



Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

On the inverse mean curvature flow by parallel hypersurfaces in space forms

by

Alancoc dos Santos Alencar

Brasília

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Thesis presented to the Graduate Program of
the Department of Mathematics of the Uni-
versidade de Brasília as part of the requisites
to obtain the degree of Ph.D. in Mathemat-
ics.

Brasília

2023

¹The author had financial support from CAPES during the elaboration of this work.

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Alancoc dos Santos Alencar*

*Tese apresentada ao Departamento de Matemática da Universidade de
Brasília, como parte dos requisitos para obtenção do grau de*

DOUTOR EM MATEMÁTICA

Brasília, 15 de fevereiro de 2023.

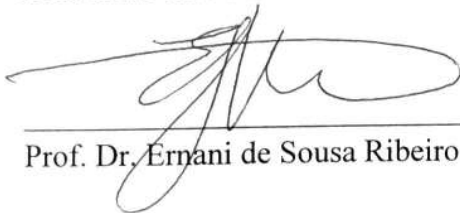
Comissão Examinadora:



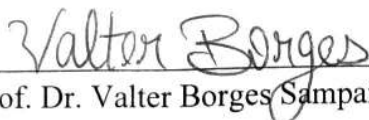
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* O autor foi bolsista da CAPES durante a elaboração desta tese.

Ficha catalográfica elaborada automaticamente,
com os dados fornecidos pelo(a) autor(a)

AAL368o Alencar, Alancoc dos Santos
On the inverse mean curvature flow by parallel
hypersurfaces in space forms / Alancoc dos Santos Alencar;
orientador Ketí Tenenblat. -- Brasília, 2023.
75 p.

Tese (Doutorado em Matemática) -- Universidade de
Brasília, 2023.

1. Space forms. 2. Parallel hypersurfaces. 3. Inverse
mean curvature flow. 4. Isoparametric hypersurfaces. I.
Tenenblat, Ketí, orient. II. Título.

Dedicatória

*Aos meus pais, Francisco e Raimunda.
Aos meus irmãos, Abraão e Ana Paula.
À minha namorada, Beatriz.*

Agradecimentos

“Se um vem a cair, o outro o levanta. Mas aí do homem só: se ele cair, não há ninguém para levantá-lo.”
(Eclesiastes 4, 10)

Em primeiro lugar, sou grato a Deus, na Pessoa de Nosso Senhor Jesus Cristo, por tudo. Passei por diversas dificuldades das quais achava que não ia conseguir dar conta de superar, mas Deus estava sempre lá para me erguer. Agradeço também pela intercessão constante de Nossa Senhora e de São José, sempre ouvindo minhas súplicas.

À minha família, que sempre me apoiou em todas as decisões que tomei ao longo da minha vida e que, apesar de estar a quase 2 mil quilômetros de distância de mim, consegue se fazer “perto”. Em especial, aos meus pais, que me acompanharam e sonharam comigo ao longo de toda a jornada.

À Beatriz, minha namorada. Nunca vou conseguir te agradecer e recompensar o suficiente por tudo o que fez por mim ao longo dos últimos anos. Obrigado, minha flor, por estar sempre lá, em todos os momentos.

A todos os meus amigos. Conheci diversas pessoas ao longo do percurso, fiz diversas amizades que foram fundamentais em minha vida. É inviável listar nomes aqui, eu acabaria por esquecer de alguém, o que não seria justo.

A todos os professores que fizeram parte de minha formação, desde a educação infantil até o doutorado. Todos vocês participaram dessa caminhada e contribuíram enormemente para que eu chegasse até aqui.

À minha orientadora, Professora Dra. Ketí Tenenblat, por toda a disposição em me ajudar, pela amizade, paciência, compreensão e pelas valiosas sugestões e ensinamentos. É uma honra ter sido seu aluno.

Aos membros da banca examinadora, por aceitarem prontamente o convite para avaliação deste trabalho e pelas valiosas sugestões.

À UnB, pela infraestrutura e recursos oferecidos para a realização deste trabalho.

À Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), pelo apoio financeiro.

Meus sinceros agradecimentos a todos.

“It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of light, it was the season of darkness, it was the spring of hope, it was the winter of despair.”

Charles Dickens, A Tale of Two Cities

Resumo

Provamos que uma hipersuperfície orientada em uma forma espacial, cuja curvatura média não se anula em nenhum ponto, é uma condição inicial para uma solução do fluxo de curvatura média inverso (FCMI) por hipersuperfícies paralelas se, e somente se, ela é isoparamétrica. Considerando as hipersuperfícies isoparamétricas nas formas espaciais, obtemos as soluções para o FCMI por hipersuperfícies paralelas explicitamente. Além disso, estudamos essas soluções em detalhes descrevendo seus comportamentos no intervalo maximal onde estão definidas. Para as hipersuperfícies isoparamétricas no espaço hiperbólico e na esfera, com duas ou quatro curvaturas principais distintas, consideramos a hipótese adicional de que essas curvaturas têm a mesma multiplicidade.

Palavras-chave: Formas espaciais; Hipersuperfícies paralelas; Fluxo de curvatura média inverso; Hipersuperfícies isoparamétricas.

Abstract

We prove that an oriented hypersurface in a space form, whose mean curvature does not vanish at any point, is an initial condition for a solution to the inverse mean curvature flow (IMCF) by parallel hypersurfaces if and only if it is isoparametric. Considering the isoparametric hypersurfaces in space forms, we obtain the solutions to the IMCF by parallel hypersurfaces explicitly. Moreover, we study these solutions describing their behavior in the maximal interval where they are defined. For the isoparametric hypersurfaces in the hyperbolic space and in the sphere, with two or four distinct principal curvatures, we consider the additional assumption that these curvatures have the same multiplicity.

Keywords: Space forms; Parallel hypersurfaces; Inverse mean curvature flow; Isoparametric hypersurfaces.

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Introduction

Let $\overline{M}^{n+1}$ be a Riemannian manifold of dimension $n + 1$ with metric $\overline{g}$. Consider $F : M^n \longrightarrow \overline{M}^{n+1}$ an oriented hypersurface with inward unit normal vector field N . A one parameter family of hypersurfaces $\hat{F} : M \times I \longrightarrow \overline{M}$, $0 \in I \subset \mathbb{R}$, is a solution to the *Inverse Mean Curvature Flow* (IMCF), with initial data $F(M)$, if

$$\begin{aligned} \frac{\partial \hat{F}}{\partial t}(x, t) &= -\frac{1}{\hat{H}(x, t)} \hat{N}(x, t) \text{ on } M \times I, \\ \hat{F}(x, 0) &= F(x) \text{ on } M, \end{aligned}$$

where $\hat{H}^t(\cdot) = \hat{H}(\cdot, t)$ is the mean curvature and $\hat{N}^t(\cdot) = \hat{N}(\cdot, t)$ is the inward unit normal vector field of $\hat{F}^t(\cdot) = \hat{F}(\cdot, t)$, with $\hat{H}^t(x) \neq 0$, $\forall t \in I$, $\forall x \in M$. We say that $\hat{F}$ is an *ancient solution* if $I = (-\infty, t^*)$, with $t^* > 0$. If $I = (t^*, +\infty)$, with $t^* < 0$, then $\hat{F}$ is called an *immortal solution*. Finally, $\hat{F}$ is an *eternal solution* if $I = \mathbb{R}$.

The inverse mean curvature flow has been studied extensively in recent decades. We start mentioning two important papers on the IMCF. The Penrose inequality was conjectured by the physicist Roger Penrose in 1973 and it relates, under some conditions, the mass of certain spacetimes to the area of their black holes. In 2001, using IMCF, Huisken and Ilmanen [19] proved the Riemannian Penrose Inequality. The semi-Riemannian case remains an open problem. On the other hand, in 2004, using IMCF, Bray and Neves [3] proved the Poincaré conjecture for 3-manifolds with Yamabe invariant greater than the Yamabe invariant of real projective 3-space. There are many papers that relate the IMCF with physical problems involving general relativity and black holes, see for example: [2], [22], [16] and [17]. Other papers that involve the study of the IMCF are the following: [5], [21], [13], [26], [15], [20], [10], [28], [9], [18] and [30].

In this thesis, motivated by the paper “*The mean curvature flow by parallel hypersurfaces*” by Reis and Tenenblat [12], we study the inverse mean curvature flow by parallel hypersurfaces in space forms. Suppose that $\overline{M}^{n+1}$ is a space form, i.e., $\overline{M}^{n+1}$ is the Euclidean space $\mathbb{E}^{n+1}$, the sphere $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ or the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$, where $\mathbb{L}^{n+2}$ is the Lorentz space of dimension $n + 2$. Consider $F : M^n \longrightarrow \overline{M}^{n+1}$ an oriented hypersurface with inward unit normal vector field N . Suppose that the mean curvature H of $F(M)$ satisfies $H(x) \neq 0$, $\forall x \in M$. Under these conditions, we prove that $F(M)$ is the initial condition of a solution to the IMCF by parallel hypersurfaces when it is

an isoparametric hypersurface. We characterize such solutions. Moreover, using the well known properties of such hypersurfaces in space forms, we study in detail the solutions to the IMCF for each isoparametric hypersurface of a space form, describing their behavior on the maximal interval where they are defined. We consider the additional assumption that the principal curvatures have the same multiplicity, whenever the number of distinct principal curvatures is 2 or 4 for hypersurfaces of $\mathbb{H}^{n+1}$ or $\mathbb{S}^{n+1}$.

We will now describe the structure of this thesis. Chapter 1 is divided into four sections, where we briefly summarize the basic concepts focusing on those that are most relevant to the next chapters. In addition, we establish the notation that is used throughout the text. In the first section we consider the space forms, which are the ambient spaces throughout the work. In the second, we comment on the isoparametric hypersurfaces in these spaces. Then, in the third section, we define parallel hypersurfaces and describe their properties. In the last section of the chapter, we introduce the IMCF by parallel hypersurfaces in space forms.

In chapter 2, we prove that $F(M)$ is an initial data of a solution to the IMCF by parallel hypersurfaces if, and only if, $F(M)$ is an isoparametric hypersurface (see Theorem 2.1.1). As a consequence, we show that the solutions to the IMCF by parallel hypersurfaces are characterized by an algebraic equation (see (2.7) in Corollary 2.1.2), which is used in the following chapters.

In chapter 3, we study the solutions to the IMCF by parallel hypersurfaces in $\mathbb{E}^{n+1}$ considering as initial condition the isoparametric hypersurfaces of the Euclidean space $\mathbb{E}^{n+1}$. We prove that, in the case of the sphere $\mathbb{S}_{R_0}^n \subset \mathbb{E}^{n+1}$ with radius $R_0 > 0$, the solution is eternal, it expands from a point and $\forall t \in \mathbb{R}$ it is a sphere with radius $R(t) = R_0 e^{t/n}$ (see Theorem 3.1.1). In the case of the cylinder $\mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}$, $m \notin \{0, n\}$, we show that the solution is eternal, it expands from an $(n - m)$ -dimensional plane of $\mathbb{E}^{n+1}$ and $\forall t \in \mathbb{R}$ it is a cylinder whose base has radius $R(t) = R_0 e^{t/m}$ (see Theorem 3.2.1).

In chapter 4, we study the solutions to the IMCF by parallel hypersurfaces in $\mathbb{H}^{n+1}$ considering as initial condition the isoparametric hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$. In the first section, we prove a lemma (see Lemma 4.1.1) that establishes sufficient conditions for the convergence of these solutions to the boundary of $\mathbb{H}^{n+1}$ ($\partial\mathbb{H}^{n+1}$). In the second section, we study the case of the horosphere. We prove that the solution is eternal and remains a horosphere for all $t \in \mathbb{R}$. Furthermore, we show that the solution expands from a point and converges to $\partial\mathbb{H}^{n+1}$ (see Theorem 4.2.1). In the third section, we study the totally umbilical case, where all the principal curvatures are equal to a constant k with $k \notin \{-1, 0, 1\}$. If $0 < |k| < 1$, we prove that the solution is immortal, it expands from a totally geodesic hypersurface of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$. If $|k| > 1$, we prove that the solution is eternal, it expands from a point and converges to $\partial\mathbb{H}^{n+1}$ (see Theorem 4.3.1). Finally, in the fourth section, we study the case of the hyperbolic cylinder ($\mathbb{S}^m \times \mathbb{H}^m$, $n = 2m$), with the additional assumption that the two

distinct principal curvatures have the same multiplicity m . We prove that the solution is eternal, it expands from an m -dimensional submanifold of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$ (see Theorem 4.4.1).

In chapter 5, we study the solutions to the IMCF by parallel hypersurfaces in $\mathbb{S}^{n+1}$ considering as initial condition the isoparametric hypersurfaces of the sphere $\mathbb{S}^{n+1}$. Münzner [25] showed that the number $\mathcal{G}$ of distinct principal curvatures, for an isoparametric hypersurface of $\mathbb{S}^{n+1}$, is restricted to be 1, 2, 3, 4 or 6. When $\mathcal{G} \in \{2, 4\}$, the multiplicities of principal curvatures can be different. For these cases, in studying the solutions to the IMCF by parallel hypersurfaces in $\mathbb{S}^{n+1}$, we additionally assume that the multiplicities of the principal curvatures of the initial hypersurface are the same. Under these conditions, we prove that, for all values of $\mathcal{G}$, the solutions to the IMCF by parallel hypersurfaces in $\mathbb{S}^{n+1}$ are ancient. When $\mathcal{G} = 1$, we prove that the solution expands from a point and converges to a totally geodesic hypersurface of $\mathbb{S}^{n+1}$ (see Theorem 5.1.1). For the cases where $\mathcal{G} \in \{2, 3, 4, 6\}$, if m is the multiplicity of principal curvatures of the initial hypersurface, we prove that the solutions expands from a $(\mathcal{G} - 1)m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ (see Theorems 5.2.1, 5.3.1, 5.4.1, 5.5.1).

Chapter 1

Preliminaries

In this chapter, throughout its four sections, we will fix the notation and establish the preliminaries for the text, presenting the definitions and results that will be used throughout the next chapters.

1.1 Space forms

The m -dimensional real space $\mathbb{R}^m$ equipped with the usual metric

$$\langle u, v \rangle_{\mathbb{R}^m} = \sum_{i=1}^m u_i v_i,$$

where $u = (u_1, u_2, \dots, u_m)$ and $v = (v_1, v_2, \dots, v_m) \in \mathbb{R}^m$, will be denoted by $\mathbb{E}^m$.

The m -dimensional real space $\mathbb{R}^m$ equipped with the metric

$$\langle u, v \rangle_{\mathbb{L}^m} = -u_1 v_1 + \sum_{i=2}^m u_i v_i,$$

where $u = (u_1, u_2, \dots, u_m)$ and $v = (v_1, v_2, \dots, v_m) \in \mathbb{R}^m$, is the *Lorentzian space*, that will be denoted by $\mathbb{L}^m$.

The *sphere*

$$\mathbb{S}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{E}^{n+1} : x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$$

and the *hyperbolic space*

$$\mathbb{H}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{L}^{n+1} : -x_1^2 + x_2^2 + \dots + x_{n+1}^2 = -1, x_1 > 0\}$$

will be considered submanifolds of $\mathbb{E}^{n+1}$ and $\mathbb{L}^{n+1}$, respectively, with the induced metric.

In what follows, we will denote by

$$\mathbb{Q}_\varepsilon^n = \begin{cases} \mathbb{S}^n, & \text{if } \varepsilon = 1, \\ \mathbb{E}^n, & \text{if } \varepsilon = 0, \\ \mathbb{H}^n, & \text{if } \varepsilon = -1, \end{cases}$$

and $\langle, \rangle$ as the metric of $\mathbb{Q}_\varepsilon^n$.

Another model of the hyperbolic space is the *Poincaré ball model* $\mathbb{B}^n$. Let us denote by B the ball with center at the origin and radius 1 in Euclidean space $\mathbb{E}^n$, i.e.,

$$B = \{(x_1, x_2, \dots, x_n) \in \mathbb{E}^n : x_1^2 + x_2^2 + \dots + x_n^2 < 1\}.$$

Let $x = (x_1, x_2, \dots, x_n) \in B$ and $u = (u_1, u_2, \dots, u_n), v = (v_1, v_2, \dots, v_n) \in T_x B$. Define, over B , the metric given by

$$\langle u, v \rangle_{x,B} = \frac{4}{[1 - (x_1^2 + x_2^2 + \dots + x_n^2)]^2} (u_1 v_1 + u_2 v_2 + \dots + u_n v_n).$$

Then, $\mathbb{B}^n := (B, \langle, \rangle_B)$ is a Riemannian manifold with constant sectional curvature -1 . Note that the boundary $\partial \mathbb{B}^n$ of $\mathbb{B}^n$ is given by

$$\partial \mathbb{B}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{E}^n : x_1^2 + x_2^2 + \dots + x_n^2 = 1\}.$$

Moreover, the application $G : \mathbb{H}^n \subset \mathbb{L}^{n+1} \longrightarrow \mathbb{B}^n \subset \mathbb{E}^n$ given by

$$G(x_1, x_2, \dots, x_{n+1}) = \frac{1}{x_1 + 1} (x_2, \dots, x_n, -x_{n+1}) \quad (1.1)$$

is an isometry.

We also have the hyperbolic space model called the *upper half-space model*. Consider the half-space of $\mathbb{R}^n$ given by

$$\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_n > 0\}.$$

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ and $u = (u_1, u_2, \dots, u_n), v = (v_1, v_2, \dots, v_n) \in T_x \mathbb{R}_+^n$. Define, over $\mathbb{R}_+^n$, the metric given by

$$\langle u, v \rangle_{x, \mathbb{R}_+^n} = \frac{1}{x_n^2} (u_1 v_1 + u_2 v_2 + \dots + u_n v_n).$$

Then, $(\mathbb{R}_+^n, \langle, \rangle_{\mathbb{R}_+^n})$ is also a Riemannian manifold with constant sectional curvature -1 .

Moreover, the application $L : \mathbb{R}_+^3 \longrightarrow \mathbb{H}^3$ given by

$$L(y_1, y_2, y_3) = \frac{1}{2y_3} \left(1 + \sum_{i=1}^3 y_i^2, 2y_1, 2y_2, 1 - \sum_{i=1}^3 y_i^2 \right),$$

is an isometry.

1.2 Isoparametric hypersurfaces in space forms

Consider $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ a hypersurface with the induced metric g given by

$$g_x(v, w) = \langle dF_x(v), dF_x(w) \rangle,$$

$\forall x \in M$ and $\forall v, w \in T_x M$. The quadratic form defined by g is called the *first fundamental form* of the hypersurface $F(M)$ in $x \in M$.

In order to simplify the notation, we will identify M with $F(M)$ and $v \in T_x M$ with $dF_x(v) \in T_{F(x)} F(M)$, where $x \in M$. If M is oriented and N is a unit normal vector field, the *second fundamental form* of M is given by

$$II_x(v, w) = -\langle dN_x(v), w \rangle,$$

$\forall v, w \in T_x M, x \in M$.

Let $E_1, E_2, \dots, E_n$ be orthonormal vector fields which are *principal directions* and let $k_1, k_2, \dots, k_n$ be the *principal curvatures* of M , i.e.,

$$g_x(E_i, E_j) = \delta_{ij} \text{ and } II_x(E_i, E_j) = k_i(x) \delta_{ij},$$

$x \in M, 1 \leq i \leq n$ and $1 \leq j \leq n$. We will denote the *mean curvature* by

$$H(x) = \sum_{i=1}^n k_i(x),$$

$x \in M$.

When the principal curvatures k_i of M are constant, for all $i = 1, \dots, n$, we say that M is an *isoparametric hypersurface*. We have the following classification results for isoparametric hypersurfaces in $\mathbb{Q}_\varepsilon^n$.

Theorem 1.2.1 (Segre [27]). *Let $F : M^n \longrightarrow \mathbb{E}^{n+1}$ be an isoparametric hypersurface in $\mathbb{E}^{n+1}$. Then, M is isometric to one of the following hypersurfaces:*

1. $\mathbb{E}^n$, with all the principal curvatures equal to $k = 0$;

2. $\mathbb{S}^n$, with all the principal curvatures equal to $k \neq 0$;

3. $\mathbb{S}^m \times \mathbb{E}^{n-m}$, with m principal curvatures equal to $k_1 \neq 0$ and $n - m$ principal curvatures equal to $k_2 = 0$.

Theorem 1.2.2 (Cartan [4]). *Let $F : M^n \longrightarrow \mathbb{H}^{n+1}$ be an isoparametric hypersurface in $\mathbb{H}^{n+1}$. Then, M is isometric to one of the following hypersurfaces:*

1. $\mathbb{H}^n$, with all the principal curvatures equal to $k = 0$;

2. $\mathbb{H}^n$, with all the principal curvatures equal to k , where $0 < |k| < 1$;

3. $\mathbb{E}^n$, with all the principal curvatures equal to $k = \pm 1$;

4. $\mathbb{S}^n$, with all the principal curvatures equal to k , where $|k| > 1$;

5. $\mathbb{S}^m \times \mathbb{H}^{n-m}$, with m principal curvatures equal to k_1 and $n - m$ principal curvatures equal to k_2 , where $k_1 k_2 = 1$.

We will now consider the isoparametric hypersurfaces of the sphere. Münzner [25] showed that the number $\mathcal{G}$ of distinct principal curvatures, for an isoparametric hypersurface $F : M^n \longrightarrow \mathbb{S}^{n+1}$, is restricted to be 1, 2, 3, 4 or 6. Cartan [4] classified these hypersurfaces when $\mathcal{G} \leq 3$:

1. If $\mathcal{G} = 1$, then $F(M)$ is a sphere obtained as the intersection of $\mathbb{S}^{n+1}$ with a hyperplane of $\mathbb{E}^{n+2}$;

2. If $\mathcal{G} = 2$, then $F(M)$ must be the standard product of spheres $\mathbb{S}_{r_1}^l \times \mathbb{S}_{r_2}^{n-l} \subset \mathbb{S}^{n+1}$, where $r_1^2 + r_2^2 = 1$;

3. If $\mathcal{G} = 3$, Cartan proved that there are only four distinct isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ with three distinct principal curvatures. Their dimensions are $n = 3m$, where $m = 1, 2, 4$ or 8 and all the principal curvatures have the same multiplicity m ;

4. If $\mathcal{G} = 4$, Münzner [25] (see also Cecil-Chi-Jensen [6] and Chi [8]) proved that the principal curvatures k_1, k_2, k_3, k_4 can be ordered so that their corresponding multiplicities satisfy $m_1 = m_3$ and $m_2 = m_4$;

6. If $\mathcal{G} = 6$, Münzner [25] (see also Dorfmeister-Neher [11], Miyaoka [23] and [24]) showed that all the principal curvatures must have the same multiplicity m and Abresch [1] showed that $m = 1$ or $m = 2$.

1.3 Parallel hypersurfaces in space forms

We will start this section with the definition of parallel hypersurfaces in space forms.

Definition 1.3.1. *Let $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ be an oriented hypersurface with unit normal vector field N . The parallel hypersurface to $F(M)$ at a distance $\mu \in \mathbb{R}$ (with sign) is the map $\hat{F}^\mu : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ such that, for each point $x \in M^n$, the point $\hat{F}^\mu(x)$ is obtained by translating the point $F(x)$ a distance μ along the geodesic in $\mathbb{Q}_\varepsilon^{n+1}$ starting from $F(x)$ with tangent vector $N(F(x))$ at that point.*

Consider the following notation

$$C_\varepsilon(\mu) = \begin{cases} \cos \mu, & \text{if } \varepsilon = 1, \\ 1, & \text{if } \varepsilon = 0, \\ \cosh \mu, & \text{if } \varepsilon = -1, \end{cases} \quad \text{and } S_\varepsilon(\mu) = \begin{cases} \sin \mu, & \text{if } \varepsilon = 1, \\ \mu, & \text{if } \varepsilon = 0, \\ \sinh \mu, & \text{if } \varepsilon = -1. \end{cases} \quad (1.2)$$

We can write the parallel hypersurface $\hat{F}^\mu(M^n)$ in $\mathbb{Q}_\varepsilon^{n+1}$ as follows

$$\hat{F}^\mu(x) = C_\varepsilon(\mu)F(x) + S_\varepsilon(\mu)N(F(x)), \quad (1.3)$$

where $\mu \in \mathbb{R}$ and $x \in M^n$.

The functions C_ε and S_ε satisfy that:

- (i) $C'_\varepsilon(\mu) = -\varepsilon S_\varepsilon(\mu)$;
- (ii) $S'_\varepsilon(\mu) = C_\varepsilon(\mu)$;
- (iii) $C_\varepsilon^2(\mu) + \varepsilon S_\varepsilon^2(\mu) = 1$.

Now, we will see some basic facts about the parallel hypersurface $\hat{F}^\mu(M^n)$.

Lemma 1.3.2. *We have that:*

- (a) *The unit normal vector field $\hat{N}^\mu$ of $\hat{F}^\mu(M)$ is given by*

$$\hat{N}^\mu(x) = -\varepsilon S_\varepsilon(\mu)F(x) + C_\varepsilon(\mu)N(F(x)).$$

- (b) *If $E_1, E_2, \dots, E_n \subset T_x M$ is an orthonormal basis of eigenvectors of the second fundamental form II_x of $F(M)$ in $x \in M$ and $k_1, k_2, \dots, k_n$ are the principal curvatures of $F(M)$ in $x \in M$ with respect to N , then*

$$\hat{g}_x^\mu(E_i, E_j) = [C_\varepsilon(\mu) - k_i(x)S_\varepsilon(\mu)]^2 \delta_{ij}$$

and the second fundamental form $\hat{I}_x^\mu$ of $\hat{F}^\mu$ in $x \in M$ is given by

$$\hat{I}_x^\mu(E_i, E_j) = [C_\varepsilon(\mu) - k_i(x)S_\varepsilon(\mu)] [\varepsilon S_\varepsilon(\mu) + k_i(x)C_\varepsilon(\mu)] \delta_{ij}.$$

Moreover, the principal curvatures $\hat{k}_i^\mu(x)$, $i = 1, 2, \dots, n$, of $\hat{F}^\mu$ in $x \in M$, with respect to $\hat{N}^\mu$, are given by

$$\hat{k}_i^\mu(x) = \frac{\varepsilon S_\varepsilon(\mu) + k_i(x)C_\varepsilon(\mu)}{C_\varepsilon(\mu) - k_i(x)S_\varepsilon(\mu)}.$$

Lemma 1.3.3 (Cartan [4]). *Let $\hat{F}^\mu : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$, $\mu \in (-\delta, \delta)$ with $\delta > 0$, a family of parallel hypersurfaces. Then, $F(M)$ is isoparametric if and only if the mean curvature $\hat{H}^\mu(x)$ of $\hat{F}^\mu(M)$ does not depend on $x \in M$, $\forall \mu \in (-\delta, \delta)$.*

1.4 The inverse mean curvature flow in space forms

We will start this section with the definition of inverse mean curvature flow in a space form.

Definition 1.4.1. *Let $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ be an oriented hypersurface with inward unit normal vector field N . A one parameter family of hypersurfaces $\hat{F} : M^n \times I \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$, $0 \in I \subset \mathbb{R}$, is a solution to the Inverse Mean Curvature Flow (IMCF), with initial data $F(M)$, if*

$$\begin{aligned} \frac{\partial \hat{F}}{\partial t}(x, t) &= -\frac{1}{\hat{H}(x, t)} \hat{N}(x, t) \text{ on } M \times I, \\ \hat{F}(x, 0) &= F(x) \text{ on } M, \end{aligned}$$

where $\hat{H}^t(\cdot) = \hat{H}(\cdot, t) = \sum_{i=1}^n \hat{k}_i^t(\cdot)$ is the mean curvature and $\hat{N}^t(\cdot) = \hat{N}(\cdot, t)$ is the inward unit normal vector field of $\hat{F}^t(\cdot) = \hat{F}(\cdot, t)$, with $\hat{H}^t(x) \neq 0$, $\forall t \in I$, $\forall x \in M$.

For solutions defined on particular intervals I , we have the following definition.

Definition 1.4.2. *Let $\hat{F} : M^n \times I \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$, $0 \in I \subset \mathbb{R}$, be a one parameter family of hypersurfaces that is a solution to the IMCF, with initial data $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$. We say that $\hat{F}$ is an*

1. *Ancient Solution*, if $I = (-\infty, t^*)$, with $t^* > 0$;
2. *Immortal Solution*, if $I = (t^*, +\infty)$, with $t^* < 0$;
3. *Eternal Solution*, if $I = \mathbb{R}$.

In the following example, we will provide a solution to the IMCF in the case where the initial hypersurface is a Euclidean sphere of radius $R_0 > 0$ and we will study the behavior of this solution over the interval where it is defined.

Example 1.4.3. Consider $F : \mathbb{S}_{R_0}^n \longrightarrow \mathbb{E}^{n+1}$ the immersion of the sphere centered at the origin with radius R_0 , in the Euclidean space $\mathbb{E}^{n+1}$. A unit normal vector field N of $\mathbb{S}_{R_0}^n$ is given by $N(x) = -\frac{1}{R_0}F(x)$ and the principal curvatures associated to N are given by $k_i = k = \frac{1}{R_0}$, $\forall i = 1, 2, \dots, n$.

Define the family $\hat{F} : \mathbb{S}_{R_0}^n \times I \longrightarrow \mathbb{E}^{n+1}$ homothetic to F by

$$\hat{F}^t(x) = \hat{F}(x, t) = \frac{R(t)}{R_0}F(x),$$

$\forall x \in \mathbb{S}_{R_0}^n$ and $\forall t \in I$, where $R : I \longrightarrow \mathbb{R}$ is a positive differentiable function such that $R(0) = R_0$. Moreover, a unit normal vector field $\hat{N}^t$ of $\hat{F}^t(\mathbb{S}_{R_0}^n)$ is given by

$$\hat{N}^t(x) = \hat{N}(x, t) = -\frac{1}{R(t)}F(x)$$

and the principal curvatures associated to $\hat{N}^t$ are given by $\hat{k}_i^t(x) = \frac{1}{R(t)}$, $\forall x \in \mathbb{S}_{R_0}^n$, $\forall t \in I$ and $\forall i = 1, 2, \dots, n$. Then, the mean curvature $\hat{H}^t$ of $\hat{F}^t(\mathbb{S}_{R_0}^n)$ is

$$\hat{H}^t(x) = \hat{H}(x, t) = \frac{n}{R(t)}.$$

Therefore, if $\hat{F}^t$ is a solution to the IMCF with initial data $F(\mathbb{S}_{R_0}^n)$, we have that

$$\begin{aligned} \frac{R'(t)}{R_0}F(x) &= \left[-\frac{R(t)}{n} \right] \left[-\frac{1}{R_0} \right] F(x) \text{ on } \mathbb{S}^n \times I, \\ \frac{R(0)}{R_0}F(x) &= F(x) \text{ on } \mathbb{S}^n. \end{aligned}$$

It follows that

$$\frac{R'(t)}{R(t)} = \frac{1}{n} \text{ on } I,$$

where $R(0) = R_0$. Thus,

$$\ln(R(t)) = \frac{t}{n} + c \text{ on } I = \mathbb{R},$$

with $R(0) = R_0$. Hence,

$$R(t) = e^c e^{t/n} \text{ on } \mathbb{R} \text{ and } R(0) = R_0.$$

So, we conclude that

$$R(t) = R_0 e^{t/n} \text{ on } \mathbb{R}.$$

Therefore,

$$\hat{F}^t(x) = e^{t/n} F(x)$$

and, then, $\hat{F}^t$ is an eternal solution that expands from a point to a sphere centered at the origin with radius $R(t) = R_0 e^{t/n}$, $\forall t \in \mathbb{R}$.

We will now define a special solution to the IMCF: the solution given by a family of parallel hypersurfaces in space forms. We will focus on this type of solution from now on.

Definition 1.4.4. Let $\hat{F} : M^n \times I \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ be a solution to the IMCF in $\mathbb{Q}_\varepsilon^{n+1}$ with initial data $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$. We say that $\hat{F}$ is a solution to the IMCF by parallel hypersurfaces if there is a function $\mu : I \longrightarrow \mathbb{R}$ such that

$$\hat{F}^t(x) = C_\varepsilon(\mu(t))F(x) + S_\varepsilon(\mu(t))N(x), \quad (1.4)$$

$\forall t \in I$, where C_ε and S_ε are the functions defined in (1.2).

The following remark will be very important for the next chapter.

Remark 1.4.5 (Sotomayor [29]). Let $f : (a_1, a_2) \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real function. Suppose that f is continuous and it does not vanish on (a_1, a_2) . Consider the following problem

$$\mu'(t) = f(\mu(t)) \text{ and } \mu(t_0) = \mu_0, \quad (1.5)$$

where $\mu_0 \in (a_1, a_2)$ and $t_0 \in I \subset \mathbb{R}$. Also, consider the function $\psi : (a_1, a_2) \subset \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\psi(\mu) = \int_{\mu_0}^{\mu} \frac{d\xi}{f(\xi)}, \quad \forall \mu \in (a_1, a_2).$$

Notice that, since f is continuous and does not vanish on (a_1, a_2) , the function ψ is well defined and differentiable by the fundamental theorem of calculus. Moreover,

$$\psi'(\mu) = \frac{1}{f(\mu)}, \quad \forall \mu \in (a_1, a_2). \quad (1.6)$$

Then, by the inverse function theorem, ψ is a diffeomorphism. Again, since f is continuous and $f(\mu) \neq 0$, $\forall \mu \in (a_1, a_2)$, we can assume that $f(\mu) > 0$, $\forall \mu \in (a_1, a_2)$. So, from (1.6), we have that $\psi'(\mu) > 0$, $\forall \mu \in (a_1, a_2)$, i.e., ψ is strictly increasing on (a_1, a_2) . It follows that ψ is injective and then, $\psi : (a_1, a_2) \subset \mathbb{R} \longrightarrow \psi(a_1, a_2) \subset \mathbb{R}$ is bijective. We conclude that $\psi : (a_1, a_2) \longrightarrow \psi(a_1, a_2)$ is a global diffeomorphism. Consider $(b_1, b_2) = \psi(a_1, a_2)$.

If φ is a solution of (1.5), then

$$\varphi'(t) = f(\varphi(t)) \text{ and } \varphi(t_0) = \mu_0. \quad (1.7)$$

From (1.6) and (1.7), we have that

$$1 = \frac{\varphi'(t)}{f(\varphi(t))} = \psi'(\varphi(t))\varphi'(t) = \frac{d}{dt}(\psi \circ \varphi)(t).$$

Hence,

$$\int_{t_0}^t \frac{d}{ds}(\psi \circ \varphi)(s)ds = \int_{t_0}^t ds,$$

i.e.,

$$(\psi \circ \varphi)(t) - (\psi \circ \varphi)(t_0) = t - t_0.$$

Since

$$(\psi \circ \varphi)(t_0) = \psi(\varphi(t_0)) = \psi(\mu_0) = 0,$$

we conclude that

$$\psi(\varphi(t)) = t - t_0.$$

Therefore, the solution of (1.5) is given by

$$\varphi(t) = \psi^{-1}(t - t_0),$$

where $t - t_0 \in (b_1, b_2)$, i.e., $t \in (b_1 + t_0, b_2 + t_0)$.

Now, we will prove the uniqueness of the solution. Consider μ_1 and μ_2 solutions of (1.5). Then,

$$\psi(\mu_1(t)) = t - t_0 = \psi(\mu_2(t)).$$

Since ψ is injective, we conclude that $\mu_1 = \mu_2$.

Chapter 2

First results

In this chapter we prove two results. The first one, Theorem 2.1.1, gives the necessary foundation for the development of the rest of the work. It shows that an oriented hypersurface in space forms, whose mean curvature does not vanish at any point, is an initial condition for a solution to the IMCF by parallel hypersurfaces if and only if it is isoparametric. The second result, which is a consequence of Theorem 2.1.1, makes it possible to explicitly write the solutions in the case where the principal curvatures of the initial hypersurface have the same multiplicity, as we will see in the following chapters.

Theorem 2.1.1. *Let $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ be an oriented hypersurface with inward unit normal vector field N . Suppose the mean curvature H of $F(M)$ satisfies $H(x) \neq 0$, $\forall x \in M$. Then, $F(M)$ is an initial data of a solution to the IMCF by parallel hypersurfaces if and only if $F(M)$ is an isoparametric hypersurface.*

Proof. Consider a one parameter family of parallel hypersurfaces $\hat{F} : M^n \times I \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$, $0 \in I \subset \mathbb{R}$, given by (1.4), i.e.,

$$\hat{F}^t(x) = C_\varepsilon(\mu(t))F(x) + S_\varepsilon(\mu(t))N(x), \quad (2.1)$$

$\forall (x, t) \in M \times I$, where C_ε , S_ε are the functions defined in (1.2) and $\mu : I \longrightarrow \mathbb{R}$ is a smooth real function such that $\hat{F}$ is a solution to the IMCF with initial data $F(M)$. Then,

$$\hat{F}(x, 0) = F(x), \quad \forall x \in M,$$

and

$$\frac{\partial \hat{F}}{\partial t}(x, t) = -\frac{1}{\hat{H}(x, t)}\hat{N}(x, t), \quad \forall (x, t) \in M \times I, \quad (2.2)$$

where $\hat{H}(., t)$ is the mean curvature of $\hat{F}(., t)$ and $\hat{N}(., t)$ is the inward unit normal vector field of $\hat{F}(., t)$, with $\hat{H}(x, t) \neq 0$, $\forall t \in I$, $\forall x \in M$.

Since $\hat{F}(x, 0) = F(x)$, $\forall x \in M$, and N is a unit normal vector field, it follows from (1.3) that $\mu(0) = 0$. Now, substituting (2.1) into (2.2), we get, $\forall(x, t) \in M \times I$,

$$\frac{\partial}{\partial t} [C_\varepsilon(\mu(t))F(x) + S_\varepsilon(\mu(t))N(x)] = -\frac{1}{\hat{H}(x, t)}\hat{N}(x, t),$$

i.e., $\forall(x, t) \in M \times I$,

$$\mu'(t) [-\varepsilon S_\varepsilon(\mu(t))F(x) + C_\varepsilon(\mu(t))N(x)] = -\frac{1}{\hat{H}(x, t)}\hat{N}(x, t).$$

Hence, from Lemma 1.3.2, $\forall(x, t) \in M \times I$,

$$\mu'(t)\hat{N}(x, t) = -\frac{1}{\hat{H}(x, t)}\hat{N}(x, t).$$

Therefore, (2.2) occurs if and only if $\forall(x, t) \in M \times I$,

$$\mu'(t) = -\frac{1}{\hat{H}(x, t)}.$$

Thus, the mean curvature $\hat{H}^t$ of $\hat{F}^t(M)$ does not depend on $x \in M$, $\forall t \in I$. It follows from Lemma 1.3.3 that $F(M)$ is an isoparametric hypersurface.

Now, let $F : M^n \rightarrow \mathbb{Q}_\varepsilon^{n+1}$ be an oriented isoparametric hypersurface with inward unit normal vector field N . Let $\mu : I \rightarrow \mathbb{R}$, $0 \in I \subset \mathbb{R}$, be the smooth real function that is the unique solution of the problem:

$$\mu'(t) = -\frac{1}{\sum_{i=1}^n \frac{\varepsilon S_\varepsilon(\mu(t)) + k_i C_\varepsilon(\mu(t))}{C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))}} \quad \text{and} \quad \mu(0) = 0, \quad (2.3)$$

where k_i , $1 \leq i \leq n$, are the principal curvatures of $F(M)$ with respect to N and C_ε , S_ε are given by (1.2). Note that the function f given by

$$f(\mu) = -\frac{1}{\sum_{i=1}^n \frac{\varepsilon S_\varepsilon(\mu) + k_i C_\varepsilon(\mu)}{C_\varepsilon(\mu) - k_i S_\varepsilon(\mu)}}$$

satisfies

$$f(0) = -\frac{1}{H} \neq 0.$$

So, f is smooth and does not vanish in a neighborhood of zero. So, the existence and uniqueness of the solution of the problem (2.3) follows from Remark 1.4.5.

Consider the family of parallel hypersurfaces $\hat{F} : M^n \times I \rightarrow \mathbb{Q}_\varepsilon^{n+1}$ given by

$$\hat{F}^t(x) = C_\varepsilon(\mu(t))F(x) + S_\varepsilon(\mu(t))N(x), \quad \forall(x, t) \in M \times I.$$

Thus, it is enough to show that $\hat{F}^t$ is a solution to the IMCF by parallel hypersurfaces with initial data $F(M)$. Indeed, note that, $\forall x \in M$,

$$\begin{aligned}\hat{F}(x, 0) &= \hat{F}^0(x) \\ &= C_\varepsilon(\mu(0))F(x) + S_\varepsilon(\mu(0))N(x) \\ &= C_\varepsilon(0)F(x) + S_\varepsilon(0)N(x) \\ &= F(x).\end{aligned}$$

From Lemma 1.3.2 we get, $\forall (x, t) \in M \times I$,

$$\begin{aligned}\frac{\partial \hat{F}}{\partial t}(x, t) &= -\varepsilon \mu'(t) S_\varepsilon(\mu(t)) F(x) + \mu'(t) C_\varepsilon(\mu(t)) N(x) \\ &= \mu'(t) [-\varepsilon S_\varepsilon(\mu(t)) F(x) + C_\varepsilon(\mu(t)) N(x)] \\ &= \mu'(t) \hat{N}(x, t).\end{aligned}\tag{2.4}$$

Moreover, it follows from Lemma 1.3.2 and (2.3) that, $\forall (x, t) \in M \times I$,

$$\begin{aligned}\hat{H}(x, t) &= \sum_{i=1}^n \hat{k}_i(x, t) \\ &= \sum_{i=1}^n \hat{k}_i^t(x) \\ &= \sum_{i=1}^n \frac{\varepsilon S_\varepsilon(\mu(t)) + k_i(x) C_\varepsilon(\mu(t))}{C_\varepsilon(\mu(t)) - k_i(x) S_\varepsilon(\mu(t))} \\ &= \sum_{i=1}^n \frac{\varepsilon S_\varepsilon(\mu(t)) + k_i C_\varepsilon(\mu(t))}{C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))} \\ &= -\frac{1}{\mu'(t)}.\end{aligned}\tag{2.5}$$

Then, from (2.4) and (2.5), we have that, $\forall (x, t) \in M \times I$,

$$\frac{\partial \hat{F}}{\partial t}(x, t) = -\frac{1}{\hat{H}(x, t)} \hat{N}(x, t),$$

which concludes the proof of the theorem. ■

Corollary 2.1.2. *Let $F : M^n \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ be an oriented isoparametric hypersurface with inward unit normal vector field N and $k_1, k_2, \dots, k_n$ the principal curvatures associated to N . Suppose the mean curvature H of $F(M)$ satisfies $H(x) \neq 0, \forall x \in M$. Then, the solution to the IMCF by parallel hypersurfaces with initial data $F(M)$ is given by*

$$\hat{F}^t(x) = C_\varepsilon(\mu(t))F(x) + S_\varepsilon(\mu(t))N(x),\tag{2.6}$$

$\forall(x, t) \in M \times I$, where $0 \in I \subset \mathbb{R}$, C_ε and S_ε are the functions given by (1.2) and $\mu : I \longrightarrow \mathbb{R}$ is a smooth real function satisfying $\mu(0) = 0$ and

$$\prod_{i=1}^n |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))| = e^t, \quad (2.7)$$

$\forall t \in I$.

Proof. It follows from the proof of Theorem 2.1.1 that $\hat{F} : M^n \times I \longrightarrow \mathbb{Q}_\varepsilon^{n+1}$ is given by (2.6), $\mu : I \longrightarrow \mathbb{R}$ is a smooth real function satisfying $\mu(0) = 0$ and

$$\mu'(t) = -\frac{1}{\sum_{i=1}^n \frac{\varepsilon S_\varepsilon(\mu(t)) + k_i C_\varepsilon(\mu(t))}{C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))}},$$

$\forall t \in I$. Thus, $\forall t \in I$,

$$\sum_{i=1}^n \frac{\varepsilon S_\varepsilon(\mu(t)) + k_i C_\varepsilon(\mu(t))}{C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))} = -\frac{1}{\mu'(t)},$$

i.e.,

$$\sum_{i=1}^n \frac{1}{C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))} \cdot [-\varepsilon S_\varepsilon(\mu(t)) - k_i C_\varepsilon(\mu(t))] \mu'(t) = 1,$$

$\forall t \in I$. It follows that

$$\sum_{i=1}^n \frac{d}{dt} (\ln |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))|) = 1,$$

$\forall t \in I$, i.e.,

$$\frac{d}{dt} \sum_{i=1}^n (\ln |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))|) = 1.$$

Then, integrating on both sides, we get

$$\sum_{i=1}^n (\ln |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))|) = t + c,$$

$\forall t \in I$ and for some constant $c \in \mathbb{R}$. We conclude that

$$\exp \left(\sum_{i=1}^n (\ln |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))|) \right) = e^c e^t,$$

$\forall t \in I$. So, one sees that

$$\prod_{i=1}^n \exp(\ln |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))|) = e^c e^t.$$

Hence,

$$\prod_{i=1}^n |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))| = e^c e^t.$$

Moreover, since $\mu(0) = 0$, we have that

$$e^c = \prod_{i=1}^n |C_\varepsilon(0) - k_i S_\varepsilon(0)| = 1.$$

Therefore, $\forall t \in I$,

$$\prod_{i=1}^n |C_\varepsilon(\mu(t)) - k_i S_\varepsilon(\mu(t))| = e^t,$$

which proves (2.7). ■

Chapter 3

On the inverse mean curvature flow by parallel hypersurfaces in the Euclidean space

In this chapter we use Corollary 2.1.2 to study the solutions to the IMCF by parallel hypersurfaces in $\mathbb{E}^{n+1}$ considering as initial condition the isoparametric hypersurfaces of the Euclidean space $\mathbb{E}^{n+1}$. We show that, in the case of the sphere $\mathbb{S}_{R_0}^n \subset \mathbb{E}^{n+1}$ with radius $R_0 > 0$, the solution is eternal, expands from a point and $\forall t \in \mathbb{R}$ it is a sphere with radius $R(t) = R_0 e^{t/n}$, which agrees with the solution in Example 1.4.3.

In the case of the cylinder $\mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}$, $m \notin \{0, n\}$, we show that the solution is eternal, expands from an $(n - m)$ -dimensional plane of $\mathbb{E}^{n+1}$ and $\forall t \in \mathbb{R}$ it is a cylinder whose base has radius $R(t) = R_0 e^{t/m}$.

3.1 The sphere

Theorem 3.1.1. *Let $F : \mathbb{S}_{R_0}^n \longrightarrow \mathbb{E}^{n+1}$ be an isometric immersion of the sphere centered at the origin with radius $R_0 > 0$ in the Euclidean space $\mathbb{E}^{n+1}$ and let $N(x) = -\frac{1}{R_0}F(x)$ be the unit normal vector field of $F(\mathbb{S}_{R_0}^n)$, with all principal curvatures equal to $k = 1/R_0$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(\mathbb{S}_{R_0}^n) \subset \mathbb{E}^{n+1}$ is*

$$\hat{F}^t(x) = e^{t/n}F(x),$$

$\forall t \in \mathbb{R}$ and $\forall x \in \mathbb{S}_{R_0}^n$. Moreover, $\hat{F}^t(\mathbb{S}_{R_0}^n)$ is an eternal solution that expands from a point and $\forall t \in \mathbb{R}$ it is a sphere centered at the origin with radius $R(t) = R_0 e^{t/n}$.

Proof. From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = F(x) + \mu(t)N(x),$$

$\forall (x, t) \in \mathbb{S}_{R_0}^n \times I$, where, from (2.7), $\mu : I \longrightarrow \mathbb{R}$ satisfies

$$\prod_{i=1}^n |1 - k_i \mu(t)| = e^t.$$

Since all principal curvatures are equal to $k = 1/R_0$, we have that

$$\left| 1 - \frac{1}{R_0} \mu(t) \right|^n = e^t,$$

i.e.,

$$\left| 1 - \frac{1}{R_0} \mu(t) \right| = e^{t/n}, \quad (3.1)$$

$\forall t \in I$.

Since $e^{t/n}$ is a smooth positive function, from (3.1) we conclude that $h : I \longrightarrow \mathbb{R}$ given by

$$h(t) = 1 - \frac{1}{R_0} \mu(t)$$

does not change sign. Moreover, since $\mu(0) = 0$, we have that $h(0) = 1$. So, $h(t) > 0$, $\forall t \in I$. Then, (3.1) becomes

$$1 - \frac{1}{R_0} \mu(t) = e^{t/n},$$

$\forall t \in I$. Therefore,

$$\mu(t) = R_0(1 - e^{t/n}),$$

$\forall t \in \mathbb{R}$. Thus, one obtains that

$$\hat{F}^t(x) = F(x) + R_0(1 - e^{t/n})N(x), \quad (3.2)$$

$\forall t \in \mathbb{R}$ and $\forall x \in \mathbb{S}_{R_0}^n$. Since $N(x) = -\frac{1}{R_0}F(x)$, $\forall x \in \mathbb{S}_{R_0}^n$, from (3.2) we have

$$\begin{aligned} \hat{F}^t(x) &= F(x) + R_0(1 - e^{t/n}) \left(-\frac{1}{R_0} \right) F(x) \\ &= F(x) - (1 - e^{t/n})F(x) \\ &= e^{t/n}F(x), \end{aligned} \quad (3.3)$$

$\forall t \in \mathbb{R}$ and $\forall x \in \mathbb{S}_{R_0}^n$. Then,

$$\lim_{t \rightarrow -\infty} \hat{F}^t(x) = \lim_{t \rightarrow -\infty} e^{t/n}F(x) = 0,$$

i.e., $\hat{F}^t$ is an eternal solution that expands from a point and $\forall t \in \mathbb{R}$ it is a sphere centered at the origin with radius $R(t) = R_0 e^{t/n}$. $\blacksquare$

Example 3.1.2. Consider the parametrized sphere $F : (0, \pi) \times (0, 2\pi) \subset \mathbb{E}^2 \longrightarrow \mathbb{E}^3$, with

$$F(\theta, \varphi) = (\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta)).$$

From (3.3), we conclude that

$$\hat{F}^t(\theta, \varphi) = e^{t/2} F(\theta, \varphi),$$

$\forall t \in \mathbb{R}, \forall(\theta, \varphi) \in (0, \pi) \times (0, 2\pi)$.

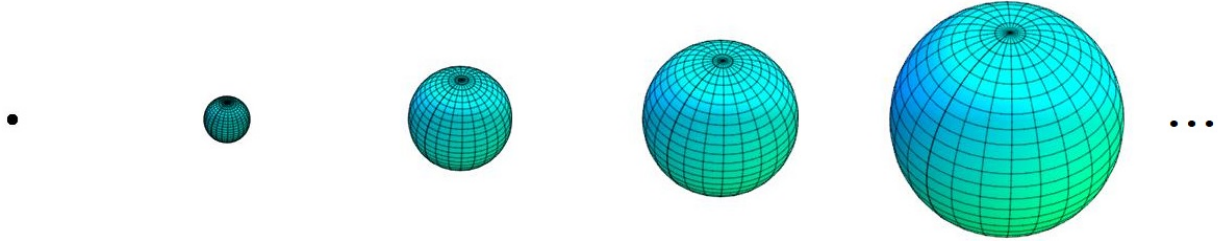


Figure 3.1: IMCF of the sphere

3.2 The cylinder

Theorem 3.2.1. Let $F : \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m} \longrightarrow \mathbb{E}^{n+1}$ be an isometric immersion of a cylinder in the Euclidean space $\mathbb{E}^{n+1}$, with inward unit normal vector field N , m principal curvatures equal to $k \neq 0$ and $n - m$ null principal curvatures, where $\mathbb{S}_{R_0}^m$ is the sphere centered at the origin with radius $R_0 > 0$ and dimension $m \notin \{0, n\}$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(\mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}) \subset \mathbb{E}^{n+1}$ is

$$\hat{F}^t(x, y) = F(x, y) + \frac{1}{k}(1 - e^{t/m})N(x, y),$$

$\forall t \in \mathbb{R}$ and $\forall(x, y) \in \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}$. Moreover, $\hat{F}^t(\mathbb{S}^m \times \mathbb{E}^{n-m})$ is an eternal solution that expands from an $(n - m)$ -dimensional plane of $\mathbb{E}^{n+1}$ and $\forall t \in \mathbb{R}$ it is a cylinder whose base has radius $R(t) = R_0 e^{t/m}$.

Proof. From Corollary 2.1.2, we have that

$$\hat{F}^t(x, y) = F(x, y) + \mu(t)N(x, y),$$

$\forall (x, y, t) \in \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m} \times I$, where $\mu : I \longrightarrow \mathbb{R}$ satisfies (2.7), i.e.,

$$\prod_{i=1}^n |1 - k_i \mu(t)| = e^t.$$

Since we have m principal curvatures equal to $k \neq 0$ and $n - m$ null principal curvatures, we get

$$|1 - k\mu(t)|^m = e^t$$

i.e.,

$$|1 - k\mu(t)| = e^{t/m}, \quad (3.4)$$

$\forall t \in I$. Since $e^{t/m}$ is a smooth positive function, from (3.4) we conclude that $h : I \longrightarrow \mathbb{R}$ given by

$$h(t) = 1 - k\mu(t)$$

does not change sign. Moreover, $\mu(0) = 0$, hence $h(0) = 1$. So, $h(t) > 0$, $\forall t \in I$. Then, (3.4) becomes

$$1 - k\mu(t) = e^{t/m},$$

$\forall t \in I$. It follows that

$$\mu(t) = \frac{1}{k}(1 - e^{t/m}),$$

$\forall t \in \mathbb{R}$. Then,

$$\hat{F}^t(x, y) = F(x, y) + \frac{1}{k}(1 - e^{t/m})N(x, y), \quad (3.5)$$

$\forall t \in \mathbb{R}$ and $\forall (x, y) \in \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}$.

Without loss of generality, consider

$$F(x, y) = (f(x), y),$$

$\forall (x, y) \in \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}$, where $f : \mathbb{S}_{R_0}^m \longrightarrow \mathbb{E}^{m+1}$ is an isometric immersion of the sphere centered at the origin with radius $R_0 > 0$ and dimension $m \notin \{0, n\}$ in the Euclidean space $\mathbb{E}^{m+1}$. Moreover,

$$N(x, y) = \left(-\frac{1}{R_0}f(x), 0 \right)$$

is the inward unit normal vector field of $F(\mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m})$, $\forall (x, y) \in \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m}$. Thus,

since $k = 1/R_0$, we have from (3.5) that

$$\begin{aligned}
\hat{F}^t(x, y) &= F(x, y) + \frac{1}{k}(1 - e^{t/m})N(x, y) \\
&= (f(x), y) + \frac{1}{k}(1 - e^{t/m})(-kf(x), 0) \\
&= (f(x) - (1 - e^{t/m})f(x), y) \\
&= (e^{t/m}f(x), y),
\end{aligned}$$

$\forall (x, y, t) \in \mathbb{S}_{R_0}^m \times \mathbb{E}^{n-m} \times \mathbb{R}$. Therefore, $\hat{F}^t(\mathbb{S}^m \times \mathbb{E}^{n-m})$ is an eternal solution that expands from an $(n - m)$ -dimensional plane of $\mathbb{E}^{n+1}$ and $\forall t \in \mathbb{R}$ it is a cylinder whose base has radius $R(t) = R_0 e^{t/m}$. ■

Example 3.2.2. Consider the parametrized cylinder $F : (0, 2\pi) \times \mathbb{R} \subset \mathbb{E}^2 \longrightarrow \mathbb{E}^3$, with

$$F(u, v) = (\cos(u), \sin(u), v) \quad \text{and} \quad N(u, v) = (-\cos(u), -\sin(u), 0).$$

The principal curvatures of $F((0, 2\pi) \times \mathbb{R})$ with respect to N are given by $k_1 = 1$ and $k_2 = 0$. So, the solution by parallel hypersurfaces to the IMCF with initial condition $F((0, 2\pi) \times \mathbb{R})$, is given by

$$\hat{F}^t(u, v) = (e^t \cos(u), e^t \sin(u), v),$$

$\forall t \in \mathbb{R}$. Moreover, $\hat{F}^t(\mathbb{S}^1 \times \mathbb{R})$ is an eternal solution that expands from a straight line and $\forall t \in \mathbb{R}$ it is a cylinder whose base has radius $R(t) = e^t$.

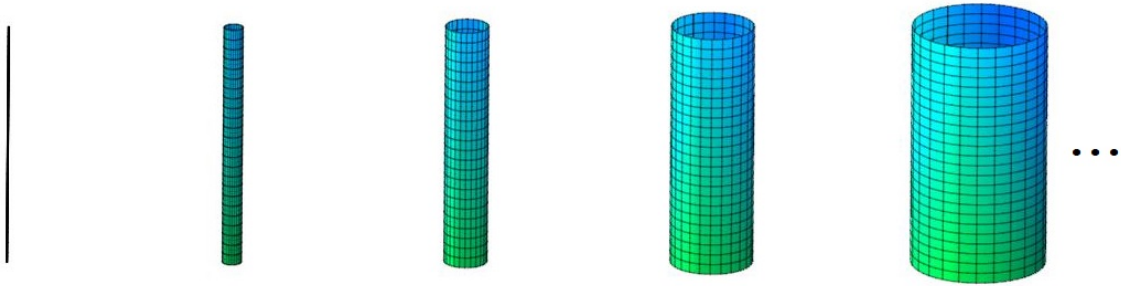


Figure 3.2: IMCF of the cylinder

Chapter 4

On the inverse mean curvature flow by parallel hypersurfaces in the hyperbolic space

In this chapter we use Corollary 2.1.2 to study the solutions to the IMCF by parallel hypersurfaces in $\mathbb{H}^{n+1}$ considering as initial condition the isoparametric hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$. In the first section, we prove a lemma that establishes sufficient conditions for the convergence of these solutions to the boundary of $\mathbb{H}^{n+1}$ ($\partial\mathbb{H}^{n+1}$). This lemma will be very important to prove the results of the next sections of this chapter.

In the second section, we study the case of the horosphere. We prove that the solution is eternal and remains a horosphere for all $t \in \mathbb{R}$. Furthermore, we show that the solution expands from a point and converges to $\partial\mathbb{H}^{n+1}$.

In the third section, we study the totally umbilical case, where all the principal curvatures are equal to a constant k with $k \notin \{-1, 0, 1\}$. If $0 < |k| < 1$, we prove that the solution is immortal, it expands from a totally geodesic hypersurface of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$. If $|k| > 1$, we prove that the solution is eternal, it expands from a point and converges to $\partial\mathbb{H}^{n+1}$.

Finally, in the fourth section, we study the case of the hyperbolic cylinder $(\mathbb{S}^m \times \mathbb{H}^m, n = 2m)$, with the additional assumption that the two distinct principal curvatures have the same multiplicity m . We prove that the solution is eternal, it expands from an m -dimensional submanifold of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$.

4.1 Convergence to the boundary

Lemma 4.1.1. *Let $F : M^n \longrightarrow \mathbb{H}^{n+1}$ be an oriented isoparametric hypersurface with inward unit normal vector field N and suppose the mean curvature H of $F(M)$ satisfies $H(x) \neq 0, \forall x \in M$. Let $\hat{F} : M^n \times I \longrightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be the solution to the IMCF by*

parallel hypersurfaces with initial condition $F(M)$. If $(0, +\infty) \subset I$ and

$$\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty,$$

then $\hat{F}^t(M)$ converges to $\partial\mathbb{H}^{n+1}$ when $t \rightarrow +\infty$.

Proof. Let $x \in M^n$, $F(x) \in \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ and $N(x) \in \mathbb{L}^{n+2}$. We can set

$$F(x) = (F_1(x), \dots, F_{n+2}(x)) \quad \text{and} \quad N(x) = (N_1(x), \dots, N_{n+2}(x)),$$

where $F_j, N_j : M^n \rightarrow \mathbb{R}$, $\forall j$ with $1 \leq j \leq n+2$, and $F_1(x) \geq 1$. Define $\hat{F}_j : M^n \times I \rightarrow \mathbb{R}$ by

$$\hat{F}_j(x, t) = \cosh(\mu(t))F_j(x) + \sinh(\mu(t))N_j(x),$$

$\forall j$ with $1 \leq j \leq n+2$. Therefore, from Corollary 2.1.2, we can write

$$\hat{F}(x, t) = \cosh(\mu(t))F(x) + \sinh(\mu(t))N(x) = (\hat{F}_1(x, t), \hat{F}_2(x, t), \dots, \hat{F}_{n+2}(x, t)),$$

$\forall t \in I$. Since $\hat{F}(x, t) \in \mathbb{H}^{n+1}$, we have that $\hat{F}_1(x, t) \geq 1$, $\forall t \in I$, and

$$-\hat{F}_1^2(x, t) + \sum_{j=2}^{n+2} \hat{F}_j^2(x, t) = -1,$$

i.e.,

$$\sum_{j=2}^{n+2} \hat{F}_j^2(x, t) = \hat{F}_1^2(x, t) - 1, \tag{4.1}$$

$\forall t \in I$.

Applying the isometry $G : \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2} \rightarrow \mathbb{B}^{n+1} \subset \mathbb{E}^{n+1}$ defined in (1.1) restricted to $\hat{F}(M \times I)$, we obtain

$$G(\hat{F}(x, t)) = \frac{1}{\hat{F}_1(x, t) + 1} (\hat{F}_2(x, t), \dots, \hat{F}_{n+1}(x, t), -\hat{F}_{n+2}(x, t)).$$

Then, $G \circ \hat{F} : M^n \times I \rightarrow \mathbb{B}^{n+1}$ is the family of hypersurfaces in the Poincaré ball model $\mathbb{B}^{n+1}$ corresponding the family of hypersurfaces $\hat{F} : M^n \times I \rightarrow \mathbb{H}^{n+1}$ in $\mathbb{H}^{n+1}$. Thus, we only need to show that

$$\lim_{t \rightarrow +\infty} \left[\sum_{j=2}^{n+2} \frac{\hat{F}_j^2(x, t)}{(\hat{F}_1(x, t) + 1)^2} \right] = 1.$$

From (4.1),

$$\sum_{j=2}^{n+2} \frac{\hat{F}_j^2(x, t)}{(\hat{F}_1(x, t) + 1)^2} = \frac{\hat{F}_1^2(x, t) - 1}{(\hat{F}_1(x, t) + 1)^2} = \frac{\hat{F}_1(x, t) - 1}{\hat{F}_1(x, t) + 1},$$

$\forall t \in I$. So, we must prove that

$$\lim_{t \rightarrow +\infty} \left[\frac{\hat{F}_1(x, t) - 1}{\hat{F}_1(x, t) + 1} \right] = 1. \quad (4.2)$$

We have that

$$\frac{\hat{F}_1(x, t) - 1}{\hat{F}_1(x, t) + 1} = \frac{F_1(x) \cosh(\mu(t)) + N_1(x) \sinh(\mu(t)) - 1}{F_1(x) \cosh(\mu(t)) + N_1(x) \sinh(\mu(t)) + 1},$$

$\forall t \in I$. Since $\cosh(\mu(t)) > 0$, $\forall t \in I$, we can divide the numerator and denominator on the right side of the above expression by $\cosh(\mu(t))$ getting

$$\begin{aligned} \frac{\hat{F}_1(x, t) - 1}{\hat{F}_1(x, t) + 1} &= \frac{F_1(x) + N_1(x) \frac{\sinh(\mu(t))}{\cosh(\mu(t))} - \frac{1}{\cosh(\mu(t))}}{F_1(x) + N_1(x) \frac{\sinh(\mu(t))}{\cosh(\mu(t))} + \frac{1}{\cosh(\mu(t))}} \\ &= \frac{F_1(x) + N_1(x) \tanh(\mu(t)) - \frac{1}{\cosh(\mu(t))}}{F_1(x) + N_1(x) \tanh(\mu(t)) + \frac{1}{\cosh(\mu(t))}}. \end{aligned} \quad (4.3)$$

Since $\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty$, taking the limit with $t \rightarrow +\infty$ on both sides of (4.3), we conclude that (4.2) holds. ■

4.2 The horosphere

Theorem 4.2.1. *Let $F : \mathbb{E}^n \rightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be an isometric immersion of a horosphere in the hyperbolic space, with inward unit normal vector field N and all principal curvatures equal to $k = \pm 1$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(\mathbb{E}^n)$ is*

$$\hat{F}^t(x) = \cosh\left(-k\frac{t}{n}\right) F(x) + \sinh\left(-k\frac{t}{n}\right) N(x), \quad (4.4)$$

$\forall t \in \mathbb{R}$. Moreover, $\hat{F}^t$ is an eternal solution and $\hat{F}^t(\mathbb{E}^n)$ is a horosphere, $\forall t \in \mathbb{R}$. Also, $\hat{F}^t(\mathbb{E}^n)$ expands from a point and converges to $\partial\mathbb{H}^{n+1}$ when $t \rightarrow +\infty$.

Proof. From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cosh(\mu(t)) F(x) + \sinh(\mu(t)) N(x),$$

$\forall (x, t) \in \mathbb{E}^n \times I$, where, from (2.7), $\mu : I \longrightarrow \mathbb{R}$ satisfies

$$\prod_{i=1}^n |\cosh(\mu(t)) - k_i \sinh(\mu(t))| = e^t.$$

Since all the principal curvatures are equal to $k = \pm 1$, we have that

$$|\cosh(\mu(t)) - k \sinh(\mu(t))|^n = e^t,$$

and hence,

$$|\cosh(\mu(t)) - k \sinh(\mu(t))| = e^{t/n}, \quad (4.5)$$

$\forall t \in I$. Since $e^{t/n}$ is a smooth positive function, we conclude from (4.5) that $h : I \longrightarrow \mathbb{R}$ given by

$$h(t) = \cosh(\mu(t)) - k \sinh(\mu(t))$$

does not change sign. Moreover, $\mu(0) = 0$ implies that

$$h(0) = \cosh(\mu(0)) - k \sinh(\mu(0)) = \cosh(0) - k \sinh(0) = 1 > 0.$$

So, $h(t) > 0$, $\forall t \in I$, and (4.5) reduces to

$$\cosh(\mu(t)) - k \sinh(\mu(t)) = e^{t/n}, \quad (4.6)$$

$\forall t \in I$. Thus, from (4.6), one sees that

$$k \sinh(\mu(t)) = \cosh(\mu(t)) - e^{t/n}, \quad (4.7)$$

$\forall t \in I$. Therefore,

$$k^2 \sinh^2(\mu(t)) = \cosh^2(\mu(t)) - 2e^{t/n} \cosh(\mu(t)) + e^{2t/n},$$

i.e.,

$$k^2 [\cosh^2(\mu(t)) - 1] - \cosh^2(\mu(t)) + 2e^{t/n} \cosh(\mu(t)) - e^{2t/n} = 0.$$

So, we have

$$(k^2 - 1) \cosh^2(\mu(t)) + 2e^{t/n} \cosh(\mu(t)) - (k^2 + e^{2t/n}) = 0, \quad (4.8)$$

$\forall t \in I$. Since $k = \pm 1$, we obtain from (4.8) that

$$2e^{t/n} \cosh(\mu(t)) = e^{2t/n} + 1$$

i.e.,

$$\cosh(\mu(t)) = \frac{e^{2t/n} + 1}{2e^{t/n}} = \frac{e^{t/n} + e^{-t/n}}{2} = \cosh(\pm t/n), \quad (4.9)$$

$\forall t \in \mathbb{R}$. Hence

$$\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty. \quad (4.10)$$

Substituting (4.9) into (4.7), we have

$$\sinh(\mu(t)) = \frac{e^{-t/n} - e^{t/n}}{2k} = -k \frac{e^{t/n} - e^{-t/n}}{2} = -k \sinh(t/n) = \sinh\left(-k \frac{t}{n}\right).$$

So, $\mu(t) = -k \frac{t}{n}$, $\forall t \in \mathbb{R}$.

Now, consider $\{E_i\}_{i=1}^n$ a local orthonormal frame of principal directions of $F(\mathbb{E}^n) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ in $x \in \mathbb{E}^n$. It follows from Lemma 1.3.2 that the principal curvatures $\hat{k}_i^t$ of $\hat{F}^t(\mathbb{E}^n)$ are given by

$$\begin{aligned} \hat{k}_i^t &= \frac{k_i \cosh(\mu(t)) - \sinh(\mu(t))}{\cosh(\mu(t)) - k_i \sinh(\mu(t))} \\ &= \frac{k \cosh(\mu(t)) - \sinh(\mu(t))}{\cosh(\mu(t)) - k \sinh(\mu(t))} \\ &= k \frac{\cosh(\mu(t)) - \frac{1}{k} \sinh(\mu(t))}{\cosh(\mu(t)) - k \sinh(\mu(t))} \\ &= k \frac{\cosh(\mu(t)) - k \sinh(\mu(t))}{\cosh(\mu(t)) - k \sinh(\mu(t))} \\ &= k, \end{aligned}$$

$\forall t \in \mathbb{R}$. Then, $\hat{F}^t(\mathbb{E}^n)$ is a horosphere, $\forall t \in \mathbb{R}$. From Lemma 1.3.2, we have the first fundamental form of $\hat{F}^t(x)$, $x \in \mathbb{E}^n$, given by

$$\begin{aligned} \hat{g}_x^t(E_i, E_j) &= [\cosh(\mu(t)) - k \sinh(\mu(t))]^2 \delta_{ij} \\ &= (e^{t/n})^2 \delta_{ij} \\ &= e^{2t/n} \delta_{ij}, \end{aligned}$$

$\forall t \in \mathbb{R}$. Then,

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) = 0.$$

It follows, from Lemma 4.1.1 and (4.10), that $\hat{F}^t(\mathbb{E}^n)$ is defined $\forall t \in \mathbb{R}$, it is an eternal solution that expands from a point and converges to $\partial\mathbb{H}^{n+1}$ when $t \rightarrow +\infty$. $\blacksquare$

Example 4.2.2. Consider the parametrized sphere $X : (0, \pi) \times (0, 2\pi) \subset \mathbb{E}^2 \rightarrow \mathbb{E}^3$, with

$$X(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).$$

Then,

$$\hat{X}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, 1 + \cos \theta)$$

is the horosphere in the half-space model $\mathbb{R}_+^3$. Using the isometry $L : \mathbb{R}_+^3 \rightarrow \mathbb{H}^3 \subset \mathbb{L}^4$ given by

$$L(y_1, y_2, y_3) = \frac{1}{2y_3} \left(1 + \sum_{i=1}^3 y_i^2, 2y_1, 2y_2, 1 - \sum_{i=1}^3 y_i^2 \right),$$

we have

$$L(\hat{X}(\theta, \varphi)) = \left(\frac{3 + 2 \cos \theta}{2(1 + \cos \theta)}, \frac{\sin \theta \cos \varphi}{1 + \cos \theta}, \frac{\sin \theta \sin \varphi}{1 + \cos \theta}, \frac{-1 - 2 \cos \theta}{2(1 + \cos \theta)} \right).$$

Denoting $F = L \circ \hat{X} : (0, \pi) \times (0, 2\pi) \subset \mathbb{E}^2 \rightarrow \mathbb{H}^3 \subset \mathbb{L}^4$, we get

$$F_\theta(\theta, \varphi) = \left(\frac{\sin \theta}{2(1 + \cos \theta)^2}, \frac{\cos \varphi}{1 + \cos \theta}, \frac{\sin \varphi}{1 + \cos \theta}, \frac{\sin \theta}{2(1 + \cos \theta)^2} \right)$$

and

$$F_\varphi(\theta, \varphi) = \left(0, -\frac{\sin \varphi \sin \theta}{1 + \cos \theta}, \frac{\cos \varphi \sin \theta}{1 + \cos \theta}, 0 \right).$$

So, the inward unit normal vector field to F is given by

$$N(\theta, \varphi) = \left(\frac{2 + \cos \theta}{2(1 + \cos \theta)}, \frac{\sin \theta \cos \varphi}{1 + \cos \theta}, \frac{\sin \theta \sin \varphi}{1 + \cos \theta}, -\frac{2 + 3 \cos \theta}{2(1 + \cos \theta)} \right).$$

Moreover,

$$|F_\theta|^2 = \frac{1}{(1 + \cos \theta)^2}, \quad |F_\varphi|^2 = \left(\frac{\sin \theta}{1 + \cos \theta} \right)^2 \quad \text{and} \quad \langle F_\theta, F_\varphi \rangle = 0.$$

We have that

$$F_{\theta\theta}(\theta, \varphi) = \left(\frac{2 + \cos \theta(1 - \cos \theta)}{2(1 + \cos \theta)^3}, \frac{\sin \theta \cos \varphi}{(1 + \cos \theta)^2}, \frac{\sin \theta \sin \varphi}{(1 + \cos \theta)^2}, \frac{2 + \cos \theta(1 - \cos \theta)}{2(1 + \cos \theta)^3} \right)$$

and

$$F_{\varphi\varphi}(\theta, \varphi) = \left(0, -\frac{\cos \varphi \sin \theta}{1 + \cos \theta}, -\frac{\sin \varphi \sin \theta}{1 + \cos \theta}, 0\right).$$

Thus,

$$\langle N, F_{\theta\theta} \rangle = -\frac{1}{(1 + \cos \theta)^2} \quad \text{and} \quad \langle N, F_{\varphi\varphi} \rangle = -\left(\frac{\sin \theta}{1 + \cos \theta}\right)^2.$$

Then,

$$k_1 = \frac{1}{|F_\theta|^2} \langle N, F_{\theta\theta} \rangle = (1 + \cos \theta)^2 \left[-\frac{1}{(1 + \cos \theta)^2} \right] = -1$$

and

$$k_2 = \frac{1}{|F_\varphi|^2} \langle N, F_{\varphi\varphi} \rangle = \frac{(1 + \cos \theta)^2}{\sin^2 \theta} \left[-\frac{\sin^2 \theta}{(1 + \cos \theta)^2} \right] = -1.$$

Therefore, from (4.4), we get

$$\begin{aligned} \hat{F}^t(\theta, \varphi) &= \cosh(t/2)F(\theta, \varphi) + \sinh(t/2)N(\theta, \varphi) \\ &= \frac{\cosh(t/2)}{1 + \cos \theta} \left(\frac{3}{2} + \cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi, -\frac{1}{2} - \cos \theta \right) \\ &\quad + \frac{\sinh(t/2)}{1 + \cos \theta} \left(1 + \frac{1}{2} \cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi, -1 - \frac{3}{2} \cos \theta \right). \end{aligned}$$

So, using the isometry $G : \mathbb{H}^3 \subset \mathbb{L}^4 \longrightarrow \mathbb{B}^3 \subset \mathbb{R}^3$ given by

$$G(y_1, y_2, y_3, y_4) = \frac{1}{1 + y_1} (y_2, y_3, -y_4),$$

we have that

$$\begin{aligned} G(\hat{F}^t(\theta, \varphi)) &= \frac{1}{2(1 + \cos \theta) + (3 + 2 \cos \theta) \cosh(t/2) + (2 + \cos \theta) \sinh(t/2)} \\ &\quad \cdot \left[(G \circ \hat{F}^t)_1(\theta, \varphi), (G \circ \hat{F}^t)_2(\theta, \varphi), (G \circ \hat{F}^t)_3(\theta, \varphi) \right], \end{aligned}$$

where

$$(G \circ \hat{F}^t)_1(\theta, \varphi) = 2e^{t/2} \sin \theta \cos \varphi,$$

$$(G \circ \hat{F}^t)_2(\theta, \varphi) = 2e^{t/2} \sin \theta \sin \varphi$$

and

$$(G \circ \hat{F}^t)_3(\theta, \varphi) = (1 + 2 \cos \theta) \cosh(t/2) + (2 + 3 \cos \theta) \sinh(t/2).$$

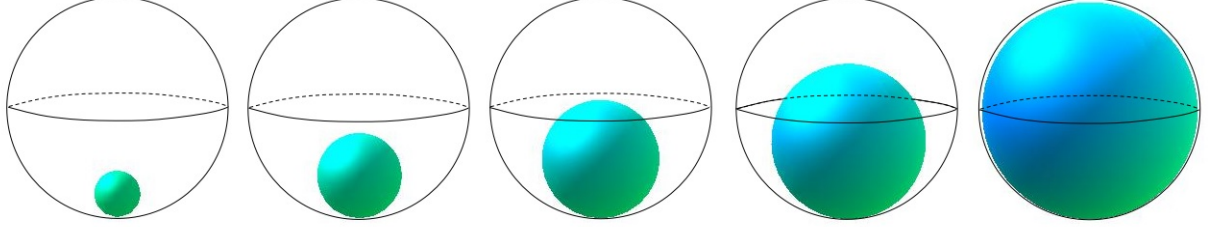


Figure 4.1: IMCF of the horosphere

4.3 The other totally umbilical case

Theorem 4.3.1. *Let $F : M^n \longrightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be an isometric immersion of a totally umbilical hypersurface in the hyperbolic space, with inward unit normal vector field N and all principal curvatures equal to k , where $k \notin \{-1, 0, 1\}$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(M^n)$ is*

$$\hat{F}^t(x) = \cosh(\mu(t))F(x) + \sinh(\mu(t))N(x),$$

where

$$\cosh(\mu(t)) = \frac{e^{t/n} - |k|\sqrt{q(t)}}{1 - k^2}$$

and

$$\sinh(\mu(t)) = \operatorname{sgn}(k) \frac{|k|e^{t/n} - \sqrt{q(t)}}{1 - k^2},$$

$\forall t \in I$ and $\forall x \in M$, with

$$q(t) = e^{2t/n} + k^2 - 1.$$

(i) If $0 < |k| < 1$, then $I = (t^*, +\infty)$, where $t^* = \frac{n}{2} \ln(1 - k^2)$. Moreover, $\hat{F}^t(M)$ is an immortal solution that expands from a totally geodesic hypersurface of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$ when $t \longrightarrow +\infty$.

(ii) If $|k| > 1$, then $I = \mathbb{R}$. Moreover, $\hat{F}^t(M)$ is an eternal solution that expands from a point and converges to $\partial\mathbb{H}^{n+1}$ when $t \longrightarrow +\infty$.

Proof. From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cosh(\mu(t))F(x) + \sinh(\mu(t))N(x),$$

$\forall (x, t) \in M^n \times I$, where, from (2.7), $\mu : I \longrightarrow \mathbb{R}$ satisfies

$$\prod_{i=1}^n |\cosh(\mu(t)) - k_i \sinh(\mu(t))| = e^t.$$

Thus, since all principal curvatures are equal to k , we have that

$$|\cosh(\mu(t)) - k \sinh(\mu(t))| = e^{t/n}, \quad (4.11)$$

$\forall t \in I$. Since $e^{t/n}$ is a smooth positive function, we conclude from (4.11) that $h : I \longrightarrow \mathbb{R}$ given by

$$h(t) = \cosh(\mu(t)) - k \sinh(\mu(t))$$

does not change sign. Moreover, $\mu(0) = 0$ implies that $h(0) = 1$. So, $h(t) > 0$, $\forall t \in I$. Then, from (4.11), we have

$$\cosh(\mu(t)) - k \sinh(\mu(t)) = e^{t/n}, \quad (4.12)$$

$\forall t \in I$. Thus, similar arguments used for (4.7) lead to

$$(k^2 - 1) \cosh^2(\mu(t)) + 2e^{t/n} \cosh(\mu(t)) - (k^2 + e^{2t/n}) = 0,$$

$\forall t \in I$. This is a second degree equation for $\cosh(\mu(t))$ and its discriminant is given by

$$\begin{aligned} \Delta &= 4e^{2t/n} + 4(k^2 - 1)(k^2 + e^{2t/n}) \\ &= 4e^{2t/n} + 4k^4 + 4k^2 e^{2t/n} - 4k^2 - 4e^{2t/n} \\ &= 4k^4 + 4k^2 e^{2t/n} - 4k^2 \\ &= 4k^2(e^{2t/n} + k^2 - 1). \end{aligned}$$

By setting

$$q(t) = e^{2t/n} + k^2 - 1,$$

we obtain,

$$\cosh(\mu(t)) = \frac{-e^{t/n} \pm |k| \sqrt{q(t)}}{k^2 - 1}.$$

Since $\mu(0) = 0$, it follows that

$$\cosh(\mu(t)) = \frac{e^{t/n} - |k|\sqrt{q(t)}}{1 - k^2}. \quad (4.13)$$

From (4.12) and (4.13), we have

$$\begin{aligned} \sinh(\mu(t)) &= \frac{1}{k} \cosh(\mu(t)) - \frac{1}{k} e^{t/n} \\ &= \frac{1}{k} \frac{e^{t/n} - |k|\sqrt{q(t)}}{1 - k^2} - \frac{1}{k} e^{t/n} \\ &= \frac{e^{t/n} - |k|\sqrt{q(t)} - e^{t/n}(1 - k^2)}{k(1 - k^2)} \\ &= \frac{k^2 e^{t/n} - |k|\sqrt{q(t)}}{k(1 - k^2)} \\ &= \operatorname{sgn}(k) \frac{|k| e^{t/n} - \sqrt{q(t)}}{1 - k^2}. \end{aligned}$$

Moreover, $\Delta \geq 0$ if and only if $q(t) \geq 0$, i.e.,

$$e^{2t/n} \geq 1 - k^2. \quad (4.14)$$

(i) If $0 < |k| < 1$, we have $1 - k^2 > 0$ and

$$\frac{2t}{n} \geq \ln(1 - k^2),$$

i.e.,

$$t \geq \frac{n}{2} \ln(1 - k^2) = t^*.$$

We have that

$$e^{t/n} - |k|\sqrt{q(t)} > e^{t/n}(1 - |k|),$$

$\forall t \geq t^*$. Therefore, it follows from (4.13) that

$$\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty. \quad (4.15)$$

Consider $\{E_j\}_{j=1}^n$ be a local orthonormal frame of principal directions of $F(M) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ in $x \in M$. It follows from Lemma 1.3.2 and (4.12) that

$$\begin{aligned} \hat{g}_x^t(E_i, E_j) &= [\cosh(\mu(t)) - k_i(x) \sinh(\mu(t))]^2 \delta_{ij} \\ &= [\cosh(\mu(t)) - k \sinh(\mu(t))]^2 \delta_{ij} \\ &= (e^{t/n})^2 \delta_{ij} \\ &= e^{2t/n} \delta_{ij}, \end{aligned}$$

$\forall t \geq t^*$. Then,

$$\hat{g}_x^{t^*}(E_i, E_j) = e^{2t^*/n} \delta_{ij} = \delta_{ij} \exp \left(\frac{2}{n} \left(\frac{n}{2} \ln(1 - k^2) \right) \right) = (1 - k^2) \delta_{ij}.$$

By using Lemma 1.3.2, we infer that the second fundamental form $\hat{I}_x^t$ of $\hat{F}^t(x)$, $x \in M$, is given by

$$\begin{aligned} \hat{I}_x^t(E_i, E_j) &= [\cosh(\mu(t)) - k \sinh(\mu(t))] [k \cosh(\mu(t)) - \sinh(\mu(t))] \delta_{ij} \\ &= \delta_{ij} e^{t/n} \left[k \frac{e^{t/n} - |k| \sqrt{q(t)}}{1 - k^2} - \frac{k^2 e^{t/n} - |k| \sqrt{q(t)}}{k(1 - k^2)} \right] \\ &= \delta_{ij} e^{t/n} \left[\frac{|k| \sqrt{q(t)} (1 - k^2)}{k(1 - k^2)} \right] \\ &= \delta_{ij} \operatorname{sgn}(k) e^{t/n} \sqrt{q(t)}, \end{aligned}$$

$\forall t \geq t^*$. Then, since $q(t^*) = 0$,

$$\hat{I}_x^{t^*}(E_i, E_j) = 0. \quad (4.16)$$

Thus, from (4.16), $t \in (t^*, +\infty)$. Moreover, by Lemma 4.1.1 and (4.15), $\hat{F}^t(M)$ is an immortal solution that expands from a totally geodesic hypersurface of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$ when $t \rightarrow +\infty$.

(ii) If $|k| > 1$, then (4.14) holds $\forall t \in \mathbb{R}$. Since $k^2 - 1 > 0$, we get

$$e^{t/n} - |k| \sqrt{e^{2t/n} + k^2 - 1} < e^{t/n} (1 - |k|), \quad (4.17)$$

$\forall t \in \mathbb{R}$. Therefore, it follows from (4.13) and (4.17) that

$$\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty. \quad (4.18)$$

Consider $\{E_j\}_{j=1}^n$ be a local orthonormal frame of principal directions of $F(M) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ in $x \in M$. By using Lemma 1.3.2 and (4.12), one sees that

$$\hat{g}_x^t(E_i, E_j) = e^{2t/n} \delta_{ij}, \quad (4.19)$$

$\forall t \in \mathbb{R}$. Moreover, from (4.19), we infer

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) = 0. \quad (4.20)$$

Therefore, from (4.18), (4.20) and Lemma 4.1.1, $\hat{F}^t(M)$ is defined for $t \in \mathbb{R}$, it is an eternal solution that expands from a point and converges to $\partial\mathbb{H}^{n+1}$ when $t \rightarrow +\infty$. ■

4.4 The hyperbolic cylinder

In this section, we study the case of the hyperbolic cylinder $\mathbb{S}^m \times \mathbb{H}^m \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$, $n = 2m$, with the additional assumption that the two distinct principal curvatures have the same multiplicity m .

Theorem 4.4.1. *Let $F : \mathbb{S}^m \times \mathbb{H}^m \longrightarrow \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be an isometric immersion of a cylinder in the hyperbolic space, with inward unit normal vector field N and two distinct principal curvatures, $k_1 > 1$ and k_2 , with the same multiplicity m , such that $k_1 k_2 = 1$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(\mathbb{S}^m \times \mathbb{H}^m) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ is given by*

$$\hat{F}^t(x) = \cosh(\mu(t))F(x) + \sinh(\mu(t))N(x),$$

where

$$\cosh(\mu(t)) = \sqrt{\frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2}} + \frac{1}{2}, \quad (4.21)$$

$$\sinh(\mu(t)) = -\operatorname{sgn}(t) \sqrt{\frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2}} - \frac{1}{2} \quad (4.22)$$

and

$$q(t) = \left(\frac{k_1 - k_2}{2} \right)^2 + e^{2t/m},$$

$\forall t \in \mathbb{R}$. Moreover, $\hat{F}^t(\mathbb{S}^m \times \mathbb{H}^m)$ is an eternal solution that expands from an m -dimensional submanifold of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$ when $t \longrightarrow +\infty$.

Proof. From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cosh(\mu(t))F(x) + \sinh(\mu(t))N(x),$$

$\forall (x, t) \in \mathbb{S}^m \times \mathbb{H}^m \times I$, where

$$\prod_{i=1}^n |\cosh(\mu(t)) - k_i \sinh(\mu(t))| = e^t.$$

Since the principal curvatures k_1 and k_2 have the same multiplicity m , we have that

$$|[\cosh(\mu(t)) - k_1 \sinh(\mu(t))][\cosh(\mu(t)) - k_2 \sinh(\mu(t))]| = e^{t/m},$$

i.e., $\forall t \in I$,

$$\left| \cosh^2(\mu(t)) - (k_1 + k_2) \sinh(\mu(t)) \cosh(\mu(t)) + k_1 k_2 \sinh^2(\mu(t)) \right| = e^{t/m}$$

and, then, using that $k_1 k_2 = 1$, we have

$$\left| \cosh(2\mu(t)) - \frac{k_1 + k_2}{2} \sinh(2\mu(t)) \right| = e^{t/m}. \quad (4.23)$$

Since $e^{t/m}$ is a smooth positive function, we conclude from (4.23) that $h : I \rightarrow \mathbb{R}$ given by

$$h(t) = \cosh(2\mu(t)) - \frac{k_1 + k_2}{2} \sinh(2\mu(t))$$

does not change sign. We claim that $h(t) > 0$, $\forall t \in I$. In fact, since $\mu(0) = 0$, we deduce that $h(0) = 1$. So, $h(t) > 0$, $\forall t \in I$. Thus, from (4.23),

$$\frac{k_1 + k_2}{2} \sinh(2\mu(t)) = \cosh(2\mu(t)) - e^{t/m}, \quad (4.24)$$

$\forall t \in I$. By (4.24) we get

$$\left[\frac{(k_1 + k_2)^2}{4} - 1 \right] \cosh^2(2\mu(t)) + 2e^{t/m} \cosh(2\mu(t)) - \left[\frac{(k_1 + k_2)^2}{4} + e^{2t/m} \right] = 0.$$

This is a second degree equation for $\cosh(2\mu(t))$ and its discriminant is given by

$$\begin{aligned} \Delta &= 4e^{2t/m} + 4 \left[\frac{(k_1 + k_2)^2}{4} - 1 \right] \left[\frac{(k_1 + k_2)^2}{4} + e^{2t/m} \right] \\ &= 4e^{2t/m} + 4 \left[\frac{(k_1 + k_2)^4}{16} + \frac{(k_1 + k_2)^2}{4} e^{2t/m} - \frac{(k_1 + k_2)^2}{4} - e^{2t/m} \right] \\ &= 4e^{2t/m} + \frac{(k_1 + k_2)^4}{4} + (k_1 + k_2)^2 e^{2t/m} - (k_1 + k_2)^2 - 4e^{2t/m} \\ &= (k_1 + k_2)^2 \left(\frac{(k_1 + k_2)^2}{4} + e^{2t/m} - 1 \right) \\ &= (k_1 + k_2)^2 \left(\frac{(k_1 + k_2)^2 - 4}{4} + e^{2t/m} \right) \\ &= (k_1 + k_2)^2 \left(\frac{(k_1 - k_2)^2}{4} + e^{2t/m} \right). \end{aligned}$$

So, $\Delta > 0$, $\forall t \in \mathbb{R}$, and $I = \mathbb{R}$. Define $q : \mathbb{R} \rightarrow \mathbb{R}$ by

$$q(t) := \left(\frac{k_1 - k_2}{2} \right)^2 + e^{2t/m}.$$

We get

$$\cosh(2\mu(t)) = \frac{-2e^{t/m} \pm (k_1 + k_2)\sqrt{q(t)}}{2\left(\frac{k_1 - k_2}{2}\right)^2} = \frac{-4e^{t/m} \pm 2(k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2}.$$

Using that $\mu(0) = 0$, we have

$$\cosh(2\mu(t)) = \frac{-4e^{t/m} + 2(k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2}, \quad (4.25)$$

$\forall t \in \mathbb{R}$. Since

$$\cosh^2(\mu(t)) = \frac{1}{2} \cosh(2\mu(t)) + \frac{1}{2},$$

it follows from (4.25) that

$$\begin{aligned} \cosh^2(\mu(t)) &= \frac{-4e^{t/m} + 2(k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2} + \frac{1}{2} \\ &= \frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{1}{2}, \end{aligned}$$

i.e.,

$$\cosh(\mu(t)) = \sqrt{\frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{1}{2}}, \quad (4.26)$$

$\forall t \in \mathbb{R}$.

We want to show that

$$\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty.$$

Indeed, by (4.26) we only need to show that

$$\lim_{t \rightarrow +\infty} \left[-2e^{t/m} + (k_1 + k_2) \sqrt{\left(\frac{k_1 - k_2}{2}\right)^2 + e^{2t/m}} \right] = +\infty. \quad (4.27)$$

Since $\left(\frac{k_1 - k_2}{2}\right)^2 > 0$, we get

$$\sqrt{\left(\frac{k_1 - k_2}{2}\right)^2 + e^{2t/m}} > e^{t/m},$$

$\forall t \in \mathbb{R}$. Hence,

$$-2e^{t/m} + (k_1 + k_2) \sqrt{\left(\frac{k_1 - k_2}{2}\right)^2 + e^{2t/m}} > e^{t/m}(k_1 + k_2 - 2),$$

$\forall t \in \mathbb{R}$. Since $k_1 k_2 = 1$ and $k_1 > 0$, we have $k_1 + k_2 - 2 > 0$ if and only if $(k_1 - 1)^2 > 0$. Hence, (4.27) holds, i.e.,

$$\lim_{t \rightarrow +\infty} \cosh(\mu(t)) = +\infty. \quad (4.28)$$

Note that

$$\sinh^2(\mu(t)) = \cosh^2(\mu(t)) - 1 = \frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} - \frac{1}{2},$$

i.e.,

$$\sinh(\mu(t)) = \pm \sqrt{\frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} - \frac{1}{2}}, \quad (4.29)$$

$\forall t \in \mathbb{R}$. We need study the sign of $\sinh(\mu(t))$. Taking into account

$$\sinh(2\mu(t)) = 2 \sinh(\mu(t)) \cosh(\mu(t))$$

and $\cosh(\mu(t)) > 0$, $\forall t \in \mathbb{R}$, we now study the sign of $\sinh(2\mu(t))$. Observe from (4.24) that

$$\begin{aligned} \sinh(2\mu(t)) &= \frac{2}{k_1 + k_2} \cosh(2\mu(t)) - \frac{2}{k_1 + k_2} e^{t/m} \\ &= \frac{2}{k_1 + k_2} \frac{-4e^{t/m} + 2(k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} - \frac{2}{k_1 + k_2} e^{t/m} \\ &= \frac{-8e^{t/m} + 4(k_1 + k_2)\sqrt{q(t)}}{(k_1 + k_2)(k_1 - k_2)^2} - \frac{2}{k_1 + k_2} e^{t/m} \\ &= \frac{-8e^{t/m} + 4(k_1 + k_2)\sqrt{q(t)} - 2(k_1 - k_2)^2 e^{t/m}}{(k_1 + k_2)(k_1 - k_2)^2} \\ &= \frac{-2[4 + (k_1 - k_2)^2] e^{t/m} + 4(k_1 + k_2)\sqrt{q(t)}}{(k_1 + k_2)(k_1 - k_2)^2} \\ &= \frac{-2(k_1 + k_2)^2 e^{t/m} + 4(k_1 + k_2)\sqrt{q(t)}}{(k_1 + k_2)(k_1 - k_2)^2} \\ &= \frac{-2(k_1 + k_2)e^{t/m} + 4\sqrt{q(t)}}{(k_1 - k_2)^2} \\ &= 2 \frac{-(k_1 + k_2)e^{t/m} + 2\sqrt{q(t)}}{(k_1 - k_2)^2}, \end{aligned}$$

$\forall t \in \mathbb{R}$. Moreover, $\sinh(2\mu(t)) > 0$ if and only if

$$2\sqrt{q(t)} > (k_1 + k_2)e^{t/m},$$

i.e.,

$$4q(t) > (k_1 + k_2)^2 e^{2t/m}.$$

This is equivalent to

$$(k_1 - k_2)^2 + 4e^{2t/m} > (k_1 + k_2)^2 e^{2t/m},$$

i.e.,

$$[(k_1 + k_2)^2 - 4] e^{2t/m} < (k_1 - k_2)^2.$$

Hence, $\sinh(2\mu(t)) > 0$ if and only if $e^{2t/m} < 1$. Therefore, (4.29) gives

$$\sinh(\mu(t)) = -\operatorname{sgn}(t) \sqrt{\frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2}} - \frac{1}{2},$$

$\forall t \in \mathbb{R}$.

Consider an orthonormal frame $\{E_1, E_2, \dots, E_m, E_{m+1}, \dots, E_{2m} = E_n\}$ of principal directions on $F(\mathbb{S}^m \times \mathbb{H}^m) \subset \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$, in $x \in \mathbb{S}^m \times \mathbb{H}^m$, such that $E_1, \dots, E_m$ are tangent to $\mathbb{S}^m$ and $E_{m+1}, \dots, E_n$ are tangent to $\mathbb{H}^m$. From Lemma 1.3.2, where $1 \leq i, j \leq m$, we then have

$$\begin{aligned} \hat{g}_x^t(E_i, E_j) &= [\cosh(\mu(t)) - k_1 \sinh(\mu(t))]^2 \delta_{ij} \\ &= [\cosh^2(\mu(t)) + k_1^2 \sinh^2(\mu(t)) - k_1 \sinh(2\mu(t))] \delta_{ij} \\ &= [\cosh^2(\mu(t)) + k_1^2 (\cosh^2(\mu(t)) - 1) - k_1 \sinh(2\mu(t))] \delta_{ij} \\ &= [(k_1^2 + 1) \cosh^2(\mu(t)) - k_1 \sinh(2\mu(t)) - k_1^2] \delta_{ij} \\ &= \left[(k_1^2 + 1) \left(\frac{-2e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{1}{2} \right) - 2k_1 \frac{-(k_1 + k_2)e^{t/m} + 2\sqrt{q(t)}}{(k_1 - k_2)^2} - k_1^2 \right] \delta_{ij} \\ &= \left[\frac{-2e^{t/m}(k_1^2 + 1) + (k_1 + k_2)(k_1^2 + 1)\sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{k_1^2 + 1}{2} \right] \delta_{ij} \\ &+ \left[\frac{2k_1(k_1 + k_2)e^{t/m} - 4k_1\sqrt{q(t)}}{(k_1 - k_2)^2} - k_1^2 \right] \delta_{ij} \\ &= \left[\frac{-2e^{t/m}(k_1^2 + 1) + (k_1 + k_2)(k_1^2 + 1)\sqrt{q(t)} + 2k_1(k_1 + k_2)e^{t/m} - 4k_1\sqrt{q(t)}}{(k_1 - k_2)^2} \right] \delta_{ij} \\ &+ \left(\frac{k_1^2 + 1}{2} - k_1^2 \right) \delta_{ij} \\ &= \left[\frac{-2e^{t/m}(k_1^2 + 1) + ((k_1 + k_2)(k_1^2 + 1) - 4k_1)\sqrt{q(t)} + 2(k_1^2 + 1)e^{t/m}}{(k_1 - k_2)^2} + \frac{k_1^2 + 1 - 2k_1^2}{2} \right] \delta_{ij} \\ &= \left[\frac{(k_1 + k_2)(k_1^2 + 1) - 4k_1}{(k_1 - k_2)^2} \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right] \delta_{ij} \\ &= \left[\frac{k_1^3 + k_1 + k_1 + k_2 - 4k_1}{(k_1 - k_2)^2} \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right] \delta_{ij}, \end{aligned}$$

i.e.,

$$\begin{aligned}
\hat{g}_x^t(E_i, E_j) &= \left[\frac{k_1^3 - 2k_1 + k_2}{(k_1 - k_2)^2} \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right] \delta_{ij} \\
&= \left[\frac{k_1^3 - 2k_1 + \frac{1}{k_1}}{(k_1 - \frac{1}{k_1})^2} \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right] \delta_{ij} \\
&= \left[\frac{\frac{k_1^4 - 2k_1^2 + 1}{k_1}}{\frac{(k_1^2 - 1)^2}{k_1^2}} \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right] \delta_{ij} \\
&= \left(k_1 \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right) \delta_{ij}.
\end{aligned}$$

Hence, for $1 \leq i, j \leq m$, we have that

$$\begin{aligned}
\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) &= \lim_{t \rightarrow -\infty} \left[k_1 \sqrt{q(t)} + \frac{1 - k_1^2}{2} \right] \delta_{ij} \\
&= \lim_{t \rightarrow -\infty} \left[k_1 \sqrt{\left(\frac{k_1 - k_2}{2} \right)^2 + e^{2t/m} + \frac{1 - k_1^2}{2}} \right] \delta_{ij} \\
&= \left[k_1 \left(\frac{k_1 - k_2}{2} \right) + \frac{1 - k_1^2}{2} \right] \delta_{ij} \\
&= \left(\frac{k_1^2 - 1}{2} + \frac{1 - k_1^2}{2} \right) \delta_{ij} \\
&= 0.
\end{aligned} \tag{4.30}$$

Similarly, for $m + 1 \leq i, j \leq n$, one obtains that

$$\begin{aligned}
\hat{g}_x^t(E_i, E_j) &= [\cosh(\mu(t)) - k_2 \sinh(\mu(t))]^2 \delta_{ij} \\
&= [\cosh^2(\mu(t)) + k_2^2 \sinh^2(\mu(t)) - k_2 \sinh(2\mu(t))] \delta_{ij} \\
&= [\cosh^2(\mu(t)) + k_2^2 (\cosh^2(\mu(t)) - 1) - k_2 \sinh(2\mu(t))] \delta_{ij} \\
&= [(k_2^2 + 1) \cosh^2(\mu(t)) - k_2 \sinh(2\mu(t)) - k_2^2] \delta_{ij} \\
&= \left[(k_2^2 + 1) \left(\frac{-2e^{t/m} + (k_1 + k_2) \sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{1}{2} \right) - 2k_2 \frac{-(k_1 + k_2)e^{t/m} + 2\sqrt{q(t)}}{(k_1 - k_2)^2} - k_2^2 \right] \delta_{ij} \\
&= \left[\frac{-2e^{t/m}(k_2^2 + 1) + (k_1 + k_2)(k_2^2 + 1)\sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{k_2^2 + 1}{2} \right] \delta_{ij} \\
&+ \left[\frac{2k_2(k_1 + k_2)e^{t/m} - 4k_2\sqrt{q(t)}}{(k_1 - k_2)^2} - k_2^2 \right] \delta_{ij} \\
&= \left[\frac{-2e^{t/m}(k_2^2 + 1) + (k_1 + k_2)(k_2^2 + 1)\sqrt{q(t)} + 2k_2(k_1 + k_2)e^{t/m} - 4k_2\sqrt{q(t)}}{(k_1 - k_2)^2} \right] \delta_{ij} \\
&+ \left(\frac{k_2^2 + 1}{2} - k_2^2 \right) \delta_{ij},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\hat{g}_x^t(E_i, E_j) &= \left[\frac{-2e^{t/m}(k_2^2 + 1) + ((k_1 + k_2)(k_2^2 + 1) - 4k_2)\sqrt{q(t)} + 2(k_2^2 + 1)e^{t/m}}{(k_1 - k_2)^2} + \frac{k_2^2 + 1 - 2k_2^2}{2} \right] \delta_{ij} \\
&= \left[\frac{(k_1 + k_2)(k_2^2 + 1) - 4k_2}{(k_1 - k_2)^2} \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \left[\frac{k_2 + k_1 + k_2^3 + k_2 - 4k_2}{(k_1 - k_2)^2} \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \left[\frac{k_2^3 - 2k_2 + k_1}{(k_1 - k_2)^2} \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right] \delta_{ij},
\end{aligned}$$

i.e., since $k_1 k_2 = 1$,

$$\begin{aligned}
\hat{g}_x^t(E_i, E_j) &= \left[\frac{k_2^3 - 2k_2 + \frac{1}{k_2}}{(\frac{1}{k_2} - k_2)^2} \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \left[\frac{\frac{k_2^4 - 2k_2^2 + 1}{k_2}}{\frac{(1 - k_2^2)^2}{k_2^2}} \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \left(k_2 \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right) \delta_{ij}.
\end{aligned}$$

Then, for $m + 1 \leq i, j \leq n$, one concludes that

$$\begin{aligned}
\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) &= \lim_{t \rightarrow -\infty} \left[k_2 \sqrt{q(t)} + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \lim_{t \rightarrow -\infty} \left[k_2 \sqrt{\left(\frac{k_1 - k_2}{2} \right)^2 + e^{2t/m}} + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \left[k_2 \left(\frac{k_1 - k_2}{2} \right) + \frac{1 - k_2^2}{2} \right] \delta_{ij} \\
&= \left(\frac{1 - k_2^2}{2} + \frac{1 - k_2^2}{2} \right) \delta_{ij} \\
&= (1 - k_2^2) \delta_{ij} \\
&= k_2(k_1 - k_2) \delta_{ij}.
\end{aligned}$$

Therefore, it follows from Lemma 4.1.1, (4.28) and (4.30), we have that $\hat{F}^t(\mathbb{S}^m \times \mathbb{H}^m)$ is defined for $t \in \mathbb{R}$, it is an eternal solution that expands from an m -dimensional submanifold of $\mathbb{H}^{n+1}$ and converges to $\partial\mathbb{H}^{n+1}$ when $t \rightarrow +\infty$. So, the proof is finished. $\blacksquare$

Example 4.4.2. Consider the hyperbolic cylinder in $\mathbb{H}^3 \subset \mathbb{L}^4$ given by

$$F(\theta, \varphi) = (\sqrt{2} \cosh \varphi, \cos \theta, \sin \theta, \sqrt{2} \sinh \varphi).$$

We get

$$F_\theta(\theta, \varphi) = (0, -\sin \theta, \cos \theta, 0)$$

and

$$F_\varphi(\theta, \varphi) = (\sqrt{2} \sinh \varphi, 0, 0, \sqrt{2} \cosh \varphi).$$

So, the unit normal vector field N of F is given by

$$N(\theta, \varphi) = (-\cosh \varphi, -\sqrt{2} \cos \theta, -\sqrt{2} \sin \theta, -\sinh \varphi).$$

Moreover, we infer

$$|F_\theta(\theta, \varphi)|^2 = 1, \quad |F_\varphi(\theta, \varphi)|^2 = 2, \quad \langle F_\theta, F_\varphi \rangle = 0,$$

$$F_{\theta\theta}(\theta, \varphi) = (0, -\cos \theta, -\sin \theta, 0)$$

and

$$F_{\varphi\varphi}(\theta, \varphi) = (\sqrt{2} \cosh \varphi, 0, 0, \sqrt{2} \sinh \varphi).$$

Thus,

$$\langle N, F_{\theta\theta} \rangle = \sqrt{2} \quad \text{and} \quad \langle N, F_{\varphi\varphi} \rangle = \sqrt{2}.$$

Hence, the principal curvatures k_1 and k_2 of F are given by

$$k_1 = \frac{1}{|F_\theta|^2} \langle N, F_{\theta\theta} \rangle = \sqrt{2} \quad \text{and} \quad k_2 = \frac{1}{|F_\varphi|^2} \langle N, F_{\varphi\varphi} \rangle = \frac{\sqrt{2}}{2}.$$

Therefore, from (4.21) and (4.22), we get

$$\begin{aligned} \cosh(\mu(t)) &= \left(\frac{-2e^t + (k_1 + k_2) \sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{1}{2} \right)^{1/2} \\ &= \left(\frac{-2e^t + \frac{3\sqrt{2}}{2} \sqrt{\frac{1}{8} + e^{2t}}}{\frac{1}{2}} + \frac{1}{2} \right)^{1/2} \\ &= \left(-4e^t + \frac{3}{2} \sqrt{1 + 8e^{2t}} + \frac{1}{2} \right)^{1/2}, \end{aligned}$$

$$\begin{aligned}
\sinh(\mu(t)) &= -\operatorname{sgn}(t) \left(\frac{-2e^t + (k_1 + k_2) \sqrt{q(t)}}{(k_1 - k_2)^2} - \frac{1}{2} \right)^{1/2} \\
&= -\operatorname{sgn}(t) \left(-4e^t + \frac{3}{2} \sqrt{1 + 8e^{2t}} - \frac{1}{2} \right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\hat{F}^t(\theta, \varphi) &= \cosh(\mu(t))F(\theta, \varphi) + \sinh(\mu(t))N(\theta, \varphi), \\
&= \cosh(\mu(t)) \left(\sqrt{2} \cosh \varphi, \cos \theta, \sin \theta, \sqrt{2} \sinh \varphi \right) \\
&\quad - \sinh(\mu(t)) \left(\cosh \varphi, \sqrt{2} \cos \theta, \sqrt{2} \sin \theta, \sinh \varphi \right) \\
&= (\hat{F}_1^t(\theta, \varphi), \hat{F}_2^t(\theta, \varphi), \hat{F}_3^t(\theta, \varphi), \hat{F}_4^t(\theta, \varphi)),
\end{aligned}$$

where

$$\begin{aligned}
\hat{F}_1^t(\theta, \varphi) &= [\sqrt{2} \cosh(\mu(t)) - \sinh(\mu(t))] \cosh \varphi, \\
\hat{F}_2^t(\theta, \varphi) &= [\cosh(\mu(t)) - \sqrt{2} \sinh(\mu(t))] \cos \theta, \\
\hat{F}_3^t(\theta, \varphi) &= [\cosh(\mu(t)) - \sqrt{2} \sinh(\mu(t))] \sin \theta
\end{aligned}$$

and

$$\hat{F}_4^t(\theta, \varphi) = [\sqrt{2} \cosh(\mu(t)) - \sinh(\mu(t))] \sinh \varphi.$$

Using the isometry $G : \mathbb{H}^3 \subset \mathbb{L}^4 \longrightarrow \mathbb{B}^3 \subset \mathbb{R}^3$ given by

$$G(y_1, y_2, y_3, y_4) = \frac{1}{1 + y_1} (y_2, y_3, -y_4),$$

we deduce that

$$\begin{aligned}
G(\hat{F}^t(\theta, \varphi)) &= G(\hat{F}_1^t(\theta, \varphi), \hat{F}_2^t(\theta, \varphi), \hat{F}_3^t(\theta, \varphi), \hat{F}_4^t(\theta, \varphi)) \\
&= \frac{1}{1 + \hat{F}_1^t(\theta, \varphi)} (\hat{F}_2^t(\theta, \varphi), \hat{F}_3^t(\theta, \varphi), -\hat{F}_4^t(\theta, \varphi)).
\end{aligned}$$

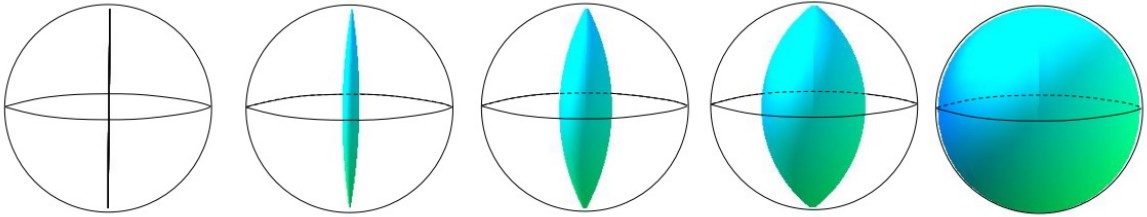


Figure 4.2: IMCF of the hyperbolic cylinder

Chapter 5

On the inverse mean curvature flow by parallel hypersurfaces in the sphere

In this chapter we use Corollary 2.1.2 to study the solutions to the IMCF by parallel hypersurfaces in $\mathbb{S}^{n+1}$ considering as initial condition the isoparametric hypersurfaces of the sphere $\mathbb{S}^{n+1}$. Münzner [25] showed that the number $\mathcal{G}$ of distinct principal curvatures, for an isoparametric hypersurface of $\mathbb{S}^{n+1}$, is restricted to be 1, 2, 3, 4 or 6. When $\mathcal{G} \in \{2, 4\}$, the multiplicities of principal curvatures can be different. For these cases, in studying the solutions to the IMCF by parallel hypersurfaces in $\mathbb{S}^{n+1}$, we additionally assume that the multiplicities of the principal curvatures of the initial hypersurface are the same. Under these conditions, we prove that, for all values of $\mathcal{G}$, the solutions to the IMCF by parallel hypersurfaces in $\mathbb{S}^{n+1}$ are ancient. When $\mathcal{G} = 1$, we prove that the solution expands from a point and converges to a totally geodesic hypersurface of $\mathbb{S}^{n+1}$. For the cases where $\mathcal{G} \in \{2, 3, 4, 6\}$, if m is the multiplicity of principal curvatures of the initial hypersurface, we prove that the solutions expands from a $(\mathcal{G} - 1)m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$.

5.1 The totally umbilical case

Theorem 5.1.1. *Let $F : M^n \longrightarrow \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ be an isometric immersion of a totally umbilical hypersurface in $\mathbb{S}^{n+1}$, with inward unit normal vector field N and all principal curvatures equal to $k > 0$. Then, the solution to the IMCF by parallel hypersurface with initial condition $F(M)$ is given by*

$$\hat{F}^t(x) = \frac{e^{t/n} + k\sqrt{q(t)}}{k^2 + 1}F(x) + \frac{-ke^{t/n} + \sqrt{q(t)}}{k^2 + 1}N(x), \quad (5.1)$$

$\forall x \in M$, where $q(t) = 1 + k^2 - e^{2t/n}$, $t \in (-\infty, t^*)$ and $t^* = \frac{n}{2} \ln(k^2 + 1)$. Moreover, $\hat{F}^t(M)$ is an ancient solution that expands from a point and converges to a totally geodesic hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$.

Proof. From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x), \quad (5.2)$$

$\forall (x, t) \in M \times I$, where

$$\prod_{i=1}^n |\cos(\mu(t)) - k_i \sin(\mu(t))| = e^t.$$

So, since $k_i = k > 0$, $\forall i$ with $1 \leq i \leq n$, we get

$$|\cos(\mu(t)) - k \sin(\mu(t))| = e^{t/n}, \quad (5.3)$$

$\forall t \in I$. Since $e^{t/n}$ is a smooth positive function, from (5.3) we conclude that

$$h(t) = \cos(\mu(t)) - k \sin(\mu(t))$$

does not change sign. We claim that $h(t) > 0$, $\forall t \in I$. In fact, since $\mu(0) = 0$, we have that $h(0) = 1$. Hence, $h(t) > 0$, $\forall t \in I$. Thus, it follows from (5.3) that

$$\cos(\mu(t)) - k \sin(\mu(t)) = e^{t/n}, \quad (5.4)$$

i.e.,

$$k \sin(\mu(t)) = \cos(\mu(t)) - e^{t/n}. \quad (5.5)$$

Taking the square on both sides of (5.5) and using that $\sin^2(\mu(t)) + \cos^2(\mu(t)) = 1$, we get the following equation

$$(k^2 + 1) \cos^2(\mu(t)) - 2e^{t/n} \cos(\mu(t)) + e^{2t/n} - k^2 = 0. \quad (5.6)$$

This is a second degree equation for $\cos(\mu(t))$. The discriminant of (5.6) is given by

$$\Delta = 4k^2(1 + k^2 - e^{2t/n}).$$

So, $\Delta \geq 0$ if and only if

$$q(t) := 1 + k^2 - e^{2t/n} \geq 0,$$

i.e.,

$$e^{2t/n} \leq k^2 + 1,$$

which is equivalent to

$$t \leq \frac{n}{2} \ln(k^2 + 1) = t^*.$$

Thus,

$$\cos(\mu(t)) = \frac{2e^{t/n} \pm 2k\sqrt{q(t)}}{2(k^2 + 1)} = \frac{e^{t/n} \pm k\sqrt{q(t)}}{k^2 + 1}, \quad (5.7)$$

$\forall t \leq t^*$. Since $\mu(0) = 0$, we obtain from (5.7) that

$$\cos(\mu(t)) = \frac{e^{t/n} + k\sqrt{q(t)}}{k^2 + 1}, \quad (5.8)$$

$\forall t \leq t^*$. Moreover, substituting (5.8) into (5.5), we get

$$k \sin(\mu(t)) = \frac{e^{t/n} + k\sqrt{q(t)}}{k^2 + 1} - e^{t/n} = \frac{-k^2 e^{t/n} + k\sqrt{q(t)}}{k^2 + 1},$$

i.e.,

$$\sin(\mu(t)) = \frac{-k e^{t/n} + \sqrt{q(t)}}{k^2 + 1}, \quad (5.9)$$

$\forall t \leq t^*$. Then, substituting (5.8) and (5.9) into (5.2), we arrive at (5.1).

Proceeding from (5.8), we have that $\cos(\mu(t)) > 0$, $\forall t \leq t^*$. Thus, from (5.4), we conclude that $\tan(\mu(t)) \neq 1/k$, $\forall t \leq t^*$. Hence, from Lemma 1.3.2, the mean curvature $\hat{H}^t$ of $\hat{F}^t(M)$ is given by

$$\hat{H}^t = \sum_{i=1}^n \frac{\sin(\mu(t)) + k_i \cos(\mu(t))}{\cos(\mu(t)) - k_i \sin(\mu(t))} = n \frac{\sin(\mu(t)) + k \cos(\mu(t))}{\cos(\mu(t)) - k \sin(\mu(t))} = n \frac{\tan(\mu(t)) + k}{1 - k \tan(\mu(t))},$$

$\forall t \leq t^*$. Therefore, $\hat{H}^t = 0$ if and only if $\tan(\mu(t)) = -k$, i.e., from (5.8) and (5.9),

$$\frac{-k e^{t/n} + \sqrt{q(t)}}{e^{t/n} + k\sqrt{q(t)}} = -k,$$

i.e.,

$$(k^2 + 1)\sqrt{q(t)} = 0.$$

Since $q(t) = 0$ if and only if $t = t^*$, we conclude that $I = (-\infty, t^*)$.

Now, consider $\{E_j\}_{j=1}^n$ be a local orthonormal frame of principal directions of $F(M) \subset$

$\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ in $x \in M$. It follows from Lemma 1.3.2 that

$$\hat{g}_x^t(E_i, E_j) = [\cos(\mu(t)) - k \sin(\mu(t))]^2 \delta_{ij} = (e^{t/n})^2 \delta_{ij} = e^{2t/n} \delta_{ij},$$

$\forall t \leq t^*$. So,

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) = 0. \quad (5.10)$$

On the other hand, observe that

$$\hat{g}_x^{t^*}(E_i, E_j) = e^{2t^*/n} \delta_{ij} = \delta_{ij} \exp\left(\frac{2}{n} \left(\frac{n}{2} \ln(k^2 + 1)\right)\right) = (k^2 + 1) \delta_{ij}.$$

Moreover, we have from Lemma 1.3.2 that the second fundamental form $\hat{I}I_x^t$ of $\hat{F}^t(x)$, $x \in M$, is given by

$$\begin{aligned} \hat{I}I_x^t(E_i, E_j) &= [\cos(\mu(t)) - k \sin(\mu(t))] [\sin(\mu(t)) + k \cos(\mu(t))] \delta_{ij} \\ &= \delta_{ij} e^{t/n} \left[\frac{-k e^{t/n} + \sqrt{q(t)}}{k^2 + 1} + k \frac{e^{t/n} + k \sqrt{q(t)}}{k^2 + 1} \right] \\ &= \delta_{ij} e^{t/n} \sqrt{q(t)}. \end{aligned}$$

Hence, since $q(t^*) = 0$,

$$\hat{I}I_x^{t^*}(E_i, E_j) = 0. \quad (5.11)$$

Thus, from (5.10) and (5.11), $\hat{F}^t(M)$ is defined for $t \in (-\infty, t^*)$, it is an ancient solution that expands from a point and converges to a totally geodesic hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$. This finishes the proof. $\blacksquare$

The case $\mathcal{G} = 1$, in Theorem 5.1.1, was also studied by Freitas and Barros in the Thesis [14]. The following observation will be very useful in the next results.

Remark 5.1.2. Let $F : M^n \rightarrow \mathbb{S}^{n+1}$ be an oriented isoparametric hypersurface with inward unit normal vector field N and $k_1, k_2, \dots, k_n$ principal curvatures of $F(M)$ associated to N . One can determine the principal curvatures of $M^n \subset \mathbb{S}^{n+1}$ up to a constant. In fact, for all $\mathcal{G} = 2, 3, 4, 6$, let $\tau \in \mathbb{R}$ with $-1 < \tau < 1$ and $\tau = \cos(\mathcal{G}s)$, i.e., $0 < s < \frac{\pi}{\mathcal{G}}$, we consider

$$k_j = \cot\left(s + \frac{j-1}{\mathcal{G}}\pi\right), \quad j = 1, \dots, \mathcal{G}. \quad (5.12)$$

The hypersurfaces for different values of the constant τ are parallel in $\mathbb{S}^{n+1}$ (see [7]).

5.2 Two distinct principal curvatures

In this section, we study the solutions to the IMCF by parallel hypersurfaces considering as initial condition the isoparametric hypersurfaces of the sphere $\mathbb{S}^{n+1}$ with two distinct principal curvatures (i.e., $\mathcal{G} = 2$). In this case, the multiplicities of the principal curvatures can be different. In the following proposition, we additionally assume that the multiplicities of principal curvatures of the initial hypersurface are equal.

Theorem 5.2.1. *Let $F : \mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m \longrightarrow \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ be an isoparametric hypersurface in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$, with inward unit normal vector field N and two distinct principal curvatures $k_1 > 1$ and k_2 with the same multiplicity m . Then, $k_1 k_2 = -1$ and the solution to the IMCF by parallel hypersurfaces with initial condition $F(\mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m)$ is given by*

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall x \in \mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m$, where

$$\cos(\mu(t)) = \sqrt{\frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}} + \frac{1}{2} \quad (5.13)$$

and

$$\sin(\mu(t)) = -\operatorname{sgn}(t) \sqrt{\frac{1}{2} - \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}}, \quad (5.14)$$

with $q(t) = (k_1 - k_2)^2 - 4e^{2t/m}$, $t \in (-\infty, t^*)$ and $t^* = m \ln\left(\frac{k_1 - k_2}{2}\right)$. Moreover, $\hat{F}^t(\mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m)$ is an ancient solution that expands from an m -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \longrightarrow t^*$.

Proof. From Remark 5.1.2, we have $k_1 = \cot(s) > 0$, where $0 < s < \pi/2$, and

$$k_2 = \cot\left(s + \frac{\pi}{2}\right) = \frac{\cot(s) \cdot \cot(\pi/2) - 1}{\cot(s) + \cot(\pi/2)} = -\frac{1}{k_1},$$

i.e., $k_1 k_2 = -1$. From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall (x, t) \in \mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m \times I$, where, from (2.7),

$$\prod_{i=1}^n |\cos(\mu(t)) - k_i \sin(\mu(t))| = e^t.$$

Thus, since the principal curvatures k_1 and k_2 have the same multiplicity m ,

$$|[\cos(\mu(t)) - k_1 \sin(\mu(t))][\cos(\mu(t)) - k_2 \sin(\mu(t))]| = e^{t/m}. \quad (5.15)$$

So, $\forall t \in I$,

$$|\cos^2(\mu(t)) - (k_1 + k_2) \sin(\mu(t)) \cos(\mu(t)) + k_1 k_2 \sin^2(\mu(t))| = e^{t/m}$$

and, hence, using $k_1 k_2 = -1$, we have

$$\left| \cos(2\mu(t)) - \frac{k_1 + k_2}{2} \sin(2\mu(t)) \right| = e^{t/m}. \quad (5.16)$$

Since $e^{t/m}$ is a smooth positive function, we conclude from (5.16) that $h : I \rightarrow \mathbb{R}$ given by

$$h(t) = \cos(2\mu(t)) - \frac{k_1 + k_2}{2} \sin(2\mu(t))$$

does not change sign. We now claim that $h(t) > 0$, $\forall t \in I$. In fact, since $\mu(0) = 0$, we have $h(0) = 1$. So $h(t) > 0$, $\forall t \in I$. Thus, from (5.16),

$$\cos(2\mu(t)) - \frac{k_1 + k_2}{2} \sin(2\mu(t)) = e^{t/m}, \quad (5.17)$$

i.e., $\forall t \in I$,

$$(k_1 + k_2) \sin(2\mu(t)) = 2 \cos(2\mu(t)) - 2e^{t/m}. \quad (5.18)$$

Taking the square on both sides of (5.18) and using that $\sin^2(2\mu(t)) + \cos^2(2\mu(t)) = 1$, we get the following equation

$$[(k_1 + k_2)^2 + 4] \cos^2(2\mu(t)) - 8e^{t/m} \cos(2\mu(t)) + [4e^{2t/m} - (k_1 + k_2)^2] = 0, \quad (5.19)$$

$\forall t \in I$. This is a second degree equation for $\cos(2\mu(t))$. Using $k_1 k_2 = -1$, the discriminant of (5.19) is given by

$$\begin{aligned} \Delta &= 64e^{2t/m} - 4[(k_1 + k_2)^2 + 4] \cdot [4e^{2t/m} - (k_1 + k_2)^2] \\ &= 64e^{2t/m} - 4[4e^{2t/m}(k_1 + k_2)^2 - (k_1 + k_2)^4 + 16e^{2t/m} - 4(k_1 + k_2)^2] \\ &= -4[4e^{2t/m}(k_1 + k_2)^2 - (k_1 + k_2)^4 - 4(k_1 + k_2)^2] \\ &= -4(k_1 + k_2)^2 [4e^{2t/m} - (k_1 + k_2)^2 - 4] \\ &= 4(k_1 + k_2)^2 [(k_1 + k_2)^2 + 4 - 4e^{2t/m}] \\ &= 4(k_1 + k_2)^2 [(k_1 - k_2)^2 - 4e^{2t/m}]. \end{aligned}$$

Define $q : I \longrightarrow \mathbb{R}$ by

$$q(t) = (k_1 - k_2)^2 - 4e^{2t/m}.$$

Hence, $\Delta \geq 0$ if and only if $q(t) \geq 0$, i.e.,

$$e^{2t/m} \leq \left(\frac{k_1 - k_2}{2} \right)^2$$

i.e.,

$$t \leq m \ln \left(\frac{k_1 - k_2}{2} \right) = t^*.$$

Using that $\mu(0) = 0$, $k_1 k_2 = -1$ and $k_1 > 1$, we have that $H = m(k_1 + k_2) > 0$ and, from (5.19),

$$\cos(2\mu(t)) = \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 + k_2)^2 + 4} = \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2}, \quad (5.20)$$

$\forall t \leq t^*$. Note that $\cos(2\mu(t)) > 0$, $\forall t \leq t^*$. Since

$$\cos^2(\mu(t)) = \frac{1}{2} \cos(2\mu(t)) + \frac{1}{2},$$

it follows from (5.20) that

$$\cos^2(\mu(t)) = \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2} + \frac{1}{2}, \quad (5.21)$$

i.e.,

$$\cos(\mu(t)) = \sqrt{\frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2} + \frac{1}{2}}, \quad (5.22)$$

$\forall t \leq t^*$. By (5.21) one sees that

$$\begin{aligned} \sin^2(\mu(t)) &= 1 - \cos^2(\mu(t)) \\ &= 1 - \left[\frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2} + \frac{1}{2} \right] \\ &= \frac{1}{2} - \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}, \end{aligned}$$

$\forall t \leq t^*$. Hence,

$$\sin(\mu(t)) = \pm \sqrt{\frac{1}{2} - \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}}, \quad (5.23)$$

$\forall t \leq t^*$. Thus, we need to determine the sign of $\sin(\mu(t))$. Since

$$\sin(2\mu(t)) = 2 \sin(\mu(t)) \cos(\mu(t))$$

and $\cos(\mu(t)) > 0, \forall t \leq t^*$, we need to study the sign of $\sin(2\mu(t))$. From (5.18) and (5.20),

$$\begin{aligned} (k_1 + k_2) \sin(2\mu(t)) &= 2 \left[\frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{(k_1 - k_2)^2} - e^{t/m} \right] \\ &= 2 \left[\frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)} - (k_1 - k_2)^2 e^{t/m}}{(k_1 - k_2)^2} \right] \\ &= 2 \left[\frac{(k_1 + k_2)\sqrt{q(t)} - (k_1 + k_2)^2 e^{t/m}}{(k_1 - k_2)^2} \right] \\ &= 2(k_1 + k_2) \left[\frac{\sqrt{q(t)} - (k_1 + k_2)e^{t/m}}{(k_1 - k_2)^2} \right]. \end{aligned}$$

Thus,

$$\sin(2\mu(t)) = 2 \frac{\sqrt{q(t)} - (k_1 + k_2)e^{t/m}}{(k_1 - k_2)^2}, \quad (5.24)$$

$\forall t \leq t^*$. It follows that $\sin(2\mu(t)) > 0$ if and only if

$$\sqrt{q(t)} > (k_1 + k_2)e^{t/m},$$

i.e.,

$$q(t) > (k_1 + k_2)^2 e^{2t/m},$$

that is equivalent to

$$4 + (k_1 + k_2)^2 - 4e^{2t/m} > (k_1 + k_2)^2 e^{2t/m}.$$

Consequently,

$$4 + (k_1 + k_2)^2 > [4 + (k_1 + k_2)^2] e^{2t/m}.$$

Hence, $\sin(2\mu(t)) > 0$ if and only if $e^{2t/m} < 1$, i.e., $2t/m < 0$ and it is equivalent to $t < 0$.

Thus,

$$\operatorname{sgn}(\sin(\mu(t))) = \operatorname{sgn}(\sin(2\mu(t))) = -\operatorname{sgn}(t).$$

Therefore, from (5.23),

$$\sin(\mu(t)) = -\operatorname{sgn}(t) \sqrt{\frac{1}{2} - \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}}, \quad (5.25)$$

$\forall t \leq t^*$.

From (5.15), we have that

$$\cos^2(\mu(t))[1 - k_1 \tan(\mu(t))][1 - k_2 \tan(\mu(t))] = e^{t/m},$$

$\forall t \leq t^*$. Thus,

$$\tan(\mu(t)) \neq \frac{1}{k_1} = -k_2 \text{ and } \tan(\mu(t)) \neq \frac{1}{k_2} = -k_1, \quad (5.26)$$

$\forall t \leq t^*$. Hence, by using (5.17) jointly Lemma 1.3.2, one concludes that the mean curvature $\hat{H}^t$ of $\hat{F}^t(\mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m)$ satisfies

$$\frac{1}{n} e^{t/m} \hat{H}^t = \sin(2\mu(t)) + \frac{k_1 + k_2}{2} \cos(2\mu(t)) = \cos(2\mu(t)) \left[\tan(2\mu(t)) + \frac{k_1 + k_2}{2} \right].$$

Therefore, since $\cos(2\mu(t)) > 0$, $\forall t \leq t^*$, from (5.20) and (5.24), we have that $\hat{H}^t = 0$ if and only if

$$\frac{2\sqrt{q(t)} - 2(k_1 + k_2)e^{t/m}}{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}} = -\frac{k_1 + k_2}{2},$$

i.e., $\hat{H}^t = 0$ if and only if $q(t) = 0$. Hence, $\hat{H}^t = 0$ if, and only if, $t = t^*$. We conclude that $\hat{H}^{t^*} = 0$ and $I = (-\infty, t^*)$.

Consider an orthonormal frame $\{E_1, E_2, \dots, E_m, E_{m+1}, \dots, E_{2m=n}\}$ of principal directions on $F(\mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m) \subset \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ such that $E_1, \dots, E_m$ are tangent to $\mathbb{S}_{r_1}^m$ and $E_{m+1}, \dots, E_n$ are tangent to $\mathbb{S}_{r_2}^m$. From Lemma 1.3.2, we have

$$\hat{g}_x^t(E_i, E_j) = [\cos(\mu(t)) - k_i \sin(\mu(t))]^2 \delta_{ij} = \cos^2(\mu(t)) [1 - k_i \tan(\mu(t))]^2 \delta_{ij}, \quad (5.27)$$

$\forall t \leq t^*$ and $\forall i, j$, where $1 \leq i, j \leq n$. Then, from (5.26), $\hat{g}_x^t(E_l, E_l) \neq 0$, $\forall t \leq t^*$ and $\forall l$, where $1 \leq l \leq n$. Moreover, from Lemma 1.3.2, the second fundamental form is given by

$$\begin{aligned} \hat{I}_x^t(E_i, E_j) &= [\cos(\mu(t)) - k_i \sin(\mu(t))] [\sin(\mu(t)) + k_i \cos(\mu(t))] \delta_{ij} \\ &= [\cos(\mu(t)) (1 - k_i \tan(\mu(t)))] [\cos(\mu(t)) (\tan(\mu(t)) + k_i)] \delta_{ij} \\ &= \cos^2(\mu(t)) [1 - k_i \tan(\mu(t))] [\tan(\mu(t)) + k_i] \delta_{ij}, \end{aligned} \quad (5.28)$$

$\forall t \leq t^*$ and $\forall i, j$, where $1 \leq i, j \leq n$. Thus, from (5.26), $\hat{I}_x^t(E_l, E_l) \neq 0$, $\forall t \leq t^*$ and $\forall l$, where $1 \leq l \leq n$.

On the other hand, by (5.22) and (5.25), one deduces that

$$\begin{aligned}
\lim_{t \rightarrow -\infty} \sin(\mu(t)) &= \lim_{t \rightarrow -\infty} \left[-\operatorname{sgn}(t) \sqrt{\frac{1}{2} - \frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}} \right] \\
&= \sqrt{\frac{1}{2} - \frac{(k_1 + k_2)(k_1 - k_2)}{2(k_1 - k_2)^2}} \\
&= \sqrt{\frac{1}{2} - \frac{k_1 + k_2}{2(k_1 - k_2)}} \\
&= \frac{1}{\sqrt{1 + k_1^2}}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t \rightarrow -\infty} \cos(\mu(t)) &= \lim_{t \rightarrow -\infty} \left[\sqrt{\frac{4e^{t/m} + (k_1 + k_2)\sqrt{q(t)}}{2(k_1 - k_2)^2}} + \frac{1}{2} \right] \\
&= \sqrt{\frac{k_1 + k_2}{2(k_1 - k_2)}} + \frac{1}{2} \\
&= \frac{k_1}{\sqrt{1 + k_1^2}}.
\end{aligned}$$

It follows that

$$\lim_{t \rightarrow -\infty} \tan(\mu(t)) = \frac{1}{k_1}.$$

Then, if $1 \leq i, j \leq m$, from (5.27), we have

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) = \lim_{t \rightarrow -\infty} \cos^2(\mu(t)) [1 - k_1 \tan(\mu(t))]^2 \delta_{ij} = 0. \quad (5.29)$$

Moreover, if $m + 1 \leq i, j \leq n$, we deduce from (5.27) that

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_i, E_j) = \lim_{t \rightarrow -\infty} \cos^2(\mu(t)) [1 - k_2 \tan(\mu(t))]^2 \delta_{ij} = \frac{1 + k_1^2}{k_1^2} \delta_{ij}.$$

Therefore, from (5.29), $\hat{F}^t(\mathbb{S}_{r_1}^m \times \mathbb{S}_{r_2}^m)$ is defined for $t \in (-\infty, t^*)$, it is an ancient solution that expands from an m -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$. ■

Example 5.2.2. Consider the Hopf Torus in $\mathbb{S}^3 \subset \mathbb{R}^4$ given by

$$F(\theta, \varphi) = \left(\frac{\sqrt{6}}{3} \cos \varphi, \frac{\sqrt{6}}{3} \sin \varphi, \frac{\sqrt{3}}{3} \cos \theta, \frac{\sqrt{3}}{3} \sin \theta \right).$$

We then get

$$F_\theta(\theta, \varphi) = \left(0, 0, -\frac{\sqrt{3}}{3} \sin \theta, \frac{\sqrt{3}}{3} \cos \theta \right)$$

and

$$F_\varphi(\theta, \varphi) = \left(-\frac{\sqrt{6}}{3} \sin \varphi, \frac{\sqrt{6}}{3} \cos \varphi, 0, 0 \right).$$

So, the unit normal vector field N of F is given by

$$N(\theta, \varphi) = \left(\frac{\sqrt{3}}{3} \cos \varphi, \frac{\sqrt{3}}{3} \sin \varphi, -\frac{\sqrt{6}}{3} \cos \theta, -\frac{\sqrt{6}}{3} \sin \theta \right).$$

Moreover,

$$|F_\theta|^2 = \frac{1}{3}, \quad |F_\varphi|^2 = \frac{2}{3} \quad \text{and} \quad \langle F_\theta, F_\varphi \rangle = 0.$$

Since

$$F_{\theta\theta}(\theta, \varphi) = \left(0, 0, -\frac{\sqrt{3}}{3} \cos \theta, -\frac{\sqrt{3}}{3} \sin \theta \right)$$

and

$$F_{\varphi\varphi}(\theta, \varphi) = \left(-\frac{\sqrt{6}}{3} \cos \varphi, -\frac{\sqrt{6}}{3} \sin \varphi, 0, 0 \right),$$

we have that

$$\langle N, F_{\theta\theta} \rangle = \frac{\sqrt{2}}{3} \quad \text{and} \quad \langle N, F_{\varphi\varphi} \rangle = -\frac{\sqrt{2}}{3}.$$

Hence, the principal curvatures k_1 and k_2 of F are given by

$$k_1 = \frac{1}{|F_\theta|^2} \langle N, F_{\theta\theta} \rangle = \sqrt{2} \quad \text{and} \quad k_2 = \frac{1}{|F_\varphi|^2} \langle N, F_{\varphi\varphi} \rangle = -\frac{\sqrt{2}}{2}.$$

Therefore, we get from (5.13) and (5.14) that

$$\cos(\mu(t)) = \left(\frac{-2e^t + (k_1 + k_2) \sqrt{q(t)}}{(k_1 - k_2)^2} + \frac{1}{2} \right)^{1/2} = \left(\frac{8e^t + \sqrt{9 - 8e^{2t}}}{18} + \frac{1}{2} \right)^{1/2},$$

$$\sin(\mu(t)) = -\operatorname{sgn}(t) \left(\frac{1}{2} - \frac{-2e^t + (k_1 + k_2) \sqrt{q(t)}}{(k_1 - k_2)^2} \right)^{1/2} = -\operatorname{sgn}(t) \left(\frac{1}{2} - \frac{8e^t + \sqrt{9 - 8e^{2t}}}{18} \right)^{1/2},$$

and

$$\begin{aligned} \hat{F}^t(\theta, \varphi) &= \cos(\mu(t))F(\theta, \varphi) + \sin(\mu(t))N(\theta, \varphi), \\ &= \cos(\mu(t)) \left(\frac{\sqrt{6}}{3} \cos \varphi, \frac{\sqrt{6}}{3} \sin \varphi, \frac{\sqrt{3}}{3} \cos \theta, \frac{\sqrt{3}}{3} \sin \theta \right) \\ &+ \sin(\mu(t)) \left(\frac{\sqrt{3}}{3} \cos \varphi, \frac{\sqrt{3}}{3} \sin \varphi, -\frac{\sqrt{6}}{3} \cos \theta, -\frac{\sqrt{6}}{3} \sin \theta \right) \\ &= (\hat{F}_1^t(\theta, \varphi), \hat{F}_2^t(\theta, \varphi), \hat{F}_3^t(\theta, \varphi), \hat{F}_4^t(\theta, \varphi)), \end{aligned}$$

where

$$\begin{aligned} \hat{F}_1^t(\theta, \varphi) &= \frac{1}{3}[\sqrt{6} \cos(\mu(t)) + \sqrt{3} \sin(\mu(t))] \cos \varphi, \\ \hat{F}_2^t(\theta, \varphi) &= \frac{1}{3}[\sqrt{6} \cos(\mu(t)) + \sqrt{3} \sin(\mu(t))] \sin \varphi, \\ \hat{F}_3^t(\theta, \varphi) &= \frac{1}{3}[\sqrt{3} \cos(\mu(t)) - \sqrt{6} \sin(\mu(t))] \cos \theta \end{aligned}$$

and

$$\hat{F}_4^t(\theta, \varphi) = \frac{1}{3}[\sqrt{3} \cos(\mu(t)) - \sqrt{6} \sin(\mu(t))] \sin \theta.$$

Using the stereographic projection $\pi : \mathbb{S}^3 - \{(0, 0, 0, 1)\} \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ given by

$$\pi(y_1, y_2, y_3, y_4) = \frac{1}{1 - y_4} (y_1, y_2, y_3),$$

we have that

$$\begin{aligned} \pi(\hat{F}^t(\theta, \varphi)) &= \pi(\hat{F}_1^t(\theta, \varphi), \hat{F}_2^t(\theta, \varphi), \hat{F}_3^t(\theta, \varphi), \hat{F}_4^t(\theta, \varphi)) \\ &= \frac{1}{1 - \hat{F}_4^t(\theta, \varphi)} (\hat{F}_1^t(\theta, \varphi), \hat{F}_2^t(\theta, \varphi), \hat{F}_3^t(\theta, \varphi)). \end{aligned}$$

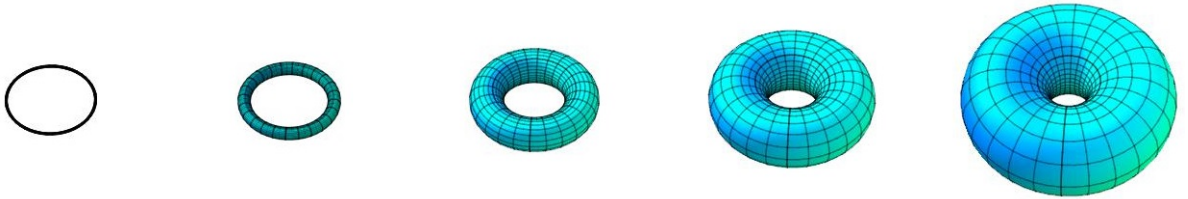


Figure 5.1: IMCF of the Hopf Torus

$\hat{F}^t$ expands from a curve of $\mathbb{S}^3$ and converges to a minimal hypersurface of $\mathbb{S}^3$ when $t \longrightarrow \ln\left(\frac{3\sqrt{2}}{4}\right)$.

5.3 Three distinct principal curvatures

Theorem 5.3.1. *Let $F : M^n \longrightarrow \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ be an isoparametric hypersurface in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$, with inward unit normal vector field N and three distinct principal curvatures k_1, k_2 and k_3 . Moreover, we may consider*

$$k_1 > \sqrt{3}, \quad k_2 = \frac{\sqrt{3}k_1 - 3}{3k_1 + \sqrt{3}} \quad \text{and} \quad k_3 = \frac{\sqrt{3}k_1 + 3}{\sqrt{3} - 3k_1},$$

all with multiplicity m , i.e., $n = 3m$, where $m \in \{1, 2, 4, 8\}$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(M)$ is given by

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall x \in M$, where

$$\cos(3\mu(t)) = \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \quad \sin(3\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{a^2 + 1},$$

$$q(t) = a^2 + 1 - e^{2t/m} \quad \text{and} \quad a = \frac{k_1 + k_2 + k_3}{3} = \frac{k_1(k_1^2 - 3)}{3k_1^2 - 1},$$

with $t \in (-\infty, t^*)$ and $t^* = \frac{m}{2} \ln(a^2 + 1)$. Moreover, $\hat{F}^t(M)$ is an ancient solution that expands from a $2m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \longrightarrow t^*$.

Proof. From Remark 5.1.2, we have $k_1 = \cot(s)$, where $0 < s < \pi/3$, and

$$k_2 = \cot\left(s + \frac{\pi}{3}\right) = \frac{\cot(s) \cdot \cot(\pi/3) - 1}{\cot(s) + \cot(\pi/3)} = \frac{k_1 \cdot \frac{\sqrt{3}}{3} - 1}{k_1 + \frac{\sqrt{3}}{3}} = \frac{\sqrt{3}k_1 - 3}{3k_1 + \sqrt{3}},$$

$$k_3 = \cot\left(s + \frac{2\pi}{3}\right) = \frac{\cot(s) \cdot \cot(2\pi/3) - 1}{\cot(s) + \cot(2\pi/3)} = \frac{k_1 \cdot (-\frac{\sqrt{3}}{3}) - 1}{k_1 + (-\frac{\sqrt{3}}{3})} = \frac{\sqrt{3}k_1 + 3}{\sqrt{3} - 3k_1},$$

where $k_3 < 0 < k_2 < k_1$, since $k_1 > \sqrt{3}$. Thus,

$$k_1 + k_2 + k_3 = \frac{3k_1(k_1^2 - 3)}{3k_1^2 - 1}, \quad k_1 k_2 k_3 = -\frac{k_1(k_1^2 - 3)}{3k_1^2 - 1} \quad (5.30)$$

and

$$k_1 k_2 + k_1 k_3 + k_2 k_3 = -3. \quad (5.31)$$

Note that

$$H = m(k_1 + k_2 + k_3) = \frac{3mk_1(k_1^2 - 3)}{3k_1^2 - 1} > 0.$$

From Corollary 2.1.2, we then have

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall (x, t) \in M \times I$, where, from (2.7),

$$\prod_{i=1}^n |\cos(\mu(t)) - k_i \sin(\mu(t))| = e^t.$$

Thus, since the principal curvatures k_1 , k_2 and k_3 have the same multiplicity m ,

$$|[\cos(\mu(t)) - k_1 \sin(\mu(t))][\cos(\mu(t)) - k_2 \sin(\mu(t))][\cos(\mu(t)) - k_3 \sin(\mu(t))]| = e^{t/m}. \quad (5.32)$$

Hence, $\forall t \in I$,

$$|[\cos^2(\mu(t)) - (k_1 + k_2) \sin(\mu(t)) \cos(\mu(t)) + k_1 k_2 \sin^2(\mu(t))][\cos(\mu(t)) - k_3 \sin(\mu(t))]| = e^{t/m},$$

i.e., from (5.30) and (5.31),

$$\begin{aligned} e^{t/m} &= |\cos^3(\mu(t)) - k_3 \sin(\mu(t)) \cos^2(\mu(t)) - (k_1 + k_2) \sin(\mu(t)) \cos^2(\mu(t)) \\ &\quad + (k_1 + k_2) k_3 \sin^2(\mu(t)) \cos(\mu(t)) + k_1 k_2 \sin^2(\mu(t)) \cos(\mu(t)) - k_1 k_2 k_3 \sin^3(\mu(t))| \\ &= |\cos^3(\mu(t)) - (k_1 + k_2 + k_3) \sin(\mu(t)) \cos^2(\mu(t)) \\ &\quad + (k_1 k_3 + k_2 k_3 + k_1 k_2) \sin^2(\mu(t)) \cos(\mu(t)) - k_1 k_2 k_3 \sin^3(\mu(t))| \\ &= \left| \cos^3(\mu(t)) - 3 \sin^2(\mu(t)) \cos(\mu(t)) - \frac{k_1(k_1^2 - 3)}{3k_1^2 - 1} [3 \sin(\mu(t)) \cos^2(\mu(t)) - \sin^3(\mu(t))] \right| \\ &= \left| \cos(3\mu(t)) - \frac{k_1(k_1^2 - 3)}{3k_1^2 - 1} \sin(3\mu(t)) \right| \\ &= |\cos(3\mu(t)) - a \sin(3\mu(t))|, \end{aligned} \quad (5.33)$$

where

$$a = \frac{k_1 + k_2 + k_3}{3} = \frac{k_1(k_1^2 - 3)}{3k_1^2 - 1}.$$

Thus, $H = 3am$ and hence, $a > 0$. Since $e^{t/m}$ is a smooth positive function, by using (5.33) we conclude that $h : I \rightarrow \mathbb{R}$ given by

$$h(t) = \cos(3\mu(t)) - a \sin(3\mu(t))$$

does not change sign. We claim that $h(t) > 0$, $\forall t \in I$. In fact, since $\mu(0) = 0$, we have

$h(0) = 1$. It follows that $h(t) > 0, \forall t \in I$. Hence, (5.33) reduces to

$$\cos(3\mu(t)) - a \sin(3\mu(t)) = e^{t/m}, \quad (5.34)$$

i.e., $\forall t \in I$,

$$a \sin(3\mu(t)) = \cos(3\mu(t)) - e^{t/m}. \quad (5.35)$$

Taking the square on both sides of (5.35) and using that $\sin^2(3\mu(t)) + \cos^2(3\mu(t)) = 1$, we get the following equation

$$(a^2 + 1) \cos^2(3\mu(t)) - 2e^{t/m} \cos(3\mu(t)) + e^{2t/m} - a^2 = 0. \quad (5.36)$$

This is a second degree equation for $\cos(3\mu(t))$. The discriminant of (5.36) is given by

$$\begin{aligned} \Delta &= 4e^{2t/m} - 4(a^2 + 1)(e^{2t/m} - a^2) \\ &= 4e^{2t/m} - 4(a^2 e^{2t/m} - a^4 + e^{2t/m} - a^2) \\ &= 4e^{2t/m} - 4a^2 e^{2t/m} + 4a^4 - 4e^{2t/m} + 4a^2 \\ &= 4a^4 + 4a^2 - 4a^2 e^{2t/m} \\ &= 4a^2(a^2 + 1 - e^{2t/m}). \end{aligned}$$

Define $q : I \longrightarrow \mathbb{R}$ by

$$q(t) = a^2 + 1 - e^{2t/m}.$$

Then, $\Delta \geq 0$ if and only if $q(t) \geq 0$, i.e.,

$$e^{2t/m} \leq a^2 + 1.$$

Consequently,

$$t \leq \frac{m}{2} \ln(a^2 + 1) = t^*.$$

Thus,

$$\cos(3\mu(t)) = \frac{2e^{t/m} \pm 2a\sqrt{q(t)}}{2(a^2 + 1)} = \frac{e^{t/m} \pm a\sqrt{q(t)}}{a^2 + 1},$$

$\forall t \leq t^*$. Since $\mu(0) = 0$, we have that

$$\cos(3\mu(t)) = \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \quad (5.37)$$

$\forall t \leq t^*$. Note that $\cos(3\mu(t)) > 0, \forall t \leq t^*$. Since

$$\cos(3\mu(t)) = 4\cos^3(\mu(t)) - 3\cos(\mu(t)), \quad (5.38)$$

we conclude that $\cos(\mu(t)) \neq 0, \forall t \leq t^*$. Moreover, since $\mu(0) = 0$, we have $\cos(\mu(0)) = 1$ and, thus, $\cos(\mu(t)) > 0, \forall t \leq t^*$.

It follows from (5.35) that

$$\begin{aligned} a \sin(3\mu(t)) &= \cos(3\mu(t)) - e^{t/m} \\ &= \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1} - e^{t/m} \\ &= \frac{e^{t/m} + a\sqrt{q(t)} - e^{t/m}(a^2 + 1)}{a^2 + 1} \\ &= \frac{e^{t/m} + a\sqrt{q(t)} - a^2 e^{t/m} - e^{t/m}}{a^2 + 1} \\ &= \frac{-a^2 e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \end{aligned}$$

i.e.,

$$\sin(3\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{a^2 + 1},$$

$\forall t \leq t^*$. This jointly with (5.37) gives

$$\tan(3\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{e^{t/m} + a\sqrt{q(t)}}. \quad (5.39)$$

Moreover, since $H > 0$, from equation (2.5), we get $\mu'(t) < 0, \forall t \leq t^*$. So, the function μ is decreasing. Note that

$$\frac{d}{dt}[\tan(\mu(t))] = \sec^2(\mu(t))\mu'(t) < 0,$$

$\forall t \leq t^*$, i.e., $\tan \circ \mu$ is decreasing as well. Also, we have from (5.32) that

$$\cos^3(\mu(t))[1 - k_1 \tan(\mu(t))][1 - k_2 \tan(\mu(t))][1 - k_3 \tan(\mu(t))] = e^{t/m}, \quad (5.40)$$

$\forall t \leq t^*$. So, one has

$$\tan(\mu(t)) \neq \frac{1}{k_1}, \tan(\mu(t)) \neq \frac{1}{k_2} \text{ and } \tan(\mu(t)) \neq \frac{1}{k_3}, \quad (5.41)$$

$\forall t \leq t^*$.

Consequently, from Lemma 1.3.2, the mean curvature $\hat{H}^t$ of $\hat{F}^t$ is given by

$$\hat{H}^t = m \left[\frac{\sin(\mu(t)) + k_1 \cos(\mu(t))}{\cos(\mu(t)) - k_1 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_2 \cos(\mu(t))}{\cos(\mu(t)) - k_2 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_3 \cos(\mu(t))}{\cos(\mu(t)) - k_3 \sin(\mu(t))} \right].$$

After reducing the above expression to the same denominator, consider $\alpha = \alpha(\mu(t))$ and $\beta = \beta(\mu(t))$ so that

$$\hat{H}^t = m \frac{\alpha(\mu(t))}{\beta(\mu(t))}.$$

Hence, from (5.34), we have that $\beta(\mu(t)) = e^{t/m}$. Moreover, (5.30) and (5.31) guarantee that

$$\begin{aligned} \alpha(\mu(t)) &= 3[3 \sin(\mu(t)) - 4 \sin^3(\mu(t))] - [2(k_1 + k_2 + k_3) - 3k_1 k_2 k_3] \cos(\mu(t)) \\ &+ 3[k_1 + k_2 + k_3 - k_1 k_2 k_3] \cos^3(\mu(t)) \\ &= 3 \sin(3\mu(t)) + 3k_1 \frac{k_1^2 - 3}{3k_1^2 - 1} [4 \cos^3(\mu(t)) - 3 \cos(\mu(t))] \\ &= 3 \sin(3\mu(t)) + 3k_1 \frac{k_1^2 - 3}{3k_1^2 - 1} \cos(3\mu(t)) \\ &= 3 [\sin(3\mu(t)) + a \cos(3\mu(t))]. \end{aligned}$$

Thus,

$$\frac{1}{n} e^{t/m} \hat{H}^t = \sin(3\mu(t)) + a \cos(3\mu(t)) = \cos(3\mu(t)) [\tan(3\mu(t)) + a].$$

It follows that $\hat{H}^t = 0$ if and only if $\tan(3\mu(t)) = -a$. Thus, from (5.54), we have that $\hat{H}^t = 0$ if and only if

$$\frac{-ae^{t/m} + \sqrt{q(t)}}{e^{t/m} + a\sqrt{q(t)}} = -a,$$

i.e.,

$$(a^2 + 1)\sqrt{q(t)} = 0.$$

Hence, $\hat{H}^t = 0$ if and only if $q(t) = 0$. Therefore, $\hat{H}^t = 0$ if and only if $t = t^*$. We conclude that $\hat{H}^{t^*} = 0$ and $I = (-\infty, t^*)$.

Consider an orthonormal frame $E_{j1}, E_{j2}, \dots, E_{jm}$, $j = 1, 2, 3$, of principal directions on $F(M) \subset \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$. Using Lemma 1.3.2, one obtains that

$$\hat{g}_x^t(E_{jr}, E_{js}) = \cos^2(\mu(t)) [1 - k_j \tan(\mu(t))]^2 \delta_{rs}, \quad (5.42)$$

$\forall t \leq t^*, \forall r, s$ with $1 \leq r, s \leq m$ and $j = 1, 2, 3$. Thus, from (5.41), $\hat{g}_x^t(E_{jl}, E_{jl}) \neq 0$, $\forall t \leq t^*, \forall l$ with $1 \leq l \leq m$ and $j = 1, 2, 3$.

On the other hand, since $k_3 < 0 < k_2 < k_1$, we have that

$$\frac{1}{k_3} < 0 < \frac{1}{k_1} < \frac{1}{k_2}. \quad (5.43)$$

Since $\tan \circ \mu$ is decreasing, $\tan(\mu(t)) > 0, \forall t < 0$. Moreover, since

$$\lim_{t \rightarrow -\infty} q(t) = \lim_{t \rightarrow -\infty} (a^2 + 1 - e^{2t/m}) = a^2 + 1,$$

from (5.37), we have that

$$\lim_{t \rightarrow -\infty} \cos(3\mu(t)) = \lim_{t \rightarrow -\infty} \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1} = a \frac{\sqrt{a^2 + 1}}{a^2 + 1} = \frac{a}{\sqrt{a^2 + 1}}.$$

Hence, from (5.38),

$$\lim_{t \rightarrow -\infty} \cos(\mu(t)) \neq 0.$$

This combined with (5.40) and (5.43) gives

$$\lim_{t \rightarrow -\infty} \tan(\mu(t)) = \frac{1}{k_1}.$$

Consequently,

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_{1r}, E_{1s}) = \lim_{t \rightarrow -\infty} \cos^2(\mu(t)) [1 - k_1 \tan(\mu(t))]^2 \delta_{rs} = 0,$$

$\forall r, s$ with $1 \leq r, s \leq m$, where we have used (5.42). Therefore, $\hat{F}^t(M)$ is defined for $t \in (-\infty, t^*)$, it is an ancient solution that expands from a $2m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$. So, the proof is completed. $\blacksquare$

5.4 Four distinct principal curvatures

In this section, we study the solutions to the IMCF by parallel hypersurfaces considering as initial condition the isoparametric hypersurfaces of the sphere $\mathbb{S}^{n+1}$ with four distinct principal curvatures (i.e., $\mathcal{G} = 4$). In this case, the multiplicities of the principal curvatures can be different. In the following proposition, we additionally assume that the multiplicities of principal curvatures of the initial hypersurface are equal.

Theorem 5.4.1. *Let $F : M^n \longrightarrow \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ be an isoparametric hypersurface in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$, with inward unit normal vector field N and four distinct principal curvatures k_1, k_2, k_3 and k_4 with the same multiplicity m , i.e., $n = 4m$. Also, we may consider*

$$k_1 > \sqrt{3 + 2\sqrt{2}}, \quad k_2 = \frac{k_1 - 1}{k_1 + 1}, \quad k_3 = -\frac{1}{k_1} \quad \text{and} \quad k_4 = \frac{k_1 + 1}{1 - k_1}.$$

Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(M)$ is given by

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall x \in M$, where

$$\cos(4\mu(t)) = \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \quad \sin(4\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{a^2 + 1},$$

$$q(t) = a^2 + 1 - e^{2t/m} \quad \text{and} \quad a = \frac{k_1 + k_2 + k_3 + k_4}{4} = \frac{k_1^4 - 6k_1^2 + 1}{4k_1(k_1^2 - 1)},$$

with $t \in (-\infty, t^*)$ and $t^* = \frac{m}{2} \ln(a^2 + 1)$. Moreover, $\hat{F}^t(M)$ is an ancient solution that expands from a $3m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \longrightarrow t^*$.

Proof. From Remark 5.1.2, we have that $k_1 = \cot(s)$, where $0 < s < \pi/4$. So, since the cotangent is decreasing,

$$k_1 = \cot(s) > \cot(\pi/4) = 1.$$

Again, by Remark 5.1.2, one sees that

$$k_2 = \cot\left(s + \frac{\pi}{4}\right) = \frac{\cot(s)\cot(\pi/4) - 1}{\cot(s) + \cot(\pi/4)} = \frac{k_1 - 1}{k_1 + 1},$$

$$k_3 = \cot\left(s + \frac{\pi}{2}\right) = \frac{\cot(s)\cot(\pi/2) - 1}{\cot(s) + \cot(\pi/2)} = -\frac{1}{k_1}$$

and

$$k_4 = \cot\left(s + \frac{3\pi}{4}\right) = \frac{\cot(s)\cot(3\pi/4) - 1}{\cot(s) + \cot(3\pi/4)} = \frac{-k_1 - 1}{k_1 - 1} = \frac{k_1 + 1}{1 - k_1},$$

where $k_4 < k_3 < 0 < k_2 < k_1$. Hence,

$$k_1 k_3 = -1, \quad k_2 k_4 = \frac{k_1 - 1}{k_1 + 1} \frac{k_1 + 1}{1 - k_1} = -1, \quad (5.44)$$

$$(k_1 + k_3)(k_2 + k_4) = \left(k_1 - \frac{1}{k_1}\right) \left(\frac{k_1 - 1}{k_1 + 1} - \frac{k_1 + 1}{k_1 - 1}\right) = -4, \quad (5.45)$$

$$k_1 k_2 k_3 k_4 = k_1 \left(\frac{k_1 - 1}{k_1 + 1}\right) \left(-\frac{1}{k_1}\right) \left(\frac{k_1 + 1}{1 - k_1}\right) = 1 \quad (5.46)$$

and

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 &= k_1 + \frac{k_1 - 1}{k_1 + 1} - \frac{1}{k_1} + \frac{k_1 + 1}{1 - k_1} \\ &= \frac{k_1^2 - 1}{k_1} + \frac{k_1 - k_1^2 - 1 + k_1 + k_1^2 + 2k_1 + 1}{k_1 + 1 - k_1^2 - k_1} \\ &= \frac{k_1^2 - 1}{k_1} + \frac{4k_1}{1 - k_1^2} \\ &= \frac{(k_1^2 - 1)(1 - k_1^2) + 4k_1^2}{k_1(1 - k_1^2)} \\ &= \frac{k_1^2 - k_1^4 - 1 + k_1^2 + 4k_1^2}{k_1(1 - k_1^2)} \\ &= \frac{-k_1^4 + 6k_1^2 - 1}{k_1(1 - k_1^2)} \\ &= \frac{k_1^4 - 6k_1^2 + 1}{k_1(k_1^2 - 1)}. \end{aligned}$$

From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall(x, t) \in M \times I$, where, from (2.7),

$$\prod_{i=1}^n |\cos(\mu(t)) - k_i \sin(\mu(t))| = e^t.$$

Thus, since the principal curvatures k_1, k_2, k_3 and k_4 have the same multiplicity m , we get

$$\begin{aligned} |[\cos(\mu(t)) - k_1 \sin(\mu(t))] \cdot [\cos(\mu(t)) - k_2 \sin(\mu(t))] \cdot [\cos(\mu(t)) - k_3 \sin(\mu(t))] \cdot [\cos(\mu(t)) - k_4 \sin(\mu(t))]| &= e^{t/m}, \end{aligned} \quad (5.47)$$

$\forall t \in I$. Now, by combining (5.44), (5.45) and (5.46), we deduce

$$\begin{aligned}
e^{t/m} &= |[\cos^2(\mu(t)) - (k_1 + k_2) \sin(\mu(t)) \cos(\mu(t)) + k_1 k_2 \sin^2(\mu(t))] \\
&\quad \cdot [\cos^2(\mu(t)) - (k_3 + k_4) \sin(\mu(t)) \cos(\mu(t)) + k_3 k_4 \sin^2(\mu(t))]| \\
&= |\cos^4(\mu(t)) + \sin^4(\mu(t)) - (k_1 + k_2 + k_3 + k_4) \sin(\mu(t)) \cos^3(\mu(t)) \\
&\quad - [k_3 k_4 (k_1 + k_2) + k_1 k_2 (k_3 + k_4)] \sin^3(\mu(t)) \cos(\mu(t)) \\
&\quad + [k_1 k_2 + k_3 k_4 + (k_1 + k_2)(k_3 + k_4)] \sin^2(\mu(t)) \cos^2(\mu(t))| \\
&= |\cos^4(\mu(t)) + \sin^4(\mu(t)) - (k_1 + k_2 + k_3 + k_4) \sin(\mu(t)) \cos^3(\mu(t)) \\
&\quad - (-k_4 - k_3 - k_2 - k_1) \sin^3(\mu(t)) \cos(\mu(t)) \\
&\quad + [(k_1 + k_3)(k_2 + k_4) - 2] \sin^2(\mu(t)) \cos^2(\mu(t))| \\
&= |\cos^4(\mu(t)) + \sin^4(\mu(t)) - (k_1 + k_2 + k_3 + k_4) \sin(\mu(t)) \cos^3(\mu(t)) \\
&\quad + (k_1 + k_2 + k_3 + k_4) \sin^3(\mu(t)) \cos(\mu(t)) - 6 \sin^2(\mu(t)) \cos^2(\mu(t))| \\
&= |\cos^4(\mu(t)) + \sin^4(\mu(t)) - 6 \sin^2(\mu(t)) \cos^2(\mu(t)) \\
&\quad - (k_1 + k_2 + k_3 + k_4) [\sin(\mu(t)) \cos^3(\mu(t)) - \sin^3(\mu(t)) \cos(\mu(t))]| \\
&= \left| \cos(4\mu(t)) - \frac{k_1 + k_2 + k_3 + k_4}{4} \sin(4\mu(t)) \right| \\
&= |\cos(4\mu(t)) - a \sin(4\mu(t))|, \tag{5.48}
\end{aligned}$$

$\forall t \in I$, where

$$a = \frac{k_1 + k_2 + k_3 + k_4}{4} = \frac{k_1^4 - 6k_1^2 + 1}{4k_1(k_1^2 - 1)}.$$

Thus, since $k_1 > \sqrt{3 + 2\sqrt{2}}$, we have that $a > 0$. Hence, $H = 4am > 0$. Since $e^{t/m}$ is a smooth positive function, we conclude from (5.48) that $h : I \rightarrow \mathbb{R}$ given by

$$h(t) = \cos(4\mu(t)) - a \sin(4\mu(t))$$

does not change sign. We claim that $h(t) > 0$, $\forall t \in I$. In fact, since $\mu(0) = 0$, we deduce $h(0) = 1$. So $h(t) > 0$, $\forall t \in I$. Thus, (5.48) reduces to

$$\cos(4\mu(t)) - a \sin(4\mu(t)) = e^{t/m}. \tag{5.49}$$

i.e., $\forall t \in I$,

$$a \sin(4\mu(t)) = \cos(4\mu(t)) - e^{t/m}. \tag{5.50}$$

Taking the square on both sides of (5.50) and using that $\sin^2(4\mu(t)) + \cos^2(4\mu(t)) = 1$, we

get the following equation

$$(a^2 + 1) \cos^2(4\mu(t)) - 2e^{t/m} \cos(4\mu(t)) + e^{2t/m} - a^2 = 0. \quad (5.51)$$

This is a second degree equation for $\cos(4\mu(t))$. The discriminant of (5.51) is given by

$$\Delta = 4a^2(a^2 + 1 - e^{2t/m}).$$

Define $q : I \longrightarrow \mathbb{R}$ by

$$q(t) = a^2 + 1 - e^{2t/m}.$$

Thus, $\Delta \geq 0$ if and only if $q(t) \geq 0$, i.e.,

$$t \leq \frac{m}{2} \ln(a^2 + 1) = t^*.$$

Hence, since $\mu(0) = 0$,

$$\cos(4\mu(t)) = \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \quad (5.52)$$

$\forall t \leq t^*$. Note that $\cos(4\mu(t)) > 0$, $\forall t \leq t^*$. Moreover,

$$\cos(4\mu(t)) = 8 \cos^4(\mu(t)) - 8 \cos^2(\mu(t)) + 1. \quad (5.53)$$

We claim that $\cos(\mu(t)) > 0$, $\forall t \leq t^*$. In fact, by contradiction, suppose that there exists $\tilde{t}$ such that $\cos(\mu(\tilde{t})) = 0$. Thus, using (5.53), we have $\cos(4\mu(\tilde{t})) = 1$. From (5.52), we conclude that $\tilde{t} = 0$. Hence, $\cos(\mu(\tilde{t})) = 1$ and we have a contradiction. Moreover, since $\mu(0) = 0$, we have $\cos(\mu(0)) = 1$ and thus, $\cos(\mu(t)) > 0$, $\forall t \leq t^*$.

It follows from (5.50) and (5.52) that

$$\sin(4\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{a^2 + 1},$$

$\forall t \leq t^*$. Thus,

$$\tan(4\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{e^{t/m} + a\sqrt{q(t)}}. \quad (5.54)$$

Moreover, one sees from (5.47) that

$$\begin{aligned} \cos^4(\mu(t)) \cdot [1 - k_1 \tan(\mu(t))] \cdot [1 - k_2 \tan(\mu(t))] &\cdot [1 - k_3 \tan(\mu(t))] \\ &\cdot [1 - k_4 \tan(\mu(t))] = e^{t/m}, \end{aligned} \quad (5.55)$$

$\forall t \leq t^*$. Thus, from (5.44) and (5.55), we infer

$$\begin{aligned} \tan(\mu(t)) &\neq \frac{1}{k_1} = -k_3, \quad \tan(\mu(t)) \neq \frac{1}{k_2} = -k_4, \\ \tan(\mu(t)) &\neq \frac{1}{k_3} = -k_1 \text{ and } \tan(\mu(t)) \neq \frac{1}{k_4} = -k_2, \end{aligned} \quad (5.56)$$

$\forall t \leq t^*$. From Lemma 1.3.2, the mean curvature $\hat{H}^t$ of $\hat{F}^t$ is given by

$$\begin{aligned} \frac{1}{m} \hat{H}^t &= \frac{\sin(\mu(t)) + k_1 \cos(\mu(t))}{\cos(\mu(t)) - k_1 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_2 \cos(\mu(t))}{\cos(\mu(t)) - k_2 \sin(\mu(t))} \\ &+ \frac{\sin(\mu(t)) + k_3 \cos(\mu(t))}{\cos(\mu(t)) - k_3 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_4 \cos(\mu(t))}{\cos(\mu(t)) - k_4 \sin(\mu(t))}, \end{aligned}$$

After reducing the above expression to the same denominator, consider $\alpha = \alpha(\mu(t))$ and $\beta = \beta(\mu(t))$ such that

$$\frac{1}{m} \hat{H}^t = \frac{\alpha(\mu(t))}{\beta(\mu(t))}.$$

Hence, from (5.49), we have that $\beta(\mu(t)) = e^{t/m}$. Moreover, by combining (5.44), (5.45) and (5.46), one deduces that

$$\begin{aligned} \alpha(\mu(t)) &= (k_1 + k_2 + k_3 + k_4) \cos^4(\mu(t)) - (k_2 k_3 k_4 + k_1 k_3 k_4 + k_1 k_2 k_4 + k_1 k_2 k_3) \sin^4(\mu(t)) \\ &+ [4 - k_1(k_2 + k_3 + k_4) - k_2(k_1 + k_3 + k_4) - k_3(k_1 + k_2 + k_4) \\ &- k_4(k_1 + k_2 + k_3)] \sin(\mu(t)) \cos^3(\mu(t)) \\ &+ 2(k_2 k_3 + k_3 k_4 + k_2 k_4 + k_1 k_3 + k_1 k_4 + k_1 k_2 - 2k_1 k_2 k_3 k_4) \sin^3(\mu(t)) \cos(\mu(t)) \\ &+ 3[k_1 k_2 k_3 + k_1 k_2 k_4 + k_1 k_3 k_4 + k_2 k_3 k_4 - (k_1 + k_2 + k_3 + k_4)] \sin^2(\mu(t)) \cos^2(\mu(t)) \\ &= 4a \cos^4(\mu(t)) - [k_2 k_4(k_1 + k_3) + k_1 k_3(k_2 + k_4)] \sin^4(\mu(t)) \\ &+ [8 - 2(k_1 + k_3)(k_2 + k_4)] \sin(\mu(t)) \cos^3(\mu(t)) \\ &+ 2((k_1 + k_3)(k_2 + k_4) - 4) \sin^3(\mu(t)) \cos(\mu(t)) \\ &- 3[k_1 + k_2 + k_3 + k_4 + 4a] \sin^2(\mu(t)) \cos^2(\mu(t)) \\ &= 4a \cos^4(\mu(t)) + (k_1 + k_2 + k_3 + k_4) \sin^4(\mu(t)) + 16 \sin(\mu(t)) \cos^3(\mu(t)) \\ &- 16 \sin^3(\mu(t)) \cos(\mu(t)) - 24a \sin^2(\mu(t)) \cos^2(\mu(t)) \\ &= 4a \cos^4(\mu(t)) + 4a \sin^4(\mu(t)) + 16 \sin(\mu(t)) \cos^3(\mu(t)) \\ &- 16 \sin^3(\mu(t)) \cos(\mu(t)) - 24a \sin^2(\mu(t)) \cos^2(\mu(t)) \\ &= 4a[\cos^4(\mu(t)) + \sin^4(\mu(t)) - 6 \sin^2(\mu(t)) \cos^2(\mu(t))] \\ &+ 16[\sin(\mu(t)) \cos^3(\mu(t)) - \sin^3(\mu(t)) \cos(\mu(t))] \\ &= 4a \cos(4\mu(t)) + 4 \sin(4\mu(t)). \end{aligned}$$

Thus,

$$\frac{1}{n}e^{t/m}\hat{H}^t = \sin(4\mu(t)) + a \cos(4\mu(t)) = \cos(4\mu(t))[\tan(4\mu(t)) + a].$$

It follows that $\hat{H}^t = 0$ if and only if $\tan(4\mu(t)) = -a$. Thus, we have from (5.54) that $\hat{H}^t = 0$ if and only if

$$\frac{-ae^{t/m} + \sqrt{q(t)}}{e^{t/m} + a\sqrt{q(t)}} = -a,$$

i.e.,

$$(a^2 + 1)\sqrt{q(t)} = 0.$$

Hence, $\hat{H}^t = 0$ if and only if $q(t) = 0$. Therefore, $\hat{H}^t = 0$ if and only if $t = t^*$. We conclude that $\hat{H}^{t^*} = 0$ and $I = (-\infty, t^*)$.

Consider an orthonormal frame $E_{j1}, E_{j2}, \dots, E_{jm}$, $j = 1, 2, 3, 4$, of principal directions on $F(M) \subset \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$. From Lemma 1.3.2, we have that

$$\hat{g}_x^t(E_{jr}, E_{js}) = \cos^2(\mu(t)) [1 - k_j \tan(\mu(t))]^2 \delta_{rs}, \quad (5.57)$$

$\forall t \leq t^*$, $\forall r, s$ with $1 \leq r, s \leq m$ and $j = 1, 2, 3, 4$. Thus, using (5.56), $\hat{g}_x^t(E_{jl}, E_{jl}) \neq 0$, $\forall t \leq t^*$, $\forall l$ with $1 \leq l \leq m$ and $j = 1, 2, 3, 4$.

At the same time, from Lemma 1.3.2, we have that the second fundamental form $\hat{I}_x^t$ of $\hat{F}^t(x)$, $x \in M$, is given by

$$\hat{I}_x^t(E_{jr}, E_{js}) = \cos^2(\mu(t)) [1 - k_j \tan(\mu(t))] [\tan(\mu(t)) + k_j] \delta_{rs},$$

$\forall t \leq t^*$, $\forall r, s$ with $1 \leq r, s \leq m$ and $j = 1, 2, 3, 4$. By (5.56), we have $\hat{I}_x^t(E_{jl}, E_{jl}) \neq 0$, $\forall t \leq t^*$, $\forall l$ with $1 \leq l \leq m$ and $j = 1, 2, 3, 4$.

On the other hand, since $k_4 < k_3 < 0 < k_2 < k_1$, we have that

$$\frac{1}{k_3} < \frac{1}{k_4} < 0 < \frac{1}{k_1} < \frac{1}{k_2}. \quad (5.58)$$

Since $H > 0$, from (2.5) we have that $\tan \circ \mu$ is decreasing. Thus, $\tan(\mu(t)) > 0$, $\forall t < 0$. Moreover, using (5.52), we have that

$$\lim_{t \rightarrow -\infty} \cos(4\mu(t)) = \frac{a}{\sqrt{a^2 + 1}}.$$

Hence, by (5.53), we conclude that

$$\lim_{t \rightarrow -\infty} \cos(\mu(t)) \neq 0. \quad (5.59)$$

It follows from (5.55), (5.58) and (5.59) that

$$\lim_{t \rightarrow -\infty} \tan(\mu(t)) = \frac{1}{k_1}.$$

Thus, using 5.57,

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_{1r}, E_{1s}) = \lim_{t \rightarrow -\infty} \cos^2(\mu(t)) [1 - k_1 \tan(\mu(t))]^2 \delta_{rs} = 0,$$

$\forall r, s$ with $1 \leq r, s \leq m$. Therefore, $\hat{F}^t(M)$ is defined for $t \in (-\infty, t^*)$, it is an ancient solution that expands from a $3m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$. So, the proof is finished. $\blacksquare$

5.5 Six distinct principal curvatures

Theorem 5.5.1. *Let $F : M^n \rightarrow \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ be an isoparametric hypersurface in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$, with inward unit normal vector field N and six distinct principal curvatures k_1, k_2, k_3, k_4, k_5 and k_6 . Also, we may consider*

$$k_1 > \sqrt{7 + 4\sqrt{3}}, \quad k_2 = \frac{\sqrt{3}k_1 - 1}{k_1 + \sqrt{3}}, \quad k_3 = \frac{\sqrt{3}k_1 - 3}{3k_1 + \sqrt{3}},$$

$$k_4 = -\frac{1}{k_1}, \quad k_5 = \frac{\sqrt{3}k_1 + 3}{\sqrt{3} - 3k_1} \quad \text{and} \quad k_6 = \frac{\sqrt{3}k_1 + 1}{\sqrt{3} - k_1},$$

all with multiplicity m , i.e., $n = 6m$, where $m \in \{1, 2\}$. Then, the solution to the IMCF by parallel hypersurfaces with initial condition $F(M)$ is given by

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall x \in M$, where

$$\cos(6\mu(t)) = \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \quad \sin(6\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{a^2 + 1},$$

$$q(t) = a^2 + 1 - e^{2t/m} \quad \text{and} \quad a = \frac{k_1 + k_2 + k_3 + k_4 + k_5 + k_6}{6} = \frac{1}{2} \frac{k_1^6 - 15k_1^4 + 15k_1^2 - 1}{3k_1^5 - 10k_1^3 + 3k_1},$$

with $t \in (-\infty, t^*)$ and $t^* = \frac{m}{2} \ln(a^2 + 1)$. Moreover, $\hat{F}^t(M)$ is an ancient solution that expands from a 5m-dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$.

Proof. From Remark 5.1.2, we have $k_1 = \cot(s)$, where $0 < s < \pi/6$. Hence, since the cotangent is decreasing,

$$k_1 = \cot(s) > \cot(\pi/6) = \sqrt{3}.$$

Again, by Remark 5.1.2, one sees that

$$k_2 = \cot\left(s + \frac{\pi}{6}\right) = \frac{\cot(s) \cot(\pi/6) - 1}{\cot(s) + \cot(\pi/6)} = \frac{\sqrt{3}k_1 - 1}{k_1 + \sqrt{3}},$$

$$k_3 = \cot\left(s + \frac{\pi}{3}\right) = \frac{\cot(s) \cot(\pi/3) - 1}{\cot(s) + \cot(\pi/3)} = \frac{\frac{\sqrt{3}}{3}k_1 - 1}{k_1 + \frac{\sqrt{3}}{3}} = \frac{\sqrt{3}k_1 - 3}{3k_1 + \sqrt{3}},$$

$$k_4 = \cot\left(s + \frac{\pi}{2}\right) = \frac{\cot(s) \cot(\pi/2) - 1}{\cot(s) + \cot(\pi/2)} = -\frac{1}{k_1},$$

$$k_5 = \cot\left(s + \frac{2\pi}{3}\right) = \frac{\cot(s) \cot(2\pi/3) - 1}{\cot(s) + \cot(2\pi/3)} = \frac{-\frac{\sqrt{3}}{3}k_1 - 1}{k_1 - \frac{\sqrt{3}}{3}} = \frac{\sqrt{3}k_1 + 3}{\sqrt{3} - 3k_1}$$

and

$$k_6 = \cot\left(s + \frac{5\pi}{6}\right) = \frac{\cot(s) \cot(5\pi/6) - 1}{\cot(s) + \cot(5\pi/6)} = \frac{-\sqrt{3}k_1 - 1}{k_1 - \sqrt{3}} = \frac{\sqrt{3}k_1 + 1}{\sqrt{3} - k_1},$$

where $k_6 < k_5 < k_4 < 0 < k_3 < k_2 < k_1$. Thus,

$$k_1 k_4 = -1, \tag{5.60}$$

$$k_3 k_6 = \frac{\sqrt{3}k_1 - 3}{3k_1 + \sqrt{3}} \frac{\sqrt{3}k_1 + 1}{\sqrt{3} - k_1} = \frac{\sqrt{3}k_1 - 3}{3k_1 + \sqrt{3}} \frac{3k_1 + \sqrt{3}}{3 - \sqrt{3}k_1} = -1, \tag{5.61}$$

$$k_2 k_5 = \frac{\sqrt{3}k_1 - 1}{k_1 + \sqrt{3}} \frac{\sqrt{3}k_1 + 3}{\sqrt{3} - 3k_1} = \frac{3k_1 - \sqrt{3}}{\sqrt{3}k_1 + 3} \frac{\sqrt{3}k_1 + 3}{\sqrt{3} - 3k_1} = -1, \tag{5.62}$$

$$k_1 k_2 k_3 k_4 k_5 k_6 = -1 \tag{5.63}$$

and

$$k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 3 \frac{k_1^6 - 15k_1^4 + 15k_1^2 - 1}{3k_1^5 - 10k_1^3 + 3k_1}.$$

From Corollary 2.1.2, we have that

$$\hat{F}^t(x) = \cos(\mu(t))F(x) + \sin(\mu(t))N(x),$$

$\forall (x, t) \in M \times I$, where by (2.7),

$$\prod_{i=1}^n |\cos(\mu(t)) - k_i \sin(\mu(t))| = e^t.$$

Hence, since the principal curvatures $k_1, k_2, k_3, k_4, k_5, k_6$ have the same multiplicity m , we get

$$\begin{aligned} e^{t/m} &= [|\cos(\mu(t)) - k_1 \sin(\mu(t))| \cdot |\cos(\mu(t)) - k_2 \sin(\mu(t))| \\ &\quad \cdot |\cos(\mu(t)) - k_3 \sin(\mu(t))| \cdot |\cos(\mu(t)) - k_4 \sin(\mu(t))| \\ &\quad \cdot |\cos(\mu(t)) - k_5 \sin(\mu(t))| \cdot |\cos(\mu(t)) - k_6 \sin(\mu(t))|], \end{aligned} \quad (5.64)$$

$\forall t \in I$. Now, by combining (5.60), (5.61), (5.62) and (5.63), we deduce

$$\begin{aligned} e^{t/m} &= \left| \cos(6\mu(t)) - \frac{k_1 + k_2 + k_3 + k_4 + k_5 + k_6}{6} \sin(6\mu(t)) \right| \\ &= |\cos(6\mu(t)) - a \sin(6\mu(t))|, \end{aligned} \quad (5.65)$$

$\forall t \in I$, where

$$a = \frac{k_1 + k_2 + k_3 + k_4 + k_5 + k_6}{6} = \frac{1}{2} \frac{k_1^6 - 15k_1^4 + 15k_1^2 - 1}{3k_1^5 - 10k_1^3 + 3k_1}.$$

Since $k_1 > \sqrt{7 + 4\sqrt{3}}$, we have that $a > 0$. Hence, $H = 6am > 0$.

Since $e^{t/m}$ is a smooth positive function, we conclude from (5.65) that $h : I \rightarrow \mathbb{R}$ given by

$$h(t) = \cos(6\mu(t)) - a \sin(6\mu(t))$$

does not change sign. We claim that $h(t) > 0, \forall t \in I$. In fact, since $\mu(0) = 0$, we deduce $h(0) = 1$. So $h(t) > 0, \forall t \in I$. Thus, (5.65) reduces to

$$\cos(6\mu(t)) - a \sin(6\mu(t)) = e^{t/m}, \quad (5.66)$$

i.e., $\forall t \in I$,

$$a \sin(6\mu(t)) = \cos(6\mu(t)) - e^{t/m}. \quad (5.67)$$

Taking the square on both sides of (5.67) and using that $\sin^2(6\mu(t)) + \cos^2(6\mu(t)) = 1$, we get the following equation

$$(a^2 + 1) \cos^2(6\mu(t)) - 2e^{t/m} \cos(6\mu(t)) + e^{2t/m} - a^2 = 0. \quad (5.68)$$

This is a second degree equation for $\cos(6\mu(t))$. The discriminant of (5.68) is given by

$$\Delta = 4a^2(a^2 + 1 - e^{2t/m}).$$

Define $q : I \longrightarrow \mathbb{R}$ by

$$q(t) = a^2 + 1 - e^{2t/m}.$$

Thus, $\Delta \geq 0$ if and only if $q(t) \geq 0$, i.e.,

$$t \leq \frac{m}{2} \ln(a^2 + 1) = t^*.$$

Since $\mu(0) = 0$, we have that

$$\cos(6\mu(t)) = \frac{e^{t/m} + a\sqrt{q(t)}}{a^2 + 1}, \quad (5.69)$$

$\forall t \leq t^*$. Note that $\cos(6\mu(t)) > 0$, $\forall t \leq t^*$. Moreover,

$$\cos(6\mu(t)) = 32 \cos^6(\mu(t)) - 48 \cos^4(\mu(t)) + 18 \cos^2(\mu(t)) - 1. \quad (5.70)$$

Thus, $\cos(\mu(t)) \neq 0$, $\forall t \leq t^*$. Moreover, since $\mu(0) = 0$, we have $\cos(\mu(0)) = 1$ and hence, $\cos(\mu(t)) > 0$, $\forall t \leq t^*$.

It follows from (5.67) and (5.69) that

$$\sin(6\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{a^2 + 1},$$

$\forall t \leq t^*$. Thus,

$$\tan(6\mu(t)) = \frac{-ae^{t/m} + \sqrt{q(t)}}{e^{t/m} + a\sqrt{q(t)}}. \quad (5.71)$$

Moreover, one sees from (5.64) that

$$\begin{aligned} \cos^6(\mu(t)) &\cdot [1 - k_1 \tan(\mu(t))] \cdot [1 - k_2 \tan(\mu(t))] \cdot [1 - k_3 \tan(\mu(t))] \\ &\cdot [1 - k_4 \tan(\mu(t))] \cdot [1 - k_5 \tan(\mu(t))] \cdot [1 - k_6 \tan(\mu(t))] = e^{t/m}, \end{aligned} \quad (5.72)$$

$\forall t \leq t^*$. Thus, from (5.60), (5.61), (5.62) and (5.72), we infer

$$\begin{aligned} \tan(\mu(t)) \neq \frac{1}{k_1} = -k_4, \quad \tan(\mu(t)) \neq \frac{1}{k_2} = -k_5, \quad \tan(\mu(t)) \neq \frac{1}{k_3} = -k_6, \\ \tan(\mu(t)) \neq \frac{1}{k_4} = -k_1, \quad \tan(\mu(t)) \neq \frac{1}{k_5} = -k_2 \quad \text{and} \quad \tan(\mu(t)) \neq \frac{1}{k_6} = -k_3, \end{aligned} \quad (5.73)$$

$\forall t \leq t^*$.

By combining Lemma 1.3.2, (5.60), (5.61), (5.62) and (5.63), one deduces that the mean curvature $\hat{H}^t$ of $\hat{F}^t$ is given by

$$\begin{aligned} \frac{1}{m} \hat{H}^t &= \frac{\sin(\mu(t)) + k_1 \cos(\mu(t))}{\cos(\mu(t)) - k_1 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_2 \cos(\mu(t))}{\cos(\mu(t)) - k_2 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_3 \cos(\mu(t))}{\cos(\mu(t)) - k_3 \sin(\mu(t))} \\ &+ \frac{\sin(\mu(t)) + k_4 \cos(\mu(t))}{\cos(\mu(t)) - k_4 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_5 \cos(\mu(t))}{\cos(\mu(t)) - k_5 \sin(\mu(t))} + \frac{\sin(\mu(t)) + k_6 \cos(\mu(t))}{\cos(\mu(t)) - k_6 \sin(\mu(t))} \\ &= 6 \frac{a \cos(6\mu(t)) + \sin(6\mu(t))}{\cos(6\mu(t)) - a \sin(6\mu(t))}, \end{aligned}$$

i.e., from (5.66),

$$\frac{1}{n} e^{t/m} \hat{H}^t = \sin(6\mu(t)) + a \cos(6\mu(t)) = \cos(6\mu(t)) [\tan(6\mu(t)) + a].$$

It follows that $\hat{H}^t = 0$ if and only if $\tan(6\mu(t)) = -a$. Thus, we have from (5.71) that $\hat{H}^t = 0$ if and only if

$$\frac{-ae^{t/m} + \sqrt{q(t)}}{e^{t/m} + a\sqrt{q(t)}} = -a,$$

i.e.,

$$(a^2 + 1)\sqrt{q(t)} = 0.$$

Hence, $\hat{H}^t = 0$ if and only if $q(t) = 0$. Therefore, $\hat{H}^t = 0$ if and only if $t = t^*$. We conclude that $\hat{H}^{t^*} = 0$ and $I = (-\infty, t^*)$.

Consider an orthonormal frame $E_{j1}, E_{j2}, \dots, E_{jm}$, $j = 1, 2, 3, 4, 5, 6$, of principal direc-

tions on $F(M) \subset \mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$. From Lemma 1.3.2, we have that

$$\hat{g}_x^t(E_{jr}, E_{js}) = \cos^2(\mu(t)) [1 - k_j \tan(\mu(t))]^2 \delta_{rs}, \quad (5.74)$$

$\forall t \leq t^*, \forall r, s$ with $1 \leq r, s \leq m$ and $j = 1, 2, 3, 4, 5, 6$. Thus, using (5.73), $\hat{g}_x^t(E_{jl}, E_{jl}) \neq 0$, $\forall t \leq t^*, \forall l$ with $1 \leq l \leq m$ and $j = 1, 2, 3, 4, 5, 6$.

At the same time, from Lemma 1.3.2, we have that the second fundamental form $\hat{I}_x^t$ of $\hat{F}^t(x)$, $x \in M$, is given by

$$\hat{I}_x^t(E_{jr}, E_{js}) = \cos^2(\mu(t)) [1 - k_j \tan(\mu(t))] [\tan(\mu(t)) + k_j] \delta_{rs},$$

$\forall t \leq t^*, \forall r, s$ with $1 \leq r, s \leq m$ and $j = 1, 2, 3, 4, 5, 6$. So, by (5.73), we have that $\hat{I}_x^t(E_{jl}, E_{jl}) \neq 0, \forall t \leq t^*, \forall l$ with $1 \leq l \leq m$ and $j = 1, 2, 3, 4, 5, 6$.

On the other hand, since $k_6 < k_5 < k_4 < 0 < k_3 < k_2 < k_1$, we have that

$$\frac{1}{k_4} < \frac{1}{k_5} < \frac{1}{k_6} < 0 < \frac{1}{k_1} < \frac{1}{k_2} < \frac{1}{k_3}. \quad (5.75)$$

Since $H > 0$, using (2.5) we have that $\tan \circ \mu$ is decreasing. Thus, $\tan(\mu(t)) > 0, \forall t < 0$. Moreover, by (5.69), we have that

$$\lim_{t \rightarrow -\infty} \cos(6\mu(t)) = \frac{a}{\sqrt{a^2 + 1}}.$$

Thus, using (5.70), we conclude that

$$\lim_{t \rightarrow -\infty} \cos(\mu(t)) \neq 0. \quad (5.76)$$

It follows from (5.72), (5.75) and (5.76) that

$$\lim_{t \rightarrow -\infty} \tan(\mu(t)) = \frac{1}{k_1}.$$

Hence, by (5.74),

$$\lim_{t \rightarrow -\infty} \hat{g}_x^t(E_{1r}, E_{1s}) = \lim_{t \rightarrow -\infty} \cos^2(\mu(t)) [1 - k_1 \tan(\mu(t))]^2 \delta_{rs} = 0,$$

$\forall r, s$ with $1 \leq r, s \leq m$. Therefore, $\hat{F}^t(M)$ is defined for $t \in (-\infty, t^*)$, it is an ancient solution that expands from a $5m$ -dimensional submanifold of $\mathbb{S}^{n+1}$ and converges to a minimal hypersurface of $\mathbb{S}^{n+1}$ when $t \rightarrow t^*$. So, the proof is completed. $\blacksquare$

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