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Quasilinear Elliptic Problems with multiple regions of singularities and convexities for the p(x)-Laplacian operator

by

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Resumo

Nesta tese estabelecemos resultados de existência, unicidade, multiplicidade e regularidade de soluções para a seguinte classe de problemas quasilineares que podem ser singulares envolvendo expoentes variáveis

$$\begin{cases} -\Delta_{p(x)}u = c(x)d(x)^{-\beta(x)}u^{-\alpha(x)} + \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega; \ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Na primeira parte, determinamos condições suficientes para existência de única solução em $W^{1,p(x)}_{loc}(\Omega)$ quando $f(x,t)$ é sublinear em $t = 0$ e $t = +\infty$ para todo $x \in \Omega$. Na segunda parte, obtemos multiplicidade de solução em $W_0^{1,p(x)}(\Omega)$ quando $f(x,t)$ é superlinear em $t = +\infty$ em algum subdomínio de Ω. Além disso, permitimos múltiplas regiões de singularidades, tanto no potencial quanto na não linearidade $u > 0$, enquanto que na segunda parte consideramos $\beta \equiv 0$. Provamos também um princípio de Comparação para sub e supersolução em $W^{1,p(x)}_{loc}(\Omega)$ para problemas sublineares em $t = 0$ e em $t = +\infty$ envolvendo o operador $p(x)-$Laplaciano.

Entre as técnicas utilizadas estão o Método de Galerkin; Técnica de regularização tipo Di Giorgi; Método de Sub-super solução e o Teorema do Passo da Montanha.

Palavras-chave: $p(x)-$Laplaciano, singular com expoentes variáveis, Princípio de Comparação, Regularidade de soluções

Abstract

In this thesis we establish results of existence, uniqueness, multiplicity and regularity of solutions for the following class of quasilinear problems that may be singular, involving variable exponents

$$\begin{cases} -\Delta_{p(x)}u = c(x)d(x)^{-\beta(x)}u^{-\alpha(x)} + \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega;\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

In the first part, we determined sufficient conditions for the existence of a unique solution in $W^{1,p(x)}_{loc}(\Omega)$ when $f(x,t)$ is sublinear in $t = 0$ and $t = +\infty$ throughout the domain. In the second part, we obtain multiplicity of solution in $W_0^{1,p(x)}(\Omega)$ when $f(x,t)$ is superlinear in $t = +\infty$ just in a subdomain of Ω in some subdomain of Ω. Besides this, we allow multiple regions of singularity, both for the potential and for the non-linearity $u > 0$, while in the second part we take $\beta \equiv 0$. In addition, we prove a Comparison principle for sub and supersolution in $W^{1,p(x)}_{loc}(\Omega)$ for sublinear problems in $t = 0$ and $t = +\infty$, involving the $p(x)-$Laplacian operator.

Among the techniques used are the Galerkin Method; the Di Giorgi regularization technique; the Sub-super solution method; the Mountain Pass Theorem.

Keywords: $p(x)-$**Laplacian, singular variable exponent, Comparison Principle, Regularity of Solutions**

Contents

Introduction

This work presents a study of questions related to existence, uniqueness, multiplicity and regularity of solutions for the following class of quasilinear problems

$$\begin{cases} -\Delta_{p(x)}u = a(x)d(x)^{-\beta(x)}u^{-\alpha(x)} + \lambda f(x,u) & \text{in } \Omega, \\ u > 0 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

involving variable exponents and powers, where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with smooth boundary, $\lambda \geq 0$ is a real parameter, $d(x) = \inf_{y\in\partial\Omega} |x-y|$, for $x \in \Omega$, is the standard distance function to the boundary of Ω, $p : \overline{\Omega} \to \mathbb{R}$ is a $C^1(\overline{\Omega})$-function that satisfies

$$1 < \min_{x\in\overline{\Omega}} p(x) = p_- \leq p_+ = \max_{x\in\overline{\Omega}} p(x) < N,$$

and $\Delta_{p(x)}$ stands for the $p(x)-$Laplacian operator, that is,

$$\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u).$$

When $p(x) \equiv p$ (a constant), we have the well known $p-$Laplacian operator. Because of the non-homogeneity of the $p(x)-$Laplacian, these kinds of problems are more delicate than ones with $p-$Laplacian. For example, the first eigenvalue of $p(x)-$Laplacian is zero in general, and only under some special conditions the positivity holds (see [32]).

By quoting [28], the study of variable exponents spaces appeared in the literature for the first time in 1931, in an article by Orlicz [58], but the field of variable exponent function spaces has witnessed an explosive growth in recent years. The developments in science lead to a period of intense study of variable exponent spaces. Also observed were problems related to modeling of so-called electrorheological fluids [63, 64], the study of thermorheological fluids [67] and image processing [19].

By going back to the problem (1), we notice that it exhibits a singular behavior at the origin when $\alpha(x) > 0$, that is, $s^{-\alpha(x)} \xrightarrow{s \to 0^+} +\infty$ for all $x \in \{\alpha(x) > 0\}$. Moreover, the weight $d(x)^{-\beta(x)}$ also presents a singular behavior near the boundary when $\beta(x) > 0$, that is, $d(x)^{-\beta(x)} \xrightarrow{d(x) \to 0} +\infty$ for all $x \in \{\beta(x) > 0\}$.

The study of singular problems relies mainly of their application to physical models such as non-Newtonian fluids [12], boundary layer phenomena for viscous fluids [11], chemical heterogenous [10] and e theory of heat conduction in electrically conducting materials [22].

Our objective in this work is exploit the variable exponent to study the problem (1) in two different ways. The first being when $f(x,t)$ is sublinear in $t = 0$ and $t = +\infty$ throughout the domain and $\alpha(x), \beta(x)$ allowing to change the signal. In second part, $f(x,t)$ is superlinear in $t = +\infty$ only in a subdomain of Ω, $\beta \equiv 0$ and $\alpha(x)$ allowing the signal to change.

Our work is divided into three chapters and two appendix. In **Chapter 1**, we are going to remember some definitions and results involving the Lebesgue and Sobolev spaces with variable exponents which will be used throughout this thesis.

In **Chapter 2** we present a Comparison principle to the problem

$$\begin{cases} -\Delta_{p(x)} u = g(x,u) \text{ in } \Omega, \\ u > 0 \ \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $p : \overline{\Omega} \to \mathbb{R}$ is a $C^1(\overline{\Omega})$-function that satisfies

$$1 < \min_{x \in \overline{\Omega}} p(x) = p_- \le p_+ = \max_{x \in \overline{\Omega}} p(x) < N,$$

and $g(x,t)$ fulfills the following conditions:

(g_1) $g : \Omega \times (0,\infty) \to [0,+\infty)$ is a function such that $t \mapsto g(x,t)$ is a continuous function a.e. $x \in \Omega$ and for each $t > 0$ the function $x \mapsto g(x,t)$ is mensurable,

(g_2) $t \mapsto \dfrac{g(x,t)}{t^{p_- - 1}}$ is strictly decreasing on $(0,\infty)$ for a.e. $x \in \Omega$,

(g_3) the functional $I_h : W_0^{1,p(x)} \to \mathbb{R}$, defined by

$$I_h(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_\Omega G_h(x,u) dx,$$

is coercive and weakly lower semicontinuous on $\{w \in W_0^{1,p(x)}(\Omega) \ / \ 0 \leq w \leq \overline{u}\}$ with respect to $W_0^{1,p(x)}(\Omega)$-norm, where

$$G_h(x,s) := \int_0^s g_h(x,t+h)dt \quad \text{and} \quad g_h(x,t) := g(x,t+h)$$

for each $h > 0$ given.

Brezis and Oswald [8] studied the semilinear case to the problem (2), that is,

$$\begin{cases} -\Delta u = g(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

with $g(x,t)$ satisfying:

$(BO)_1$ $t \mapsto g(x,t)$ is a continuous function a.e. $x \in \Omega$ and for each $t \geq 0$ the function $x \mapsto g(x,t)$ belongs to $L^\infty(\Omega)$ a.e. $x \in \Omega$,

$(BO)_2$ $t \mapsto \dfrac{g(x,t)}{t}$ is strictly decreasing on $(0,\infty)$ for a.e. $x \in \Omega$,

$(BO)_3$ there is a constant $C > 0$ such that $g(x,t) \leq C(1+t)$ for a.e. $x \in \Omega$ and $t \geq 0$.

Besides this, they introduce the extended functions

$$a_0(x) = \lim_{t\to 0^+} \frac{g(x,t)}{t} \quad \text{and} \quad a_\infty(x) = \lim_{t\to\infty} \frac{g(x,t)}{t} \quad \text{for } x \in \Omega$$

and the quantity

$$\lambda(a) = \inf_{v\in H^1(\Omega),\ ||v||_2=1} \left\{ \int_\Omega |\nabla v|^2 - \int_{\{v\neq 0\}} a|v|^2 dx \right\}$$

for any extended function $a : \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ given. With this set of hypotheses, they showed that the problem (3) has at most one weak solution. Moreover, a weak solution of (3) exists if and only if $\lambda(a_0) < 0 < \lambda(a_\infty)$. Later, Diaz and Saá [27] extended the result of Brezis and Oswald for $p-$Laplacian operator with $p > 1$ and similar hypotheses. The fundamental tool to prove the uniqueness of solution is the following inequality

$$\int_\Omega \left[|\nabla w_1^{\frac{1}{p}}|^{p-2} \nabla w_1^{\frac{1}{p}} \nabla \left(\frac{w_1 - w_2}{w_1^{\frac{p-1}{p}}} \right) - |\nabla w_2^{\frac{1}{p}}|^{p-2} \nabla w_2^{\frac{1}{p}} \nabla \left(\frac{w_1 - w_2}{w_2^{\frac{p-1}{p}}} \right) \right] dx \geq 0, \tag{4}$$

which became known as Diaz-Saá Inequality [27].

Going back to our problem, the hypotheses $(g_1)-(g_3)$ do not imply any growth restriction on g with respect to the variable t and allow $g(x,t)$ to have singular behavior at 0. Furthermore, the technical hypothesis (g_3) is not a "strange assumption". In fact, with the hypothesis $\lambda(a_\infty)>0$ considered in Brezis-Oswald [8] (Laplacian operator) or Diaz-Saa [27] ($p-$Laplacian operator) together with $(BO)_1-(BO)_3$ and $p(x)\equiv p$, lead us to show that functional I_h is coercive and sequentially weakly lower semicontinuous on $\{w\in W_0^{1,p(x)}(\Omega)\ /\ 0\le w\le \overline{u}\}$. On the other side, in a Sobolev variable exponent space, the amount

$$\inf_{v\in W^{1,p(x)},\ ||v||_{p(x)}=1}\left\{\int_\Omega |\nabla v|^{p(x)}-\int_{\{v\neq 0\}}\frac{a_\infty(x)|v|^{p(x)}}{p(x)}dx\right\} \tag{5}$$

may not be positive, see for instance [32]. We also point out that if we take $g(x,t)=a(x)/t^{\alpha(x)}$, for some $\alpha(x)>1-p_-$ and $a\in L^{r(x)}(\Omega)$ with a suitable choice of $r(x)\ge 1$, and, in particular, $p(x)$ has a strictly local minimum (or maximum) in Ω, then the infima in (5) is null, but the hypothesis $(g_1)-(g_3)$ are still satisfied.

Before enunciating our first result, we need of the next definition.

Definition 0.0.1 *We say that $\underline{u}\in W_{loc}^{1,p(x)}(\Omega)$ is a subsolution of* (2) *if $\underline{u}\ge 0$, $g(x,\underline{u})\in L_{loc}^1(\Omega)$ and*

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla\underline{u}\nabla\phi dx-\int_\Omega g(x,\underline{u})\phi dx\le 0,\ \forall\ \phi\in C_0^\infty(\Omega),\ \phi\ge 0.$$

Analogously, $\overline{u}\in W_{loc}^{1,p(x)}(\Omega)$ is a supersolution of (2) *if $\overline{u}\ge 0$, $g(x,\overline{u})\in L_{loc}^1(\Omega)$ and*

$$\int_\Omega |\nabla \overline{u}|^{p(x)-2}\nabla\overline{u}\nabla\phi dx-\int_\Omega g(x,\overline{u})\phi dx\ge 0,\ \forall\ \phi\in C_0^\infty(\Omega),\ \phi\ge 0.$$

Now, we have.

Theorem 0.0.2 *Assume that $(g_1)-(g_3)$ holds true for each $h>0$ given. If $\underline{u},\overline{u}$ are subsolution and supersolution for* (2)*, respectively, such that $\underline{u}\in L_{loc}^\infty(\Omega)$ with $(\underline{u}-\epsilon)^+\in W_0^{1,p(x)}(\Omega)$ for each $\epsilon>0$ given and $\operatorname*{ess\,inf}_{x\in U}\overline{u}(x)>0$ for each $U\subset\subset\Omega$, then $\underline{u}\le\overline{u}$ a.e. in Ω.*

The importance of our first result is principally because it may be applied to subsolutions and supersolutions just in $W_{loc}^{1,p(x)}(\Omega)$. The proof is quite technical, because we have to keep away from the boundary of Ω to avoid the possible singularity of $g(x,t)$ at $t=0$. The first part of our proof is inspired on ideas in [16] that show the comparison between a sub and a supersolution for a nonlocal and singular problem by truncating the singularity in an suitable way. The second part is inspired on ideas of [43], to take advantage to the convexity of the functional $J = J_K : L^1_{loc}(\Omega) \to (-\infty, \infty]$ be given by

$$J(u) = \begin{cases} \displaystyle\int_K \frac{|\nabla u^{\frac{1}{p_-}}|^{p(x)}}{p(x)} dx, & u \geq 0,\ u^{\frac{1}{p_-}} \in W^{1,p(x)}_{loc}(\Omega), \\ +\infty, \text{ otherwise}, \end{cases}$$

for each $K \subset\subset \Omega$ given to derive a Diaz-Saá type inequality

$$\int_K \left[|\nabla w_1^{\frac{1}{p_-}}|^{p(x)-2} \nabla w_1^{\frac{1}{p_-}} \nabla \left(\frac{w_1 - w_2}{w_1^{\frac{p_- - 1}{p_-}}} \right) - |\nabla w_2^{\frac{1}{p_-}}|^{p(x)-2} \nabla w_2^{\frac{1}{p_-}} \nabla \left(\frac{w_1 - w_2}{w_2^{\frac{p_- - 1}{p_-}}} \right) \right] dx \geq 0,$$

where $w_1, w_2 \in L^\infty_{loc}(\Omega) \cap \{u \in L^1_{loc}(\Omega) \ / \ u \geq 0,\ u^{\frac{1}{p_-}} \in W^{1,p(x)}_{loc}(\Omega)\}$ with $w_i/w_j \in L^\infty_{loc}(\Omega)$, $i \neq j$. To our knowledge, this result is new even for Laplacian operator.

In the **Chapter 3**, we study issues about existence, regularities and uniqueness of solutions to the problem (1). To enunciate these results, let us denote the δ-strip around to the boundary of Ω by

$$\Omega_\delta := \{x \in \Omega \ / \ d(x) < \delta\}$$

and consider the numbers

$$\theta_1 = \begin{cases} \displaystyle\max_{x \in \overline{\Omega}_\delta} \frac{p(x) - \beta(x)}{p(x) + \alpha(x) - 1} & \text{if } \beta(x) + \alpha(x) > 1 \text{ in } \Omega_\delta, \\ 1 & \text{if } \beta(x) + \alpha(x) \leq 1 \text{ in } \Omega_\delta, \end{cases} \qquad \theta_2 = \min_{x \in \overline{\Omega}_\delta} \frac{p(x) - \beta(x)}{p(x) + \alpha(x) - 1},$$

for each $\delta > 0$ given.

So, let us assume that there exists a $\delta > 0$ such that:

(H_1) $\alpha : \overline{\Omega} \to \mathbb{R}$ is a $C^{0,1}(\overline{\Omega})$-function that satisfies $\alpha(x) \geq \min_{x \in \overline{\Omega}} \alpha(x) := \alpha_- > 1 - p_-$,

(H_2) $f : \Omega \times [0, \infty) \to [0, \infty)$ is a Carathéodory function such that

$$f(x,t) \leq b(x)(1 + t^{q(x)-1}) \text{ for all } x \in \Omega$$

holds true, for some functions $q \in C^1(\overline{\Omega})$ and $0 \leq b \in L^{s(x)}(\Omega) \cap L^\infty(\Omega_\delta)$ with $1 < q_- \leq q_+ \leq p_-$ and $s(x) > N/p_-$ for $x \in \Omega$,

(H_3) (i) $\beta : \overline{\Omega} \to \mathbb{R}$ is a $C^{0,1}(\overline{\Omega})$-function that satisfies $\beta_+ < p_-$,
(ii) $0 < c \in L^{r(x)}(\Omega) \cap L^\infty(\Omega_\delta)$ for some $r \in C^1(\overline{\Omega})$ with $1 \leq r(x) \leq +\infty$,
(iii) $c(x)/(1 - \alpha(x)) \in L^{r(x)}(\Omega) \cap L^\infty(\Omega_\delta)$,

(H_4) $\dfrac{f(x,t)}{t^{p_- -1}}$ is strictly decreasing on $(0,\infty)$ for a.e. $x \in \Omega$.

Under the condition (H_1), the problem (1) may be singular at $u = 0$ in multiple regions of the domain. For example, if $\Omega = B_R(0)$ is the ball centered at origin of $\mathbb{R}^N$ with radius $R = 10\pi$, then the problem (1) oscillates from singular in the rings $B_{(2k+1)\pi}(0)\backslash B_{(2k)\pi}(0)$ to non-singular one in $B_{(2k)\pi}(0)\backslash B_{(2k-1)\pi}(0)$ for $k = 1, \cdots, 5$. Beside this, we allow the signal of $\alpha(x)$ oscillates from a sub-linearity $(1-p_- < \alpha(x) \leq 0)$ passing through an weak singularity $(0 < \alpha(x) \leq 1)$, to reach a strong singularity $(\alpha(x) \geq 1)$ both in the domain and its boundary.

Before sharing our principal results, we here briefly recall the literature about related singular problems. Crandall, Rabinowitz and Tartar [24] have considered a class of singular problems which included, as special model,

$$-\Delta u = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{6}$$

for some $0 < a \in C^1(\overline{\Omega})$ and $\alpha > 0$ being a real number, showing not only existence of classical and weak solutions but also some boundary regularity.

A broad literature on problems like (6) is available to this date. Since then, many authors have considered the above problem with other operators.

In a famous paper, Lazer and McKenna [51] studied the problem

$$-\Delta u = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{7}$$

where $a \in C(\overline{\Omega})$ with $a > 0$ in $\overline{\Omega}$, and $\alpha > 0$ is a real constant. They proved that (7) has a solution in $H_0^1(\Omega)$ if and only if $0 < \alpha < 3$, while for $\alpha > 1$ the solutions are not in $C^1(\overline{\Omega})$. An extension of the Lazer and McKenna's obstruction was proved by Zhang and Cheng [72] when $a(x)$ is like $d(x)^\beta$ with $\beta \in \mathbb{R}$ (i.e., $\exists\ c, C > 0$ s.t. $cd(x)^\beta \leq a(x) \leq Cd(x)^\beta$ on $\overline{\Omega}$), where they showed that (7) has a solution still in $H_0^1(\Omega)$ if, and only if, $\alpha - 2\beta < 3$.

Boccardo and Orsina [6], by combining the technique of truncation with some necessary apriori estimates on the solutions of the corresponding approximation problem, showed existence and regularity results for

$$-div\,(M(x)\nabla u) = \frac{a(x)}{u^\alpha} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\alpha > 0$ is a real constant, and $0 < a \in L^r(\Omega)$. In particular, they showed that if $\alpha \leq 1$ and $r = 2^*/(2^* + \alpha - 1)$, then their solution $u \in H_0^1(\Omega)$, while $u \in H_{loc}^1(\Omega)$ if $\alpha > 1$ and $r = 1$. Following these ideas, Chu, Gao and Gao [20] have generalized the main result in [6] for the case when $\alpha > 0$ is a variable power, by considering three cases: $0 < \alpha_- < \alpha_+ < 1$, $\alpha_- < 1 < \alpha_+$ and $1 < \alpha_- < \alpha_+$ in Ω.

Carmona and Aparicio [17] also considered $\alpha > 0$ as a variable power that may have a region inside Ω with $\alpha(x) \leq 1$ and another one with $\alpha(x) > 1$. They proved the existence of solution in $H_0^1(\Omega)$ when $\alpha(x) \leq 1$ in a strip around the boundary and belongs to the $H_{loc}^1(\Omega)$ with zero on the boundary in a general sense for the other cases. Most of these results was generalized for different operators. We would like to mention [7, 20, 26, 57] and their references.

Results for $p(x)-$Laplace equations with pure singular non-linearity have been recently explored. In [71], Zhang has studied the problem

$$-\Delta_{p(x)} u = \frac{\lambda}{u^{\alpha(x)}} \quad \text{in} \quad \Omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

with $\alpha(x) > 0$. By using the sub-solution method, he has obtained the existence result of solutions in $W_{loc}^{1,p(x)}(\Omega) \cap C(\overline{\Omega})$ and has presented an asymptotic behavior of these positive solutions when $\lambda > 0$ is large enough. The same author in [71] has improved his above existence result by considering the problem

$$-\Delta_{p(x)} u = \lambda K(x) f(x, u) + \beta u^{\gamma(x)} \quad \text{in} \quad \Omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

where $\gamma \in C(\overline{\Omega})$ with $\gamma(x) < p_-$, $f(x,t) \in C(\overline{\Omega} \times (0, \infty), (0, \infty))$ is a decreasing and singular function at the origin, $0 \lneq K \in L^{q(x)}$ for some $q(x) > N$, and $\lambda, \beta > 0$ are real parameters.

After Lazer and Mackenna [51], it is well known that our problem may not have solutions with zero-boundary value in the sense of the trace function. Along this chapter, we are going to consider the next one.

Definition 0.0.3 *Let $u \in W_{loc}^{1,p(x)}(\Omega)$. We say that $u \leq 0$ on $\partial\Omega$ if $(u-\epsilon)^+ \in W_0^{1,p(x)}(\Omega)$ for every $\epsilon > 0$ given. Furthermore, we also say that $u \geq 0$ on $\partial\Omega$ if $-u \leq 0$ on $\partial\Omega$, and $u = 0$ on $\partial\Omega$ if $u \leq 0$ and $u \geq 0$ on $\partial\Omega$, simultaneously.*

It is readily seen that if $u \in W_0^{1,p(x)}(\Omega)$, then $u = 0$ on $\partial\Omega$ in the sense of above definition. Moreover, for each small $\delta > 0$ given, the function

$$u(x) = \begin{cases} \sigma d(x)^{\theta} \ \text{ if } d(x) < \delta, \\ \sigma\delta^{\theta} + \displaystyle\int_{\delta}^{d(x)} \sigma\theta\delta^{\theta-1}\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p_- - 1}} dt \ \text{ if } \delta \leq d(x) < 2\delta, \\ \sigma\delta^{\theta} + \displaystyle\int_{\delta}^{2\delta} \sigma\theta\delta^{\theta-1}\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p_- - 1}} dt \ \text{ if } 2\delta \geq d(x), \end{cases}$$

does not belong to $W_0^{1,p(x)}(\Omega)$ if $\theta > 1 - 1/p_+$, but $(u - \epsilon)^+ \in W_0^{1,p(x)}(\Omega)$ for each $\epsilon > 0$ given.

Inspired by the ideas in [17], for each $\Gamma \subset \partial\Omega$ smooth enough, and $h \in C^1(\overline{\Omega})$ given, let us denote by

$$W_{\Gamma}^{1,h(x)}(\Omega) = \left\{u \in W^{1,h(x)}(U) \ / \ u\big|_{\Gamma} = 0 \ \text{ in the trace sense}\right\}$$

for all open set $U \subseteq \Omega$ such that $\overline{U} \cap \partial\Omega = \Gamma$. In particular, we notice that

$$W_{\Gamma}^{1,h(x)}(\Omega) = \begin{cases} W_{loc}^{1,h(x)}(\Omega) \text{ if } \Gamma = \emptyset, \\ \\ W_0^{1,h(x)}(\Omega) \text{ if } \Gamma = \partial\Omega. \end{cases}$$

We notice that the trace over Γ is well defined if, for example, Ω is Lipschitz continuous (see [28, Chapter 12])

Definition 0.0.4 *A positive function $u \in W_{\Gamma}^{1,p(x)}(\Omega)$ is a solution to problem* (1) *if $u = 0$ on $\partial\Omega$ in the sense of Definition* 0.0.3, *and*

(i) $c(x)d(x)^{-\beta(x)}u(x)^{-\alpha(x)} \in L^1_{loc}(\Omega)$,

(ii) $\operatorname*{ess\,inf}_{x\in K} u(x) > 0$ *for all* $K \subset\subset \Omega$,

(iii) $\displaystyle\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla\phi dx = \int_{\Omega} c(x)d(x)^{-\beta(x)}u^{-\alpha(x)}\phi dx + \lambda\int_{\Omega} f(x,u)\phi dx, \ \forall\ \phi \in C_0^{\infty}(\Omega).$

From the results in [6, 17, 26, 51, 72], it is reasonable to expect that (1) admits a solution that fulfills the boundary datum in the sense of the trace function when the trio $(c(x), \alpha(x), \beta(x))$ satisfies some "compatibility condition". For this reason, let us consider the $C^{0,1}$-manifold

$$\Gamma_t = \{x \in \partial\Omega \ / \ [-\beta(x) + t(1 - \alpha(x))]\frac{1}{1 - 1/r(x)} + 1 > 0\},$$

and the number

$$\sigma = \max\left\{\frac{p_- + (\beta_+ - 1)/\theta_2 + \alpha_+ - 1}{p_-}, \frac{p_- + \alpha_+ - 1}{p_-}\right\}.$$

Our first result is related to existence of solutions and it is formulated as follows.

Theorem 0.0.5 *Assume* $(H_1)-(H_4)$. *If*

$$r(x)=\begin{cases}\left(\frac{\sigma p_-^*}{p_-(\sigma-1)+1-\alpha(x)}\right)' & \textit{if } |\beta(x)+\alpha(x)>1|>0 \textit{ in } \Omega_\delta,\\ \left(\frac{p^*(x)}{1-\alpha_-}\right)' & \textit{if } |\beta(x)+\alpha(x)>1|=0 \textit{ in } \Omega_\delta,\end{cases}$$

then there exists a $0<\lambda^*\leq\infty$ *such that the problem* (1) *admits a solution* $u=u_\lambda\in W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$ *with* $u(x)\geq Cd(x)$, $x\in\Omega$ *for each* $0\leq\lambda<\lambda^*$ *given. In addition:*

(i) *if* $q_+<p_-$ *in* (H_2), *then* $\lambda^*=\infty$,

(ii) *if* $c(x)\geq c_\delta$ *in* Ω_δ *for some* $c_\delta>0$, *then* $u(x)\geq Cd(x)^{\theta_1}$ *for* $x\in\Omega_\delta$ *and* $u\in W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega)$.

The variable exponents considered on our problem implied two integrability conditions when were seeking solutions still in $W_0^{1,p(x)}(\Omega)$. This happened because we were not able to show the boundedness $C_1d(x)^{\theta(x)}\leq u(x)\leq C_2d(x)^{\theta(x)}$ a.e. $x\in\overline{\Omega}$ for $\theta(x)=(p(x)-\beta(x))/(p(x)+\alpha(x)-1)$ for $x\in\overline{\Omega}$. For p,β and α constants and for some particular cases, this inequality is true (see for instance Bougherara, Giacomoni and Hernandez [7]). In fact, we prove (see Proposition 3.2.6) that $C_1d(x)^{\theta_1}\leq u(x)\leq C_2d(x)^{\theta_2}$ for $x\in\overline{\Omega_\delta}$ for c like that one considered in [7]. For a general c, we proved an inequality with 1 in the place of θ_1 (see Propositions 3.2.5 and 3.2.6). This fact has great influence in the final shape of the solution, that is, in this case the solution u belongs just to $W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$, where

$$W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)=\begin{cases}W^{1,p(x)}_{\Gamma_1}(\Omega)\text{ if }(\alpha(x)-1)(\beta(x)+\alpha(x)-1)\leq 0,\\ \\ W^{1,p(x)}_{\Gamma_{\theta_2}}(\Omega)\text{ if }(\alpha(x)-1)(\beta(x)+\alpha(x)-1)>0,\end{cases}$$

that is, $W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)=W_0^{1,p(x)}(\Omega)$ if and only if

$$\alpha(x)<\max\left\{2-\frac{1}{r(x)}-\beta(x),1+\frac{1}{\theta_2}\left(1-\frac{1}{r(x)}-\beta(x)\right)\right\}\text{ for all }x\in\partial\Omega. \tag{8}$$

More, as claimed in Theorem 0.0.5, if $c(x)\geq c_\delta$ in Ω_δ for some $c_\delta>0$, then

$$u\in W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega)=\begin{cases}W^{1,p(x)}_{\Gamma_{\theta_1}}(\Omega)\text{ if }\alpha(x)\leq 1,\\ \\ W^{1,p(x)}_{\Gamma_{\theta_2}}(\Omega)\text{ if }\alpha(x)>1,\end{cases}$$

that is, $u \in W_0^{1,p(x)}(\Omega)$ if either

$$\alpha(x) \leq \min\left\{1, 1+\frac{1}{\theta_1}\left(1-\frac{1}{r(x)}-\beta(x)\right)\right\}$$
$$\text{or } 1<\alpha(x)<1+\frac{1}{\theta_2}\left(1-\frac{1}{r(x)}-\beta(x)\right) \text{ for all } x \in \partial\Omega. \tag{9}$$

Returning to constant exponents that were treated in literature up to now, it follows from Theorem 0.0.5 and above informations that the solution u still belongs to $W_0^{1,p(x)}(\Omega)$ if either $W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega) = W_0^{1,p(x)}(\Omega)$ or $W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega) = W_0^{1,p(x)}(\Omega)$. In particular, Theorem 0.0.5-(ii) includes the main results found in the literature about this issue up to now:

(i) (Lazer and McKenna - 1991 [51]) Let $p(x)\equiv 2$, $\alpha(x)=\alpha$, $\beta(x)\equiv 0$ and $c(x)\in C^1(\overline{\Omega})$ with $c>0$ in $\overline{\Omega}$. Thus, it follows from (8) and (9), that

$$W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega) = W_0^{1,p(x)}(\Omega) \Leftrightarrow 0<\alpha<3,$$

(ii) (Zhang and Cheng - 2004 [72]) Let $p(x)\equiv 2$, $\alpha(x)=\alpha$, $\beta(x)=\beta\in(0,2)$ and $c(x)=c$ with $c>0$ in $\overline{\Omega}$. Thus,

$$W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega) = W_0^{1,p(x)}(\Omega) \Leftrightarrow 0<\alpha<3-2\beta,$$

(iii) (Mohammed - 2009 [55]) Let $p(x)\equiv p$, $\alpha(x)=\alpha$, $\beta(x)\equiv 0$ and $c(x)\in L^\infty(\overline{\Omega})$ with $c>0$ in $\overline{\Omega}$. Thus,

$$W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega) = W_0^{1,p(x)}(\Omega) \Leftrightarrow 0<\alpha<\frac{2p-1}{p-1},$$

(iv) (Giacomoni, Bougherara and Hernandez - 2015 [7]) Let $p(x)\equiv p$, $\alpha(x)=\alpha$, $\beta(x)<p$ and $c(x)\in L^\infty(\overline{\Omega})$ with $c>0$ in $\overline{\Omega}$. Thus,

$$W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega) = W_0^{1,p(x)}(\Omega) \Leftrightarrow \alpha<\frac{2p-1}{p-1}-\frac{p}{p-1}\beta,$$

(v) (Yijing and Duanzhi - 2013 [66]) The problem

$$\begin{cases} -\Delta_{p(x)}u = c(x)d(x)^{q(x)}u^{-1-q(x)}, & \Omega \\ u=0, & \partial\Omega \end{cases} \tag{10}$$

possesses a solution in $W_0^{1,p(x)}(\Omega)$ for any $0<c\in L^1(\Omega)$ and $0<q\in C^{0,1}(\overline{\Omega})$ due to (8). It includes the example presented in [66] that considered $p(x)=q(x)=2$ and $c(x)=c>0$ for some real constant $c>0$.

The proof of Theorem 0.0.5 relies on a generalized Galerkin Method, which consists in finding one solution to a "regularized problem" and then to perform an apriori uniform estimates. From the hypothesis (H_1), the term $u^{\alpha(x)}$ may not be monotone in $u > 0$ anymore, this makes it more difficult to show uniform positivity estimates for an approximate sequence in the interior of Ω (condition (ii) in Definition 0.0.4).

The importance of our next result is principally because it may be applied to different types of problems, namely: purely singular with weak singularity (i.e., $f(x,t) \equiv 0$ and $0 < \alpha(x) \leq 1$), purely singular with strong singularity (i.e., $f(x,t) \equiv 0$ and $\alpha(x) > 1$), singular-sublinear ($\alpha(x) > 0$), purely sublinear ($\alpha(x) < 0$) and to oscillated problems (i.e. $\alpha(x)$ changing its signal). Moreover, since $c(x)d(x)^{-\beta(x)}$ may not lie in $L^1(\Omega)$, we emphasize that the analysis of the behavior of the trio (c, α, β) only near the boundary is very essential.

The third result deals with regularity of solutions for Problem (1). It is stated as a combination of Theorem 3.1.4 and the Corollary 3.1.5 of the Chapter 3.

Theorem 0.0.6 *Assume* $(H_1)-(H_4)$. *Let* $u \in W^{1,p(x)}_{\Gamma_1 \cup \Gamma_{\theta_2}}(\Omega)$ *be the solution for* (1) *given by Theorem* 0.0.5. *Then there exists a* $0 < \lambda_* \leq \infty$ (λ_* *is possibly less than* λ^* *given in Theorem* 0.0.5) *such that for all* $0 \leq \lambda < \lambda_*$, *we have:*

(i) $u \in L^\infty(\Omega)$ *if* $r(x) > N/p_-$,

(ii) $u \in C^{0,\gamma}(\overline{U})$ *for all open set* $U \subseteq \Omega$ *such that* $\overline{U} \cap \partial\Omega = \Gamma_1 \cup \Gamma_{\theta_2}$ *if* $r(x) > N/p_-$,

(iii) $u \in L^{\frac{Nr_-(p_-+\alpha_--1)}{N-r_-p_-}}(\Omega)$ *if* $|\beta(x) + \alpha(x) > 1| > 0$ *in* Ω_δ *and* $r_- < N/p_-$ *with and*

$$\max\left\{\frac{N(p_- + \alpha_+ - 1)}{(N-p)(p_- + \alpha_- - 1) + p_-(p_- + \alpha_+ - 1)}, \frac{N(p_- + \frac{\beta_+ - 1}{\theta_2} + \alpha_+ - 1)}{(N-p)(p_- + \alpha_- - 1) + p_-(p_- + \frac{\beta_+ - 1}{\theta_2} + \alpha_+ - 1)}\right\} \leq r_-,$$

(iv) $u \in L^{\frac{Nr_-(p_-+\alpha_--1)}{N-r_-p_-}}(\Omega)$ *if* $|\beta(x) + \alpha(x) > 1| = 0$ *in* Ω_δ *and*

$$\frac{Np_-}{Np_- - (N-p_-)(1-\alpha_-)} \leq r_- < \frac{N}{p_-}.$$

for $\delta > 0$ *as in Theorem* 0.0.5. *In addition, if* $q_+ < p_-$ *in* (H_2), *then* $\lambda_* = \infty$. *Besides this, the same conclusions hold true if we change* Γ_1 *for* Γ_{θ_1}.

The boundedness and regularity of the solutions depending on the trio (c, α, β) have been considered in [6, 17, 26] for particular cases. We establishes similar results and prove the Hölder continuity adopting the method of De Giorgi developed by Ladyzhenskaya and Ural'tseva and derived the suitable Caccioppoli type inequality [31].

To conclude, we present a sufficient condition for uniqueness of solution for 1.

Theorem 0.0.7 *Assume that* $(H_1)-(H_4)$ *holds true with* $r(x) > N/p_-$. *If* $\beta(x) < 1$ *on* $\partial\Omega$, *then there exists a* $0 < \lambda_{**} \leq \infty$ (λ_{**} *is possibly less than* λ^* *given in Theorem* 0.0.5) *such that for all* $0 \leq \lambda < \lambda_{**}$ *the problem* (1) *admits an* ***only*** *solution in* $W^{1,p(x)}_{loc}(\Omega)$ *in sense of Definition* 0.0.4. *Beside this,* $\lambda_{**} = +\infty$ *if* $p_- = q_+$.

Theorem 0.0.2 will be fundamental to prove uniqueness of $W^{1,p(x)}_{loc}(\Omega)$-solutions for (1). In the last years, some papers have proved uniqueness of solutions for purely-singular problems for different operators. The semilinear case $p(x) \equiv 2$ and $g(x,t) = t^{-\alpha}$, for $\alpha > 0$ being a real constant, was proved in [14]. This result was recently extended for $g(x,t) = a(x)t^{-\alpha}$ with $0 \lneq a(x) \in L^1(\Omega)$ for $p(x) = 2$ in [13], for $p(x) \equiv p$ in [15], while in [16] the fractional $p-$Laplacian operator was considered. Our result generalizes and complements these results, since we consider the variable exponent $p(x)$-Laplacian and the variable power $\alpha(x)$, which can oscillates from negative to positive values. That is, we do not require $g(x,t)$ being monotone and do not impose $g(x,t) = f(x)h(t)$.

As a novelty in this chapter, we point out that we took advantage as most as possible of the variability of the exponents and powers. As a consequence of this, we have shown that the difficulty in answering the principal issues about this kind of problem is concentrated in understanding the behaviors of the powers and exponents just near to the boundary of the domain where the singularity is really triggered for Dirichlet boundary conditions problems. For instance, the "integrability condition" of trio (c, α, β) only near the boundary of the domain is sufficient to obtain existence of solutions still in $W^{1,p(x)}_0(\Omega)$. We conjecture that the converse claim is true as well.

In **Chapter 4**, we study the problem

$$\begin{cases} -\Delta_{p(x)} u = a(x)u^{-\alpha(x)} + \lambda f(x,u) & \text{in } \Omega, \\ u > 0 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{11}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $0 < a \in L^{r(x)}(\Omega)$ for some $r \in C(\overline{\Omega})$, $p \in C^1(\overline{\Omega})$, $\lambda > 0$ is a real parameter and $f(x,t)$ has a superlinear local behavior at $t = +\infty$ to be presented below.

Coclite and Palamieri [21] consider the problem

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda u^p = g(x,u) \text{ in } \Omega, \\ u > 0 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{12}$$

with $\alpha, p > 0$ and $\lambda > 0$ and showed that there exists $\lambda^* > 0$ such that (12) has a solution for all $0 < \lambda < \lambda^*$ and no classical solution for $\lambda > \lambda^*$.

Long, Sun and Wu [53] studied (12) with $0 < \alpha < 1$ and $1 < p < 2^* - 1$ to obtain the existence of a $\lambda^* > 0$ such that (12) has at least two weak solutions for all $0 < \lambda < \lambda^*$. Later, Sun and Wu [54] returned to the problem (12) and obtained an exact result value for $\lambda^* > 0$. After these works, a broad literature has been accumulating in relation to Laplacian operator with $g(x,s)$ in different kinds of hypotheses, see for instance [1, 44, 46, 47] and their references.

More general operators have been considered recently, as well. For the $p-$Laplacian

$$\begin{cases} -\Delta_p u = a(x) u^{-\alpha} + \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{13}$$

Perera and Zhang [62] obtained multiplicity of solutions for (13) combining a cutoff argument, variational methods, results relating to $W^{1,p}$ versus C^1 minimizers, with $p \geq 2$, $\alpha > 0$ and $f(x,t)$ satisfying the classical Ambrosetti-Rabinowitz condition and

(H_α) there are $\phi_0 \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > n$ such that $a\phi_0^{-\alpha} \in L^q(\Omega)$.

Later, Perera and Silva [60] improved above result by considering stronger hypotheses. For instance, they did not assume $p \geq 2$ or any stronger regularity assumption on f. Giacomoni, Schindler and Takac [41] established the global multiplicity results of the above problem for a certain range of λ by considering $0 < \alpha < 1$ and $f(x,t) = t^q$. Still related to problem 13, we may also cite [39, 59, 61] and their references.

In context of $p(x) - Laplacian$, to our knowledge, there exists few works treating the problem like (11). Byun and Ko [9] and Ghamni and Saoudi [40]

$$\begin{cases} -\Delta_{p(x)} u = \lambda u^{-\alpha(x)} + f(x,u) \text{ in } \Omega, \\ u > 0 \ \text{ in } \Omega, \qquad u = 0, \ \text{ on } \partial\Omega, \end{cases} \tag{14}$$

improved the principal result in [41] by considering variable $0 < \alpha(x) < 1$ and $f(x,t)$ superlinear at $+\infty$ throughout the domain. Their proof is variational and the fundamental tool used in their approach is an extension for the $p(x)-$Laplacian context of the local minimization C^1 versus $W_0^{1,p(x)}$.

To state ours results, let us first define a solution to Problem (11).

Definition 0.0.8 *A positive function $u \in W_0^{1,p(x)}(\Omega)$ is a solution to (4.1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \phi dx = \int_\Omega a(x)u^{-\alpha(x)}\phi dx + \lambda \int_\Omega f(x,u)\phi dx$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$.

Besides this, let us remind that $\Omega_\delta := \{x \in \Omega \ / \ d(x) < \delta\}$, for each $\delta > 0$, stands for the interior δ-strip around the boundary of the domain,

$$\Gamma_t = \{x \in \partial\Omega \ / \ [t(1-\alpha(x))]\frac{1}{1-1/r(x)} + 1 > 0\}, \text{ for } t \in \{1, \theta_1, \theta_2\},$$

is a subset of the boundary of the domain and the numbers θ_1 and θ_2 are defined by

$$\theta_1 = \begin{cases} \max\limits_{x \in \overline{\Omega}_\delta} \dfrac{p(x)}{p(x)+\alpha(x)-1} & \text{if } \alpha(x) > 1, \\ 1 & \text{if } \alpha(x) \leq 1, \end{cases} \quad \text{and} \quad \theta_2 = \min_{x \in \overline{\Omega}_\delta} \frac{p(x)}{p(x)+\alpha(x)-1}.$$

Related to the functions $\alpha(x), a(x)$ and $f(x,t)$, we make the following general assumptions. Assume that there exists a $\delta > 0$ such that:

(H_1) $\alpha : \overline{\Omega} \to \mathbb{R}$ is a $C^{0,1}(\overline{\Omega})$-function that satisfies $\alpha_- > 1 - p_-$,

(H_2) $0 < a \in L^{r(x)}(\Omega)$ with $r(x) > N/p_-$ and one of the items below:

- (i) $a \in L^\infty(\Omega_\delta)$ and $\Gamma_1 \cup \Gamma_{\theta_2} = \partial\Omega$,
- (ii) $a(x) \geq a_\delta > 0$ in Ω_δ, $a \in L^\infty(\Omega_\delta)$ and $\Gamma_{\theta_1} \cup \Gamma_{\theta_2} = \partial\Omega$,

(H_3) $\frac{a(x)}{1-\alpha(x)} \in L^{r(x)}(\{\alpha(x) \neq 1\})$.

(f_1) $f : \Omega \times [0,\infty) \to [0,\infty)$ is a Caratheodory function such that for each $M > 0$ given there exists $c_1 = c_1(M) > 0$ satisfying

$$0 \leq f(x,s) \leq c_1 \text{ for every } 0 \leq s \leq M \text{ and a.e. } x \text{ in } \Omega,$$

where the last hypothesis was inspired on a hypotheses in [60].

Our first result is.

Theorem 0.0.9 *Suppose* (H_1), (H_2) *and* (f_1) *are satisfied. Then there exist* $\lambda_0 > 0$ *such that the problem* (11) *has a positive weak solution* $u_\lambda \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ *for each* $0 < \lambda < \lambda_0$ *given satisfying* $u_\lambda \geq m_0 d(x)$ *in* Ω *for some* $m_0 > 0$. *In addition, there exists* $M_0, M_1, m_1 > 0$ *such that:*

(i) $m_0 d(x) \leq u_\lambda \leq M_0 d(x)^{\theta_2}$ *for* $x \in \Omega_\delta$ *if* $(H_2)(i)$ *holds,*

(ii) $m_1 d(x)^{\theta_1} \leq u_\lambda \leq M_1 d(x)^{\theta_2}$ *for* $x \in \Omega_\delta$ *if* $(H_2)(ii)$ *holds.*

We can also consider a setting in what $f(x,s)$ is allowed to change its signal and is bounded from below by integrable functions on bounded intervals of the variable $s > 0$, that is:

(f_2) $f : \Omega \times [0, \infty) \to \mathbb{R}$ is a Caratheodory function such that for each $M > 0$ given there exists $c_2 = c_2(M) > 0$ and $0 \leq h = h_M \in L^1(\Omega)$ satisfying

$$-h(x) \leq f(x,s) \leq c_2 \text{ for every } 0 \leq s \leq M \text{ and a.e. } x \in \Omega,$$

(f_3) there are $\zeta > 0$ and $c_3 > 0$ such that

$$f(x,s) \geq -c_3 a(x) \text{ for every } 0 \leq s \leq \zeta \text{ and a.e. } x \in \Omega.$$

Our second result is.

Theorem 0.0.10 *Suppose* (H_1), (H_2), (f_2) *and* (f_3) *are satisfied. If* $\alpha(x) \geq 0$ *in* $\overline{\Omega}$ *with* $\alpha(x) < 1$ *on* $\partial\Omega$, *then there exist* $\lambda_1 > 0$ *such that the problem* (11) *has a positive weak solution* $u_\lambda \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ *for each* $0 < \lambda < \lambda_1$ *given satisfying* $u_\lambda \geq C d(x)$ *in* Ω *for some* $C > 0$.

The proofs of Theorem 0.0.9 and 0.0.10 relied upon finding a sub and a supersolution, say $\underline{u}, \overline{u}$, for (11) in $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, minimizing an appropriated C^1 energy functional in $W_0^{1,p(x)}(\Omega)$ and showing that this minimum belongs to the cone $[\underline{u}, \overline{u}]$ and is an weak solution for (11). To do this, the results obtained in Chapter 3 were determinants, since $\underline{u}$ and $\overline{u}$ are solutions for an appropriated singular-sublinear problems. More, the Comparison principle demonstrated in Chapter 2 was a fundamental tool used to show that $\underline{u} \leq \overline{u}$. However, when $f(x,t)$ is allowed to change its signal, the restriction $\alpha(x) \geq 0$ was necessary in order to obtain a comparison between the sub and supersolution.

In order to establish the existence of at least two solutions for the problem (11), we also assume:

(f_4) there exists $C > 0$ such that

$$|f(x,t)| \leq C(1 + t^{q(x)-1}) \text{ for } t > 0 \text{ and a.e. } x \in \Omega,$$

with $1 < q \in C(\overline{\Omega})$ and $p_+ < q_+ < p_-^*$,

(f_5) there exists a subdomain $\emptyset \neq D \subset \Omega$ such that

$$\lim_{t\to\infty} \frac{F(x,t)}{t^{p_+}} = +\infty \text{ uniformly on } x \in D,$$

where $F(x,t) = \int_0^t f(x,s)ds$ for $t > 0$ and $x \in \Omega$,

(f_6) there exist $t_0, \beta_0 \geq 0$ and $\tau \in C(\overline{\Omega})$ with $1 < \tau(x) < p_-$, $x \in \overline{\Omega}$ such that

$$p_+F(x,t) - f(x,t)t \leq \beta_0 t^{\tau(x)} \text{ for all } t > t_0 \text{ and a.e. } x \in \Omega.$$

This set of hypotheses was inspired in [37]. Note that (f_4) does not impose $q(x) > p(x)$ in Ω and (f_5) implies that $f(x,s)$ is just locally $(p_+ - 1)-$superlinear at infinity just in D, that is,

$$\lim_{t\to+\infty} f(x,t)/t^{p_+-1} = +\infty \text{ uniformly } in\ D.$$

The hypothesis (f_6), as pointed out in [37] for constants functions $s(x)$ and $q(x)$, is a weaker form of the classical condition of Ambrosetti-Rabinowitz. For instance, the function

$$F(x,t) = b(x)t^{s(x)} + c(x)t^{q(x)}$$

with $1 < s_- \leq s_+ < p_- \leq p_+ < q_+ < p_-^*$, $b \geq 0$, $c \in \mathbb{R}$, satisfies $(f_4) - (f_6)$, but do not satisfy Ambrosetti-Rabinowitz condition if $b \equiv 0$ in D and $c \equiv 0$ in some $K \subset \Omega/D$.

Theorem 0.0.11 *Suppose $(H_1) - (H_3), (f_4) - (f_6)$ are satisfied. There exists $\lambda_* > 0$ such that the problem (11) has at least two different solutions $u_\lambda, v_\lambda \in W_0^{1,p(x)}(\Omega)$ for each $0 < \lambda < \lambda_*$ given. In addition, $u_\lambda \leq v_\lambda$ and u_λ has negative energy while v_λ is a positive energy solution.*

The proof of Theorem 0.0.11 rely heavily on perturbation arguments and on the variational method employed by Perera and Silva [60]. We verify that the cut-off functional associated with the problem satisfy the geometric hypotheses of the Mountain Pass Theorem and the Cerami condition. However, the changing of signal $\alpha(x)$ provides an obstacle in estimates and, with our set of hypotheses, we do not know that the solutions (11) belongs to C^1, which prevent us use results relating of $W^{1,p(x)}$ versus C^1 minimizers.

The importance of our result, related to other works involving variable exponents, is principally due to the fact that we do not impose $0 < \alpha(x) < 1$ in $\overline{\Omega}$ when $f(x,t)$ is nonnegative and we just demand $\alpha(x) < 1$ on $\partial\Omega$ if $f(x,t)$ is allowed to change its signal. Moreover, we do not require in the hypothesis (f_4) that $q_- > p_+$ in whole Ω, as in the former works. We take advantage of the result obtained in Chapter 3 to obtain the existence of solutions in $W_0^{1,p(x)}(\Omega)$ with a different hypothesis that (H_α), as considered in [39, 60, 62], namely, $ad^{t(1-\alpha(x))} \in L^1(\Omega_\delta)$, for $t = \{1, \theta_1, \theta_2\}$ given in Theorem 0.0.5. Besides these, the results obtained in Chapter 4 complements the results in Chapter 3 in the sense that a local superlinearity at infinity of $f(x,t)$ implies multiplicity of solutions, in contrast to uniqueness obtained if $f(x,t)$ is sublinear in whole Ω.

In order to make the chapters self-sufficient, we will state once again, in each chapter, the main results as well as the problems and hypotheses considered in the introduction.

Notation and Terminology

- C and C_i denote positive constants.
- $\mathbb{R}^N$ denote the N-dimensional Euclidean Space.
- $B_r(x)$ is the open ball centered in x radius $r > 0$.
- If $\Omega \subset \mathbb{R}^N$ is Lebesgue mensurable, then $|\Omega|$ denote the Lebesgue measure of Ω.
- The notation $x_n \to x$ mean strongly convergence.
- The notation $x_n \rightharpoonup x$ mean weakly convergence.
- $X \hookrightarrow Y$ denote that X is continuously embedded in Y.
- $X \hookrightarrow\hookrightarrow Y$ denote that X is compactly embedded in Y.
- If $u : \Omega \to \mathbb{R}$ is mensurable, then $u^- = -\min\{u(x), 0\}$ and $u^+ = \max\{u(x), 0\}$ denote the negative and positive part, respectively.
- If $u : \Omega \to \mathbb{R}$ is mensurable, then $u_- = \operatorname*{ess\,inf}_{\Omega} u(x)$ and $u_+ = \operatorname*{ess\,sup}_{\Omega} u(x)$.
- $d(x) = \inf_{y \in \partial\Omega} |x - y|$, for $x \in \Omega$, the standard distance function to $\partial\Omega$,
- $\Omega_s = \{x \in \overline{\Omega} \ / \ d(x) \leq s\}$, for $s > 0$, the strip around of the boundary of Ω.

0.0.1 Space of Functions

- $C(\Omega)$ denote the space of continuous functions in Ω and $C_0(\Omega)$ the continuous functions with compact support in Ω.

- $C^k(\Omega)$ consists of those functions on Ω having continuous derivatives up to order k and $C^\infty(\Omega) = \cap_{k\geq 1} C^k(\Omega)$.

- $C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$.

- $L^p(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is mensurable } / \int_\Omega |u(x)|^p dx < \infty\right\}$, endowed with the norm

$$||u||_p = \inf \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

- $L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is mensurable } / \ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty \right\}$, endowed with the norm

$$||u||_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

- $L_+^\infty(\Omega) = \left\{ p \in L^\infty(\Omega) \ / \ \operatorname*{ess\,inf}_{x\in\Omega} p > 1 \right\}$.

- If $p \in L_+^\infty(\Omega)$, we define the the variable exponent space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is mensurable } / \ \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

  endowed with the Luxemburg norm

$$||u||_{p(x)} = \inf \left\{ \lambda > 0 \ / \ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

- If $p \in L_+^\infty(\Omega)$, we define the space

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \ / \ |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

  endowed with the norm

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)}.$$

- The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect of the norm $|| \cdot ||_{1,p(x)}$ endowed with the norm

$$||u|| = ||\nabla u||_{p(x)}.$$

Chapter 1

About variable exponent spaces

In this chapter, let us present some properties and results about the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded open set. For the interested reader in more information about these spaces, including the proofs omitted in this chapter, we refer the papers Fan and Zhao [34], Kovácik and Rákosník [49] and the book of Diening et al. [28].

1.1 Lebesgue spaces with variable exponents

Let us denote by $L^\infty_+(\Omega)$ the set

$$L^\infty_+(\Omega) = \left\{ u \in L^\infty(\Omega) \ / \ \operatorname{ess}\inf_{x\in\Omega} u \geq 1 \right\}$$

and by $L^{p(x)}(\Omega)$ the variable exponent space defined by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is mensurable } / \ \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$||u||_{p(x)} = \inf \left\{ \lambda > 0 \ / \ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

for each $p \in L^\infty_+(\Omega)$ given. This space is well-known as the variable exponent Lebesgue space.

It is well-known that when $p(x) = p$ is constant the Luxemburg norm coincides with the usual norm in $L^p(\Omega)$, that is, the variable exponent Lebesgue space turns into the classical Lebesgue space.

Now, given a $p \in L^{\infty}_{+}(\Omega)$, let us we denote by p_- and p_+ the following real numbers

$$p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

and define the modular function $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_{\Omega} |u(x)|^{p(x)}.$$

Proposition 1.1.1 ([34], Theorems 1.2 **and** 1.3**)** *Let $u \in L^{p(x)}(\Omega)$. Then:*

1. $||u||_{p(x)} < 1 \ (= 1, > 1) \Leftrightarrow \rho(u) < 1 \ (= 1, > 1)$,
2. $||u||_{p(x)} > 1 \Rightarrow ||u||^{p_-}_{p(x)} \leq \rho(u) \leq ||u||^{p_+}_{p(x)}$,
3. $||u||_{p(x)} < 1 \Rightarrow ||u||^{p_+}_{p(x)} \leq \rho(u) \leq ||u||^{p_-}_{p(x)}$,
4. $||u||_{p(x)} = a$ *if, and only if,* $\displaystyle\int_{\Omega} \left(\frac{|u|}{a}\right)^{p(x)} dx = 1.$

Proposition 1.1.2 ([34], Theorem 1.4**)** *Let $(u_n) \subset L^{p(x)}(\Omega)$. Then,*

1. $\lim_{n\to\infty} ||u_n||_{p(x)} = 0$ *if, and only if,* $\lim_{n\to\infty} \rho(u_n) = 0.$
2. $\lim_{n\to\infty} ||u_n||_{p(x)} = +\infty$ *if, and only if,* $\lim_{n\to\infty} \rho(u_n) = +\infty.$

In special, for some $u \in L^{p(x)}(\Omega)$,

$$\lim_{n\to\infty} ||u_n - u||_{p(x)} = 0 \ \textit{if, and only if,} \ \lim_{n\to\infty} \rho(u_n - u) = 0.$$

As a Corollary of the above result, we have.

Corollary 1.1.3 ([34]) *Let $(u_n) \subset L^{p(x)}(\Omega)$ with $u_n \to u$ in $L^{p(x)}(\Omega)$. Then there exists a subsequence (u_{n_k}) such that*

1. $u_{n_k}(x) \to u(x)$ *a.e. in* Ω,
2. $|u_{n_k}(x)| \leq h(x)$ *for all* $k \geq 1$ *and a.e. in* Ω *with* $h \in L^{p(x)}(\Omega)$.

An important estimate that will be frequently used in this work is given in the next proposition.

Proposition 1.1.4 ([34]) *Let $h, p \in L^{\infty}_{+}(\Omega)$ with $h(x) \leq p(x)$ a.e in Ω, and $u \in L^{p(x)}(\Omega)$. Then, $|u|^{h(x)} \in L^{\frac{p(x)}{h(x)}}(\Omega)$ and*

$$\left\| |u|^{h(x)} \right\|_{\frac{p(x)}{h(x)}} \leq ||u||^{h_+}_{p(x)} + ||u||^{h_-}_{p(x)}$$

or

$$\left\| |u|^{h(x)} \right\|_{\frac{p(x)}{h(x)}} \leq \max\left\{ ||u||^{h_+}_{p(x)}, ||u||^{h_-}_{p(x)} \right\}.$$

Reciprocally, if $|u|^{h(x)} \in L^{\frac{p(x)}{h(x)}}(\Omega)$ with $h(x) \leq p(x)$, then $u \in L^{p(x)}(\Omega)$, and there is a number $h_0 \in [h_-, h_+]$ such that $\left\| |u|^{h(x)} \right\|_{\frac{p(x)}{h(x)}} = ||u||^{h_0}_{p(x)}$.

Proposition 1.1.5 ([34], Theorems 1.6 and 1.10) *Let $p \in L^\infty_+(\Omega)$. Then the space $(L^{p(x)}(\Omega), ||\cdot||_{p(x)})$ is separable. In addiction, if $p_- > 1$, then $L^{p(x)}(\Omega)$ is uniform convex and thus is reflexive.*

Given $p \in L^\infty_+(\Omega)$, we denote by $p'(x)$ the conjugate function of $p(x)$, that is,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

with the convention that $1/\infty = 0$. Now we present the generalization of Hölder's Inequality.

Proposition 1.1.6 ([49], Theorem 2.1) *Let $p \in L^\infty_+(\Omega)$ with $p_- > 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have*

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) ||u||_{p(x)}||v||_{p'(x)} \leq 2||u||_{p(x)}||v||_{p'(x)}.$$

In addition, if $1/p(x) + 1/p'(x) + 1/p''(x) = 1$ holds true, then

$$\left|\int_\Omega uvwdx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} + \frac{1}{p''_-}\right) ||u||_{p(x)}||v||_{p'(x)}||w||_{p''(x)} \leq 3||u||_{p(x)}||v||_{p'(x)}||w||_{p''(x)}$$

for all $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$ and $w \in L^{p''(x)}(\Omega)$.

To end, we present the natural inclusion of variable exponent Lebesgue's spaces.

Proposition 1.1.7 ([34], Theorem 1.11) *Let $h, p \in L^\infty_+(\Omega)$ with $1 \leq h(x) \leq p(x)$ a.e in Ω. Then $L^{p(x)}(\Omega)$ is continuously embedding into $L^{h(x)}(\Omega)$.*

1.2 Sobolev spaces with variable exponents

In this section, we consider only the space of Sobolev $W^{1,p(x)}(\Omega)$. The definition and properties of the spaces $W^{k,p(x)}(\Omega)$, with $k > 1$, can be found in the references quoted above in this chapter.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) \ / \ |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

endowed with the norm

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)}. \tag{1.1}$$

Similarly to $L^{p(x)}(\Omega)$, the Banach space $W^{1,p(x)}(\Omega)$ is separable and, if $p_- > 1$, it is also a reflective space. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect of the norm defined in (1.1). It is also worth to point out that unlike of the validity of the density of $C^\infty(\Omega)$ in $W^{1,p}(\Omega)$ when $p(x) = p > 1$, in the context of the variable exponent space $W_0^{1,p(x)}(\Omega)$ this will be true if we require additional conditions on the domain and exponent $p(x)$, for instance, if $\partial\Omega$ is Lipschitz continuous and $p(x)$ satisfies the log Hölder condition, that is,

$$\log|x-y|^{-1}|p(x)-p(y)| \leq C \quad \text{for all } x, y \in \overline{\Omega} \text{ with } 0 < |x-y| < 1$$

for some $C > 0$, then $C^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$ with respect of the norm defined in (1.1). See for instance [28].

By using similar arguments like those used in [48, Lemma 1.25], we obtain.

Proposition 1.2.1 *Let $v \in W^{1,p(x)}(\Omega)$.*

(i) If v has compact support, then $v \in W_0^{1,p(x)}(\Omega)$.

(ii) If $u \in W_0^{1,p(x)}(\Omega)$ and $0 < v < u$ a.e. in Ω, then $v \in W_0^{1,p(x)}(\Omega)$.

(iii) If $u \in W_0^{1,p(x)}(\Omega)$ and $|v| < |u|$ a.e. in $\Omega \setminus K$, where K is a compact subset of Ω, then $v \in W_0^{1,p(x)}(\Omega)$.

In this setting, the variable critical function-exponent for embedding of Sobolev to Lebesgue with variable exponents is defined by

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \leq p(x), \end{cases}$$

and it is called as the critical function with respect to $p(x)$.

Proposition 1.2.2 ([34], Theorem 2.3**)** *Let $p \in L_+^\infty(\Omega)$. If $p, q \in C(\overline{\Omega})$ with $1 < p_- \leq p_+ < N$, then:*

(i) $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ if $1 \leq q(x) \leq p^*(x)$ a.e. in Ω,

(ii) $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ if $1 \leq q(x) < p^*(x)$ a.e. in Ω.

More, there exists a constant $C > 0$ such that

$$||u||_{p(x)} \leq C||\nabla u||_{p(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

The last inequality is well known as Poincaré's inequality. As a consequence of it, we infer that $||\nabla u||_{p(x)}$ define on $W_0^{1,p(x)}(\Omega)$ an equivalent norm to $||u||_{1,p(x)}$. From now on, we are going to denote this norm by $||u||$ and we will use it for the whole paper.

Proposition 1.2.3 *Let $u \in W_0^{1,p(x)}(\Omega)$ and the modular function $\rho_0 : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by $\rho_0(u) = \int_\Omega |\nabla u(x)|^{p(x)}$. Then the same conclusion of Proposition 1.1.1 holds if we consider $||\cdot||$ and ρ_0.*

The notion of a map of (S_+)-type is useful to help us to prove that a sequence converge strongly in $W_0^{1,p(x)}(\Omega)$ under appropriate assumptions.

We say that a function is (S_+)-type

$$\text{if } u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega) \text{ and } \limsup_{n\to\infty} \langle Lu_n, u_n - u\rangle \leq 0, \text{ then } u_n \to u \text{ in } W_0^{1,p(x)}(\Omega).$$

Proposition 1.2.4 ([33], Theorem 3.1) *The map $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ defined by*

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx$$

is:

(i) continuous,

(ii) bounded,

(iii) strictly monotone, that is,

$$\langle Lu - Lv, u - v\rangle > 0 \text{ for all } u, v \in W_0^{1,p(x)}(\Omega),\ u \neq v,$$

(iv) $(S)_+$-type.

Below, we present two inequalities that will be useful in parts of this thesis.

Lemma 1.2.5 ([45], Hardy's Inequality) *Let Ω be an open and bounded subset of $\mathbb{R}^N$. Assume that there exists a constant $\theta > 0$ such that*

$$|B_r(y) \cap \Omega^c| \geq \theta |B_r(y)|$$

for every $y \in \partial\Omega$ and $r > 0$. Then there exists positive constants c and a_0 depending only on p, N and θ such that the inequality

$$\left\| \frac{u}{d(x)^{1-a}} \right\|_{p(x)} \leq c||d(x)^a \nabla u||_{p(x)},$$

holds for all $u \in W_0^{1,p(x)}(\Omega)$ and $a \in [0, a_0)$.

Lemma 1.2.6 (Simon's Inequality) *For all $1 < p \in C(\overline{\Omega})$ there exists a positive constants $C = C(p)$ such that*

$$\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla(u-v) \geq \begin{cases} (p_- - 1)\dfrac{|\nabla(u-v)|}{(|\nabla u| + |\nabla v|)^{2-p(x)}} & \text{if } 1 < p(x) < 2, \\ \dfrac{2^{3-p_+}}{p_+}|\nabla(u-v)|^{p(x)} & \text{if } p(x) \geq 2 \end{cases}$$

for all $u, v \in W^{1,p(x)}(\Omega)$.

1.3 Regularity results in variable exponents spaces

In this section, we establish some results concerning regularity of bounded functions satisfying some relations involving the $p(x)-$Laplace operator. The first one is an improvement of a result of Fan and Zhao [31] that was the first result in this direction n the context of variable exponents.

We highlight that for our purposes, we need of the below Proposition 1.3.5 instead of the classical Fan's result [31], because ours weights $a(x), b(x)$ in (1) are not in $L^\infty(\Omega)$. Despite of this generality, we are to show that our solution u of the Problem (1) satisfies $C_1 d(x) \leq u(x) \leq C_2 d(x)^{\theta_2}$ close to the boundary of Ω for some $C_1, C_2 > 0$, where $d(x)$ is the standard distance function to the boundary of Ω. This makes possible to verify (1.3) and apply Proposition 1.3.5.

Definition 1.3.1 *Let $M, \gamma, \gamma_1, \delta, r, R$ be positive constants with $\delta \leq 2$, $r > 1$ and $B_R(y) \subset \Omega$. We say that a function v belongs to class $\mathcal{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$ if $v \in W^{1,p(x)}(\Omega)$ with $\max\limits_{B_R}|v(x)| \leq M$ and the functions $w(x) = \pm v(x)$ satisfy the inequalities,*

$$\int_{A_{k,\tau}} |\nabla w|^{p(x)}dx \leq \gamma \int_{A_{k,t}} \left(\frac{w(x)-k}{t-\tau}\right)^{p(x)} dx + \gamma_1 |A_{k,t}|^{1-\frac{1}{r}} \tag{1.2}$$

for arbitrary $0 < \tau < t < R$ and k such that $k \geq \max\limits_{B_t(y)} w(x) - \sigma M$, where $A_{k,\rho} = \{x \in B_\rho \,:\, w(x) > k\}$. In a analogue way, we say that a function v belongs to class $\mathcal{B}_{p(\cdot)}(B_R(z) \cap \partial\Omega, M, \gamma, \gamma_1, \delta, 1/r)$ if $v \in W^{1,p(x)}(B_R(z) \cap \Omega)$ with $\max\limits_{B_R(z)\cap\Omega} |v(x)| \leq M$, $\max\limits_{B_R(z)\cap\partial\Omega} |v(x)| < \infty$ and (1.2) holds for $k \geq \max\{\max\limits_{B_R(z)\cap\Omega} w(x) - \delta M, \max\limits_{B_R(z)\cap\partial\Omega} w(x)\}$.

Definition 1.3.2 *We say that Ω satisfies an exterior cone condition at a point $x \in \partial\Omega$ if there exists a finite right circular cone V_x with vertex x such that $\overline{\Omega} \cap V_x = x$, in particular, say that Ω satisfes a uniform exterior cone condition on $\partial\Omega$ if Ω satisfes an exterior cone condition at every $x \in \partial\Omega$ and the cones V_x are all congruent to some fixed cone V.*

Lemma 1.3.3 (Lemma 4.5, [68]) *Let $p \in C(\overline{\Omega})$, $1 < p_- \leq p_+ < \infty$ and be log-Hölder continuous in Ω and let $R_0 \in (0,1)$, $\sigma_0 > 1$ be numbers such that $p_0\sigma_0 > N$, where $p_0 = \min_{B_{R_0}(x_0)\cap\Omega} p(x)$ and $x_0 \in \Omega$. Let $B_{R'}(y) \subset B_{R_0}(x_0)\cap\Omega$ and $u \in W^{1,p(x)}(B_{R'}) \cap L^\infty(B_{R'})$. Suppose that, for arbitrary $R \leq R'$, there exists a number $r \geq \sigma_0$ such that $u \in \mathcal{B}_{p(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r)$, where M is a positive number satisfying $||u||_{L^\infty(B_{R'})} \leq M$. Then there exists a constant $s = s(N, p_0, \sigma_0, \max_{B_{R_0}(x_0)\cap\Omega} p(x), M, \gamma, L) > 2$ such that, for arbitrary $R \leq R'$,*

$$\sup_{x\in B_R(y)} u(x) - \inf_{x\in B_R(y)} u(x) \leq cR'^{-\alpha}R^\alpha,$$

where c, α are constants independent of M.

Lemma 1.3.4 (Lemma 4.10, [68]) *Let $p \in C(\overline{\Omega})$, $1 < p_- \leq p_+ < \infty$ and be log-Hölder continuous in Ω and let $R_0 \in (0,1)$, $\sigma_0 > 1$ be numbers such that $p_0\sigma_0 > N$, where $p_0 = \min_{B_{R_0}(x_0)\cap\Omega} p(x)$ and $x_0 \in \overline{\Omega}$. Suppose that Ω satisfes an exterior cone condition at $z \in \partial\Omega$. Let $B_{R'}(z) \subset B_{R_0}(x_0)$ and $u \in W^{1,p(x)}(B_{R'}(z)\cap\Omega) \cap L^\infty(B_{R'}(z)\cap\Omega)$. Suppose that, for arbitrary $R \leq R'$, there exists a number $r \geq \sigma_0$ such that $u \in \mathcal{B}_{p(\cdot)}(B_R(y)(z)\cap\Omega, M, \gamma, \gamma_1, \delta, 1/r)$ and satisfies*

$$\sup_{x\in B_R(z)\cap\partial\Omega} u(x) - \inf_{x\in B_R(z)\cap\partial\Omega} u(x) \leq \beta_0 R^{\alpha_0},$$

where β_0, α_0 are positive constants and M is a positive number satisfying $||u||_{L^\infty(B_{R'}(z)\cap\Omega)} \leq M$. Then there exists a constant $s = s(N, p_0, \sigma_0, \max_{B_{R_0}(x_0)\cap\Omega} p(x), M, \gamma, L, V_z) > 2$ such that, for arbitrary $R \leq R'$,

$$\sup_{x\in B_R(z)} u(x) - \inf_{x\in B_R(z)} u(x) \leq cR'^{-\alpha}R^\alpha,$$

where c, α are constants independent of M.

As a consequence of above Lemmas, we have the result.

Proposition 1.3.5 *If $p \in C(\overline{\Omega})$, $1 < p_- \leq p_+ < \infty$ and be log-Hölder continuous in Ω, then $\mathcal{B}_{\rho(\cdot)}(B_R(y), M, \gamma, \gamma_1, \delta, 1/r) \subset C^{0,\beta_1}(\Omega)$, where the constant $\beta_1 \in (0,1]$ is independent of M and γ. In addiction, if Ω satisfes an exterior cone condition at $z \in \partial\Omega$ and $u \in \mathcal{B}_{p(\cdot)}(B_R(z)\cap\partial\Omega, M, \gamma, \gamma_1, \delta, 1/r)$ with $u|_{\partial\Omega} \in C^{0,\alpha_1}(\partial\Omega)$ satisfies*

$$\sup_{x\in B_R(z)\cap\partial\Omega} u(x) - \inf_{x\in B_R(z)\cap\partial\Omega} u(x) \leq CR^{\alpha_1} \tag{1.3}$$

for some $C, \alpha > 0$ constants, then $u \in C^{0,\beta_2}(\overline{\Omega})$, where the constant $\beta_2 \in (0,1]$ is independent of M and γ.

The following Lemma is due to Ladyzhenskaya and Uraltseva [50] and will be fundamental to apply the above Proposition.

Lemma 1.3.6 ([50], Lemma 4.7**)** *Let* (x_n) *be a sequence such that* $x_0 \leq \lambda^{-\frac{1}{\eta}}\mu^{-\frac{1}{\eta^2}}$ *and* $x_{n+1} \leq \lambda\mu^n x_n^{1+\eta}$ *for any* $n \in \mathbb{N}$ *with* λ, η *and* μ *being positive constants and* $\mu > 1$*. Then* (x_n) *converges to* 0 *as* $n \to \infty$*.*

Another application of the above Lemma is the next result.

Lemma 1.3.7 ([71], Lemma 2.4**)** *Suppose* $0 < b_0 \leq b(\cdot) \in L^{\alpha(x)}(\Omega)$ *with* $\alpha(x) > N$ *on* Ω*. Let* $M > 0$ *and* u *is the unique solution of the problem*

$$\begin{cases} -\Delta_{p(x)}u = Mb(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, $||u||_\infty \leq C_1 M^{\frac{1}{(p_- - 1)}}$ *for* $M \geq 1$*, and* $||u||_\infty \leq C_2 M^{\frac{1}{(p_+ - 1)}}$ *for* $M < 1$*, where* C_1, C_2 *are positive constants depending on* $p_+, p_-, N, ||b||_{L^{\alpha_-}(\Omega)}$ *and* $|\Omega|$*.*

The next C^1-regularity result is due to Fan.

Theorem 1.3.8 (Theorem 1.2**, [29])** *If* p *is Hölder continuous on* Ω *and*

$$|f(x,t)| \leq c_1 + c_2|t|^{q(x)-1} \text{ for all } x \in \Omega \text{ and } t \in \mathbb{R},$$

where $q \in C(\overline{\Omega})$ *and* $1 < q(x) < p^*(x)$ *for* $x \in \Omega$*, then every solution* $u \in W_0^{1,p(x)}(\Omega)$ *of*

$$\begin{cases} -\Delta_{p(x)}u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

belongs to $C^{1,\gamma}(\overline{\Omega})$ *for some* $\gamma \in (0,1)$*.*

To end, we present a strong maximum principle for the $p(x)-$Laplacian operator due to Fan, Zhang and Zhao.

Proposition 1.3.9 (Theorems 1 **and** 2**, [69])** *Suppose that* $p \in C^1(\overline{\Omega})$*,* $p_- > 1$*,* $u \in W^{1,p(x)}(\Omega)$*,* $u \geq 0$ *and* $u \neq 0$ *in* Ω*. If* $-\Delta_{p(x)}u + h(x)u^{q(x)-1} \geq 0$ *in* Ω*, where* $h \in L^\infty(\Omega)$*,* $h \geq 0$ *and* $p(x) \leq q(x) \leq p^*(x)$*, then* $u > 0$ *in* Ω*, and when* $u \in C^1(\overline{\Omega})$*, then* $\frac{\partial u}{\partial \eta} > 0$ *on* $\partial\Omega$*, where* η *is the inward unit normal on* $\partial\Omega$*.*

Chapter 2

A Comparison Principle for a kind of $(p(x)-1)-$sublinear problems

2.1 Introduction

In this chapter we present a Comparison principle for sub and super solutions in $W^{1,p(x)}_{loc}(\Omega)$ to a kind of $(p(x)-1)-$sublinear problems, which will be so useful in several points of this thesis.

Consider the problem

$$\begin{cases} -\Delta_{p(x)}u = g(x,u) \text{ in } \Omega, \\ u>0 \ \text{ in } \Omega, \qquad u=0 \ \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $g:\Omega\times(0,\infty)\to[0,+\infty)$ is a function satisfying:

(g_1) $t\mapsto g(x,t)$ is a continuous function a.e. $x\in\Omega$ and for each $t>0$ the function $x\mapsto g(x,t)$ is mensurable,

(g_2) $t\mapsto \dfrac{g(x,t)}{t^{p_- -1}}$ is strictly decreasing on $(0,\infty)$ for a.e. $x\in\Omega$.

Note that under the above hypotheses, we do not impose growth restriction just on g with respect to the variable t and allow $g(x,t)$ to be singular at the origin, that is, $g(x,t)\to+\infty$ as $t\to 0^+$ a.e. $x\in\Omega$. For instance, the function $g(x,t)=t^{-\alpha(x)}+t^{\beta(x)}$, $t>0$, with $\alpha(x)>1-p_-$ and $\beta(x)<p_- -1$ on Ω satisfies $(g_1)-(g_2)$.

From Lazer and Mckenna [51], the existence of weak solutions with zero-boundary value in the sense of the trace function to singular problems is possible just in some cases. For example, if $p(x) \equiv 2$ and $g(x,t) = t^{-\alpha}$, $t > 0$, then there exists a solution still in $H_0^1(\Omega)$ if, and only if, $0 < \alpha < 3$. Therefore, the way of understanding the boundary condition will be the following:

Definition 2.1.1 *Let $u \in W_{loc}^{1,p(x)}(\Omega)$. We say that $u \leq 0$ on $\partial\Omega$ if $(u-\epsilon)^+ \in W_0^{1,p(x)}(\Omega)$ for every $\epsilon > 0$. Furthermore $u = 0$ on $\partial\Omega$ if u is nonnegative and $u \leq 0$ on $\partial\Omega$.*

It is readily seen that if $u \in W_0^{1,p(x)}(\Omega)$, then $u = 0$ on $\partial\Omega$ in the sense of the above definition. Moreover, the function

$$u(x) = \begin{cases} \sigma d(x)^\theta \quad \text{if } d(x) < \delta, \\ \sigma\delta^\theta + \displaystyle\int_\delta^{d(x)} \sigma\theta\delta^{\theta-1}\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p_- -1}} dt \ \ \text{if } \delta \leq d(x) < 2\delta, \\ \sigma\delta^\theta + \displaystyle\int_\delta^{2\delta} \sigma\theta\delta^{\theta-1}\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p_- -1}} dt \ \ \text{if } 2\delta \geq d(x), \end{cases}$$

where $d(x)$ is the distance function in Ω, does not belong to $W_0^{1,p(x)}(\Omega)$ if $\theta > 1 - 1/p_+$, but $(u-\epsilon)^+ \in W_0^{1,p(x)}(\Omega)$ for each $\epsilon > 0$ given.

Definition 2.1.2 *We say that $\underline{u} \in W_{loc}^{1,p(x)}(\Omega)$ is a subsolution of (2.1) if $\underline{u} \geq 0$, $g(x,\underline{u}) \in L_{loc}^1(\Omega)$ and*

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla\underline{u}\nabla\phi dx - \int_\Omega g(x,\underline{u})\phi dx \leq 0 \ \forall \ \phi \in C_0^\infty(\Omega), \ \ \phi \geq 0.$$

Analogously, $\overline{u} \in W_{loc}^{1,p(x)}(\Omega)$ is a supersolution of (2.1) if $\overline{u} \geq 0$, $g(x,\overline{u}) \in L_{loc}^1(\Omega)$ and

$$\int_\Omega |\nabla \overline{u}|^{p(x)-2}\nabla\overline{u}\nabla\phi dx - \int_\Omega g(x,\overline{u})\phi dx \geq 0 \ \forall \ \phi \in C_0^\infty(\Omega), \ \ \phi \geq 0.$$

The main result of this chapter is the following Comparison Principle.

Theorem 2.1.3 *Assume that $(g_1)-(g_2)$ hold and suppose that for each $h > 0$*

(g_3) *the functional $I_h :\to \mathbb{R}$, defined by*

$$I_h(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_\Omega G_h(x,u)dx,$$

is coercive and weakly lower semicontinuous on

$$\mathcal{K} := \{w \in W_0^{1,p(x)}(\Omega) \ / \ 0 \leq w \leq \overline{u}\}$$

with respect to $W_0^{1,p(x)}(\Omega)$-norm, where

$$G_h(x,s) := \int_0^s g_h(x,t+h)dt, \ \ s \geq 0 \text{ and } g_h(x,t) := g(x,t+h) \text{ for } t \geq 0$$

Let $\underline{u}, \overline{u} \in W^{1,p(x)}_{loc}(\Omega)$ be a subsolution and a supersolution of Problem (2.1)*, respectively. If $\underline{u} \in L^{\infty}_{loc}(\Omega)$ with $\underline{u} \leq 0$ on $\partial\Omega$ and $\operatorname*{ess\,inf}_{x\in U} \overline{u}(x) > 0$ for each $U \subset\subset \Omega$, then $\underline{u} \leq \overline{u}$ a.e. in Ω.*

The hypotheses (g_1) and (g_2) are used to derive a type of Diaz-Saá's Inequality (see (2.2) below) in variable exponents context. Due to the absence of growth condition and the lack of positivity of the first eigenvalue in the setting of $W^{1,p(x)}(\Omega)$, we have to consider the assumption (g_3) that will be used to obtain a fundamental estimate in our proof. More details are presented in the next section.

The importance of our first result is principally because it may be applied to subsolutions and supersolutions just in $W^{1,p(x)}_{loc}(\Omega)$. To our knowledge, this result is new even for Laplacian operator.

2.2 Auxiliary results

In this section we present the results that will be useful in the proof of Theorem 2.1.3. Inspired by the ideas in [43], let $\mathcal{D} = \{u \in L^1_{loc}(\Omega) \;/\; u \geq 0,\; u^{\frac{1}{p_-}} \in W^{1,p(x)}_{loc}(\Omega)\}$. Fixed $\phi \in C^\infty_0(\Omega)$, consider the functional $J = J_\phi : L^1_{loc}(\Omega) \to (-\infty, \infty]$ given by

$$J(u) = \begin{cases} \displaystyle\int_\Omega \frac{|\nabla u^{\frac{1}{p_-}}|^{p(x)}}{p(x)} \phi dx \text{ if } u \in \mathcal{D}, \\ +\infty \text{ otherwise.} \end{cases}$$

Lemma 2.2.1 *Let J be the above functional. Then J is convex and $J \not\equiv +\infty$.*

Proof. Let us begin our proof showing that $J \not\equiv +\infty$. To this end, fixed $x_0 \in \Omega$, take $R > 0$ such that $B_R(x_0) \subset \Omega$ is the closed ball centered in x_0 with radius R. Now, given $\theta > p_-$ let

$$v(x) = \begin{cases} \overline{v}^{\theta}_R(|x - x_0|) \text{ if } x \in B_R(x_0), \\ 0 \text{ otherwise.} \end{cases}$$

where $\overline{v}_R : [0, R] \to \mathbb{R}$ is defined by

$$\overline{v}_R(t) = \begin{cases} 1 \text{ if } \; t = 0, \\ \text{linear } \text{ if } \; 0 < t < R/2, \\ 0 \text{ if } \; R/2 \leq t \leq R. \end{cases}$$

Evidently $v \in \mathcal{D}$. Moreover,

$$\begin{aligned}\int_\Omega \frac{|\nabla\left(v^{\frac{1}{p_-}}\right)|^{p(x)}}{p(x)}\phi dx &= \int_{B_R(x_0)} \frac{\left(\frac{\theta}{p_-} v^{\frac{\theta}{p_-}-1}|\nabla \overline{v}_R|\right)^{p(x)}}{p(x)}\phi dx \\ &\leq \frac{1}{p_-}\left(\frac{\theta}{p_-}\right)^{p_+}\int_{B_R(x_0)} v^{\left(\frac{\theta}{p_-}-1\right)p_-}|\nabla \overline{v}_R|^{p(x)}\phi dx \\ &\leq \frac{1}{p_-}\left(\frac{\theta}{p_-}\right)^{p_+}\int_{B_R(x_0)} |\nabla \overline{v}_R|^{p(x)}\phi dx < +\infty,\end{aligned}$$

that is, $J(v) \not\equiv +\infty$.

Now we are going to show that J is convex. As in proof of [27, Lemma 1], we have

$$|\nabla(sw_1 + (1-s)w_2)^{1/p_-}|^{p_-} \leq s|\nabla w_1^{1/p_-}|^{p_-} + (1-s)|\nabla w_2^{1/p_-}|^{p_-} \text{ for all } s \in [0,1],$$

where $w_1, w_2 \in \mathcal{D}$. Since the function $s \mapsto s^{p(x)/p_-}$ is convex on $[0,\infty)$, it follows from the above inequality that

$$\begin{aligned} J(sw_1 + (1-s)w_2) &= \int_\Omega \frac{|\nabla(sw_1+(1-s)w_2)^{\frac{1}{p_-}}|^{p(x)}}{p(x)}\phi dx \\ &\leq \int_\Omega \frac{\left(s|\nabla w_1^{\frac{1}{p_-}}|^{p_-} + (1-s)|\nabla w_2^{\frac{1}{p_-}}|^{p_-}\right)^{\frac{p(x)}{p_-}}}{p(x)}\phi dx \\ &\leq \int_\Omega \left(\frac{s|\nabla w_1^{\frac{1}{p_-}}|^{p(x)}}{p(x)} + \frac{(1-s)|\nabla w_2^{\frac{1}{p_-}}|^{p(x)}}{p(x)}\right)\phi dx \\ &= sJ(w_1) + (1-s)J(w_2), \end{aligned}$$

for each $s \in [0,1]$ given. This shows the Lemma. ■

The next result will be fundamental for our purposes.

Lemma 2.2.2 (Diaz-Saá's type Inequality) *Assume that $w_1, w_2 \in L^\infty_{loc}(\Omega) \cap \mathcal{D}$. If $w_i/w_j \in L^\infty_{loc}(\Omega)$, $i \neq j$, then*

$$\int_\Omega \left[|\nabla w_1^{\frac{1}{p_-}}|^{p(x)-2}\nabla w_1^{\frac{1}{p_-}}\nabla\left(\frac{w_1-w_2}{w_1^{(p_- -1)/p_-}}\right) - |\nabla w_2^{\frac{1}{p_-}}|^{p(x)-2}\nabla w_2^{\frac{1}{p_-}}\nabla\left(\frac{w_1-w_2}{w_2^{(p_- -1)/p_-}}\right)\right]\phi dx \geq 0 \tag{2.2}$$

holds, for all $\phi \in C_0^\infty(\Omega)$.

Proof. To obtain the inequality (2.2) it suffices to show

$$\langle J'(u), v\rangle = \int_\Omega \left|\nabla u^{\frac{1}{p_-}}\right|^{p(x)-2}\nabla u^{\frac{1}{p_-}}\nabla\left(u^{\frac{1-p_-}{p_-}}v\right)\phi dx, \tag{2.3}$$

for all $u, v \in L^\infty_{loc}(\Omega) \cap \mathcal{D}$ with $w_1/w_2 \in L^\infty_{loc}(\Omega)$ and apply Lemma 2.2.1. In fact, admitting it by now, it follows from Lemma 2.2.1, that

$$0 \le p_- \langle J'(w_1) - J'(w_2), w_1 - w_2 \rangle$$
$$= \int_\Omega \left[|\nabla w_1^{\frac{1}{p_-}}|^{p(x)-2} \nabla w_1^{\frac{1}{p_-}} \nabla \left(\frac{w_1 - w_2}{w_1^{(p_--1)/p_-}} \right) - |\nabla w_2^{\frac{1}{p_-}}|^{p(x)-2} \nabla w_2^{\frac{1}{p_-}} \nabla \left(\frac{w_1 - w_2}{w_2^{(p_--1)/p_-}} \right) \right] \phi dx.$$

Now, we are going to prove that (2.3) holds true. First, we notice that if $u \in L^\infty_{loc}(\Omega) \cap \mathcal{D}$, then $u \in W^{1,p(x)}_{loc}(\Omega)$. In fact, by denoting $w = u^{1/p_-}$, we have that

$$|\nabla u| = |\nabla (w^{p_-})| = p_- |w^{p_- -1}| |\nabla w| = p_- |u|^{\frac{p_- -1}{p_-}} |\nabla (u^{\frac{1}{p_-}})| \in L^{p(x)}_{loc}(\Omega).$$

Let $u, v \in L^\infty_{loc}(\Omega) \cap \mathcal{D}$. Thus,

$$\begin{aligned} \langle J'(u), v \rangle &= \lim_{t \to 0} \frac{J(u+tv) - J(u)}{t} = \lim_{t \to 0} \int_\Omega \frac{\left| \nabla \left((u+tv)^{\frac{1}{p_-}} \right) \right|^{p(x)} - |\nabla u^{\frac{1}{p_-}}|^{p(x)}}{tp(x)} \phi dx \\ &= \lim_{t \to 0} \int_\Omega \frac{h(x,t)}{tp(x)} \phi dx = \lim_{t \to 0} \int_\Omega \frac{dh}{dt}(x,t) dx, \end{aligned} \tag{2.4}$$

where

$$h(x,t) = \left(\left| \nabla \left((u+tv)^{\frac{1}{p_-}} \right) \right|^{p(x)} - |\nabla u^{\frac{1}{p_-}}|^{p(x)} \right) \phi,$$

and

$$\frac{dh}{dt}(x,t) = \frac{p(x)}{p_-} \left| \nabla \left((u+tv)^{\frac{1}{p_-}} \right) \right|^{p(x)-2} \nabla \left((u+tv)^{\frac{1}{p_-}} \right) \nabla \left((u+tv)^{\frac{1-p_-}{p_-}} v \right) \phi$$

for $x \in \Omega$ and $t > 0$.

The last equality at follows from Mean Value Theorem, that is, there exists an $0 < s = s(x,t) < t < 1$ such that $h(x,t)/t = \frac{dh}{dt}(x,s)$ for $x \in \Omega$. Since

$$\begin{aligned} \left| \frac{dh}{dt}(x,s) \right| \le & \left| \frac{1}{p_-} (u+sv)^{\frac{1-p_-}{p_-}} \nabla (u+sv) \right|^{p(x)-1} \\ & \left| \frac{1-p_-}{p_-^2} (u+sv)^{\frac{1-2p_-}{p_-}} v \nabla (u+sv) + \frac{1}{p_-} (u+sv)^{\frac{1-p_-}{p_-}} \nabla v \right| \phi \\ \le & \left(\frac{1}{p_-} \frac{(u+sv)^{\frac{1}{p_-}}}{(u+sv)} (|\nabla u| + s|\nabla v| \right)^{p(x)-1} \\ & \left(\frac{p_- - 1}{p_-^2} \frac{(u+sv)^{\frac{1}{p_-}}}{(u+sv)^2} |v| (|\nabla u| + s|\nabla v|) + \frac{1}{p_-} \frac{(u+sv)^{\frac{1}{p_-}}}{(u+sv)} |\nabla v| \right) \phi \\ \le & \left(\frac{1}{p_-} \frac{(u+v)^{\frac{1}{p_-}}}{|u|} (|\nabla u| + |\nabla v|) \right)^{p(x)-1} \\ & \left(\frac{p_- - 1}{p_-^2} \frac{(u+v)^{\frac{1}{p_-}}}{u^2} v (|\nabla u| + |\nabla v|) + \frac{1}{p_-} \frac{(u+v)^{\frac{1}{p_-}}}{u} |\nabla v| \right) \phi, \end{aligned}$$

where we used $u, v > 0$ in the last inequality.

So, it follows from the above information and hypotheses $v, u, u/v, v/u \in L^{\infty}_{loc}(\Omega)$, that

$$\begin{aligned}
\left|\frac{dh}{dt}(x,s)\right| &\leq \frac{p(x)}{p_-^{p(x)}}\frac{(u+v)^{\frac{p(x)}{p_-}}}{u^{p(x)}}(|\nabla u|+|\nabla v|)^{p(x)}\left(\frac{v}{u}+1\right)||\phi||_\infty \\
&\leq C_1||\phi||_\infty\left(\left\|\frac{v}{u}\right\|_{L^\infty(supp\ (\phi))}+1\right)\left(|u|^{\frac{p(x)}{p_-}-p(x)}+v^{\frac{p(x)}{p_-}}u^{-p(x)}\right)(|\nabla u|^{p(x)}+|\nabla v|^{p(x)}) \\
&\leq C_2||\phi||_\infty|u|^{\frac{p(x)}{p_-}-p(x)}\left(1+\left\|\left(\frac{v}{u}\right)^{\frac{p(x)}{p_-}}\right\|_{L^\infty(supp\ (\phi))}\right)(|\nabla u|^{p(x)}+|\nabla v|^{p(x)}) \qquad (2.5)\\
&\leq C_3||\phi||_\infty\left[\left(u^{\frac{1}{p_-}-1}|\nabla u|\right)^{p(x)}+\left(v^{\frac{1}{p_-}-1}|\nabla v|\right)^{p(x)}\left(\frac{u}{v}\right)^{\frac{p(x)}{p_-}-p(x)}\right] \\
&\leq C_4||\phi||_\infty\left[|\nabla u^{\frac{1}{p_-}}|^{p(x)}+|\nabla v^{\frac{1}{p_-}}|^{p(x)}\right]\in L^1_{loc}(\Omega),
\end{aligned}$$

where $C_3 = C_3(||u/v||_{L^\infty(\Omega)}, ||u||_{L^\infty(\Omega)}, p_+, p_-) > 0$ is a real constant.

Thus, the Lebesgue Dominated Convergence theorem implies that

$$\langle J'(u), v\rangle = \frac{1}{p_-}\int_\Omega \frac{h(x,t)}{tp(x)}dx = \int_\Omega |\nabla u^{\frac{1}{p_-}}|^{p(x)-2}\nabla u^{\frac{1}{p_-}}\nabla(u^{\frac{1-p_-}{p_-}}v)\phi dx$$

holds. This ends our proof. ■

In [15], the authors showing the comparison between a sub and a supersolution for the problem (2.1) with $p(x) = p$ and $g(x,t) = a(x)t^{-\alpha}$, where $\alpha > 0$ is a real constant, by truncating the singularity in an suitable way. Inspired in these ideas, let us define

$$g_h(x,t) = g(x, t+h) \text{ for } (x,t) \in \Omega \times (0,\infty),$$

for each $h > 0$ given, and consider the problem

$$\begin{cases} -\Delta_{p(x)}u = g_h(x,u) \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.6}$$

So, we have.

Lemma 2.2.3 *Assume* (g_1) *and* (g_3) *hold. Then there exists a* $w \in \mathcal{K}$ *(defined at (2.1.3)) such that*

$$\int_\Omega |\nabla w|^{p(x)-2}\nabla w\nabla\psi dx - \int_\Omega g_h(x, w+h)\psi dx \geq 0,\ \forall\ \psi \in W_0^{1,p(x)}(\Omega),\ \psi \geq 0. \tag{2.7}$$

Proof. It follows by hypothesis (g_3), that the functional

$$I_h(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \int_\Omega G_h(x,u)dx,\ u \in \mathcal{K}$$

is coercive and weakly lower semicontinuous on the closed and convex set $\mathcal{K}$. So, there exists a $w = w_h$ such that $I_h(w) = \inf_{u\in\mathcal{K}} I_h(u)$.

Now, given $0 \le \psi \in C_0^\infty(\Omega)$, set

$$v_t = \min\{w + t\psi, \overline{u}\} \ \text{ and } \ w_t = (w + t\psi - \overline{u})^+,$$

for $t > 0$ such that $t\psi \le \overline{u}$. Since $v_t, w_t \in W^{1,p(x)}(\Omega)$, $0 \le v_t \le w + t\psi$, $0 \le w_t \le t\psi$ and $0 \le w_t \le \overline{u}$ with *supp* $(w_t) \subset$ *supp* (ψ), it follows by Proposition 1.2.1 in that $v_t, w_t \in W_0^{1,p(x)}(\Omega)$, that is, $v_t, w_t \in \mathcal{K}$.

For $y \in \mathcal{K}$, let us define

$$\sigma(t) = I_h(ty + (1-t)w) \text{ for } t \in [0,1]$$

and deduce from $I_h(w) = \min_{\mathcal{K}} I_h$ that $\sigma(0) \le \sigma(t)$ for all $t \in [0,1]$, that is,

$$0 \le \sigma'(0) = \langle I_h'(w), y - w\rangle\,.$$

Now, by taking $y = v_t$ and noticing that $v_t - w = t\psi - w_t$, we obtain

$$\begin{aligned} 0 &\le \int_\Omega |\nabla w|^{p(x)-2}\nabla w\nabla(t\psi - w_t)dx - \int_\Omega g_h(x, w+h)(t\psi - w_t)dx \\ &= t\left(\int_\Omega |\nabla w|^{p(x)-2}\nabla w\nabla\psi dx - \int_\Omega g_h(x,w+h)\psi dx\right) \\ &- \int_\Omega |\nabla w|^{p(x)-2}\nabla w\nabla w_t dx + \int_\Omega g_h(x,w+h)w_t dx. \end{aligned} \tag{2.8}$$

On the other hand, since $0 \le w_t \in W_0^{1,p(x)}(\Omega)$, there exists a sequence $(\zeta_n) \subset C_0^\infty(\Omega)$ with $\zeta_n \ge 0$, *supp* $(\zeta_n) \subset$ *supp* (w_t) and $\zeta_n \to w_t$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$. Now, by using the fact that $\overline{u} \in W_{loc}^{1,p(x)}(\Omega)$ is a supersolution of problem 2.6, with ζ_n as a test function, we obtain

$$\int_\Omega |\nabla\overline{u}|^{p(x)-2}\nabla\overline{u}\nabla\zeta_n dx - \int_\Omega g_h(x,\overline{u})\zeta_n dx \ge 0 \text{ for all } n \in \mathbb{N},$$

that lead us to

$$\int_\Omega |\nabla\overline{u}|^{p(x)-2}\nabla\overline{u}\nabla w_t dx - \int_\Omega g_h(x,\overline{u})w_t dx \ge 0 \tag{2.9}$$

by the using of the Lebesgue's convergence theorem together with the fact that $|\nabla\overline{u}|^{p(x)}$ is integrable on the support of w_t and $0 \leq w_t \leq t\psi$ for each $x \in \Omega$.

So, it follows from (2.8) and (2.9), that

$$0 \leq t\left(\int_\Omega |\nabla w|^{p(x)-2}\nabla w \nabla\psi dx - \int_\Omega g_h(x, w+h)\psi dx\right) \tag{2.10}$$

$$+ \int_\Omega \left(|\nabla\overline{u}|^{p(x)-2}\nabla\overline{u} - |\nabla w|^{p(x)-2}\nabla w\right)\nabla w_t dx + \int_\Omega \left(g_h(x, w+h) - g_h(x, \overline{u})\right) w_t dx$$

for all $t > 0$ enough small.

Since

$$\left(g_h(x, w+h) - g_h(x, \overline{u})\right) w_t \leq g_h(x, w+h)t\psi \text{ on } supp\ (w_t),$$

it follows from (2.10), by dividing (2.10) by $t > 0$, that

$$\begin{aligned} 0 \quad &\leq \quad \int_\Omega |\nabla w|^{p(x)-2}\nabla w\nabla\psi dx - \int_\Omega g_h(x, w+h)\psi dx \\ &+ \quad \int_{\{w+t\psi\geq\overline{u}\}} \left(|\nabla\overline{u}|^{p(x)-2}\nabla\overline{u} - |\nabla w|^{p(x)-2}\nabla w\right)\nabla\psi + \int_{\{w+t\psi\geq\overline{u}\}} g_h(x, w+h)\psi dx, \end{aligned}$$

that is, by doing $t \to 0$, using Proposition 1.2.6, and applying Lebesgue's Convergence Theorem, we conclude that (2.7) is true for all $0 \leq \psi \in C_0^\infty(\Omega)$. The result follows by a standard density of $C_0^\infty(\Omega)$ in $W_0^{1,p(x)}(\Omega)$. ■

Now we are able to prove the Theorem 2.1.3.

2.3 Proof of Theorem 2.1.3 - Completed

Let us do the proof by an contradiction argument by combining the above results together with a very fine analysis.

Proof. Consider the set

$$\Omega^h := \{x \in \Omega \ / \ \underline{u}(x) - h > w(x) + h\}$$

and assume, by contradiction, that $|\Omega^h| > 0$ for some $h > 0$. From compactness of $\overline{\Omega^h}$, there exists a $x_0 \in \overline{\Omega^h}$ and $R > 0$ such that $|K_R| = |B_R \cap \overline{\Omega^h}| > 0$, where B_R be the ball of radius R centered in x_0.

We can assume, without loss of generality, that $B_R \subset \Omega$. In fact, since Ω is smooth, then $|\partial\Omega| = 0$. In particular, there exits $\delta > 0$ such that the set $\Omega_\delta = \{x \in$

$\Omega \ / \ d(x, \partial\Omega) < \delta\}$ satisfies $|\Omega_\delta| < |\Omega^h|/4$. Moreover, by compactness of $\overline{\Omega \setminus \Omega_\delta}$ there exists a finite cover $\cup_{i=0}^m B_{r_i}(x_i)$ with $x_i \in \Omega$ and $r_i \leq \delta/4$ such that $d(B_{r_i}(x_i), \partial\Omega) \geq (3\delta)/4$, for all $i = 1, 2, ..., m$. Thus, $\left|(\Omega \setminus \Omega_\delta) \cap \Omega^h\right| \geq (3|\Omega^h|)/4$. So there exists $B_R := B_{r_i}(x_i) \subset \Omega$ such that $\left|B_R \cap \Omega^h\right| > 0$ for some $1 \leq i \leq m$.

Fix $0 < t < R < \delta/4$ and take $0 < s < t$ such that $K_s := B_s \cap \Omega^h$ and $K_t := B_t \cap \Omega^h$ have positive Lebesgue measure. Define $\phi_s \in C_0^\infty(\Omega)$ such that $0 \leq \phi_s \leq 1$, $\phi_s \equiv 1$ in B_s, *supp* $(\phi_s) \subset B_t$ and $|\nabla \phi_s| \leq C(t-s)^{-1/(2p_+)}$. Now defining

$$\phi_1 = \frac{\phi_s\left[((\underline{u}-h)^+)^{p_-} - (w+h)^{p_-}\right]^+}{((\underline{u}-h)^+)^{p_--1}} \quad \text{and} \quad \phi_2 = \frac{\phi_s\left[((\underline{u}-h)^+)^{p_-} - (w+h)^{p_-}\right]^+}{(w+h)^{p_--1}},$$

we obtain

$$0 \leq \phi_1 \leq (\underline{u}-h)^+ \quad \text{and} \quad 0 \leq \phi_2 \leq C_{s,h}(\underline{u}-h)^+. \tag{2.11}$$

Moreover,

$$\begin{aligned}\nabla\phi_1 = \ &\phi_s\left[\left(1 + (p_- - 1)\left(\frac{w+h}{\underline{u}-h}\right)^{p_-}\right)\nabla(\underline{u}-h)^+ - p_-\left(\frac{w+h}{\underline{u}-h}\right)^{p_--1}\nabla w\right] \\ &+ \nabla\phi_s\left(\frac{(\underline{u}-h)^{p_-} - (w+h)^{p_-}}{(\underline{u}-h)^{p_--1}}\right)\end{aligned}$$

and

$$\begin{aligned}\nabla\phi_2 = \ &\phi_s\left[\left(1 + (p_- - 1)\left(\frac{\underline{u}-h}{w+h}\right)^{p_-}\right)\nabla w - p_-\left(\frac{\underline{u}-h}{w+h}\right)^{p_--1}\nabla(\underline{u}-h)^+\right] \\ &+ \nabla\phi_s\left(\frac{(\underline{u}-h)^{p_-} - (w+h)^{p_-}}{(w+h)^{p_--1}}\right).\end{aligned}$$

Since $\underline{u} \in L^\infty_{loc}(\Omega)$ and $w > h$ in K_t, we obtain

$$|\nabla\phi_1| \leq p_-||\phi_s||_\infty\left(|\nabla\underline{u}| + |\nabla w|\right) + ||\nabla\phi_s||_\infty\left(|\underline{u}| + |w|\right)$$

and

$$|\nabla\phi_2| \leq \frac{p_-||\phi_s||_\infty||\underline{u}||^{p_-}_{L^\infty(K_R)}}{h^{p_-}}\left(|\nabla\underline{u}| + |\nabla w|\right) + ||\nabla\phi_s||_\infty\frac{||\underline{u}||^{p_-}_{L^\infty(K_R)}}{h^{p_-}},$$

that is, $\phi_1, \phi_2 \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ with *supp* (ϕ_1), *supp* $(\phi_2) \subset K_t \subset\subset \Omega$. Besides that we infer that $\phi_1, \phi_2 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, by the using (2.11) and Proposition 1.2.1.

By taking ϕ_2 as a test function in Lemma 2.2.3, we obtain

$$\begin{aligned}\int_{K_t} |\nabla w|^{p(x)-2}\nabla w\nabla\left(\phi_s\frac{(\underline{u}-h)^{p_-} - (w+h)^{p_-}}{(w+h)^{p_--1}}\right)dx& \\ \geq \int_{K_t} g_h(x, w+h)\frac{(\underline{u}-h)^{p_-} - (w+h)^{p_-}}{(w+h)^{p_--1}}\phi_s dx& \end{aligned} \tag{2.12}$$

and by repeating the density arguments used in Lemma 2.2.3, we can take ϕ_1 as test function in Definition 2.1.2 to obtain

$$\int_{K_t} |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla\left(\phi_s \frac{(\underline{u}-h)^{p_-}-(w+h)^{p_-}}{(\underline{u}-h)^{p_--1}}\right)dx$$
$$\leq \int_{K_t} g(x,\underline{u})\frac{(\underline{u}-h)^{p_-}-(w+h)^{p_-}}{(\underline{u}-h)^{p_--1}}\phi_s dx. \quad (2.13)$$

Now, let us do the next two estimates to come back in (2.12) and (2.13) for further information. First, by using the definition of ϕ_s and the fact that $\underline{u} > 2h$ in Ω_h, we get

$$\begin{aligned}
&\left|\int_{K_t} \left((\underline{u}-h)^{p_-}-(w+h)^{p_-}\right)\left(\frac{|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}}{(\underline{u}-h)^{p_--1}} - \frac{|\nabla w|^{p(x)-2}\nabla w}{(w+h)^{p_--1}}\right)\nabla\phi_s dx\right| \\
= &\left|\int_{K_t\setminus K_s} \left((\underline{u}-h)^{p_-}-(w+h)^{p_-}\right)\left(\frac{|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}}{(\underline{u}-h)^{p_--1}} - \frac{|\nabla w|^{p(x)-2}\nabla w}{(w+h)^{p_--1}}\right)\nabla\phi_s dx\right| \\
\leq\ & 2h^{1-p_-}\frac{||\underline{u}-h||^{p_-}_{L^\infty(K_R)}}{(t-s)^{\frac{1}{2p_+}}}\int_{K_t\setminus K_s}\left(|\nabla \underline{u}|^{p(x)-1}+|\nabla w|^{p(x)-1}\right)dx \\
\leq\ & 2h^{1-p_-}\frac{||\underline{u}-h||^{p_-}_{L^\infty(K_R)}}{(t-s)^{\frac{1}{2p_+}}}\left|\left||\nabla \underline{u}|^{p(x)-1}+|\nabla w|^{p(x)-1}\right|\right|_{L^{\frac{p(x)}{p(x)-1}}(K_t\setminus K_s)}||1||_{L^{p(x)}(K_t\setminus K_s)} \\
\leq\ & 2h^{1-p_-}\frac{||\underline{u}-h||^{p_-}_{L^\infty(K_R)}}{(t-s)^{\frac{1}{2p_+}}}\left|\left||\nabla \underline{u}|^{p(x)-1}+|\nabla w|^{p(x)-1}\right|\right|_{L^{\frac{p(x)}{p(x)-1}}(K_R)}||1||_{L^{p_+}(K_t\setminus K_s)} \\
\leq\ & C(t-s)^{\frac{1}{p_+}}.
\end{aligned} \quad (2.14)$$

where $C = C\left(||w||_{W^{1,p(x)}(K_R)}, ||\underline{u}||_{W^{1,p(x)}(K_R)}, ||\underline{u}||_{L^\infty(K_R)}, R\right)$is a real constant.

Second, by using Lemma 2.2, we obtain that

$$\int_{K_t}\left[|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla\left(\frac{(\underline{u}-h)^{p_-}-(w+h)^{p_-}}{(\underline{u}-h)^{p_--1}}\right)\right.$$
$$\left. - |\nabla w|^{p(x)-2}\nabla w\nabla\left(\frac{(\underline{u}-h)^{p_-}-(w+h)^{p_-}}{(w+h)^{p_--1}}\right)\right]\phi_s dx \quad (2.15)$$

is non-negative.

So, by subtracting (2.13) by (2.12) and using (2.14) and (2.15), we obtain

$$\begin{aligned}
0 \leq &\int_{K_t}\left[|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla\left(\frac{(\underline{u}-h)^{p_-}-(w+h)^{p_-}}{(\underline{u}-h)^{p_--1}}\right)\right. \quad (2.16)\\
&\left.- |\nabla w|^{p(x)-2}\nabla w\nabla\left(\frac{(\underline{u}-h)^{p_-}-(w+h)^{p_-}}{(w+h)^{p_--1}}\right)\right]\phi_s dx \\
\leq &\int_{K_t}\left(\frac{g(x,\underline{u})}{(\underline{u}-h)^{p_--1}} - \frac{g_h(x,w+h)}{(w+h)^{p_--1}}\right)\left((\underline{u}-h)^{p_-}-(w+h)^{p_-}\right)\phi_s dx + C(t-s)^{\frac{1}{p_+}} \\
= &\int_{K_t}\left(\frac{g_h(x,\underline{u}-h)}{(\underline{u}-h)^{p_--1}} - \frac{g_h(x,w+h)}{(w+h)^{p_--1}}\right)\left((\underline{u}-h)^{p_-}-(w+h)^{p_-}\right)\phi_s dx + C(t-s)^{\frac{1}{p_+}}
\end{aligned}$$

for each $0 < s < t$ given. That is, by using that $\phi_s(x) \to 1$ as $s \to t$ a.e. in Ω, it follows from (2.16) and Fatou's Lemma, that

$$0 \leq \int_{K_t} \left(\frac{g_h(x, \underline{u} - h)}{(\underline{u} - h)^{p_- - 1}} - \frac{g_h(x, w + h)}{(w + h)^{p_- - 1}} \right) ((\underline{u} - h)^{p_-} - (w + h)^{p_-}) \, dx. \tag{2.17}$$

As the hypothesis (g_2) implies that

$$\frac{g_h(x, t)}{t^{p_- - 1}} \text{ is strictly decreasing for } t > 0,$$

we obtain from (2.17) that

$$0 \leq \int_{K_t} \left(\frac{g_h(x, \underline{u} - h)}{(\underline{u} - h)^{p_- - 1}} - \frac{g_h(x, w + h)}{(w + h)^{p_- - 1}} \right) ((\underline{u} - h)^{p_-} - (w + h)^{p_-}) \, dx < 0,$$

but this is impossible. Then Ω_h has null Lebesgue measure for all $h > 0$, that is,

$$\underline{u} \leq w + 2h \leq \overline{u} + 2h \quad \text{a.e. in } \Omega$$

for all $h > 0$. So, letting $h \to 0$, we obtain $\underline{u} \leq \overline{u}$ a.e. in Ω, as desired. This ends our proof. ■

Chapter 3

Uniqueness of $W_{loc}^{1,p(x)}(\Omega)$-solution for a oscillating-singular-concave problem

3.1 Introduction

In this chapter we study the following quasilinear elliptic singular-concave problem with variable exponents and powers

$$\begin{cases} -\Delta_{p(x)}u = c(x)d(x)^{-\beta(x)}u^{-\alpha(x)} + \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with smooth boundary, $\lambda \geq 0$ is a real parameter, $p : \overline{\Omega} \to \mathbb{R}$ is a $C^1(\overline{\Omega})$-function that satisfies

$$1 < p_- = \min_{x\in\overline{\Omega}} p(x) \leq p_+ = \max_{x\in\overline{\Omega}} p(x) < N$$

and $d(x) = \inf_{y\in\partial\Omega} |x-y|$ for $x \in \Omega$ is the standard distance function to the boundary of Ω.

Inspired on ideas of [17], for each $\Gamma \subset \partial\Omega$ smooth enough and $h \in C^1(\overline{\Omega})$ given, let us define

$$W_\Gamma^{1,h(x)}(\Omega) = \big\{u \in W^{1,h(x)}(U) \ / \ u\big|_\Gamma = 0 \ \text{ in the trace sense}\big\} \tag{3.2}$$

for all open sets $U \subseteq \Omega$ such that $\partial U \cap \partial\Omega = \Gamma$. In special, we notice that

$$W_\Gamma^{1,h(x)}(\Omega) = \begin{cases} W_{loc}^{1,h(x)}(\Omega) \text{ if } \Gamma = \emptyset, \\ \\ W_0^{1,h(x)}(\Omega) \text{ if } \Gamma = \partial\Omega. \end{cases}$$

The trace over Γ is well defined if, for example, $\partial\Omega$ is Lipschitz continuous (see [28, Chapter 12]).

Throughout this chapter we adopt the following definition of solution:

Definition 3.1.1 *A positive function $u \in W_{\Gamma}^{1,p(x)}(\Omega)$ is a solution to problem* (3.1) *if $u \leq 0$ on $\partial\Omega$ in sense of Definition* 2.1.1 *and*

(i) $a(x)u(x)^{-\alpha(x)} \in L^1_{loc}(\Omega)$;

(ii) $\operatorname*{ess\,inf}_{x\in K} u(x) > 0$ *for all* $K \subset\subset \Omega$;

(iii) for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\phi dx = \int_\Omega c(x)d(x)^{-\beta(x)}u^{-\alpha(x)}\phi dx + \lambda\int_\Omega f(x,u)\phi dx.$$

To state ours results, let us denote the interior strip around of the boundary of Ω by Ω_δ, that is,

$$\Omega_\delta := \{x \in \Omega \ / \ d(x) < \delta\} \text{ for each } \delta > 0 \text{ given}$$

and define the numbers

$$\theta_1 = \begin{cases} \max\limits_{x\in\overline{\Omega}_\delta} \dfrac{p(x)-\beta(x)}{p(x)+\alpha(x)-1} & \text{if } \beta(x)+\alpha(x) > 1 \text{ in } \Omega_\delta, \\ 1 & \text{if } \beta(x)+\alpha(x) \leq 1 \text{ in } \Omega_\delta, \end{cases} \qquad \theta_2 = \min_{x\in\overline{\Omega}_\delta} \frac{p(x)-\beta(x)}{p(x)+\alpha(x)-1}. \tag{3.3}$$

Related to the functions $\alpha(x), \beta(x), c(x)$ and $f(x,t)$, we make the following assumptions: Assume that there exists a $\delta > 0$ such that:

(H_1) $\alpha : \overline{\Omega} \to \mathbb{R}$ is a $C^{0,1}(\overline{\Omega})$-function that satisfies $\alpha(x) \geq \min_{x\in\overline{\Omega}} \alpha(x) := \alpha_- > 1-p_-$,

(H_2) $f : \Omega \times [0,\infty) \to [0,\infty)$ is a Carathéodory function such that

$$f(x,t) \leq b(x)(1+t^{q(x)-1}) \text{ for all } x \in \Omega$$

holds true, for some functions $q \in C^1(\overline{\Omega})$ and $0 \leq b \in L^{s(x)}(\Omega) \cap L^\infty(\Omega_\delta)$ with $1 < q_- \leq q_+ \leq p_-$ and $s(x) > N/p_-$ for $x \in \Omega$, where

(H_3) (i) $\beta : \overline{\Omega} \to \mathbb{R}$ is a $C^{0,1}(\overline{\Omega})$-function that satisfies $\beta_+ < p_-$,

(ii) $0 < c \in L^{r(x)}(\Omega) \cap L^\infty(\Omega_\delta)$ for some $r \in C^1(\overline{\Omega})$ with $1 \leq r(x) \leq +\infty$,

(iii) $c(x)/(1-\alpha(x)) \in L^{r(x)}(\Omega) \cap L^\infty(\Omega_\delta)$,

(H_4) $\dfrac{f(x,t)}{t^{p_- -1}}$ is strictly decreasing on $(0,\infty)$ for a.e. $x \in \Omega$.

The main objective of this chapter is provide sufficient conditions for existence, regularity and uniqueness of $W^{1,p(x)}_{loc}(\Omega)$-solutions to the problem (3.1) in sense of Definition 3.1.1. For this reason, let us consider the $C^{0,1}$-manifold

$$\Gamma_t = \{x \in \partial\Omega \ / \ [-\beta(x) + t(1-\alpha(x))]\frac{1}{1-1/r(x)} + 1 > 0\},$$

and the number

$$\sigma = \max\left\{\frac{p_- + (\beta_+ - 1)/\theta_2 + \alpha_+ - 1}{p_-}, \frac{p_- + \alpha_+ - 1}{p_-}\right\}.$$

Our first result is related to existence of solutions and it is formulated as follows.

Theorem 3.1.2 *Assume $(H_1) - (H_4)$. If*

$$r(x) = \begin{cases} \left(\frac{\sigma p^*_-}{p_-(\sigma-1)+1-\alpha(x)}\right)' & \text{if } |\beta(x)+\alpha(x) > 1| > 0 \text{ in } \Omega_\delta, \\ \left(\frac{p^*(x)}{1-\alpha_-}\right)' & \text{if } |\beta(x)+\alpha(x) > 1| = 0 \text{ in } \Omega_\delta, \end{cases}$$

then there exists a $0 < \lambda^* \leq \infty$ such that the problem (3.1) admits a solution $u = u_\lambda \in W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$ with $u(x) \geq Cd(x)$, $x \in \Omega$ for each $0 \leq \lambda < \lambda^*$ given and for some $C > 0$. In addition:

(i) if $q_+ < p_-$ in (H_2), then $\lambda^* = \infty$,

(ii) if $c(x) \geq c_\delta$ in Ω_δ for some $c_\delta > 0$, then there exists a $c > 0$ such that $u(x) \geq cd(x)^{\theta_1}$ for $x \in \Omega_\delta$ and, in particular, $u \in W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega)$.

When $\beta \equiv 0$, we are able to highlight how the regularity of $c(x)$ influences the behavior of the solution for (3.1) close to the boundary of Ω.

Corollary 3.1.3 *Assume $(H_1), (H_2)$, (H_4) and $\beta \equiv 0$. If*

$$r(x) = \begin{cases} \left(\frac{p^*_-(p_-+\alpha_+-1)}{(\alpha_+-\alpha(x))p_-}\right)' & \text{if } |\alpha(x) > 1| > 0 \text{ in } \Omega_\delta, \\ \left(\frac{p^*(x)}{1-\alpha_-}\right)' & \text{if } |\alpha(x) > 1| = 0 \text{ in } \Omega_\delta, \end{cases}$$

then there exists a $0 < \lambda^* \leq \infty$ such that the problem (3.1) admits a solution $u = u_\lambda \in W^{1,p(x)}_{\Gamma_1}(\Omega)$ for each $0 \leq \lambda < \lambda^*$ given, with $u(x) \geq Cd(x)$, $x \in \Omega$ for some $C > 0$. In addition:

(i) if $c(x) \in L^\infty(\Omega_\delta)$, then $u(x) \leq Md(x)^{\theta_2}$ for $x \in \Omega_\delta$ and some $M > 0$ and, in particular, $u \in W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$.

(ii) *if* $c(x) \in L^\infty(\Omega_\delta)$ *and* $c(x) \geq c_\delta$ *in* Ω_δ *for some* $c_\delta > 0$, *then there exists a* $m > 0$ *such that* $md(x)^{\theta_1} \leq u(x) \leq Md(x)^{\theta_2}$ *for* $x \in \Omega_\delta$. *In particular,* $u \in W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega)$.

In any case, if $q_+ < p_-$ *in* (H_2), *then* $\lambda^* = \infty$.

The second result deals with regularity of solutions obtained in Theorem 3.1.2

Theorem 3.1.4 *Assume that* $(H_1) - (H_4)$ *hold true. Let* $u \in W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$ *be the solution of Problem* (3.1) *given by Theorem* 3.1.2. *Then there exists a* $0 < \lambda_* \leq \infty$, *possibly smaller than* λ^* *given in Theorem* 3.1.2, *such that for all* $0 \leq \lambda < \lambda_*$, *we have:*

(i) $u \in L^\infty(\Omega)$ *if* $r(x) > N/p_-$,

(ii) $u \in L^{\frac{Nr_-(p_-+\alpha_--1)}{N-r_-p_-}}(\Omega)$ *if* $|\beta(x) + \alpha(x) > 1| > 0$ *in* Ω_δ *and* $r_- < N/p_-$ *with and*

$$\max\left\{\frac{N(p_- + \alpha_+ - 1)}{(N-p)(p_- + \alpha_- - 1) + p_-(p_- + \alpha_+ - 1)}, \frac{N(p_- + \frac{\beta_+-1}{\theta_2} + \alpha_+ - 1)}{(N-p)(p_- + \alpha_- - 1) + p_-(p_- + \frac{\beta_+-1}{\theta_2} + \alpha_+ - 1)}\right\} \leq r_-.$$

(iii) $u \in L^{\frac{Nr_-(p_-+\alpha_--1)}{N-r_-p_-}}(\Omega)$ *if* $|\beta(x) + \alpha(x) > 1| = 0$ *in* Ω_δ *and*

$$\frac{Np_-}{Np_- - (N-p_-)(1-\alpha_-)} \leq r_- < \frac{N}{p_-}.$$

In addition if $q_+ < p_-$ *in* (H_2), *then* $\lambda_* = \infty$. *Besides this, the same conclusions hold true if we change* Γ_1 *by* Γ_{θ_1}.

As a consequence of Theorem 3.1.4(i), we get the Hölder continuity up to the boundary of with some restriction is placed on the domain. We say that Ω satisfies an exterior cone condition at a point $x \in \partial\Omega$ if there exists a finite right circular cone V_x with vertex x such that $\overline{\Omega} \cap V_x = x$, in particular, say that Ω satisfes an uniform exterior cone condition on $\partial\Omega$ if Ω satisfes an exterior cone condition at every $x \in \partial\Omega$ and the cones V_x are all congruent to some fixed cone V (see Section 8.10 of [42]).

Corollary 3.1.5 *Assume that* $(H_1) - (H_4)$ *holds true with* $r(x) > N/p_-$. *If* u *is a solution of* (3.1) *given in Theorem* 3.1.2, *then* $u \in C^{0,\gamma}(\overline{U})$ *for all open set* $U \subset \Omega$ *with* $\partial U \cap \partial\Omega = \Gamma_1 \cup \Gamma_{\theta_2}$ *satisfying a uniform exterior cone condition on* $\partial U \cap \partial\Omega$.

To end, we present a sufficient condition for uniqueness of solution for (3.1).

Theorem 3.1.6 *Assume that* $(H_1)-(H_4)$ *holds true with* $r(x) > N/p_-$. *If* $\beta(x) < 1$ *on* $\partial\Omega$, *then there exists a* $0 < \lambda_{**} \leq \infty$, *possibly smaller than* λ^* *given in Theorem* 3.1.2, *such that for all* $0 \leq \lambda < \lambda_{**}$ *the problem* (3.1) *admits an* ***only*** *solution in* $W^{1,p(x)}_{loc}(\Omega)$ *in sense of Definition* 2.1.2. *Beside this,* $\lambda_{**} = +\infty$ *if* $p_- = q_+$ *in* (H_2).

Before going on to the proofs, a comment on the powers α and β should be done. We will assume that all the sets $\{0 < \alpha(x) < 1\}, \{\alpha(x) > 1\}, \{\alpha(x) \leq 0\}, \{\beta(x) > 0\}$ and $\{\beta(x) \leq 0\}$ have a positive Lebesgue measure. We emphasize that if one or more of them has null measure, the result will still be valid and the proofs become simpler.

The chapter is organized as follows. In section 3.2, we consider the approximated problems and prove the existence of the approximated solutions in $W_0^{1,p(x)}(\Omega)$ satisfying the Definition 3.1.1. Moreover, we prove the boundedness of these solutions in $W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$ and the asymptotic behavior depending of the trio $(c(x), \alpha(x), \beta(x))$ on the boundary. In section 3.3 is devoted to prove our results by using all the properties that we have proved in the previous sections.

3.2 A family of auxiliary problems

In order to prove our results, we inspired in some ideas of Boccardo and Orsina [6] who work by "regularizing" the singular term by a small perturbation $1/n$ and studying the behavior of a sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ of solutions for this approximated problems. In general, that sequence is obtained by a fixed point argument. We shall employ a different approach based on a Generalized Galerkin method.

From now on, we will understand that $f(x,t)$ has been extended for $t < 0$ by putting $f(x,t) = f(x,0)$.

Let us consider the family of regularized problems

$$\begin{cases} -\Delta_{p(x)}u = c_n(x)\left(d(x) + \dfrac{1}{n}\right)^{-\beta(x)}\left(u + \dfrac{1}{n}\right)^{-\alpha(x)} + \lambda f_n(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.4}$$

where $c_n(x) = \min\{c(x), n\}$ and $f_n(x,t) = \min\{f(x,t), n\}$. We note that $u \in W_0^{1,p(x)}(\Omega)$ is a solution of (3.4) if, and only if, u is such that

$$A(u,v) = 0 \text{ for all } v \in W_0^{1,p(x)}(\Omega),$$

where the functional $A : W_0^{1,p(x)}(\Omega) \times W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is defined by

$$A(u,v) = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v - c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)}\left(|u| + \frac{1}{n}\right)^{-\alpha(x)} v + \lambda f_n(x,|u|)v \right) dx.$$

Lemma 3.2.1 *Assume that (H_1) and (H_3) holds true. The operator $T := T_{n,\lambda}$, defined by $\langle T(u), v\rangle = A(u,v)$ for all $v \in W_0^{1,p(x)}(\Omega)$ and for each $u \in W_0^{1,p(x)}(\Omega)$, is linear and continuous, that is, $T(u) \in W^{-1,p'(x)}(\Omega)$ for each $u \in W_0^{1,p(x)}(\Omega)$ given.*

Proof. The linearity is obvious. To show the continuity, first we notice that the hypothesis (H_1) implies that

$$\begin{aligned} c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(|u|+\frac{1}{n}\right)^{-\alpha(x)} &\le n\left(n^{\beta(x)}+(d(x)+1)^{-\beta_-}\right)\left(n^{\alpha(x)}+(|u|+1)^{-\alpha_-}\right) \\ &\le C_1(n)\left(1+(|u|+1)^{-\alpha_-}\right). \end{aligned} \tag{3.5}$$

So, it follows from (3.5), Hölder's Inequality and Sobolev embeddings, that

$$\begin{aligned} |A(u,v)| &\le \int_\Omega |\nabla u|^{p(x)-1}|\nabla v|dx + C_2(n)\int_\Omega \left(1+(|u|+1)^{-\alpha_-}\right)|v|dx \\ &\le C_3(n)\left[\left\||\nabla u|^{p(x)-1}\right\|_{\frac{p(x)}{p(x)-1}}||\nabla v||_{p(x)} + \left(1+||1||_{\frac{p(x)}{p(x)+\alpha_--1}}\left\|(1+|u|)^{-\alpha_-}\right\|_{\frac{p(x)}{-\alpha_-}}\right)||v||_{p(x)}\right] \\ &\le C_4(n)\left(||u||^{-\alpha_-}+1\right)||v||, \end{aligned} \tag{3.6}$$

recalling that we are assuming that $\alpha_- < 0$. Thus,

$$|\langle T(u),v\rangle| = |A(u,v)| \le C(n,||u||)\|v\| \text{ for all } v \in W_0^{1,p(x)}(\Omega),$$

showing the continuity of $T(u)$ for each $u \in W_0^{1,p(x)}(\Omega)$ given. This ends the proof of Lemma 3.2.1. ■

As a consequence of the above Lemma, we note to find a weak solution to problem (3.4) is equivalent to obtain an $u_n \in W_0^{1,p(x)}(\Omega)$ such that $T(u_n) = 0$. To do this, we begin by fixing a $0 < \psi \in C_0^\infty(\Omega)$ such that

$$c(x)d(x)^{-\beta(x)}\psi \neq 0 \quad\text{and}\quad c(x)d(x)^{-\beta(x)}\psi \in L^1(\Omega). \tag{3.7}$$

Let $F \subset W_0^{1,p(x)}(\Omega)$ be a finite dimensional subspace with $\psi \in F$ and $T_F : F \to F^*$ a function defined by $T_F = I_F^\star \circ T \circ I_F$, where

$$\begin{aligned} I_F : (F, ||\cdot||) &\to \left(W_0^{1,p(x)}(\Omega)\right) \\ u &\mapsto I_F(u) = u. \end{aligned}$$

and $I_F^\star$ is an adjoint operator of I_F. We note that $T_F = T|_F$, that is, for all $u, v \in F$, we have

$$\begin{aligned} \langle T_F(u),v\rangle = &\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx \\ &-\int_\Omega\left[c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(|u|+\frac{1}{n}\right)^{-\alpha(x)}v+\lambda f_n(x,|u|)v\right]dx \end{aligned} \tag{3.8}$$

Below, let us find a zero of T_F for each finite dimensional subspace $F \subset W_0^{1,p(x)}(\Omega)$ given with $\psi \in F$.

Lemma 3.2.2 *Assume (H_1) and (H_3). Then there exists an $0 \neq u_F = u_{n,\lambda,F} \in F$ such that $T_F(u_F) = 0$.*

Proof. We claim that T_F is a continuous operator. In fact, let $(u_j) \subset F$ with $u_j \to u$ in F. From Proposition 1.2.4, the operator $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ given by

$$\langle L(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx,$$

is continuous.

Now, from Proposition 1.1.3 and the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we have that $u_j(x) \to u(x)$ a.e. in Ω and exists $h \in L^{p(x)}(\Omega)$ such that $|u_j| \leq h$, unless of subsequence. As a consequence,

$$\begin{aligned} c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(|u_j|+\frac{1}{n}\right)^{-\alpha(x)} v \to c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(|u|+\frac{1}{n}\right)^{-\alpha(x)} v \\ f_n(x,|u_j|)v \to f_n(x,|u|)v \end{aligned} \tag{3.9}$$

a.e. in Ω for each $v \in W_0^{1,p(x)}(\Omega)$. So, the informations at (3.5) and (3.9) permit us to use Lebesgue's Dominated Convergence Theorem to conclude that T_F is continuous.

Now, let $m = dim(F)$ be the dimension of F and $(e_n)_{n=1}^m$ be an orthonormal basis of F, that is, each $u \in F$ is uniquely expressed as

$$\sum_{i=1}^m \eta_i e_i \text{ for some } \eta = (\eta_1, \eta_2, ..., \eta_m) \in \mathbb{R}^m.$$

This permit us to define $i = i_F : (\mathbb{R}^m, |\cdot|) \to (F, ||\cdot||)$ by $i(\eta) = u$ and set $|\eta| = ||u||$. By using this and the continuity of T_F, we obtain that the operator $S_F : \mathbb{R}^s \to \mathbb{R}^s$ defined by $S_F = i^\star \circ T_F \circ i$ is continuous, where $i^\star$ stands for the adjoint operator of i. Let $u = i(\eta)$ for $\eta \in \mathbb{R}^m$. So, it follows from (3.5), Proposition 1.1.1, Hölder's inequality and the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, that

$$\begin{aligned} \langle S_F(\eta), \eta\rangle &= \langle i^\star \circ T_F \circ i(\eta), \eta\rangle = \langle T_F(u), u\rangle \\ &\geq \int_\Omega |\nabla u|^{p(x)}dx - C(n)\int_\Omega \left[|u| + (|u|+1)^{1-\alpha_-}\right]dx - \lambda n \int_\Omega |u|dx \\ &\geq \max\{||u||^{p_-}, ||u||^{p_+}\} - C_5(n)\left(||u||_{p(x)} + ||1+|u|||_{p(x)}^{1-\alpha_-}\right) \\ &\geq \max\{||u||^{p_-}, ||u||^{p_+}\} - C_6(n)\left(||u|| + ||1+|u|||^{1-\alpha_-}\right). \end{aligned} \tag{3.10}$$

Now, if $p_- > 1-\alpha_-$, then we are able to choose an $r_0 = r_0(n) > 1$ such that $(S_F(\eta), \eta) > 0$ for each $|\eta| = ||u|| = r_0$.

So, by using Lemma A.1.4, there exists $\eta_F \in \overline{B}_{r_0}(0)$ such that $S_F(\eta_F) = 0$, that is, by letting $u_F = i(\eta_F)$ and $v = i(\nu)$, we conclude that

$$\langle T_F(u_F), v\rangle = \langle S_F(\eta_F), \nu\rangle = 0 \text{ for all } v \in F,$$

which implies, for all $v \in F$, that

$$\begin{aligned} &\int_\Omega |\nabla u_F|^{p(x)-2}\nabla u_F \nabla v dx \\ &\quad = \int_\Omega c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(|u_F|+\frac{1}{n}\right)^{-\alpha(x)} v dx - \lambda \int_\Omega f_n(x,|u_F|)v dx. \end{aligned}$$

Finally, assume by contradiction that $u_F = 0$. By (3.7) and taking $v = \psi$, we obtain

$$0 \leq \int_\Omega c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)} n^{\alpha(x)}\psi dx = -\lambda \int_\Omega f_n(x,0)\psi \leq 0,$$

since $f_n(x,t) \geq 0$ for all $t \geq 0$. So, would follow that $\int_\Omega c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)} \psi dx \leq 0$. Thus, by Fatou's Lemma,

$$0 \leq \int_\Omega c(x)d(x)^{-\beta(x)}\psi dx \leq \liminf_{n\to\infty} \int_\Omega c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)} \psi dx = 0$$

but this is impossible, since $c(x)d^{-\beta(x)}\psi \neq 0$. So, $u_F \neq 0$. This finish the proof. ∎

Proposition 3.2.3 *Assume that* (H_1) *and* (H_3) *holds true. Then the problem* (3.4) *has a weak solution* $u_n \in W_0^{1,p(x)}(\Omega)$ *for each* $n \in \mathbb{N}$ *given.*

Proof. Let ψ as in (3.7) and set

$$\mathcal{A} = \left\{F \subset W_0^{1,p(x)}(\Omega) \ / \ \psi \in F \text{ and } F \text{ is a finite dimensional subspace of } W_0^{1,p(x)}\right\}.$$

Given $F_0 \in \mathcal{A}$, let

$$V_{F_0} = \{u_F \in F \ / \ F \in \mathcal{A},\ F_0 \subset F,\ T_F(u_F) = 0 \text{ e } ||u_F|| \leq r_0\}$$

and note that $V_{F_0} \neq \emptyset$, as a consequence of Lemma 3.2.2. Since $V_{F_0} \subset B_{r_0}(0)$, we have $\overline{V}^\sigma_{F_0} \subset B_{r_0}(0)$, where $\overline{V}^\sigma_{F_0}$ is the weak closure of V_{F_0} and $B_{r_0}(0)$ is the closed ball on F. So $\overline{V}^\sigma_{F_0}$ is weakly compact.

Now, consider the set

$$\mathcal{B} = \left\{\overline{V_F}^\sigma \mid F \in \mathcal{A}\right\},$$

and a finite subfamily

$$\{\overline{V}^\sigma_{F_1}, \overline{V}^\sigma_{F_2}, ..., \overline{V}^\sigma_{F_n}\} \subset \mathcal{B},$$

where $F := \text{span}\{F_1, F_2, ...F_n\}$. By definition of V_{F_i}, we have $u_F \in \overline{V}^\sigma_{F_i}$ for $i = 1, 2, ..., n$, that is,

$$\bigcap_{i=1}^n \overline{V}^\sigma_{F_i} \neq \emptyset$$

showing that $\mathcal{B}$ has the finite intersection property. Since $B_{r_0}(0)$ is weakly compact, it follows from Proposition A.1.5, that

$$W = \bigcap_{F\in\mathcal{A}} \overline{V_F}^\sigma \neq \emptyset.$$

Let $u_n \in W$. Given $\phi \in W_0^{1,p(x)}(\Omega)$, take $F_0 \in \mathcal{A}$ such that $\text{span}\{\psi, u_n, \phi\} \subset F_0$. Since $u_n \subset B_{r_0}$, it follows from Proposition A.1.6 that there exists $(u_{n,j}) \subset V_{F_0}$ and $F_j = F_{n,j} \subset \mathcal{A}$ such that $u_{n,j} \rightharpoonup u_n$ in $W_0^{1,p(x)}(\Omega)$ with $||u_{n,j}|| \leq r_0$ and

$$\int_\Omega |\nabla u_{n,j}|^{p(x)-2}\nabla u_{n,j}\nabla v dx$$
$$= \int_\Omega \left(c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(|u_{n,j}|+\frac{1}{n}\right)^{-\alpha(x)} + \lambda f_n(x,|u_{n,j}|)\right) v dx. \tag{3.11}$$

for all $v \in F_j$. By passing to a subsequence if necessary, we have $u_{n,j} \to u_n$ in $L^{h(x)}(\Omega)$ for all $1 < h(x) < p^*(x)$ and $u_{n,j}(x) \to u_n(x)$ a.e. in Ω. So, by taking $v = u_{n,j} - u_n \in F_j$ in (3.11), it follows from (3.5), that

$$\begin{aligned}&\int_\Omega |\nabla u_{n,j}|^{p(x)-2}\nabla u_{n,j}(\nabla u_{n,j} - \nabla u_n)dx \\ &= \int_\Omega \left(c_n(x) \left(d(x) + \frac{1}{n} \right)^{-\beta(x)} \left(|u_{n,j}| + \frac{1}{n} \right)^{-\alpha(x)} + \lambda f_n(x, |u_{n,j}|) \right) (u_{n,j} - u_n)dx \\ &\leq C(n) \left(||u_{n,j} + 1||_{p(x)}^{-\alpha_-} + 1 \right) ||u_{n,j} - u_n||_{p(x)},\end{aligned}$$

that is,

$$\limsup_{j\to\infty} \int_\Omega |\nabla u_{n,j}|^{p(x)-2}\nabla u_{n,j}(\nabla u_{n,j} - \nabla u_n)dx \leq 0.$$

and a consequence of this, we have that $u_{n,j} \to u_n$ in $W_0^{1,p(x)}(\Omega)$, by using Proposition 1.2.4. So, passing to a subsequence if necessary, we have that $\nabla u_{n,j}(x) \to \nabla u_n(x)$ a.e. in Ω, which lead us to conclude that

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v dx = \int_\Omega \left(c_n(x) \left(d(x) + \frac{1}{n} \right)^{-\beta(x)} \left(|u_n| + \frac{1}{n} \right)^{-\alpha(x)} + \lambda f_n(x, |u_n|) \right) v dx$$

holds true for all $v \in F_j$. Since we can take $v = \phi$ and ϕ was taken arbitrary, then we obtain

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla \phi dx = \int_\Omega \left(c_n(x) \left(d(x) + \frac{1}{n} \right)^{-\beta(x)} \left(|u_n| + \frac{1}{n} \right)^{-\alpha(x)} + \lambda f_n(x, |u_n|) \right) \phi dx.$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$.

Now, let us show that $u_n > 0$ in Ω. Arguing as in the proof of Lemma 3.2.2 we infer that $u_n \neq 0$. More, by taking $\phi = -u_n^-$, we obtain

$$\int_\Omega |\nabla u_n^-|^{p(x)}dx \leq -\int_\Omega c_n(x) \left(d(x) + \frac{1}{n} \right)^{-\beta(x)} \left(u_n^- + \frac{1}{n} \right)^{-\alpha(x)} u_n^- dx - \lambda \int_\Omega f(x, u_n^-)u_n^- dx \leq 0,$$

which implies that $u_n^- \equiv 0$. So,

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla \phi dx \\ &\qquad = \int_\Omega \left(c_n(x) \left(d(x) + \frac{1}{n} \right)^{-\beta(x)} \left(u_n + \frac{1}{n} \right)^{-\alpha(x)} + \lambda f_n(x, u_n) \right) \phi dx. \quad (3.12)\end{aligned}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$. By using that Theorem 1.3.9, it follows that $u_n > 0$ in Ω. To end, by Proposition 1.3.8, $u_n \in C^{1,\gamma_n}(\overline{\Omega})$ for some $\gamma_n \in (0, 1)$, finishing the proof. ■

Now, let us verify an assumption of the our Comparison Principle holds true for our problem.

Lemma 3.2.4 *Assume* $(H_1), (H_3)$ *and* (H_4) *holds true. The functional* $I_h : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, *defined by*

$$I_h(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\{\alpha(x)=1\}} c_n(x) \left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \ln\left(\frac{\frac{1}{n} + 2h}{u^+ + \frac{1}{n} + 2h}\right) dx$$
$$+ \int_{\{\alpha(x)<1\}} \frac{c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left[\left(\frac{1}{n} + 2h\right)^{1-\alpha(x)} - \left(u^+ + \frac{1}{n} + 2h\right)^{1-\alpha(x)}\right]}{1 - \alpha(x)} dx$$
$$+ \int_{\{\alpha(x)>1\}} \frac{c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left[\left(u^+ + \frac{1}{n} + 2h\right)^{1-\alpha(x)} - \left(\frac{1}{n} + 2h\right)^{1-\alpha(x)}\right]}{\alpha(x) - 1} dx$$
$$+ \lambda \int_\Omega [F_n(x, 2h) - F_n(x, u^+ + 2h)] dx$$

is coercive and weakly lower semicontinuous for each $h > 0$ *and* $n \in \mathbb{N}$ *given, where* $F_n(x,t) = \int_0^t f_n(x,s)ds$.

Proof. By using $\ln s \le s$ for all $s > 0$, Hölder's Inequality and Sobolev embedding, we obtain that

$$I_h(u) \;\ge\; \frac{1}{p^+} \min\left\{||u||^{p^-}, ||u||^{p^+}\right\} - C\left(||u||^{1-\alpha_-} + ||u|| + 1\right),$$

where $C = C(n, h, \Omega, \alpha_+)$ is a positive real constant. Since $p_- > 1 - \alpha_-$, it follows that I_h is coercive.

To prove the weakly lower semicontinuity of I_h, let $u_j \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$. So, it is well known that $u_j \to u$ in $L^{t(x)}(\Omega)$ for all $1 \le t(x) < p^*(x)$, $u_j(x) \to u(x)$ a.e. in Ω and there exists $\Theta \in L^{t(x)}(\Omega)$ such that $u_n \le \Theta$. Below, let us consider each integral in the definition of I_h. First, by using these informations and Fatou's Lemma, we obtain

$$\liminf_{j\to\infty}\left(\int_{\{\alpha(x)>1\}} \frac{c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(u_j^+ + \frac{1}{n} + 2h\right)^{1-\alpha(x)}}{\alpha(x) - 1} dx\right)$$
$$\ge \int_{\{\alpha(x)>1\}} \frac{c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(u^+ + \frac{1}{n} + 2h\right)^{1-\alpha(x)}}{\alpha(x) - 1} dx.$$

More. Since

$$\int_{\{\alpha(x)=1\}} c_n(x) \left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \ln\left(\frac{\frac{1}{n} + 2h}{u_j^+ + \frac{1}{n} + 2h}\right) dx$$
$$\le \int_{\{\alpha(x)=1\}} c_n(x) \left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(\frac{\frac{1}{n} + 2h}{u_j^+ + \frac{1}{n} + 2h}\right) dx$$
$$\le \int_{\{\alpha(x)=1\}} c_n(x) \left(d(x) + \frac{1}{n}\right)^{-\beta(x)} dx < \infty$$

holds, we obtain by the hypothesis (H_3), that

$$\int_{\{\alpha(x)<1\}} \frac{c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left[\left(\frac{1}{n} + 2h\right)^{1-\alpha(x)} - \left(u_j^+ + \frac{1}{n} + 2h\right)^{1-\alpha(x)}\right]}{1 - \alpha(x)} dx$$
$$\le \int_{\{\alpha(x)<1\}} \frac{c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(\frac{1}{n} + 2h\right)^{1-\alpha(x)}}{1 - \alpha(x)} dx < \infty$$

holds as well. Now, we are able to apply Lebegue's Theorem to obtain

$$\lim_{j\to\infty}\left(\int_{\{\alpha(x)<1\}}\frac{c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left[\left(\frac{1}{n}+2h\right)^{1-\alpha(x)}-\left(u_j^++\frac{1}{n}+2h\right)^{1-\alpha(x)}\right]}{1-\alpha(x)}dx\right)$$
$$=\int_{\{\alpha(x)<1\}}\frac{c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left[\left(\frac{1}{n}+2h\right)^{1-\alpha(x)}-\left(u^++\frac{1}{n}+2h\right)^{1-\alpha(x)}\right]}{1-\alpha(x)}dx,$$

and

$$\lim_{j\to\infty}\left(-\int_{\{\alpha(x)=1\}}c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\ln\left(u_j+\frac{1}{n}+2h\right)dx\right)$$
$$=\int_{\{\alpha(x)=1\}}c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\ln\left(u+\frac{1}{n}+2h\right)dx.$$

Again, to finish our analysis, we just note that $f_n(x,t)\le n$ that implies that

$$|F_n(u_j+2h)|\le n(\Theta+2h)$$

holds. So, once using Lebesgue's Theorem, we obtain

$$\lim_{j\to\infty}\int_\Omega F_n(x,u_j+2h)dx=\int_\Omega F_n(x,u+2h)dx.$$

that is, $I_h(u)\le\liminf I_h(u_j)$, as desired. This ends the proof. ■

The next result is fundamental in our approach.

Proposition 3.2.5 *Assume that $(H_1),(H_3)$ and (H_4) holds true. For each $U\subset\subset\Omega$ given there exists a $C_U>0$, independent of n, such that*

$$u_n(x)\ge C_U>0 \text{ for every } x\in U \text{ and for all } n\ge 1,$$

where $u_n\in C^1(\overline{\Omega})$ is the solution of Problem (3.4) given by Proposition 3.2.3. In addiction, there exists $\delta_1>0$ such that $u_n(x)\ge Cd(x)$ for $x\in\Omega_{\delta_1}$, for some $C>0$ independent of n.

Proof. Fixed $n\in\mathbb{N}$, let $g_n:\Omega\times\mathbb{R}\to\mathbb{R}^+$ be defined by

$$g_n(x,t)=\begin{cases} t^{-\alpha(x)} & \text{if } \alpha(x)>0 \text{ and } t>1/n,\\ n^{\alpha(x)} & \text{if } \alpha(x)\le 0 \text{ or } t\le 1/n\end{cases}$$

and the problem

$$\begin{cases} -\Delta_{p(x)}w=c_n(x)d_n(x)g_n\left(x,|w|+\dfrac{1}{n}\right) \text{ in } \Omega,\\ v\ge 0 \text{ in } \Omega;\quad v=0 \text{ on } \partial\Omega,\end{cases}\tag{3.13}$$

where $c_n(x)=\min\{c(x),n\}$ and $d_n(x)=\min\left\{\left(d(x)+\frac{1}{n}\right)^{-\beta(x)},n^{\beta(x)}\right\}$. Now, by arguing as in Proposition 3.2.3, the problem (3.13) admits a positive solution $w_n\in C^{1,\gamma_n}(\overline{\Omega})$, for some $\gamma_n\in(0,1)$ and for each $n\in\mathbb{N}$ given. Moreover, since

$$c_n(x)d_n(x)g_n\left(x,w_n+\frac{1}{n}\right)\le c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(w_n+\frac{1}{n}\right)^{-\alpha(x)}+\lambda f_n(x,w_n),$$

holds true, we obtain that w_n is a subsolution of Problem (3.4).

We claim that w_n is increasing in n. In fact, defining

$$\Theta(x) = g\left(x, w_n + \frac{1}{n}\right) - g\left(x, w_{n+1} + \frac{1}{n+1}\right),$$

we obtain

$$\Theta(x) = \begin{cases} \left(w_n + \frac{1}{n}\right)^{-\alpha(x)} - \left(w_{n+1} + \frac{1}{n+1}\right)^{-\alpha(x)} & \text{if } \alpha(x) > 0 \text{ and } w_{n+1} + \frac{1}{n+1} > \frac{1}{n}, \\ \left(w_n + \frac{1}{n}\right)^{-\alpha(x)} - n^{\alpha(x)} & \text{if } \alpha(x) > 0 \text{ and } w_{n+1} + \frac{1}{n+1} < \frac{1}{n}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.14}$$

that is, $\Theta(x)(w_n - w_{n+1})^+ \leq 0$ in Ω.

Now, by using that w_n and w_{n+1} are solutions of (3.13) and taking $(w_n - w_{n+1})^+$ as test functions to them, it follows from (3.14), that

$$\begin{aligned} &\int_\Omega \left(|\nabla w_n|^{p(x)-2}\nabla w_n - |\nabla w_{n+1}|^{p(x)-2}\nabla w_{n+1}\right)\nabla(w_n - w_{n+1})^+ dx \\ &= \int_\Omega \left(c_n(x)d_n(x)g\left(x, w_n + \frac{1}{n}\right) - c_{n+1}(x)d_{n+1}(x)g\left(x, w_{n+1} + \frac{1}{n+1}\right)\right)(w_n - w_{n+1})^+ dx \\ &\leq \int_\Omega c_{n+1}(x)d_{n+1}(x)\Theta(x)(w_n - w_{n+1})^+ dx \leq 0, \end{aligned}$$

that lead us to infer that $||(w_n - w_{n+1})^+|| = 0$ thanks to Lemma 1.2.6. In particular, $w_{n+1} \geq w_n$, as claimed.

Let $\tilde{g}_n(x,t) = c_n(x)\left(d(x) + 1/n\right)^{-\beta(x)}(t + 1/n)^{-\alpha(x)} + \lambda f_n(x,t)$. It follows from hypotheses (H_1) and (H_4) that $\tilde{g}_n(x,t)/t^{p_- - 1}$ is strictly decreasing for $t > 0$ and a.e. $x \in \Omega$. More, by Proposition 3.2.4, the functional

$$I_h(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_\Omega \int_0^u \tilde{g}_n(x, s + 2h) ds dx,$$

is coercive and weakly lower semicontinuous for each $h > 0$ and $n \in \mathbb{N}$ given. Then, by using Theorem 2.1.3, we have that $u_n \geq w_n$ in Ω. In particular, it follows from the monotonicity of w_n and continuity of w_1 that there exists a $C_U > 0$ such that $u_n \geq w_1 \geq C_U > 0$ for each $U \subset\subset \Omega$ given.

On the other hand, we know by [42, Lemma 14.16] that $d \in C^2(\overline{\Omega}_{\delta_1})$ and $\frac{\partial d}{\partial \nu}(x) < 0$ in $\partial\Omega$, where ν is the outward unit normal on $\partial\Omega$. Since $w_1 \in C^{1,\gamma_1}(\overline{\Omega})$, it follows by Proposition 1.3.9 that $\frac{\partial w_1}{\partial \nu}(x) < 0$. So, by compactness of $\overline{\Omega}_{\delta_1}$, $C^1(\overline{\Omega})$-regularities of the solution w_1 and of the distance function d, and the boundary conditions $w_1 = d = 0$ on $\partial\Omega$, there exists a constant $C_{\delta_1} > 0$ such that

$$\frac{\partial w_1}{\partial \nu}(x) \leq C_{\delta_1}\frac{\partial d}{\partial \nu}(x) \text{ for all } x \in \Omega_{\delta_1}.$$

that is,

$$C_{\delta_1} d(x) \leq w_1(x) \text{ for all } x \in \Omega_{\delta_1},$$

and, in particular, $u_n \geq w_1 \geq Cd(x)$ in Ω_{δ_1} finishing the proof. ■

We are able to obtain more accurate asymptotic behavior than the above one for u_n if we request more restrictions on the function $c(x)$. To do this, let $\delta_1 > 0$ be that one given in Lemma 3.2.5, and remember the numbers

$$\theta_1 = \begin{cases} \max\limits_{x\in\overline{\Omega}_\delta} \dfrac{p(x)-\beta(x)}{p(x)+\alpha(x)-1} & \text{if } \beta(x)+\alpha(x) > 1 \text{ in } \Omega_\delta, \\ 1 & \text{if } \beta(x)+\alpha(x) \le 1 \text{ in } \Omega_\delta, \end{cases} \qquad \theta_2 = \min_{x\in\overline{\Omega}_\delta} \frac{p(x)-\beta(x)}{p(x)+\alpha(x)-1},$$

for some $0 < \delta \le \delta_1$ small enough.

Proposition 3.2.6 *Assume that $(H_1)-(H_4)$ hold true. Then there exists $0 < \delta \le \delta_1$, $n_0 > 1$, and*

(i) an $m > 0$, independent of n, such that

$$u_n + 1/n \ge m\left[\left(d(x)+\frac{1}{n}\right)^{\theta_1} - \frac{1}{n^{\theta_1}}\right] \quad \textit{in } \Omega_{n,\delta} \textit{ for all } n \ge n_0$$

if $c(x) \ge c_\delta > 0$ in Ω_δ,

(ii) an $M > 0$, independent of n, such that

$$u_n + 1/n \le M\left[\left(d(x)+\frac{1}{n}\right)^{\theta_2}\right] \quad \textit{in } \Omega_{n,\delta} \textit{ for all } n \ge n_0,$$

where $\Omega_{n,\delta} = \{x \in \Omega \ / \ d(x)+1/n < \delta\}$.

Proof. The proof is inspired on ideas contained in [70, Theorem 4.1]. Since Ω is smooth, we can consider $d \in C^2(\overline{\Omega}_{3\delta_2})$ with $|\nabla d(x)| \equiv 1$ in $\Omega_{3\delta_2}$ for some $\delta_1 \ge \delta_2 > 0$. Fix $n_0 > 1$ large enough such that $c_{n_0}(x) \ge (c_{\delta_2})/2$ and let $\delta \in (1/n_0, \delta_2/3)$ be a small constant to be fix later. For $n \ge n_0$ and $\sigma > 0$, $\theta \in (0,1]$ positive constants, defining

$$z_n(x) = \begin{cases} \sigma\left[\left(d(x)+\dfrac{1}{n}\right)^{\theta} - \dfrac{1}{n^{\theta}}\right] \ \text{if } \left(d(x)+\dfrac{1}{n}\right) < \delta, \\ \sigma\left(\delta^{\theta} - \dfrac{1}{n^{\theta}}\right) + \displaystyle\int_{\delta}^{\left(d(x)+\frac{1}{n}\right)} \sigma\theta\delta^{\theta-1}\left(\frac{2\delta-t}{\delta}\right)^{\frac{2}{p_--1}} dt \ \text{ if } \delta \le \left(d(x)+\dfrac{1}{n}\right) < 2\delta, \\ \sigma\left(\delta^{\theta} - \dfrac{1}{n^{\theta}}\right) + \displaystyle\int_{\delta}^{2\delta} \sigma\theta\delta^{\theta-1}\left(\frac{2\delta-t}{\delta}\right)^{\frac{2}{p_--1}} dt \ \text{ if } 2\delta \ge \left(d(x)+\dfrac{1}{n}\right), \end{cases} \tag{3.15}$$

we infer that $z_n \in C^1(\Omega) \cap C(\overline{\Omega})$.

Proof of (i): We prove the result just considering that $\beta(x)+\alpha(x) > 1$ in Ω_δ, because the other situation is treated in a similar way. Let us show that $z_n(x)$ is a subsolution for (3.4), that is,

$$\begin{aligned} &\int_\Omega c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(z_n+\frac{1}{n}\right)^{-\alpha(x)}\phi dx + \lambda\int_\Omega f_n(x,z_n)\phi dx \\ &\ge \int_\Omega |\nabla z_n|^{p(x)-2}\nabla z_n \nabla\phi dx = \int_{\{d(x)+\frac{1}{n}<\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n\nabla\phi dx \\ &+ \int_{\{\delta\le d(x)+\frac{1}{n}<2\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n\nabla\phi dx + \int_{\{d(x)+\frac{1}{n}\ge 2\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n\nabla\phi dx \end{aligned} \tag{3.16}$$

for all $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. Once this is done, it follows from Theorem 2.1.3 that $u_n + 1/n \geq z_n$ in Ω and, in particular, we obtain the claim (i).

Initially we note that

$$|\nabla z_n|^{p(x)-2}\nabla z_n = \begin{cases} (\sigma\theta)^{p(x)-1}\left(d(x)+\frac{1}{n}\right)^{(\theta-1)(p(x)-1)}\nabla d(x), & \left(d(x)+\frac{1}{n}\right) < \delta \\ \left[\sigma\theta\delta^{\theta-1}\left(\frac{2\delta-\left(d(x)+\frac{1}{n}\right)}{\delta}\right)^{\frac{2}{p_--1}}\right]^{p(x)-1}\nabla d(x), & \delta \leq \left(d(x)+\frac{1}{n}\right) < 2\delta \\ 0, & \left(d(x)+\frac{1}{n}\right) \geq 2\delta \end{cases}$$

that lead us to conclude that

$$\int_{\partial\left\{\left(d(x)+\frac{1}{n}\right)<\delta\right\}} |\nabla z_n|^{p(x)-2}\frac{\partial z_n}{\partial \eta_1}\phi dx + \int_{\partial\left\{\delta<\left(d(x)+\frac{1}{n}\right)<2\delta\right\}} |\nabla z_n|^{p(x)-2}\frac{\partial z_n}{\partial \eta_2}\phi dx + \int_{\partial\left\{\left(d(x)+\frac{1}{n}\right)\geq 2\delta\right\}} |\nabla z_n|^{p(x)-2}\frac{\partial z_n}{\partial \eta_3}\phi dx = 0 \quad (3.17)$$

where η_i, $i = 1,2,3$ are the normal unit outward vectors to the sets $\{\left(d(x)+\frac{1}{n}\right) < \delta\}$, $\{\delta < \left(d(x)+\frac{1}{n}\right) \leq \delta\}$ and $\{\left(d(x)+\frac{1}{n}\right) \geq \delta\}$, respectively .

Thus, by using integration by parts and (3.17), we obtain

$$\begin{aligned} &\int_{\{d(x)+\frac{1}{n}<\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n \nabla\phi dx + \int_{\{\delta\leq d(x)+\frac{1}{n}<2\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n\nabla\phi dx \\ &= -\int_{\{d(x)+\frac{1}{n}<\delta\}} (\sigma\theta)^{p(x)-1}(\theta-1)(p(x)-1)\left(d(x)+\frac{1}{n}\right)^{(\theta-1)(p(x)-1)-1}(1+\Pi_1(x))\phi dx \\ &+ \int_{\left\{\delta\leq d(x)+\frac{1}{n}<2\delta\right\}} \left(\sigma\theta\delta^{\theta-1}\right)^{p(x)-1}\delta^{-1}\left(\frac{2\delta-\left(d(x)+\frac{1}{n}\right)}{\delta}\right)^{\frac{2(p(x)-1)}{p_--1}-1}\Pi_2(x)\phi dx, \qquad (3.18) \end{aligned}$$

where

$$\Pi_1(x) = \left(d(x)+\frac{1}{n}\right)\left(\frac{\nabla p(x)\nabla d(x)\ln(\sigma\theta)}{(\theta-1)(p(x)-1)} + \frac{\nabla p(x)\nabla d(x)\ln\left(d(x)+\frac{1}{n}\right)}{p(x)-1} + \frac{\nabla d\nabla\phi}{\phi(\theta-1)(p(x)-1)}\right)$$

and

$$\begin{aligned} \Pi_2(x) &= \frac{2(p(x)-1)}{(p_--1)} \\ &- \left(2\delta-\left(d(x)+\frac{1}{n}\right)\right)\left[\phi\ln\left(\sigma\theta\delta^{\theta-1}\left(\frac{2\delta-\left(d(x)+\frac{1}{n}\right)}{\delta}\right)^{\frac{2}{p_--1}}\right)\nabla d\nabla p + \frac{\nabla d\nabla\phi}{\phi}\right]. \end{aligned}$$

To show that (3.16) holds true, it suffices to estimate the two integral of the right side in (3.18). Let us begin with the first one. Since

$$|\Pi_1(x)| \leq \delta\left(\frac{|||\nabla p(x)|||_\infty\,\sigma\theta}{(\theta-1)(p_--1)} + \frac{|||\nabla p(x)|||_\infty\,\delta}{p_--1} + \frac{||\nabla\phi||_\infty}{\phi(\theta-1)(p_--1)}\right),$$

we have that $|\Pi_1(x)| < 1/2$ for $\delta > 0$ small enough. So, by using this and $\theta = \theta_1$, we obtain

$$
\begin{aligned}
\Big|(\sigma\theta_1)^{p(x)-1} &\ (\theta_1 - 1)(p(x)-1)(1+\Pi_1(x))| \left(d(x) + \frac{1}{n}\right)^{(\theta_1-1)(p(x)-1)-1} \\
&\leq \frac{c_\delta}{2}\left(d(x) + \frac{1}{n}\right)^{-\beta(x)-\theta_1\alpha(x)} \\
&\leq c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(\sigma\left(\left(d(x)+\frac{1}{n}\right)^{\theta_1} - \frac{1}{n^{\theta_1}}\right) + \frac{1}{n}\right)^{-\alpha(x)} \qquad (3.19)\\
&\leq c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(z_n + \frac{1}{n}\right)^{-\alpha(x)} + \lambda f(x, z_n)
\end{aligned}
$$

holds true in $d(x) + 1/n < \delta$, when $\alpha(x) > 0$, and $\sigma, \delta > 0$ are small enough, since $(\theta_1 - 1)(p_- - 1) - 1 \geq -\beta(x) - \theta_1\alpha(x)$.

For the case $\alpha(x) \leq 0$, we have

$$
\begin{aligned}
\Big|(\sigma\theta_1)^{p(x)-1} &\ (\theta_1 - 1)(p(x)-1)(1+\Pi_1(x))| \left(d(x) + \frac{1}{n}\right)^{(\theta_1-1)(p(x)-1)-1} \\
&\leq \sigma^{-\alpha(x)}\frac{c_\delta}{2}\left(d(x) + \frac{1}{n}\right)^{-\beta(x)-\theta_1\alpha(x)} \\
&\leq c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(\sigma\left(\left(d(x)+\frac{1}{n}\right)^{\theta_1} - \frac{1}{n^{\theta_1}}\right) + \frac{1}{n}\right)^{-\alpha(x)} \qquad (3.20)\\
&\leq c_n(x)\left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(z_n + \frac{1}{n}\right)^{-\alpha(x)} + \lambda \int_\Omega f_n(x, z_n)
\end{aligned}
$$

is true in $d(x) + 1/n < \delta$ for some $\sigma, \delta > 0$ small enough.

Hence, it follows from (3.19) – (3.20), that

$$
\begin{aligned}
&\int_{\{d(x)+\frac{1}{n}<\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n \nabla\phi dx \qquad (3.21)\\
&\leq \int_{\{d(x)+\frac{1}{n}<\delta\}} c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(z_n+\frac{1}{n}\right)^{-\alpha(x)}\phi dx + \lambda\int_{\{d(x)+\frac{1}{n}<\delta\}} f_n(x,z_n)\phi dx
\end{aligned}
$$

for $\delta, \sigma > 0$ small enough.

Now, we going to evaluate the integral in (3.18) in the strip $\delta \leq d(x) + 1/n < 2\delta$. Since

$$
|\Pi_2(x)| \leq \frac{2(p_+ - 1)}{(p_- - 1)} + \delta\left(\theta_1\delta^{\theta_1 - 1}\,|||\nabla p(x)|||_\infty + \frac{||\nabla\phi||_\infty}{\phi}\right)
$$

holds true in $\delta \leq d(x) + 1/n < 2\delta$, we can use the boundedness of Π_2 to obtain

$$
\left(\sigma\theta_1\delta^{\theta_1-1}\right)^{p(x)-1}\delta^{-1}\left(\frac{2\delta - \left(d(x)+\frac{1}{n}\right)}{\delta}\right)^{\frac{2(p(x)-1)}{p_--1}-1}\Pi_2(x) \leq m_1\delta^{(\theta_1-1)(p(x)-1)-1},
$$

where $m_1 = m_1(\delta, p_+, p_-, \sigma, \theta_1)$.

After this, we can use similar arguments used to obtain (3.19) and (3.20) to infer that

$$
\left(\sigma\theta_1\delta^{\theta-1}\right)^{p(x)-1}\left(\frac{2\delta - \left(d(x)+\frac{1}{n}\right)}{\delta}\right)^{\frac{2(p(x)-1)}{p_--1}-1}\Pi_2(x) \leq c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(z_n+\frac{1}{n}\right)^{-\alpha(x)}
$$

holds true in $\delta \leq d(x)+1/n < 2\delta$ for $\delta, \sigma > 0$ small enough, that is,

$$\int_{\{\delta \leq d(x)+\frac{1}{n}<2\delta\}} |\nabla z_n|^{p(x)-2}\nabla z_n \nabla\phi dx \tag{3.22}$$
$$\leq \int_{\{\delta \leq d(x)+\frac{1}{n}<2\delta\}} c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(z_n+\frac{1}{n}\right)^{-\alpha(x)}\phi dx + \lambda\int_{\{\delta \leq d(x)+\frac{1}{n}<2\delta\}} f_n(x,z_n)\phi dx$$

for $\delta, \sigma > 0$ small enough.

So, it follows from $(3.21)-(3.22)$ the inequality in (3.16), that is, $z_n(x)$ is a subsolution for (3.4). This finishes the proof of the claim (i).

Proof of (ii)**:** Consider the function

$$\tilde{z}_n(x) = \sigma\left(d(x)+\frac{1}{n}\right)^{\theta_2},\ d(x)+\frac{1}{n} < \delta.$$

It is easy to see that $\tilde{z}_n \in W^{1,p(x)}_{loc}(\Omega_{n,\delta}) \cap C^1(\Omega_{n,\delta})$. Similarly to (i), we will show that $\tilde{z}_n$ is a supersolution for (3.4) in $\Omega_{n,\delta}$, that is,

$$\int_{\Omega_{n,\delta}} |\nabla \tilde{z}_n|^{p(x)-2}\nabla \tilde{z}_n \nabla\phi dx \tag{3.23}$$
$$\geq \int_{\Omega_{n,\delta}} c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(\tilde{z}_n+\frac{1}{n}\right)^{-\alpha(x)}\phi dx + \lambda\int_{\Omega_{n,\delta}} f_n(x,\tilde{z}_n)\phi dx,$$

for all $\phi \in C_0^\infty(\Omega_{n,\delta})$ with $\phi \geq 0$.

As in (3.18), we have that

$$\int_{\Omega_{n,\delta}} |\nabla \tilde{z}_n|^{p(x)-2}\nabla \tilde{z}_n \nabla\phi dx$$
$$= -\int_{\Omega_{n,\delta}} (\sigma\theta_2)^{p(x)-1}(\theta_2-1)(p(x)-1)\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(p(x)-1)-1}(1+\Pi_1(x))\phi dx.$$

To obtain (3.23), we initially infer from (H_2), that

$$c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\tilde{z}_n^{-\alpha(x)} + \lambda f_n(x,\tilde{z}_n)$$
$$\leq \sigma^{-\alpha(x)}||c||_\infty d(x)^{-\beta(x)}\left(d(x)+\frac{1}{n}\right)^{-\theta_2\alpha(x)} + \lambda||b||_\infty\left(1+\sigma^{q(x)-1}\left(d(x)+\frac{1}{n}\right)^{\theta_2(q(x)-1)}\right)$$

holds true. If $\beta(x)+\theta_2\alpha(x) \geq 0$, then

$$\sigma^{-\alpha(x)}||c||_\infty d(x)^{-\beta(x)}\left(d(x)+\frac{1}{n}\right)^{-\theta_2\alpha(x)} + \lambda||b||_\infty\left(1+\sigma^{q(x)-1}\left(d(x)+\frac{1}{n}\right)^{\theta_2(q(x)-1)}\right)$$
$$\leq d(x)^{-\beta(x)-\theta_2\alpha(x)}\left(||c||_\infty\sigma^{-\alpha(x)} + \lambda||b||_\infty\left(1+\sigma^{q(x)-1}\delta^{\theta_2(q(x)-1)}\right)d(x)^{\beta(x)+\theta_2\alpha(x)}\right)$$
$$\leq d(x)^{-\beta(x)-\theta_2\alpha(x)}\left(||c||_\infty\sigma^{-\alpha(x)} + \lambda||b||_\infty\left(1+\sigma^{q(x)-1}\delta^{\theta_2(q(x)-1)}\right)\delta^{\beta(x)+\theta_2\alpha(x)}\right)$$
$$:= E_1. \tag{3.24}$$

On the other hand, if $\beta(x)+\theta_2\alpha(x)<0$, then

$$\begin{aligned}
&\sigma^{-\alpha(x)}||c||_\infty d(x)^{-\beta(x)}\left(d(x)+\frac{1}{n}\right)^{-\theta_2\alpha(x)}+\lambda||b||_\infty\left(1+\sigma^{q(x)-1}\left(d(x)+\frac{1}{n}\right)^{\theta_2(q(x)-1)}\right)\\
&\quad\le d(x)^{\beta(x)+\theta_2\alpha(x)}\left(||c||_\infty\sigma^{-\alpha(x)}\delta^{-\beta(x)-\theta_2\alpha(x)}+\lambda||b||_\infty\left(1+\sigma^{q(x)-1}\delta^{\theta_2(q(x)-1)}\right)\right)\\
&\quad:=E_2.
\end{aligned}\tag{3.25}$$

Since $(\theta_2-1)(p(x)-1)-1\le\min\{-\beta(x)-\theta_2\alpha(x),\beta(x)+\theta_2\alpha(x)\}$, it follows (3.24) and (3.25), that we can choose $\sigma>0$ large enough such that

$$\begin{aligned}
\left|(\sigma\theta_2)^{p(x)-1}(\theta_2-1)(p(x)-1)(1+\Pi_1(x))\right|\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(p(x)-1)-1}&\ge\max\{E_1,E_2\}\\
&\ge c(x)d(x)^{-\beta(x)}\left(d(x)+\frac{1}{n}\right)^{-\theta_2\alpha(x)}+\lambda f(x,\tilde z_n)
\end{aligned}\tag{3.26}$$

So, it follows from (3.23) and (3.26), that $\tilde z_n$ is a supersolution for (3.4) in $\Omega_{n,\delta}$. Thus, it follows from Theorem 2.1.3 that $u_n+1/n\le\tilde z_n$ in $\Omega_{n,\delta}$, as desired. ■

3.2.1 Estimates in the variable exponents spaces

Throughout this section we will fix

$$\omega_{n,\delta}=\Omega\setminus\Omega_{n,\delta}\ \text{ where }\ \Omega_{n,\delta}=\{x\in\Omega\ :\ d(x)+\frac{1}{n}<\delta\}.$$

for $\delta>0$ as in Proposition 3.2.6.

Proposition 3.2.7 *Assume $(H_1)-(H_4)$ hold true. If $|\beta(x)+\alpha(x)>1|>0$ in $\Omega_{n,\delta}$ and*

$$r(x)=\left(\frac{\sigma p_-^*}{p_-(\sigma-1)+1-\alpha(x)}\right)'\quad\textit{with}\quad\sigma=\max\left\{\frac{p_-+\frac{\beta_+-1}{\theta_2}+\alpha_+-1}{p_-},\frac{p_-+\alpha_+-1}{p_-}\right\},$$

then there exists a $0<\lambda_1\le\infty$ such that the sequence u_n is bounded in $L^{\sigma p^*(x)}(\Omega)$ for all $0\le\lambda<\lambda_1$. Besides this, $\lambda_1=+\infty$ if $q^+<p_-$.

Proof. Let $u_n\in W_0^{1,p(x)}(\Omega)$ be the solution of the problem 3.4. We have that

$$\begin{aligned}
\int_\Omega|\nabla u_n^\sigma|^{p_-}dx&=\int_\Omega\sigma^{p_-}u_n^{p_-(\sigma-1)}|\nabla u_n|^{p_-}dx\\
&\le\sigma^{p_-}\left(\int_\Omega u_n^{p_-(\sigma-1)}dx+\int_\Omega u_n^{p_-(\sigma-1)}|\nabla u_n|^{p(x)}dx\right)
\end{aligned}\tag{3.27}$$

holds true for each $\sigma\ge1$ given. So, by fixing the number

$$\sigma=\max\left\{\frac{p_-+\frac{\beta_+-1}{\theta_2}+\alpha_+-1}{p_-},\frac{p_-+\alpha_+-1}{p_-}\right\}\ge1\tag{3.28}$$

and taking $u_n^{p_-(\sigma-1)+1} \in W_0^{1,p(x)}(\Omega)$ as a test function in (3.12), it follows from (H_2) that

$$
\begin{aligned}
(p_-(\sigma-1)+1)\int_\Omega u_n^{p_-(\sigma-1)}|\nabla u_n|^{p(x)}dx &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n^{p_-(\sigma-1)+1})dx \\
\leq \int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)} \left(u_n+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)} dx \qquad (3.29)\\
+\lambda\int_\Omega b(x)(1+u_n^{q(x)-1})u_n^{p_-(\sigma-1)+1}dx.
\end{aligned}
$$

Below, we are going to estimate the two integrals of the right side in (3.29). To begin, since

$$
s(x) > \frac{N}{p_-} \geq \frac{\sigma p_-^*}{\sigma p_-^* - p_-(\sigma-1)-q(x)} = \left(\frac{\sigma p_-^*}{p_-(\sigma-1)+q(x)}\right)',
$$

then, by using Hölder's Inequality, we obtain

$$
\begin{aligned}
\int_\Omega b(x)(1+u_n^{q(x)-1})u_n^{p_-(\sigma-1)+1}dx &\leq M_1\left(\int_\Omega b(x)dx + \int_\Omega b(x)u_n^{p_-(\sigma-1)+q(x)}dx\right)\\
&\leq M_2\left(1+\left\|u_n^{p_-(\sigma-1)+q(x)}\right\|_{\frac{\sigma p_-^*}{p_-(\sigma-1)+q(x)}}\right) \qquad (3.30)\\
&\leq M_2\left(1+\|u_n\|_{\sigma p_-^*}^{p_-(\sigma-1)+q_-} + \|u_n\|_{\sigma p_-^*}^{p_-(\sigma-1)+q_+}\right),
\end{aligned}
$$

where we used $u_n^{p_-(\sigma-1)+1} \leq 1+u_n^{p_-(\sigma-1)+q(x)}$ to obtain the first inequality.

On the other hand, it follows from $(H_3)(ii)$ that $c \in L^\infty(\Omega_{n,\delta})$, because $\Omega_{n,\delta} \subset \Omega_\delta$. By using Proposition 3.2.6(ii), we obtain

$$
\begin{aligned}
&\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)} u_n^{p_-(\sigma-1)+1}dx\\
&\leq \int_{\Omega_{n,\delta}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx \qquad (3.31)\\
&\quad + M_3\int_{\omega_{n,\delta}} c(x)\left(u_n+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx\\
&\leq M_4\int_{\Omega_{n,\delta}}\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left[\left(d(x)+\frac{1}{n}\right)^{\theta_2}\right]^{p_-(\sigma-1)+1-\alpha(x)}dx\\
&\quad + M_3\int_{\omega_\delta} c(x)\left(u_n+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx\\
&\leq M_5\left(\int_\Omega\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+\theta_2[p_-(\sigma-1)+1-\alpha(x)]}dx + \int_\Omega c(x)u_n^{p_-(\sigma-1)+1-\alpha(x)}dx\right).
\end{aligned}
$$

To estimate the first integral in (3.31), we note that (3.28) implies that $t(x) = -\beta(x) + \theta_2[p_-(\sigma-1)+1-\alpha(x)] > -1$ and, as a consequence of this, we obtain

$$
\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+\theta_2[p_-(\sigma-1)+1-\alpha(x)]} \leq \begin{cases} (d(x)+1)^{t(x)}, & \text{if } t(x)\geq 0\\ d(x)^{t(x)}, & \text{if } -1<t(x)<0. \end{cases}
$$

holds true. Thus, by Lazer and Mckenna [51], we have

$$
\int_\Omega\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+\theta_2[p_-(\sigma-1)+1-\alpha(x)]}dx \leq M_6, \qquad (3.32)
$$

with M_6 independent of n.

To estimate the last term of (3.31), we use Hölder's Inequality to obtain

$$\begin{aligned}\int_\Omega c(x)u_n^{p_-(\sigma-1)+1-\alpha(x)}dx &= M_7||c||_{r(x)}\left\|u_n^{p_-(\sigma-1)+1-\alpha(x)}\right\|_{\frac{\sigma p_-^*}{p_-(\sigma-1)+1-\alpha(x)}} \\ &\leq M_7\left(||u_n||_{\sigma p_-^*}^{p_-(\sigma-1)+1-\alpha_-}+||u_n||_{\sigma p_-^*}^{p_-(\sigma-1)+1-\alpha_+}\right).\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33) in (3.31) we conclude that

$$\begin{aligned}&\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx \\ &\qquad\leq M_8\left(1+||u_n||_{\sigma p_-^*}^{p_-(\sigma-1)+1-\alpha_-}+\|u_n\|_{\sigma p_-^*}^{p_-(\sigma-1)+1-\alpha_+}\right).\end{aligned} \tag{3.34}$$

Now, by Sobolev embedding $W_0^{1,p_-}(\Omega)\hookrightarrow L^{p_-^*}(\Omega)$ we have

$$M_{10}||u_n||_{\sigma p_-^*}^{\sigma p_-}=M_{10}||u_n^\sigma||_{p_-^*}^{p_-}\leq||u_n^\sigma||_{W_0^{1,p_-}(\Omega)}^{p_-}=\int_\Omega|\nabla u_n^\sigma|^{p_-}dx. \tag{3.35}$$

So, by using (3.30), (3.34) and (3.35) in (3.27), we obtain that

$$||u_n||_{\sigma p_-^*}^{\sigma p_-}\leq M_{11}\left(1+||u_n||_{\sigma p_-^*}^{p_-(\sigma-1)+1-\alpha_-}+\|u_n\|_{\sigma p_-^*}^{p_-(\sigma-1)+1-\alpha_+}+\lambda||u_n||_{\sigma p_-^*}^{p_-(\sigma-1)+q_+}\right), \tag{3.36}$$

holds true for some $M_{11}>0$ independent of n. Thus, we are able to choose a $\lambda_1>0$ small enough in the case $q_+=p_-$ or $\lambda_1=\infty$ if $q_+<p_-$ holds to conclude that u_n is bounded in $L^{\sigma p_-^*}(\Omega)$. ■

Proposition 3.2.8 *Assume that $(H_1)-(H_4)$ hold true. If $|\beta(x)+\alpha(x)>1|=0$ in $\Omega_{n,\delta}$ and*

$$r(x)=\left(\frac{p^*(x)}{1-\alpha_-}\right)',$$

then there exists a $0<\lambda_2\leq\infty$ such that the sequence (u_n) is bounded in $L^{p^*(x)}(\Omega)$ for all $0\leq\lambda<\lambda_2$. Besides this, $\lambda_2=+\infty$ if $q^+<p_-$.

Proof. Taking $u_n\in W_0^{1,p(x)}(\Omega)$ as a test function in (3.12) and using (H_2), we get

$$\int_\Omega|\nabla u_n|^{p(x)}dx\leq\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}dx+\lambda\int_\Omega b(x)(1+u_n^{q(x)-1})u_ndx.$$

As in proof of Proposition 3.2.7, we will estimate the integrals above. Initially, note that

$$\begin{aligned}\int_\Omega b(x)(1+u_n^{q(x)-1})u_ndx &\leq M_1\left(\int_\Omega b(x)dx+\int_\Omega b(x)u_n^{q(x)}dx\right) \\ &\leq M_2\left(||b||_1+||b(x)||_{s(x)}||u_n||_{p^*(x)}^{q-}+||b(x)||_{s(x)}||u_n||_{p^*(x)}^{q+}\right)\end{aligned} \tag{3.37}$$

and for the singular term, we have

$$
\begin{aligned}
&\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}dx\\
&\quad=\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{1-\alpha(x)-\beta(x)}\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{1-\alpha(x)}dx \qquad (3.38)\\
&\quad\le M_1\left(\int_{\{\alpha(x)>1\}}c(x)\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{1-\alpha(x)}dx+\int_{\{\alpha(x)\le 1\}}c(x)\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{1-\alpha(x)}dx\right)\\
&\quad\le M_2\left(\int_\Omega c(x)dx+\int_{\{\alpha(x)\le 1\}}c(x)\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{1-\alpha(x)}dx\right),
\end{aligned}
$$

where we used that $u_n+1/n\ge C_2(d(x)+1/n)$ in Ω to obtain the last inequality, as claimed in Proposition 3.2.5. Now, by using Proposition 3.2.6(ii) and $c\in L^\infty(\Omega_\delta)$, we obtain

$$
\begin{aligned}
&\int_{\{\alpha(x)\le 1\}}c(x)\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{1-\alpha(x)}dx \qquad (3.39)\\
&\le M_3\left(\int_{\Omega_{n,\delta}\cap\{\alpha(x)\le 1\}}\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(1-\alpha(x))}dx+\int_{\omega_{n,\delta}\cap\{\alpha(x)\le 1\}}c(x)(u_n+1)^{1-\alpha(x)}\right)\\
&\le M_4\left(\int_{\Omega_{n,\delta}\cap\{\alpha(x)\le 1\}}\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(1-\alpha(x))}dx+\int_{\omega_{n,\delta}\cap\{\alpha(x)\le 1\}}c(x)u_n^{1-\alpha_-}+\int_\Omega c(x)dx\right).
\end{aligned}
$$

To finish, it follows from Lazer and Mckenna [51], that

$$
\int_\Omega\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(1-\alpha(x))}dx\le M_5, \qquad (3.40)
$$

since $(\theta_2-1)(1-\alpha(x))>-1$. To the last term in (3.39), we use Hölder's Inequality once, to obtain

$$
\int_\Omega c(x)u_n^{1-\alpha_-}dx\le ||c||_{r(x)}||u_n||_{p^*(x)}^{1-\alpha_-}. \qquad (3.41)
$$

So, by combining (3.29) with (3.39) – (3.41) and following the same lines of the proof of Proposition 3.2.7, we conclude that

$$
\max\{||u_n||_{p^*(x)}^{p_-},||u_n||_{p^*(x)}^{p_+}\}\le M_6\left(1+\|u_n\|_{p^*(x)}^{1-\alpha_-}+\lambda||u_n||_{p^*(x)}^{q_+}\right), \qquad (3.42)
$$

hold true for some $M_6>0$ independent of n. Thus, again we are able to choose a $\lambda_2>0$ small enough if $q_+=p_-$ holds or $\lambda_2=+\infty$ when $q_+<p_-$ occurs to infer that (u_n) is bounded in $L^{p(x)}(\Omega)$. ■

Below, let us prove that the sequence (u_n) converges to a solution of (3.1). To do this, we begin by proving a priori estimate on the sequence (u_n) in $W_\Gamma^{1,p(x)}(\Omega)$. The role played by the trio $(c(x),\alpha(x),\beta(x))$ near the boundary is determinant. Let us remember the $C^{0,1}$-manifold

$$
\Gamma_t=\{x\in\partial\Omega\ /\ [-\beta(x)+t(1-\alpha(x))]\frac{1}{1-1/r(x)}+1>0\}. \qquad (3.43)
$$

Definition 3.2.9 *We say that (u_n) is bounded in $W^{1,p(x)}_{\Gamma}(\Omega)$ if (u_n) is bounded in $W^{1,p(x)}(U)$ for all open set $U \subset \Omega$ given such that $\partial U \cap \partial\Omega = \Gamma$.*

Proposition 3.2.10 *Assume that $(H_1) - (H_4)$ hold with $r(x)$ as in Propositions* 3.2.7 *or* 3.2.8*. Then (u_n) is bounded in $W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$ for all $0 \le \lambda < \min\{\lambda_1, \lambda_2\}$. In addiction, if $c(x) \ge c_\delta$ in Ω_δ, then (u_n) is bounded in $W^{1,p(x)}_{\Gamma_{\theta_1}\cup\Gamma_{\theta_2}}(\Omega)$ for all $0 \le \lambda < \min\{\lambda_1, \lambda_2\}$.*

Proof. Given an open set $U \subset \Omega$ such that $\partial U \cap \partial\Omega = \Gamma_1 \cup \Gamma_{\theta_2}$, let $\psi \in C^\infty(\overline{U})$ with *supp* $(\psi) \subset U \cup \Gamma$. Denoting by *supp* $(\psi) = S_\psi$, consider the sets

$$\Omega_{n,\delta,\psi,\Gamma} = \left\{x \in S_\psi \ / \ d(x, \partial S_\psi \cap \Gamma) + \frac{1}{n} < \delta\right\} \quad \text{and} \quad \omega_{n,\delta,\psi,\Gamma} = S_\psi \setminus \omega_{n,\delta,\psi,\Gamma}.$$

We get that $\omega_{n,\delta,\psi,\Gamma} \subset\subset \Omega$, where $\Gamma = \Gamma_1 \cup \Gamma_{\theta_2}$. By taking $u_n\psi^{p_+} \in W^{1,p(x)}_0(\Omega)$ as a test function in (3.12), we obtain that

$$\begin{aligned}\int_U |\nabla u_n|^{p(x)}\psi^{p_+}dx &+ p_+\int_U u_n\psi^{p_+-1}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi dx \\ &\le \int_U \left(c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)} + \lambda f(x,u_n)\right)u_n\psi^{p_+}dx\end{aligned} \tag{3.44}$$

Now we will estimate each integral in (3.44). First, we notice that $\{\alpha(x) > \alpha_\Gamma\} \subset \omega_{n,\delta,\psi,\Gamma} \subset\subset \Omega$, where $\alpha_\Gamma = \max_{x\in\Omega_{n,\delta,\psi,\Gamma}} \alpha(x)$. To estimate the first integral after the inequality, we need consider two cases. Initially, let us assume that $\alpha_\Gamma > 1$. From Proposition 3.2.5, it follows that

$$\int_{\{\alpha(x)>\alpha_\Gamma\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx \le M_1||c||_1. \tag{3.45}$$

For the complimentary case, we will split the integral in two new ones, that is,

$$\begin{aligned}&\int_{\{\alpha(x)\le\alpha_\Gamma\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}u_n\psi^{p_+}dx \\ &\le \int_{\{1<\alpha(x)\le\alpha_\Gamma\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx \\ &\qquad + \int_{\{\alpha(x)\le 1\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx.\end{aligned} \tag{3.46}$$

Firstly we notice that,

$$\alpha(x) < \max\left\{2 - \frac{1}{r(x)} - \beta(x), 1 + \frac{1}{\theta_2}\left(1 - \frac{1}{r(x)} - \beta(x)\right)\right\}, \quad x \in \Gamma$$

and thus,

$$\min\left\{(-\beta(x) + 1 - \alpha_\Gamma)r'(x), (-\beta(x) + \theta_2(1-\alpha_\Gamma))r'(x)\right\} > -1, \quad x \in \Gamma,$$

that is, it follows from a Lazer and Mckenna's result [51], that

$$\max\left\{\int_{\Omega_\delta} d(x)^{[-\beta(x)+1-\alpha_\Gamma]r'(x)}dx, \int_{\Omega_\delta} d(x)^{[-\beta(x)+\theta_2(1-\alpha_\Gamma)]r'(x)}dx\right\} < \infty. \tag{3.47}$$

For the first integral of (3.46), by using Proposition 3.2.5 and Hölder's Inequality, we obtain

$$
\begin{aligned}
&\int_{\{1<\alpha(x)\leq\alpha_\Gamma\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx \\
&\leq M_2\int_{\{1<\alpha(x)\leq\alpha_\Gamma\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+1-\alpha(x)}\psi^{p_+}dx \\
&\leq M_3\int_{\{1<\alpha(x)\leq\alpha_\Gamma\}} c(x)d(x)^{-\beta(x)+1-\alpha(x)}dx + M_4||c||_1 \\
&\leq M_3\int_{\Omega_{n,\delta,\psi,\Gamma}\cap\{1<\alpha(x)\leq\alpha_\Gamma\}} c(x)d(x)^{-\beta(x)+1-\alpha(x)}dx + M_5||c||_1 \\
&\leq M_6\left(||c||_{r(x)}\left\|d(x)^{-\beta(x)+1-\alpha(x)}\right\|_{L^{r'(x)}(\Omega_\delta)}+1\right).
\end{aligned}
\tag{3.48}
$$

From Proposition 1.1.1 and (3.47)

$$
\left\|d(x)^{-\beta(x)+1-\alpha(x)}\right\|_{L^{r'(x)}(\Omega_\delta)} \leq \left(\int_{\Omega_\delta} d(x)^{(-\beta(x)+1-\alpha_\Gamma)r'(x)}dx\right)^{\gamma}<\infty,
$$

where $\gamma\in\{1/r_+,1/r_-\}$. Thus,

$$
\int_{\{1<\alpha(x)\leq\alpha_\Gamma\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx\leq M_7 \tag{3.49}
$$

for some M_7 independent of n;

For the second integral of (3.46), we should analyze more sub cases. From Propositition 3.2.7 or 3.2.8, we have $r(x)\geq p^*_-/(p^*_-+\alpha_- - 1)$. By applying Proposition 3.2.6(ii) and Hölder's Inequality, we obtain

$$
\begin{aligned}
&\int_{\{\alpha(x)\leq 1\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx \\
&\leq M_7\int_{\Omega_{n,\delta,\psi,\Gamma}\cap\{\alpha(x)\leq1\}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+\theta_2(1-\alpha(x))}dx \\
&\qquad\qquad + M_8\int_{\omega_{n,\delta,\psi,\Gamma}\cap\{\alpha(x)\leq1\}} c(x)\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}dx \\
&\leq M_9\left(||c||_{r(x)}\left\|d(x)^{-\beta(x)+\theta_2(1-\alpha(x))}\right\|_{L^{r'(x)}(\Omega_{n,\delta,\psi,\Gamma})}+||c||_{\frac{p^*_-}{p^*_-+\alpha_--1}}||u_n||_{p^*_-}^{1-\alpha_-}+||c||_1\right) \\
&\leq M_{10}\left[\left(\int_{\Omega_\delta} d(x)^{[-\beta(x)+\theta_2(1-\alpha_\Gamma)]r'(x)}dx\right)^{\gamma}+1\right]
\end{aligned}
\tag{3.50}
$$

for some $M_{10}>0$ independent of n, since $||u_n||_{p^*_-}$ is uniformly bounded, by Propositions 3.2.7 or 3.2.8.

By combining (3.45),(3.48) and (3.50) we conclude that

$$
\int_U c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}u_n\psi^{p_+}dx\leq M_{11} \tag{3.51}
$$

holds for some $M_{11}>0$ independent of n.

Now, let us assume the opposite case $\alpha_\Gamma \leq 1$. Thus, by arguing as in (3.50), we obtain

$$
\begin{aligned}
&\int_U c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}\psi^{p_+}dx \\
&\leq M_{12}\left(\int_{\Omega_{n,\delta,\psi,\Gamma}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+\theta_2(1-\alpha(x))}dx+\int_{\omega_{n,\delta,\psi,\Gamma}} c(x)\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)}dx\right) \\
&\leq M_{13}\left(||c||_{r(x)}\left\|d(x)^{-\beta(x)+\theta_2(1-\alpha(x))}\right\|_{L^{r'(x)}(\Omega_{n,\delta,\psi,\Gamma})}+||c||_{\frac{p_-^*}{p_-^*+\alpha_--1}}||u_n||_{p_-^*}^{1-\alpha_-}+||c||_1\right) \\
&\leq M_{14}\left[\left(\int_{\Omega_\delta} d(x)^{[-\beta(x)+\theta_2(1-\alpha_\Gamma)]r'(x)}dx\right)^{\gamma}+1\right].
\end{aligned}
$$

So, this information together with our assumptions on U and (3.43), we obtain again (3.51).

Besides these, it follows from hypothesis (H_2), that

$$
\begin{aligned}
\lambda\int_U f(x,u_n)u_n\psi^{p_+}dx &\leq \lambda\int_U b(x)(u_n+u_n^{q(x)})\psi^{p_+}dx \\
&\leq M_{16}\lambda||b||_{s_-}\left(||u_n||_{p_-^*}||1||_{\frac{p_-^*}{p_--1}}+\left\|u_n^{q(x)}\right\|_{\frac{p_-^*}{q(x)}}||1||_{\frac{p_-^*}{p_-^*-q(x)}}\right) \\
&\leq \lambda M_{17}\left(||u_n||_{p_-^*}+\|u_n\|_{p_-^*}^{q_-}+\|u_n\|_{p_-^*}^{q_+}\right)\leq M_{18}.
\end{aligned}
\tag{3.52}
$$

for some $M_{18}>0$ independent of n, since $||u_n||_{p_-^*}$ is uniformly bounded, by applying Propositions 3.2.7 or 3.2.8 again.

On the other side, it follows from the Young's Inequality and the boundedness of (u_n) in $L^{p(x)}(\Omega)$, see Propositions 3.2.7 or 3.2.8 again, that

$$
\begin{aligned}
\left|\int_U |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi u_n\psi^{p_+-1}dx\right| &\leq \epsilon\int_U |\nabla u_n|^{p(x)}\psi^{\frac{p_+-1}{p(x)-1}p(x)}dx+C_\epsilon\int_U |u_n\nabla\psi|^{p(x)}dx \\
&\leq \epsilon\int_U |\nabla u_n|^{p(x)}\psi^{p_+}dx+C_\epsilon||\nabla\psi||_\infty\int_U |u_n|^{p(x)}dx \\
&\leq \epsilon\int_U |\nabla u_n|^{p(x)}\psi^{p_+}dx+M_{19}.
\end{aligned}
\tag{3.53}
$$

After all these, by taking $\epsilon>0$ small enough in (3.53) and combining the informations given at (3.51) in (3.44), we deduce that

$$\int_U |\nabla u_n|^{p(x)}\psi^{p_+}dx\leq M_{20}$$

holds for all open set $U\subset\Omega$ such that $\partial U\cap\partial\Omega=\Gamma_1\cup\Gamma_{\theta_2}$, that is, (u_n) is bounded in $W^{1,p(x)}_{\Gamma_1\cup\Gamma_{\theta_2}}(\Omega)$.

We also notice that

- If $\Gamma_1\cup\Gamma_{\theta_2}=\partial\Omega$, then we can take $\psi\equiv 1$ and $U=\Omega$ to conclude that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$.

- If $\Gamma_1\cup\Gamma_{\theta_2}=\emptyset$, then $S_\psi\subset\subset\Omega$. Thus, (u_n) is bounded in $W^{1,p(x)}_{loc}(\Omega)$.

To the end, if $c(x)\geq c_\delta$ in the set Ω_δ, we can redo the above arguments with the estimate $u_n+1/n\geq m[(d(x)+1/n)^{\theta_1}-1/n^{\theta_1}]$ (see Proposition 3.2.6) in the place of $u_n\geq Cd(x)$ to obtain the claim. These finishes the proof. ■

3.3 Proof of main results

In this section, let us complete the proof of Theorem 3.1.2 from the sequence we have obtained in the last section. Besides this, we will prove regularities results for this solution.

3.3.1 Proof of Theorem 3.1.2 - Completed

Proof. Let $(u_n) \subset W_0^{1,p(x)}(\Omega)$ be the sequence of solutions of the problem (3.4) given by Proposition 3.2.3. As proved in Proposition 3.2.10, we have that the sequence (u_n) is bounded in $W^{1,p(x)}_{\Gamma_1 \cup \Gamma_{\theta_2}}(\Omega)$. So, given an open set $U \subset \Omega$ with $\partial U \cap \partial\Omega = \Gamma_1 \cup \Gamma_{\theta_2}$we have that, up to subsequence, that $u_n \rightharpoonup u$ in $W^{1,p(x)}(U)$, $u_n \to u$ in $L^{t(x)}(U)$ for any $1 \leq t(x) < p^*(x)$ given, $u_n(x) \to u(x)$ a.e. in U and there exists $h_U \in L^{t(x)}(U)$ such that $u_n \leq h_U$.

Let $\phi \in C_0^\infty(U)$. By using $\phi(u_n - u)$ as a test function for the problem (3.4), we have

$$\begin{aligned}&\int_{S_\phi} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla[(u_n-u)\phi]dx \\ &= \int_{S_\phi} \left(c_n(x) \left(d(x) + \frac{1}{n}\right)^{-\beta(x)} \left(u_n + \frac{1}{n}\right)^{-\alpha(x)} + \lambda f(x,u_n)\right)(u_n - u)\phi dx. \qquad (3.54)\end{aligned}$$

First, by using Proposition 3.2.5 and 3.2.10, standard embedding, and splitting the $S_\phi = supp\ (\phi)$ in the regionof singularity and non-singularity, we get to

$$\begin{aligned}&\left| \int_{S_\phi} c_n(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)} \left(u_n + \frac{1}{n}\right)^{-\alpha(x)} (u_n-u)\phi dx \right| \\ &\leq M_1 \left(\int_{\{\alpha(x)>0\}} c(x)(u_n-u)dx + \int_{\{\alpha(x)\leq 0\}} c(x)(u_n+1)^{-\alpha(x)}(u_n-u)dx \right) \qquad (3.55) \\ &\leq M_2 \left(||u_n-u||_{r'_-} + ||u_n-u||_{\frac{\sigma p^*_-}{p(\sigma-1)+1}} ||u_n+1||^{-\alpha_-}_{\sigma p^*_-} \right),\end{aligned}$$

recalling that we are assuming that $\alpha_- < 0$.

More, by using the hypothesis (H_2), we have

$$|f(x,u_n)(u_n-u)\phi| \leq ||\phi||_\infty b(x)\left(h_U + h_U^{q(x)}\right) \in L^1(U). \qquad (3.56)$$

So, by taking the limit in (3.54), it follows from (3.55), (3.56) combined with Lebesgue's theorem, that

$$\int_{S_\phi} \phi|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u)dx = \int_{S_\phi} (u_n-u)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi dx + o_n(1).$$

Since

$$\left| \int_{S_\phi} (u_n-u)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi dx \right| \leq ||\nabla\phi||_\infty \left\| \nabla u_n^{p(x)-1} \right\|_{\frac{p(x)}{p(x)-1}} ||u_n-u||_{p(x)} = o_n(1),$$

we obtain that

$$\int_{S_\phi} \phi|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx = o_n(1) \qquad (3.57)$$

and recalling that $u_n \rightharpoonup u$ in $W^{1,p(x)}(U)$, we have

$$\int_{S_\phi} \phi |\nabla u|^{p(x)-2} \nabla u \nabla (u_n - u) dx = o_n(1). \tag{3.58}$$

So, it follows from (3.57) and (3.58), that

$$0 \leq \int_{S_\phi} \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \nabla (u_n - u) dx \to 0$$

and as a consequence of this together with Proposition A.1.7, we obtain that $\nabla u_n(x) \to \nabla u(x)$ a.e. in U and

$$t_n := (|\nabla u_n| + |\nabla u|)^{p(x)-2} \nabla (u_n - u) \to 0 \ \text{ a.e. in } \ U.$$

By using the Hölder's Inequality, we get

$$\begin{aligned}
&\int_{S_\phi} (|\nabla u_n| + |\nabla u|)^{p(x)-2} |\nabla (u_n - u)| |\nabla \phi| dx \leq \int_{S_\phi} (|\nabla u_n| + |\nabla u|)^{p(x)-1} |\nabla \phi| dx \\
\leq \ & C ||\nabla \phi||_\infty \max \left\{ |||\nabla u_n| + |\nabla u|||_{p(x)}^{p_+ - 1}, |||\nabla u_n| + |\nabla u|||_{p(x)}^{p_- - 1} \right\} |S_\phi|^{\frac{1}{p_+}} \\
\leq \ & C |S_\phi|^{\frac{1}{p_+}} ||\nabla \phi||_\infty \max \left\{ \left(||\nabla u_n||_{p(x)} + ||\nabla u||_{p(x)} \right)^{p_+ - 1}, \left(||\nabla u_n||_{p(x)} + ||\nabla u||_{p(x)} \right)^{p_- - 1} \right\} \\
\leq \ & C_1 |S_\phi|^{\frac{1}{p_+}},
\end{aligned} \tag{3.59}$$

where $C_1 = C_1(p_-, p_+, \phi) > 0$ is a real constant.

Exploiting Vitali's Theorem and the estimate

$$\left| |x|^{p-2} x - |y|^{p-2} y \right| \leq C_2 (|x| + |y|)^{p-2} |x - y| \text{ for all } x, y \in \mathbb{R}^N \text{ with } |x| + |y| > 0,$$

we get

$$\begin{aligned}
&\left| \int_{S_\phi} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) \nabla \phi dx \right| \\
&\qquad\qquad \leq \int_{S_\phi} (|\nabla u_n| + |\nabla u|)^{p(x)-2} |\nabla (u_n - u)| |\nabla \phi| dx \to 0,
\end{aligned}$$

and then

$$\int_{S_\phi} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi dx \to \int_{S_\phi} |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx. \tag{3.60}$$

Finally, it follows from the hypothesis (H_2), Proposition 3.2.5, the convergence (3.60), by passing the limit at (3.4), we obtain that $u \in W^{1,p(x)}(U)$ satisfies

$$\int_U |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx = \int_U \left(a(x) u^{-\alpha(x)} + \lambda f(x, u) \right) \phi dx \text{ for all } \phi \in C_0^\infty(U),$$

for all $U \subset \Omega$ with $\partial U \cap \partial \Omega = \Gamma_1 \cup \Gamma_{\theta_2}$, that is, $u \in W^{1,p(x)}_{\Gamma_1 \cup \Gamma_{\theta_2}}(U)$ is a solution of Problem (3.1). Moreover, by Propositions 3.2.5 and 3.2.6, we obtain that $C_1 d(x) \leq u(x) \leq C_2 d(x)^{\theta_2}$ or $C_1 d(x)^{\theta_1} \leq u(x) \leq C_2 d(x)^{\theta_2}$ for a.e. $x \in \Omega$.

To finish the proof, we just need to show that the boundary condition. For each $\epsilon > 0$ given, we may argue as in Proposition 3.2.10 to show that $((u_n - \epsilon)^+)$ is bounded in $W_0^{1,p(x)}(\Omega)$, and hence it has a subsequence that converges weakly in $W_0^{1,p(x)}(\Omega)$ and a.e. in Ω to some $v \in W_0^{1,p(x)}(\Omega)$. Then $v = (u-\epsilon)^+$ since $(u_{n_k}-\epsilon)^+ \to (u-\epsilon)^+$ a.e. in Ω. ■

As a consequence of the proof, we have:

- if $\Gamma_1 \cup \Gamma_{\theta_2} = \partial\Omega$, then we can take $U = \Omega$ to conclude that $u \in W_0^{1,p(x)}(\Omega)$,
- if $\Gamma_1 \cup \Gamma_{\theta_2} = \emptyset$, then $S_\phi \subset\subset \Omega$. Thus, $u \in W_{loc}^{1,p(x)}(\Omega)$.

Proof of Corollary 3.1.3 **- Completed.** The proof of is identical to the corresponding one for Theorem 3.1.2, by noticing that Propositions 3.2.7 or 3.2.8, holds with the assumptions on $c(x)$. ■

3.3.2 Proof of Theorem 3.1.4 - Completed

In order to prove the Theorem 3.1.4(i), we will follow some ideas found in Fan [29] and Fusco and Sbordone [38] to the problem (3.4).

Proof of Theorem 3.1.4:.

Proof of (i)**:** For each $x_0 \in \overline{\Omega}$ and $R > 0$ given, set $K_R = B_R(x_0) \cap \overline{\Omega}$ and

$$\tilde{p}_- = \min_{K_R} p(x), \quad \tilde{p}_+ = \max_{K_R} p(x), \quad \tilde{p}_-^* = \frac{N\tilde{p}_-}{N-\tilde{p}_-}.$$

From now on, let us take this $R > 0$ small enough such that $\tilde{p}_+ < \tilde{p}_-^*$. Let $0 < r_1 < r_2 < R$ such that $K_{r_1} \subset K_{r_2} \subset K_R$ and take $\xi \in C^\infty(\Omega)$ with $0 \le \xi \le 1$, $\xi \equiv 1$ in K_{r_1}, *supp* $(\xi) \subset K_{r_2}$ and $|\nabla\xi| \le (r_2-r_1)^{-1}$. Given $k \le 1$, define

$$A_{n,k,i} = K_i \cap \{x \in \Omega \;/\; u_n(x) > k\}, \quad i = \{r_1, r_2, R\}.$$

Since $u_n \in W_0^{1,p(x)}(\Omega)$ is the sequence of solutions for (3.4), we can take the function $\xi^{\tilde{p}_+}(u_n-k)^+ \in W_0^{1,p(x)}(\Omega)$ as a test function for (3.4) to infer, by using (H_2), that

$$\begin{aligned}&\int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)}\xi^{\tilde{p}_+}dx + \tilde{p}_+\int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\xi \xi^{\tilde{p}_+-1}(u_n-k)^+dx \qquad (3.61)\\ &\le \int_{A_{n,k,r_2}} \left(c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)} + \lambda b(x)\left(1+u_n^{q(x)-1}\right)\right)\xi^{\tilde{p}_+}(u_n-k)^+dx,\end{aligned}$$

holds true for each $k \ge 1$ given.

Below, let us evaluate each integral of the inequality in (3.61). First, it follows by Young's Inequality that

$$\begin{aligned}&\int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)-1}|\nabla\xi|(u_n-k)^+\xi^{\tilde{p}_+-1}dx = \int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)-1}\xi^{\tilde{p}_+-1}|\nabla\xi|(u_n-k)^+dx\\ &\le \int_{A_{n,k,r_2}} \frac{1}{p'(x)}\epsilon^{p'(x)}|\nabla u_n|^{p(x)}\xi^{p'(x)(\tilde{p}_+-1)}dx + \int_{A_{n,k,r_2}} \frac{1}{p(x)}\epsilon^{-p(x)}|\nabla\xi|^{p(x)}|u_n-k|^{p(x)}dx\\ &\le C\left(\epsilon^{\tilde{p'}_-}\int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)}\xi^{\tilde{p}_+}dx + \epsilon^{-\tilde{p}_+}\int_{A_{n,k,r_2}} \left(\frac{u_n-k}{r_2-r_1}\right)^{\tilde{p}_-^*}dx + \epsilon^{-\tilde{p}_+}|A_{n,k,r_2}|\right). \qquad (3.62)\end{aligned}$$

holds true for each $\epsilon > 0$ given, since $|\nabla\xi| \leq (r_2 - r_1)^{-1}$ and $[(u_n - k)/(r_2 - r_1)]^{p(x)} \leq 1 + [(u_n - k)/(r_2 - r_1)]^{\tilde{p}_-^*}$.

In the sequel, let us estimate the integral involving the $b(x)$. Since $s(x) > (p_-^*/p_-)' \geq (\tilde{p}_-^*/\tilde{p}_-)'$ holds, it follows by Hölder's inequality and from the embedding $L^{s'(x)}(\Omega) \hookrightarrow L^{s'_-}(\Omega)$, that

$$
\begin{aligned}
\int_\Omega b(x)\left(1 + u_n^{q(x)-1}\right)\xi^{\tilde{p}_+}(u_n - k)^+ dx &\leq \int_{A_{n,k,r_2}} b(x)(u_n - k)dx + \int_{A_{n,k,r_2}} b(x)u_n^{q(x)}dx \\
&\leq M_1\left(\int_{A_{n,k,r_2}} b(x)u_n^{p_-}dx + \int_{A_{n,k,r_2}} b(x)dx\right) \\
&\leq M_2\left(||b||_{L^{s_-}(A_{n,k,r_2})}||u_n^{p_-}||_{L^{s'_-}(A_{n,k,r_2})} + ||b||_{L^{s_-}(A_{n,k,r_2})}||1||_{L^{s'_-}(A_{n,k,r_2})}\right) \qquad (3.63) \\
&\leq M_3\left(||u_n^{\tilde{p}_-}||_{L^{s'_-}(A_{n,k,r_2})} + |A_{n,k,r_2}|^{\frac{1}{s'_-}}\right) \\
&\leq M_4\left(||(u_n - k + k)^{\tilde{p}_- s'_-}||^{\frac{1}{s_-}}_{L^{\left(\frac{\tilde{p}_-^*}{s'_-\tilde{p}_-}\right)}(A_{n,k,r_2})}||1||^{\frac{1}{s'_-}}_{L^{\frac{\tilde{p}_-^*}{\tilde{p}_-^* - s'_-\tilde{p}_-}}(A_{n,k,r_2})} + |A_{n,k,r_2}|^{\frac{1}{s'_-}}\right) \\
&= M_4\left[\left(\int_{A_{n,k,r_2}}\left((u_n - k)^{\tilde{p}_-^*} + k^{\tilde{p}_-^*}\right)dx\right)^{\frac{\tilde{p}_-}{\tilde{p}_-^*}}|A_{n,k,r_2}|^{\frac{1}{s'_-} - \frac{\tilde{p}_-}{\tilde{p}_-^*}} + |A_{n,k,r_2}|^{\frac{1}{s'_-}}\right] \\
&\leq M_5\left[\left(\int_{A_{n,k,r_2}}(u_n - k)^{\tilde{p}_-^*}dx\right)^{\frac{\tilde{p}_-}{\tilde{p}_-^*}}|A_{n,k,r_2}|^{\frac{1}{s'_-} - \frac{\tilde{p}_-}{\tilde{p}_-^*}} + k^{\tilde{p}_-}|A_{n,k,r_2}|^{\frac{1}{s'_-}}\right] \\
&= M_5\left[\left(\int_{A_{n,k,r_2}}\left(\frac{u_n - k}{r_2 - r_1}\right)^{\tilde{p}_-^*}(r_2 - r_1)^{\tilde{p}_-^*}dx\right)^{\frac{\tilde{p}_-}{\tilde{p}_-^*}}|A_{n,k,r_2}|^{\frac{1}{s'_-} - \frac{\tilde{p}_-}{\tilde{p}_-^*}} + k^{\tilde{p}_-}|A_{n,k,r_2}|^{\frac{1}{s'_-}}\right] \\
&\leq M_6\left[\left(\int_{A_{n,k,r_2}}\left(\frac{u_n - k}{r_2 - r_1}\right)^{\tilde{p}_-^*}dx\right)^{\frac{\tilde{p}_-}{\tilde{p}_-^*}}|A_{n,k,r_2}|^{\frac{1}{s'_-} - \frac{\tilde{p}_-}{\tilde{p}_-^*}} + k^{\tilde{p}_-}|A_{n,k,r_2}|^{\frac{1}{s'_-}}\right].
\end{aligned}
$$

where we used the inequality $u_n^{p_-} \leq u_n^{\tilde{p}_-} + 1$ to obtain the fourth inequality.

About the possible singular integral in (3.61), we need consider other sub cases. Define the sets $A^+_{n,k,r_2} = A_{n,k,r_2} \cap \{\beta(x) > 0\}$ and $A^-_{n,k,r_2} = A_{n,k,r_2} \cap \{\beta(x) \leq 0\}$. So, by arguing as in (3.63), we can conclude

$$
\begin{aligned}
&\int_{A^-_{n,k,r_2}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}(u_n-k)\xi^{p_+}dx \\
&\leq \left\|(d(x)+1)^{-\beta(x)}\right\|_{\{\beta(x)\leq 0\}}\int_{A^-_{n,k,r_2}} c(x)\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}(u_n-k)dx \\
&\leq M_6\left(\int_{A^-_{n,k,r_2}\cap\{\alpha(x)>0\}} c(x)(u_n-k)dx+\int_{A^-_{n,k,r_2}\cap\{\alpha(x)\leq 0\}} c(x)(u_n+1)^{p_--1}(u_n-k)dx\right) \\
&\leq M_7\left(\int_{A_{n,k,r_2}} c(x)(u_n-k)dx+\int_{A_{n,k,r_2}} c(x)u_n^{p_-}dx\right) \\
&\leq M_8\left[\left(\int_{A_{n,k,r_2}}\left(\frac{u_n-k}{r_2-r_1}\right)^{\tilde{p}^*_-}dx\right)^{\frac{p_-}{\tilde{p}^*_-}}|A_{n,k,r_2}|^{\frac{1}{r'_-}-\frac{p_-}{\tilde{p}^*_-}}+k^{p_-}|A_{n,k,r_2}|^{\frac{1}{r'_-}}\right].
\end{aligned} \tag{3.64}
$$

To another term, we notice that

$$
\begin{aligned}
&\int_{A^+_{n,k,r_2}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}(u_n-k)\xi^{p_+}dx \\
&\leq \int_{\Omega_\delta\cap A^+_{n,k,r_2}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}(u_n-k)dx \\
&+\int_{\omega_\delta\cap A^+_{n,k,r_2}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}(u_n-k)dx.
\end{aligned}
$$

To the first integral after the inequality above, it follows from Propositions 3.2.5 and 3.2.6 (ii) and repeating the arguments used to obtain used in (3.47), that

$$
\begin{aligned}
&\int_{\Omega_\delta\cap A^+_{n,k,r_2}} c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}(u_n-k)dx \\
&\leq M_9||c||_{L^\infty(\Omega_\delta)}\int_{\Omega_\delta\cap A^+_{n,k,r_2}\cap\{\alpha(x)>1\}}\left(d(x)+\frac{1}{n}\right)^{1-\alpha(x)-\beta(x)}dx \\
&+M_{10}||c||_{L^\infty(\Omega_\delta)}\int_{\Omega_\delta\cap A^+_{n,k,r_2}\cap\{\alpha(x)\leq 1\}}\left(d(x)+\frac{1}{n}\right)^{-\beta(x)+\theta_2(1-\alpha(x))}dx \\
&\leq M_{11}\left(\left\|d(x)^{[-\beta(x)+1-\alpha(x)]}\right\|_{L^{r(x)}(A^+_{n,k,r_2})}+\left\|(d(x)+1)^{[-\beta(x)+1-\alpha(x)]}\right\|_{L^{r(x)}(A^+_{n,k,r_2})}\right. \\
&\left.+\left\|d(x)^{[-\beta(x)+\theta_2(1-\alpha(x))]}\right\|_{L^{r(x)}(A^+_{n,k,r_2})}+\left\|(d(x)+1)^{[-\beta(x)+\theta_2(1-\alpha(x))]}\right\|_{L^{r(x)}(A^+_{n,k,r_2})}\right)||1||_{L^{r'(x)}(A^+_{n,k,r_2})} \\
&\leq M_{12}|A_{n,k,r_2}|^{\frac{1}{r'_-}}\leq M_{12}k^{\tilde{p}_-}|A_{n,k,r_2}|^{\frac{1}{r'_-}}.
\end{aligned} \tag{3.65}
$$

To the second one, by following the same lines as in (3.64), we obtain that

$$
\begin{aligned}
\int_{\omega_\delta \cap A^+_{n,k,r_2}} & c(x) d(x)^{-\beta(x)} u_n^{-\alpha(x)} (u_n - k) dx \\
& \leq \left\| d(x)^{-\beta(x)} \right\|_{L^\infty(\omega_\delta)} \int_{A^+_{n,k,r_2}} c(x) u_n^{-\alpha(x)} (u_n - k) dx \\
& \leq M_{13} \left(\int_{A_{n,k,r_2} \cap \{\alpha > 0\}} c(x)(u_n - k) dx + \int_{A_{n,k,r_2} \cap \{\alpha \leq 0\}} c(x) u_n^{\tilde{p}_-} dx \right) \\
& \leq M_{13} \left[\left(\int_{A_{n,k,r_2}} \left(\frac{u_n - k}{r_2 - r_1} \right)^{\tilde{p}^*_-} dx \right)^{\frac{p_-}{\tilde{p}^*_-}} |A_{n,k,r_2}|^{\frac{1}{r'_-} - \frac{p_-}{\tilde{p}^*_-}} + k^{\tilde{p}_-} |A_{n,k,r_2}|^{\frac{1}{r'_-}} \right]
\end{aligned}
\tag{3.66}
$$

holds true.

After these estimates, it follows from (3.62) – (3.66), that

$$
\begin{aligned}
\int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)} \xi^{\tilde{p}_+} dx \leq M_{14} & \left(\int_{A_{n,k,r_2}} \left(\frac{u_n - k}{r_2 - r_1} \right)^{\tilde{p}^*_-} dx + |A_{n,k,r_2}| + k^{\tilde{p}_-} |A_{n,k,r_2}|^\zeta \right. \\
& \left. + \left(\int_{A_{n,k,r_2}} \left(\frac{u_n - k}{r_2 - r_1} \right)^{\tilde{p}^*_-} dx \right)^{\frac{\tilde{p}_-}{\tilde{p}^*_-}} |A_{n,k,r_2}|^{\zeta - \frac{\tilde{p}_-}{\tilde{p}^*_-}} \right).
\end{aligned}
\tag{3.67}
$$

is true for some $\epsilon > 0$ small enough, where $|A_{n,k,r_2}|^\zeta := \max\left\{ |A_{n,k,r_2}|^{1/s'_-}, |A_{n,k,r_2}|^{1/r'_-} \right\}$. That is, by definition of ξ, we obtain that

$$
\begin{aligned}
\int_{A_{n,k,r_1}} |\nabla u_n|^{\tilde{p}_-} dx & \leq \int_{A_{n,k,r_2}} |\nabla u_n|^{p(x)} \xi^{\tilde{p}_+} dx + |A_{n,k,r_2}| \\
& \leq M_{15} \left(\int_{A_{n,k,r_2}} \left(\frac{u_n - k}{r_2 - r_1} \right)^{\tilde{p}^*_-} dx + |A_{n,k,r_2}| + k^{\tilde{p}_-} |A_{n,k,r_2}|^\zeta \right. \\
& \quad \left. + \left(\int_{A_{n,k,r_2}} \left(\frac{u_n - k}{r_2 - r_1} \right)^{\tilde{p}^*_-} dx \right)^{\frac{\tilde{p}_-}{\tilde{p}^*_-}} |A_{n,k,r_2}|^{\zeta - \frac{\tilde{p}_-}{\tilde{p}^*_-}} \right)
\end{aligned}
\tag{3.68}
$$

holds true, where $M_{15} > 0$ is a real constant independent of n and k.

Now, set

$$
R_h = \frac{R}{2} + \frac{R}{2^{h+1}}, \quad \tilde{R}_h = \frac{R_h + R_{h+1}}{2} \quad \text{and} \quad k_h = k \left(1 - \frac{1}{2^{h+1}} \right) \quad \text{for } h \in \mathbb{N} \cup \{0\},
$$

and note that

$$
R_h \searrow \frac{R}{2}, \quad k_h \nearrow \frac{k}{2} \quad \text{and} \quad R_{h+1} < \tilde{R} < R_h < R.
$$

Define

$$
J_{n,h} = \int_{A_{n,k_h,R_h}} |u_n(x) - k_h|^{\tilde{p}^*_-} dx
$$

and consider $\phi \in C^1([0,\infty))$ satisfying $0 \leq \phi(t) \leq 1$, $\phi(t) = 1$ for $t \leq \frac{1}{2}$ and $\phi(t) = 0$ for $t \geq \frac{3}{4}$, $|\phi(t)| \leq C$. Set $\phi_h(x) = \phi\left(\frac{2^{h+1}}{R} \left(|x| - \frac{R}{2} \right) \right)$. Hence $\phi_h = 1$ in $K_{R_{h+1}}$ and $\phi_h = 0$ in

$\mathbb{R}^N \setminus K_{\tilde{R}_{h+1}}$. Thus

$$\begin{aligned} J_{n,h+1} &= \int_{A_{n,k_{h+1},R_{h+1}}} |(u_n(x)-k_{h+1})\phi_h|^{\tilde{p}_-^*}dx \le \int_{A_{n,k_{h+1},\tilde{R}_h}} |(u_n(x)-k_{h+1})\phi_h|^{\tilde{p}_-^*}dx \\ &\le \int_{K_R} |(u_n(x)-k_{h+1})\phi_h|^{\tilde{p}_-^*}dx \end{aligned}.$$

Since $\phi_h(v-k_{h+1})^+ \in W_0^{1,\tilde{p}_-}(K_R)$, it follows from the Sobolev inequality that

$$\begin{aligned} J_{n,h+1}^{\frac{\tilde{p}_-}{\tilde{p}_-^*}} &\le M_{14}\left(\int_{A_{n,k_{h+1},\tilde{R}_h}} |\nabla\big((u_n(x)-k_{h+1})\phi_h\big)|^{\tilde{p}_-}dx\right) \\ &\le M_{14}\left(\int_{A_{n,k_{h+1},\tilde{R}_h}} |\nabla u_n|^{\tilde{p}_-}dx + \int_{A_{k_{h+1},\tilde{R}_h}} |\nabla\phi_h|^{\tilde{p}_-}(u_n-k_{h+1})^{\tilde{p}_-}dx\right) \\ &\le M_{15}\left(\int_{A_{n,k_{h+1},\tilde{R}_h}} |\nabla u_n|^{\tilde{p}_-}dx + 2^{h\tilde{p}_-}\int_{A_{k_{h+1},\tilde{R}_h}} (u_n-k_{h+1})^{\tilde{p}_-}dx\right) \end{aligned}$$

holds.

By using (3.68) with $\mathbf{r_1} = \tilde{\mathbf{R}}_\mathbf{h} < \mathbf{R_h} = \mathbf{r_2}$ and $J_{n,h} \ge J_{n,h+1}$, we obtain

$$\begin{aligned} J_{n,h+1}^{\frac{\tilde{p}_-}{\tilde{p}_-^*}} &\le M_{16}\Bigg[\int_{A_{n,k_{h+1},R_h}} \left(\frac{u_n-k_{h+1}}{R_h-\tilde{R}_h}\right)^{\tilde{p}_-^*}dx + |A_{n,k_{h+1},R_h}| + k_{h+1}^{\tilde{p}_-}|A_{n,k_{h+1},R_h}|^{\zeta} \\ &+ \left(\int_{A_{n,k_{h+1},R_h}} \left(\frac{u_n-k_{h+1}}{R_h-\tilde{R}_h}\right)^{\tilde{p}_-^*}dx\right)^{\frac{\tilde{p}_-}{\tilde{p}_-^*}} |A_{n,k_{h+1},R_h}|^{\zeta-\frac{\tilde{p}_-}{\tilde{p}_-^*}} + 2^{h\tilde{p}_-}\int_{A_{n,k_{h+1},R_h}} (u_n-k_{h+1})^{\tilde{p}_-^*}dx\Bigg] \\ &\le M_{17}\Bigg[\int_{A_{n,k_{h+1},R_h}} (u_n-k_{h+1})^{\tilde{p}_-^*}dx + |A_{n,k_{h+1},R_h}| + k^{\tilde{p}_-}|A_{n,k_{h+1},R_h}|^{\zeta} \\ &+ \left(\int_{A_{n,k_{h+1},R_h}} (u_n-k_{h+1})^{\tilde{p}_-^*}dx\right)^{\frac{\tilde{p}_-}{\tilde{p}_-^*}} |A_{n,k_{h+1},R_h}|^{\zeta-\frac{\tilde{p}_-}{\tilde{p}_-^*}} + 2^{h\tilde{p}_-}\int_{A_{n,k_{h+1},R_h}} (u_n-k_{h+1})^{\tilde{p}_-^*}dx\Bigg] \\ &\le M_{17}\left[J_{n,h+1} + |A_{n,k_{h+1},R_h}| + |A_{n,k_{h+1},R_h}|^{\zeta} + J_{n,h+1}^{\frac{\tilde{p}_-}{\tilde{p}_-^*}}|A_{n,k_{h+1},R_h}|^{\zeta-\frac{\tilde{p}_-}{\tilde{p}_-^*}} + 2^{h\tilde{p}_-}J_{n,h}\right] \\ &\le M_{17}\left[J_{n,h} + |A_{n,k_{h+1},R_h}| + |A_{n,k_{h+1},R_h}|^{\zeta} + J_{n,h}^{\frac{\tilde{p}_-}{\tilde{p}_-^*}}|A_{k_{h+1},R_h}|^{\zeta-\frac{\tilde{p}_-}{\tilde{p}_-^*}} + 2^{h\tilde{p}_-}J_{n,h}\right], \end{aligned} \tag{3.69}$$

where $M_{17} = M_{17}(p_-, N, k)$.

Besides this, since $k_h \le k_{h+1}$ for any h, we have

$$\begin{aligned} J_{n,h} &= \int_{A_{n,k_h,R_h}} (u_n-k_h)^{\tilde{p}_-^*}dx \ge \int_{A_{n,k_{h+1},R_h}} (u_n-k_h)^{\tilde{p}_-^*}dx \\ &\ge \int_{A_{n,k_{h+1},R_h}} (k_{h+1}-k_h)^{\tilde{p}_-^*}dx \\ &= \big|A_{n,k_{h+1},R_h}\big|\,|k_{h+1}-k_h|^{\tilde{p}_-^*} = \big|A_{n,k_{h+1},R_h}\big|\left(\frac{k}{2^{h+1}}\right)^{\tilde{p}_-^*}, \end{aligned} \tag{3.70}$$

that is, by using (3.70) in (3.69), we obtain

$$J_{n,h+1}^{\frac{\tilde{p}_-}{\tilde{p}_-^*}} \le M_{18}\left[J_{n,h} + 2^{h\tilde{p}_-^*}J_{n,h} + 2^{h\zeta\tilde{p}_-^*}J_{n,h}^{\zeta} + 2^{h(\zeta\tilde{p}_-^*-\tilde{p}_-)}J_{n,h}^{\zeta} + 2^{h\tilde{p}_-^*}J_{n,h}\right].$$

By setting $M = \max(\tilde{p}_-^*, \zeta\tilde{p}_-^*, \zeta\tilde{p}_-^* - \tilde{p}_-)$, it follows from the above inequality that

$$J_{n,h+1}^{\frac{\tilde{p}_-}{\tilde{p}_-^*}} \leq M_{19} 2^{hM} J_{n,h}^{\zeta} \left[J_{n,h}^{1-\zeta} + 1 \right]$$

holds true for some $M_{19} = M_{19}(p_-, N, k)$. Now, by using Proposition 3.2.7 or 3.2.8, we have

$$J_{n,h} \leq \int_{K_R} (|u_n - k_h|^+)^{\tilde{p}_-^*} dx \leq ||u_n||_{L^{\tilde{p}_-^*}(K_R)} < M_{20},$$

where M_{20} is independent of n. Thus, by above estimate and $\zeta \in \{1/r'_-, 1/s'_-\}$ we obtain

$$J_{n,h+1} \leq CB^h J_{n,h}^{1+\eta} \tag{3.71}$$

holds for some C independent of n, $B := 2^{M\frac{\tilde{p}_-^*}{\tilde{p}_-}}$ and $\eta = (\zeta\tilde{p}_-^*/\tilde{p}_-) - 1 > 0$ since we are using the hypotheses $r(x), s(x) > N/p_- = (p_-^*/p_-)' \geq (\tilde{p}_-^*/\tilde{p}_-)'$.

Now, we claim that

$$J_{n,h} = \int_{A_{n,k_h,R_h}} |u_n(x) - k_h|^{\tilde{p}_-^*} dx \to \int_{A_{k_h,R_h}} |u(x) - k_h|^{\tilde{p}_-^*} dx := J_h \tag{3.72}$$

as $n \to \infty$, where $A_{k,i} := K_i \cap \{x \in \Omega \ : \ u(x) > k\}$. In fact, since

$$(u_n(x) - k_h)\mathcal{X}_{A_{n,k_h,R_h} \cap \Omega_{n,\delta}}(x) \to (u(x) - k_h)\mathcal{X}_{A_{k_h,R_h} \cap \Omega_\delta}(x) \text{ a.e. in in } \Omega,$$

and Proposition 3.2.6 implies that

$$0 \leq (u_n(x) - k_h)\mathcal{X}_{A_{n,k_h,R_h} \cap \Omega_{n,\delta}} \leq M_{21} \left(d(x) + \frac{1}{n} \right)^{\theta_2} \leq M_{21}(d(x) + 1)^{\theta_2} \in L^1(\Omega)$$

holds, we are able to apply Lebesgue's Theorem to obtain

$$\lim_{n \to \infty} \int_{A_{n,k_h,R_h} \cap \Omega_{n,\delta}} |u_n(x) - k_h|^{\tilde{p}_-^*} dx = \int_{A_{k_h,R_h} \cap \Omega_\delta} |u(x) - k_h|^{\tilde{p}_-^*} dx.$$

Besides this, by using that $u_n \to u$ in $W_{loc}^{1,p(x)}(\Omega)$, we have

$$\lim_{n \to \infty} \int_{A_{n,k_h,R_h} \cap \omega_{n,\delta}} |u_n(x) - k_h|^{\tilde{p}_-^*} dx = \int_{A_{k_h,R_h} \cap \omega_\delta} |u(x) - k_h|^{\tilde{p}_-^*} dx,$$

that is, (3.72) holds. As a consequence of this, by passing the limit $n \to \infty$ in (3.71), we obtain

$$J_{h+1} \leq CB^h J_h^{1+\eta}$$

holds for all $h \in \mathbb{N} \cup \{0\}$.

To finish, remembering that u satisfies $C_1 d(x) \leq u(x) \leq C_2 d(x)^{\theta_2}$ in Ω_δ and $u \in L_{loc}^{p_-^*}(\Omega)$, we are able to apply once Lebesgue's Theorem to conclude that

$$J_0 = \int_{A_{\frac{k}{2},r_2}} \left| u - \frac{k}{2} \right|^{p_-^*} dx = \int_{A_{\frac{k}{2},r_2} \cap \Omega_\delta} \left| u - \frac{k}{2} \right|^{p_-^*} dx + \int_{A_{\frac{k}{2},r_2} \cap \omega_\delta} \left| u - \frac{k}{2} \right|^{p_-^*} dx \to 0$$

as $k \to \infty$. So, by taking $k \geq k_0$ large enough such that $J_0 \leq C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}}$ and applying Lemma 1.3.6, we have that J_h converges to 0 as $h \to \infty$, that is,

$$\int_{A_{k_0,\frac{R}{2}}} |u - k_0|^{\tilde{p}^*_-} dx = 0$$

Since $u - k_0 > 0$ in $A_{k_0,\frac{R}{2}}$ and $x_0 \in \overline{\Omega}$ was taken arbitrary, the last integral implies $\left|A_{k_0,\frac{R}{2}}\right| = 0$ for all $x_0 \in \overline{\Omega}$. Thus, $0 \leq u \leq k_0$ on $K_{k_0,\frac{R}{2}}$ for all $x_0 \in \overline{\Omega}$, that is, $u \in L^\infty(\Omega)$. This finishes the proof of (i)

Proof of (ii) Let $u_n \in W_0^{1,p(x)}(\Omega)$ be the solution of the problem (3.4) and $u_n^{p_-(\sigma-1)+1} \in W_0^{1,p(x)}(\Omega)$ a test function in (3.12) for $\sigma \geq 1$ given. So, we have

$$\begin{aligned}(p_-(\sigma-1)+1)\int_\Omega u_n^{p_-(\sigma-1)}|\nabla u_n|^{p(x)}dx &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n^{p_-(\sigma-1)+1})dx \\ &\leq \int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)} u_n^{p_-(\sigma-1)+1}dx + \lambda\int_\Omega b(x)(1+u_n^{q(x)-1})u_n^{p_-(\sigma-1)+1}dx.\end{aligned} \tag{3.73}$$

Since

$$\begin{aligned}\int_\Omega |\nabla u_n^\sigma|^{p_-}dx &= \int_\Omega \sigma^{p_-}u_n^{p_-(\sigma-1)}|\nabla u_n|^{p_-}dx \\ &\leq \sigma^{p_-}\left(\int_\Omega u_n^{p_-(\sigma-1)}dx + \int_\Omega u_n^{p_-(\sigma-1)}|\nabla u_n|^{p(x)}dx\right)\end{aligned}$$

holds, we obtain

$$\begin{aligned}\int_\Omega |\nabla u_n^\sigma|^{p_-}dx \leq C\Big[&\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)} u_n^{p_-(\sigma-1)+1}dx \\ &+\lambda\int_\Omega b(x)(1+u_n^{q(x)-1})u_n^{p_-(\sigma-1)+1}dx + \int_\Omega u_n^{p_-(\sigma-1)}dx\Big].\end{aligned}$$

holds true for each $\sigma \geq 1$ given.

Below, let us evaluate each integral in the above inequality. Let us begin by considering the parameter $\sigma \geq 1$ satisfying

$$\sigma \geq \max\left\{\frac{p_- + (\beta_+ - 1)/\theta_2 + \alpha_+ - 1}{p_-}, \frac{p_- + \alpha_+ - 1}{p_-}\right\}. \tag{3.74}$$

To the first integral, we note that following the same lines used in (3.31) and (3.32), we obtain

$$\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)} u_n^{p_-(\sigma-1)+1}dx \leq M_1\left(1+\int_\Omega c(x)u_n^{p_-(\sigma-1)+1-\alpha(x)}dx\right).$$

Now, by Hölder Inequality,

$$\begin{aligned}\int_\Omega c(x)u_n^{p_-(\sigma-1)+1-\alpha(x)}dx &\leq \int_\Omega c(x)u_n^{p_-(\sigma-1)+1-\alpha_+}dx + \int_\Omega c(x)u_n^{p_-+(\sigma-1)+1-\alpha_-}dx \\ &\leq ||c||_{r_-}\left\|u_n^{p_-(\sigma-1)+1-\alpha_+}\right\|_{r'_-} + ||c||_{r_-}\left\|u_n^{p_-(\sigma-1)+1-\alpha_-}\right\|_{r'_-} \\ &\leq ||c||_{r_-}\left(\left\|u_n^{p_-(\sigma-1)+1-\alpha_-}\right\|_{r'_-}^{\frac{p_-(\sigma-1)+1-\alpha_+}{(p_-(\sigma-1)+1-\alpha_-)r'_-}} + \left\|u_n^{p_-(\sigma-1)+1-\alpha_-}\right\|_{r'_-}\right),\end{aligned} \tag{3.75}$$

that is, it follows from the last two inequalities that

$$\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{-\alpha(x)}u_n^{p_-(\sigma-1)+1}dx$$
$$\leq M_3\left(\left\|u_n^{p_-(\sigma-1)+1-\alpha_-}\right\|_{r'_-}^{\frac{p_-(\sigma-1)+1-\alpha_+}{(p_-(\sigma-1)+1-\alpha_-)r'_-}}+\left\|u_n^{p_-(\sigma-1)+1-\alpha_-}\right\|_{r'_-}+1\right). \tag{3.76}$$

For the second integral,

$$\int_\Omega b(x)(1+u_n^{q(x)-1})u_n^{p_-(\sigma-1)+1}dx \leq M_4\left(\int_\Omega b(x)dx+\int_\Omega b(x)u_n^{p_-(\sigma-1)+q(x)}dx\right)$$
$$\leq M_5\left(1+\|u_n^\sigma\|_{p^*_-}^{\frac{p_-(\sigma-1)+q_+}{\sigma}}\right)=M_5\left(1+\|u_n\|_{\sigma p^*_-}^{p_-(\sigma-1)+q_+}\right). \tag{3.77}$$

To the last integral in, we have

$$\int_\Omega u_n^{p_-(\sigma-1)}dx \leq \int_\Omega u_n^{p_-(\sigma-1)+1-\alpha_-}dx+\int_\Omega 1dx$$
$$\leq M_6\left(\left\|u_n^{p_-(\sigma-1)+1-\alpha_-}\right\|_{r'_-}+1\right) \tag{3.78}$$

Now, let us choose $\sigma \geq 1$ such that $\sigma p^*_- := (p_-(\sigma-1)+1-\alpha_-)r'_-$. That is,

$$\sigma=\frac{r_-(p_-+\alpha_--1)(N-p_-)}{p_-(N-r_-p_-)}.$$

Since $(N-r_-p_-)>0$, then σ is well defined. Also, $\sigma \geq (p_-+\alpha_+-1)/p_-$ if, and only if,

$$\frac{N(p_-+\alpha_+-1)}{(N-p)(p_-+\alpha_--1)-p_-(p_-+\alpha_+-1)}\leq r_-,$$

and $\sigma \geq \Big(p_-+(\beta_+-1)/\theta_2+\alpha_+-1\Big)/p_-$ if, and only if,

$$\frac{N(p_-+\frac{\beta_+-1}{\theta_2}+\alpha_+-1)}{(N-p)(p_-+\alpha_--1)-p_-(p_-+\frac{\beta_+-1}{\theta_2}+\alpha_+-1)}\leq r_-,$$

holds true. Thus, it follows from (3.75) – (3.78), that

$$\sigma^{p_-}\int_\Omega u_n^{p_-(\sigma-1)}|\nabla u_n|^{p(x)}dx \leq M_7\left(1+||u_n||_{\sigma p^*_-}^{p_-(\sigma-1)+1-\alpha_+}+||u_n||_{\sigma p^*_-}^{p_-(\sigma-1)+1-\alpha_-}+\lambda||u_n||_{\sigma p^*_-}^{p_-(\sigma-1)+q_+}\right). \tag{3.79}$$

Now, by Sobolev embedding $W_0^{1,p_-}(\Omega)\hookrightarrow L^{p^*_-}(\Omega)$ we have

$$M_5||u_n^\sigma||_{p^*_-}^{p_-}\leq ||u_n^\sigma||_{W_0^{1,p_-}(\Omega)}^{p_-}=\int_\Omega|\nabla u_n^\sigma|^{p_-}dx. \tag{3.80}$$

So, it follows from (3.79) and (3.80) that

$$||u_n||_{\sigma p^*_-}^{\sigma p_-}\leq M_6\left[|u_n||_{\sigma p^*_-}^{p_-(\sigma-1)+1-\alpha_+}+||u_n||_{\sigma p^*_-}^{p_-(\sigma-1)+1-\alpha_-}+||u_n||_{\sigma p^*_-}^{p_-(\sigma-1)+q_+}+1\right]$$

for some $M_6>0$ independent of n. Thus, we are able to choose a $\lambda_*>0$ small enough of $q_+=p_-$ or $\lambda_*=\infty$ if $q_+<p_-$, such that u_n is bounded in $L^{\sigma p^*_-}(\Omega)=L^{\frac{Nr_-(p_-+\alpha_--1)}{N-r_-p_-}}(\Omega)$.

Proof of (iii)**:** In this case, we need just to estimate the below integral in (3.73), because the estimate to the other one is already done in (3.77). Let us procedure. By splitting the domain Ω and using $1-\alpha(x)-\beta(x)>0$ in $\Omega_{n,\delta}$, we obtain

$$\begin{aligned}
&\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)} u_n^{p_-(\sigma-1)+1}dx\\
&\quad=\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)-\beta(x)}\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx\\
&\quad=\int_{\Omega_{n,\delta}} c(x)\left(d(x)+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)-\beta(x)}\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx\\
&\quad+\int_{\omega_{n,\delta}} c(x)\left(d(x)+\frac{1}{n}\right)^{p_-(\sigma-1)+1-\alpha(x)-\beta(x)}\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx\\
&\quad\leq M_1\int_\Omega c(x)\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx.
\end{aligned}$$

To this last integral, by arguing as in (3.38) and (3.39), we get

$$\begin{aligned}
&\int_\Omega c(x)\left(\frac{u_n+\frac{1}{n}}{d(x)+\frac{1}{n}}\right)^{p_-(\sigma-1)+1-\alpha(x)}dx\leq M_2\int_{\{p_-(\sigma-1)+1-\alpha(x)\leq 0\}} c(x)dx\\
&\quad+\int_{\Omega_{n,\delta}\cap\{p_-(\sigma-1)+1-\alpha(x)>0\}} c(x)\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(p_-(\sigma-1)+1-\alpha(x))}dx\\
&\quad+\int_{\omega_{n,\delta}\cap\{p_-(\sigma-1)+1-\alpha(x)>0\}} c(x)(u_n+1)^{p_-(\sigma-1)+1-\alpha(x)}dx.
\end{aligned}$$

Around to the boundary of Ω, we have

$$\begin{aligned}
&\int_{\Omega_{n,\delta}\cap\{p_-(\sigma-1)+1-\alpha(x)>0\}} c(x)\left(d(x)+\frac{1}{n}\right)^{(\theta_2-1)(p_-(\sigma-1)+1-\alpha(x))}dx\\
&\leq\int_{\Omega_{n,\delta}\cap\{p_-(\sigma-1)+1-\alpha(x)>0\}} c(x)(d(x)+1)^{(\theta_2-1)(p_-(\sigma-1)+1-\alpha(x))}dx\leq M_4,
\end{aligned}$$

where we used that $\theta_2>1$ in $\Omega_{n,\delta}$, since $1-\alpha(x)-\beta(x)>0$ in $\Omega_{n,\delta}$. That is, by above inequalities, we have

$$\begin{aligned}
&\int_\Omega c(x)\left(d(x)+\frac{1}{n}\right)^{-\beta(x)}\left(u_n+\frac{1}{n}\right)^{1-\alpha(x)} u_n^{p_-(\sigma-1)+1}dx\\
&\qquad\leq M_3\left(1+\int_\Omega c(x)u_n^{p_-(\sigma-1)+1-\alpha(x)}dx\right).
\end{aligned}\tag{3.81}$$

Since by hypotheses $N-r_-p_->0$ and

$$\frac{Np_-}{Np_--(N-p_-)(1-\alpha_-)}\leq r_-$$

are true, we are able to fix $\sigma\geq 1$ satisfying $\sigma p_-^*:=(p_-(\sigma-1)+1-\alpha_-)r_-'$, that is,

$$\sigma=\frac{r_-(p_-+\alpha_--1)(N-p_-)}{p_-(N-r_-p_-)}.$$

So, by using (3.81) and (3.77) in (3.73), we can repeat the same lines as the final part of the proof of (ii) to choose a $\lambda_* > 0$ small enough if $q_+ = p_-$ or $\lambda_* = \infty$ if $q_+ < p_-$, to conclude that u_n is bounded in $L^{\sigma p_-^*}(\Omega) = L^{\frac{Nr_-(p_-+\alpha_--1)}{N-r_-p_-}}(\Omega)$. $\blacksquare$

Remark 3.3.1 *Let $u \in W^{1,p(x)}_{loc}(\Omega)$ an arbitrary solution for (3.1). If we repeat the proof of Theorem 3.1.4(i) with u in the place of u_n, $x_0 \in \Omega$ and $\xi \in C_0^\infty(\Omega)$, then we are able to conclude that $u \in L^\infty_{loc}(\Omega)$, that is, any $W^{1,p(x)}_{loc}(\Omega)-$solution for (3.1) belongs to $L^\infty_{loc}(\Omega)$.*

Proof of Corollary 3.1.5. For each $x_0 \in \Omega$ and $R > 0$ given, set $B_R(x_0)$ the ball centered in x_0 with radius R. Let $0 < r_1 < r_2 < R$ such that $\overline{B}_{r_1} \subset B_{r_2} \subset B_R$ and take $\xi \in C_0^\infty(\overline{\Omega})$ with $0 \le \xi \le 1$, $\xi \equiv 1$ in $B_{r_1}(x_0)$, *supp* $(\xi) \subset B_{r_2}(x_0)$ and $|\nabla \xi| \le (r_2 - r_1)^{-1}$. For $k \ge 1$, consider the function $\psi = \xi^{\tilde{p}_+}(u-k)^+$ and note that $\psi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. By taking ψ as a test function for (3.1) and using (H_2) we have

$$\int_\Omega |\nabla u|^{p(x)} \xi^{\tilde{p}_+} dx + \tilde{p}_+ \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \xi \xi^{\tilde{p}_+-1}(u-k)^+ dx \qquad (3.82)$$
$$\le \int_\Omega \left(c(x) d(x)^{-\alpha(x)} u^{-\alpha(x)} + \lambda b(x) \left(1 + u^{q(x)-1}\right)\right) \xi^{\tilde{p}_+}(u-k)^+ dx.$$

Now, by arguing as in (3.61), (3.62), and (3.64), we obtain

$$\int_\Omega |\nabla u|^{p(x)-1} |\nabla \xi| (u-k)^+ \xi^{\tilde{p}_+-1} dx \qquad (3.83)$$
$$\le \epsilon \int_{A_{k,r_2}} |\nabla u|^{p(x)} \xi^{\tilde{p}_+} dx + C_\epsilon \int_{A_{k,r_2}} \left(\frac{u-k}{r_2 - r_1} \right)^{\tilde{p}_-^*} dx,$$

where $A_{k,i} = B_i \cap \{x \in \Omega \ : \ u(x) > k\}, \quad i = \{r_1, r_2, R\}$.

For the singular integral, using $(H_3)(ii)$ and Lemma 3.2.6(ii) we obtain

$$\int_\Omega c(x) d(x)^{-\beta(x)} u^{-\alpha(x)} \xi^{\tilde{p}_+}(u-k)^+ dx$$
$$\le \left\| d(x)^{-\beta(x)} u^{1-\alpha(x)} \xi^{\tilde{p}_+} \right\|_{L^\infty(\Omega)} ||c||_{r_-} |A_{k,r_2}|^{1-\frac{1}{r_-}}. \qquad (3.84)$$

More,

$$\int_{A_{k,r_2}} b(x) \left(1 + u^{q(x)-1}\right) \xi^{\tilde{p}_+}(u-k)^+ dx \le \left\| u + u^{q(x)} \right\|_{L^\infty(\Omega)} ||b||_{s_-} |A_{k,r_2}|^{1-\frac{1}{s_-}}. \qquad (3.85)$$

From (3.82) – (3.86)

$$\int_{A_{k,r_1}} |\nabla u|^{p(x)} dx \le C \left(\int_{A_{k,r_2}} \left(\frac{u-k}{r_2 - r_1}\right)^{p(x)} dx + \max\left\{ |A_{k,r_2}|^{1-\frac{1}{r_-}}, |A_{k,r_2}|^{1-\frac{1}{s_-}} \right\}\right)$$

holds true for each $\epsilon > 0$ small enough given, that is, $u \in C^{0,\gamma}(\Omega)$ for some $0 < \gamma < 1$, by using Lemma 1.3.5.

Now, let us prove the Hölder continuity up to the boundary of U, for all open sets $U \subseteq \Omega$ such that $\partial U \cap \partial \Omega = \Gamma_1 \cup \Gamma_{\theta_2}$. For each $x_0 \in \overline{U}$ set $K_R = B_R(x_0) \cap \overline{U}$. Let $0 < r_1 < r_2 < R$ such that $\overline{K}_{r_1} \subset K_{r_2} \subset K_R$ and take $\xi \in C^\infty(\overline{U})$ with $0 \le \xi \le 1$, $\xi \equiv 1$ in

K_{r_1}, $supp\ (\xi) \subset K_{r_2}$ and $|\nabla\xi| \leq (r_2 - r_1)^{-1}$. For $k \geq \max_{K_{r_2}} u(x) - \sigma ||u||_{L^\infty(U)}$, $\sigma \leq 2$, consider the function $\psi = \xi^{\tilde{p}_+}(u-k)^+$ and note that $\psi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ with $\psi = 0$ in $\partial U \cap \partial\Omega$ in trace sense. By taking ψ as a test function for (3.1) and using (H_2) we have

$$\begin{aligned}\int_U |\nabla u|^{p(x)} \xi^{\tilde{p}_+} dx + \tilde{p}_+ \int_U |\nabla u|^{p(x)-2} \nabla u \nabla \xi \xi^{\tilde{p}_+ - 1}(u-k)^+ dx \\ \leq \int_U \left(c(x) d(x)^{-\alpha(x)} u^{-\alpha(x)} + \lambda b(x) \left(1 + u^{q(x)-1}\right)\right) \xi^{\tilde{p}_+}(u-k)^+ dx.\end{aligned} \tag{3.86}$$

For the singular integral, using that

$$C_1 d(x) \leq u(x) \leq C_2 d(x)^{\theta_2} \text{ in } \Omega_\delta \cap \overline{U}, \tag{3.87}$$

we obtain

$$\begin{aligned}&\int_{A'_{k,r_2}} c(x) d(x)^{-\beta(x)} u^{-\alpha(x)} \xi^{\tilde{p}_+}(u-k)^+ dx \leq \int_{A'_{k,r_2}} c(x) d(x)^{-\beta(x)} u^{1-\alpha(x)} \xi^{\tilde{p}_+} dx \\ &\leq M_1 \left(\int_{\Omega_\delta \cap A'_{k,r_2} \cap \{\alpha(x) > 1\}} c(x) d(x)^{1-\alpha(x)-\beta(x)} \xi^{\tilde{p}_+} dx + \int_{\Omega_\delta \cap A'_{k,r_2} \cap \{\alpha(x) \leq 1\}} c(x) d(x)^{-\beta(x)+\theta_2(1-\alpha(x))} \xi^{\tilde{p}_+} dx \right) \\ &+ \left\| d(x)^{-\beta(x)} u^{1-\alpha(x)} \xi^{\tilde{p}_+} \right\|_{L^\infty(\omega_\delta)} \int_\Omega c(x) dx,\end{aligned}$$

where $A'_{k,i} = K_i \cap \{x \in U \ : \ u(x) > k\}$, $\ i = \{r_1, r_2, R\}$. Since $\partial U \cap \partial\Omega = \Gamma_1 \cup \Gamma_{\theta_2}$, then by Lazer and McKenna [51]

$$\max \left\{ \int_{\Omega_\delta} d(x)^{[-\beta(x)+1-\alpha_+]r'(x)} dx, \int_{\Omega_\delta} d(x)^{[-\beta(x)+\theta_2(1-\alpha_+)]r'(x)} dx \right\} < \infty.$$

Thus, by the above inequalities

$$\int_{A'_{k,r_2}} c(x) d(x)^{-\beta(x)} u^{-\alpha(x)} \xi^{\tilde{p}_+}(u-k)^+ dx \leq M_2 |A'_{k,r_2}|^{1-\frac{1}{r_-}} \tag{3.88}$$

More,

$$\int_{A'_{k,r_2}} b(x) \left(1 + u^{q(x)-1}\right) \xi^{\tilde{p}_+}(u-k)^+ dx \leq \left\| u + u^{q(x)} \right\|_{L^\infty(\Omega)} ||b||_{s_-} |A'_{k,r_2}|^{1-\frac{1}{s_-}}.$$

which lead us to conclude, as done in first part, that

$$\int_{A'_{k,r_1}} |\nabla u|^{p(x)} dx \leq C \left(\int_{A'_{k,r_2}} \left(\frac{u-k}{r_2 - r_1} \right)^{p(x)} dx + \max \left\{ |A'_{k,r_2}|^{1-\frac{1}{r_-}}, |A'_{k,r_2}|^{1-\frac{1}{s_-}} \right\} \right).$$

Beside this, by using (3.87) and the fact of $u \in C^{0,\gamma}(\Omega)$, we conclude that there exists $C > 0$ such that

$$\sup_{x \in K_R \cap \partial U} u(x) - \inf_{x \in K_R \cap \partial U} u(x) \leq C R^\gamma.$$

Thus, by Proposition 1.3.5 we conclude that $u \in C^{0,\gamma}(\overline{U})$. ■

3.3.3 Proof of Theorem 3.1.6 - Completed

Finally, let us prove the uniqueness result of $W^{1,p(x)}_{loc}(\Omega)$-solutions to the problem (3.1).

Proof. Let $u_1, u_2 \in W^{1,p(x)}_{loc}(\Omega)$ be two solutions of the problem (3.1). By Remark 3.3.1 we have that $u_1, u_2 \in L^\infty_{loc}(\Omega)$. Now, set $g(x,t) = c(x)d(x)^{-\beta(x)}t^{-\alpha(x)} + \lambda f(x,t)$ for $x \in \Omega$ and $t > 0$. We claim that the hypotheses of Theorem 2.1.3 holds true on the cone

$$[0, u_1] = \{w \in W^{1,p(x)}_0(\Omega) \ / \ 0 \leq w \leq u_1\}.$$

Admitting this by now, we are able to apply Theorem 2.1.3 to conclude that $u_1 \leq u_2$ in Ω. In the same way, we obtain that $u_1 \geq u_2$ in Ω, that is, $u_1 = u_2$ in Ω.

Now, we will prove the claim. First, the hypothesis (g_1) is immediate. Second, from hypotheses (H_1) and (H_4) we have that $g(x,t)/t^{p_-}$ is strictly decreasing in $t > 0$ for a.e. $x \in \Omega$, showing (g_2). To show (g_3), given $h > 0$, define the functional $I_h : W^{1,p(x)}_0(\Omega) \to \mathbb{R}$

$$\begin{aligned}
I_h(u) = & \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\{\alpha(x)=1\}} c(x)d(x)^{-\beta(x)} \ln\left(\frac{2h}{u^+ + 2h}\right) dx \\
& + \int_{\{\alpha(x)\neq 1\}} \frac{c(x)d(x)^{-\beta(x)}\left[(u^+ + 2h)^{1-\alpha(x)} - (2h)^{1-\alpha(x)}\right]}{\alpha(x) - 1} dx \\
& - \lambda \int_\Omega (F(x, u + 2h) - F(x, 2h)) dx.
\end{aligned}$$

The weakly lower semicontinuity of I_h on $[0, u_1]$ with respect to $W^{1,p(x)}_0(\Omega)$ follows by the same arguments used in Proposition 3.2.4. Below, we are going to show that I_h is coercive on $[0, u_1]$. To this, we notice that

$$\begin{aligned}
\int_{\{\alpha(x)=1\}} c(x)d(x)^{-\beta(x)} \ln(u^+ + 2h)dx & \leq \int_{\{\alpha(x)=1\}} c(x)d(x)^{-\beta(x)}(u^+ + 2h)dx \\
& \leq \int_\Omega c(x)d(x)^{1-\beta(x)} \frac{u^+}{d(x)} dx + 2h \int_\Omega c(x)d(x)^{-\beta(x)} dx \\
& \leq M_1 \left(\int_{\Omega_\delta} \frac{u^+}{d(x)} dx + \int_{\omega_\delta} c(x)u^+ dx + \int_\Omega c(x)d(x)^{-\beta(x)} dx\right),
\end{aligned} \tag{3.89}$$

that is, by using that $\beta(x) < 1$ on $\partial\Omega$, it follows from Lazer and McKenna [51], that

$$\int_\Omega c(x)d(x)^{-\beta(x)} \leq ||c||_{L^\infty(\Omega_\delta)} \int_\Omega d(x)^{-\beta(x)} dx + \left\| d(x)^{-\beta(x)} \right\|_{L^\infty(\omega_\delta)} \int_\Omega c(x) dx < \infty. \tag{3.90}$$

To others integrals, by applying Hardy's Inequality, Hölder's Inequality and the embedding $W^{1,p(x)}(\Omega) \to L^{r'(x)}(\Omega)$, we obtain

$$\int_{\{\alpha(x)=1\}} c(x)d(x)^{-\beta(x)} \ln(u + 2h)dx \leq M_2 \left(||u|| + 1\right). \tag{3.91}$$

To the complementary sets. First, we have

$$\int_{\{\alpha(x)<1\}} \frac{c(x)d(x)^{-\beta(x)}\,(u+2h)^{1-\alpha(x)}}{1-\alpha(x)}dx \leq \int_{\{\alpha(x)<1\}} \frac{c(x)d(x)^{-\beta(x)}(1+(u+2h))}{1-\alpha(x)}dx$$
$$\leq \int_{\Omega} \frac{c(x)}{1-\alpha(x)}d(x)^{1-\beta(x)}\frac{1+u+2h}{d(x)}dx$$
$$\leq M_3\left(\int_{\Omega_\delta\cap\{\alpha(x)<1\}} \frac{1+u+2h}{d(x)}dx + \int_{\omega_\delta\cap\{\alpha(x)<1\}} \frac{c(x)}{1-\alpha(x)}(1+u+2h)dx\right).$$

So, by using hypothesis $(H_3)(iii)$, Hölder's Inequality, Hardy's Inequality and again the embedding $W^{1.p(x)}(\Omega) \to L^{r'(x)}(\Omega)$, we obtain

$$\int_{\{\alpha(x)<1\}} \frac{c(x)d(x)^{-\beta(x)}\,(u+2h)^{\alpha(x)}-1}{\alpha(x)-1}dx \leq M_4(||u||+1). \tag{3.92}$$

To another one, by arguing as in (3.90), we obtain

$$\int_{\{\alpha(x)>1\}} \frac{c(x)d(x)^{-\beta(x)}\,(u+2h)^{1-\alpha(x)}}{\alpha(x)-1}dx \leq \int_{\{\alpha(x)>1\}} \frac{c(x)d(x)^{-\beta(x)}(2h)^{1-\alpha(x)}}{\alpha(x)-1}dx \quad < \infty.$$

To end, by using the hypothesis (H_2), $s(x) > N/p_- \geq p_-^*/q(x)$ and Hölder's inequality, we have

$$\begin{aligned}\left|\int_\Omega F(x,u+2h)dx\right| &\leq M_5\int_\Omega b(x)\left(|u+2h| + |u+2h|^{q(x)}\right)dx\\ &\leq M_6\left(||u+2h||_{s'(x)} + ||(u+2h)^{q(x)}||_{\frac{p_-^*}{q(x)}}\right)\\ &\leq M_6\left(||u+2h||_{s'(x)} + ||(u+2h)||_{p_-^*}^{q_-} + ||(u+2h)||_{p_-^*}^{q_+}\right)\\ &\leq M_7\left(||u+2h|| + ||(u+2h)||^{q_-} + ||(u+2h)||^{q_+}\right).\end{aligned} \tag{3.93}$$

So, we obtain from (3.91) – (3.93) in (3.89), that

$$I_h(u) \geq \frac{1}{p_+}||u||^{p_-} - C_4(1+||u||+||u||^{q_-}+\lambda||u||^{q_+})$$

for all $u \in W_0^{1,p(x)}(\Omega)$ with $||u|| > 1$.

Thus, we are able to choose a $\lambda_{**} > 0$ small enough f $q_+ = p_-$ or $\lambda_{**} = \infty$ if $q_+ < p_-$, such that I_h is coercive for all $0 < \lambda < \lambda_{**}$, proving the claim and finishing the proof. ■

Chapter 4

Multiplicity of $W_0^{1,p(x)}(\Omega)$-solutions for local-singular-convex problem

4.1 Introduction

In this chapter we study the following quasilinear elliptic local-singular-convex problem with variable exponents and powers

$$\begin{cases} -\Delta_{p(x)}u = a(x)u^{-\alpha(x)} + \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega;, \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $0 < a \in L^{r(x)}(\Omega)$ for some $1 < p \in C^1(\overline{\Omega})$ and $\lambda > 0$ is a real parameter.

Throughout this chapter we adopt the following definition of solution:

Definition 4.1.1 *A positive function $u \in W_0^{1,p(x)}(\Omega)$ is a solution to* (4.1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \phi dx = \int_\Omega a(x)u^{-\alpha(x)}\phi dx + \lambda \int_\Omega f(x,u)\phi dx$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$.

To state ours results, let us remind that:

$$\Omega_\delta := \{x \in \Omega \ / \ d(x) < \delta\}, \text{ for each } \delta > 0,$$

stands for the interior δ-strip around the boundary of the domain,

$$\Gamma_t = \{x \in \partial\Omega \ / \ [t(1-\alpha(x))]\frac{1}{1-1/r(x)} + 1 > 0\}, \text{ for } t \in \{1, \theta_1, \theta_2\}$$

is a subset of the boundary of the domain and the numbers

$$\theta_1 = \begin{cases} \max\limits_{x\in\overline{\Omega}_\delta} \dfrac{p(x)}{p(x)+\alpha(x)-1} & \text{if } \alpha(x) > 1, \\ 1 & \text{if } \alpha(x) \le 1, \end{cases} \quad \text{and} \quad \theta_2 = \min_{x\in\overline{\Omega}_\delta} \frac{p(x)}{p(x)+\alpha(x)-1}$$

will be important to establish behaviors of the solutions around the boundary.

Related to the functions $\alpha(x), a(x)$ and $f(x,t)$, we make the following general assumptions. Assume that there exists a $\delta > 0$ such that:

(H_1) $\alpha : \overline{\Omega} \to \mathbb{R}$ is a $C^{0,1}(\overline{\Omega})$-function that satisfies $\alpha_- > 1 - p_-$,

(H_2) $0 < a \in L^{r(x)}(\Omega)$ with $r(x) > N/p_-$ and one of the items below:

- (i) $a \in L^\infty(\Omega_\delta)$ and $\Gamma_1 \cup \Gamma_{\theta_2} = \partial\Omega$,
- (ii) $a(x) \geq a_\delta > 0$ in Ω_δ, $a \in L^\infty(\Omega_\delta)$ and $\Gamma_{\theta_1} \cup \Gamma_{\theta_2} = \partial\Omega$,

(H_3) $\frac{a(x)}{1-\alpha(x)} \in L^{r(x)}(\{\alpha(x) \neq 1\})$,

(f_1) $f : \Omega \times [0, \infty) \to [0, \infty)$ is a Caratheodory function such that for each $M > 0$ given there exists $c_1 = c_1(M) > 0$ satisfying

$$0 \leq f(x,s) \leq c_1 \text{ for every } 0 \leq s \leq M \text{ and a.e. } x \text{ in } \Omega$$

We would like to notice that the hypothesis (H_1) and (H_2) will be used to guarantee the existence of a positive subsolution for (4.1) that belongs to $W_0^{1,p(x)}(\Omega)$, via Corollary 3.1.3, while the condition (f_1) will be used to establish the existence of a positive supersolution for (4.1), without any additional growth condition on $f(x,t)$ in $t > 0$.

From now on, whenever we use the hypothesis (f_1), we will understand that $f(x,s)$ has been extended for $s < 0$ by putting $f(x,s) = f(x,0)$.

Our first result is.

Theorem 4.1.2 *Suppose (H_1), (H_2) and (f_1) are satisfied. Then there exist $\lambda_0 > 0$ such that the problem (4.1) has a weak solution $u_\lambda \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ for each $0 < \lambda < \lambda_0$ given satisfying $u_\lambda \geq m_0 d(x)$ in Ω for some $m_0 > 0$. In addition, there exist $M_0, M_1, m_1 > 0$ such that:*

- (i) *$m_0 d(x) \leq u_\lambda \leq M_0 d(x)^{\theta_2}$ for $x \in \Omega_\delta$ if $(H_2)(i)$ holds,*
- (ii) *$m_1 d(x)^{\theta_1} \leq u_\lambda \leq M_1 d(x)^{\theta_2}$ for $x \in \Omega_\delta$ if $(H_2)(ii)$ holds.*

We can also consider a setting in what $f(x,s)$ is allowed to change its sign if we replace (f_1) for the following couple of assumptions:

(f_2) $f : \Omega \times [0, \infty) \to \mathbb{R}$ is a Caratheodory function such that for each $M > 0$ given there exists $c_2 = c_2(M) > 0$ and $0 \leq h = h_M \in L^1(\Omega)$ satisfying

$$-h(x) \leq f(x,s) \leq c_2 \text{ for all } 0 \leq s \leq M \text{ and a.e. } x \in \Omega,$$

(f_3) there are $\zeta > 0$ and $c_3 > 0$ such that

$$f(x,s) \geq -c_3 a(x) \text{ for all } 0 \leq s \leq \zeta \text{ and a.e. } x \in \Omega.$$

So, for $f(x,t)$ changing the signal, we have.

Theorem 4.1.3 *Suppose* (H_1), (H_2), (f_2) *and* (f_3) *are satisfied. If* $\alpha(x) \geq 0$ *in* $\overline{\Omega}$ *with* $\alpha(x) < 1$ *on* $\partial\Omega$, *then there exist* $\lambda_1 > 0$ *such that the problem* (4.1) *has a weak solution* $u_\lambda \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ *for each* $0 < \lambda < \lambda_1$ *given satisfying* $u_\lambda \geq Cd(x)$ *in* Ω *for some* $C > 0$.

In order to establish the existence of at least two solutions for the problem (4.1), we also assume:

(f_4) there exists $C > 0$ such that

$$|f(x,t)| \leq C(1 + t^{q(x)-1}) \text{ for } t > 0 \text{ and a.e. } x \in \Omega,$$

with $1 < q \in C(\overline{\Omega})$ and $p_+ < q_+ < p_-^*$,

(f_5) there exists a subdomain $\emptyset \neq D \subset \Omega$ such that

$$\lim_{t \to \infty} \frac{F(x,t)}{t^{p_+}} = +\infty \text{ uniformly on } x \in D,$$

where $F(x,t) = \int_0^t f(x,s)ds$ for $t > 0$ and $x \in \Omega$,

(f_6) there exist $\tau \in C(\overline{\Omega})$ with $\tau < p_-$ such that

$$p_+ F(x,t) - f(x,t)t \leq \beta_0 t^{\tau(x)} \text{ for } a.e. \ x \in \Omega \text{ and all } t \geq t_0$$

for some $t_0 \geq 0$ and $\beta_0 \geq 0$.

So, we have existence of two ordered weak solutions

Theorem 4.1.4 *Suppose* $(H_1) - (H_3), (f_4) - (f_6)$ *are satisfied. There exists* $\lambda_* > 0$ *such that the problem* (4.1) *has at least two different solutions* $u_\lambda, v_\lambda \in W_0^{1,p(x)}(\Omega)$ *for each* $0 < \lambda < \lambda_*$ *given. In addition,* $u_\lambda \leq v_\lambda$ *and* u_λ *has negative energy while* v_λ *is a positive energy solution.*

The chapter is organized as follows. The section 4.2 is dedicated to obtain a weak solution for the problem (4.1) by using a sub-solution method and the results of Chapters 2 and 3. In section 4.3 we present the multiplicity of weak solutions via Mountain Pass Theorem.

4.2 Existence of a first solution

We start defining a sub and a supersolution to problema (4.1).

Definition 4.2.1 *A function* $\underline{u} \in W^{1,p(x)}(\Omega)$ *is a subsolution to* (4.1) *if* $\underline{u} > 0$ *in* Ω, $a(x)\underline{u}^{-\alpha(x)} \in L^1_{loc}(\Omega)$, $\underline{u}^+ \in W_0^{1,p(x)}(\Omega)$ *and*

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2} \nabla \underline{u} \nabla \phi dx \leq \int_\Omega a(x)\underline{u}^{-\alpha(x)} \phi dx + \lambda \int_\Omega f(x,\underline{u})\phi dx$$

holds for all $\phi \in C_0^\infty(\Omega)$ *with* $\phi \geq 0$ *a.e. in* Ω. *Analogously,* $\overline{u} \in W^{1,p(x)}(\Omega)$ *is a supersolution to* (4.1) *if* $\overline{u} > 0$ *in* Ω, $a(x)\overline{u}^{-\alpha(x)} \in L^1_{loc}(\Omega)$, $\overline{u}^- \in W_0^{1,p(x)}(\Omega)$ *and*

$$\int_\Omega |\nabla \overline{u}|^{p(x)-2} \nabla \overline{u} \nabla \phi dx \geq \int_\Omega a(x)\overline{u}^{-\alpha(x)} \phi dx + \lambda \int_\Omega f(x,\overline{u})\phi dx$$

holds true for all $\phi \in C_0^\infty(\Omega)$ *with* $\phi \geq 0$ *a.e. in* Ω.

Lemma 4.2.2 *Assume (H_1), $(H_2)(i)$ and (f_1) hold. Then there exists $\lambda_0 > 0$ the problem* (4.1) *admits a subsolution and a supersolution $\underline{u}, \overline{u} \in W_0^{1,p(x)}(\Omega)$ for each $0 < \lambda < \lambda_0$ given satisfying $\overline{u} \geq \underline{u} > 0$ for all $x \in \Omega$.*

Proof. Since we are assuming that (H_1) and (H_2) hold true, let $\underline{u} \in W_0^{1,p(x)}(\Omega)$ be the unique solution of the singular-concave problem

$$\begin{cases} -\Delta_{p(x)} u = a(x) u^{-\alpha(x)} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

given by Corollary 3.1.3. In particular, by non-negativity of $f(x,t)$, we have that $\underline{u}$ is a subsolution of the problem (4.1).

Now, let us construct a supersolution of (4.1). Again, by applying Corollary 3.1.3, we obtain an only $W_0^{1,p(x)}(\Omega)$-solution to the problem

$$\begin{cases} -\Delta_{p(x)} u = a(x) u^{-\alpha(x)} + 1 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.3}$$

Let us denote it by $\overline{u} \in W_0^{1,p(x)}(\Omega)$. Once using (H_1) and (H_2), it follows by Theorem 3.1.4, that $\underline{u}, \overline{u} \in L^\infty(\Omega)$.

Now, it follows from the hypothesis (f_1) with $M = ||\overline{u}||_\infty$, that

$$\int_\Omega |\nabla \underline{u}|^{p(x)-2} \nabla \underline{u} \nabla \phi dx - \int_\Omega a(x) \underline{u}^{-\alpha(x)} \phi dx - \lambda \int_\Omega f(x, \underline{u}) \phi dx \geq \int_\Omega (1 - \lambda c_1) a(x) \phi \geq 0,$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$, whenever $0 < \lambda < \lambda_0$, where $\lambda_0 = 1/c_1 > 0$. This shows that $\overline{u}$ is a supersolution for (4.1).

To end, we point out that $\underline{u}$ and $\overline{u}$ are also subsolution and supersolution to the problem (4.2). Thus, we can apply Theorem 2.1.3, to conclude that $\overline{u} \geq \underline{u} > 0$ for all $x \in \Omega$. This complete the proof. ■

Now we will study the case when $f(x,s)$ may change the signal.

Lemma 4.2.3 *Assume that $(H_1), (H_2), (f_2)$ and (f_3) holds true. If $\alpha(x) \geq 0$ in $\overline{\Omega}$ with $\alpha(x) < 1$ on $\partial\Omega$, then exists $\lambda_1 > 0$ such that the problem* (4.1) *admits a subsolution and a supersolution $\underline{v}, \overline{v} \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ for each $x \in \Omega$ and $0 < \lambda < \lambda_1$ given satisfying $\overline{v} \geq \underline{v} > 0$.*

Proof. First, let us build a subsolution. Given $\epsilon > 0$, consider the problem

$$\begin{cases} -\Delta_{p(x)} u = \epsilon a(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Since that the map $v \mapsto \int_\Omega \epsilon a(x) v dx$ defines a continuous linear functional on $W_0^{1,p(x)}(\Omega)$ and L is an homeomorphism, as shown at Lemma 1.2.4, then the problem (4.4) admits an unique weak solution $0 \lneqq \underline{v} = \underline{v}_\epsilon \in W_0^{1,p(x)}(\Omega)$. Also, it follows from Proposition 1.3.9 that $\underline{v} > 0$ in Ω and, from Proposition 1.3.8, we obtain that $\underline{v} \in C^{1,\gamma}(\overline{\Omega})$. In particular we obtain from Lemma 1.3.7 that

$$||\underline{v}||_\infty \leq C \epsilon^{\frac{1}{p_+ - 1}} \text{ for } 0 < \epsilon < 1. \tag{4.5}$$

Now, by taking $0 < \epsilon < 1$ so small, we obtain $0 < ||\underline{v}||_\infty \leq \min\{\zeta, 1\}$, where $\zeta > 0$ is given at (f_3). So, we are able to use the hypothesis (f_3) to obtain

$$\int_\Omega |\nabla \underline{v}|^{p(x)-2} \nabla \underline{v} \nabla \phi dx - \int_\Omega a(x) \underline{v}^{-\alpha(x)} \phi dx - \lambda \int_\Omega f(x, \underline{v}) \phi dx$$
$$\leq - \int_\Omega (1 - \epsilon - \lambda c_1)\, a(x) \phi \leq 0,$$

whenever $0 < \lambda < \lambda'$ for some $\lambda' > 0$ sufficiently small, that is, $\underline{v} > 0$ is a subsolution to the problem (4.1).

About the supersolution. By following the same arguments as done in the proof of Lemma 4.2.2, we obtain a $\overline{v} \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ that is a supersolution to the problem (4.1), whenever $0 < \lambda < \lambda''$ for some $\lambda'' > 0$.

To end, by defining $\lambda_1 = \min\{\lambda', \lambda''\}$ and noticing that $\overline{v}, \underline{v}$ are also a subsolution and a supersolution to the problem (4.2), we are able to apply Theorem 2.1.3 to deduce that $\overline{v} \geq \underline{v}$ in $x \in \Omega$. This finish the proof. ■

4.2.1 Proof of Theorems 4.1.2 and 4.1.3- Completed

Below, let us minimize an appropriated energy functional in $W_0^{1,p(x)}(\Omega)$ and show that this minimum belongs to the cone $[\underline{u}, \overline{u}]$.

Proof of Theorem 4.1.2 - Completed. Consider the following truncation

$$\overline{f}(x,t) = \begin{cases} a(x)\underline{u}^{-\alpha(x)} + \lambda f(x, \underline{u}) & \text{if } \ t \leq \underline{u}, \\ a(x)t^{-\alpha(x)} + \lambda f(x,t) & \text{if } \ \underline{u} < t < \overline{u}, \\ a(x)\overline{u}^{-\alpha(x)} + \lambda f(x, \overline{u}) & \text{if } \ t \geq \overline{u}. \end{cases} \tag{4.6}$$

So, $\overline{f}(x,t)$ is a Carathéodory function. We set $\overline{F}(x,t) = \int_0^t \overline{f}(x,s)ds$ and consider the functional $\overline{J} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\overline{J}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_\Omega \overline{F}(x,u)dx. \tag{4.7}$$

From Lemma $B.0.3$, $\overline{J}$ belongs to $C^1\left(W_0^{1,p(x)}(\Omega)\right)$, is coercive and sequentially weakly lower semi-continuous. Then it has a global minimizer $u_\lambda \in W_0^{1,p(x)}(\Omega)$, that is,

$$\overline{J}(u_\lambda) = \inf_{v \in W_0^{1,p(x)}(\Omega)} \overline{J}(v) \quad \text{ and } \quad \overline{J}'(u_\lambda) = 0.$$

In particular, by using $(u_\lambda - \overline{u})^+$ as a test function, we obtain

$$\begin{aligned} \int_\Omega |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda \nabla (u_\lambda - \overline{u})^+ dx &= \int_\Omega \overline{f}(x, u_\lambda)(u_\lambda - \overline{u})^+ dx \\ &= \int_\Omega \left(a(x)\overline{u}^{-\alpha(x)} + \lambda f(x, \overline{u})\right)(u_\lambda - \overline{u})^+ dx \\ &\leq \int_\Omega |\nabla \overline{u}|^{p(x)-2} \nabla \overline{u} \nabla (u_\lambda - \overline{u})^+ dx, \end{aligned}$$

where the last inequality is obtained by using the fact that $\overline{u}$ is a supersolution for the problem (4.1), that is,

$$\int_\Omega \left(|\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda - |\nabla \overline{u}|^{p(x)-2}\nabla \overline{u}\right)\nabla(u-\overline{u})^+ dx \leq 0,$$

holds true.

So, it follows from Lemma 1.2.6, that $|\{u_\lambda > \overline{u}\}| = 0$, that is, $u_\lambda \leq \overline{u}$ a.e. in Ω. In a analogous way, we have $\underline{u} \leq u_\lambda$ a.e. in Ω and thus

$$\int_\Omega |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda \nabla v dx = \int_\Omega a(x)u_\lambda^{-\alpha(x)} v dx + \lambda \int_\Omega f(x,u_\lambda) v dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$, that is, u_λ is a weak solution for (4.1).

To end, the asymptotic behavior follows directly from Corollary 3.1.3, since $\underline{u}$ and $\overline{u}$ satisfies the hypotheses considered. This finish the proof. ■

Proof of Theorem 4.1.3 - Completed. The proof follows the same lines of the proof of Theorem 4.1.2 by changing $\underline{u}, \overline{u}$ used in the proof of Theorem 4.1.2 by new ones $\underline{v}, \overline{v}$ given by Lemma 4.2.3 and finally using Lemma B.0.4 instead of Lemma B.0.3. ■

4.3 Existence of a second solution

Now, we are able to show the existence of a second solution to problem (4.1) by using the Mountain Pass Theorem. For convenience, throughout this section we are going to denote by $\underline{u}, \overline{u}$ the subsolution and supersolution obtained both in Lemma 4.2.2 and Lemma 4.2.3 and by u_λ the solution obtained both in Theorem 4.1.2 and 4.1.3.

Let the Carathéodory function defined by

$$\hat{f}(x,t) = \begin{cases} a(x)u_\lambda^{-\alpha(x)} + \lambda f(x,u_\lambda) & \text{if } t \leq u_\lambda, \\ a(x)t^{-\alpha(x)} + \lambda f(x,t) & \text{if } t > u_\lambda, \end{cases} \tag{4.8}$$

for $0 < \lambda < \min\{\lambda_1, \lambda_2\}$, where λ_1, λ_2 were given in Lemmas 4.2.2 and 4.2.3, respectively. Now, consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta_{p(x)} u = \hat{f}(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.9}$$

The functional associated to (4.9) is defined by

$$\hat{J}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_\Omega \hat{F}(x,u) dx, \quad u \in W_0^{1,p(x)}(\Omega), \tag{4.10}$$

where

$$\begin{aligned} \hat{F}(x,u) &= \int_0^u \hat{f}(x,s) ds \\ &= \int_0^{u_\lambda} \hat{f}(x,s) ds + \int_{u_\lambda}^u \hat{f}(x,s) ds \\ &= (a(x)u_\lambda^{-\alpha(x)} + \lambda f(x,u_\lambda))u_\lambda + a(x)\left(\mathcal{X}_{\{\alpha(x)\neq 1\}}(x)\frac{u^{1-\alpha(x)}}{1-\alpha(x)} + \mathcal{X}_{\{\alpha(x)=1\}}(x)\ln u\right) \\ &- a(x)\left(\mathcal{X}_{\{\alpha(x)\neq 1\}}(x)\frac{u_\lambda^{1-\alpha(x)}}{1-\alpha(x)} + \mathcal{X}_{\{\alpha(x)=1\}}(x)\ln u_\lambda\right) + \lambda(F(x,u) - F(x,u_\lambda)). \end{aligned} \tag{4.11}$$

So, defined like this, it follows from Lemma B.0.5 that $\hat{J} \in C^1\left(W_0^{1,p(x)}(\Omega)\right)$. Let

$$K_{\hat{J}} = \left\{u \in W_0^{1,p(x)}(\Omega) \ / \ \hat{J}'(u) = 0\right\}$$

be the set of critical points of $\hat{J}$. We claim that

$$K_{\hat{J}} \subseteq \left\{u \in W_0^{1,p(x)}(\Omega) \ / \ u(x) \geq u_\lambda(x) \ \text{ a.e. in } \Omega\right\}, \tag{4.12}$$

that is, any critical point of $\hat{J}$ is a weak solution of (4.1). Indeed, if $v \in K_{\hat{J}}$, it follows from the fact that u_λ is a weak solution to problem (4.1), that

$$\begin{aligned}
\int_\Omega |\nabla v|^{p(x)-2}\nabla v \nabla (u_\lambda - v)^+ dx &= \int_\Omega \hat{f}(x,v)(u_\lambda - v)^+ dx \\
&= \int_\Omega (a(x)u_\lambda^{-\alpha(x)} + \lambda f(x,u_\lambda))(u_\lambda - v)^+ dx \\
&= \int_\Omega |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda \nabla (u_\lambda - v)^+ dx,
\end{aligned}$$

that is,

$$\int_\Omega \left(|\nabla v|^{p(x)-2}\nabla v - |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda\right) \nabla (u_\lambda - v)^+ dx \leq 0.$$

So, it follows from Lemma 1.2.6 that $|\{u_\lambda > v\}| = 0$, proving the claimed. After this, to prove Theorem 4.1.4, it suffices to show that $\hat{J}$ has a critical point other than u_λ for $0 < \lambda < \min\{\lambda_1, \lambda_2\}$ sufficiently small.

4.3.1 Mountain Pass Geometry

Lemma 4.3.1 *Assume* $(H_1) - (H_3)$ *and* (f_4) *holds true. Then there exist* $R, \beta > 0$ *and* $0 < \lambda_* < \min\{\lambda_1, \lambda_2\}$ *such that*

$$\inf\left\{\hat{J}(u) \ / \ ||u|| = R\right\} \geq \beta > 0 \tag{4.13}$$

for each $0 < \lambda < \lambda_*$ *given.*

Proof. To begin, we claim that

$$\int_\Omega \hat{F}(x,u)dx \leq M_1\left(1 + ||u|| + ||u||^{1-\alpha_-} + \lambda ||u||^{q_+}\right)$$

holds true for all $u \in W_0^{1,p(x)}(\Omega)$ with $||u|| > 1$ and for each $0 < \lambda < \min\{\lambda_1, \lambda_2\}$ given. By admitting this from now, we obtain that

$$\begin{aligned}
\hat{J}(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \int_\Omega \hat{F}(x,u)dx \\
&\geq M_2\left(||u||^{p_+} - 1 - ||u|| - ||u||^{1-\alpha_-} - \lambda ||u||^{q_+}\right)
\end{aligned}$$

for all $u \in W_0^{1,p(x)}(\Omega)$ with $||u|| \geq 1$.

Now, let $\beta > 0$ and $R = ||u||$ be such that $R^{p_+} - R - R^{1-\alpha_-} - 1 \geq 2\beta/M_2$. So, by taking $0 < \lambda_* < \min\{\lambda_1, \lambda_2\}$ such that $\lambda_* R^{q_+} \leq \beta/M_2$, we conclude that

$$\hat{J}(u) \geq \beta \text{ for all } u \in W_0^{1,p(x)}(\Omega) \text{ with } ||u|| = R,$$

that is, $\inf \left\{ \hat{J}(u) \ / \ ||u|| = R \right\} \geq \beta > 0$ holds true for each $0 < \lambda < \lambda_*$ given.

Now, let us prove the claim. It follows from (4.11) and from the fact that u_λ being a solution of the problem (4.1), that

$$\int_\Omega \hat{F}(x,u)dx \leq \int_\Omega (a(x)u_\lambda^{-\alpha(x)} u dx + \int_{\{u \leq u_\lambda\}} \lambda f(x,u_\lambda) u dx + \int_{\{u > u_\lambda\} \cap \{\alpha(x)=1\}} a(x)(\ln u - \ln u_\lambda) dx$$

$$+ \int_{\{u>u_\lambda\}\cap\{\alpha(x)\neq 1\}} a(x) \left(\frac{u^{1-\alpha(x)}}{1-\alpha(x)} - \frac{u_\lambda^{1-\alpha(x)}}{1-\alpha(x)} \right) dx + \lambda \int_{\{u>u_\lambda\}} (F(x,u) - F(x,u_\lambda)) dx$$

$$\leq \int_\Omega |\nabla u_\lambda|^{p(x)} dx + 2\lambda \int_\Omega |f(x,u_\lambda)| u_\lambda dx + \int_\Omega a(x) \frac{u}{u_\lambda} dx + \int_{\{u>u_\lambda\}\cap\{\alpha(x)<1\}} \frac{a(x)u^{1-\alpha(x)}}{1-\alpha(x)} dx$$

$$+ \int_{\{\alpha(x)>1\}} a(x) \frac{u_\lambda^{1-\alpha(x)}}{\alpha(x)-1} dx + \lambda \int_\Omega |F(x,u)| dx + \lambda \int_\Omega |F(x,u_\lambda)| dx.$$

By virtue of hypothesis (f_4) we obtain

$$|F(x,t)| \leq M_3 \left(|t| + |t|^{q(x)} \right) \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R},$$

By using the above informations, once the hypotheses (f_4) and (H_3), the fact that $u_\lambda \geq Cd(x)$, $r(x) > (p_-^*/p_-)' > (p_-^*/(1-\alpha_-))'$, Hölder's Inequality and Hardy's Inequality we conclude that

$$\int_\Omega \hat{F}(x,u)dx \leq \int_\Omega |\nabla u_\lambda|^{p(x)} dx + 2\lambda \int_\Omega |f(x,u_\lambda)| u_\lambda dx + M_4 \left(\int_{\Omega_\delta} \frac{u}{d(x)} dx + \int_{\omega_\delta} a(x) u dx \right)$$

$$+ M_5 \int_{\{\alpha(x)>1\}} a(x) \frac{d(x)^{1-\alpha(x)}}{\alpha(x)-1} dx + \int_{\{u>u_\lambda\}\cap\{\alpha(x)<1\}} \frac{a(x)u^{1-\alpha(x)}}{1-\alpha(x)} dx$$

$$+ \lambda M_6 \left(\int_{\{u>u_\lambda\}} (|u| + |u|^{q(x)}) dx + \int_{\{u>u_\lambda\}} (u_\lambda + u_\lambda^{q(x)}) dx \right)$$

$$\leq M_5 \Bigg(||u_\lambda||^{p_+} + \lambda ||u_\lambda|| + \lambda ||u_\lambda||^{q_+} + \lambda ||u|| + \lambda ||u||^{q_+} + \left\| \frac{a}{1-\alpha(x)} \right\|_{L^{r(x)}(\{\alpha(x)<1\})} ||u||^{1-\alpha_-}$$

$$+ \int_{\{\alpha(x)>1\}} a(x) \frac{d(x)^{1-\alpha(x)}}{\alpha(x)-1} dx \Bigg).$$

where $\omega_\delta = \Omega \setminus \Omega_\delta$. Also, by using (H_3) and Hölder's Inequality, we conclude that

$$\int_{\{\alpha(x)>1\}} a(x) \frac{d(x)^{1-\alpha(x)}}{\alpha(x)-1} dx \leq M_6 \left\| \frac{a}{\alpha(x)-1} \right\|_{L^{r(x)}(\{\alpha(x)>1\})} \left\| d(x)^{1-\alpha(x)} \right\|_{L^{r'(x)}(\{\alpha(x)>1\})}$$

and thus, by hypothesis (H_2) and Lazer and McKenna [51] result,

$$\left\| d(x)^{1-\alpha(x)} \right\|_{L^{r'(x)}(\{\alpha(x)>1\})} \leq \left(\int_{\{\alpha(x)>1\}} d(x)^{(1-\alpha(x))r'(x)} dx \right)^\gamma < \infty,$$

where $\gamma \in \{1/r_+, 1/r_-\}$, proving the claim and finishing the proof. ■

Lemma 4.3.2 *Assume that $(H_1), (H_2)$ and (f_5) holds true. Then $\hat{J}(t\phi) \to -\infty$ as $t \to \infty$ for some $\phi \in C_0^\infty(\Omega)$.*

Proof. By using the hypothesis (f_5), there exists $s_0 = s_0(\epsilon) > 0$ such that

$$F(x,s) \geq \frac{1}{\epsilon p^+} s^{p^+} \text{ for } x \in D \quad \text{and} \quad s \geq s_0$$

for each $\epsilon > 0$ given. Now, take $\phi \in C_0^\infty(D)$ with $\phi \geq 0, \phi \not\equiv 0$, and $t > 1$ large enough such that the set $\{x \in D \ / \ t\phi(x) \geq s_0\}$ has positive Lebesgue measure. It follows from the above estimate, by taking $s = t\phi$, that

$$\int_\Omega \frac{F(x,t\phi)}{t^{p^+}} dx \geq \frac{1}{p^+\epsilon} \int_D \phi^{p^+} dx$$

holds true, that is,

$$\liminf_{t\to\infty} \int_\Omega \frac{F(x,t\phi)}{t^{p^+}} dx \geq \frac{1}{p_+\epsilon} \int_D \phi^{p^+} dx,$$

which lead us to conclude

$$\liminf_{t\to\infty} \int_\Omega \frac{F(x,t\phi)}{t^{p^+}} = \infty, \tag{4.14}$$

by doing $\epsilon \to 0^+$.

Besides this, it follows from (4.11) that

$$\begin{aligned}
\frac{\hat{F}(x,t\phi)}{t^{p_+}} &= (a(x)u_\lambda^{-\alpha(x)} + \lambda f(x,u_\lambda))\frac{\phi}{t^{p_+-1}} \\
&+ a(x)\left(\mathcal{X}_{\{\alpha(x)\neq 1\}}(x)\frac{\phi^{1-\alpha(x)}}{1-\alpha(x)}\frac{1}{t^{p_+ + \alpha(x)-1}} + \mathcal{X}_{\{\alpha(x)=1\}}(x)\frac{\ln t\phi}{t^{p_+}}\right) \\
&- \frac{a(x)}{t^{p_+}}\left(\mathcal{X}_{\{\alpha(x)\neq 1\}}(x)\frac{u_\lambda^{1-\alpha(x)}}{1-\alpha(x)} + \mathcal{X}_{\{\alpha(x)=1\}}(x)\ln u_\lambda\right) + \lambda\left(\frac{F(x,t\phi)}{t^{p_+}} - \frac{F(x,u_\lambda)}{t^{p_+}}\right).
\end{aligned}$$

holds true. By using the above expression and (4.14), we obtain

$$\liminf_{t\to\infty} \int_\Omega \frac{\hat{F}(x,t\phi)}{t^{p^+}} dx = \infty.$$

Hence,

$$\limsup_{t\to\infty} \frac{\hat{J}(t\phi)}{t^{p^+}} \leq \frac{1}{p^-} \int_\Omega |\nabla\phi|^{p(x)} dx - \liminf_{t\to\infty} \lambda \int_\Omega \frac{\hat{F}(x,t\phi)}{t^{p^+}} dx = -\infty,$$

that is, $\hat{J}(t\phi) \to -\infty$ as $t \to \infty$, finishing the proof. ■

4.3.2 The Cerami Condition

Lemma 4.3.3 *If hypotheses $(H_1) - (H_3), (f_4)$ and (f_6) holds, then $\hat{J}$ satisfies the Cerami condition.*

Proof. Let $(u_n) \subset W_0^{1,p(x)}(\Omega)$ be a sequence such that

(a) $|\hat{J}(u_n)| \leq M$,

(b) $(1 + ||u_n||)||\hat{J}'(u_n)|| \to 0$ as $n \to \infty$.

We are going to prove the result in two steps. In the first one, it will be shown that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. To do this, we begin using the item (b) to conclude that

$$\left|\left\langle \hat{J}'(u_n), w\right\rangle\right| \leq \frac{\epsilon_n ||w||}{1+||u_n||} \text{ for all } w \in W_0^{1,p(x)}, \tag{4.15}$$

for some $\epsilon_n \searrow 0$ as $n \to \infty$. So, by choosing $w = -u_n^-$ in (4.15), we obtain

$$\left|\int_\Omega |\nabla u_n^-|^{p(x)} dx + \int_\Omega a(x) u_\lambda^{-\alpha(x)} u_n^- dx + \lambda \int_\Omega f(x, u_\lambda) u_n^- dx\right| \leq \epsilon_n,$$

which lead us to conclude, by using the above inequality, Proposition 1.1.1 and Hölder's inequality, that

$$\begin{aligned}
\min\left\{||u_n^-||^{p_-}, ||u_n^-||^{p_+}\right\} &\leq \int_\Omega |\nabla u_n^-|^{p(x)} dx \\
&\leq \epsilon_n - \int_\Omega a(x) u_\lambda^{-\alpha(x)} u_n^- dx - \lambda \int_\Omega f(x, u_\lambda) u_n^- dx \\
&= \epsilon_n - \int_\Omega |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda \nabla u_n^- dx \\
&\leq \epsilon_n + C \left\| |\nabla u_\lambda|^{p(x)-1} \right\|_{p'(x)} ||u_n^-||
\end{aligned}$$

for some $C > 0$. So,

$$(u_n^-) \text{ is bounded in } W_0^{1,p(x)}(\Omega). \tag{4.16}$$

Let us show that (u_n^+) is bounded in $W_0^{1,p(x)}(\Omega)$. By taking $w = u_n^+$ in (4.15), it follows that

$$-\int_\Omega |\nabla u_n^+|^{p(x)} dx + \int_\Omega \hat{f}(x, u_n^+) u_n^+ dx = o_n(1). \tag{4.17}$$

Now, by using the item (a) above, we have

$$\begin{aligned}
p_+ M &\geq p_+ \int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx - p_+ \int_\Omega \hat{F}(x, u_n) dx \\
&= p_+ \int_{\{u_n \geq 0\}} \left(\frac{|\nabla u_n^+|^{p(x)}}{p(x)} - \hat{F}(x, u_n^+)\right) dx + p_+ \int_{\{u_n \leq 0\}} \left(\frac{|\nabla u_n^-|^{p(x)}}{p(x)} - \hat{F}(x, -u_n^-)\right) dx,
\end{aligned}$$

which lead us, by using (4.11) and the boundedness of u_n^- given in (4.16), to

$$p_+ \int_\Omega \frac{|\nabla u_n^+|^{p(x)}}{p(x)} dx - p_+ \int_\Omega \hat{F}(x, u_n^+) dx \leq p_+ M_1. \tag{4.18}$$

By summing (4.17)and (4.18), we obtain

$$\begin{aligned}
\int_\Omega \left(\frac{p^+}{p(x)} - 1\right) |\nabla u_n^+|^{p(x)} dx &\leq M_2 - \int_\Omega \left(\hat{f}(x, u_n^+) u_n^+ - p^+ \hat{F}(x, u_n^+)\right) dx \qquad (4.19) \\
&= M_2 - \int_{\{u_n^+ \leq u_\lambda\}} \left(\hat{f}(x, u_n^+) u_n^+ - p^+ \hat{F}(x, u_n^+)\right) dx - \int_{\{u_n^+ > u_\lambda\}} \left(\hat{f}(x, u_n^+) u_n^+ - p^+ \hat{F}(x, u_n^+)\right) dx.
\end{aligned}$$

Below, let us estimate both integrals in the last line above. For the first one, by using (4.8), (f_4) and Hölder's Inequality, we have

$$\begin{aligned}
&\left|\int_{\{u_n^+\le u_\lambda\}}\left(\hat{f}(x,u_n^+)u_n^+ - p^+\hat{F}(x,u_n^+)\right)dx\right| \\
&\qquad \le (p_+-1)\int_{\{u_n^+\le u_\lambda\}}\left(a(x)u_\lambda^{-\alpha(x)}u_n^+ + \lambda|f(x,u_\lambda)|u_n^+\right)dx \\
&\qquad \le (p_+-1)\int_{\{u_n^+\le u_\lambda\}}\left(a(x)u_\lambda^{-\alpha(x)}u_\lambda + \lambda|f(x,u_\lambda)|u_\lambda\right)dx \\
&\qquad \le (p_+-1)\left(\int_\Omega a(x)u_\lambda^{-\alpha(x)}u_\lambda dx + \lambda\int_\Omega |f(x,u_\lambda)|u_\lambda dx\right) \\
&\qquad \le (p_+-1)\left(\int_\Omega |\nabla u_\lambda|^{p(x)}dx + \lambda\int_\Omega |f(x,u_\lambda)|u_\lambda dx + \lambda\int_\Omega |f(x,u_\lambda)|u_\lambda dx\right) \\
&\qquad \le M_3.
\end{aligned} \tag{4.20}$$

For the second integral, by letting

$$\begin{aligned}
A^1 &= \{u_n^+ > u_\lambda\}\cap\{\alpha(x)=1\}, \\
A^+ &= \{u_n^+ > u_\lambda\}\cap\{\alpha(x)>1\}, \\
A^- &= \{u_n^+ > u_\lambda\}\cap\{\alpha(x)<1\},
\end{aligned} \tag{4.21}$$

and using (4.21) and (4.11), we obtain

$$\begin{aligned}
&\int_{\{u_n^+>u_\lambda\}}\left(p_+\hat{F}(x,u_n^+) - \hat{f}(x,u_n^+)u_n^+\right)dx = \int_{A^1} a(x)(p_+\ln u_n^+ - p_+\ln u_\lambda - 1)dx \\
&+ \int_{A^+\cup A^-} a(x)\left(\frac{p_+}{1-\alpha(x)}-1\right)(u_n^+)^{1-\alpha(x)}dx - \int_{A^+\cup A^-} a(x)\frac{p_+}{1-\alpha(x)}u_\lambda^{1-\alpha(x)}dx \\
&+ \lambda\int_{\{u_n^+>u_\lambda\}}\mathcal{F}(x,u_n^+)dx - \lambda p_+\int_{\{u_n^+>u_\lambda\}}F(x,u_\lambda)dx \\
&\le p_+\int_{A^1} a(x)\ln\frac{u_n^+}{u_\lambda}dx + \int_{A^-} a(x)\left(\frac{p_+}{1-\alpha(x)}-1\right)(u_n^+)^{1-\alpha(x)}dx + \lambda\int_{\{u_n^+>u_\lambda\}}\mathcal{F}(x,u_n^+)dx \\
&\le p_+\int_\Omega a(x)\frac{u_n^+}{u_\lambda}dx + \int_{A^-} a(x)\left(\frac{p_+}{1-\alpha(x)}-1\right)(u_n^+)^{1-\alpha(x)}dx + \lambda\int_{\{u_n^+>u_\lambda\}}\mathcal{F}(x,u_n^+)dx,
\end{aligned} \tag{4.22}$$

where $\mathcal{F}(x,t) = p_+F(x,t) - f(x,t)t$. More, by using that $r(x) > (p_-^*/p_-)' > (p_-^*/(1-\alpha_-))'$, $u_\lambda \ge Cd(x)$, (H_2) and (H_3), the integrals in the last line of (4.22) can be estimate by

$$\begin{aligned}
&\int_\Omega a(x)\frac{u_n^+}{u_\lambda}dx + \int_{A^-} a(x)\left(\frac{p^+}{1-\alpha(x)}-1\right)(u_n^+)^{1-\alpha(x)}dx \\
&\qquad \le M_5\left(\int_\Omega a(x)\frac{u_n^+}{d(x)}dx + p_+\int_{A^-}\frac{a(x)}{1-\alpha(x)}(u_n^+)^{1-\alpha(x)}dx\right) \\
&\qquad \le M_6\left(||a||_{L^\infty(\Omega_\delta)}\int_{\Omega_\delta}\frac{u_n^+}{d(x)}dx + \frac{1}{\delta}\int_{\omega_\delta} a(x)u_n^+dx + \int_{A^-}\frac{a(x)}{1-\alpha(x)}(u_n^+)^{1-\alpha(x)}dx\right) \\
&\qquad \le M_6\left(||u_n^+|| + ||u_n^+||^{1-\alpha_-}\right),
\end{aligned} \tag{4.23}$$

where we used that Hardy's, Hölder's Inequality and the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$.

After this, by combining (4.22) and (4.23), we obtain

$$\int_{\{u_n^+>u_\lambda\}}\left(p_+\hat{F}(x,u_n^+)-\hat{f}(x,u_n^+)u_n^+\right)dx\le\lambda\int_{\{u_n^+>u_\lambda\}}\mathcal{F}(x,u_n^+)dx+M_7\left(1+||u||+||u_n^+||^{1-\alpha_-}\right). \tag{4.24}$$

To conclude that (u_n^+) is bounded in $W_0^{1,p(x)}(\Omega)$, let us use (f_6) to find $s_0=s_0(\beta_0)>0$ such that

$$p_+F(x,s)-f(x,s)s\le\beta_0 s^{\tau(x)}\text{ for a.e. }x\in\Omega\text{ and for all }s\ge s_0.$$

Adding to this information, a consequence of hypothesis (f_4) that

$$p_+F(x,s)-f(x,s)s\le M_8\text{ for }x\in\Omega\text{ and }s<s_0,$$

holds true for some $M_8>0$, we obtain that

$$\mathcal{F}(x,u_n^+)\le p^+F(x,u_n^+)-f(x,u_n^+)u_n^+\le\beta_0(u_n^+)^{\tau(x)}+M_8,\ x\in\Omega. \tag{4.25}$$

So, it follows from (4.24), (4.25) and Hölder's Inequality, that

$$\int_{\{u_n^+>u_\lambda\}}\left(p_+\hat{F}(x,u_n^+)-\hat{f}(x,u_n^+)u_n^+\right)dx\le M_9\left(1+||u_n^+||^{\tau_-}+||u_n^+||^{\tau_+}+||u_n^+||^{1-\alpha_-}\right) \tag{4.26}$$

holds true. Now, combining (4.20) and (4.26) in (4.19), we obtain

$$\min\left\{||u_n^+||^{p_-},||u_n^+||^{p_+}\right\}\le M_{10}\left(1+||u_n^+||^{\tau_-}+||u_n^+||^{\tau_+}+||u_n^+||^{1-\alpha_-}\right),$$

that is, u_n^+ is bounded in $W_0^{1,p(x)}(\Omega)$.

Summarizing, since we already know that u_n^- is bounded in $W_0^{1,p(x)}(\Omega)$ (see (4.16)) and the boundedness of u_n^+ as just shown, we have that u_n is bounded in $W_0^{1,p(x)}(\Omega)$, finishing the first step.

In this last step, let us complete the proof that $\hat{J}$ satisfies the Cerami condition. To do this, since u_n is bounded $W_0^{1,p(x)}(\Omega)$, then there exists $u\in W_0^{1,p(x)}(\Omega)$ such that, unless to a subsequence, $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, $u_n\to u$ in $L^{t(x)}(\Omega)$ for $1\le t(x)<p^*(x)$ and $u_n(x)\to u(x)$ a.e. in Ω. Now, by taking $w=u_n-u$ in (4.15)

$$\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx-\int_\Omega\hat{f}(x,u_n)(u_n-u)dx\le\frac{\epsilon_n||u_n-u||}{1+||u_n||}. \tag{4.27}$$

Now, we are going to show that

$$\left|\int_\Omega\hat{f}(x,u_n)(u_n-u)dx\right|\to 0 \tag{4.28}$$

as $n\to\infty$. As a consequence, we can obtain that

$$\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx\le 0, \tag{4.29}$$

that is, $u_n\to u$ in $W_0^{1,p(x)}(\Omega)$, by using Proposition 1.2.4, which lead us to conclude that $\hat{J}$ satisfies the Cerami condition.

To proof (4.27), using the hypothesis (f_4), the fact of u_λ is a weak solution for (4.1) and Hölder's Inequality, we notice that

$$\begin{aligned} &\left|\int_{\{u_n\le u_\lambda\}}\hat{f}(x,u_n)(u_n-u)dx\right| \le \left|\int_\Omega (a(x)u_\lambda^{-\alpha(x)}+\lambda f(x,u_\lambda))(u_n-u)dx\right| \\ &\le \int_\Omega |\nabla u_\lambda|^{p(x)-1}|u_n-u|dx + 2\lambda\int_\Omega |f(x,u_\lambda)||u_n-u|dx \\ &\le M_{11}\left(||u_n-u||_{p(x)} + ||u_\lambda^{q(x)-1}||_{\frac{q_+}{q(x)-1}}||u_n-u||_{\frac{q_+}{q_+-q(x)+1}}\right). \end{aligned} \tag{4.30}$$

On the other hand, by using again that u_λ is a weak solution for (4.1), hypothesis (f_2) and Hölder's Inequality

$$\left|\int_{\{u_n>u_\lambda\}}\hat{f}(x,u_n)(u_n-u)dx\right| = \left|\int_{\{u_n>u_\lambda\}}(a(x)u_n^{-\alpha(x)}+\lambda f(x,u_n))(u_n-u)dx\right| \tag{4.31}$$

$$\begin{aligned} &\le \int_{\{u_n>u_\lambda\}\cap\{\alpha(x)>0\}} a(x)u_\lambda^{-\alpha(x)}(u_n-u_\lambda)dx + \int_{\{u_n>u_\lambda\}\cap\{\alpha(x)\le 0\}} a(x)(1+u_n^{p_--1})(u_n-u_\lambda)dx \\ &+\lambda\int_\Omega |f(x,u_n)||u_n-u|dx \\ &\le \int_{\{u_n>u_\lambda\}\cap\{\alpha(x)>0\}} a(x)u_\lambda^{-\alpha(x)}(u_n-u_\lambda)dx + M_{12}||a||_{r(x)}\left(\left\|u_n^{p_--1}+1\right\|_{\frac{p_-r'(x)}{p_--1}}\|u_n-u_\lambda\|_{p_-r'(x)}\right) \\ &+M_{13}\left(||u_n-u||_{p(x)} + ||u_n^{q(x)-1}||_{\frac{q_+}{q(x)-1}}||u_n-u||_{\frac{q_+}{q_+-q(x)+1}}\right). \end{aligned}$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and

$$\frac{q_+}{q_+-q(x)+1}, p_-r'(x) < p_-^* \le p^*(x),$$

then by using (4.30) and (4.31) we conclude that (4.28) holds. This finish the proof. ■

4.3.3 Proof of Theorem 4.1.4 - Completed

Proof. Now, we are going to complete the proof of Theorem 4.1.4. To establish this, we begin noticing that

$$\int_\Omega |\nabla u_\lambda|^{p(x)}dx = \int_\Omega a(x)u_\lambda^{1-\alpha(x)}dx + \int_\Omega \lambda f(x,u_\lambda)u_\lambda dx.$$

holds true, since u_λ is a weak solution of (4.1).

Beside this, by using (4.8) and (4.10), we obtain that

$$\hat{J}(u_\lambda) = \int_\Omega \frac{|\nabla u_\lambda|^{p(x)}}{p(x)}dx - \int_\Omega a(x)u_\lambda^{1-\alpha(x)}dx + \int_\Omega \lambda f(x,u_\lambda)u_\lambda dx,$$

that is,

$$\hat{J}(u_\lambda) = \int_\Omega\left(\frac{1}{p(x)}-1\right)|\nabla u_\lambda|^{p(x)}dx < 0.$$

Finally, by using Lemmas 4.3.1, 4.3.2 and 4.3.3, we can apply the Theorem Mountain Pass theorem (see A.1.9) to obtain a function $v_\lambda \in W_0^{1,p(x)}(\Omega)$ that is critical point of $\hat{J}$ satisfying

$$\hat{J}(u_\lambda) < 0 < \beta = \inf \left\{ \hat{J}(u) \ / \ ||u|| = r \right\} \leq \hat{J}(v_\lambda),$$

that is, $u_\lambda \neq v_\lambda$ and

$$\int_\Omega \left(|\nabla v_\lambda|^{p(x)-2} \nabla v_\lambda \nabla \xi - a(x) v_\lambda^{-\alpha(x)} \xi - \lambda f(x, v_\lambda) \xi \right) dx = 0,$$

for all $\xi \in W_0^{1,p(x)}(\Omega)$, which lead us to conclude that v_λ is a weak solution of (4.1) with $v_\lambda \geq u_\lambda$, because of (4.12). ■

Chapter 5

Open problems and future work

We left some open questions in this work. Here we summarize our contribution and what remains open.

Chapter 2

In Theorem 2.1.3, we consider $\underline{u} \in W^{1,p(x)}_{loc}(\Omega)$ subsolution of (2.1) with $\underline{u} \in L^{\infty}_{loc}(\Omega)$. In our proof, this locally boundedness was fundamental to obtain that the set $|\Omega_h| = 0$. We tried some techniques to prove it without this assumption, but we could not solve the problem. It remains as an open question.

On the other hand, the proofs based in Diaz-Saá Inequality demands, in general, $\overline{u}/\underline{u}, \underline{u}/\overline{u} \in L^{\infty}(\Omega)$. In this sense, we have a contribution.

Chapter 3

In Theorem 3.1.2, we show that the "integrability condition" of trio (c, α, β) just near the boundary of the domain is sufficient to obtain existence of solutions in $W^{1,p(x)}_0(\Omega)$. We conjecture that the converse claim is true as well.

On the other hand, we present sufficient conditions for that the solution for (3.1) be Holder continuous on the boundary. As a future work, we want to find conditions to obtain solutions in $C^{1,\alpha}$. The work of Lazer and Mckenna suggest that the right answer of this question is $\alpha(x) < 1$ on $\partial\Omega$. A future work is studying it.

Chapter 4

In this chapter, we prove that just a locally $(p_+ - 1) - superlinear$ perturbation of the singularity is suffices to obtain multiplicity of solutions for small $\lambda^* > 0$. If we define

$$\Lambda = \sup\{\lambda \ / \ \text{The problem (4.1) has a solution}\}$$

then $\Lambda > 0$. The next step is try to show under which hypothesis that $\Lambda < \infty$ to obtain a global multiplicity result. As a future work, we pretend to give a positive answer for this question.

Appendix A

Algebraic tools

In this Chapter, let us enunciate some very classical and well known results in order to ease the lecture of reader.

A.1 Algebraic tools

Proposition A.1.1 (Fatou's Lemma, [3]) *Let $f_n : \mathbb{R} \to [0, \infty]$ be (nonnegative) Lebesgue measurable functions. Then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n dx \geq \int_{\mathbb{R}} \liminf_{n\to\infty} f_n dx.$$

Proposition A.1.2 (Lebesgue dominated convergence theorem, [3]) *Suppose $f_n : \mathbb{R} \to [-\infty, \infty]$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume there is an integrable $g : \mathbb{R} \to [0, \infty]$ with $|f_n(x)| \geq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n dx \geq \int_{\mathbb{R}} \lim_{n\to\infty} f_n dx = \int_{\mathbb{R}} f dx.$$

Proposition A.1.3 ([65], Vitali's convergence Theorem) *Let μ be a finite positive measure on a measure space X. If f_n has uniformly absolutely continuous integrals and $f_n(x) \to f(x)$ a.e. in X, then $f \in L^1(\mu)$ and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Lemma A.1.4 ([52]) *Assume that $S : \mathbb{R}^s \to \mathbb{R}^s$ is a continuous map such that $(S(\eta), \eta) > 0$ for all $\eta \in \mathbb{R}^s$ such that $|\eta| = r$ for some $r > 0$, where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^s$. Then, there is $\eta_0 \in \overline{B}_r(0)$ such that $S(\eta_0) = 0$.*

Proposition A.1.5 ([56], Theorem 26.9) *Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, i.e., the intersection $\cap_{C\in\mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.*

Proposition A.1.6 ([36], Theorem 1.5) *Let X be a reflexive Banach space, A a bounded subset of X, and x_0 a point in the weak closure of A. Then there exists an infinite sequence (x_k) in A converging weakly to x_0 in X.*

Proposition A.1.7 ([25], Lemma 6) *Let X be a finite dimensional real Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. Let (β_k) be a sequence of functions from X into X which converges uniformly on compact subsets of X to a continuous function β. Assume that the functions β, are monotone and the β is strictly monotone, i.e.*

$$(\beta_k(x)-\beta_k(y),x-y)\geq 0, \qquad (\beta(x)-\beta(y),x-y)>0,$$

for every k and for every $x,y\in X$ with $x\neq y$. Let (η_k) be a sequence in X and let η be an element of X such that

$$\lim_{k\to\infty}(\beta_k(\eta_k)-\beta_k(\eta),\eta_k-\eta)=0.$$

Then (η_k) converges to η in X.

Due to Ambrosetti and Rabinowitz [67], the Mountain Pass Theorem is a fundamental result in Critical Point Theory and whose development was strongly related to the search for saddle-type critical points. In this section, X denotes a space of Banach real, $\phi: X\to\mathbb{R}$ is a functional and (u_n) is a sequence in X.

Below, we present a condition of compactness on the functional ϕ due to Cerami [18].

Definition A.1.8 *We say that ϕ satisfies the Cerami condition at level c, if every sequence $(x_n)\subset W_0^{1,p(x)}(\Omega)$ such that $\phi(x_n)\to c$ in $\mathbb{R}$ and*

$$(1+||x_n||)\phi'(x_n)\to 0 \quad in \quad W_0^{1,p(x)}(\Omega) \quad as \quad n\to\infty,$$

admits a strongly convergent subsequence. We say that ϕ satisfies the Cerami condition, if it satisfies the Cerami condition at every level $c\in\mathbb{R}$.

This compactness type condition on ϕ is weaker than the usual Palais-Smale condition. However, as it has shown in [4], the deformation theorem and consequently the minimax theory of the critical values of ϕ is still valid if the Palais-Smale condition is replaced by the Cerami condition. In particular, we have the following form of the well-known "Mountain Pass theorem".

Proposition A.1.9 *Suppose $\phi\in C^1(X)$ satisfies the geometric condition*

$$\max\{\phi(0),\phi(e)\}\leq 0<\beta=\inf\{\phi(x)\ :\ ||x||=\rho\},$$

for some $0<\beta$, $\rho>0$ and $e\in X$ with $||e||>\rho$. If c is defined by

$$c=\inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\phi(\gamma(t)),$$

where $\Gamma=\{\gamma\in C([0,1];X)\ /\ \gamma(0)=0,\gamma(1)=e\}$, and ϕ satisfies the Cerami condition, then $c\geq\beta$ and c is a critical value of ϕ. Moreover, if $c=\eta$, then there exists a critical point $x\in X$ of ϕ with $\phi(x)=c$ and $||x||=\rho$.

Appendix B

Auxiliary Results

We will now enunciate some lemmas that have helped us in the results tests presented in the thesis.

In Chapter 4, we introduced some functionals that was useful in ours proofs. In this appendix we present and prove their properties. We also enunciate a Lemma that guarantee when the test functions in $C_0^\infty(\Omega)$ can be change for test functions in $W_0^{1,p(x)}(\Omega)$ in problem (4.1).

Lemma B.0.1 *Assume that* (f_4) *holds true. If* $u \in W_0^{1,p(x)}(\Omega)$ *be a solution of* (4.1) *in sense of Definition* 2.1.2*, then*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega a(x)u^{-\alpha(x)}v dx + \lambda \int_\Omega f(x,u)v dx, \quad \forall v \in W_0^{1,p(x)}(\Omega),$$

that is, u *is an weak solution of* (4.1).

Proof. The proof is inspired in an argument of Boccardo and Casado-Díaz [5]. Let v be an arbitrary function in $W_0^{1,p(x)}(\Omega)$ and take $(v_n) \subset C_0^\infty(\Omega)$ such that $v_n \to v$ in $W_0^{1,p(x)}(\Omega)$ and also pointwise almost everywhere. So, given $\epsilon > 0$, by taking $\sqrt[\theta]{\epsilon^\theta + |v_n - v_k|^\theta} - \epsilon$ as a test function, for some $\theta \in \mathbb{N}$, we obtain that

$$
\begin{aligned}
&\int_\Omega a(x)u^{-\alpha(x)}\left(\sqrt[\theta]{\epsilon^\theta + |v_n - v_k|^\theta} - \epsilon\right)dx \\
&\quad \leq \left(\int_\Omega |\nabla u|^{p(x)-1}\frac{|v_n - v_k|^{\theta-1}|\nabla(v_n - v_k)|}{(\epsilon^\theta + |v_n - v_k|^\theta)^{\frac{\theta-1}{\theta}}}dx + \lambda\int_\Omega |f(x,u)|\left(\sqrt[\theta]{\epsilon^\theta + |v_n - v_k|^\theta} - \epsilon\right)dx\right) \\
&\quad \leq \left(\int_\Omega |\nabla u|^{p(x)-1}\frac{|v_n - v_k|^{\theta-1}|\nabla(v_n - v_k)|}{(\epsilon^\theta + |v_n - v_k|^\theta)^{\frac{\theta-1}{\theta}}}dx + \lambda C_1\int_\Omega (1 + |u|^{q(x)-1})\left(\sqrt[\theta]{\epsilon^\theta + |v_n - v_k|^\theta} - \epsilon\right)dx\right) \\
&\quad \leq C_2\left(\left\||\nabla u|^{p(x)-1}\right\|_{\frac{p(x)}{p(x)-1}}\left\|\frac{|v_n - v_k|^{\theta-1}|\nabla(v_n - v_k)|}{(\epsilon^\theta + |v_n - v_k|^\theta)^{\frac{\theta-1}{\theta}}}\right\|_{p(x)}\right. \\
&\qquad\qquad \left. + \left\|1 + u^{q(x)-1}\right\|_{\frac{p^*(x)}{q(x)-1}}\left\|\sqrt[\theta]{\epsilon^\theta + |v_n - v_k|^\theta} - \epsilon\right\|_{\frac{p^*(x)}{p^*(x)-q(x)+1}}\right) \\
&\quad \leq C_3(||u||^{p_--1} + ||u||^{p_+-1} + ||u||^{q_--1} + ||u||^{q_+-1} + 1)||v_n - v_k||.
\end{aligned}
$$

Letting $\epsilon \to 0$ and using the Fatou's lemma, we derive that

$$
\left(\frac{a(x)v_n}{u^{\alpha(x)}}\right) \text{ is a Cauchy sequence in } L^1(\Omega), \tag{B.1}
$$

so that $a(x)u^{-\alpha(x)}v_n \to a(x)u^{-\alpha(x)}v$ in $L^1(\Omega)$ taking into account that $v_n(x) \to v(x)$ a.e. in Ω. Thus we can make $n \to \infty$ in the inequality

$$
\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v_n dx = \int_\Omega a(x)u^{-\alpha(x)}v_n dx + \lambda\int_\Omega f(x,u)v_n dx,
$$

in order to obtain

$$
\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx = \int_\Omega a(x)u^{-\alpha(x)}v dx + \lambda\int_\Omega f(x,u)v dx.
$$

■

Remark B.0.2 *If $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, then we can replace (f_4) for (f_1) in the statement of Lemma $B.0.1$ to obtain the same conclusion.*

Lemma B.0.3 *Assume (H_1), (H_2) and (f_1) holds. Then the functional $\overline{J}$ defined at (4.7) belongs to $C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, is coercive and weakly lower semicontinuous.*

Proof. We start showing that $\hat{J}$ has Gateaux derivative for each $u \in W_0^{1,p(x)}(\Omega)$. Let $u, v \in W_0^{1,p(x)}(\Omega)$ and $\epsilon > 0$ small. So, we have,

$$
\int_\Omega \frac{\overline{F}(x, u + \epsilon v) - \overline{F}(x,u)}{\epsilon} = \int_\Omega \left(\int_0^1 \overline{f}(x, u + s\epsilon v)ds\right)v dx. \tag{B.2}
$$

First we notice,

$$
\int_0^1 \overline{f}(x, u + s\epsilon v)ds \to \overline{f}(x,u) \text{ as } \epsilon \to 0 \text{ a.e } x \in \Omega. \tag{B.3}
$$

Besides this, it follows from (4.6) that,

$$\begin{aligned}\overline{f}(x,t) &\leq \left(a(x)\underline{u}^{-\alpha(x)}+\lambda f(x,\underline{u})\right)\mathcal{X}_{\{t<\underline{u}\}}(x) \\ &\quad+\left(a(x)t^{-\alpha(x)}+\lambda f(x,t)\right)\mathcal{X}_{\{\underline{u}\leq t\leq\overline{u}\}}(x)+\left(a(x)\overline{u}^{-\alpha(x)}+\lambda f(x,\overline{u})\right)\mathcal{X}_{\{t>\underline{u}\}}(x) \\ &\leq 2a(x)\left(\underline{u}^{-\alpha(x)}+\overline{u}^{-\alpha(x)}\right)+\lambda\left(f(x,\underline{u})+f(x,\overline{u})+\mathcal{X}_{\{\underline{u}\leq t\leq\overline{u}\}}(x)f(x,t)\right).\end{aligned} \tag{B.4}$$

In particular,

$$\begin{aligned}&\left(\int_0^1\overline{f}(x,u+s\epsilon v)ds\right)v=\left(\mathcal{X}_{\{\underline{u}\leq u+s\epsilon v\leq\overline{u}\}}(x)\int_0^1\left(a(x)(u+s\epsilon v)^{-\alpha(x)}+\lambda f(x,u+s\epsilon v)\right)ds\right)v \\ &+\mathcal{X}_{\{u+s\epsilon v<\underline{u}\}}(x)\left(a(x)\underline{u}^{-\alpha(x)}+\lambda f(x,\underline{u})\right)v+\mathcal{X}_{\{u+s\epsilon v>\overline{u}\}}(x)\left(a(x)\overline{u}^{-\alpha(x)}+\lambda f(x,\overline{u})\right)v \\ &\leq\left[2a(x)\left(\underline{u}^{-\alpha(x)}+\overline{u}^{-\alpha(x)}\right)+\lambda\left(f(x,\underline{u})+f(x,\overline{u})+\mathcal{X}_{\{\underline{u}\leq u+s\epsilon v\leq\overline{u}\}}(x)\int_0^1 f(x,u+s\epsilon v)ds\right)\right]|v|.\end{aligned}$$

By using the hypothesis (f_1), with $M=||\overline{u}||_\infty$, we have

$$\lambda\left(f(x,\underline{u})+f(x,\overline{u})+\mathcal{X}_{\{\underline{u}\leq u+s\epsilon v\leq\overline{u}\}}(x)\int_0^1 f(x,u+s\epsilon v)ds\right)|v|\leq 3\lambda c_1|v|. \tag{B.5}$$

for all $0<t<M$, that is, to apply Lebesgue's dominated convergence theorem just remains to show that

$$2a(\cdot)\left(\underline{u}^{-\alpha(\cdot)}+\overline{u}^{-\alpha(\cdot)}\right)|v|\in L^1(\Omega). \tag{B.6}$$

Since $\underline{u}\in W_0^{1,p(x)}(\Omega)$ is a solution for the problem (4.2), it follows by Lemma B.0.1, with $|v|$ as test function and Hölder's Inequality that

$$\begin{aligned}\int_\Omega a(x)\underline{u}^{-\alpha(x)}|v|dx &= \int_\Omega|\nabla\underline{u}|^{p(x)-2}\nabla\underline{u}\nabla|v|dx \\ &\leq 2\left(\left\||\nabla\underline{u}|^{p(x)-1}\right\|_{p'(x)}||v||\right)<\infty.\end{aligned}$$

In a analogue way we conclude that $a(x)\overline{u}^{-\alpha(x)}|v|\in L^1(\Omega)$. Then, by Lebesgue's dominated convergence theorem, the Gâteaux derivative $\overline{J}'(u)$ exists and is given by

$$\left\langle\overline{J}'(u),v\right\rangle=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx-\int_\Omega\overline{f}(x,u)vdx,\quad\text{for all } u,v\in W_0^{1,p(x)}(\Omega).$$

To show the continuity of $\hat{J}'$, let $w_k,w\in W_0^{1,p(x)}(\Omega)$ be such that $w_k\to w$ in $W_0^{1,p(x)}(\Omega)$. By using Hölder's Inequality, we obtain

$$\begin{aligned}&\left|\left\langle\overline{J}'(w_k)-\overline{J}'(w),v\right\rangle\right| \\ &=\int_\Omega(|\nabla w_k|^{p(x)-2}\nabla w_k-|\nabla w|^{p(x)-2}\nabla w)\nabla v dx-\int_\Omega(\overline{f}(x,w_k)-\overline{f}(x,w))vdx \\ &\leq\left\||\nabla w_k|^{p(x)-2}\nabla w_k-|\nabla w|^{p(x)-2}\nabla w\right\|_{\frac{p(x)}{p(x)-1}}||v||+\int_\Omega(\overline{f}(x,w_k)-\overline{f}(x,w))vdx.\end{aligned} \tag{B.7}$$

It follows from (4.6), that

$$\overline{f}(x,w_k)-\overline{f}(x,w)=\begin{cases}0 & \text{if } w_k\leq\underline{u}, \\ a(x)\left(w_k^{-\alpha(x)}-w^{-\alpha(x)}\right)-\lambda(f(x,w_k)-f(x,w)) & \text{if } \underline{u}<w_k<\overline{u}, \\ 0 & \text{if } w_k\geq\overline{u},\end{cases}$$

and, as a consequence of this and following similar arguments used to obtain $(B.5)$ and $(B.6)$, we can conclude

$$\int_\Omega (\overline{f}(x,w_k)-\overline{f}(x,w))vdx \leq C||v||.$$

So, it follow from above arguments, by using Lebesgue's dominated convergence theorem in $(B.7)$, that the Gâteaux derivative of $\overline{J}'$ is continuous.

For the coercivity, let $v \in W_0^{1,p(x)}(\Omega)$ with $||v||>1$. By using $(B.4)$, $(B.5)$ and $\underline{u},\overline{u}$ are solutions of 4.2 and 4.3, respectively, we obtain

$$\begin{aligned}
\overline{J}(v) &= \int_\Omega \frac{|\nabla v|^{p(x)}}{p(x)}dx - \int_\Omega \overline{F}(x,v)dx \\
&\geq \frac{||v||^{p_-}}{p_+} - C\int_\Omega (a(x)\underline{u}^{-\alpha(x)} + a(x)\overline{u}^{-\alpha(x)}+1)vdx \\
&\geq \frac{||v||^{p_-}}{p_+} - C\left(\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla\underline{u}\nabla v dx - \int_\Omega |\nabla \overline{u}|^{p(x)-2}\nabla\overline{u}\nabla v dx\right) \\
&\geq \frac{||v||^{p_-}}{p_+} - C\left(\left\|\nabla\underline{u}^{p(x)-1}\right\|_{\frac{p(x)}{p(x)-1}} - \left\|\nabla\overline{u}^{p(x)-1}\right\|_{\frac{p(x)}{p(x)-1}}\right)||v||,
\end{aligned}$$

showing that $\overline{J}$ is coercive, since $p_->1$.

To finish, we note that the weakly lower semi continuity follows from continuity and convexity of the map $s\mapsto |s|^{p(x)}$, $(B.5)$ and $(B.6)$. ■

Lemma B.0.4 *Assume $(H_1),(H_2)(ii),(f_2)$ and (f_3). If $\alpha(x)\geq 0$ and $\alpha(x)<1$ on $\partial\Omega$, then the functional*

$$\tilde{J}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \int_\Omega \tilde{F}(x,u)dx. \tag{B.8}$$

belongs to $C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$, is coercive and weakly lower semicontinuous, where

$$\tilde{f}(x,t) = \begin{cases} a(x)\underline{v}^{-\alpha(x)} + \lambda f(x,\underline{v}) & t\leq \underline{v}, \\ a(x)t^{-\alpha(x)}+\lambda f(x,t) & \underline{v}<t<\overline{v}, \\ a(x)\overline{v}^{-\alpha(x)}+\lambda f(x,\overline{v}) & t\geq\overline{v}. \end{cases} \tag{B.9}$$

Proof. We start showing that $\hat{J}$ has Gateaux derivative for each $u\in W_0^{1,p(x)}(\Omega)$. Let $u,v\in W_0^{1,p(x)}(\Omega)$ and $\epsilon>0$ small. So, we have,

$$\int_\Omega \frac{\tilde{F}(x,u+\epsilon v)-\tilde{F}(x,u)}{\epsilon} = \int_\Omega\left(\int_0^1 \tilde{f}(x,u+s\epsilon v)ds\right)vdx. \tag{B.10}$$

Again,

$$\int_0^1 \tilde{f}(x,u+s\epsilon v)ds \to \tilde{f}(x,u) \text{ as } \epsilon\to 0 \text{ a.e } x\in\Omega, \tag{B.11}$$

To use Lebesgue's dominated convergence theorem we need to show that

$$\left(\int_0^1 \tilde{f}(x,u+s\epsilon v)ds\right)|v|\in L^1(\Omega).$$

Initially, by using (B.9), we obtain

$$\left(\int_0^1 \tilde{f}(x,u+s\epsilon v)ds\right)|v| \leq 2a(x)\left(\underline{v}^{-\alpha(x)}+\overline{u}^{-\alpha(x)}\right)|v|$$
$$+\lambda\left(f(x,\underline{v})+f(x,\overline{v})+\mathcal{X}_{\{\underline{v}\leq u+s\epsilon v\leq\overline{u}\}}(x)\int_0^1 f(x,u+s\epsilon v)ds\right)|v|.$$

We only need to estimate the singular term $a(x)\underline{v}^{-\alpha(x)}$, because the estimate of other terms follows from (B.5) and (B.6). To this end, by following the same arguments used in proof of Lemma 3.2.5 we conclude that $\underline{v}(x)\geq Cd(x)$ for all $x\in\Omega$, for some $C>0$. More, from continuity of $\alpha(x)$, there exits $0<\delta_1<\delta$ such that $\alpha(x)<1$ in Ω_{δ_1}, with δ as in hypothesis (H_2). By using this and $(H_2)(ii)$ we have

$$\int_\Omega a(x)\underline{v}^{-\alpha(x)}vdx \leq \left\|C^{-\alpha(x)}\right\|_\infty \int_\Omega a(x)d(x)^{-\alpha(x)}vdx \tag{B.12}$$
$$\leq \left\|C^{-\alpha(x)}\right\|_\infty\left(||a||_{L^\infty(\Omega_{\delta_1})}\int_{\Omega_{\delta_1}} d(x)^{1-\alpha(x)}\frac{v}{d(x)}dx+\left\|\delta_1^{-\alpha(x)}\right\|_\infty\int_{\Omega\setminus\Omega_{\delta_1}} a(x)vdx\right).$$

Now, from Hölder's inequality and Lemma 1.2.5, we obtain

$$\int_{\Omega_{\delta_1}} d(x)^{1-\alpha(x)}\frac{v}{d(x)}dx+\int_{\Omega\setminus\Omega_{\delta_1}} a(x)vdx$$
$$\leq C\left(\left\|\frac{v}{d(x)}\right\|_{L^{p(x)}(\Omega_{\delta_1})}\left\|d(x)^{1-\alpha(x)}\right\|_{L^{p'(x)}(\Omega_{\delta_1})}+||a||_{r(x)}||v||_{r'(x)}\right)$$
$$\leq C_1\left(||\nabla v|||_{L^{p(x)}(\Omega_{\delta_1})}+||v||_{r'(x)}\right)\leq C_2||v||. \tag{B.13}$$

From $(B.12)$ and $(B.13)$ we deduce that $a(x)\underline{v}^{-\alpha(x)}v\in L^1(\Omega)$.Then, by Lebesgue's dominated convergence theorem, the Gâteaux derivative $\tilde{J}'(u)$ exists and is given by

$$\left\langle \tilde{J}'(u),v\right\rangle=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla vdx-\int_\Omega\tilde{f}(x,u)vdx,\quad\text{for all } u,v\in W_0^{1,p(x)}(\Omega).$$

Now, by following the same arguments as done in the proof of Lemma B.0.3 we obtain the continuity of $\tilde{J}'$ and coercivity and weakly lower semi continuity of $\tilde{J}$. ■

Lemma B.0.5 *Assume* $(H_1)-(H_3)$ *and* (f_1) *holds. The functional* $\hat{J}$ *defined at* (4.10) *belongs to* $C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$ *and is weakly lower semicontinuous. The same result is valid if we consider the set of hypothesis* $(H_1),(H_2)(ii),(f_2)$, (f_3), $\alpha(x)<1$ *in* Ω *and* $\alpha(x)<1$ *on* $\partial\Omega$.

Proof. The proof is analogous to that made in Lemmas B.0.3 and B.0.4. ■

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