci solitons with compact base have constant warping function. Since Ricci solitons are special solutions of the Ricci Flow, it would be interesting to know whether this result extends to solutions of the Ricci Flow other than solitons. The result below asserts that for warped product ancient solutions this still true.

Theorem 9 *Let $(M^{n+m} = B^n \times F^m, g(t))$ be an ancient Ricci Flow that is warped product along time t and that has compact base. Then $(F^m, g_F(t))$ is an Einstein manifolds for each time t with positive Einstein constant.*

To prove Theorem 9 we will compute the evolution equation satisfied by $|\nabla_B u|^2$ and then use the Parabolic Maximum Principle. The result below is an immediate consequence of Proposition 2.

Proposition 22 *Let $(M^{n+m}, g(t) = g_B(t) + e^{2u(t)}g_F^0)$, $t \in [0, \varepsilon)$, $\varepsilon \in (0, \infty]$, be a Ricci flow that preserves warped product. Then R_F^0 is constant and, if it is zero, then $|\nabla_{g_B(t)} u|^2$ satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta_{g_B(t)}\right)|\nabla_{g_B(t)} u|^2 \leq m \left\langle \nabla_{g_B(t)} u, \nabla_{g_B(t)}(|\nabla_{g_B(t)} u|^2)\right\rangle - 2m|\nabla_{g_B(t)} u|^4 \tag{2.25}$$

Proof: It follows from Proposition 2, item (1.9), that if $r = R_F^0 = 0$, then we get (2.25). ∎

Proof of Theorem 9: Let $(M^{n+m}, g(t))$, $t \in [0, T)$, be a Ricci Flow that preserves warped product structure. It follows from Theorem 8 that we can write $g(t) = g_B(t) + e^{2u(t)}g_F^0$, where (F^m, g_F^0) is an Einstein manifold with constant scalar curvature R_F^0 and the families $g_B(t)$ and $u(t)$ satisfy (2.3)-(2.4).

Suppose that B is compact. Then there are $c_1, c_2 \in \mathbb{R}$ so that $u : B^n \times [0, T) \to \mathbb{R}$ satisfies

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \Delta_B\right) u = m|\nabla_B u|^2 - \dfrac{R_F^0}{m} e^{-2u(t)} \\ c_1 \leq u(0) \leq c_2. \end{cases} \tag{2.26}$$

By the Maximum Principle, Corollary 1, if we take $c_j(t)$ solving

$$\begin{cases} \dfrac{d}{dt} c_j(t) = -\dfrac{R_F^0}{m} e^{-2c_j(t)} \\ c_j(0) = c_j, \end{cases} \tag{2.27}$$

$j \in \{1, 2\}$, it follows that $c_1(t) \leq u(t) \leq c_2(t)$, as long as they exist. Solving (2.27) we get for $h = e^u$,

$$e^{2c_1} - \frac{2R_F^0}{m}t = c_1(t) \leq h(t)^2 \leq c_2(t) = e^{2c_2} - \frac{2R_F^0}{m}t. \tag{2.28}$$

Suppose that $(M^{n+m}, g(t))$, $t \in (-\infty, T)$, is an ancient solution that is a warped product along the time. Consider the family of metrics given by $\tilde{g}(t) = g(t - \alpha)$, $t - \alpha \in (-\infty, T)$. It is easy to see that $\tilde{g}$ is a solution of the Ricci Flow that preserves warped product, and has warping function given by $\tilde{h}(t) = h(t-\alpha)$. Notice that taking t and α so that $-\alpha < t < T$, we get $\tilde{h}(t + \alpha) = h(t)$, with $0 < t + \alpha < T + \alpha$.

If $R_F^0 < 0$, it follows from (2.28) by taking the limit when $\alpha \to \infty$ we get

$$h(t) = \lim_{\alpha\to\infty} \tilde{h}(t + \alpha) \geq \lim_{\alpha\to\infty} \left(-\frac{2R_F^0}{m}(t + \alpha) \right) = \infty,$$

which is clearly a contradiction.

If $R_F^0 = 0$, using the Maximum Principle it follows from (2.25) that

$$|\nabla_{g_B(t)} u(t)|^2 \leq \frac{c}{1 + 2mct}, \tag{2.29}$$

where c is the positive constant so that $|\nabla_{g_B(0)} u(0)|^2 \leq c$ and the right hand side of (2.29) is the solution of the ODE

$$\begin{cases} \frac{d}{dt}c(t) = -2mc(t)^2 \\ c(0) = c. \end{cases}$$

Applying (2.29) to $\tilde{u}(t + \alpha) = u(t)$ and taking the limit when $\alpha \to \infty$ we get

$$|\nabla_{g_B(t)} u(t)|^2 \leq \lim_{\alpha\to\infty} \left(\frac{c}{1 + 2mc(t + \alpha)} \right) = 0. \tag{2.30}$$

This is a contradiction, because $u(t) : B \to \mathbb{R}$ is not constant. ■

Theorem 9 can be seen as a parabolic version of Corollary 1 in [22]. Using the fact that Ricci solitons are self similar solutions of the Ricci Flow, we can derive this result as a corollary, as we will see in the next result.

Corollary 5 *There is no gradient Ricci soliton, either shrinking or steady, on a warped product with compact base and fiber with negative scalar curvature somewhere.*

Proof: Assume that f is not constant, B is compact and $\lambda \geq 0$. We can generate the following solutions $g(t)$ for the Ricci flow equation, defined on an interval I either of the form $I = (-\infty, \infty)$, when $\lambda = 0$, or $I = (-\infty, 2/\lambda)$, when $\lambda > 0$. Note that $0 \in I$ and $g(0) = g_0$. For each $t \in I$,

$$\begin{aligned} g(t) &= (1-2\lambda t)\psi_t^* g_0 \\ &= (1-2\lambda t)\psi_t^* g_B + (\sqrt{(1-2\lambda t)}\psi_t^* h)^2 g_F, \end{aligned}$$

that is, $g(t)$ is an ancient solution that is warped product along the time, where ψ_t is the flow of a field suitably chosen (see [20], page 154). By Theorem 9 it follows that $(F^n, g_F(t))$ is Einstein and its Einstein constant is positive, for each t. The result follows from Theorem 9 by taking $t = 0$, since B is compact. ■

It follows from Theorem 8 that the fiber of a Ricci Flow that preserves warped product is an Einstein manifold. It is a corollary of the proof of Theorem 9 that when the scalar curvature of its fiber is positive, the solution develops singularity in finite time. Below we will see an estimate for the singular time in terms of the warping function and the scalar curvature of the fiber. More precisely we have

Corollary 6 *Let $(M^{n+m}, g(t)) = (B^n \times_{h(t)} F^m, g(t) = g_B(t) + e^{2u(t)} g_F^0)$, be a Ricci Flow that preserves warped product and that has compact base. If the scalar curvature of g_F^0 is positive, then the flow develops singularity in a finite time T, where*

$$T \leq \frac{m e^{2max_B u(0)}}{2R_F^0},$$

and R_F^0 is the constant scalar curvature of the fiber (F^m, g_F^0).

Proof: If $R_F^0 > 0$ and $g(t)$ is defined for

$$t_0 = \frac{m e^{2max_B u(0)}}{2R_F^0},$$

it follows from (2.28) with $c_2 = max_{_B} u(0)$ that

$$0 < e^{2u(t_0)} \leq e^{2max_{_B} u(0)} - \frac{2R_F^0}{m} t_0 = 0,$$

which is a contradiction. Therefore $g(t)$ develops singularity before t_0. ■

Chapter 3

Ricci Almost Solitons on semi-Riemannian Warped Products

In this chapter we consider Ricci almost solitons on warped products $B^n \times_h F^m$ and we assume that they are not trivial, in the sense that $h : B \to (0, \infty)$ is not a constant function. In the first section we characterize the fundamental equation (1.35) taking into account the warped product property. It is made by considering two complementary cases depending whether the potential function $f : B^n \times F^m \to \mathbb{R}$ depends on the fiber F or not. The second section deals with the additional assumption that f depends not trivially of the fiber F. From this new assumption we derive rigidity, in the sense that either λ is not constant and $(B^n \times F^m, g)$ is Einstein or $(B^n \times F^m, g, f, \lambda)$ is a Ricci soliton. The third and last section is addressed to the classification under completeness of not trivial Ricci almost solitons for which its potential function depends not trivially on the fiber F.

3.1 Characterization

We start with an important decomposition property of the potential function of a Ricci almost soliton on a warped product. Roughly speaking, it says that under the warped product property, the potential function of a Ricci almost soliton must decompose (see (3.1) below).

Proposition 23 *Let $(B^n \times_h F^m, g, f, \lambda)$ be a Ricci almost soliton defined on a semi-Riemannian warped product manifold, where the base (B^n, g_B) or the fiber (F^m, g_F)*

are either Riemannian or semi-Riemannian manifolds, $h : B \to \mathbb{R}$ is a positive smooth function and $g = g_B + h^2 g_F$. Then the potential function f can be decomposed as

$$f = \beta + h\varphi, \tag{3.1}$$

where $\beta : B \to \mathbb{R}$ and $\varphi : F \to \mathbb{R}$ are smooth functions. Then the fundamental equation (1.35) is equivalent to the system

$$\begin{cases} Ric_B + \nabla_B\nabla_B\beta + (\varphi - mh^{-1})\nabla_B\nabla_B h = \lambda g_B, \\ Ric_F + h\nabla_F\nabla_F\varphi = [h\Delta_B h + (m-1)|\nabla_B h|^2 - h(\nabla_B h)\beta - \varphi h(\nabla_B h)h + \lambda h^2]g_F. \end{cases} \tag{3.2}$$

Proof: In view of Proposition 10 and Proposition 12 we can rewrite the fundamental equation (1.35) as follows

$$\begin{cases} Ric_B(X,Y) - mh^{-1}\nabla_B\nabla_B h(X,Y) + \nabla_B\nabla_B f(X,Y) = \lambda g_B(X,Y), \\ Ric_F(U,V) + \nabla_F\nabla_F f(U,V) = [\lambda h^2 + (m-1)h^{-2}|\nabla_B h|^2 + h^{-1}\Delta_B h - h(\nabla_B h)f]g_F(U,V), \\ X(U(f)) = h^{-1}X(h)U(f). \end{cases} \tag{3.3}$$

Observe that $X(U(f)) - h^{-1}U(f)X(h) = 0$ implies

$$\begin{aligned} X(U(fh^{-1})) &= X(U(f)h^{-1}) \\ &= X(U(f))h^{-1} - U(f)h^{-2}X(h) \\ &= 0, \end{aligned}$$

for all $X \in \mathfrak{L}(B)$ and all $U \in \mathfrak{L}(F)$. Therefore, there are smooth functions $\beta : B \to \mathbb{R}$ and $\varphi : F \to \mathbb{R}$ such that the potential function f decomposes as in (3.1).

Substituting (3.1) in the first two equations of (3.3), a straighforward computation implies that (3.2) holds. ■

In order to analyse the system (3.2), we will consider separately the cases where the potential function f depends on the fiber or not. We observe that when the warping function h is constant, the warped product reduces to the semi-Riemannian product. In this case, the base and the fiber must be Ricci solitons, as we can easily see from

(3.2). So, from now on, we will assume that h is not constant.

The next two theorems show how to separate the fundamental equation of a Ricci almost soliton on a warped product, into equations on the base and on the fiber. First we deal with the case where the potential function depends on the fiber. In this case equation (1.35) reduces as in the following theorem:

Theorem 10 *Let $B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton, with f non constant on F if, and only if, $f = \beta + h\varphi$, where $\varphi : F \to \mathbb{R}$ is not constant and $\beta : B \to \mathbb{R}$ are differentiable functions such that*

$$\begin{cases} \nabla_B \nabla_B h + ahg_B = 0, \\ Ric_B + \nabla_B \nabla_B \beta = [h^{-1}(\nabla_B h)\beta - bh^{-1} + (n-1)a] g_B, \\ \nabla_F \nabla_F \varphi + (c\varphi + b) g_F = 0, \\ Ric_F = (m-1) c g_F, \end{cases} \tag{3.4}$$

for some constants a, b, $c \in \mathbb{R}$, the function λ is given by

$$\lambda = h^{-1}(\nabla_B h)\beta - bh^{-1} + (m+n-1)a - ah\varphi, \tag{3.5}$$

and the constants a and c are related to h by the equation

$$|\nabla_B h|^2 + ah^2 = c. \tag{3.6}$$

Proof: If $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton then it follows from Theorem 23 that $f = \beta + h\varphi$ and the system (3.2) is satisfied. We are assuming that h is not constant and f depends on the fibers. Hence φ is not constant.

Considering the system (3.2) evaluated at pairs of orthogonal vector fields (X, Y), $X, Y \in \mathfrak{X}(B)$ and (U, V), $U, V \in \mathfrak{X}(F)$ locally defined on a neighborhood of any point $(p, q) \in B \times F$, we have

$$\begin{cases} Ric_B(X,Y) + \nabla_B \nabla_B \beta(X,Y) + (\varphi - mh^{-1}) \nabla_B \nabla_B h(X,Y) = 0, \\ Ric_F(U,V) + h \nabla_F \nabla_F \varphi(U,V) = 0. \end{cases} \tag{3.7}$$

Fix $p_1 \in B$ and consider an open neighborhood $G_1 \subset B$ of regular points q of φ and W a vector field such that $W(\varphi) \neq 0$ in G_1. Considering the first equation of (3.7) at the points (p_1, q) and applying W to this equation, we get that

$$\begin{aligned} &\nabla_B \nabla_B h(X,Y)(p_1) = 0, \\ &Ric_B(X,Y)(p_1) + \nabla_B \nabla_B \beta(X,Y)(p_1) = 0 \end{aligned} \qquad \forall p_1 \in B.$$

Similarly, by fixing $q_1 \in F$ and considering an open neighborhood $D_1 \subset B$, of regular points p of h, we obtain from the second equation of (3.7) that

$$\begin{aligned} &\nabla_F \nabla_F \varphi(U,V)(q_1) = 0, \\ &Ric_F(U,V)(q_1) = 0 \end{aligned} \qquad \forall q_1 \in F.$$

Therefore, for any pairs of orthogonal vector fields (X, Y) and (U, V), locally defined in $B \times F$, we have

$$\begin{cases} \nabla_B \nabla_B h(X,Y) = 0, \\ Ric_B(X,Y) + \nabla_B \nabla_B \beta(X,Y) = 0, \\ \nabla_F \nabla_F \varphi(U,V) = 0, \\ Ric_F(U,V) = 0. \end{cases} \tag{3.8}$$

Let $(p_0, q_0) \in B \times F$ such that p_0 and q_0 are regular points of the functions h and φ respectively. Then there exist vector fields X_1 and U_1 defined one open connected sets $D \subset B$ and $G \subset F$ with $p_0 \in D$ and $q_0 \in G$, such that

$$X_1(h)(p) \neq 0, \quad \forall p \in D, \qquad U_1(\varphi)(q) \neq 0, \quad \forall q \in G. \tag{3.9}$$

Let $\{X_1, X_j\}_{j=2}^{n}$ and $\{U_1, U_\alpha\}_{\alpha=2}^{m}$ be orthogonal vector fields on D and G respectively. Without loss of generality we may consider

$$\begin{cases} g_B(X_j, X_k) = \epsilon_j \delta_{jk} h^2, & \forall j,\ k \in \{1, \ldots, n\}, \\ g_F(U_\alpha, U_\gamma) = \varepsilon_\alpha \delta_{\alpha\gamma}, & \forall \alpha,\ \gamma \in \{1, \ldots, m\}, \end{cases} \tag{3.10}$$

where ϵ_j and ε_α denote the signatures of the vector fields.

Now we consider the system (3.2) evaluated at the pairs (X_j, X_j) and (U_α, U_α). Subtracting the first equation multiplied by ϵ_j from the second one mutiplied by ε_α, we get the following expression

$$\varphi(q)\mu_{1j}(p) + h(p)\mu_{2\alpha}(q) = \rho_{1j}(p) + \rho_{2\alpha}(q), \qquad \forall (p,q) \in D \times G, \tag{3.11}$$

where $1 \leq j \leq n, \quad 1 \leq \alpha \leq m$ and

$$\begin{cases} \mu_{1j} = -\epsilon_j \nabla_B \nabla_B h(X_j, X_j) + h|\nabla_B h|^2, \\ \rho_{1j} = h\Delta_B h + (m-1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \epsilon_j [Ric_B + \nabla_B \nabla_B \beta - mh^{-1}\nabla_B \nabla_B h](X_j, X_j), \\ \mu_{2\alpha} = \varepsilon_\alpha \nabla_F \nabla_F \varphi(U_\alpha, U_\alpha), \\ \rho_{2\alpha} = -\varepsilon_\alpha Ric_F(U_\alpha, U_\alpha). \end{cases} \tag{3.12}$$

In view of Lemma 1, it follows from (3.11) that, for each pair (j, α), there exist contants $a_{j\alpha}, b_{j\alpha}, c_{j\alpha}, d_{j\alpha}$, such that

$$\begin{cases} \mu_{1j} = c_{j\alpha} h + \tilde{c}_{j\alpha}, \\ \rho_{1j} = -b_{j\alpha} h + \tilde{b}_{j\alpha}, \\ \mu_{2\alpha} = -c_{j\alpha}\varphi - b_{j\alpha}, \\ \rho_{2\alpha} = \tilde{c}_{j\alpha}\varphi - \tilde{b}_{j\alpha}. \end{cases} \tag{3.13}$$

Therefore,

$$\frac{X_1(\mu_{1j})}{X_1(h)} = c_{j\alpha}, \qquad \frac{X_1(\rho_{1j})}{X_1(h)} = -b_{j\alpha},$$
$$\frac{U_1(\mu_{2\alpha})}{U_1(\varphi)} = -c_{j\alpha}, \qquad \frac{U_1(\rho_{2\alpha})}{U_1(\varphi)} = \tilde{c}_{j\alpha},$$

i.e., $c_{j\alpha}$, $b_{j\alpha}$ do not depend on α, and $c_{j\alpha}$ and $\tilde{c}_{j\alpha}$ do not depend on j. Hence we denote $c_{j\alpha} = c$, $b_{j\alpha} = b_j$ and $\tilde{c}_{j\alpha} = \tilde{c}_\alpha$. Moreover, it follows from (3.13) that

$$\mu_{1j} - ch = \tilde{c}_\alpha \qquad \text{and} \qquad \mu_{2\alpha} + c\varphi = -b_j.$$

Therefore, $\tilde{c}_\alpha$ does not depend on α and b_j does not depend on j. Hence we may denote $\tilde{c}_\alpha = \tilde{c}$, $b_j = b$ and

$$\rho_{1j} + bh = \tilde{b}_{j\alpha}, \qquad \rho_{2\alpha} - \tilde{c}\varphi = -\tilde{b}_{j\alpha}.$$

We conclude that $\tilde{b}_{j\alpha}$ does not depend on j and α and we can denote $\tilde{b}_{j\alpha} = \tilde{b}$. Therefore, it follows from (3.12) and (3.13) that in $D \times G$ we have

$$\begin{cases} -\epsilon_j \nabla_B\nabla_B h(X_j,X_j) + h|\nabla_B h|^2 = ch + \tilde{c}, \\ h\Delta_B h + (m-1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \epsilon_j[Ric_B + \nabla_B\nabla_B\beta - mh^{-1}\nabla_B\nabla_B h](X_j,X_j) = -bh + \tilde{b}, \\ \varepsilon_\alpha \nabla_F\nabla_F\varphi(U_\alpha,U_\alpha) = -c\varphi - b \\ -\varepsilon_\alpha Ric_F(U_\alpha,U_\alpha) = \tilde{c}\varphi - \tilde{b}. \end{cases} \tag{3.14}$$

Considering (3.8) for the orthogonal vector fields $\{X_j\}_{j=1}^n$, $\{U_\alpha\}_{\alpha=1}^m$ it follows from (3.14) that in $D \times G$ we have

$$\begin{cases} \nabla_B\nabla_B h + [ch^{-1} + \tilde{c}h^{-2} - h^{-1}|\nabla_B h|^2]\, g_B = 0, \\ Ric_B + \nabla_B\nabla_B\beta + \left\{h\Delta_B h - |\nabla_B h|^2 - h(\nabla_B h)\beta - \tilde{b} + bh + m\tilde{c}h^{-1} + mc\right\} h^{-2} g_B = 0, \\ \nabla_F\nabla_F\varphi + (c\varphi + b)g_F = 0, \\ Ric_F + (\tilde{c}\varphi - \tilde{b})g_F = 0, \end{cases} \tag{3.15}$$

We will now prove that (3.15) holds in $B \times F$. Let $p_1 \in B$ and $X \in T_{p_1}B$ such that $g_B(X,X) = \epsilon_X h^2(p_1)$, where $\epsilon_X = \pm 1$. Consider $q \in G$ and the system (3.2) at the pair of vectors (X,X) and the pair of vectors fields (U_1,U_1) at (p_1,q), $q \in G$. Multiplying the first equation by $-\epsilon_X$ and adding to the second equation multiplied by ε_1, we get

$$\varphi(q)\mu_{1X}(p_1) + h(p_1)\mu_{21}(q) = \rho_{1X}(p_1) + \rho_{21}(q), \qquad \forall q \in G, \tag{3.16}$$

where

$$\begin{cases} \mu_{1X} = -\epsilon_X \nabla_B \nabla_B h(X,X) + h|\nabla_B h|^2, \\ \rho_{1X} = h\Delta_B h + (m-1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \epsilon_j [Ric_B + \nabla_B \nabla_B \beta - mh^{-1}\nabla_B \nabla_B h](X,X), \\ \mu_{21} = -c\varphi - b, \\ \rho_{21} = \tilde{c}\varphi - \tilde{b}, \end{cases} \tag{3.17}$$

where the last two equalities follow from (3.14) and the fact that $q \in G$. Therefore, (3.16) reduces to

$$[-ch(p_1) + \mu_{1X}(p_1) - \tilde{c}]\varphi(q) = bh(p_1) + \rho_{1X}(p_1) - \tilde{b}, \qquad \forall q \in G.$$

Applying the vector field U_1 to this equation, we conclude that

$$\mu_{1X}(p_1) = ch(p_1) + \tilde{c}, \qquad \rho_{1X}(p_1) = -bh(p_1) + \tilde{b}. \tag{3.18}$$

Similarly, considering $q_1 \in F$ and $U \in T_{q_1}F$ such that $g_F(U,U) = \varepsilon_U = \pm 1$, for all $p \in D$ the equations of (3.2) evaluated at the pairs (X_1, X_1) and (U,U) will imply that

$$\varphi(q_1)\mu_{11}(p) + h(p)\mu_{2U}(q_1) = \rho_{11}(p) + \rho_{2U}(q_1),$$

where

$$\mu_{2U} = \varepsilon_U \nabla_F \nabla_F \varphi(U,U), \qquad \rho_{2U} = -\varepsilon_U Ric_F(U,U).$$

Analogue arguments as before will imply that

$$\mu_{2U}(q_1) = -c\varphi(q_1) - b, \qquad \rho_{2U}(q_1) = \tilde{c}\varphi(q_1) - \tilde{b}. \tag{3.19}$$

Since $p_1 \in B$ and $q_1 \in F$ are arbitrary, we conclude that for any locally defined vector fields $X \in \mathfrak{X}(B)$ and $U \in \mathfrak{X}(F)$, such that $g_B(X,X) = \epsilon_X h^2$ and $g_F(U,U) = \varepsilon_U$ we have that (3.18) and (3.19) hold. We now consider any point $(p_1, q_1) \in B \times F$ and orthogonal fields locally defined $Y_1, ..., Y_n$ in $\mathfrak{X}(B)$, $V_1, ...V_m$ in $\mathfrak{X}(F)$ such that

$g_B(Y_j, Y_j) = \epsilon_j h^2$ and $g_F(V_\alpha, V_\alpha) = \varepsilon_\alpha$. Then

$$\begin{cases} -\epsilon_j \nabla_B \nabla_B h(Y_j, Y_j) + h|\nabla_B h|^2 = ch + \tilde{c}, \\ h\Delta_B h + (m-1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \epsilon_j[Ric_B + \nabla_B \nabla_B \beta - mh^{-1}\nabla_B \nabla_B h](Y_j, Y_j) = -bh + \tilde{b}, \\ \varepsilon_\alpha \nabla_F \nabla_F \varphi(V_\alpha, V_\alpha) = -c\varphi - b \\ -\varepsilon_\alpha Ric_F(V_\alpha, V_\alpha) = \tilde{c}\varphi - \tilde{b}. \end{cases}$$

Considering (3.8) for the orthogonal vector fields $\{Y_j\}_{j=1}^n$ and $\{V_\alpha\}_{\alpha=1}^m$ it follows that (3.15) holds in $B \times F$.

We will now use Bochner formula (1.52) to prove that

$$\tilde{c} = 0 \qquad \text{and} \qquad \tilde{b} = (m-1)c. \tag{3.20}$$

In fact, it follows from the third equation of (3.15) that

$$U_1(\Delta_F \varphi) = -cmU_1(\varphi).$$

From the fourth equation, we have

$$Ric_F(\nabla_F \varphi, U_1) = (-\tilde{c}\varphi + \tilde{b})U_1(\varphi).$$

Moreover,

$$\begin{aligned} div(\nabla_F \nabla_F \varphi)(U_1) &= \sum_{\alpha=1}^m (\nabla_{F\,U_\alpha} \nabla_F \nabla_F \varphi)(U_1, U_\alpha) \\ &= \sum_{\alpha=1}^m (\nabla_{F\,U_\alpha}(-(c\varphi + b)g_F))(U_1, U_\alpha) \\ &= -cg_B(U_1, \sum_{\alpha=1}^m U_\alpha(\varphi)U_\alpha) \\ &= -cU_1(\varphi). \end{aligned}$$

Now Bochner formula implies that

$$[\tilde{c}\varphi - \tilde{b} + c(m-1)]U_1(\varphi) = 0.$$

Since $U_1(\varphi) \neq 0$, we conclude that (3.20) holds.

Therefore, on $B \times F$, the system (3.15) reduces to

$$\begin{cases} \nabla_B \nabla_B h + (c - |\nabla_B h|^2) h^{-1} g_B = 0, \\ Ric_B + \nabla_B \nabla_B \beta + \{h^{-1}[\Delta_B h - (\nabla_B h)\beta + b] + h^{-2}(c - |\nabla_B h|^2)\} g_B = 0, \\ \nabla_F \nabla_F \varphi + (c\varphi + b) g_F = 0, \\ Ric_F - (m-1) c g_F = 0. \end{cases} \tag{3.21}$$

Observe that for any $X \in \mathfrak{L}(B)$, we have the following expressions

$$\nabla_B \nabla_B h(X, \nabla_B X) = g_B(\nabla_X \nabla_B h, \nabla_B h) = \tfrac{1}{2} X(|\nabla_B h|^2),$$
$$\nabla_B \nabla_B h(X, \nabla_B X) = (|\nabla_B h|^2 - c)\, h^{-1} X(h),$$

where the second equality follows from (3.21). Therefore,

$$\frac{X(|\nabla_B h|^2)}{2} - \left(|\nabla_B h|^2 - c\right) h^{-1} X(h) = 0,$$

which implies that

$$X\left[\left(c - |\nabla_B h|^2\right) h^{-2}\right] = 0.$$

Hence there exists a constant a such that

$$\left(c - |\nabla_B h|^2\right) h^{-2} = a,$$

i.e., (3.6) holds. Moreover, the first equation of (3.21) reduces to

$$\nabla_B \nabla_B h + a h g_B = 0$$

and $\Delta_B h = -anh$. Hence, the second equation of (3.21) reduces to

$$Ric_B + \nabla_B \nabla_B \beta = \left[(n-1)a + h^{-1}(\nabla_B h)\beta - b h^{-1}\right] g_B.$$

Finally, it follows from these two last equations that the first equation of (3.2) provides

$$\lambda = h^{-1}(\nabla_B h)\beta + (m+n-1)a - b h^{-1} - a h \varphi.$$

Therefore, the functions f, h and λ satisfy the system (3.4). The converse is a straightforward computation. This concludes the proof of Theorem 10. ■

Remark 2 Equations such as the first or third equations of (3.4) have appeared in many contexts in semi-Riemannian geometry. They appeared for example in concircular transformations [50], in conformal transformations between Einstein spaces [37] and in conformal vector fields on Einstein manifolds [38].

Remark 3 A function satisfying equation (3.6) is said to have constant energy, following [13], where the author investigated properties of such functions. Equation (3.6) also appeared in the Critical Point Equation conjecture [40].

As an application of Theorem 10 we will prove that for a complete warped product Ricci solitons (that is, when λ is a constant) the potential function does not depend on the fiber.

Corollary 7 *Let $(B \times_h F, g, f, \lambda)$ be a Ricci soliton on a complete non trivial semi-Riemannian warped product. Then f does not depend on the fiber.*

Proof: Suppose by contradiction that f depends on the fiber, then it follows from Theorem 10 that $f = \beta + h\varphi$ where φ is not constant. Moreover, β, h, φ and λ satisfy (3.4)-(3.6). Hence there exists a vector field $U \in \mathfrak{L}(F)$ such that $U(\varphi) \neq 0$ on an open subset of F. Since λ is constant, taking the derivative of (3.5) with respect to U, we obtain $0 = U(\lambda) = -ahU(\varphi)$. Hence $a = 0$ and the first equation of (3.4) reduces to $\nabla_B \nabla_B h = 0$. However, it follows from Proposition 14 that if $B \times_h F$ is complete then $\nabla_B h$ is not parallel, which is a contradiction. ■

Remark 4 Corollary 7 was also considered in [47] by a different approach. It shows that examples of Ricci solitons on complete semi-Riemannian warped product occur when the potential function depends only on the base.

Now we consider the case where the potential function does not depend on the fiber.

Theorem 11 *Let $B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton, with f constant on F if, and only if,*

$$\begin{cases} Ric_B + \nabla_B \nabla_B f - mh^{-1} \nabla_B \nabla_B h = \lambda g_B, \\ \lambda h^2 = h(\nabla_B h) f - (m-1)|\nabla_B h|^2 - h\Delta_B h + c(m-1), \\ Ric_F = c(m-1) g_F, \end{cases} \tag{3.22}$$

for some constant $c \in \mathbb{R}$.

Proof: It follows from Proposition 23 that if $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton and f is constant on F, then in the decomposition of f given by (3.1) we may consider $\varphi = 0$. Therefore, from the first equation of (3.2) we get that the first equation of (3.22) holds and that λ is a function constant on F, hence it depends only on B. In order to obtain the other equations of (3.22), we observe that if $U \in \mathfrak{L}(F)$ is a unitary vector field satisfying $g_F(U, U) = \epsilon \in \{-1, 1\}$ we obtain from the second equation of (3.2) :

$$\epsilon Ric_F(U, U) = h\Delta_B h + (m-1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \lambda h^2.$$

Since the left hand side is a function defined only on F and the right hand side is a function defined only on B, there is a constant $\tilde{c} \in \mathbb{R}$ independent of the fixed field U, (as we can see using the right hand side of the above equality), such that

$$\lambda h^2 = h(\nabla_B h)\beta - (m-1)|\nabla_B h|^2 - h\Delta_B h + \tilde{c},$$

and

$$Ric_F = \tilde{c} g_F.$$

In order to normalize the Einstein constant, we consider $\tilde{c} = (m-1)c$. This proves that (3.22) holds. The converse is a simple calculation. ■

The Riemannian version of Theorem 11 was considered in [23], where the authors gave some explicit solutions to the system.

The essence of both Theorems 10 and 11 is to express the condition for a warped product to be a Ricci almost soliton in terms of conditions on the base and on the fiber.

Note that the first and third equations in Theorem 10 say that the corresponding gradient vector fields are conformal (see Section 1.3 for definitions). In addition, the fourth equation of Theorem 10 and the third equation of Theorem 11 show that the fiber is an Eisntein manifold in both cases.

3.2 Rigidity when the potential function depends on the Fiber

We start this section stating its main result, which says that when the potential function f depends on the fiber F then the Ricci almost soliton must be somehow rigid.

Theorem 12 *If $(B^n \times_h F^m, g, f, \lambda)$ is a non-trivial warped product Ricci almost soliton with f non constant on F, then either λ is not constant and $(B^n \times_h F^m, g)$ is an Einstein manifold or $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci soliton.*

The proof of Theorem 12 will follow from the proofs of Theorem 14 and Theorem 13 (see below for their statements). In order, for stating these theorems we will introduce some important notions in what follows. Among all conformal transformations, we stress the following ones.

Definition 10 A vector field X is *homothetic* if its local flow acts by translations. Otherwise, it is called *non homothetic.*

Now we will give the notion of Brinkmann space, which plays an important role in General Relativity [9] and was introduced by Brinkmann [9] when the author studied conformal transformations between Einstein manifolds.

Definition 11 We say that a semi-Riemannian manifold (M, g) is a *Brinkmann space* if it admits a parallel light like vector field X, called a *Brinkmann field.*

Our next result characterizes Ricci almost solitons on semi-Riemannian warped products, when the potential function depends on the fiber and $\nabla_B h$ is an improper vector field on B (see Definitioin 6 to recall this notion).

Theorem 13 *Let $B^n \times_h F^m$, $n \geq 2$, be a non trivial warped product where the base (B^n, g_B) is a semi-Riemannian manifold and the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost* *soliton, with f non constant on F and $\nabla_B h$ an improper vector field on B if, and only if, λ is constant and $f = \beta + h\varphi$, where $\varphi : F \to \mathbb{R}$ non constant and $\beta : B \to \mathbb{R}$ are smooth functions satisfying*

$$g(\nabla_B h, \nabla_B \beta) = \lambda h + b, \qquad Ric_B + \nabla_B \nabla_B \beta = \lambda g_B, \qquad \nabla_F \nabla_F \varphi + b g_F = 0$$

for a constant $b \in \mathbb{R}$, B is a Brinkmann space with $\nabla_B h$ as a Brinkmann field and F is Ricci flat. If in addition F is complete, then it is isometric to

1. $\pm\mathbb{R} \times \bar{F}^{m-1}$, *where $\bar{F}$ is Ricci flat, if $b = 0$;*
2. $\mathbb{R}^m_\epsilon$, *if $b \neq 0$.*

Proof: From Theorem 10, we have that $f = \beta + h\varphi$ and equations (3.4)-(3.6) are satisfied. If $\nabla_B h$ is an improper vector field on B, it follows from equation (3.6) that $a = c = 0$. Hence, (3.4) and (3.5) imply that $\nabla_B h$ is a parallel light like vector field, (F, g_F) is Ricci flat and

$$\begin{cases} Ric_B + \nabla_B \nabla_B \beta = \lambda g_B, \\ \lambda = h^{-1}(\nabla_B h)\beta - bh^{-1}, \\ \nabla_F \nabla_F \varphi + b g_F = 0. \end{cases} \tag{3.23}$$

Now we will prove that λ is constant. If $\lambda = 0$ there is nothing to prove. Otherwise, there is an open set $U \subset M$ where λ does not vanish. Then it follows from the second equation of (3.23) that

$$\begin{aligned} \frac{1}{2}X(\ln(\lambda^2)) &= \frac{1}{2}X(\ln(h^{-2}(g(\nabla_B h, \nabla_B \beta) - b)^2)) \\ &= -h^{-1}X(h) + (g(\nabla_B h, \nabla_B \beta) - b)^{-1} X(g(\nabla_B h, \nabla_B \beta)) \\ &= -h^{-1}X(h) + (g(\nabla_B h, \nabla_B \beta) - b)^{-1} \nabla_B \nabla_B \beta(X, \nabla h). \end{aligned} \tag{3.24}$$

Since $\nabla_B h$ is a parallel vector field, Bochner's Formula implies that $Ric(X, \nabla_B h) = 0$, hence from the first equation of (3.23), we get that $\nabla_B \nabla_B \beta(X, \nabla_B h) = \lambda g_B(X, \nabla_B h)$. We conclude, using the second equation of (3.23) that (3.24) reduces to

$$\frac{1}{2}X(\ln(\lambda^2)) = -h^{-1}X(h) + h^{-1}X(h) = 0,$$

which proves that λ is constant. The converse is immediate.

Now suppose that (F, g_F) is complete. Since $\nabla_F \nabla_F \varphi + b g_F = 0$, the result follows from Theorem 5, if $b \neq 0$ and from Theorem 6 if $b = 0$. ■

The next result deals with the rigidity of a Ricci almost soliton on a warped product when the potential function depends on the fiber and $\nabla_B h$ is a proper vector field.

Theorem 14 *Let $B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold and suppose that $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton with f non constant on F and $\nabla_B h$ a proper vector field. Then*

1. *If $\nabla_B h$ is homothetic, then λ is constant, i.e., it is a Ricci soliton;*
2. *If $\nabla_B h$ is non-homothetic, then λ is not constant, B, F and $B^n \times_h F^m$ are Einstein manifolds such that*

$$\begin{aligned} Ric_{B\times_h F} &= (n+m-1)ag, \\ Ric_B &= (n-1)ag_B, \\ Ric_F &= (m-1)cg_F, \end{aligned} \tag{3.25}$$

where the constants $a \neq 0$ and c are related to h by $|\nabla_B h|^2 + ah^2 = c$. Moreover, ∇f and $\nabla_B h$ are conformal gradient fields on $B^n \times_h F^m$ and on B^n, respectively, satisfying

$$\begin{aligned} \nabla\nabla f + (af + a_0)g &= 0, \\ \nabla_B \nabla_B h + ahg_B &= 0, \end{aligned} \tag{3.26}$$

and

$$\lambda = -af + a(m+n-1) - a_0, \tag{3.27}$$

for some constant $a_0 \in \mathbb{R}$.

Proof: If $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci almost soliton with h non constant and f depending on the fiber then, it follows from Theorem 10 that there are functions $\beta : B \to \mathbb{R}$ and $\varphi : F \to \mathbb{R}$ and constants $a, b, c \in \mathbb{R}$, such that $f = \beta + h\varphi$ where β, h, φ and λ satisfy (3.4)-(3.6).

If $\nabla_B h$ is a homothetic vector field, then $a = 0$. It means that this vector field is parallel, and by the same argument as in the proof of Theorem 13, we see that λ is constant, which proves that $(B^n \times_h F^m, g, f, \lambda)$ is a Ricci soliton.

From now on we will suppose that $\nabla_B h$ is a non homothetic vector field, that is, $a \neq 0$.

If $n = 1$, from these equations we get

$$\begin{cases} h'' \pm ah = 0, \\ \pm(h')^2 + ah^2 = c, \\ Ric_F = (m-1)cg_F, \end{cases}$$

where $g_{B^1} = \pm dt^2$. Therefore, $B^1 \times_h F^m$ is an Einstein manifold with normalized Einstein constant a, as a consequence of Corollary 2.

If $n \geq 2$, it follows from the second equation of (3.4) that $(B, g_B, \beta, \bar{\lambda})$ is a Ricci almost soliton, i.e.,

$$Ric_B + \nabla_B \nabla_B \beta = \bar{\lambda} g_B \tag{3.28}$$

where

$$\bar{\lambda} = h^{-1}(\nabla_B h)\beta - bh^{-1} + (n-1)a. \tag{3.29}$$

From the first equation of (3.4), we get that $\nabla_B h$ is a gradient conformal field satisfying

$$\nabla_B \nabla_B h + ahg_B = 0, \tag{3.30}$$

i.e., $(B, g_B, \beta, \bar{\lambda})$ is a Ricci almost soliton. Moreover, $\nabla_B \nabla_B h - \Delta_B h/ng = 0$. By hypothesis, $\nabla_B h$ is a non homothetic vector field hence $\nabla_B h$ is a proper vector field, and therefore $a \neq 0$ and h admits regular points. Fixing a regular point of h, $p \in B$, it follows from Proposition 21 that there exists a connected open set $D \subset B$, containing p, such that D is diffeomorphic to $(-\varepsilon, \varepsilon) \times N^{n-1}$ for $\varepsilon > 0$ and a regular level N^{n-1} of h, in such a way that h does not depend on N^{n-1} and (D, g_D) is isometric to $((-\varepsilon, \varepsilon) \times N^{n-1}, \pm dt^2 + h'(t)^2 g_N)$, where $g_D = g_B|_D$ and $g_N = g_B|_N$. By restricting β and $\bar{\lambda}$ to D, we have that $(D, g_D, \beta, \bar{\lambda})$ is a Ricci almost soliton, therefore

$$((-\varepsilon, \varepsilon) \times_{h'} N^{n-1}, \pm dt^2 + h'(t)^2 g_N, \beta, \bar{\lambda}), \tag{3.31}$$

is also a Ricci almost soliton. We are going to use this coordinate system to conclude that (D, g_D) is an Einstein manifold with normalized Einstein constant a. This is equivalent to proving that the following equations hold

$$\begin{cases} h''' \pm ah' = 0, \\ \pm(h'')^2 + a(h')^2 = \bar{c}, \\ Ric_N = (n-2)\bar{c}g_N, \end{cases} \tag{3.32}$$

as one can see from Corollary 2. In order to do so, we must consider two cases whether β depends on N^{n-1} or not.

Suppose that β depends on N^{n-1}, then we can apply Theorem 10 to (3.28), when restricted to D, given as in 3.31. From the first and fourth equations of (3.4) we get that the following equations hold

$$\begin{cases} h''' \pm \bar{a}h' = 0, \\ Ric_N = (n-2)\bar{c}g_N, \end{cases} \tag{3.33}$$

for some constants $\bar{a}, \bar{c} \in \mathbb{R}$. Moreover, from (3.6) the constants $\bar{a}$ and $\bar{c}$ are related to h' by the equation $\pm(h'')^2 + \bar{a}(h')^2 = \bar{c}$. It follows from the first equation of (3.4) and (3.33) that $a = \bar{a}$. This proves (3.32) for this case.

Suppose that β does not depend on N^{n-1}, then since (3.28) holds, we can apply Theorem 11 to D given as in (3.31). Then (3.22) reduces to

$$\begin{aligned} &\beta'' - (n-1)(h')^{-1}h''' = \pm\bar{\lambda}, \\ &(h')^2\bar{\lambda} = \pm h'h''\beta' \mp (n-2)(h'')^2 \mp h''' + \bar{c}(n-2), \\ &Ric_N = \bar{c}(n-2)g_N, \end{aligned} \tag{3.34}$$

for some constant $\bar{c} \in \mathbb{R}$. Moreover, the first equation of (3.4) restricted to D gives $h'' \pm ah = 0$ and hence $h''' \pm ah' = 0$. These two equations substituted into the first two equations of (3.34) implies that

$$\begin{aligned} &\beta'' \pm (n-1)a = \pm\bar{\lambda}, \\ &(h')^2\bar{\lambda} = -ahh'\beta' \mp (n-2)a^2h^2 + a(h')^2 + \bar{c}(n-2). \end{aligned} \tag{3.35}$$

Substituting (3.29) into both equations of (3.35), and using (3.6) we conclude that the following equations hold

$$\begin{cases} (\beta' h^{-1})' = \mp b h^{-2}, \\ c\beta' h^{-1} = b h^{-1} h' + (n-2)(\bar{c} \mp ac)(h')^{-1}, \\ h''' \pm a h' = 0, \\ \pm (h'')^2 + a(h')^2 = \pm ac, \\ Ric_N = \bar{c}(n-2) g_N \end{cases} \tag{3.36}$$

Therefore, in order to prove that (3.32) holds, we need to show the equality $\bar{c} = \pm ac$. If $c = 0$ it follows from the second equation of (3.36) that $\bar{c} = 0$. If $c \neq 0$, then we substitute the second equation of (3.36) into the first one to obtain

$$a(\bar{c} \mp ac) = 0.$$

which implies $\bar{c} = \pm ac$, since $a \neq 0$. Therefore, we have proved that (3.32) also holds when β does not depend on N^{n-1}.

Now from Proposition 20, we know that the set of regular points of h is a dense subset of B, and the argument above implies that (B, g_B) is an Einstein manifold with normalized Einstein constant a. As a consequence we have

$$\begin{cases} Ric_B = a(n-1) g_B, \\ \nabla_B \nabla_B h + a h g_B = 0, \\ Ric_F = c(m-1) g_F, \end{cases} \tag{3.37}$$

which implies from Proposition 11 that $B \times_h F$ is itself Einstein with normalized Einstein constant a.

From the fundamental equation (1.35), we obtain that

$$\nabla \nabla f + ((m+n-1)a - \lambda) g = 0,$$

and ∇f is a gradient conformal field on an Einstein manifold. Proposition 18 says that there is a constant a_0 such that

$$\lambda = -af + a(m + n - 1) + a_0,$$

in view of $n + m \geq 2$. Hence $\nabla\nabla f + (af - a_0)g = 0$. This concludes the proof of Theorem 14. ∎

The proof of Theorem 12 is a direct consequence of both Theorem 13 and Theorem 14.

3.3 Classification when the potential function depends on the Fiber

In this section, we classify complete Ricci almost solitons on warped product Riemannian or semi-Riemannian manifolds in the case where f depends on the fiber. In order to state our classification result for Ricci almost solitons on complete semi-Riemannian warped products, we consider the following classes of n-dimensional complete semi-Riemannian Einstein manifolds (see Theorems 6 and 7) :

Class I

1. $\mathbb{R} \times N^{n-1}$ where (N, g_N) is a complete semi-Riemannian Einstein manifold.

2. A Brinkman space of dimension $n \geq 3$, i.e. a semi-Riemannian manifold (M^n, g) admitting a parallel light like vector field.

Class II

1. $\mathbb{S}^n_\varepsilon(1/\sqrt{c})$, when $0 \leq \varepsilon \leq n-2$; the covering of $\mathbb{S}^n_{n-1}(1/\sqrt{c})$ when $\varepsilon = n-1$ and the upper part of $\mathbb{S}^n_n(1/\sqrt{c})$ when $\varepsilon = n$ with $c > 0$.

2. $\mathbb{H}^n_\varepsilon(1/\sqrt{|c|})$, when $2 \leq \varepsilon \leq n-1$; the covering of $\mathbb{H}^n_1(1/\sqrt{|c|})$ when $\varepsilon = 1$ and the upper part of $\mathbb{H}^n_0(1/\sqrt{|c|})$ when $\varepsilon = 0$, with $c < 0$.

3. $(\mathbb{R} \times N^{n-1}, \pm dt^2 + \cosh^2(\sqrt{|c|}\,t)g_N)$, where (N^{n-1}, g_N) is a semi-Riemannian Einstein manifold.

4. $(\mathbb{R} \times N^{n-1}, \pm dt^2 \pm e^{2\sqrt{|c|}\,t} g_N)$, where (N^{n-1}, g_N) is a Riemannian Einstein manifold,

The following result classifies the complete Ricci almost solitons on warped products, whose potential functions depend on the fiber.

Theorem 15 *Let $M^{n+m} = B^n \times_h F^m$ be a non trivial warped product where the base (B^n, g_B) or the fiber (F^m, g_F) can be either a Riemannian or a semi-Riemannian manifold. Then $(B^n \times_h F^m, g, f, \lambda)$ is a complete Ricci almost soliton with f non constant on F if, and only if, there exist constants $a \neq 0$, $a_0, c \in \mathbb{R}$ such that $f = a^{-1}(-\lambda + a(m+n-1) - a_0)$ and*

1. *if $n = 1$ then B^1 is isometric to $(\mathbb{R}, sgn\, a\, dt^2)$*

$$h = \begin{cases} Ae^{\sqrt{|a|}t} & \text{if} \quad c = 0, \\ \sqrt{|\frac{c}{a}|}[\cosh(\sqrt{|a|}t + B)] & \text{if} \quad c \neq 0, \end{cases} \tag{3.38}$$

   where $A \neq 0$ and $B \in \mathbb{R}$. Moreover, M is an Einstein manifold satisfying $Ric_M = (m+n-1)ag$ and if $m \geq 2$, F is an Einstein manifold satisfying $Ric_F = (m-1)cg_F$.

2. *If $n \geq 2$ and $m \geq 2$ then*

   - *M^{n+m} is an Eisntein manifold isometric either to a manifold of Class II.1 (resp. II.2) when $a > 0$ (resp. $a < 0$) and f has some critical point or it is isometric to a manifold of Class II.3 or II.4 if f has no critical points.*

   - *B is a complete Einstein manifold isometric either to a manifold of Class II.1 (resp. Class II.2) and index $\varepsilon_B = n$ (resp. $\varepsilon_B = 1$) if $a > 0$ (resp. $a < 0$) and h has critical points or to a manifold of Class II.3 or II.4 if h has no critical points.*

   - *F is a complete Einstein manifold isometric to either $\mathbb{R}^n_\varepsilon$, or to a manifolds of Class I when $c = 0$ and it is isometric to a manifold of Class II when $c \neq 0$.*

3. *Moreover, F^m, $m \geq 1$ is positive definite (resp. negative definite) if B^n, $n \geq 1$ is positive definite (resp. negative definite).*

Proof: Suppose that $(B^n \times_h F^m, g, f, \lambda)$ is a complete Ricci almost soliton, with h non constant and f depending on F. Then it follows from Theorem 10 that there are functions $\beta : B \to \mathbb{R}$ and $\varphi : F \to \mathbb{R}$ and constants $a, b, c \in \mathbb{R}$ such that $f = \beta + h\varphi$, where β, h, φ and λ satisfy (3.4)-(3.6). From Proposition 14 the completeness of $B^n \times_h F^m$ implies that $\nabla_B h$ is not a parallel vector field on B and hence it follows from the first equation of (3.4) that $a \neq 0$, therefore $\nabla_B h$ is not homothetic. Applying Theorem 14 we have that $B^n \times_h F^m$, B and F are Einstein manifolds satisfying (3.25) for constants $a \neq 0$ and $c, a_0 \in \mathbb{R}$, $\nabla_B h$, $\nabla_F \varphi$ and ∇f are conformal vector fields satisfying (3.26) and λ is given by (3.27).

If $n = 1$ then $g_B = \pm dt^2$ and from the first equation of (3.4) and (3.6) we have that $h'' \pm ah = 0$ and $\pm(h')^2 + ah^2 = c$. Since B is not compact it follows that $B^1 = \mathbb{R}$ and the non vanishing of h implies that $\pm a < 0$. Therefore h satisfies

$$
\begin{aligned}
&h'' - |a|h = 0,\\
&(h')^2 - |a|h^2 = \pm c,
\end{aligned}
$$

and hence (3.38) holds i.e.

$$
h = \begin{cases} Ae^{\sqrt{|a|}t} & \text{if} \quad c = 0, \\ \sqrt{|\frac{c}{a}|}[\cosh(\sqrt{|a|}t + \theta)] & \text{if} \quad c \neq 0, \end{cases}
$$

where $A \neq 0$ and $\theta \in \mathbb{R}$.

If $n \geq 2$ and $m \geq 2$, it follows that B^n and F^m are complete Einstein manifolds satisfying (3.25).

Since f satisfies the first equation of (3.26) it follows that $\tilde{f} = f - a_0/a$ is a solution of $\nabla\nabla\tilde{f} + a\tilde{f}g = 0$, therefore from Theorem 7 we conclude that when f has some critical point then $B \times_h F$ is isometric to a manifold of Class II 1 (resp Class II 2) when $a > 0$ (resp. $a < 0$) and f is a height function on $S^n_\varepsilon(1/\sqrt{a})$ (resp. $H^n_\varepsilon(1/\sqrt{|a|})$ (see Examples 2 and 3); when f has no critical points then $B \times_h F$ is isometric to a manifold of Class II 3 or 4.

Since h satisfies the second equation of (3.26) then it follows from Theorem 7 that if h has no critical points then B is isometric to one of the manifolds of Class II 3 or 4 and if h has some critical point then B is isometric to a manifold of Class II 1 or 2 according to the sign of a moreover, h is a height function. However, since h does not vanish it induces a restriction on the index of B, in fact, it follows from Proposition 19 that when $a > 0$ (resp. $a < 0$) B is isometric to $\mathbb{S}^n_n(1/\sqrt{a})$ (resp. $\mathbb{H}^n_1(1/\sqrt{|a|})$.

Since φ satisfies the third equation of (3.4), i.e. $\nabla_F \nabla_F \varphi + (c\varphi + b)g_F = 0$, it follows from Theorem 5 that if $c = 0$ and $b \neq 0$, then F is isometric to a semi Eulidean space $\mathbb{R}^m_\varepsilon$. If $c = b = 0$ then Theorem 6 implies that F is isometric to a manifold of Class I. Finally, if $c \neq 0$ then Theorem 7 implies that F is isometric to a manifold of Class II 1 (resp. Class II 2) when $c > 0$ (resp. $c < 0$) and φ has some critical point while F is isometric to a manifold of Class II 3 or 4 when φ has no critical points.

We conclude by observing that, since $B \times F$ is complete, in order to avoid the phenomena of Been-Buseman example one must have F^m, $m \geq 1$ positive definite (resp. negative definite) if B is positive definite (resp. negative definite). ■

Remarks:

1. The proofs of our main results rely strongly on an important property of the potential function f that decomposes as $f = \beta + h\varphi$, where β and h are defined on the base and φ is defined on the fiber F (see Proposition 23). By considering this decomposition, in Theorem 15 item 2, when $c \neq 0$, the fiber F is isometric to a manifold of Class II 1 (resp. Class II 2) when $c > 0$ (resp. $c < 0$) and φ has some critical point, while F is isometric to a manifold of Class II 3 or 4 when φ has no critical points (see proof of Theorem 15).

2. We observe that, when we are in the Riemannian setting, Theorem 13 does not occur, Class I only contains the product of $\mathbb{R} \times N^{n-1}$, where (N, g_N) is a complete Riemannian Einstein manifold and Class II is restricted to the Riemannian manifolds.

Chapter 4

Hamilton-Ivey Estimate for the Ricci-Bourguignon Flow

Recall that a Ricci-Bourguignon Flow on a manifold M is a family of Riemannian metrics $g(t)$, $t \in [0,T)$, satisfying equation (1.1), that is,

$$\frac{\partial}{\partial t} g = -2(Ric - \rho R g),$$

where ρ is a given constant.

In this chapter, we will prove that for a three dimensional manifold, ancient solutions of the Ricci-Bourguignon Flow have nonnegative sectional curvature for $\rho \in (-1/2, 1/4)$, extending results previously proved by Hamilton ($\rho = 0$, [28]) and Catino and coworkers ($\rho \in (0, 1/6)$, [7]). Ancient solutions are important in the process of understanding and classifying singular solutions of this flow.

The strategy to prove this result is similar to the one established by Hamilton for $\rho = 0$. The first step is to provide an estimate for the scalar curvature in terms of the smallest sectional curvature, known as Hamilton-Ivey estimate. This estimate, in turn, has its own interest since it has a clear geometric interpretation (see [20], page 243).

In Section 4.1, we introduce and start investigating a set that depends on time and on 2 parameters, one of them being ρ, to which we will apply the Tensor Maximum Principle. In Section 4.2, we prove the invariance of this set by the ODE (1.29) when $\rho \in (-1/2, 0]$ and prove the Hamilton-Ivey estimate in this case. In Section 4.3, we prove analogous results for the case where $\rho \in [0, 1/4)$. In the last section, Section 4.4, we demonstrate that ancient solutions of the Ricci-Bourguignon Flow have positive sectional curvature.

4.1 The set $K_p^{\eta,\rho}(t)$ and its Properties

To prove the Hamilton-Ivey estimate, we will consider a subset of the bundle of self adjoint endomorphisms of $\wedge^2 V$, $End_{SA}(\wedge^2 T_p M)$, for which we intend to use the Vectorial Maximum Principle (Theorem 3 or Theorem 4). The idea is to investigate the solutions of the PDE (1.17) by studying the solutions of its associated ODE (1.29).

Given an element $\mathcal{O}_p \in End_{SA}(\wedge^2 T_p M)$, we denote its smallest eigenvalue by $\gamma(\mathcal{O}_p)$. Let us consider real numbers η and ρ such that $1 + \eta\rho \geq 0$. We consider the following time-dependent set $K_p^{\eta,\rho}(t)$, $t \in [0, T)$, so that for $t = 0$ it is defined as

$$K_p^{\eta,\rho}(0) = \left\{ \ \mathcal{O}_p \in End_{SA}(\wedge^2 T_p M) : \gamma(\mathcal{O}_p) \geq -1, \ \mathcal{O}_p \text{ satisfies } (P_1)_0 \text{ and } (P_2)_0 \ \right\},$$

and for $t > 0$ it is defined as

$$K_p^{\eta,\rho}(t) = \left\{ \ \mathcal{O}_p \in End_{SA}(\wedge^2 T_p M) : \mathcal{O}_p \text{ satisfies } (P_1)_t \text{ and } (P_2)_t \ \right\},$$

where p is a point in M^3 and the properties $(P_1)_t$ and $(P_2)_t$ are given by

$(P_1)_t$ $\quad tr(\mathcal{O}_p) \geq -\dfrac{3}{1 + 2(1 + \eta\rho)t}$,

$(P_2)_t$ $\quad$ There are constants $\theta_1 \geq \theta_2 > 0$ such that if $\gamma(\mathcal{O}_p) \leq -\dfrac{1}{1 + 2(1 + \eta\rho)t}$, then

$$tr(\mathcal{O}_p) \geq -\gamma(\mathcal{O}_p)(\theta_1 \log(-\gamma(\mathcal{O}_p)) + \theta_2 \log(1 + 2(1 + \eta\rho)t) - 3)$$

Since we plan to use Theorem 4, we need to check the following properties for $K_p^{\eta,\rho}(t)$:

1. Convexity;

2. Invariance under parallel translations;

3. Closedness of its track (1.34);

4. Invariance under ODE (1.29).

These properties will be proved in several steps, the last one being the hardest to show when $\rho \leq 0$. When $\rho \in [0, 1/4)$ the proof is similar to Hamilton's original proof, when $\rho = 0$.

Lemma 2 (Invariance by Parallel Transport and Track Closedness) *The track of $K^{\eta,\rho}$ is closed. For each t, the set $K^{\eta,\rho}(t)$ is invariant by parallel transport with respect to $D(t)$.*

Proof: The track $\{(\mathcal{O}, t) \in End_{SA}(\wedge^2 TM) \times \mathbb{R} : t \in [0, T),\ \mathcal{O} \in K^{\eta,\rho}(t)\}$ is closed since $K^{\eta,\rho}(t)$ is closed for each t; it depends continuously on t; for $t \in (0, T)$, the smallest eigenvalue function $\gamma(\mathcal{O})$ is continuous; $\mathcal{O}$ is a smooth section of $End_{SA}(\wedge^2 TM)$.

To see that $K^{\eta,\rho}(t)$ is invariant by parallel translations, fix η and ρ so that $1 + \eta\rho > 0$ and $t \in [0, T)$. Consider θ_1 and θ_2 determined in property $(P_2)_t$, which depend only on η, ρ and t. Consider the set

$$\Gamma = \{(x, y, z) \in \mathbb{R}; x \geq y \geq z\},$$

and let $G_1^{\eta,\rho,t}$, $G_2^{\eta,\rho,t}$, $G_3^{\eta,\rho,t}$, $G_4 : \Gamma \to \mathbb{R}$ be functions defined as

$$\begin{aligned}
G_1^{\eta,\rho,t}(x, y, z) &= (1 + 2(1 + \eta\rho)t)z + 1, \\
G_2^{\eta,\rho,t}(x, y, z) &= \left[x + y + z - |z|\Big(\theta_1 \log(|z|) + \theta_2 \log(1 + 2(1 + \eta\rho)t)\Big)x\right] G_1^t(x, y, z), \\
G_3^{\eta,\rho,t}(x, y, z) &= (1 + 2(1 + \eta\rho)t)[x + y + z] + 3, \\
G_4(x, y, z) &= -z.
\end{aligned}$$

Let $\alpha(\mathcal{O}) \geq \beta(\mathcal{O}) \geq \gamma(\mathcal{O})$ be the eigenvalues of $\mathcal{O} \in End_{SA}(\wedge^2 TM)$ and let

$$\mathfrak{L}^{\eta,\rho}(t) = \{\mathcal{O} \in End_{SA}(\wedge^2 TM) : G_1^{\eta,\rho,t}(\alpha(\mathcal{O}), \beta(\mathcal{O}), \gamma(\mathcal{O})) \leq 0\},$$

$$\mathfrak{M}^{\eta,\rho}(t) = \{\mathcal{O} \in End_{SA}(\wedge^2 TM) : G_2^{\eta,\rho,t}(\alpha(\mathcal{O}), \beta(\mathcal{O}), \gamma(\mathcal{O})) \leq 0\},$$

$$\mathfrak{N}^{\eta,\rho}(t) = \{\mathcal{O} \in End_{SA}(\wedge^2 TM) : G_3^{\eta,\rho,t}(\alpha(\mathcal{O}), \beta(\mathcal{O}), \gamma(\mathcal{O})) \leq 0\},$$

$$\mathfrak{O} = \{\mathcal{O} \in End_{SA}(\wedge^2 TM) : G_4(\alpha(\mathcal{O}), \beta(\mathcal{O}), \gamma(\mathcal{O})) \leq 1\},$$

be subsets of $End_{SA}(\wedge^2 TM)$. It follows from Proposition 7 that $\mathfrak{L}^{\eta,\rho}(t)$, $\mathfrak{M}^{\eta,\rho}(t)$, $\mathfrak{N}^{\eta,\rho}(t)$ and $\mathfrak{O}$ are invariant by parallel translations with respect to the connection $D(t)$. On the other hand, one can see that

$$K^{\eta,\rho}(0) = \mathfrak{L}^{\eta,\rho}(0) \cap \mathfrak{M}^{\eta,\rho}(0) \cap \mathfrak{N}^{\eta,\rho}(0) \cap \mathfrak{O}$$

and if $t \in (0, T)$, then

$$K^{\eta,\rho}(t) = \mathfrak{L}^{\eta,\rho}(t) \cap \mathfrak{M}^{\eta,\rho}(t) \cap \mathfrak{N}^{\eta,\rho}(t),$$

which shows that $K^{\eta,\rho}(t)$ is invariant by parallel translations with respect to $D(t)$, for all $t \in [0, T)$. ■

Lemma 3 (Convexity) *The set $K_p^{\eta,\rho}(t)$ is convex.*

Proof: Consider the map $\phi : End_{SA}(\wedge^2 T_p M) \to \mathbb{R}^2$ given by

$$\phi(\mathcal{O}_p) = (|\gamma(\mathcal{O}_p)|, tr(\mathcal{O}_p)) = (x(\mathcal{O}_p), y(\mathcal{O}_p)), \tag{4.1}$$

where $\gamma(\mathcal{O}_p)$ is the smallest eigenvalue of $\mathcal{O}_p$, which is a concave function of $\mathcal{O}_p$. Consider the set

$$A^{\eta,\rho}(t) = \left\{ \ (x, y) \in \mathbb{R}^2 : (x, y) \text{ satisfies } (\tilde{P}_1)_t,\ (\tilde{P}_2)_t \text{ and } (\tilde{P}_3)_t \ \right\}, \tag{4.2}$$

where

$(\tilde{P}_1)_t$ $\quad y \geq -\dfrac{3}{1 + 2(1 + \eta\rho)t}$,

$(\tilde{P}_2)_t$ $\quad y \geq -3x$,

$(\tilde{P}_3)_t$ $\quad$ There are constants $\theta_1 \geq \theta_2 > 0$ such that if $x \geq \dfrac{1}{1 + 2(1 + \eta\rho)t}$, then

$$y \geq x(\theta_1 \log(x) + \theta_2 \log(1 + 2(1 + \eta\rho)t) - 3).$$

Then $A^{\eta,\rho}(t)$ is convex for each fixed η, ρ and t.

With this notation, for all $t \in (0, T)$, we have $\mathcal{O}_p \in K_p^{\eta,\rho}(t)$ if, and only if, $\phi(\mathcal{O}_p) \in A^{\eta,\rho}(t)$. Consider $t \in (0, T)$ and $\mathcal{O}_p(s) = s\mathcal{O}_p + (1 - s)\mathcal{O}'_p$, where $\mathcal{O}_p$ and $\mathcal{O}'_p$ are in $K_p^{\eta,\rho}(t)$ and $s \in [0, 1]$. Since $A^{\eta,\rho}(t)$ is convex, all we need to prove is that

$\phi(\mathcal{O}_p(s)) \in A^{\eta,\rho}(t)$. The first condition is satisfied, once

$$\begin{aligned} y(\mathcal{O}_p(s)) &= sy(\mathcal{O}_p) + (1-s)y(\mathcal{O}_p') \\ &\geq s\Big(-\frac{3}{1+2(1+\eta\rho)t}\Big) + (1-s)\Big(-\frac{3}{1+2(1+\eta\rho)t}\Big) \\ &= -\frac{3}{1+2(1+\eta\rho)t}. \end{aligned}$$

The inequality $y(\mathcal{O}_p(s)) \geq -3x(\mathcal{O}_p(s))$ is trivially satisfied and it implies that

$$\begin{aligned} s\Big(-\tfrac{1}{3}y(\mathcal{O}_p)\Big) + (1-s)\Big(-\tfrac{1}{3}y(\mathcal{O}_p')\Big) &= -\tfrac{1}{3}y(s\mathcal{O}_p + (1-s)\mathcal{O}_p') \\ &\leq x(s\mathcal{O}_p + (1-s)\mathcal{O}_p') \\ &\leq sx(\mathcal{O}_p') + (1-s)x(\mathcal{O}_p), \end{aligned}$$

where in the last inequality we have used that $x(\mathcal{O}_p) = -\gamma(\mathcal{O}_p)$ is a concave function. These inequalities imply that $\phi(s\mathcal{O}_p + (1-s)\mathcal{O}_p')$ is contained in the trapezium $\mathfrak{T}$ of vertices

$$(u(\mathcal{O}_p), v(\mathcal{O}_p)), \quad (u(\mathcal{O}_p'), v(\mathcal{O}_p')), \quad (-1/3v(\mathcal{O}_p), v(\mathcal{O}_p)), \quad (-1/3v(\mathcal{O}_p'), v(\mathcal{O}_p')),$$

where each vertex is in $A^{\eta,\rho}(t)$. Since $\mathfrak{T}$ is convex and is contained in $A^{\eta,\rho}(t)$, which in turn is convex, it follows that $\phi(\mathcal{O}_p(s)) \in A^{\eta,\rho}(t)$, and then $\mathcal{O}_p(s) \in K_p^{\eta,\rho}(t)$, for each $t \in (0,T)$. It shows that $K_p^{\eta,\rho}(t)$ is convex for $t \in (0,T)$.

Let us prove the convexity when $t = 0$. In fact, consider the sets

$$\tilde{K}_p^{\eta,\rho}(0) = \Big\{ \ \mathcal{O}_p \in End_{SA}(\wedge^2 T_pM): \ \mathcal{O}_p \text{ satisfies } (P_1)_0 \text{ and } (P_2)_0, \ \Big\},$$

and

$$\bar{K}_p^{\eta,\rho}(0) = \{\mathcal{O}_p \in End_{SA}(\wedge^2 T_pM) : \gamma(\mathcal{O}_p) \geq -1\}.$$

Note that

$$K_p^{\eta,\rho}(0) = \tilde{K}_p^{\eta,\rho}(0) \cap \bar{K}_p^{\eta,\rho}(0). \tag{4.3}$$

With this notation we have $\mathcal{O}_p \in \tilde{K}_p^{\eta,\rho}(0)$ if, and only if, $\phi(\mathcal{O}_p) \in A^{\eta,\rho}(0)$. By using the same argument as for $t \in (0, T)$, one shows that $\tilde{K}_p^{\eta,\rho}(0)$ is convex. On the other hand, if $\mathcal{O}_p$ $\mathcal{O}'_p \in \bar{K}_p^{\eta,\rho}(0)$, by using the concavity of γ,

$$\begin{aligned}\gamma(\mathcal{O}_p(s)) &\geq s\gamma(\mathcal{O}_p) + (1-s)\gamma(\mathcal{O}'_p)\\ &\geq s(-1) + (1-s)(-1)\\ &= -1.\end{aligned}$$

This shows that $\bar{K}_p^{\eta,\rho}(0)$ is convex. Using (4.3) we see that $K_p^{\eta,\rho}(0)$ is convex, which completes the proof. ■

We need to proof that $K_p^{\eta,\rho}(t)$ is invariant by (1.29). Since the proof is long, it will be presented in two sections, treating the cases $\rho \in (-1/2, 0]$ and $\rho \in [0, 1/4)$ separately, in sections 4.2 and 4.3, respectively. At the end of each section we present the corresponding Hamilton-Ivey estimate.

4.2 Hamilton-Ivey Estimate for $\rho \in (-1/2, 0]$

By considering $\eta = 2$, in this section we prove that the system (1.29) leaves invariant the set $K_p^{2,\rho}(t)$. In the sequence we prove the Hamilton-Ivey estimate for $\rho \in (-1/2, 0]$.

Lemma 4 *The set $K_p^{2,\rho}(t)$ is invariant under system (1.29).*

Proof: Let $Q_p(t) = Q(t) \in End_{SA}(\wedge^2 T_p M)$ be a solution to the ODE (1.28) with $Q(0) \in K_p^{2,\rho}(0)$ and fix $t_0 \in (0, T)$. We would like to show that $Q(t_0) \in K_p^{2,\rho}(t_0)$.

Note that, defining the sets

$$U_p(t) = \left\{ \mathcal{O}_p \in K_p^{2,\rho}(t) : \mathcal{O}_p \text{ satisfies } (P_1)_t \right\},$$

and

$$V_p(t) = \left\{ \mathcal{O}_p \in K_p^{2,\rho}(t) : \mathcal{O}_p \text{ satisfies } (P_2)_t \right\},$$

we obviously have $K_p^{\eta,\rho}(t) = U_p(t) \cap V_p(t)$.

We will first show that $Q(t_0) \in U_p(t_0)$. If $tr(Q(t_0))$ is nonnegative, then

$$tr(Q(t_0)) \geq 0 > -\frac{3}{1+2(1+2\rho)t_0},$$

which gives the result in this case. Now suppose that $tr(Q(t_0)) < 0$. By Proposition 9, we have that $tr(Q)$ satisfies

$$(tr(Q))' \geq \frac{4}{3}(1-3\rho)tr(Q)^2 \geq 0, \tag{4.4}$$

for each $t \in [0, T)$, which gives $tr(Q(0)) \leq tr(Q(t_0)) < 0$. Integrating (4.4), we have

$$\begin{aligned} -\frac{1}{tr(Q(t_0))} + \frac{1}{tr(Q(0))} &= \int_0^{t_0} \frac{[tr(Q(s))]'}{[tr(Q(s))]^2} ds \\ &\geq \frac{4}{3}(1-3\rho)t_0. \end{aligned}$$

Taking into account that $tr(Q(0)) \geq -3$, we have

$$-\frac{1}{tr(Q(t_0))} - \frac{1}{3} \geq \frac{4}{3}(1-3\rho)t_0,$$

which gives

$$tr(Q_p(t_0)) \geq -\frac{3}{1+4(1-3\rho)t_0} > -\frac{3}{1+2(1+2\rho)t_0},$$

where in the last inequality we have used that $\rho \leq 0$. This shows that $Q(t_0) \in U_p(t_0)$.

Now we will prove that $Q(t_0) \in V_p(t_0)$. In order to do so, assume that

$$\nu(t_0) \leq -\frac{1}{1+2(1+2\rho)t_0}, \tag{4.5}$$

and consider

$$\theta_1 = 4\theta_2 = \frac{1}{2(2\rho^2 - 2\rho + 1)}. \tag{4.6}$$

If the equality holds in (4.5), i.e., $-\nu(t_0)(1+2(1+2\rho)t_0) = 1$, then we get:

$$\begin{aligned} tr(Q(t_0)) \geq& 3\nu(t_0) \\ =& -\nu(t_0)(\underbrace{\theta_1 \ln(-\nu(t_0)) + \theta_1 \ln(1+(1+2\rho)t_0)}_{=0} - 3) \end{aligned} \tag{4.7}$$

$$\geq -\nu(t_0)(\theta_1 \ln(-\nu(t_0)) + \theta_2 \ln(1 + (1+2\rho)t_0) - 3),$$

which gives $Q(t_0) \in V_p(t_0)$ in this case.

If, on the other hand, we have the strict inequality in (4.5), i.e.

$$\nu(t_0) < -\frac{1}{1 + 2(1+2\rho)t_0},$$

then consider the smallest number $\tilde{t} \in [0, t_0)$ such that

$$\nu(t) < -\frac{1}{1 + 2(1+2\rho)t}, \ \forall t \in (\tilde{t}, t_0]. \tag{4.8}$$

It follows from this choice that

$$\nu(\tilde{t}) = -\frac{1}{1 + 2(1+2\rho)\tilde{t}}. \tag{4.9}$$

In fact, if $\tilde{t} > 0$, then (4.9) is obvious. If $\tilde{t} = 0$, it follows from the definition of $K(0)$ that $\nu(0) \geq -1$. Since $\nu(0) = \nu(\tilde{t}) \leq -1$, we get (4.9) with $\tilde{t} = 0$.

Consider the function $f : [\tilde{t}, t_0] \to \mathbb{R}$ defined by

$$f(t) = \frac{tr(Q)}{-\nu} - \theta_1 \ln(-\nu) - \theta_2 \ln(1 + 2(1+2\rho)t). \tag{4.10}$$

We claim that

Claim 1 *If $\nu(t) < 0$, for all $t \in [\tilde{t}, t_0]$, then $f(t_0) \geq -3$.*

Assuming Claim 1 we have at t_0 that

$$tr(Q(t_0)) \ \geq \ -\nu(t_0)(\theta_1 \ln(-\nu(t_0)) + \theta_2 \ln(1 + 2(1+2\rho)t_0) - 3),$$

which gives $Q(t_0) \in V_p(t_0)$ in this case.

To conclude the proof of Lemma 4, we still need to prove Claim 1.

Proof of Claim 1: Consider the auxiliary function $\xi : [\tilde{t}, t_0] \to \mathbb{R}$ defined in the following way

$$\xi(t) = \frac{a\lambda + \mu + b\nu}{-\nu} - \ln(-\nu) - \frac{1}{4} \ln(1 + 2(1+2\rho)t), \tag{4.11}$$

where

$$\begin{cases} a = 1 - 2\rho \\ b = 3(a^2+1) - a - 1 = 2(6\rho^2 - 5\rho + 2), \end{cases} \tag{4.12}$$

and $[\tilde{t}, t_0]$ is defined as in (4.8). Taking the derivative of $\xi(t)$ we get

$$\begin{aligned} \xi'(t) =& \left(\frac{a\lambda + \mu + b\nu}{-\nu} - \ln(-\nu) - \frac{1}{4}\ln(1 + 2(1+2\rho)t) \right)' \\ =& \nu^{-2}\left(-\nu(a\lambda' + \mu') + (a\lambda + \mu)\nu' - \nu\nu' - \frac{1+2\rho}{2(1+2(1+2\rho)t)}\nu^2 \right), \end{aligned}$$

and using (4.8) for estimating the last term of $\xi'(t)$ on the second line we have

$$\xi'(t) > \nu^{-2} \underbrace{\left(-\nu(a\lambda' + \mu') + (a\lambda + \mu)\nu' - \nu\nu' + \frac{(1+2\rho)}{2}\nu^3 \right)}_{=I}. \tag{4.13}$$

We will use the hypothesis that λ, μ and ν satisfy ODE (1.28) to get an expression for I with no derivatives of these functions. It follows from (1.29) that one has:

$$\begin{aligned} & -\nu(a\lambda' + \mu') + (a\lambda + \mu)\nu' = \\ =& -2\nu[a\lambda^2 + a\mu\nu - 2a\rho\lambda(\lambda + \mu + \nu) + \mu^2 + \lambda\nu - 2\rho\mu(\lambda + \mu + \nu)] \\ & + 2(a\lambda + \mu)[\nu^2 + \lambda\mu - 2\rho\nu(\lambda + \mu + \nu)] \\ =& 2[-a\lambda^2\nu - a\mu\nu^2 - \mu^2\nu - \lambda\nu^2 + a\lambda\nu^2 + a\lambda^2\mu + \mu\nu^2 + \lambda\mu^2] \\ & + 4\rho[\underbrace{\nu(a\lambda + \mu)(\lambda + \mu + \nu) - \nu(a\lambda + \mu)(\lambda + \mu + \nu)}_{=0}] \\ =& 2a(\mu - \nu)\lambda^2 + 2((a-1)\nu^2 + \mu^2)\lambda + 2((1-a)\mu\nu^2 - \nu\mu^2). \end{aligned} \tag{4.14}$$

On the other hand:

$$\begin{aligned} \nu\nu' =& 2\nu^3 + 2\lambda\mu\nu - 4\rho\nu^2(\lambda + \mu + \nu) \\ =& 2(\mu\nu - 2\rho\nu^2)\lambda + 2((1-2\rho)\nu^3 - 2\rho\mu\nu^2). \end{aligned} \tag{4.15}$$

Using (4.13), (4.14) and (4.15) we can write I as

$$\frac{1}{2}I = a(\mu - \nu)\lambda^2 + (\mu^2 - \mu\nu + (\underbrace{a - 1 + 2\rho}_{=0})\nu^2)\lambda$$

$$\begin{aligned}
& -\mu^2\nu + (1-a+2\rho)\mu\nu^2 + \left(-1+2\rho+\frac{1}{4}+\frac{1}{2}\rho\right)\nu^3 \\
=& a(\mu-\nu)\lambda^2 + (\mu^2-\mu\nu)\lambda - \mu^2\nu + 4\rho\mu\nu^2 + \left(\frac{5}{2}\rho-\frac{3}{4}\right)\nu^3 \qquad (4.16)
\end{aligned}$$

It follows from (4.13) that if we prove that $I > 0$, then $\xi'(t) > 0$, and we finish the proof of Claim 1. To prove that $I > 0$, we will fix any $t_1 \in [T, 0)$ so that $\nu(t_1) < 0$ and consider some possibilities accordingly to the sign of the quantities involving $\lambda(t_1)$, $\mu(t_1)$ and $\nu(t_1)$. The cases and subcases are the following:

(1) $\mu(t_1) = \nu(t_1)$;

(2) $\mu(t_1) > \nu(t_1)$

- (2.1) $\mu(t_1) > 0$
  - (2.1.i) $\mu(t_1) + \nu(t_1) > 0$
  - (2.1.ii) $\mu(t_1) + \nu(t_1) \leq 0$
- (2.2) $\mu(t_1) \leq 0$

Case (1): $\mu(t_1) = \nu(t_1)$.

It follows from (4.16) that

$$\frac{1}{2}I = \left(-1+4\rho+\left(\frac{5}{2}\rho-\frac{3}{4}\right)\right)\nu^3 > 0$$

Case (2.1.i): $\lambda(t_1) > \mu(t_1) > 0 > \nu(t_1)$ and $\mu(t_1) + \nu(t_1) < 0$.

It follows from the hypothesis that $-\nu > \mu$. This together with the expression (4.16) gives:

$$\begin{aligned}
\frac{1}{2}I =& \underbrace{a(\mu-\nu)\lambda^2 + (\mu^2-\mu\nu)\lambda}_{>0} \underbrace{-\mu^2\nu}_{>0} + 4\rho\mu\nu^2 + \left(\frac{5}{2}\rho-\frac{3}{4}\right)\nu^3 \\
>& 4\rho\mu\nu^2 + \left(\frac{3}{4}-\frac{5}{2}\rho\right)\nu^2(\underbrace{-\nu}_{>\mu}) \\
>& \left(4\rho+\frac{3}{4}-\frac{5}{2}\rho\right)\mu\nu^2 \\
=& \frac{3}{4}(2\rho+1)\mu\nu^2 > 0
\end{aligned}$$

Case (2.1.ii): $\lambda(t_1) > \mu(t_1) > 0 > \nu(t_1)$ and $\mu(t_1) + \nu(t_1) \geq 0$.

The hypothesis gives $\lambda > \mu > -\nu$. Therefore,

$$\begin{aligned}\frac{1}{2}I =& a(\mu-\nu)\lambda^2 + \underbrace{(\mu^2-\mu\nu)\lambda - \mu^2\nu}_{>0} + 4\rho\mu\nu^2 + \underbrace{\left(\frac{5}{2}\rho - \frac{3}{4}\right)\nu^3}_{>0} \\ >& a(\underbrace{\mu}_{>-\nu} - \nu)\underbrace{\lambda}_{>-\nu}\underbrace{\lambda}_{>\mu} + 4\rho\mu\nu^2 \\ >& 2a\mu\nu^2 + 4\rho\mu\nu^2 \\ =& (2-4\rho+4\rho)\mu\nu^2 \\ =& 2\mu\nu^2 > 0\end{aligned}$$

Case (2.2): $0 > \mu(t_1) > \nu(t_1)$.

It follows from (4.16) that:

$$\begin{aligned}\frac{1}{2}I =& \underbrace{a}_{>1}(\mu-\nu)\lambda^2 + (\mu^2-\mu\nu)\lambda - \mu^2\nu + \underbrace{4\rho\mu\nu^2 + \left(\frac{5}{2}\rho - \frac{3}{4}\right)\nu^3}_{>0} \\ >& (\mu-\nu)\lambda^2 + (\mu^2-\mu\nu)\lambda - \mu^2\nu \\ =& (\mu-\nu)(\lambda+\mu)\lambda - \mu^2\nu \qquad (4.17) \\ =& \mu\lambda^2 - \nu\lambda^2 + \lambda\mu^2 - \lambda\mu\nu - \mu^2\nu \\ =& \mu(\lambda-\nu)(\lambda+\mu) - \nu\lambda^2. \qquad (4.18)\end{aligned}$$

We have two cases to analyze according to the sign of $\lambda + \mu$ at t_1. If it is positive, then $\lambda(t_1) > -\mu(t_1) > 0$ and it follows from (4.17) that $I > 0$. If on the other hand $\lambda(t_1) + \mu(t_1) \le 0$, then it follows from (4.18) that $I > 0$ as well.

Now recall that at $\tilde{t}$ the function ν satisfies (4.9), and then

$$\begin{aligned}\xi(\tilde{t}) =& \frac{a\lambda(\tilde{t}) + \mu(\tilde{t}) + b\nu(\tilde{t})}{-\nu(\tilde{t})} - \ln(-\nu(\tilde{t})) - \frac{1}{4}\ln(\underbrace{1 + 2(1+2\rho)\tilde{t}}_{-\nu(\tilde{t})^{-1}}) \\ =& \frac{a\lambda(\tilde{t}) + \mu(\tilde{t}) + b\nu(\tilde{t})}{-\nu(\tilde{t})} - \ln(-\nu(\tilde{t})) - \frac{1}{4}\ln(-\nu(\tilde{t})^{-1}) \\ =& \frac{a\lambda(\tilde{t}) + \mu(\tilde{t}) + b\nu(\tilde{t})}{-\nu(\tilde{t})} \underbrace{- \frac{3}{4}\ln(-\nu(\tilde{t}))}_{\ge 0}\end{aligned}$$

$$\geq \frac{(a+1+b)\nu(\tilde{t})}{-\nu(\tilde{t})}$$
$$= -3(a^2+1),$$

where in the last line we have used the expression of b in terms of a (4.12). Since we have proved that $\xi(t)$ has positive slope on $[\tilde{t}, t_0]$, it follows that $\xi(t_0) > -3(a^2+1)$, or equivalently that

$$a\lambda(t_0) + \mu(t_0) + b\nu(t_0) \geq -\nu(t_0)(\ln(-\nu(t_0)) + \frac{1}{4}\ln(1+2(1+2\rho)t_0) - 3(a^2+1)),$$

which dividing by a^2+1 gives at t_0 that ,

$$\frac{a}{a^2+1}\lambda + \frac{1}{a^2+1}\mu + \frac{b}{a^2+1}\nu \geq -\nu(\theta_1 \ln(-\nu(t_0)) + \theta_2 \ln(1+2(1+2\rho)t_0) - 3). \quad (4.19)$$

On the other hand, note that

$$\begin{aligned}
&\frac{a}{a^2+1}\lambda + \frac{1}{a^2+1}\mu + \frac{b}{a^2+1}\nu = \\
&\frac{a}{a^2+1}\lambda + \frac{1}{a^2+1}\mu + \left(3 - \frac{a}{a^2+1} - \frac{1}{a^2+1}\right)\nu = \\
&\frac{a}{a^2+1}(\lambda-\nu) + \frac{1}{a^2+1}(\mu-\nu) + 3\nu < \qquad (4.20)\\
&\lambda - \nu + \mu - \nu + 3\nu = \\
&tr(Q(t_0)).
\end{aligned}$$

Putting (4.19) and (4.20) together we get

$$tr(Q(t_0)) > -\nu(t_0)(\theta_1 \ln(-\nu(t_0)) + \theta_2 \ln(1+2(1+2\rho)t_0) - 3),$$

which finishes the proof since it is equivalent to $f(t_0) > -3$. ■

■

Combining Lemma 2, Lemma 3 and Lemma 4 we have:

Theorem 16 *Let M^3 be a compact three manifold, $\rho \in (-1/2, 0]$, $\theta_1 = 4\theta_2 = 1/2(2\rho^2 - 2\rho + 1)$ and let g_0 be a Riemannian metric on M satisfying the normalized assumption*

$\min_{p\in M} \nu_0(p) \geq -1$, *where* ν_0 *is the smallest sectional curvature of* g_0. *If* $g(t)$, $t \in [0, T)$, *is the solution of the Ricci-Bourguignon Flow corresponding to* ρ *satisfying* $g(0) = g_0$, *then the scalar curvature* $R(t)$ *of* $g(t)$ *satisfies*

$$R \geq -\nu(\theta_1 \log(-\nu) + \theta_2 \log(1 + 2(1 + 2\rho)t) - 3), \tag{4.21}$$

at any point (p, t) *where the smallest sectional curvature* $\nu(p, t)$ *of* $g_p(t)$ *is negative.*

Proof: Let $Rm(t)$, $t \in [0, T)$, be the curvature operator of a Ricci-Bourguignon Flow $g(t)$, $t \in [0, T)$. Fix t_0 so that $\nu(t_0) < 0$.

First we will assume that

$$\nu(t_0) > -\frac{1}{1 + 2(1 + 2\rho)t_0}. \tag{4.22}$$

In this case one has

$$\begin{aligned} \theta_1 \ln(-\nu(t_0)) + \theta_2 \ln(1 + (1 + 2\rho)t_0) &\leq \theta_1 \ln(-\nu(t_0)) + \theta_1 \ln(1 + (1 + 2\rho)t_0) \\ &= \theta_1 \ln(-\nu(t_0)(1 + (1 + 2\rho)t_0)) \\ &\leq \theta_1 \ln \left(\frac{1}{1 + (1 + 2\rho)t_0} (1 + (1 + 2\rho)t_0) \right) \\ &= 0, \end{aligned}$$

where from the second to the third line we used (4.22). Therefore,

$$\begin{aligned} R(t_0) &\geq 3\nu(t_0) \\ &= -\nu(t_0)(-3) \\ &\geq -\nu(t_0)(\theta_1 \ln(-\nu(t_0)) + \theta_2 \ln(1 + (1 + 2\rho)t_0) - 3), \end{aligned}$$

which gives the estimate in this case.

Now assume that

$$\nu(t_0) \leq -\frac{1}{1 + 2(1 + 2\rho)t_0}. \tag{4.23}$$

Let $\varphi(t) : (TM, g_0) \to (TM, g(t))$ be the family of isometries which satisfy (1.16). Let $P(t) = \varphi(t)^* Rm(t)$. It follows from Proposition 5 that $P(t)$ is a solution of (1.17). Since $R(t) \geq 3\nu(t)$, for all t, it follows from the normalizing assumption and from the fact that $P(t)$ and $Rm(t)$ have the same eigenvalues (Proposition 4) that $P(0) \in K_p^{2,\rho}(0)$.

We have seen that the set $K_p^{2,\rho}(t)$ is invariant under parallel translations, it has a closed track (Lemma 2), it is convex (Lemma 3) and it is invariant under the system (1.29) (Lemma 4). On the other hand, the fiber preserving map $F(Q, t) = 2Q^2 + 2Q^{\#} - 4\rho\, tr_{g_0}(Q)Q$ is continuous and locally Lipschitz (See [14] for the details). The Tensorial Maximum Principle (Theorem 4) now assures that $P(t) \in K_p^{2,\rho}(t)$, for all $t \in [0, T)$. Since $P(t_0)$ and $Rm(t_0)$ have the same eigenvalues, we have (4.21) at t_0, since (4.23) is satisfied. This concludes the proof of the theorem. ■

4.3 Hamilton-Ivey Estimate for $\rho \in [0, 1/4)$

In this section we will consider $\eta = -4$, and prove that the system (1.29) leaves invariant the set $K_p^{-4,\rho}(t)$. In the sequence we prove the Hamilton-Ivey estimate for $\rho \in [0, 1/4)$. Similar to Lemma 4 we have the following lemma.

Lemma 5 *The set $K_p^{-4,\rho}(t)$ is invariant under system (1.29).*

Proof: Let $Q_p(t) = Q(t) \in End_{SA}(\wedge^2 T_p M)$ be a self-adjoint endomorphism so that its eigenvalues satisfy system (1.29). Assume $Q(0) \in K_p^{-4,\rho}(0)$ and fix $t_0 \in (0, T)$. We want to show that $Q(t_0) \in K_p^{-4,\rho}(t_0)$.

Consider the sets

$$A_p(t) = \left\{ \mathcal{O}_p \in K_p^{-4,\rho}(t) : \mathcal{O}_p \text{ satisfies } (P_1)_t \right\},$$

and

$$B_p(t) = \left\{ \mathcal{O}_p \in K_p^{-4,\rho}(t) : \mathcal{O}_p \text{ satisfies } (P_2)_t \right\},$$

and note that $K_p^{-4,\rho}(t) = A_p(t) \cap B_p(t)$. Proceeding as in the first part of Lemma 4, we conclude that

$$tr(Q_p(t_0)) \geq -\frac{3}{1 + 4(1 - 3\rho)t_0} \geq -\frac{3}{1 + 2(1 - 4\rho)t_0},$$

which implies that $Q(t_0) \in A_p(t_0)$.

Now let us show that $Q(t_0) \in B_p(t_0)$. To do so, let us assume that

$$\nu(t_0) \leq -\frac{1}{1 + 2(1 - 4\rho)t_0}, \tag{4.24}$$

and take $\theta_1 = \theta_2 = 1$. If the equality holds in (4.24) we proceed as in Lemma 4 to conclude that $Q(t_0) \in B(t_0)$. Now assume that

$$\nu(t_0) < -\frac{1}{1 + 2(1 - 4\rho)t_0}, \tag{4.25}$$

and let $\tilde{t} \in [0, t_0)$ be the smallest number such that

$$\nu(t) < -\frac{1}{1 + 2(1 - 4\rho)t}, \quad \forall t \in (\tilde{t}, t_0).$$

Arguing as in Lemma 4, we have

$$\nu(\tilde{t}) = -\frac{1}{1 + 2(1 - 4\rho)\tilde{t}}.$$

Consider the function $f : [\tilde{t}, t_0] \to \mathbb{R}$ given by

$$f(t) = \frac{tr(Q)}{-\nu} - \ln(-\nu) - \ln(1 + 2(1 - 4\rho)t).$$

We claim the following:

Claim 2 *For all $t \in [\tilde{t}, t_0]$, $f'(t) \geq 0$ and $f(\tilde{t}) \geq -3$.*

Let us assume that the claim is true. It is an immediate consequence of the claim that $f(t_0) \geq -3$, which is equivalent to

$$tr(Q_p(t_0)) \geq -\nu(t_0)(\log(-\nu(t_0)) + \log(1 + 2(1 - 4\rho)t_0) - 3), \tag{4.26}$$

that concludes the proof of Lemma 5'.

Below we prove Claim 2.

Proof of Claim 2: In order, to prove the claim, we first note that

$$\left(\frac{\lambda + \mu + \nu}{-\nu}\right)' - (\ln(-\nu))' \geq -2(1 - 4\rho)\nu, \tag{4.27}$$

provided $\nu < 0$. First of all, note that (1.29) implies

$$\begin{cases} \nu^2 \left(\dfrac{\lambda + \mu + \nu}{-\nu}\right)' = -2\nu(\lambda^2 + \mu^2) + 2\mu\lambda(\mu + \lambda) \\ \nu^2(\ln(-\nu))' = 2\nu^3 + 2\nu\mu\lambda - 4\rho\nu^2(\lambda + \mu + \nu), \end{cases} \tag{4.28}$$

and then

$$\nu^2 \left(\frac{\lambda+\mu+\nu}{-\nu}\right)' - \nu^2(\ln(-\nu))' + 2(1-4\rho)\nu^3 =$$

$$-2\nu^3 - 2\nu(\lambda^2+\mu^2) + 2\mu\lambda(\mu+\lambda) - 2\nu\mu\lambda + 4\rho\nu^2(\lambda+\mu+\nu) + 2(1-4\rho)\nu^3 =$$

$$\underbrace{-2\nu(\lambda^2+\mu^2) + 2\mu\lambda(\mu+\lambda) - 2\nu\mu\lambda + 4\rho\nu^2(\lambda+\mu) - 4\rho\nu^3}_{I}.$$

Therefore, to prove (4.27), all we need is to prove that $I \geq 0$. To show this we are going to divide it into 2 cases according to the sign of λ, and then subdivide each case into two other subcases.

Case 1. Assume that $\lambda \geq 0$. We will divide this case into two further subcases, namely when $\mu \leq 0$ or $\mu > 0$.

Subcase 1.1. $\lambda \geq 0$ and $\mu \leq 0$. In this case

$$\begin{aligned} I &= 2(\mu-\nu)(\lambda^2+\lambda\mu+\mu^2) - 2\mu(\lambda^2+\lambda\mu+\mu^2) + 2\mu\lambda(\mu+\lambda) + 4\rho\nu^2\lambda + 4\rho\nu^2(\mu-\nu) \\ &= 2(\mu-\nu)(\lambda^2+\lambda\mu+\mu^2) - 2\mu^3 + 4\rho\nu^2\lambda + 4\rho\nu^2(\mu-\nu) \\ &\geq 0. \end{aligned}$$

Subcase 1.2. $\lambda \geq 0$ and $\mu > 0$. In this case

$$\begin{aligned} I &= -2\nu(\lambda^2+\mu^2) + 2\mu\lambda(\mu+\lambda) - 2\nu\mu\lambda + 4\rho\nu^2\lambda \\ &\geq 0. \end{aligned}$$

Case 2. Assume that $\lambda < 0$. We will divide this case into two further subcases, namely when $2\mu \leq \nu$ or $2\mu > \nu$.

Subcase 2.1. $\lambda < 0$ and $2\mu \leq \nu$. In this case we have

$$\begin{aligned} I &= -2\nu(\lambda^2+\mu^2) + 2\mu\lambda(\mu+\lambda) - 2\nu\mu\lambda + 8\rho\nu\mu\lambda - 8\rho\nu\mu\lambda + 4\rho\nu^2\lambda \\ &= 2\lambda^2(\mu-\nu) + 2\mu^2(\lambda-\nu) + 2(4\rho-1)\nu\mu\lambda + 4\rho\nu\lambda(\nu-2\mu) \\ &\geq 0. \end{aligned}$$

Subcase 2.2. $\lambda < 0$ and $2\mu > \nu$. In this case we have

$$\begin{aligned} I &= -2\nu(\lambda^2+\mu^2) + 2\mu\lambda(\mu+\lambda) - 2\nu\mu\lambda + 4\rho\nu^2(\lambda+\mu) - 4\rho\nu^3 \\ &= 2\lambda^2(\mu-\nu) + 2\mu^2(\lambda-\nu) - 2\nu\mu\lambda + 4\rho\nu^2(\lambda+\mu-\nu) \\ &\geq 2\lambda^2(\mu-\nu) + 2\mu^2(\lambda-\nu) - 2\nu\mu\lambda + 4\rho\nu^2(2\mu-\nu) \\ &\geq 0. \end{aligned}$$

To finish the proof of the claim, since $\nu(t) \leq -1/(1+2(1-4\rho)t)$, (4.27) gives

$$\begin{aligned} f'(t) &= \left(\frac{tr(Q(t))}{-\nu} - \ln(-\nu) - \ln(1+2(1-4\rho)t)\right)' \\ &= \left(\frac{tr(Q(t)}{-\nu}\right)' - (\ln(-\nu))' - \frac{2(1-4\rho)}{1+2(1-4\rho)t} \\ &\geq 0, \end{aligned} \tag{4.29}$$

$\forall t \in [\tilde{t}, t_0]$. On the other hand, at $\tilde{t}$ we have $f(\tilde{t}) \geq -3$. This implies that $f(t_0) \geq -3$, that concludes the proof. ■

■

In the next result we will show that the Hamilton-Ivey inequality holds when $\rho \in [0, 1/4)$.

Theorem 17 *Let $(M^3, g(t))$ be a solution of the Ricci-Bourguignon flow on a compact three manifold such that the initial metric satisfies the normalized assumption* $\min_{p\in M} \nu_p(0) \geq -1$. *If $\rho \in [0, 1/4)$, then the scalar curvature satisfies*

$$R \geq |\nu|(\log|\nu| + \log(1+2(1-4\rho)t) - 3), \tag{4.30}$$

at any point (p, t) where $\nu_p(t) < 0$,

Proof: Let $Rm(t)$, $t \in [0, T)$, be the curvature operator of a Ricci-Bourguignon Flow $g(t)$, $t \in [0, T)$. Fix t_0 so that $\nu(t_0) < 0$.

First we will assume that

$$\nu(t_0) > -\frac{1}{1+2(1-4\rho)t_0}. \tag{4.31}$$

In this case one has

$$\begin{aligned} \ln(-\nu(t_0)) + \ln(1+(1-4\rho)t_0) &= \ln(-\nu(t_0)(1+(1-4\rho)t_0)) \\ &< \ln\left(\frac{1}{1+(1-4\rho)t_0}(1+(1-4\rho)t_0)\right) \\ &= 0, \end{aligned}$$

where from the second to the third line we used (4.31). Therefore,

$$\begin{aligned} R(t_0) &\geq 3\nu(t_0) \\ &= -\nu(t_0)(-3) \\ &\geq -\nu(t_0)(\ln(-\nu(t_0)) + \ln(1+(1-4\rho)t_0) - 3), \end{aligned}$$

which gives the estimate in this case.

Now assume that

$$\nu(t_0) \leq -\frac{1}{1+2(1-4\rho)t_0}. \tag{4.32}$$

Let $\varphi(t) : (TM, g_0) \to (TM, g(t))$ be the family of isometries which satisfy (1.16). Let $P(t) = \varphi(t)^* Rm(t)$. It follows from Proposition 5 that $P(t)$ is a solution of (1.17). Since $R(t) \geq 3\nu(t)$, for all t, it follows from the normalizing assumption and from the fact that $P(t)$ and $Rm(t)$ have the same eigenvalues (Proposition 4) that $P(0) \in K_p^{-4,\rho}(0)$. We have seen that the set $K_p^{-4,\rho}(t)$ is invariant under parallel translations, it has a closed track (Lemma 2), it is convex (Lemma 3) and it is invariant under the system (1.29) (Lemma 5). On the other hand, the fiber preserving map $F(Q,t) = 2Q^2 + 2Q^\# - 4\rho\, tr_{g_0}(Q)Q$ is continuous and locally Lipschitz (See [14] for the details). The Tensorial Maximum Principle (Theorem 4) now assures that $P(t) \in K_p^{-4,\rho}(t)$, for all $t \in [0,T)$. Since $P(t_0)$ and $Rm(t_0)$ have the same eigenvalues, we have (4.30) at t_0, since (4.32) is satisfied. This concludes the proof of the theorem. ■

4.4 Ancient Solutions have Nonnegative Curvature

Theorem 16 and Theorem 17 can be interpreted as the following: for solutions of the Ricci-Bourguignon flow, with $\rho \in (-1/2, 1/4)$, negative sectional curvature occurs only in the presence of larger positive sectional curvature [20]. To see this, assume that

$g(t)$, $t \in [0,T)$, is a solution of the Ricci-Bourguignon Flow, with $\rho \in (-1/2, 0]$ and $\min_{p\in M} \nu_p(0) \geq -1$. Let $(p_0, t_0) \in M^3 \times [0,T)$ and assume that there is a constant $C > 0$ such that $\nu(p_0, t_0) < -e^{2(C+3)(2\rho^2-2\rho+1)}$. Using Theorem 16 one can see that

$$\begin{aligned}
\lambda(t_0) &\geq \frac{1}{3}R(t_0) \\
&\geq -\frac{1}{3}\nu(t_0)(\theta_1 \log(-\nu(t_0)) + \theta_2 \underbrace{\log(1+2(1+2\rho)t_0)}_{\geq 0} - 3) \\
&\geq \frac{1}{3}e^{2(C+3)(2\rho^2-2\rho+1)}(C+3-3) \\
&= \frac{C}{3}e^{2(C+3)(2\rho^2-2\rho+1)},
\end{aligned}$$

which implies that $R(p_0, t_0) \geq Ce^{2(C+3)(2\rho^2-2\rho+1)} > 0$, showing that positive sectional curvature wins in the average. Below we state one of the main consequences of the Hamilton-Ivey estimate, which asserts that for certain solutions, not only the average of curvatures, but the sectional curvatures themselves are nonnegative.

Theorem 18 *Let $(M^3, g(t))$ be a compact ancient solution of the Ricci-Bourguignon flow with uniformly bounded scalar curvature. Then $g(t)$ has nonnegative sectional curvature, for as long as it exists.*

Proof: Let $(M^3, g(t))$ be an ancient solution of the Ricci-Bourguignon flow, with $\rho \in (-\infty, 0]$, that has uniformly bounded scalar curvature. Consider $\eta > 0$ so that $\rho \in (-1/\eta, 0]$. Assume that for a certain $t_0 \in (-\infty, T)$, $\nu_0 = \inf_p \nu_p(t_0) < 0$ and fix $\alpha > 0$ so that $-\alpha < t_0$. Consider the solution

$$\tilde{g}(t) = |\nu_0| g\left(\frac{t-\alpha}{|\nu_0|}\right),$$

where $|\nu_0|t + \alpha \in (-\infty, T)$. Since $\tilde{\nu}(|\nu_0|t_0 + \alpha) > -1$, it follows from Theorem 16 that

$$\begin{aligned}
\tilde{R}(|\nu_0|t+\alpha) \quad \geq \quad & -\tilde{\nu}(|\nu_0|t+\alpha)\Big(\theta_1 \log(-\tilde{\nu}(|\nu_0|t+\alpha)) \\
& +\theta_2 \log(1+2(1+2\rho)(|\nu_0|t+\alpha)) - 3\Big),
\end{aligned} \tag{4.33}$$

wherever $\tilde{\nu}(|\nu_0|t + \alpha) < 0$. Rewriting (4.33) in terms of $g(t)$, we have

$$\begin{aligned} R(t) &\geq -\nu(t)(\theta_1 \log(-|\nu_0|^{-1}\nu(t)) + \theta_2 \log(1 + 2(1+2\rho)(|\nu_0|t + \alpha)) - 3) \\ &\geq -\nu(t)(\theta_1 \log(-\nu(t)) + \theta_2 \log(|\nu_0|^{-2} + 2(1+2\rho)(|\nu_0|^{-1}t + |\nu_0|^{-2}\alpha)) - 3). \end{aligned}$$

Since

$$\lim_{\alpha\to\infty} \log(|\nu_0|^{-2} + 2(1+2\rho)(|\nu_0|^{-1}t + |\nu_0|^{-2}\alpha)) = \infty,$$

we get a contradiction. It implies that $\nu(p,t) \geq 0$, $\forall (p,t) \in M^3 \times (-\infty, T]$ and the result follows. ■

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