Universidade de Brasília
Faculdade de Economia, Administração e Contabilidade
Departamento de Economia

Matheus Schmeling Costa

ESSAYS ON DECISION THEORY

Brasília
2022

Matheus Schmeling Costa

ESSAYS ON DECISION THEORY

Tese de Doutorado apresentada como parte dos requisitos necessários à obtenção do Título de Doutor em Economia pelo Programa de Pós-Graduação em Economia da Universidade de Brasília.

Orientador: Prof. Leandro Golçalves do Nascimento, PhD

Brasília

2022

Matheus Schmeling Costa

ESSAYS ON DECISION THEORY

Trabalho aprovado. Brasília, 04 de Março de 2022:

Prof. Leandro Gonçalves do Nascimento, PhD
Universidade de Brasília
(Orientador)

Prof. José Guilherme de Lara Resende, PhD
Universidade de Brasília

Prof. Gil Riella, PhD
EPPG - FGV

Prof. José Heleno Faro, PhD
Insper

Brasília
Março de 2022

FICHA CATALOGRÁFICA

C837

Costa, Matheus Schmeling

Essays on Decision Theory / Matheus Schmeling Costa; orientador: Leandro Golçalves do Nascimento – Brasília, 2022. XXX p.41

Tese (Doutorado - Doutorado em Economia) -— Universidade de Brasília/ Faculdade de Administração, Contabilidade e Economia / Programa de Pós-Graduação em Economia, 2022.

1. Choice Theory. 2. Microeconomic Theory. 3. Social Choice. 4. Preferences over Menus. 5. Random Choice. I. Nascimento, Leandro Gonçalves do, orient. II. Título.

Acknowledgements

First and foremost I thank my parents Griseldes Costa and Marco Antônio Costa for the lifelong support and stimulus so that I could focus on delving into my interests and achieving the best possible education even when this would not be translated immediately into better financial prospects.

I am profoundly grateful to my advisor, Leandro Gonçalves do Nascimento, and to professor Gil Riella, for the inspiration to pursue both this field of research and the academic role in general. The many talks and meetings along these years were not only intellectually stimulating and essential to this work, but also quite enjoyable.

I also thank my former partner, Karla Queiroz, who was by my side during most of the path that brought me to concluding this thesis, and my friends and colleagues who, I am grateful to notice, are too many to name individually, for making this last few years far more pleasurable than they would otherwise have been.

Finally I am grateful to acknowledge the financial support of the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES).

RESUMO

Este trabalho é composto de dois capítulos, independentes entre si, que tem como objetivo aprofundar a literatura econômica que versa sobre escolhas individuais e coletivas. O primeiro capítulo versa sobre o processo de racionalização de indivíduos que apresentam um comportamento de escolhas não transitivo. Tomando como dada uma relação de preferências completa, porém não necessariamente transitiva, é proposta uma família de representações de escolha inspirada no procedimento *king-chicken*, de acordo com o qual uma alternativa x é escolhida do conjunto de alternativas A se, para cada outra alternativa y em A, ou x é preferido a y ou existe uma outra alternativa z em A tal que x é preferido a z e z é preferido a y. Mostra-se que é possível generalizar este processo para permitir um caminho com mais de uma alternativa entre x e y e caracteriza-se todas as correspondências de escolhas que emergem deste processo. Duas das mais proeminentes soluções de torneios, o *uncovered set* e o *top-cycle*, são casos especiais deste procedimento de *king-chicken* generalizado. Este trabalho, portanto, avança resultados anteriores da literatura de teoria da escolha ao apresentar a axiomatização destes modelos em espaços de escolhas genéricos, não necessariamente finitos. O segundo capítulo explora o processo de atualização bayesiana de uma *Random Choice Rule* com representação por *Finite Random Expected Utility*. O capítulo apresenta uma condição necessária e suficiente, chamada de *Random Consistency*, para que uma *Random Choice Rule* seja a atualização bayesiana de outra após o agente aprender novas informações e contrair ou expandir seu espaço de estados subjetivo. É apresentada uma extensão a trabalhos já publicados através da caracterização da direção oposta da representação por *unforeseen contingencies*, quando o espaço de estados subjetivos de uma representação por *Finite Random Expected Utility* está contido no espaço de estados subjetivo da representação de uma preferência sobre menus. O capítulo ainda apresenta uma discussão sobre as condições sob as quais uma coleção de *Random Choice Rules* representa uma partição de uma *Random Choice Rule* mais abrangente ou de uma preferência sobre menus.

Palavras-chave: Teoria da Decisão, microeconomia teórica, escolha social, preferências sobre menus, escolhas aleatórias.

Brasília

Março de 2022

ABSTRACT

This work is composed of two independent chapters that focus on deepening the economic literature on individual and collective choice. The first chapter explores the process of rationalization for agents that reveal a nontransitive behavior. Given a complete, though not necessarily transitive, preference relation, it is proposed a family of choice representations inspired by the king-chicken procedure, according to which an alternative x is chosen among a set of alternatives A if, for every other alternative y in A, either x is preferred to y or there is another alternative z in A such that x is preferred to z, and z is preferred to y. It is shown that it is possible to generalize this process by allowing the path from x to y to include more than one alternative z and to fully characterize the choice correspondences that can be achieved through it. Two of the most relevant tournament solutions, the uncovered set and the top-cycle, are special cases of this generalized king-chicken choice procedure, so this work improves previous results that have appeared in the choice theory literature by delivering axiomatizations for those models in generic (not necessarily finite) choice spaces. The second chapter explores the process of bayesian updating of a Random Choice Rule with a Finite Random Expected Utility representation. This chapter presents the necessary and sufficient condition, which we call Random Consistency, for a Random Choice Rule to be a update of another after the Decision Maker learns new information and contracts or expands her subjective state spaces. It is also presented an extension to previous works by characterizing the opposite direction of the unforeseen contingencies representation, when the subjective states of the Finite Random Expected Utility representation of a Random Choice Rule is contained in the subjective state space of the representation of a Preference Over Menus. This chapter also presents a discussion on the conditions under which a collection of Random Choice Rules represent a partition of a broader Random Choice Rule or of a Preference Over Menus.

Keywords: Choice Theory, microeconomic theory, social choice, preferences over menus, random choice.

Brasília

Março de 2022

Contents

Chapter 1

King-Chicken Choice Correspondences

1.1 Introduction

Tournaments, or complete and not necessarily transitive binary relations, are a matter of great relevance in the context of the social choice literature. That is because, as discussed in Miller [22], far above the implications for sports competition the name might suggest, tournaments represent the outcome of majority voting processes even when all individual voters have complete and transitive preferences over the set of alternatives under consideration. Since the lack of transitivity may imply the existence of preference cycles, a problem known as the Condorcet Paradox referring to the early work of Condorcet [6], there may not exist a maximal element in the consideration set, implying that the chosen outcome may depend strongly on the voting process applied.

The study of nontransitive rationalization is also relevant in the context of individual choice. Both in cases where the individual has multiple selves, understood as multiple preferences over the same set of alternatives, or when the alternatives themselves vary along multiple attributes, the individual choice procedure may give raise to the same voting paradox mentioned above.

Several tournament solution concepts have been proposed seeking to narrow down which alternatives in the feasible set could, or should, be deemed as possible winners. Among them, two are of special relevance, both in general and to the scope of this paper: the uncovered set and the top-cycle choice rules.

Given a set of available alternatives A, the covering relation, which has multiple closely related definitions, usually states that an alternative x covers another alternative y if, and only if, for every z such that $y \succsim z$, meaning that y is considered at least as good as z, we also have $x \succsim z$. The set of maximal elements of this relation is known as the uncovered set. When the preference relation is antisymmetrical, the uncovered set may also be understood as the set of alternatives that beat any other alternative in at most two steps, meaning that, if x is in the uncovered set of A, then, for every other $z \in A$, either $x \succsim z$ or there is some $y \in A$ such that $x \succsim y$ and $y \succsim z$.

The top-cycle of a set A, usually defined as the minimal set of A for which every alternative in it beats every alternative outside of it, always includes the uncovered set and may be quite bigger. In a similar way, when A is finite, the top-cycle may be defined as those alternatives that beat any other alternative in a finite number of steps, meaning that alternative x is in the top-cycle of A if, and only if, for every $z \in A$ there is a chain of alternatives $y_0 \succsim y_1 \succsim \cdots \succsim y_n$, such that $x = y_0$ and $z = y_n$.

In this paper, we investigate the *generalized king-chicken procedure*, also known as *k-kings* solution, and fully characterize the choice correspondences that can be represented by it. Under the generalized king-chicken procedure, the winning alternatives, for a given k, are those that beat any other available alternative in at most k steps. In this sense, we show that this procedure encompasses both the uncovered set and the top-cycle tournament solutions when $k = 2$ and $k = \infty$, respectively.

Our characterization is based on two known and fixed axioms, Sen's Gamma, proposed in Sen [33], and Tournament Consistency, discussed in Smith [34] and Fishburn [12], and two axioms depending on k, (k+1)-Bounded Beta Plus, an adaptation of the Beta Plus axiom from Bordes [3], and (k+1)-Bounded Weakened Chernoff, which is a version of a postulate that has appeared in Lombardi [16]. Considering previous characterizations of choice correspondences that represent the uncovered set and the top-cycle rules proposed in the literature, we believe that the one presented here has some advantages. Firstly, because it builds a stepping bridge between these two well known solution concepts, and secondly because, as far as we know, it is the first general characterization for these concepts that accommodate the possibility of infinite choice problems.

We dedicate the remainder of this section to presenting some of the related literature and a discussion on the relevance of the generalized king-chicken procedure. In the second section we present the setup that will be used for our results, while in the third we state the main theorem and the axioms supporting the general king-chicken representation. In Section 1.4, we explore the limit cases of $k = 1$ and k unbounded and in Section 5 we relate our results to some other solution concepts that have previously appeared in the literature. We conclude in Section 1.6. To improve readability we leave the proof of the theorems to section 1.7.

1.1.1 Related Literature

The literature on tournaments begins with the very early work of Condorcet [6], which points out the Condorcet Paradox, the possibility of cycles in the outcomes of a majoritarian decision process even when all voters have transitive and complete preferences. McGarvey [20] extends this point showing that for every complete binary relation there is a set of voters, with complete and transitive preferences, for which the outcome of a pairwise majority voting process is this given relation. Therefore, the study of the outcomes of voting procedures is deeply connected to the study of tournament solutions.

The top-cycle was originally proposed in Good [13], there called Condorcet Set, which would be the minimal set of alternatives that dominates all the available alternatives outside of it. Schwartz [31] is also concerned with the problem of nontransitive choice behaviors, being them a consequence of pairwise majority voting or of individual choices, as found in Tversky [35]. Though he does not assume the completeness of preferences, the solution there proposed is equivalent to the top-cycle in the presence of completeness.

Smith [34] studies the ranking of alternatives in a voting procedure and its stability to variations in the body of voters. He proposes a point system procedure, in which the candidates are ranked according to their positions in voters' preferences. More relevant to the work developed here, Smith proposes the Condorcet Criterion, according to which if every alternative in a set A is majority preferred to each alternative in a set B, disjoint of A, then every alternative in A must be ranked above each alternative in B when $A \cup B$ is the set of feasible candidates. This property is a ranking version of the axiom of Tournament Consistency used in our theorems and is closely related to the definition of the top-cycle.

Bordes [3] presents the first proper characterization of the top-cycle choice rule using the axioms Beta Plus and Minimality. Beta Plus is a strengthening of the well known axiom Beta from Sen [33]. The (k+1)-Bounded Beta Plus axiom we present in this paper is a restriction of Beta Plus to sets of k+1 or less elements. Minimality assures that only the alternatives in the most preferred cycle are included in the choice from the tournament. Since this condition is strongly related to the definition of the top-cycle itself, the characterization we present here may be preferred to the original one from Bordes [3] to some readers.

The same might be said in favor of the more recent characterization of the top-cycle proposed in Ehlers and Sprumont [9], where they use a weakened version of the classical

WARP axiom introduced in Samuelson [30] to rule out context-dependence in choices. A choice behavior is context-independent whenever the choice of x in detriment of y from a set A containing both alternatives precludes the choice of y in detriment of x from any other set B containing both alternatives. As we discuss in Section 1.5, combining Weakened WARP with the axioms of Binary Dominance Consistency and Weak Contraction Consistency, Ehlers and Sprumont [9] arrive at a characterization of the top-cycle tournament solution.

While the top-cycle tournament solution is always context-independent, the same is not true for the uncovered set. The idea of uncovered set was first reached in Miller [22]. Studying the process of pairwise majority voting with an uneven number of sophisticated voters with transitive preferences, which leads to a strong (asymmetrical) tournament, they show that the chosen alternatives must always be uncovered, in the sense that if y is the winning alternative, there must not exist another alternative x such that $y \succ z$ implies $x \succ z$, for any z in the considered set. The sophistication of voters means that voters are aware of each other's preferences and can anticipate the result of the decision process given the order of voting and act accordingly to adjust their votes in each step to achieve their preferred feasible alternative. This discussion, including the formal definition of the uncovered set, was further developed in Miller [23], where it is shown that a number of other voting processes also arrive in decisions contained in the uncovered set.

McKelvey [21] works with a similar question to that of Miller [23], seeking to narrow down the achievable outcomes of a social choice procedure. The fundamental difference is that McKelvey studies an universe of infinite multi-dimensional alternatives. He shows that the possible emerging choices in three different contexts of voting are contained in the uncovered set, including an amendment procedure similar to that of Miller [23], but with endogenous agenda formation. Although McKelvey also works with infinite choice problems, his setup is fairly different from ours, as he imposes conditions of continuity and convexity over his multidimensional alternative space.

Moulin [24] proposes an early characterization for the uncovered set solution when the tournament relation is given. This characterization is based on the Gamma axiom, also used in our results, and the well known social choice axioms of Neutrality, which implies the nondiscrimination among outcomes, and Arrow's Independence of Irrelevant Alternatives. His characterization differs from ours as the tournament relation is given, while our results focus on the choice correspondences that may emerge from an underlying, not previously known, tournament. Moulin [24] also shows that the set of Copeland winners, the alternatives that beat the largest number of other alternatives, is contained in the uncovered set.

A full characterization of the uncovered set for finite choice problems is provided in Lombardi [16]. As in this paper, his axiomatization, discussed in Section 1.5, is based only on observed choices across feasible sets, meaning that it does not require a predefined tournament relation.

Though the idea remains essentially unchanged, the literature on the uncovered set has posed a few different definitions for the covering relation that converge or diverge depending on the asymmetry and completeness of the base relation considered. Penn [27] and Duggan [8] bring forth a discussion on these definitions. Duggan [8], particularly, provides exhaustive references to previous works and considers almost every imaginable extension of the uncovered set, including two boundary concepts, the deep and the shallow uncovered sets, that encompass the other uncovered set definitions in between them.

A recent development in nontransitive choice behavior was proposed in Nishimura [25], where the violations of transitivity are understood as possible choice mistakes by the decision maker. He discusses the *transitive core* subrelation, where $x$trancore$(\succsim)y$ if, and only if, $y \succsim z$ implies $x \succsim z$ and $z \succsim x$ implies $z \succsim y$ for every $z \in X$, as a way of extracting the agent's true preferences from observed choice. The transitive core is connected to the deep covering relation from Duggan [8] as deep covering implies the transitive core when the base relation is complete and the transitive core implies deep covering when the base relation is anti-symmetric. In Section 1.5, we discuss the connection to our own results and argue that the transitive core is equivalent to the covering relation in the presence of completeness and anti-symmetry of the base relation.

The idea of a *k*-step procedure to choose the winner of a tournament, in which the winner alternatives, or *kings*, are those that beat any other alternative in at most *k* steps, appears in Maurer [19]. Maurer discusses the emergence of dominant chicken from a tournament over an hypothetical flock of hens and calls the dominant chicken in a *k*-step procedure as *k-kings*. In reference to Maurer's work, we name this k-step approach, that bridges the uncovered set to the top-cycle, as the *generalized king-chicken procedure*.

Saile and Suksompong [29] studies the decisiveness of the uncovered set and the top-cycle solutions in a setup of large random tournaments, tournaments in which the binary relation between each pair of alternatives has a probability of being reversed, and prove that, for large enough tournaments and probabilities of reversion, even the uncovered set is likely to include every alternative under consideration. Manurangsi and Suksompong [18] extends this result to the *k*-kings approach and shows that in this random setting the 3-kings solutions is already far more indecisive than the uncovered set (2-kings) and resembles more the top-cycle solution.

Another consideration on the *k*-kings approach as a social choice decision process, and on any other tournament solution more encompassing than the uncovered set, is presented in Brandt et al. [4], where the authors show that any alternative not included in the uncovered set may be Pareto dominated in the voters' underlying preference profile. This means that for any $k > 2$, the generalized king-chicken procedure may arrive at inefficient outcomes. Still, more indecisive approaches, as the top-cycle, remain of relevance in the literature on collective and individual choice, as in the recent development of the concept of *preference structures* presented in Nishimura and Ok [26] and Evren et al. [11]. Moreover, the guarantee that choices are going to belong to the uncovered set after a sequence of pairwise majority voting contests only exists if voters are fully sophisticated. If voters are not entirely sophisticated nor naive, it is conceivable that the possible choices belong to some k-king class different from 2 and ∞. The consequences are twofold. First, whoever controls the agenda may induce voters to choose inefficient outcomes. Second, it may be necessary an agenda consisting of more than two majority voting contests in order to induce a given option.

1.2 Setup

We will follow the setup and notation in Eliaz and Ok [10].

1.2.1 Choice Correspondences

Let X be an arbitrary nonempty set. We interpret X as the set of all (mutually exclusive) alternatives.

Definition 1. A *choice field* on X, hereafter denoted by Ω_X, is any subset of $2^X \backslash \{\emptyset\}$ with the following properties:

(I) $\{x\} \in \Omega_X$, for all $x \in X$;

(II) $\bigcup_{i=1}^{n} A_i \in \Omega_X$ whenever $A_i \in \Omega_X,\ i = 1, 2, ..., n,\ n \in \mathbb{N}$.

We interpret Ω_X as the set of all possible choice problems. Note that the above definition implies that all finite subsets of X are in Ω_X. We refer to any pair (X, Ω_X) as a *choice space*.

Definition 2. Given a choice space (X, Ω_X), we define a *choice correspondence* on Ω_X as a correspondence $c : \Omega_X \rightrightarrows X$ which satisfies

(I) $c(S) \neq \emptyset$, for all $S \in \Omega_X$;

(II) $c(S) \subseteq S$, for all $S \in \Omega_X$.

1.2.2 Binary Relations

Given any set X, a binary relation on X is simply a subset of $X \times X$. We adopt the standard notation $x \succsim y$ to represent the fact that $(x, y) \in \succsim$. We define the *symmetric* part of a relation $\succsim$ by $\backsim := \{(x, y) \in X \times X : \; x \succsim y$ and $y \succsim x\}$. The *asymmetric* part of a relation $\succsim$ is defined by $\succ := \succsim \setminus \backsim$.

We say that a binary relation $\succsim$ is *reflexive* when $x \succsim x$, for every $x \in X$, *anti-symmetric* if $\sim \subseteq \{(x, x) : x \in X\}$, and *complete* if $x \succsim y$ or $y \succsim x$ hold for every $x, y \in X$. Finally, if $x \succsim y$ and $y \succsim z$ imply that $x \succsim z$, for every $x, y, z \in X$, then we say that $\succsim$ is *transitive*.

The relation $\succsim$ is a *preorder* when it is reflexive and transitive. If $\succsim$ is a preorder and $\sim = \{(x, x) : x \in X\}$, we say it is a partial order.

For any relation $\succsim \subseteq X \times X$ and $\emptyset \neq S \subseteq X$, the set of $\succsim$-maximal elements of S is denoted by MAX$(S, \succsim)$, that is

$$\text{MAX}(S, \succsim) := \{x \in S : y \succ x \text{ for no } y \in S\}.$$

Furthermore, the set of $\succsim$-maximum elements of S will be denoted by $\max(S, \succsim)$, i. e.

$$\max(S, \succsim) := \{x \in S : x \succsim y, \text{ for all } y \in S\}.$$

Because we will mostly work with nontransitive binary relations, the composition of binary relations will be helpful. We say $x \succsim^n y$, for some $n \in \mathbb{N}$, if there exist $x_0, \ldots, x_n \in X$ such that $x = x_0 \succsim x_1 \succsim \cdots \succsim x_n = y$. Given any binary relation $\succsim$, we denote by $tran(\succsim)$ the transitive closure of $\succsim$. That is, $tran(\succsim)$ is the smallest relation $\succsim'$ such that $\succsim \subseteq \succsim'$ and $\succsim'$ is transitive. We note that $tran(\succsim) = \succsim^\infty := \cup_{i=1}^\infty \succsim^i$.

Given a binary relation $\succsim$ and a subset A of X, we define the restriction of $\succsim$ to A by $\succsim_A := \succsim \cap (A \times A)$.

1.3 General Representation

Let (X, Ω_X) be a generic choice space, c be a choice correspondence on (X, Ω_X) and fix some $k \in \mathbb{N}$. We begin with a basic postulate.

Axiom 1 (Tournament Consistency)**.** *If A and B are nonempty subsets of X such that $A \cup B \in \Omega_X$ and $\{x\} = c(\{x, y\})$ for every $x \in A$ and $y \in B$, then $c(A \cup B) \subseteq A$.*

In the postulate above, when facing a choice between any alternative x from the set A and any distinct alternative y from the set B, the individual always chooses x. It is only natural, then, that when making a choice from $A \cup B$ the individual chooses only alternatives that belong to A. We label it Tournament Consistency because virtually all choice correspondences rationalized by some tournament solution concept satisfy it. The idea of Tournament Consistency was first presented in a setting of ranking of the alternatives in Smith [34] and further developed, in the context of social choice functions, in Fishburn [12] and Moulin [24].

The following is a standard postulate in choice theory.

Axiom 2 (Gamma)**.** *If $\mathcal{A} \subseteq \Omega_X$ is such that $\cup\mathcal{A} \in \Omega_X$ and $x \in c(A)$ for every $A \in \mathcal{A}$, then $x \in c(\cup\mathcal{A})$.*

This is a very well known property presented in Sen [33]. The next two postulates make use of the exogenous integer k we have fixed above.

Axiom 3 ((k+1)-Bounded Beta Plus)**.** *If $A, B \in \Omega_X$ are such that $|A| \leq k+1$, $B \subseteq A$, $x \in c(B)$ and $B \cap c(A) \neq \emptyset$, then $x \in c(A)$.*

Axiom 4 ((k+1)-Bounded Weakened Chernoff)**.** *For any choice problem $A \in \Omega_X$ and $x, y \in A$, if $x \notin c(B)$ for every subset B of A with $y \in B$ and $|B| \leq k+1$, then $x \notin c(A)$.*

The (k+1)-Bounded Beta Plus postulate is a version of the Beta Plus postulate from Bordes [3], itself a strengthening of the classical Beta from Sen [33]. The difference is that (k+1)-Bounded Beta Plus restricts the application of Beta Plus to sets of cardinality at most $k+1$. Bordes [3] shows that Beta Plus, together with Inclusion Minimality, characterizes the top-cycle rule. Here we prove that by restricting Beta Plus we are capable of characterizing all the generalized king-chicken solutions from the uncovered set to the top-cycle.

While (k+1)-Bounded Beta Plus represents an expansion consistency condition, in the sense that it imposes some structure from smaller to bigger sets, as the axioms of Gamma and Beta, (k+1)-Bounded Weakened Chernoff resembles more the Alpha Axiom from Sen [33], which had already been proposed in Chernoff [5], in the sense that it is a contraction consistency condition, bringing the structure from bigger to smaller sets.[1] It is a version that limits the cardinality of the set B in the Weakened Chernoff postulate of Lombardi [16]. In

[1] Both the concepts of contraction and expansion conditions were developed in Sen [32].

words, (k+1)-Bounded Weakened Chernoff states that, for a choice problem A and alternatives $x, y \in A$, if x is never chosen in the presence of y and at most other $k-1$ distinct alternatives in A, then x is not chosen from A. Intuitively, once y is better than x in the sense that $x \notin c(\{x, y\})$, any reasoning that justifies the choice of x in the presence of y must involve at most $k-1$ other alternatives. That is, (k+1)-Bounded Weakened Chernoff may be understood as a restriction on the complexity of the choice procedure.

The four postulates above deliver the following result:

Theorem 1. *The choice correspondence c satisfies Tournament Consistency, Gamma, (k+1)-Bounded Beta Plus and (k+1)-Bounded Weakened Chernoff if, and only if, there exists a complete binary relation $\succsim$ such that, for every $A \in \Omega_X$, $c(A) = \max(A, \succsim^k_A)$.*

We call the class of representations above the class of *generalized king-chicken choice correspondences*. Recall that in the original king-chicken story (see Maurer [19]), a chicken is a king if it directly beats or beats some chicken that beats any other chicken. That is, a king-chicken is maximal with respect to the $\succsim^2$ relation.

When $k \geq 2$, it is possible to prove the tightness of the axioms in Theorem 1. We show the independence of the axioms in Theorem 1 in Section 1.7.2.

In the next section, we discuss how some of the limit values of k are related to other tournament based representations that have previously appeared in the literature.

1.4 Limit Cases

1.4.1 Binary Choice Correspondences

When $k = 1$, the (k+1)-Bounded Beta Plus postulate is satisfied by any choice correspondence and is, therefore, irrelevant. We also note that 2-Bounded Weakened Chernoff implies Tournament Consistency. Therefore, when $k = 1$, Theorem 1 reduces to:

Theorem 2. *The choice correspondence c satisfies Gamma and 2-Bounded Weakened Chernoff if, and only if, there exists a complete binary relation $\succsim$ such that, for every $A \in \Omega_X$, $c(A) = \max(A, \succsim)$.*

Of course, Theorem 2 is well-known (see Theorem 3 in Sen [33]), although it is usually stated with the standard Alpha (Chernoff) postulate replacing 2-Bounded Weakened Chernoff. Formally, consider the following postulate:

Axiom 5 (Alpha)**.** *For every choice problem $A \in \Omega_X$, if $x \in c(A)$, then $x \in c(B)$ for every $B \in \Omega_X$ with $x \in B$ and $B \subseteq A$.*

We note that 2-Bounded Weakened Chernoff is simply the postulate above when the choice problem B is restricted to have at most 2 elements, so it is, in general, weaker than the Alpha postulate. Of course, they are both equivalent in the presence of Gamma.

1.4.2 k-Unbounded Representations

We shall now think of the representation in Theorem 1 for unbounded k. In this case, (k+1)-Bounded Beta Plus becomes simply:

Axiom 6 (Finite Beta Plus)**.** *If $A, B \in \Omega_X$ are such that $|A| < \infty$, $B \subseteq A$, $x \in c(B)$ and $B \cap c(A) \neq \emptyset$, then $x \in c(A)$.*

The (k+1)-Bounded Weakened Chernoff postulate becomes:

Axiom 7 (Finite Weakened Chernoff)**.** *For any choice problem $A \in \Omega_X$ and $x, y \in A$, if $x \notin c(B)$ for every finite subset B of A with $y \in B$, then $x \notin c(A)$.*

We can now state the following result:

Theorem 3. *The choice correspondence c satisfies Tournament Consistency, Gamma, Finite Beta Plus and Finite Weakened Chernoff if, and only if, there exists a complete binary relation $\succsim$ such that, for every $A \in \Omega_X$, $c(A) = \max(A, \succsim_A^\infty)$.*

Suppose now that c is a choice correspondence that satisfies Gamma, Finite Beta Plus and Finite Weakened Chernoff. Fix two choice problems A and B with $B \subseteq A$. Suppose also that $x \in c(B)$ and $B \cap c(A) \neq \emptyset$. Fix $y \in B \cap c(A)$ and pick any $z \in A$. By Finite Weakened Chernoff, there exist a finite subset D of B and a finite subset E of A such that $x \in c(D)$, $y \in D$, $y \in c(E)$ and $z \in E$. By Finite Beta Plus, if $c(D \cup E) \cap E \neq \emptyset$, then $y \in c(D \cup E)$. Now another application of Finite Beta Plus gives us that $x \in c(D \cup E)$. Otherwise, we must have $c(D \cup E) \cap D \neq \emptyset$ and Finite Beta Plus immediately gives us that $x \in c(D \cup E)$. This shows that for every $z \in A$ there exists a finite subset F of A with $x \in c(F)$ and $z \in F$. Now Gamma implies that $x \in c(A)$. This shows that c satisfies the following postulate:

Axiom 8 (Beta Plus)**.** *For any two choice problems A and B in Ω_X with $B \subseteq A$, if $x \in c(B)$ and $B \cap c(A) \neq \emptyset$, then $x \in c(A)$.*

It is obvious that Beta Plus is stronger than Finite Beta Plus. Now suppose that $\mathcal{A} \subseteq \Omega_X$ is such that $\cup\mathcal{A} \in \Omega_X$ and $x \in c(A)$ for every $A \in \mathcal{A}$. There must exist $A \in \mathcal{A}$ with $A \cap c(\cup\mathcal{A}) \neq \emptyset$. Now Beta Plus implies that $x \in c(\cup\mathcal{A})$. That is, c satisfies Gamma. This discussion can be summarized by the following lemma:

Lemma 1. *Let c be a choice correspondence that satisfies Finite Weakened Chernoff. Then c satisfies Gamma and Finite Beta Plus if, and only if, it satisfies Beta Plus.*

We have, thus, the following corollary:

Corollary 1. *The choice correspondence c satisfies Tournament Consistency, Beta Plus and Finite Weakened Chernoff if, and only if, there exists a complete binary relation $\succsim$ such that, for every $A \in \Omega_X$, $c(A) = \max(A, \succsim_A^\infty)$.*

1.5 Tournaments and Other Solution Concepts

1.5.1 Transitive Core and the Uncovered Set

Let X be any set. Following Nishimura [25], given a reflexive binary relation $\succsim \subseteq X \times X$, define the *transitive core* of $\succsim$ by $x$trancore$(\succsim)y$ if, and only if, $y \succsim z$ implies $x \succsim z$ and $z \succsim x$ implies $z \succsim y$ for every $z \in X$. It is easy to see that trancore$(\succsim)$ is always a preorder such that $x$trancore$(\succsim)y$ implies $x \succsim y$. Alternatively, we can define, for each $A \subseteq X$, a binary relation $\trianglerighteq_A \subseteq A \times A$ by $x \trianglerighteq_A y$ iff $y \succsim z$ implies $x \succsim z$ for every $z \in A$. In the tournament literature, the set MAX$(A, \trianglerighteq_A)$ is known as the uncovered set of A with respect to $\succsim$. It is clear that $x$trancore$(\succsim_A)y$ implies that $x \trianglerighteq_A y$, but the converse is not usually true. It is true, however, whenever $\succsim$ is complete and anti-symmetric. When $\succsim$ is complete and anti-symmetric, it is also well-known that MAX$(A, \trianglerighteq_A) = \max(A, \succsim_A^2)$, for any $A \subseteq X$. This discussion can be summarized by the following lemma:

Lemma 2. *Let $\succcurlyeq \subseteq X \times X$ be a complete and anti-symmetric relation. Then, for any choice problem A,* MAX$(A,$ *trancore*$(\succcurlyeq_A))$ = MAX$(A, \trianglerighteq_A) = \max(A, \succcurlyeq_A^2)$.

Now let c be a choice correspondence over an arbitrary choice space (X, Ω_X). Consider the following postulate:

Axiom 9 (Resoluteness)**.** *For every $x, y \in X$, $|c(\{x, y\})| = 1$.*

Lemma 2 and Theorem 1 have the following corollary:

Corollary 2. *The choice correspondence c satisfies Tournament Consistency, Gamma, 3-Bounded Beta Plus, 3-Bounded Weakened Chernoff and Resoluteness if, and only if, there exists a complete and anti-symmetric binary relation $\succcurlyeq \subseteq X \times X$ such that, for every choice problem A,*

$$c(A) = \text{MAX}(A, \textit{trancore}(\succcurlyeq_A)) = \text{MAX}(A, \trianglerighteq_A).$$

Choice correspondences that can be axiomatized by the uncovered set of some complete and asymmetric relation were also axiomatized, under the restriction of a finite space of alternatives X, by Lombardi [16]. Besides Resoluteness, Lombardi imposed also the following postulates:

Axiom 10 (Weak Expansion)**.** *If $A_i \in \Omega_X$, $i = 1, \ldots, n$, are such that $\bigcup_{i=1}^n A_i \in \Omega_X$, then $\bigcap_{i=1}^n c(A_i) \subseteq c\left(\bigcup_{i=1}^n A_i\right)$.*

Axiom 11 (Binary Dominance Consistency)**.** *If $A \in \Omega_X$, $x \in A$ and $c(\{x, y\}) = \{x\}$ for every $y \in A$, then $c(A) = \{x\}$.*

Axiom 12 (Weakened Chernoff)**.** *For every $A \in \Omega_X$ with $|A| \geq 3$, if $x \in c(A)$ and $y \in A \setminus \{x\}$, then $x \in \bigcup_{B \subsetneq A : x, y \in B} c(B)$.*

Axiom 13 (Non-Discrimination)**.** *For every distinct $x, y, z \in X$, if $c(\{x, y\}) = \{x\}$, $c(\{y, z\}) = \{y\}$ and $c(\{x, z\}) = \{z\}$, then $c(\{x, y, z\}) = \{x, y, z\}$.*

Weak Expansion is just a restatement of the Gamma postulate. Binary Dominance Consistency says that if a set has a Condorcet winner, then it has to be the only choice from the set. We note that this is equivalent to applying Tournament Consistency only to problems A such that $|A| = 1$. Therefore, Binary Dominance Consistency is a weakening of Tournament Consistency. As the name suggests, Weakened Chernoff is a weakening of the alpha axiom, also known as Chernoff's postulate. We note that it is equivalent to imposing $(|A| - 1)$-Weakened Chernoff on each menu A with $|A| \geq 3$. Finally, Non-Discrimination imposes that if there exists a cycle involving three alternatives in binary choices, then all of then have to be chosen when the three of them are available. We see the axiomatization in Corollary 2 as complementary to Lombardi's, but with three possible benefits. First, it is obtained for an arbitrary choice space (X, Ω_X) with X not being necessarily finite. Second, even in the absence of Resoluteness, we know from Theorem 1 that a choice correspondence that satisfies the postulates in Corollary 2 admits a representation in terms of the king-chicken orders $\succsim_A^2$. Finally, because the axiomatization in Corollary 2 follows the structure in Theorem 1, it makes the comparison with other forms of representation easier. See Section 1.5.2, for example.

1.5.2 Top-Cycle Rule

Let X be any set and let $\succcurlyeq \subseteq X \times X$ be a complete and anti-symmetric relation. The set MAX$(X, \succcurlyeq^\infty)$ is known in the tournament literature as the top-cycle solution. Choice correspondences that always choose the top-cycle elements of some complete binary relation were axiomatized, under the restriction of a finite space of alternatives X, by Ehlers and Sprumont [9], both in the case when the relation is anti-symmetric and when it is not. Their characterization uses the Binary Dominance Consistency axiom above plus the following postulates:

Axiom 14 (Weakened Weak Axiom of Revealed Preference)**.** *If $A, B \in \Omega_X$, $x, y \in A \cap B$, $x \in c(A)$ and $y \in A \setminus c(A)$ then we must not have that $y \in c(B)$ and $x \in B \setminus c(B)$.*

Axiom 15 (Weak Contraction Consistency)**.** *If $A \in \Omega_X$ and $|A| \geq 2$, then $c(A) \subseteq \bigcup_{x \in A} c(A \setminus \{x\})$.*

The Weakened Weak Axiom of Revealed Preference rules out choices that are unambiguously context dependent, meaning that y cannot be rejected towards x in one context and x rejected and y chosen in another. It is interesting to notice that a king-chicken choice rule will be context independent as long as $k+1$ is larger than the number of alternatives in the biggest preference cycle, making the procedure equivalent to the top-cycle choice rule. For any k smaller than that and greater than one, this will no longer be the case, meaning we will have some degree of context dependence.

Weak Contraction Consistency is again a weakening of the alpha postulate. Since Beta Plus implies Weakened Weak Axiom of Revealed Preference, Corollary 1 above shows that we can replace Binary Dominance Consistency and Weak Contraction Consistency in Ehlers and Sprumont's characterization by Tournament Consistency. Moreover, in order to extend the characterization to arbitrary choice spaces all one need is to add the Finite Weakened Chernoff postulate.

Finally, from Theorem 1, we know that in the presence of Tournament Consistency, Gamma and Resoluteness, the difference between the choice correspondences that admit a top-cycle representation and an uncovered set representation is that the ones that have a top-cycle representation satisfy the stronger Finite Beta Plus and the weaker Finite Weakened Chernoff postulates instead of the weaker 3-Bounded Beta Plus and the stronger 3-Bounded Weakened Chernoff postulates, satisfied by the choice correspondences that admit an uncovered set representation.

1.6 Conclusion

In this paper we have characterized a choice correspondence derived from non-standard preferences, transcending the traditional requirement of transitivity. Though other instances of such choice rules have already been studied in the literature, as the uncovered set choice rule, studied by Lombardi [16], the transitive core, from Nishimura [25], and the top-cycle choice rule studied by Ehlers and Sprumont [9], the class of generalized king-chicken choice correspondences has the advantage of allowing the comparison of them under a unified set of axioms. We also note that the characterizations here are obtained for general, not necessarily finite, choice spaces. This may be specially useful if one wish to use tournament solution concepts as part of more sophisticated choice models, as the preference structure model of Nishimura and Ok [26] and Evren et al. [11].

1.7 Proofs

1.7.1 Proof of Theorem 1

[Necessity] Suppose that there exists a complete binary relation $\succsim$ such that, for every choice problem $A \in \Omega_X$, $c(A) = \max(A, \succsim^k_A)$. It is clear that, for every pair $x, y \in X$, $x \succsim y \iff x \in c(\{x, y\})$. Suppose now that A and B in $2^X \setminus \{\emptyset\}$ are such that $\{x\} = c(\{x, y\})$ for every $x \in A$ and $y \in B$. This implies that for every distinct $x \in A$ and $y \in B$ we have $x \succ y$. It is clear, then, that for no distinct $x \in A$ and $y \in B$ we have $y \succsim^k_{A \cup B} x$. Therefore, if $A \cup B \in \Omega_X$, we must necessarily have $c(A \cup B) \subseteq A$. This shows that c satisfies Tournament Consistency.

Now suppose that $\mathcal{A} \subseteq \Omega_X$ is such that $\cup\mathcal{A} \in \Omega_X$ and $x \in c(A)$ for every $A \in \mathcal{A}$. This implies that, for every $A \in \mathcal{A}$ and every $y \in A$, $x \succsim^k_A y$. It is clear that this implies that $x \succsim^k_{\cup\mathcal{A}} y$ for every $y \in \cup\mathcal{A}$ and, consequently, $x \in c(\cup\mathcal{A})$. We conclude that c satisfies Gamma.

Now let $A \in \Omega_X$ be such that $|A| \leq k + 1$. Let $B \in \Omega_X$ be a proper subset of A and suppose that $x \in c(B)$ and $B \cap c(A) \neq \emptyset$. Let $l := |B|$ and fix any $z \in A \setminus B$. Since

$|A \setminus B| \leq k+1-l$ and $B \cap c(A) \neq \emptyset$, there must exist $y \in B$ with $y \succsim_A^{k+1-l} z$. Since $x \in c(B)$ and $y \in B$, we must have $x \succsim_A^{l-1} y$. This implies that $x \succsim_A^k z$. Since this is true for every $z \in A \setminus B$ and $x \succsim^{l-1} y$ for every $y \in B$, we conclude that $x \in c(A)$ and, consequently, c satisfies (k+1)-Bounded Beta Plus.

Finally, let $A \in \Omega_X$ and fix x and y in A. Suppose that for every $B \subseteq A$ with $|B| \leq k+1$ and $y \in B$ we have $x \notin c(B)$. It is easy to see that this implies that it is not true that $x \succsim_A^k y$ and, consequently, $x \notin c(A)$. We learn that c satisfies (k+1)-Bounded Weakened Chernoff.

[Sufficiency] Suppose now that c is a choice correspondence that satisfies Tournament Consistency, Gamma, (k+1)-Bounded Beta Plus and (k+1)-Bounded Weakened Chernoff. Define the binary relation $\succsim \subseteq X \times X$ by $x \succsim y \iff x \in c(\{x,y\})$. Notice that $\succsim$ is complete. Fix $A \in \Omega_X$ and suppose that $x \in \max(A, \succsim_A^k)$. Fix any $y \in A$. This implies that there exists $\{y_0, \ldots, y_k\} \subseteq A$ with $x = y_0 \succsim \cdots \succsim y_k = y$. Since $|\{y_0, \ldots, y_k\}| \leq k+1$ and $y_i \in c(\{y_i, y_{i+1}\})$ for $i = 0, \ldots k-1$, (k+1)-Bounded Beta Plus implies that $y_i \in c(\{y_0, \ldots, y_k\})$ whenever $y_{i+1} \in c(\{y_0, \ldots, y_k\})$, for $i = 0, \ldots, k-1$. This, in turn, implies that $x = y_0 \in c(\{y_0, \ldots, y_k\})$. Since $y \in A$ was arbitrarily chosen, this shows that for every $y \in A$ there exists $B \subseteq A$ with $|B| \leq k+1$, $y \in B$ and $x \in c(B)$. Now Gamma implies that $x \in c(A)$. We learn that $\max(A, \succsim_A^k) \subseteq c(A)$ for every $A \in \Omega_X$.

Now suppose that $x \in c(A)$ for some $A \in \Omega_X$ and fix any $y \in A$. By (k+1)-Bounded Weakened Chernoff, there must exist $B \subseteq A$ with $|B| \leq k+1$, $y \in B$ and $x \in c(B)$. Since $x \in c(B)$, Tournament Consistency implies that there must exist $y_1 \in B \setminus \{x\}$ with $x \succsim y_1$. Applying Tournament Consistency again, we learn that there must exist $y_2 \in B \setminus \{x, y_1\}$ such that $x \succsim y_2$ or $y_1 \succsim y_2$. This implies that $x \succsim_B^2 y_2$. Proceeding this way, we obtain distinct elements $y_1, \ldots, y_{|B|-1} \in (B \setminus \{x\})$ such that $x \succsim_B^i y_i$ for $i = 1, \ldots, |B|-1$. Since $|B| \leq k+1$, this shows that $x \succsim_B^k z$ for every $z \in B$. In particular, we have $x \succsim_B^k y$, which implies $x \succsim_A^k y$. Since y was arbitrarily chosen, we conclude that $x \in \max(A, \succsim_A^k)$. Therefore, for every $A \in \Omega_X$, $c(A) = \max(A, \succsim_A^k)$, which completes the proof of the theorem.

1.7.2 Independence of the Axioms

We show through four simple examples that, whenever $k \geq 2$, the axioms in Theorem 1 are independent. The label of each subsection indicates the postulate violated by the example.

Tournament Consistency

Fix some $k \geq 2$ and let X be such that $|X| \geq 3$. Fix some $x \in X$ and suppose c is given by $c(\{x, y\}) = \{x\}$ for any $y \in X$ and $c(A) = A$ otherwise. It is easy to see that c satisfies all axioms in Theorem 1, but Tournament Consistency.

Gamma

Again, fix some $k \geq 2$ and let X be such that $|X| \geq k + 2$. Fix some $A \in \Omega_X$ with $|A| \geq k + 2$ and some $x \in A$. Let c be such that $c(B) = B$ for every $B \in \Omega_X$, except for A where $c(A) = A \setminus \{x\}$. Note that c satisfies all postulates in Theorem 1, but Gamma.

(k+1)-Bounded Beta Plus

Now, let $k \geq 2$ and X be such that $|X| \geq 3$. Fix distinct $x, y \in X$ and let c be such that $c(A) = A$ if $x \notin A$ and $c(A) = A \setminus \{y\}$ if $x \in A$. The choice correspondence c satisfies all postulates in Theorem 1, but (k+1)-Bounded Beta Plus.

(k+1)-Bounded Weakened Chernoff

Finally, let $k \geq 2$ and define $X := \{1, 2, \dots, k + 2\}$. Let $\succcurlyeq$ be the reflexive binary relation on X such that $1 \succ 2 \succ \cdots \succ k+2$ and, for $i = 3, \dots, k+2$, $i \succ j$ for every $j \leq i-2$. It can be verified that the choice correspondence c such that $c(A) = \max(A, \succcurlyeq_A^{k+1})$ for every $A \in \Omega_X$ satisfies all the axioms in Theorem 1, but (k+1)-Bounded Weakened Chernoff. To see that c violates (k+1)-Bounded Weakened Chernoff, notice that $1 \in c(X)$, but $1 \notin c(A)$ for every $A \in 2^X$ with $k + 2 \in A$ and $|A| \leq k + 1$.

1.7.3 Proof of Theorem 3

It is easy to show that the axioms are necessary for the representation, so we will show only that they are sufficient. For that, suppose c is a choice correspondence that satisfies Tournament Consistency, Gamma, Finite Beta Plus and Finite Weakened Chernoff. Again, define the binary relation $\succsim \subseteq X \times X$ by $x \succsim y \iff x \in c(\{x, y\})$. Now fix any finite choice problem A and let $k := |A| - 1$. It is easy to see that the restriction of c to $2^A \setminus \{\emptyset\}$ satisfies all postulates in the statement of Theorem 1. Applying that theorem we obtain that $c(A) = \max(A, \succsim_A^k) = \max(A, \succsim_A^\infty)$, where the last equality is a consequence of the cardinality of A. Now pick a choice problem A that is not finite and fix $x \in c(A)$. By Finite

Weakened Chernoff, for each $y \in A$ there exists a finite subset B of A with $x \in c(B)$ and $y \in B$. By our previous observation, this implies that $x \succsim_B^{|B|-1} y$ and, consequently, $x \succsim_A^\infty y$. We learn that $c(A) \subseteq \max(A, \succsim_A^\infty)$. Now fix $x \in \max(A, \succsim_A^\infty)$. This implies that, for each $y \in A$, there exist a finite sequence $y_1, \ldots, y_n \in A$ such that $x \succsim y_1 \succsim \cdots \succsim y_n \succsim y$. But then $x \in \max(\{x, y_1, \ldots, y_n, y\}, \succsim_{\{x,y_1,\ldots,y_n,y\}}^{n+1}) = c(\{x, y_1, \ldots, y_n, y\})$. Since this is true for every $y \in A$, Gamma now implies that $x \in c(A)$. We conclude that $c(A) = \max(A, \succsim_A^\infty)$, which concludes the proof of the theorem.

Chapter 2

Updating With a Subjective State Space

2.1 Introduction

The discussion concerning how agents react to new information is a topic extensively studied in individual decision theory. While in a Savagean framework the state space is regarded as an objective reality, exogenously given to the decision maker, in the context of preferences over menus, as developed in Dekel et al. [7], the state space is subjective and endogenous to the decision process. A connection between these two frameworks was proposed in Ahn and Sarver [1], where they axiomatize the relations between the objective and subjective state spaces, using the objective states from the Random Expected Utility model proposed in Gul and Pesendorfer [14] to uniquely identify the subjective states from the Dekel et al. [7] framework.

This connection has an important intuitive meaning, as individuals are expected to make coherent choices transitioning between the choice over menus, as in which restaurant to dine, to the actual choice of what to eat in the restaurant when the moment arrives. The framework in Ahn and Sarver [1] also allow for the identification of a lack of sophistication in the preference over menus when the decision maker realizes new states in the second choice, meaning that they choose, with positive probability, an alternative previously considered irrelevant.

Riella [28] extends this analysis to the learning process between distinct preferences over menus. Assume the Decision Maker is choosing where to dine in a given night and reveal a preference during the morning and a different one after lunch. When do these two preferences reveal that the agent has learned new information about her future preferences and dropped some subjective states during lunch? They answer this question with the property of Flexibility Consistency, which uses a set of additional alternatives to identify which subjective states were dropped.

A similar question to that of Ahn and Sarver [1] is discussed in Lu [17], where the the first stage is a choice over a menu of acts, in the fashion of Anscombe and Aumann [2], and the second stage is a Random Choice Rule, henceforth RCR, generated by a stochastic signal received by the agent between the two stages. Their results allow for an outside observer, unaware of the signal received, to evaluate the agent's decision process and the level of informativeness of the signal received.

In this paper we study some remaining processes of learning, to the best of our knowledge not previously developed. Particularly we focus on recognizing updates in Random Choice patterns that follow a Finite Random Expected Utility procedure, which is an adaptation of the model in Gul and Pesendorfer [14] with finite state spaces. As done in Ahn and Sarver [1], the finiteness of the state space requires the use of a tie-breaking rule with infinite support. We also study the conditions under which a collection of RCRs may be understood as emerging from a partition of the state space that defines either another Finite Random Expected Utility model or a Preference Over Menus that admits a Dekel et al. [7], henceforth DLR, representation.

Our results allow us to recognize when different patterns of choice may be understood as resulting from an updating process between them. Suppose an individual makes a daily choice of what to eat for lunch, but at some periods the revealed preference changes. Maybe the Decision Maker has started a diet or choose to adopt vegetarianism and, because of this, some of the alternatives stop being chosen or are chosen less frequently. The question we seek to answer is when will this new revealed pattern, which we understand as a new RCR, is actually a Bayesian update from their previously revealed pattern. This would mean that some alternatives are chosen less frequently, but the relative frequency, or probabilities, between some other choices remains unchanged.

For the main theorem in this paper the order of the updating is not really relevant, meaning we can explain both when the learning process leads to a drop of states or to the enrichment of the original set of states. In this sense, the theorem accommodates both the traditional notion of Bayesian Updating, in which the state space shrinks, and that of Reverse Bayesianism, discussed in Karni and Viero [15], in which the state space expands. As the characterization of Reverse Bayesianism in the transition from menus to random choice is already discussed in Ahn and Sarver [1], in this paper we also propose a characterization of the other direction.

The remainder of this paper is organized as follow. In the next section we discuss the primitives and main definitions necessary to our work. In Section 2.3 we pose our main results, characterizing the updating between RCRs. We also provide the conditions under which a collection of RCRs build a partition of the original one after the updating. In Section 2.4 we study the regular updating from Menus to Random Choice and provide a partitioning result in this setting, similar to that of the previous section. The last section presents our conclusions.

2.2 Setup

Let Z be a finite set with $|Z| \geq 2$, $\Delta(Z)$ be the space of probability measures on Z and $\mathcal{A} \subset \Delta(Z)$ be the collection of all nonempty, finite subsets of $\Delta(Z)$. We call an arbitrary element $A \in \mathcal{A}$ a choice problem or, similarly, a menu. Let $\Delta(\Delta(Z))$ denote the space of all probability distributions over $\Delta(Z)$. We will denote by S the set of possible states of nature that will influence the individual's preferences over $\Delta(Z)$. Let $n := |Z|$, since $\Delta(Z) \subset \mathbb{R}^n$ we use the euclidean distance over lotteries and denote by $B_\epsilon(z)$ the open ball with radius ϵ and center z and by $\overline{B_\epsilon}(z)$ its closure. We denote by int$A$ the interior of a set A and by ri$A$ its relative interior.

Definition 3. A *random choice rule* (RCR) is a function $\rho : \mathcal{A} \to \Delta(\Delta Z)$ that associates to each choice problem A a probability measure ρ^A on A, meaning, for any $A \in \mathcal{A}$, $\rho^A(A) = 1$ and, if $B, C \in \mathcal{A}$ are such that $B \cap C = \emptyset$, then $\rho^A(B \cup C) = \rho^A(B) + \rho^A(C)$.

In this setup, an expected-utility function on $\Delta(Z)$ is equivalent to a vector in $\mathbb{R}^Z$, therefore, we denote expected-utility functions interchangeably as vectors and functions by u, meaning that $u(p) = u \cdot p$. We define by

$$\mathcal{U} := \left\{ u \in \mathbb{R}^Z : \sum_{z \in Z} u_z = 0, \sum_{z \in Z} u_z^2 = 1 \right\},$$

the set of all normalized (nonconstant) expected-utility functions over $\Delta(Z)$, and by $\Delta^f(\mathcal{U})$ the space of finitely additive probability measures over $\mathcal{U}$. Given an expected-utility function $u \in \mathbb{R}^Z$, we let $M(A, u)$ denote the maximizers of u in A:

$$M(A, u) := \left\{ p \in A : u(p) = \max_{q \in A} u(q) \right\}.$$

Let $U : S \times \Delta(Z) \mapsto \mathbb{R}$ be the agent's utility function across states, such that $U_s \in \mathbb{R}^Z$ is the expected-utility function that represents the agents preferences over $\Delta(Z)$ upon the realization of $s \in S$. If we have that, for some $A \in \mathcal{A}$, $|M(A, U_s)| = 1$ for every $s \in S$, then the Random Expected Utility representation on A would be resumed to

$$\rho^A(p) = \mu\left(\{s \in S : p \in M(A, U_s)\}\right),$$

where μ is a probability distribution over S.

As we work with a finite state space, though, we will need a rule to deal with situations where a state with positive probability leads to a tie among available alternatives. In the original set-up of Gul and Pesendorfer [14] this problem is averted as the authors show that it is always possible to achieve an infinite state space representation where each individual state has zero probability.[1] We follow Ahn and Sarver [1] by defining a tie-breaking rule.

Definition 4. Given a finite state space S, a *tie-breaking rule* for S is a map $\tau : S \to \Delta^f(\mathcal{U})$ that satisfies the following regularity condition for all $A \in \mathcal{A}$, $p \in A$ and $s \in S$:

$$\tau_s\left(\{u \in \mathcal{U} : u(p) > u(q), \forall q \in A \setminus \{p\}\}\right) = \tau_s\left(\left\{u \in \mathcal{U} : u(p) = \max_{q \in A} u(q)\right\}\right).$$

Note that, despite having finite states in S, the regularity condition implies that the tie-breaking rule τ_s cannot have a finite support on $\mathcal{U}$, otherwise it would itself lead back into ties among lotteries.

With this we can define the Finite Random Expected Utility representation.

Definition 5. A *Finite Random Expected Utility representation* (FREU) is a tuple (S, U, μ, τ), where S is a finite state space, $U : S \times \Delta(Z) \to \mathbb{R}$, μ is a probability distribution on S, and τ is a tie-breaking rule over S such that the following statements hold:

(i) For every $A \in \mathcal{A}$ and $p \in A$,

$$\rho^A(p) = \sum_{s \in S} \mu(s)\tau_s\left(\{u \in \mathcal{U} : p \in M(M(A, U_s), u)\}\right)$$

(ii) For any two distinct states $s, s' \in S$, U_s and $U_{s'}$ do not represent the same von Neumann-Morgestern (vNM) preference on $\Delta(Z)$.

(iii) For every $s \in S$, $\mu(s) > 0$ and U_s is nonconstant.

In this setup every FREU representation is essentially unique. Meaning that, if (S, U, μ, τ) and (S', U', μ', τ') represent the same Random Choice Rule, then it must be the case that for any $s \in S$ there is a unique $s' \in S'$ such that, for every $A \subset \Delta(Z)$, $\arg\max U_s(A) =$

[1]They deal with nonregular random utility functions, requiring a tie-breaking rule in the supplemental material to their paper.

$\arg\max U'_{s'}(A)$, $\mu(s) = \mu'(s')$ and $\tau_s = \tau'_{s'}$, meaning that, essentially, $S = S'$. Through the remainder of this paper, whenever we say two subjective states, s, s' are equal ($s = s'$), we mean that the utilities they imply, say U_s and $U'_{s'}$, represent the same vNM preferences. Whenever two RCRs, ρ and ρ', have FREU representations (S, U, μ, τ) and (S', U', μ', τ') such that $S' \subseteq S$ and, for every $s \in S'$, $\mu'(s) = \frac{\mu(s)}{\mu(S')}$ and $\tau'_s = \tau_s$, we abuse notation by saying $(S', U, \mu_{S'}, \tau)$ is a FREU representation of ρ'. Note that $S' \subseteq S$ already implies that, for every $s \in S'$, U_s and U'_s represent the same vNM preferences over $\Delta(Z)$.

In section 2.4 we work with preferences over menus and its relations to RCRs. For that we will need the following definitions.

Definition 6. A *preference over menus* is a binary relation $\succsim \subseteq \mathcal{A} \times \mathcal{A}$.

Definition 7. A preference over menus $\succsim$ has a DLR representation if there is a tuple (S, U, μ), where S is a finite state space, $U : S \times \Delta(Z) \to \mathbb{R}$ is a state-dependent expected-utility function, and μ is a probability distribution on S, such that the following statements hold:

(i) $A \succsim B$ if and only if $V(A) \geq V(B)$, where $V : \mathcal{A} \to \mathbb{R}$ is defined by $V(A) = \sum_{s \in S} \mu(s) \max_{p \in A} U_s(p)$.

(ii) For any two distinct states $s, s' \in S$, U_s and $U_{s'}$ do not represent the same von Neumann-Morgenstern preference on $\Delta(Z)$.

(iii) For every $s \in S$, $\mu(s) > 0$ and U_s is nonconstant.

For the results in 2.4 we endow $\mathcal{A}$ with the Hausdorff metric:

Definition 8. Let $A, B \in \mathcal{A}$, we denote by d_h the Hausdorff Metric given by

$$d_h(A, B) := \max \left\{ \max_{p \in A} \min_{q \in B} d(p, q), \max_{q \in B} \min_{p \in A} d(p, q) \right\}.$$

2.3 Updating Finite Random Expected Utility representations

2.3.1 Main Result: Updating Between FREU representations

We proceed by stating our result of updating between FREU representations. Let ρ_1 and ρ_2 be two RCRs, (S, U, μ, τ) and (T, U', μ', τ') its respective FREU representations. Our main theorem is based upon the following axiom.

Axiom 16 (Random Consistency)**.** *For any choice problem $A \in \mathcal{A}$ and $p, q \in A$, if $\rho_1^A(p)\rho_2^A(q) > \rho_1^A(q)\rho_2^A(p)$, then there exists a set $B \in \mathcal{A}$ and a radius $\delta > 0$ such that, $\rho_2^{A \cup B \cup \{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(p) \cap \Delta(Z)$, but $\rho_1^{A \cup B}(p) > 0$.*

Theorem 4. *Let ρ_1 and ρ_2 be two stochastic choice functions that admit finite random expected utility representations. The following statements are equivalent:*

1. *The stochastic choice functions ρ_1 and ρ_2 satisfy Random Consistency;*
2. *either $S \cap T = \emptyset$ or $T \subseteq S$ and (T, U, μ_T, τ) is a random expected utility representation of ρ_2, where μ_T is the Bayesian update of μ after the observation of T.*

Proof. Suppose ρ_1 and ρ_2 satisfy Random Consistency and $S \cap T \neq \emptyset$, so that $\mu(T) > 0$. Fix any menu A and $p \in A$. Without loss of generality, we may assume that $A \subseteq \text{ri}\Delta(Z)$.[2] Let $E \subseteq \text{ri}\Delta(Z)$ be any sphere. Let C be the subset of E that includes only the maximizers of s for all $s \in S \setminus T$ and D be the subset of the maximizers of s for all $s \in T$. For each $\lambda \in (0,1)$, define $A_\lambda := C \cup (\lambda D + (1-\lambda)A)$. Let λ be large enough so that $\arg\max U(A_\lambda, s) \subseteq C$ for every $s \in S \setminus T$ and $\arg\max U_s(A_\lambda) \subseteq \lambda D + (1-\lambda)A$ for every $s \in T$.[3] Suppose $q \in \lambda D + (1-\lambda)A$ is such that $\rho_2^{A^\lambda}(q) > 0$ and fix $p \in \lambda D + (1-\lambda)A$ with $\rho_1^{A^\lambda}(p) > 0$.[4] We note that this implies that there exist unique $s, s' \in T$ with $p \in \arg\max U(A^\lambda, s)$ and $q \in \arg\max U(A^\lambda, s')$. Suppose now that either $\rho_1^{A^\lambda}(q) = 0$ or $\rho_2^{A^\lambda}(p) = 0$. By Random Consistency, there must exist a finite set $B \subseteq \Delta(Z)$ and $\delta > 0$ such that $\rho_1^{A^\lambda \cup B}(p) > 0$, but $\rho_2^{A^\lambda \cup B \cup \{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(p) \cap \Delta(Z)$. However, $\rho_2^{A^\lambda \cup B \cup \{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(p)$ can happen only if $\max_{r \in B} U(r, s) > U(p, s)$, which would imply that $\rho_1^{A^\lambda \cup B}(p) = 0$. We conclude that, for any $p \in \lambda D + (1-\lambda)A$, $\rho_1^{A^\lambda}(p) > 0$ if and only if $\rho_2^{A^\lambda}(p) > 0$. We note that this implies that $T \subseteq S$. Fix any $p, q \in \lambda D + (1-\lambda)A$ with $\rho_1^{A^\lambda}(p) > 0$ and $\rho_1^{A^\lambda}(q) > 0$. Assume, without loss of generality, that $\rho_1^{A^\lambda}(p)\rho_2^{A^\lambda}(q) \geq \rho_1^{A^\lambda}(q)\rho_2^{A^\lambda}(p)$. There must exist a unique $s \in T$ such that $p \in \arg\max_{r \in A^\lambda} U(r, s)$. But then, there exists a finite set B and $\delta > 0$ with $\rho_2^{A^\lambda \cup B \cup \{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(p)$ only if $\max_{r \in B} U(r, s) > U(p, s)$. This implies that $\rho_1^{A^\lambda \cup B}(p) = 0$, so that,

[2] Otherwise, just work with $\frac{1}{2}A + \frac{1}{2}\{p\}$, where p is any lottery with full support.

[3] Such λ must exist since, if we took $\lambda = 1$, then $A_\lambda = C \cup D$, and, for any $s \in S \cup T$, $|\arg\max U_s(A_\lambda)| = 1$. If $s \in S \setminus T$, then $\arg\max U_s(A_\lambda) \subseteq C$, and $\arg\max U_s(A_\lambda) \subseteq D$, otherwise. Therefore, for every $\lambda \in (0,1)$ close enough to 1, $\max U_s(C) > \max U_s(\lambda D + (1-\lambda)A)$, if $s \in S \setminus T$, and $\max U_s(C) < \max U_s(\lambda D + (1-\lambda)A)$, if $s \in T$.

Note that, in this context, if $a \in \arg\max U_s(A)$ and $q_s = \arg\max U_s(D)$ for some $s \in T$, then $U_s(\lambda q_s + (1-\lambda)a) > U_s(\lambda q + (1-\lambda)a)$, for any $q \in D \setminus \{q_s\}$, $U_s(\lambda q_s + (1-\lambda)a) \geq U_s(\lambda q_s + (1-\lambda)a')$ for any $a' \in A \setminus \{a\}$ and $U_s(\lambda q_s + (1-\lambda)a) = U_s(\lambda q_s + (1-\lambda)a')$ if, and only if, $\{a, a'\} \subseteq \arg\max U_s(A)$.

[4] Such a p is guaranteed to exist because $\mu(T) > 0$.

by Random Consistency, we must have $\rho_1^{A^\lambda}(p)\rho_2^{A^\lambda}(q) = \rho_1^{A^\lambda}(q)\rho_2^{A^\lambda}(p)$. We conclude that, for any $p, q \in \lambda D + (1-\lambda)A$ with $\rho_1^{A^\lambda}(q) > 0$, we must have

$$\frac{\rho_1^{A^\lambda}(p)}{\rho_1^{A^\lambda}(q)} = \frac{\rho_2^{A^\lambda}(p)}{\rho_2^{A^\lambda}(q)}.$$

Now note that

$$\sum_{p\in\lambda D+(1-\lambda)A} \rho_1(p) = \mu(T)$$

and

$$\sum_{p\in\lambda D+(1-\lambda)A} \rho_2(p) = \mu'(T) = 1.$$

But then, for any $p \in \lambda D + (1-\lambda)A$,

$$\begin{aligned}
\frac{\rho_1^{A^\lambda}(p)}{\mu(T)} &= \frac{\rho_1^{A^\lambda}(p)}{\sum_{q\in\lambda D+(1-\lambda)A}\rho_1(q)} \\
&= \frac{\rho_2^{A^\lambda}(p)}{\sum_{q\in\lambda D+(1-\lambda)A}\rho_2(q)} \\
&= \rho_2^{A^\lambda}(p)
\end{aligned}$$

And, therefore, for any $p \in A$,

$$\begin{aligned}
\rho_2^A(p) &= \sum_{q\in D}\rho_2^{A^\lambda}(\lambda q + (1-\lambda)p) \\
&= \frac{1}{\mu(T)}\sum_{q\in D}\rho_1^{A^\lambda}(\lambda q + (1-\lambda)p) \\
&= \frac{1}{\mu(T)}\sum_{q\in D}\sum_{s\in S}\mu(s)\tau_s\left(\{u\in\mathcal{U} : \lambda q + (1-\lambda)p \in M(M(A^\lambda, U_s), u)\}\right) \\
&= \frac{1}{\mu(T)}\sum_{q\in D}\sum_{s\in T}\mu(s)\tau_s\left(\{u\in\mathcal{U} : \lambda q + (1-\lambda)p \in M(M(A^\lambda, U_s), u)\}\right) \\
&= \frac{1}{\mu(T)}\sum_{s\in T}\mu(s)\tau_s\left(\{u\in\mathcal{U} : p \in M(M(A, U_s), u)\}\right)
\end{aligned}$$

This proves statement 2. Conversely, suppose there exists a set $T \subseteq S$, such that (T, U, μ_T, τ) is a FREU representation of ρ_2. Fix a menu $A \subseteq \text{ri}\Delta(Z)$ and $p, q \in A$ with $\rho_1^A(p)\rho_2^A(q) > \rho_1^A(q)\rho_2^A(p)$. For that to happen, we must have $\rho_2^A(q) > 0$, which also implies that $\rho_1^A(q) > 0$. Therefore, the previous condition can be written as

$$\frac{\rho_1^A(p)}{\rho_1^A(q)} > \frac{\rho_2^A(p)}{\rho_2^A(q)}.$$

It is clear that this can happen only if there exists $s^* \in S\backslash T$ with $\tau_{s^*}(\{u \in \mathcal{U} : p \in M(M(A, U_{s^*}), u)\}) > 0$. Following the same steps as in the proof of the main result in Riella [28], we can find a finite set B such that $\max_{q\in B} U(q, s) > \max_{q\in A} U(q, s)$ for every $s \in S \setminus \{s^*\}$, but

$U(p, s^*) > \max_{q \in B} U(q, s^*)$. Let δ be small enough so that $\max_{q \in B} U(q, s) > U(p^\delta, s)$ for every $s \in S \setminus \{s^*\}$ and $p^\delta \in B_\delta(p)$. Note that this implies that $\rho_2^{A \cup B \cup \{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(p)$, but $\rho_1^{A \cup B}(p) > 0$. That is, ρ_1 and ρ_2 satisfy Random Consistency. Finally, if $S \cap T = \emptyset$, it is clear that $\rho_1^A(p)\rho_2^A(q) > \rho_1^A(q)\rho_2^A(p)$ implies that there exists $s^* \in S \setminus T$ with $\tau_{s^*}(\{u \in \mathcal{U} : p \in M(M(A, U_{s^*}), u)\}) > 0$. We may now follow the same steps as above to find a menu B and a $\delta > 0$ such that $\rho_2^{A \cup B \cup \{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(p)$, but $\rho_1^{A \cup B}(p) > 0$. Again, this shows that ρ_1 and ρ_2 satisfy Random Consistency. □

If we want to make sure that at least one state is shared among S and T, meaning $\mu(T) > 0$, we can add the following axiom:

Axiom 17. *For every $A \in \mathcal{A}$, $supp(\rho_1^A) \cap supp(\rho_2^A) \neq \emptyset$.*

Corollary 3. *ρ_1 and ρ_2 satisfy Random Consistency and Axiom 17 if, and only if, $T \subseteq S$, μ_2 is the Bayesian update of μ after the observation of T and they share the same tie breaking rule ($\tau = \tau'$).*

Proof. Take some $p \in$ ri$\Delta(Z)$ and $\epsilon > 0$ such that $\overline{B_\epsilon}(p) \cap \Delta(Z) \subset$ ri$\Delta(Z)$ and define, for each $s \in S \cup T$, $q_s =$ argmax$_{q \in \overline{B_\epsilon}(p)} U_s(q)$. Note that $q_s = q_{s'}$ implies that U_s and $U_{s'}$ represent the same vNM preference over lotteries. Take now $A := \{q_s \in \Delta(Z) : s \in S \cup T\}$. We must have that, if $supp(\rho_1^A) \cap supp(\rho_2^A) \neq \emptyset$, then argmax$_{q \in \overline{B_\epsilon}(p)} U_s(q) =$ argmax$_{q \in \overline{B_\epsilon}(p)} U_{s'}(q)$ meaning that there are $s \in S$ and $s' \in T$ such that U_s and $U_{s'}$ represent the same vNM preferences, implying $s = s'$ and $s \in S \cap T$. □

2.3.2 Multiple Signals and Partitions

Suppose now that the information received comes from a set o signals that is sufficiently informative so that each subjective state is only realized after one possible signal, though the same signal may still lead to different subjective states in the second stage. In this case the collection of RCRs after the updating build a partition of the broader RCR from the first stage.

To characterize the relations between the original RCR an the collection formed after the signal in this setting, consider a finite collection of $I + 1$ random choice rules, ρ and $\{\rho_i\}_{i \in I}$, with FREU representations (S, U, μ, τ) and $(S_i, U^i, \mu^i, \tau^i)$, such that, for each $i \in I$, ρ and ρ_i satisfy Random Consistency, Axiom 17 and the following axioms.

Axiom 18. *For every* $i, j \in I$*, and* $A \in \mathcal{A}$*, if, for some* $p \in ri\Delta(Z)$ *and* $\delta > 0$*, we have* $\rho_i^{A\cup\{p^\delta\}}(p^\delta) > 0$ *for every* $p^\delta \in B_\delta(p) \cap \Delta(Z)$*, then there is* $D \in \mathcal{A}$ *with* $\rho_i^{A\cup D\cup\{p^\delta\}}(p^\delta) > 0$*, but* $\rho_j^{A\cup D\cup\{p^\delta\}}(p^\delta) = 0$*, for every* $p^\delta \in B_\delta(p) \cap \Delta(Z)$*.*

Axiom 19. *For any choice problem* $A \in \mathcal{A}$*, supp*$(\rho^A) = \bigcup_{i\in I}$ *supp*(ρ_i^A)*.*

To prove Proposition 1 we first state and prove the following Lemma:

Lemma 3. *A collection of RCRs* $\{\rho_i\}_{i\in I}$ *with FREU representations* $\{(S_i, U^i, \mu^i, \tau^i)\}_{i\in I}$ *satisfies Axiom 18 if, and only if, for any* $i, j \in I, i \neq j$*, we have* $S_i \cap S_j = \{\emptyset\}$*.*

Proof. As stated in Section 2.2, $S_i \cap S_j = \{\emptyset\}$ means that for any $s \in S_i$ and $s' \in S_j$, U_s^i and $U_{s'}^j$ do not represent the same vNM preferences. Fix any arbitrary $i, j \in I$ with $i \neq j$.

[$\Longrightarrow$] Suppose $\{\rho_i, \rho_j\}$ satisfies Axiom 18. Take some $p \in$ ri$\Delta(Z)$ and $\epsilon > 0$ such that $\overline{B_\epsilon}(p) \cap \Delta(Z) \subset$ ri$\Delta(Z)$. Define $\hat{U} : S_i \cup S_j \times \Delta(Z) \mapsto \mathbb{R}$ as

$$\hat{U}_s := \begin{cases} U_s^i & \text{if } s \in S_i \\ U_s^j & \text{if } s \in S_j \setminus S_i \end{cases}.$$

For each $s \in S_i \cup S_j$, define $q_s = \text{argmax}_{q\in\overline{B_\epsilon}(p)\cap\Delta(Z)} \hat{U}_s(q)$ and let $A := \{q_s \in \Delta(Z) : s \in S_i \cup S_j\}$. Now choose some $q \in A$ such that $\rho_i^A(q) > 0$. By construction, there is an unique $s \in S_i$ with $\hat{U}_s(q) > \max_{p\in A\setminus\{q\}} \hat{U}_s(p)$ and $\hat{U}_{s'}(q) < \max_{p\in A\setminus\{q\}} \hat{U}_{s'}(p)$ for every $s' \in (S_i \cup S_j) \setminus s$. We must then have that, for some $\delta > 0$, $\rho_i^{(A\setminus\{q\})\cup\{q^\delta\}}(q^\delta) > 0$ for every $q^\delta \in B_\delta(q) \cap \Delta(Z)$ which, by 18, implies the existence of some $D \in \mathcal{A}$ such that $\rho_i^{(A\setminus\{q\})\cup D\cup\{q^\delta\}}(q^\delta) > 0$, but $\rho_j^{(A\setminus\{q\})\cup D\cup\{q^\delta\}}(q^\delta) = 0$, for every $q^\delta \in B_\delta(q) \cap \Delta(Z)$. For $\rho_j^{(A\setminus\{q\})\cup D\cup\{q^\delta\}}(q^\delta) = 0$ to be true for every $q^\delta \in B_\delta(q) \cap \Delta(Z)$, it must be the case that $\hat{U}_{s'}(q) < \max_{p\in(A\setminus\{q\})\cup D} \hat{U}_{s'}(p)$ for every $s' \in S_j$, which can only happen if $s \notin S_j$. Since we took $s \in S_i$ arbitrarily, we must have that $S_i \cap S_j = \{\emptyset\}$.

[$\Longleftarrow$] Suppose that ρ_i and ρ_j have FREU representations such that $S_i \cap S_j = \{\emptyset\}$. Take some $A \in \mathcal{A}$, $p \in$ ri$\Delta(Z)$ and $\delta > 0$ such that $\rho_i^{A\cup\{p^\delta\}}(p^\delta) > 0$ for every $p^\delta \in B_\delta(p) \cap \Delta(Z)$. Fix some $s \in S_i$ such that, $\mu^i(s)\tau_s^i(\{u \in \mathcal{U} : p \in M(M(A, U_s^i), u)\}) > 0$. Let $\hat{v}$ be the vector in $\mathbb{R}^Z$ such that $\hat{v} \cdot q = U_s^i(q)$ for every $q \in \Delta(Z)$ and

$$v := \hat{v} - \left(\frac{1}{|Z|}\sum_{z\in Z} \hat{v}_z\right)\mathbf{1},$$

where $\mathbf{1} = (1, \ldots, 1)$ is the unit vector of size $|Z|$. Take $\epsilon > 0$ so that $\overline{B_{d(p,p-\epsilon v)}}(p - \epsilon v) \cap$ Span$(\Delta(Z)) \subset \Delta(Z)$. Now, for each $s' \in S_j$, let $q_{s'} := \arg\max_{q\in\overline{B_{d(p,p-\epsilon v)}}(p-\epsilon v)} U_{s'}^j(q)$ and $D := \{q_{s'} : s' \in S^j\}$. Note that, since $s \notin S_j$, $p \notin D$, $U_s^i(p) > \max_{q\in D} U_s^i(q)$ and, for each

$s' \in S_j$, $U^j_{s'}(p) < U^j_{s'}(q_{s'})$. Therefore, choosing $\delta' \in (0, \delta]$ small enough, we must have that $\rho_i^{A\cup D\cup\{p^{\delta'}\}}(p^{\delta'}) > 0$, but $\rho_j^{A\cup D\cup\{p^{\delta'}\}}(p^{\delta'}) = 0$, for every $p^{\delta'} \in B_{\delta'}(p) \cap \Delta(Z)$, proving that Axiom 18 is satisfied. $\square$

Proposition 1. *Let I be a finite set of indices and suppose ρ and $\{\rho_i\}_{i\in I}$ are random choice rules with FREU representations such that, for each $i \in I$, ρ and ρ_i satisfy Random Consistency and Axiom 17. Then, the collection $\{\rho_i\}_{i\in I}$ satisfy Axioms 18 and 19 if, and only if, the collection $\{S_i\}_{i\in I}$ is a partition of S and each ρ_i has a FREU representation $(S_i, U, \mu_{S_i}, \tau)$.*

Proof. [$\Longrightarrow$] Suppose the collection $\{\rho_i\}_{i\in I}$ satisfies Axioms 18 and 19. Since, for each $i \in I$, ρ and ρ_i satisfy Random Consistency and Axiom 17, Theorem 4 implies that $S_i \subset S$, $\mu_i = \mu_{S_i}$ and, for each $s \in S_i$, $\tau_s = \tau^i_s$. Therefore, for each $i \in I$, $(S_i, U, \mu_{S_i}, \tau)$ is a FREU representation of ρ_i. Since the collection $\{\rho_i\}_{i\in I}$ satisfies Axiom 18, Lemma 3 implies that, for each $i, j \in I, i \neq j$, $S_i \cap S_j = \{\emptyset\}$. It remains for us to show that $S \subseteq \bigcup_{i\in I} S_i$. To see that take some $p \in$ ri$\Delta(Z)$ and $\epsilon > 0$ such that $\overline{B_\epsilon}(p) \cap \Delta(Z) \subset$ ri$\Delta(Z)$ and, as we did in the proof of Lemma 3, for each $s \in S$, define $q_s =$ argmax$_{q\in\overline{B_\epsilon}(p)\cap\Delta(Z)} U_s(q)$ and let $A := \{q_s \in \Delta(Z) : s \in S\}$. Now, suppose there is $s' \in S \setminus \bigcup_{i\in I} S_i$. But then we should have $q_{s'} \in$ supp$(\rho^A) \setminus \bigcup_{i\in I}$ supp(ρ^A_i), which contradicts Axiom 19.

[$\Longleftarrow$] Conversely, suppose $\{S_i\}_{i\in I}$ is a partition of S. Axiom 19 is an immediate consequence of this fact, and Lemma 3 implies that Axiom 18 holds. $\square$

2.4 Updating from Menus to Random Choice Rules

Here we develop the traditional Bayesian updating direction between Preferences Over Menus and Random Choice Rules. This is the opposite direction of the unforeseen contingencies representations from Ahn and Sarver [1] and a straightforward application of the second part of the Proposition 2 in their paper. We also explore the partitioning of a Preference Over Menus into a collection of RCRs, similarly to what we have done in the previous section between RCRs. The following axiom is a restatement of Axiom 2 in Ahn and Sarver [1].

Axiom 20. *Let $A \in \mathcal{A}$ be a menu and $p \in \Delta(Z) \setminus A$ an arbitrary lottery. If there is $\delta > 0$ such that $\rho^{D\cup\{p^\delta\}}(p^\delta) > 0$ for every $p^\delta \in B_\delta(p) \cap \Delta(Z)$ and $D \in \mathcal{A}$ with $d_h(A, D) < \delta$, then $A \cup \{p\} \succ A$.*

Proposition 2. *Let ρ be a RCR that admits a FREU representation and $\succsim$ a preference over menus that admits a DLR representation, then ρ and $\succsim$ satisfy axiom 20 if, and only if, $\succsim$ has a DLR representation (S, U, μ) such that (T, U, μ_T, τ), $T \subseteq S$, is a FREU representation of ρ.*

Proof. Let (S, U', μ') be a DLR representation of $\succsim$ and $(T, \hat{U}, \hat{\mu}, \tau)$ a representation of ρ. The second part of the Proposition 2 in Ahn and Sarver [1] assures us that ρ and $\succsim$ satisfy Axiom 20 if, and only if, $T \subseteq S$. This means that, for each $t \in T$ there is an unique $s \in S$ such that $\hat{U}_t$ and U'_s represent the same vNM preference over $\Delta(Z)$. Therefore, the necessity of Axiom 20 follows directly. It remains for us to show that there is a state dependent utility function U and a probability distribution over states μ such that (S, U, μ) is a DLR representation of $\succsim$ and (T, U, μ_T, τ) is a FREU representation of ρ.

For that, define

$$\mu(s) := \begin{cases} \mu'(T)\hat{\mu}(s) & \text{if } s \in T \\ \mu'(s) & \text{if } s \in S \setminus T \end{cases},$$

and

$$U_s := \frac{\mu'(s)}{\mu(s)} U'_s.$$

Notice this implies that, for any $A \in \mathcal{A}$ and $p \in A$,

$$\begin{aligned}\sum_{s \in S} \mu(s) \max_{p \in A} U_s(p) &= \sum_{s \in S \setminus T} \mu'(s) \max_{p \in A} \left[\frac{\mu'(s)}{\mu'(s)} U'_s(p)\right] + \sum_{s \in T} \mu'(T)\hat{\mu}(s) \max_{p \in A} \left[\frac{\mu'(s)}{\mu'(T)\hat{\mu}(s)} U'_s(p)\right] \\ &= \sum_{s \in S} \mu'(s) \max_{p \in A} U'_s(p),\end{aligned}$$

and

$$\begin{aligned}\rho^A(p) &= \sum_{s \in T} \hat{\mu}(s)\tau_s\left(\left\{u \in \mathcal{U} : p \in M(M(A, \hat{U}_s), u)\right\}\right) \\ &= \sum_{s \in T} \frac{\mu(s)}{\mu'(T)} \tau_s\left(\{u \in \mathcal{U} : p \in M(M(A, U_s), u)\}\right) \\ &= \sum_{s \in T} \mu_T(s)\tau_s\left(\{u \in \mathcal{U} : p \in M(M(A, U_s), u)\}\right).\end{aligned}$$

Therefore we have that (S, U, μ) is a DLR representation of $\succsim$ and (T, U, μ_T, τ) is a representation of ρ. $\square$

We now turn to the question of when a collection of Random Choice Rules, $\{\rho_i\}_{i \in I}$, with FREU representations $(S_i, U^i, \mu^i, \tau^i)$, is a partition of the subjective state space from a DLR representation of a preference over menus $\succsim$.

Axiom 21. *$A \cup \{p\} \succ A$ if, and only if, there is $i \in I$ and $\delta > 0$ such that, $\rho_i^{D\cup\{p^\delta\}}(p^\delta) > 0$ for every $p^\delta \in B_\delta(p) \cap \Delta(Z)$ and $D \in \mathcal{A}$ with $d_h(A, D) < \delta$.*

Proposition 3. *Let $\succsim$ be a preference over menus that admits a DLR representation and $\{\rho_i\}_{i\in I}$ be a collection of Random Choice Rules with FREU representations. Then, $\{\rho_i\}_{i\in I}$ satisfies Axiom 18 and $(\succsim, \{\rho_i\}_{i\in I})$ satisfy Axiom 21 if, and only if, $\succsim$ has a DLR representation (S, U, μ) such that, for each $i \in I$, $(S_i, U, \mu_{S_i}, \tau_i)$, is a FREU representation of ρ_i and $\{S_i\}_{i\in I}$ is a partition of S.*

Proof. [$\Longrightarrow$] Let (S, U', μ') be any DLR representation for $\succsim$ and for each ρ_i let $(S_i, \hat{U}^i, \hat{\mu}^i, \tau^i)$ be its FREU representation. By Lemma 3, we know that for each $i, j \in I$, with $i \neq j$, $S_i \cap S_j = \{\emptyset\}$. Fix some $i \in I$. Axiom 21 implies that, if there is some $\delta > 0$ such that, $\rho_i^{D\cup\{p^\delta\}}(p^\delta) > 0$ for every $p^\delta \in B_\delta(p) \cap \Delta(Z)$ and $D \in \mathcal{A}$ with $d_h(A, D) < \delta$, then we must have $A \cup \{p\} \succ A$, meaning that $\succsim$ and ρ_i satisfy Axiom 20. Therefore, by Proposition 2, for each $i \in I$, we have that $S_i \subseteq S$. Now, similarly to what we did in the proof of Proposition 1, take some $p \in \text{ri}\Delta(Z)$ and $\epsilon > 0$ such that $\overline{B_\epsilon}(p) \cap \Delta(Z) \subset \text{ri}\Delta(Z)$ and for each $s \in S$, define $q'_s = \text{argmax}_{q\in\overline{B_\epsilon}(p)\cap\Delta(Z)} U'_s(q)$ and, for each $i \in I$ and $\hat{s} \in S_i$, $q^i_{\hat{s}} = \text{argmax}_{q\in\overline{B_\epsilon}(p)\cap\Delta(Z)} \hat{U}_{\hat{s}}(q)$. Let $A' := \{q_{s'} \in \Delta(Z) : s' \in S\}$ and $\hat{A} := \{q_{\hat{s}} \in \Delta(Z) : \hat{s} \in S_i, i \in I\}$. Since $S_i \subseteq S$ for every $i \in I$, we must have that $\hat{A} \subseteq A'$. If there was some $s \in S \setminus \bigcup_{i\in I} S_i$, we should have that $q'_s \notin \hat{A}$, $\hat{U}^i_{\hat{s}}(q'_s) < \max_{q\in\hat{A}} \hat{U}^i_{\hat{s}}(q)$ and, for some $\delta > 0$ small enough, for every $i \in I$, $\rho_i^{D\cup\{p^\delta\}}(p^\delta) = 0$ for every $p^\delta \in B_\delta(q'_s) \cap \Delta(Z)$, $D \in \mathcal{A}$ with $d_h(A', D) < \delta$. This contradicts Axiom 21, since $\{q'_s\} = \arg\max_{q\in A'} U'_s(q)$, implying $A' \succ A' \setminus \{q'_s\}$. Therefore, we must have $S = \bigcup_{i\in I} S_i$ and $\{S_i\}_{i\in I}$ is a partition of S.

Now define $\mu := \sum_{i\in I} \mu'(S_i)\hat{\mu}^i$ and $U_s := \frac{\mu'(s)}{\mu(s)} U'_s$, for every $s \in S$, and note that, for any $A \in \mathcal{A}$ and $s \in \hat{S}$,

$$\begin{aligned}
\mu'(s) \max_{p\in A} U'_s(p) &= \frac{\sum_{i\in I} \mu'(S'_i)\hat{\mu}^i(s)}{\sum_{i\in I} \mu'(S'_i)\hat{\mu}^i(s)} \mu'(s) \max_{p\in A} U'_s(p) \\
&= \sum_{i\in I} \mu'(S'_i)\hat{\mu}^i(s) \max_{p\in A} \frac{\mu'(s)}{\sum_{i\in I} \mu'(S'_i)\hat{\mu}^i(s)} U'_s(p) \\
&= \mu(s) \max_{p\in A} U_s(p),
\end{aligned}$$

meaning that (S, U, μ) is a DLR representation for $\succsim$. Since, for each $i \in I$ and $s \in S$, $\mu_{S_i}(s) = \hat{\mu}^i(s)$ and $\hat{U}^i_s$ and U_s represent the same vNM preferences on $\Delta(Z)$, we have that, for each $i \in I$, $(S_i, U, \mu_{S_i}, \tau)$ is a FREU representation of ρ_i.

[$\Longleftarrow$] Suppose now $\succsim$ has a DLR representation (S, U, μ) such that for each $i \in I$, $(S_i, U, \mu_{S_i}, \tau)$ is a FREU representation of ρ_i and $\{S_i\}_{i\in I}$ is a partition of S. Lemma 3 assures us that Axiom 18 is satisfied, so we only need to show that Axiom 21 also holds. Fix some arbitrary $A \in \mathcal{A}$, $p \in \Delta(Z)$ and suppose $A \cup \{p\} \succ A$, this implies that, for some $s \in S$, $U_s(p) > \max_{a \in A} U_s(a)$. Since A is finite and U_s is continuous, we must have that there is $\delta > 0$ such that, for every $p^\delta \in B_\delta(p)$, $U_s(p^\delta) > \max_{a \in A} U_s(a)$. Since $s \in S_i$ for some $i \in I$ and, we must have $\mu_{S_i}(s) > 0$ and $\rho_i^{A \cup (p^\delta)}(p^\delta) > 0$, for every $p^\delta \in B_\delta(p)$. $\square$

2.5 Conclusion

In this paper we extended the theory of Bayesian and Reverse Bayesian updating to the learning revealed between Random Choice Rules. We worked in a framework of Finite Random Expected Utilities already developed in Gul and Pesendorfer [14] and Ahn and Sarver [1]. We proposed the property of Random Consistency that is closely related to the Axiom 2 in Ahn and Sarver [1] and to the property of Flexibility Consistency in Riella [28], which apply to the transition from menus to random choice and between menus, respectively. We also developed the characterization of when a collection of Random Choice Rules represents a partition of the state space from another broader Random Choice Rule or from a Preference Over Menus.

A possible extension from the work in this paper would be to study a way to recognize when two Random Choice Rules may, or may not, represent the updating of an unknown broader Random Choice Rule or Preference Over Menus. Another way forward is to prove the equivalence of the updating between worlds, meaning that, if $(\succsim_1, \rho_1)$ and $(\succsim_2, \rho_2)$ have DLR-GP representations $(S_1, U_1, \mu_1, \tau_1)$ and $(S_2, U_2, \mu_2, \tau_2)$, then $\succsim_1$ and $\succsim_2$ satisfy Random Consistency, as proposed by Riella [28], if, and only if, ρ_1 and ρ_2 satisfy Random Consistency.

Bibliography

[1] David S. Ahn and Todd Sarver. Preference for flexibility and random choice. *Econometrica*, 81(1):341–361, 2013. doi: 10.3982/ecta10431.

[2] F. J. Anscombe and R. J. Aumann. A definition of subjective probability. *Annals of Mathematical Statistics*, 34(1):199–205, 1963.

[3] Georges Bordes. Consistency, rationality and collective choice. *The Review of Economic Studies*, 43(3):451, oct 1976. doi: 10.2307/2297222.

[4] Felix Brandt, Christian Geist, and Paul Harrenstein. A note on the McKelvey uncovered set and pareto optimality. *Social Choice and Welfare*, 46(1):81–91, aug 2015. doi: 10.1007/s00355-015-0904-5.

[5] Herman. Chernoff. Rational selection of decision functions. *Econometrica*, 22(4):422–443, 1954.

[6] Marquis de Condorcet. *Essai sur l'application de l'analyse la probabilit des dcisions rendues la pluralit des voix.* Paris, l'Imprimerie Royale, 1785.

[7] Eddie Dekel, Barton L. Lipman, and Aldo Rustichini. Representing preferences with a unique subjective state space. *Econometrica*, 69(4):891–934, 2001.

[8] John Duggan. Uncovered sets. *Social Choice and Welfare*, 41(3):489–535, oct 2013. doi: 10.1007/s00355-012-0696-9.

[9] Lars Ehlers and Yves Sprumont. Weakened warp and top-cycle choice rules. *Journal of Mathematical Economics*, 44:87–94, 2008.

[10] Kfir Eliaz and Efe A. Ok. Indifference or indecisiveness? choice-theoretic foundations of incomplete preferences. *Games and Economic Behavior*, 56:61–86, 2006.

[11] Özgür Evren, Hiroki Nishimura, and Efe A. Ok. Top-cycles and revealed preference structures. Working Paper, December 2019.

[12] Peter C. Fishburn. Condorcet social choice functions. *SIAM Journal of Applied Math*, 33(3):469–489, 1977.

[13] I. J. Good. A note on condorcet sets. *Public Choice*, 10(1):97–101, mar 1971. doi: 10.1007/bf01718625.

[14] Faruk Gul and Wolfgang Pesendorfer. Random expected utility. *Econometrica*, 74(1): 121–146, 2006.

[15] Edi Karni and Marie-Louise Viero. Reverse bayesianism: A choice-based theory of growing awareness. *American Economic Review*, page forthcoming, 2013.

[16] Michele Lombardi. Uncovered set choice rules. *Social Choice and Welfare*, 31:271–279, 2008.

[17] Jay Lu. Random choice and private information. *Econometrica*, 84(6):1983–2027, 2016. doi: 10.3982/ecta12821.

[18] Pasin Manurangsi and Warut Suksompong. Generalized kings and single-elimination winners in random tournaments. https://arxiv.org/abs/2105.00193, May 2021.

[19] Stephen B. Maurer. The king chicken theorems. *Mathematics Magazine*, 53(2), 1980.

[20] David C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21(4):608, oct 1953. doi: 10.2307/1907926.

[21] Richard D. McKelvey. Covering, dominance, and institution-free properties of social choice. *American Journal of Political Science*, 30(2):283, may 1986. doi: 10.2307/2111098.

[22] Nicholas R. Miller. Graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, 21(4):769–803, November 1977.

[23] Nicholas R. Miller. A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, 24(1):68, feb 1980. doi: 10.2307/2110925.

[24] H. Moulin. Choosing from a tournament. *Social Choice and Welfare*, 3(4):271–291, dec 1986. doi: 10.1007/bf00292732.

[25] Hiroki Nishimura. The transitive core: Inference of welfare from nontransitive preference relations. *Theoretical Economics*, 13(2):579–606, 2018. doi: 10.3982/TE1769. URL https://onlinelibrary.wiley.com/doi/abs/10.3982/TE1769.

[26] Hiroki Nishimura and Efe A. Ok. Preference structures. Working Paper, July 2021.

[27] Elizabeth Maggie Penn. Alternate definitions of the uncovered set and their implications. *Social Choice and Welfare*, 27(1):83–87, mar 2006. doi: 10.1007/s00355-006-0114-2.

[28] Gil Riella. Preference for flexibility and dynamic consistency. *Journal of Economic Theory*, 148(6):2467–2482, nov 2013. doi: https://doi.org/10.1016/j.jet.2013.09.001.

[29] Christian Saile and Warut Suksompong. Robust bounds on choosing from large tournaments. *Social Choice and Welfare*, 54(1):87–110, aug 2019. doi: 10.1007/s00355-019-01213-6.

[30] Paul Anthony Samuelson. A note on the pure theory of consumer's behaviour. *Economica*, 5(17):61–71, 1938.

[31] Thomas Schwartz. Rationality and the myth of the maximum. *Nos*, 1972.

[32] Amartya Sen. Social choice theory: A re-examination. *Econometrica*, 45(1):53, jan 1977. doi: 10.2307/1913287.

[33] Amartya K. Sen. Choice functions and revealed preference. *Review of Economic Studies*, 38(3):307–317, 1971.

[34] John H. Smith. Aggregation of preferences with variable electorate. *Econometrica*, 41 (6):1027, nov 1973. doi: 10.2307/1914033.

[35] Amos Tversky. Intransitivity of preferences. *Psychological Review*, 76(1):31–48, 1969. doi: 10.1037/h0026750.