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Nonlocal pseudosymmetries and Bäcklund transformations as $\mathcal{C}$-morphisms

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Abstract

In this paper, we show how factorisation with respect to nonlocal pseudosymmetries enables us to obtain Bäcklund transformations, which are described as nonlocal $\mathcal{C}$-morphisms of differential equations. It follows that, in this approach, Bäcklund transformations are determined by basic invariants of the exploited nonlocal pseudosymmetries. As illustrated in several representative examples, including one involving a new integrable equation, our resulting framework enables a very general structural approach, distinct from the case-by-case ones often adopted in the literature—especially in the analysis of cases where the sought Bäcklund transformations also affect the independent variables.

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1. Introduction

Bäcklund transformations, although originally arising within the realm of classical differential geometry, have come to play a significant role in the modern analysis of nonlinear phenomena. Indeed, this type of transformation is named after the mathematician traditionally associated with a theorem that studies the transformation of surfaces with constant negative curvature in $\mathbb{R}^3$, realizing them as focal surfaces of a pseudospherical congruence of lines. In practice, starting from a surface with constant negative Gaussian curvature, Bäcklund's theorem allows for the construction of a one-parameter family of surfaces by integrating only a system of ordinary differential equations (ODEs). Since such surfaces correspond, from an analytical point of view, to solutions of the sine-Gordon equation $z_{xt} = \sin z$, Bäcklund's theorem provides a method for generating new solutions of the sine-Gordon equation from a given one, again through the integration of a system of ODEs. In doing so, this transformation acts as an automorphism of the sine-Gordon solution space — a property that justifies the term auto-Bäcklund transformation.

In its modern formulation, an auto-Bäcklund transformation for a differential equation $\mathcal{E}$ refers to a map that associates, to any given solution of $\mathcal{E}$, a family of new solutions of $\mathcal{E}$, typically by integrating a system of ODEs. Then, more generally, a Bäcklund transformation is understood as a transformation that, via the integration of a system of ordinary differential equations, maps solutions of one differential equation $\mathcal{E}$ to solutions of another equation $\mathcal{E}'$.

Given a nonlinear differential equation, the problem of determining the existence of its auto-Bäcklund transformations—or, more generally, of Bäcklund transformations between this equation and other equations of interest—is an important problem that has attracted the attention of many mathematicians (see, for instance [4,27,28,36,39,41,50] and references therein). However, it is a difficult problem which is still open in its full generality, despite having been studied for a long time and, in recent years, interesting progress having been made in some particular cases (see, for instance, the works [2,3,18–20,47,48]).

On the other hand, a notion closely related to that of a Bäcklund transformation is that of a differential substitution. In fact, a differential substitution from an equation $\mathcal{E}$ to an equation $\mathcal{E}'$ is a transformation that maps solutions of $\mathcal{E}$ to solutions of $\mathcal{E}'$, without requiring the integration of a system of ordinary differential equations. Classical examples of differential substitutions include the Cole–Hopf transformation, which maps solutions of the heat equation to solutions of Burgers equation, and the Miura transformation, which maps solutions of the modified Korteweg-de Vries (mKdV) equation to solutions of the Korteweg-de Vries (KdV) equation.

As with Bäcklund transformations, given a differential equation $\mathcal{E}$, it is often of great interest to know whether a differential substitution exists that maps its solutions into those of another equation $\mathcal{E}'$. Regarding this issue, in the context of evolutionary equations, Sokolov took a significant step forward in [42], highlighting a particularly important relationship between factorisation via pseudosymmetries and differential substitutions. However, although Sokolov underscored the importance of pseudosymmetries through this discovery, these have been surprisingly underexplored in the literature. Indeed, aside from the works of Sokolov himself, pseudosymmetries have been little studied in their full generality. In a simplified form, they have been used in several works mainly to reduce and integrate ordinary differential equations (see, for example, [10,11,13,35] and the references therein). While for the particular case of integrable pseudosymmetries, some possible uses have been discussed in the study of differential coverings [16,17] (see [26] for the notion of differentiable covering) or as a tool to explore the effectiveness and generating power of an already known Bäcklund transformation [32]. Therefore, no previous work on using pseudosymmetries addresses the important problem of determining Bäcklund transformations.

A particularly convenient approach to Bäcklund transformations is based on the notion of $\mathcal{C}$-morphism. Indeed, all Bäcklund transformations described in the literature so far can be treated as $\mathcal{C}$-morphisms connecting differential equations. After discussing this approach to Bäcklund transformations, this paper presents a previously unexplored connection between Bäcklund transformations and a type of pseudosymmetry, hereafter referred to as nonlocal pseudosymmetries. Indeed, we demonstrate how factorisation by nonlocal pseudosymmetries enables the derivation of Bäcklund transformations, determined by the basic invariants of the exploited nonlocal pseudosymmetries. This is done in the case of equations admitting zero-curvature representations (ZCRs), where it is natural to consider the nonlocal pseudosymmetries defined in a kind of Riccati-type differentiable coverings, which are shown to be determined by such ZCRs.

As illustrated by the application to several representative examples, including one involving a novel integrable equation, our framework allows for a very general structural approach, distinct from the case-by-case ones often adopted in the literature—especially in the analysis of cases where the sought Bäcklund transformations also affect the independent variables.

The paper is organised as follows. Section 2 is a section of preliminaries, devoted to some general aspects of the geometry of differential equations and a discussion of more specific notions that will play a particularly significant role in the remainder of the paper. This section is divided into the following four subsections: the first is devoted to a brief introduction to differential equations as submanifolds of jet spaces; the second deals with $\mathcal{C}$-morphisms; the third, after a short review of the notion of symmetry, provides a discussion of the notion of pseudosymmetry together with a brief description of how this notion leads to a factorisation procedure; finally the fourth subsection reviews the theory of differential coverings. In particular, some significant examples are included in this section to illustrate the central notions discussed here. Next, starting from ZCRs of $\mathcal{E}$, in Section 3 we show how to determine Riccati-type differential coverings of $\mathcal{E}$ and corresponding nonlocal conservation laws. This type of covering, with the related conservation laws, is particularly useful for determining possible nonlocal pseudosymmetries of $\mathcal{E}$. Finally, Section 4 shows how factorisation using nonlocal pseudosymmetries of an equation $\mathcal{E}$ can provide Bäcklund transformations for $\mathcal{E}$. This approach to determining Bäcklund transformations is illustrated here through several representative examples, including one involving a novel integrable equation, which demonstrate not only the method's degree of generality but also its applicability.

Throughout the paper the following main notations and conventions are adopted:

- all objects in the paper, e.g., manifolds, mappings, functions, vector fields, etc, are supposed to be smooth;
- in addition to vector fields, intended as derivations of the algebras of smooth functions, we will also consider *relative vector fields* (also referred to in the literature as vector fields along maps) [7,24] that naturally occur when considering projections of vector fields in a fiber bundle;
- for jet bundles $J^k(\pi)$ and $J^\infty(\pi)$ of a fiber bundle $\pi : E \to M$ (in particular for total derivatives, Cartan distributions, prolongations, symmetries, horizontal and vertical differentials, conservation laws, zero-curvature representations, and differential coverings) we will use the notations and conventions revised in Subsections 2.1, 2.3.1 and 2.4;
- (x_i, u^j_σ) usually denote the *canonical coordinates* in a k-order jet bundle $J^k(\pi)$;

- the algebra of smooth functions on a manifold N is usually denoted by $C^\infty(N)$, whereas $\mathcal{F}_k(\pi)$ and $\mathcal{F}(\pi)$ will denote the algebra of *smooth functions on* $J^k(\pi)$ and $J^\infty(\pi)$, respectively;
- by considering a system of differential equations $\mathcal{E} \subset J^k(\pi)$, we usually describe it as $\mathcal{E} = \{F^1(x, u_\sigma) = 0, \ldots, F^r(x, u_\sigma) = 0\}$;
- when $\mathcal{E}, \mathcal{V} \subset J^k(\pi)$ are such that $\mathcal{V}$ is a q-dimensional differentiable extension of $\mathcal{E}$, i.e., $\mathcal{V}$ defines a q-dimensional differentiable covering $\mathcal{V}^{(\infty)} \to \mathcal{E}^{(\infty)}$, we will usually distinguish the fiber coordinates $\{u^1, \ldots, u^m\}$ of π into the two subsets $\{z^1, \ldots, z^{m-q}\}$ (also referred to as *local variables*) and $\{v^1, \ldots, v^q\}$ (also referred to as *nonlocal variables*), such that $\mathcal{E} = \{F^1(x, z_\sigma) = 0, \ldots, F^r(x, z_\sigma) = 0\}$ and $\mathcal{V} = \{F^h(x, z_\sigma) = 0,\ v^s_i = X^s_i(x, z_\sigma, v)\}$, where $h = 1, \ldots, r$, $s = 1, \ldots q$ and i enumerates the independent variables $x_1, \ldots, x_n$;
- for Riccati-type differentiable coverings of an equation $\mathcal{E}$, we will usually denote nonlocal variables by $\{\rho^j\}$ and the differentiable extension of $\mathcal{E}$ by $\mathcal{Y}$;
- by a "basic system of invariants", we mean a system of invariants that generates all invariants through prolongations and functional combinations, whereas by a "complete system of invariants", we mean a system of invariants that generates all invariants only through functional combinations.

2. Preliminaries

We assume the reader is familiar with the geometric theory of differential equations. However, to make the main results accessible to a wide range of readers, we collected here some notations and basic facts of this theory used throughout the paper. We also introduce lesser-known topics that will be key in the forthcoming sections. The reader is referred to [8,25,38,44] and [4,12,21, 23,31,37,42], as well as references therein, for further details.

2.1. *Differential equations as submanifolds of jet spaces*

Consider a fiber bundle $\pi : E \to M$, with $dim\, M = n$ and $dim\, E = n + m$. Given a smooth (local) section s of π at $a \in M$, for any $k \in \mathbb{N}$, we denote by $[s]^k_a$ the k-th order jet of s at a; by definition $[s]^k_a$ is the equivalence class of (local) smooth sections of π that are k-order tangent (or, have k-fold contact) to s at a. The space $J^k(\pi)$ of k-th order jets of sections of π is naturally equipped with a differentiable manifold structure, induced by bundle atlas of π. If $\{x_1, \ldots, x_n\}$ are local coordinates on M and $\{u^1, \ldots, u^m\}$ local fiber coordinates of π, the induced *canonical coordinates* on $J^k(\pi)$ will be denoted by $\{x_i, u^j_\sigma\}$, where $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multi-index of order $|\sigma| = \sigma_1 + \ldots + \sigma_n$ such that $0 \le |\sigma| \le k$; by definition, if $\theta = [s]^k_a$, then $x_i(\theta) := x_i(a)$ and $u^j_\sigma(\theta) := \frac{\partial^{|\sigma|} s^j}{\partial x_1^{\sigma_1} \ldots \partial x_n^{\sigma_n}}(a)$. The manifold $J^k(\pi)$ is usually referred to as the *k-order jet bundle* of sections of π, since for any $k \in \mathbb{N}$ the natural projection $\pi_k : J^k(\pi) \to M$, $[s]^k_a \mapsto a$, is a fiber bundle; in canonical coordinates, π_k has the form $(x_i, u^j_\sigma) \mapsto (x_i)$. In particular, $J^0(\pi)$ can be identified with E and π_0 with π.

Throughout the paper, by $\mathcal{F}_k(\pi)$ we will denote the algebra of *smooth functions on* $J^k(\pi)$, $k \ge 1$. Moreover, to emphasize that $f \in \mathcal{F}_k(\pi)$ depends only on (x_i, u^j_σ), with $|\sigma| \le k$, we sometimes use the notation $f = f(x, u^{(k)})$. Also, for lower order k, notations like $u^j_{x_i}, u^j_{x_i x_j}, \ldots$ will usually be preferred to multi-index notation in concrete computations.

Now, since for any $h > k$ the *natural projections* $\pi_{h,k} : J^h(\pi) \to J^k(\pi)$, $[s]^h_a \to [s]^k_a$, are fiber bundles, one can also define the *infinite jet space* $J^\infty(\pi)$ as the inverse limit of the sequence

$M \overset{\pi}{\longleftarrow} J^0(\pi) \overset{\pi_{1,0}}{\longleftarrow} \ldots \overset{\pi_{k,k-1}}{\longleftarrow} J^k(\pi) \overset{\pi_{k+1,k}}{\longleftarrow} \ldots$. By definition, $J^\infty(\pi)$ is the space of sequences $\theta = \{\theta_i\}_{i\in\mathbb{N}}$ with $\theta_i \in J^i(\pi)$ and such that $\pi_{h,k}(\theta_h) = \theta_k$, for all $h > k$. Despite it is not a finite dimensional manifold, one can still introduce a differential calculus on $J^\infty(\pi)$ by making use of standard constructions of differential calculus over commutative algebras [8,25]. For instance, one can define the algebra $\mathcal{F}(\pi)$ of *smooth functions on* $J^\infty(\pi)$ as the filtered algebra given by direct limit of the sequence of inclusions $C^\infty(M) \overset{\pi^*}{\longrightarrow} \mathcal{F}_0(\pi) \overset{\pi^*_{1,0}}{\longrightarrow} \ldots \overset{\pi^*_{k,k-1}}{\longrightarrow} \mathcal{F}_k(\pi) \overset{\pi^*_{k+1,k}}{\longrightarrow} \ldots$. Analogously, one can define the exterior algebra $\Lambda^*(\pi)$ of *differential forms on* $J^\infty(\pi)$ as the filtered exterior algebra provided by the direct limit of the sequence of inclusions $\Lambda^*(M) \overset{\pi^*}{\longrightarrow} \Lambda^*\left(J^0(\pi)\right) \overset{\pi^*_{1,0}}{\longrightarrow} \ldots \overset{\pi^*_{k,k-1}}{\longrightarrow} \Lambda^*\left(J^k(\pi)\right) \overset{\pi^*_{k+1,k}}{\longrightarrow} \ldots$. Thus, being direct limits, any smooth function or form on an infinite jet space is nothing but a smooth function or form on some finite order jet space. Therefore, the *exterior differential* d naturally extends to differentiable forms on $J^\infty(\pi)$. On the other hand, using $\mathcal{F}(\pi)$, one can also think about *vector fields* on $J^\infty(\pi)$ as derivations of $\mathcal{F}(\pi)$. Throughout the paper the $\mathcal{F}(\pi)$-module of vector fields on $J^\infty(\pi)$ will be denoted by $\mathcal{D}(\pi)$. In canonical coordinates these vector fields can be identified with formal series $Z = \sum_i \alpha_i \partial_{x_i} + \sum_\sigma \sum_j \beta^j_\sigma \partial_{u^j_\sigma}$, with $\alpha_i, \beta^j_\sigma \in \mathcal{F}(\pi)$. In particular, one can say that a vector field Z has *filtration degree* r when it is the smallest $r \in \mathbb{N}$ such that $Z(\mathcal{F}_k(\pi)) \subseteq \mathcal{F}_{k+r}(\pi)$, $\forall k \in \mathbb{N}$, $k \geq 1$. Since in general vector fields on $J^\infty(\pi)$ do not have an associated flow, a particularly important case is that of vector fields with zero filtration degree, because any such field Z admits a flow that can be seen as an inverse limit of a sequence of flows on finite order jet spaces. Also the *Lie derivative* of functions, vector fields or forms on $J^\infty(\pi)$ can be defined in a completely algebraic way. For instance, the Lie derivative of a function $f \in \mathcal{F}(\pi)$ along a vector field $Z \in \mathcal{D}(\pi)$ is $L_Z(f) := Z(f)$, and the Lie derivative of $Y \in \mathcal{D}(\pi)$ along Z is $L_Z Y := [Z, Y] = Z \circ Y - Y \circ Z$. Whereas, the Lie derivative of a form $\omega \in \Lambda^*(\pi)$ along Z is defined as $L_Z\omega := i_Z(d\omega) + d(i_Z\omega)$, where i_Z denotes the *insertion operator* $i_Z : \Lambda^h(\pi) \longrightarrow \Lambda^{h-1}(\pi)$.

Jets spaces are naturally equipped with a tangent distribution which is referred to as *Cartan distribution*, or contact distribution. Indeed, if s is a (local) section of π, then the *k-th order jet prolongation* $s^{(k)}$ of s is the (local) section of π_k defined by $s^{(k)}(a) = [s]^k_a$, for any a in the domain of s. Then the *Cartan distribution* $\mathcal{C}^k(\pi) = \cup_{\theta\in J^k(\pi)} \mathcal{C}^k_\theta(\pi)$ on the k-th order jet space $J^k(\pi)$ can be point-wise defined by the spans $\mathcal{C}^k_\theta(\pi)$ of the tangent planes at $\theta = [s]^k_a$ to the graphs of k-th order jet prolongations $s'^{(k)}$ of sections s' such that $[s']^k_a = [s]^k_a$. In terms of canonical coordinates, the Cartan distribution $\mathcal{C}^k(\pi)$ is described by the annihilator of the Pfaffian system $\{\omega^j_\sigma : 0 \leq |\sigma| \leq k-1,\ j = 1, \ldots, m\}$, with $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma+1_i} dx_i$ denoting the so called Cartan forms. Dually, $\mathcal{C}^k(\pi)$ can also be described as the distribution generated by the system of vector fields $\{\partial_{u^j_\sigma}, D^{(k)}_i : |\sigma| = k,\ j = 1, \ldots, m,\ i = 1, \ldots, n\}$, with $D^{(k)}_i := \partial_{x_i} + \sum_{|\sigma|\leq k-1} u^j_{\sigma+1_i} \partial_{u^j_\sigma}$ denoting the k-th order *truncated total derivative*s.

Then by taking the inverse limit of the sequence of surjections $\mathcal{C}^1(\pi) \overset{\pi_{2,1*}}{\longleftarrow} \mathcal{C}^2(\pi) \overset{\pi_{3,2*}}{\longleftarrow} \ldots \overset{\pi_{k,k-1*}}{\longleftarrow} \mathcal{C}^k(\pi) \overset{\pi_{k+1,k*}}{\longleftarrow} \ldots$, one defines the *Cartan distribution* $\mathcal{C}(\pi)$ of $J^\infty(\pi)$. One can see $\mathcal{C}(\pi)$ as the distribution annihilating all Cartan forms $\{\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma+1_i} dx_i : |\sigma| \geq 0,\ j = 1, \ldots, m\}$, or equivalently the distribution generated by all *total derivative*s

$$D_i := \partial_{x_i} + \sum_{|\sigma|\geq 0} \sum_{j=1}^{m} u^j_{\sigma+1_i} \partial_{u^j_\sigma}, \qquad i = 1, \ldots, n.$$

It is easy to show that integral manifolds Σ of Cartan distributions with *independence condition* $\Omega = dx_1 \wedge \ldots \wedge dx_n \neq 0$ (i.e., such that $\Omega|_\Sigma \neq 0$) are prolongations of sections of π (see for instance [44]).

Geometrically, the solutions of a k-th order differential equation (or system) $\mathcal{E} = \{\mathbf{F} = \mathbf{0}\} \subset J^k(\pi)$ are just sections s of π whose k-order prolongations $j_k(s)$ lay on $\mathcal{E}$. Under regularity conditions for $\mathbf{F}$, if $\mathcal{E}$ is a submanifold of $J^k(\pi)$, it is naturally equipped with the *induced Cartan distribution* $\mathcal{C}^k(\mathcal{E}) := \mathcal{C}^k(\pi) \cap T\mathcal{E}$ and the solutions of $\mathcal{E}$ are sections of π whose k-order prolongation are integral manifolds of $\mathcal{C}^k(\mathcal{E})$.

On the other hand, under further regularity conditions for $\mathcal{E}$, for any $r \in \mathbb{N}$ one may also consider the *r-th order prolongation* $\mathcal{E}^{(r)} = \{D_\mu \mathbf{F} = \mathbf{0} : \ 0 \leq |\mu| \leq r\}$, where $D_\mu := (D_1)^{\mu_1} \circ \ldots \circ (D_n)^{\mu_n}$. One says that $\mathcal{E}$ is *formally integrable* if and only if for any $r \in \mathbb{N}$ the prolongations $\mathcal{E}^{(r)}$ are submanifolds of $J^{k+r}(\pi)$ and the maps $\pi_{k+r+1,k+r} : \mathcal{E}^{(r+1)} \to \mathcal{E}^{(r)}$ are smooth fiber bundles.

Then the *infinite prolongation* $\mathcal{E}^{(\infty)}$, of a formally integrable equation $\mathcal{E}$, is defined as the inverse limit of the sequence of fiber bundles $\pi_{k+r+1,k+r} : \mathcal{E}^{(r+1)} \to \mathcal{E}^{(r)}$. Since each $\mathcal{E}^{(r)}$ is naturally equipped with the induced Cartan distribution $\mathcal{C}^{k+r}(\mathcal{E}^{(r)})$, also $\mathcal{E}^{(\infty)}$ is equipped with an induced *Cartan distribution* $\mathcal{C}(\mathcal{E})$ defined by the inverse limit of the sequence of surjections $\pi_{k+r+1,k+r*} : \mathcal{C}^{k+r+1}\left(\mathcal{E}^{(r+1)}\right) \to \mathcal{C}^{k+r}\left(\mathcal{E}^{(r)}\right)$. One has that $\mathcal{E}^{(\infty)} = \{D_\mu F = 0 : \ |\mu| \geq 0\} \subset J^\infty(\pi)$ and $\mathcal{C}(\mathcal{E}) = \left\langle \bar{D}_1, \ldots, \bar{D}_n \right\rangle = \operatorname{Ann}\left\{\bar{\omega}^j_\sigma : \ j = 1, \ldots, m, \quad |\sigma| \geq 0\right\}$, where $\bar{D}_i$ and $\bar{\omega}^j_\sigma$ are the restrictions to $\mathcal{E}^{(\infty)}$ of the total derivatives and Cartan forms, respectively. Moreover, by restricting $\Lambda^*(\pi)$ to $\mathcal{E}^{(\infty)}$ one gets the exterior algebra $\Lambda^*(\mathcal{E})$ of *differential forms on* $\mathcal{E}^{(\infty)}$ and in particular the algebra $\mathcal{F}(\mathcal{E})$ of *smooth functions on* $\mathcal{E}^{(\infty)}$.

Now, since $\mathcal{C}(\pi)$ is totally horizontal with respect to the mapping $\pi_\infty : J^\infty(\pi) \to M$, the tangent bundle $\mathcal{T}(\pi)$ on $J^\infty(\pi)$ decomposes as $\mathcal{T}(\pi) = \mathcal{V}(\pi) \oplus \mathcal{C}(\pi)$, where $\mathcal{V}(\pi) := Ker\,(\pi_\infty)_*$. Dually one has $\Lambda^1(\pi) = \Lambda^{(1,0)}(\pi) \oplus \Lambda^{(0,1)}(\pi)$, where $\Lambda^{(1,0)}(\pi) := Ann\,(\mathcal{V}(\pi))$ and $\Lambda^{(0,1)}(\pi) := Ann\,(\mathcal{C}(\pi))$ are the $\mathcal{F}(\pi)$-modules of horizontal and vertical 1-forms on $J^\infty(\pi)$ locally generated by $\{dx_i\}$ and Cartan forms $\left\{\omega^j_\sigma\right\}$, respectively. More in general, by considering $\Lambda^{(p,q)}(\pi) = \left(\bigwedge^p \Lambda^{(1,0)}(\pi)\right) \bigwedge \left(\bigwedge^q \Lambda^{(0,1)}(\pi)\right)$, the $\mathcal{F}(\pi)$-module of r-forms on $J^\infty(\pi)$ decomposes as $\Lambda^r(\pi) = \bigoplus_{p+q=r} \Lambda^{(p,q)}(\pi)$. By definition we set $\mathcal{F}(\pi) = \Lambda^{(0,0)}(\pi)$. Accordingly, the exterior differential splits into the sum $d = d_H + d_V$ of the horizontal and vertical differentials $d_H : \Lambda^{(p,q)}(\pi) \to \Lambda^{(p+1,q)}(\pi)$ and $d_V : \Lambda^{(p,q)}(\pi) \to \Lambda^{(p,q+1)}(\pi)$, satisfying $d_H^2 = d_V^2 = 0$ and $d_H \circ d_V = -d_V \circ d_H$. In coordinates, these differentials can be easily computed since they act as graded derivations on $\Lambda^*(\pi)$ and for any function $f \in \mathcal{F}(\pi)$ one has $d_H f := \sum_i D_i f\, dx_i$ and $d_V f := \sum_\sigma \sum_j \frac{\partial f}{\partial u^j_\sigma} \omega^j_\sigma$.

Analogously, given a formally integrable equation $\mathcal{E}$, since $\mathcal{C}(\mathcal{E})$ is totally horizontal with respect to the mapping $\bar{\pi}_\infty : \mathcal{E}^{(\infty)} \to M$, the tangent bundle $\mathcal{T}(\mathcal{E})$ on $\mathcal{E}^{(\infty)}$ decomposes as $\mathcal{T}(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \oplus \mathcal{C}(\mathcal{E})$, where $\mathcal{V}(\mathcal{E}) := Ker\,(\bar{\pi}_{\infty *})$ is the vertical bundle on $\mathcal{E}^{(\infty)}$. Hence the $\mathcal{F}(\mathcal{E})$-modules $\Lambda^{(1,0)}(\mathcal{E})$ and $\Lambda^{(0,1)}(\mathcal{E})$ of horizontal and vertical 1-forms on $\mathcal{E}^{(\infty)}$, locally generated by $\{dx^i\}$ and restricted Cartan forms $\left\{\bar{\omega}^j_\sigma := \omega^j_\sigma\Big|_{\mathcal{E}^{(\infty)}}\right\}$, can be used to decompose the $\mathcal{F}(\mathcal{E})$-module of r-forms on $\mathcal{E}^{(\infty)}$ as $\Lambda^r\left(\mathcal{E}^{(\infty)}\right) = \bigoplus_{p+q=r} \Lambda^{(p,q)}(\mathcal{E})$; in particular one has $\mathcal{F}(\mathcal{E}) = \Lambda^{(0,0)}(\mathcal{E})$. Accordingly, on $\mathcal{E}^{(\infty)}$ the exterior differential d (still denoted by d, for ease of notation) splits into the sum $d = \bar{d}_H + \bar{d}_V$ of the horizontal and vertical differentials $\bar{d}_H : \Lambda^{(p,q)}(\mathcal{E}) \to \Lambda^{(p+1,q)}(\mathcal{E})$ and $\bar{d}_V : \Lambda^{(p,q)}(\mathcal{E}) \to \Lambda^{(p,q+1)}(\mathcal{E})$, which satisfy $\bar{d}_H^2 = \bar{d}_V^2 = 0$ and $\bar{d}_H \circ \bar{d}_V = -\bar{d}_V \circ \bar{d}_H$. Also in this case, these differentials can be easily computed in coordinates, since they act as graded derivations on $\Lambda^*(\mathcal{E}^{(\infty)})$ and for any function $f \in \mathcal{F}(\mathcal{E})$ one has that $\bar{d}_H f := \sum_i \bar{D}_i f \, dx_i$ and $\bar{d}_V f := \sum_\sigma \sum_j \frac{\partial f}{\partial u^j_\sigma} \bar{\omega}^j_\sigma$, where $\bar{D}_i$ denote the total derivatives restricted to $\mathcal{E}^{(\infty)}$. For notational convenience, from now on we will denote $\Lambda^{(p,0)}(\pi)$ and $\Lambda^{(p,0)}(\mathcal{E})$ by $\bar{\Lambda}^p(\pi)$ and $\bar{\Lambda}^p(\mathcal{E})$, respectively.

Starting from these algebraic structures in the algebra of differential forms on $\mathcal{E}$, new constructions and new notions can be introduced further. For instance, using the horizontal differentials $\bar{d}_H : \overline{\Lambda}^p(\mathcal{E}) \to \overline{\Lambda}^{p+1}(\mathcal{E})$, one can consider the notion of *conservation law* for $\mathcal{E}$, which is a closed horizontal form $\mu \in \overline{\Lambda}^{n-1}(\mathcal{E})$, i.e., a horizontal $(n-1)$-form on $\mathcal{E}^{(\infty)}$ satisfying $\bar{d}_H \mu = 0$. On the other hand, since an exact horizontal form is trivially closed, one can naturally limit itself to consider closed horizontal $(n-1)$-forms up to exact horizontal forms. Thus, a conservation law can also be understood as a cohomology class $[\mu] \in \overline{H}^{n-1}(\mathcal{E})$, i.e., $[\mu] = \left\{ \mu + \bar{d}_H \rho : \rho \in \overline{\Lambda}^{n-2}(\mathcal{E}) \right\}$. For instance, when $n = 2$, a conservation law is locally described by an horizontal 1-form $A\, dx_1 + B\, dx_2$ on $\mathcal{E}^{(\infty)}$ such that $\bar{D}_2 A - \bar{D}_1 B = 0$, i.e., $D_2 A - D_1 B = 0$ on $\mathcal{E}^{(\infty)}$.

Moreover, given a matrix Lie algebra $\mathfrak{g}$, one may consider the exterior algebras $\mathfrak{g} \otimes \Lambda^*(\pi)$ and $\mathfrak{g} \otimes \Lambda^*(\mathcal{E})$ of $\mathfrak{g}$-valued forms on $J^\infty(\pi)$ and $\mathcal{E}^{(\infty)}$, respectively. Also, one can consider the graded algebra of $\mathfrak{g}$-valued horizontal forms on $J^\infty(\pi)$ and $\mathcal{E}^{(\infty)}$, that will be denoted here by $\mathfrak{g} \otimes \bar{\Lambda}^*(\pi) = \bigoplus_p \mathfrak{g} \otimes \bar{\Lambda}^p(\pi)$ and $\mathfrak{g} \otimes \bar{\Lambda}^*(\mathcal{E}) = \bigoplus_p \mathfrak{g} \otimes \bar{\Lambda}^p(\mathcal{E})$, respectively. By definition, $\mathfrak{g}$-valued horizontal p-forms on $J^\infty(\pi)$ (resp., $\mathcal{E}^{(\infty)}$) are generated by $\mathfrak{g}$-valued p-forms $A\omega$, with A a $\mathfrak{g}$-valued functions on $J^\infty(\pi)$ (resp., $\mathcal{E}^{(\infty)}$). Then, one may define a bilinear product $[\ ,\ ]$ by linearly extending the product $[A_1\omega_1, A_2\omega_2] := [A_1, A_2]\, \omega_1 \wedge \omega_2$, between generators. One can check that $[\ ,\ ]$ satisfies the following properties: (i) $[\rho, \sigma] = -(-1)^{rs}[\sigma, \rho]$; (ii) $(-1)^{rt}[\rho, [\sigma, \tau]] + (-1)^{sr}[\sigma, [\tau, \rho]] + (-1)^{ts}[\tau, [\rho, \sigma]] = 0$; (iii) $d_H[\rho, \sigma] = [d_H\rho, \sigma] + (-1)^r[\rho, d_H\sigma]$, analogously for $\bar{d}_H$. Where r, s and t are the degrees of the $\mathfrak{g}$-valued horizontal forms ρ, σ and τ, respectively. Also, one may define an exterior product $\wedge$ on $\mathfrak{g} \otimes \bar{\Lambda}^*(\pi)$, or $\mathfrak{g} \otimes \bar{\Lambda}^*(\mathcal{E})$, by linearly extending the product $A_1\, \omega_1 \wedge A_2\, \omega_2 = A_1 A_2\, \omega_1 \wedge \omega_2$.

This allows one to introduce the notion of *zero-curvature representation* (ZCR) of $\mathcal{E}$, that is a $\mathfrak{g}$-valued non-vanishing 1-form $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ such that

$$\bar{d}_H \alpha - \frac{1}{2}[\alpha, \alpha] = 0. \tag{1}$$

This is a very important notion in the theory of integrable equations with 2 independent variables [1,12,21,45,49]. In such a case, by taking $\alpha = A\, dx_1 + B\, dx_2$, condition (1) reads $D_2 A - D_1 B + [A, B] = 0$ on $\mathcal{E}^{(\infty)}$.

Here we notice that (1) can also be equivalently written as $\bar{d}_H \alpha - \alpha \wedge \alpha = 0$. Moreover, since $\mathcal{E}^{(\infty)} \subset J^\infty(\pi)$, any element of $\mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ can be identified with an element of $\mathfrak{g} \otimes \bar{\Lambda}^1(\pi)$. Hence, in the outer geometry, (1) can also be rewritten as $d_H \alpha - \frac{1}{2}[\alpha, \alpha] = 0 \mod \mathcal{E}^{(\infty)}$. In particular, when $\mathcal{E} = \{F^j = 0,\ j = 1, \ldots, h\}$, under regularity assumptions (i.e., if any prolongation $\mathcal{E}^{(h)}$, $h \geq 0$, is totally non-degenerating [38]) equation (1) can also be rewritten in "characteristic" form $d_H \alpha - \frac{1}{2}[\alpha, \alpha] = \sum D_\sigma(F^j)\gamma^\sigma_j$, where $\gamma^\sigma_j \in \mathfrak{g} \otimes \bar{\Lambda}^2(\pi)$. Hence in general (1) holds modulo differential consequences of $\mathcal{E}$, thus (1) is not equivalent to $\{F^j = 0,\ j = 1, \ldots, h\}$.

2.2. $\mathcal{C}$-morphisms (or Lie-Bäcklund maps)

In this subsection $J^\infty(\pi)$ and $J^\infty(\pi')$ will denote the infinite jet spaces of sections of two fiber bundles $\pi : E \to M$ and $\pi' : E' \to M'$ with *canonical coordinates* $\{x_i, u^j_\sigma\}$ and $\{x'_i, u'^j_\sigma\}$, respectively. In particular we assume that $n = \dim M = \dim M'$, since in the paper we are mainly concerned with this case.

A $\mathcal{C}$-morphism, also referred to as Lie-Bäcklund transformations [4,25], is a smooth map $\mathcal{B} : J^\infty(\pi) \to J^\infty(\pi')$, such that $\mathcal{B}_*\mathcal{C}_\theta(\pi) \subseteq \mathcal{C}_{\mathcal{B}(\theta)}(\pi')$, for any $\theta \in J^\infty(\pi)$, i.e.,

$$\mathcal{B}_*\mathcal{C}(\pi) \subseteq \mathcal{C}(\pi'). \tag{2}$$

Notice that $\mathcal{B}_*\mathcal{C}(\pi) \subseteq \mathcal{C}(\pi')$ is equivalent to $\mathcal{B}^*\left(\mathcal{I}_{\mathcal{C}(\pi')}\right) \subseteq \mathcal{I}_{\mathcal{C}(\pi)}$, where $\mathcal{I}_{\mathcal{C}(\pi)}$ is the EDS generated by Cartan (or multi-contact) forms $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma+1_i}dx_i$ on $J^\infty(\pi)$, analogously $\mathcal{I}_{\mathcal{C}(\pi')}$.

Notice that in general these transformations are generalizations of Lie transformations (i.e., point and contact transformations) that need not to be diffeomorphisms. Indeed, as shown by Bäcklund [6] (see also [4,5,30]), only Lie transformations are invertible. Moreover, by $\mathcal{B}^*\left(\mathcal{I}_{\mathcal{C}(\pi')}\right) \subseteq \mathcal{I}_{\mathcal{C}(\pi)}$ it follows that under such a transformation the image of an integral manifolds of Cartan distribution is still an integral manifold.

In canonical coordinates, a $\mathcal{C}$-morphism (Lie-Bäcklund transformation) has the form

$$\begin{cases} x'_i = \xi^i\left(x, u^{(k)}\right), \\ u'^j = \nu^j\left(x, u^{(k)}\right), \\ \vdots \\ u'^j_\sigma = \nu^j_\sigma\left(x, u^{(k+|\sigma|)}\right), \qquad |\sigma| \geq 0 \end{cases} \tag{3}$$

where in view of $\mathcal{B}^*\left(\mathcal{I}_{\mathcal{C}(\pi')}\right) \subseteq \mathcal{I}_{\mathcal{C}(\pi)}$ the functions ξ^i and ν^j_σ are such that $d\nu^j_\sigma - \sum_i \nu^j_{\sigma+1_i}d\xi^i = 0 \mod \mathcal{I}_{\mathcal{C}}$.

Like in the finite order case, integral manifolds Σ of $\mathcal{C}(\pi)$ with independence condition $\Omega{=}dx_1 \wedge \ldots \wedge dx_n \neq 0$ (i.e., such that $\Omega|_\Sigma \neq 0$) are ∞-th order prolongations of sections of π. One has the following

Proposition 1. *A $\mathcal{C}$-morphism maps an integral manifold Σ of $\mathcal{C}(\pi)$ with independence condition $dx_1 \wedge \ldots \wedge dx_n \neq 0$ to an integral manifold Σ' of $\mathcal{C}(\pi')$ with independence condition $dx'_1 \wedge \ldots \wedge dx'_n \neq 0$ whenever the regularity assumption*

$$\det\left(D_s\xi^i\right) \neq 0 \tag{4}$$

is satisfied on Σ.

Proof. Indeed, being Σ the infinite prolongation of a section $\mathfrak{s}$ locally described as $\mathfrak{s}(x) = (x, u(x))$, one has $\left(\mathcal{B} \circ \mathfrak{s}^{(\infty)}\right)^* (\omega'^j_\sigma) = \left(\mathfrak{s}^{(\infty)}\right)^* \circ \mathcal{B}^*\left(\omega'^j_\sigma\right) = 0 \mod \mathcal{I}_{\mathcal{C}(\pi)}$, since the graph of $\mathfrak{s}^{(\infty)}$ is an integral manifold of $\mathcal{I}_{\mathcal{C}(\pi)}$. Thus the graph Σ' of $\mathcal{B} \circ \mathfrak{s}^{(\infty)}$ describes another integral manifold of $\mathcal{I}_{\mathcal{C}(\pi')}$. On the other hand whenever $\det\left(D_s\xi^i\right) \neq 0$ along the graph of $\mathfrak{s}^{(k)}$ (i.e., $\mathfrak{s}^{(k)*}\left(\det\left(D_s\xi^i\right)\right) \neq 0$), the first n-equations $\{x'_i = \xi^i(x, u^{(k)}(x)),\ i = 1, ..., n\}$ of (3) restricted to $\mathfrak{s}^{(k)}$ can be locally solved with respect to $(x_1, ..., x_n)$. Thus, there exists a local diffeomorphism $x' = \mathcal{B}_{\mathfrak{s}}(x) = \left(\xi \circ \mathfrak{s}^{(k)}\right)(x)$ that allows one to pass from the parametrization $\left(\mathcal{B} \circ \mathfrak{s}^{(\infty)}\right)(x)$ of Σ' to the new parametrization $\left(\mathcal{B} \circ \mathfrak{s}^{(\infty)} \circ \mathcal{B}_{\mathfrak{s}}^{-1}\right)(x')$.

Then, since

$$\begin{aligned}
\left(\mathcal{B} \circ \mathfrak{s}^{(\infty)} \circ \mathcal{B}_{\mathfrak{s}}^{-1}\right)^* \left(dx'_1 \wedge ... \wedge dx'_n\right) &= \left(\left(\mathcal{B}_{\mathfrak{s}}^{-1}\right)^* \circ s^{(\infty)*} \circ \mathcal{B}^*\right)\left(dx'_1 \wedge ... \wedge dx'_n\right) \\
&= \left(\left(\mathcal{B}_{\mathfrak{s}}^{-1}\right)^* \circ s^{(\infty)*}\right)\left(d\left(\xi^1(x, u^{(k)})\right) \wedge ... \wedge d\left(\xi^n\left(x, u^{(k)}\right)\right)\right) \\
&= \left(\mathcal{B}_{\mathfrak{s}}^{-1}\right)^*\left(d\left(\xi^1 \circ s^{(k)}(x)\right) \wedge ... \wedge d\left(\xi^n \circ s^{(k)}(x)\right)\right) \\
&= dx'_1 \wedge ... \wedge dx'_n,
\end{aligned}$$

it turns out that $\left(\mathcal{B} \circ \mathfrak{s}^{(\infty)} \circ \mathcal{B}_{\mathfrak{s}}^{-1}\right)(x')$ is the infinite prolongation of the section of π'

$$\mathfrak{s}'(x') = \left(\mathcal{B}^{(0)} \circ \mathfrak{s}^{(k)} \circ \mathcal{B}_{\mathfrak{s}}^{-1}\right)(x') = \left(x', \nu\left(x, u^{(k)}\left(\mathcal{B}_{\mathfrak{s}}^{-1}(x')\right)\right)\right). \quad \square$$

Remark 2. Proposition 1 entails that a $\mathcal{C}$-morphism sends infinite prolongations of (local) sections to infinite prolongations of (local) sections, whenever (4) is satisfied. Indeed, it is noteworthy to stress that the action of such a $\mathcal{C}$-morphism $\mathcal{B}$ on the infinite prolongation $\mathfrak{s}^{(\infty)}(x)$ of a (local) section $\mathfrak{s}(x)$ is not necessarily defined for any x, because $\det\left(D_s\xi^i\right)(\mathfrak{s}^{(\infty)}(x))$ could be zero at some points. Also, we stress that (4) entails that the push-forward $\mathcal{B}_{*\theta} : \mathcal{C}_\theta(\pi) \to \mathcal{C}_{\mathcal{B}(\theta)}(\pi')$ establishes an isomorphism for any $\theta \in J^\infty(\pi)$.

We will refer to a $\mathcal{C}$-morphism (3) satisfying regularity assumption (4) as a *regular $\mathcal{C}$-morphism*. One also have the following

Proposition 3. *A regular $\mathcal{C}$-morphism (3) is completely determined by its lower components*

$$\begin{cases} x'_i = \xi^i\left(x, u^{(k)}\right), \\ u'^j = \nu^j\left(x, u^{(k)}\right), \end{cases}$$

through the prolongation formulas

$$\begin{pmatrix} \nu^1_{\sigma+1_1} & \cdots & \nu^m_{\sigma+1_1} \\ \vdots & & \vdots \\ \nu^1_{\sigma+1_n} & \cdots & \nu^m_{\sigma+1_n} \end{pmatrix} = \begin{pmatrix} D_1\xi^1 & \cdots & D_1\xi^n \\ \vdots & & \vdots \\ D_n\xi^1 & \cdots & D_n\xi^n \end{pmatrix}^{-1} \cdot \begin{pmatrix} D_1\nu^1_\sigma & \cdots & D_1\nu^m_\sigma \\ \vdots & & \vdots \\ D_n\nu^1_\sigma & \cdots & D_n\nu^m_\sigma \end{pmatrix}. \tag{5}$$

Proof. Indeed, in view of the decomposition $d = d_H + d_V$, one readily gets that

$$\begin{aligned}\mathcal{B}^*\left(\omega'^j_\sigma\right) &= \mathcal{B}^*\left(du'^j_\sigma - \sum_i u'^j_{\sigma+1_i} dx'_i\right) = d v^j_\sigma - \sum_i v^j_{\sigma+1_i} d\xi^i \\ &= d_H v^j_\sigma - \sum_i v^j_{\sigma+1_i} d_H \xi^i + d_V v^j_\sigma - \sum_i v^j_{\sigma+1_i} d_V \xi^i \\ &= \sum_s \left(D_s v^j_\sigma - \sum_i v^j_{\sigma+1_i} D_s \xi^i\right) dx_s \mod \mathcal{I}_{\mathcal{C}(\pi)},\end{aligned}$$

in view of the identity $d_V v^j_\sigma - \sum_i v^j_{\sigma+1_i} d_V \xi^i = 0 \mod \mathcal{I}_{\mathcal{C}(\pi)}$. Thus $\mathcal{B}^*\left(\mathcal{I}_{\mathcal{C}(\pi')}\right) \subseteq \mathcal{I}_{\mathcal{C}(\pi)}$ if and only if $D_s v^j_\sigma - \sum_i v^j_{\sigma+1_i} D_s \xi^i = 0$, that is

$$\begin{pmatrix} D_1\xi^1 & \cdots & D_1\xi^n \\ \vdots & & \vdots \\ D_n\xi^1 & \cdots & D_n\xi^n \end{pmatrix} \cdot \begin{pmatrix} v^1_{\sigma+1_1} & \cdots & v^m_{\sigma+1_1} \\ \vdots & & \vdots \\ v^1_{\sigma+1_n} & \cdots & v^m_{\sigma+1_n} \end{pmatrix} = \begin{pmatrix} D_1 v^1_\sigma & \cdots & D_1 v^m_\sigma \\ \vdots & & \vdots \\ D_n v^1_\sigma & \cdots & D_n v^m_\sigma \end{pmatrix}.$$

Hence, in view of (4) one gets (5). □

Moreover, in view of (2), one also has the following

Lemma 4. *For any $\mathcal{C}$-morphism $\mathcal{B} : J^\infty(\pi) \to J^\infty(\pi')$ one has*

$$D_i \circ \mathcal{B}^* = \sum_j \alpha_{ij} \mathcal{B}^* \circ D'_j, \tag{6}$$

for some smooth functions α_{ij} on $J^\infty(\pi)$. In particular, if $\mathcal{B}$ is regular, one has $\det\left(\alpha_{ij}\right) \neq 0$ *and*

$$\mathcal{B}^* \circ D'_j = \sum_i \alpha^{ij} D_i \circ \mathcal{B}^*, \tag{7}$$

with $(\alpha^{ij}) = (\alpha_{ij})^{-1}$.

Proof. The invariance condition (2) is equivalent to say that for any $\theta \in J^\infty(\pi)$ one has $\mathcal{B}_*(D_i|_\theta) = \sum_j a_{ij} \left. D_j \right|_{\mathcal{B}(\theta)}$, where a_{ij} are some constants. Hence, for any function f on $J^\infty(\pi')$ one has that $D_i\left(\mathcal{B}^*(f)\right)(\theta) = \sum_j a_{ij} D_j f\left(\mathcal{B}(\theta)\right) = \sum_j a_{ij} \mathcal{B}^*\left(D_j f\right)(\theta)$. Then, since this holds for any θ one readily gets (6).

Now, applying (6) to $f = x'_s$ and using the fact that $\mathcal{B}^*\left(x'_s\right) = \xi^s\left(x, u^{(k)}\right)$ and $\mathcal{B}^*\left(D'_j x'_s\right) = \mathcal{B}^*\delta_{js} = \delta_{js}$, one readily gets $D_i \xi^s = \alpha_{is}$ and hence (7) readily follows by (4) and (6). □

We will use the following

Definition 5. Let $\mathcal{E}^{(\infty)}$ and $\mathcal{E}'^{(\infty)}$ be the infinite prolongations of two formally integrable equations $\mathcal{E} \subset J^k(\pi)$ and $\mathcal{E}' \subset J^l(\pi')$, respectively. By a (regular) $\mathcal{C}$-morphism from $\mathcal{E}$ to $\mathcal{E}'$ we mean a regular $\mathcal{C}$-morphism $\mathcal{B} : J^\infty(\pi) \to J^\infty(\pi')$ such that $\mathcal{B}(\mathcal{E}^{(\infty)}) \subseteq \mathcal{E}'^{(\infty)}$.

In view of Proposition 1 and Remark 2, a regular $\mathcal{C}$-morphism from $\mathcal{E}$ to $\mathcal{E}'$ transforms solutions of $\mathcal{E}$ to solutions of $\mathcal{E}'$. Moreover, one has the following

Proposition 6. *A regular $\mathcal{C}$-morphism $\mathcal{B}: J^\infty(\pi) \to J^\infty(\pi')$ is a $\mathcal{C}$-morphism from $\mathcal{E} = \{\mathbf{F} = \mathbf{0}\}$ to $\mathcal{E}' = \{\mathbf{F}' = \mathbf{0}\}$ if and only if*

$$\mathcal{B}^*(\mathbf{F}') = 0 \mod \{D_\sigma \mathbf{F} = \mathbf{0} : |\sigma| \geq 0\}. \tag{8}$$

Proof. Since $\mathcal{E}^{(\infty)} = \{D_\sigma \mathbf{F} = \mathbf{0} : |\sigma| \geq 0\}$ and $\mathcal{E}'^{(\infty)} = \{D'_\rho \mathbf{F}' = \mathbf{0} : |\rho| \geq 0\}$, the condition $\mathcal{B}(\mathcal{E}^{(\infty)}) \subseteq \mathcal{E}'^{(\infty)}$ is equivalent to $\mathcal{B}^*\left(D'_\rho\left(\mathbf{F}'\right)\right) = 0 \mod \mathcal{E}^{(\infty)}$, for any multi-index ρ. Hence, in view of (7), $\mathcal{B}(\mathcal{E}^{(\infty)}) \subseteq \mathcal{E}'^{(\infty)}$ is equivalent to

$$D_\mu \mathcal{B}^*\left(\mathbf{F}'\right) = 0 \mod \{D_\sigma \mathbf{F} = \mathbf{0} : |\sigma| \geq 0\}, \tag{9}$$

for any multi-index μ. Thus (9) implies $\mathcal{B}^*(\mathbf{F}') = 0 \mod \{D_\sigma \mathbf{F} = \mathbf{0} : |\sigma| \geq 0\}$, for $\mu = 0$. Conversely, by totally deriving $\mathcal{B}^*(\mathbf{F}') = 0 \mod \{D_\sigma \mathbf{F} = \mathbf{0} : |\sigma| \geq 0\}$ one gets (9). □

It is noteworthy to remark that, being the order of $\mathcal{B}^*(\mathbf{F}')$ always finite, the analysis of condition (8) in practice only requires the consideration of a finite number of differential consequences of $\mathbf{F} = \mathbf{0}$.

Example 7. An example of regular $\mathcal{C}$-morphism between two equations is the Cole-Hopf transformation from the heat equation $\mathcal{E} = \{u_t - u_{xx} = 0\}$ to the Burgers equation $\mathcal{E}' = \{u'_t - u'_{xx} - 2u'u'_x = 0\}$, where $u = u(x,t)$ and $u' = u'(x,t)$. Indeed the Cole-Hopf transformation is defined by $u' = u_x/u$, at the points where $u \neq 0$. In this case, by repeatedly applying (5), one can readily check that

$$u'_x = \frac{u_{xx}}{u} - \left(\frac{u_x}{u}\right)^2, \qquad u'_{xx} = \frac{u_{xxx}}{u} - 3\left(\frac{u_x}{u}\right)\left(\frac{u_{xx}}{u}\right) + 2\left(\frac{u_x}{u}\right)^3, \quad \ldots .$$

On the other hand

$$u'_t = \frac{u_{xt} - u_t\, u'}{u} = \frac{u_{xxx}}{u} - \left(\frac{u_{xx}}{u}\right)\left(\frac{u_x}{u}\right) \mod \mathcal{E}^{(1)},$$

hence $u'_t - u'_{xx} - 2u'u'_x = 0 \mod \mathcal{E}^{(1)}$.

Example 8. Another example of regular $\mathcal{C}$-morphism between two equations is the Miura transformation from the mKdV equation $\mathcal{E} = \{u_t - u_{xxx} + 6u^2 u_x = 0\}$ to the KdV equation $\mathcal{E}' = \{u'_t - u'_{xxx} - 6u'u'_x = 0\}$, where $u = u(x,t)$ and $u' = u'(x,t)$. For ease of comparison, here and in Example 14, it will be used the form of KdV adopted in [42] where the Miura transformation is defined by $u' = u_x - u^2$. In this case, by repeatedly applying (5), one can readily check that

$$\begin{aligned} &u'_x = -2\,u u_x + u_{xx}, \qquad u'_{xx} = -2\,u_x^2 - 2\,u u_{xx} + u_{xxx}, \\ &u'_{xxx} = -6\,u_x u_{xx} - 2\,u u_{xxx} + u_{xxxx}, \quad \ldots . \end{aligned}$$

On the other hand

$$u'_t = u_{xt} - 2uu_t = u_{xxxxx} - 2uu_{xxx} - 6u^2u_{xx} - 12u^3u_x - 12uu_x^2 \mod \mathcal{E}^{(1)},$$

hence $u'_t - u'_{xxx} - 6u'u'_x = 0 \mod \mathcal{E}^{(1)}$.

2.3. Symmetries and pseudosymmetries

In this subsection, after reviewing the notion of symmetry of a differential equation [8,25,38], we will give an introduction to pseudosymmetries, one of its possible generalizations proposed by Sokolov in the paper [42], together with some of its key properties, which will be particularly important in the forthcoming parts of the paper.

2.3.1. Symmetries of differential equations

Finite symmetries of a smooth distribution $\mathcal{D}$ on a manifold N are diffeomorphisms $\psi : N \to N$ such that $\psi_*\mathcal{D} \subseteq \mathcal{D}$. Analogously, from the infinitesimal point of view, infinitesimal symmetries of a smooth distribution $\mathcal{D}$ on a manifold N are smooth vector fields Y on N such that $L_Y\mathcal{D} \subseteq \mathcal{D}$. Hence, the flow of an infinitesimal symmetry of $\mathcal{D}$ is a 1-parameter local group of finite symmetries of $\mathcal{D}$. If $\mathcal{D}$ is generated by a system of vector fields, i.e., $\mathcal{D} =< X_1, ..., X_n >$, the symmetry condition is equivalent to $[Y, X_i] = \sum_s \alpha_i^s X_s$, for any $i \in \{1, ..., n\}$ and some smooth functions α_i^s.

Now, given a k-th order equation (or system) $\mathcal{E} = \{\mathbf{F} = \mathbf{0}\} \subset J^k(\pi)$, the *classical finite symmetries* of $\mathcal{E}$ are finite symmetries of the distribution $\mathcal{C}^k(\pi)$ which leave invariant the submanifold $\mathcal{E}$. Analogously, *classical infinitesimal symmetries* of $\mathcal{E}$ are vector fields on $J^k(\pi)$ which are infinitesimal symmetries of $\mathcal{C}^k(\pi)$ and are tangent to $\mathcal{E}$. A finite symmetry ψ is called *projectable* if $\psi^*(C^\infty(M)) \subseteq C^\infty(M)$. Analogously, an infinitesimal symmetry X is called projectable if $X(C^\infty(M)) \subseteq C^\infty(M)$.

It can be seen (see [8] for details) that infinitesimal symmetries X of $\mathcal{C}^k(\pi)$ are completely described by a *generating function* $\varphi = (\varphi^j)$, defined as $\varphi^j := X \lrcorner\, \omega_0^j$ (where $\omega_0^j = du^j - \sum_i u_i^j dx_i$). This function is always of first order (i.e., only depend on x_i, u^j, u_i^j) and, whenever $m > 1$, it is always of the form $\varphi^j = \sum_i a_i(x,u)u_i^j + b^j(x,u)$. In particular, in canonical coordinates $\{x_i, u_\sigma^j\}$, one has

$$X = -\sum_i \frac{\partial \varphi^s}{\partial u_i^s} D_i^{(k)} + \sum_{|\sigma|=0}^{k-1} \sum_{j=1}^{m} D_\sigma^{(k)} \varphi^j \, \partial_{u_\sigma^j}, \tag{10}$$

where $D_\sigma^{(k)} := \left(D_1^{(k)}\right)^{\sigma_1} \circ ... \circ \left(D_n^{(k)}\right)^{\sigma_n}$ and s is any fixed integer in $\{1, ..., m\}$.

Thus, in view of (10), the higher order components of an infinitesimal symmetry can be obtained from lower order ones by means of a recurrence formula. In this sense one says that the infinitesimal symmetries of $\mathcal{C}^k(\pi)$ are *prolongations* of lower order vector fields. For instance, when $m > 1$ any infinitesimal symmetry X of $\mathcal{C}^k(\pi)$ is the prolongation of a vector field on $J^0(\pi)$; in such a case, X is usually referred to as a *point infinitesimal symmetry*. On the other hand, when $m = 1$ an infinitesimal symmetry X of $\mathcal{C}^k(\pi)$ is in general the prolongation of a symmetry of $\mathcal{C}^1(\pi)$, that is not necessarily the prolongation of a vector field on $J^0(\pi)$. In particular, when $m = 1$, X is usually referred to as a *contact infinitesimal symmetry* whenever it is not a point symmetry.

Thus, the computation of infinitesimal classical symmetries X of an equation $\mathcal{E} \subset J^k(\pi)$ reduces to the determination of functions $\varphi = (\varphi^j) = (X \lrcorner \omega_0^j)$ such that X is tangent to $\mathcal{E}$. This condition provides typically an overdetermined linear system of PDEs for φ that can be analyzed and explicitly solved by considering its integrability conditions. The analysis of such a system is generally more feasible if one uses some symbolic manipulation package of the type developed in a computer algebra system like Maple.

On the other hand, when $\mathcal{E}$ is formally integrable, a noteworthy extension of the notion of symmetry is possible. Indeed, one can say that a vector field $X \in \mathcal{D}(\pi)$ is a *generalised symmetry* of $\mathcal{E}$ if, and only if, X is a symmetry of $\mathcal{C}(\pi)$ which is tangent to $\mathcal{E}^{(\infty)}$. However, since $\mathcal{C}(\pi)$ is Frobenius, the Lie algebra $\mathcal{DC}(\pi)$ of vector fields inscribed in $\mathcal{C}(\pi)$ is an ideal of trivial (or characteristic) symmetries, that one wants preliminarily gauge out from symmetries of $\mathcal{C}(\pi)$. Thus, denoting by $\mathrm{Sym}(\pi)$ the full Lie algebra of symmetries of $\mathcal{C}(\pi)$, one can define the Lie algebra of *higher symmetries of* $\mathcal{C}(\pi)$ as the quotient algebra $\mathrm{sym}(\pi) = \mathrm{Sym}(\pi)/\mathcal{DC}(\pi)$. Hence, since any field of $\mathcal{DC}(\pi)$ is already tangent to $\mathcal{E}^{(\infty)}$, all elements of a coset $[X] \in \mathrm{sym}(\pi)$ (i.e., $[X] = X \mod \mathcal{DC}(\pi)$, with $X \in \mathrm{Sym}(\pi)$) are tangent to $\mathcal{E}^{(\infty)}$, or neither are they. Thus, since it make sense to say that an element of $\mathrm{sym}(\pi)$ is tangent or not to $\mathcal{E}^{(\infty)}$, one defines the Lie algebra of *higher symmetries of* $\mathcal{E}$ as the sub-algebra $\mathrm{sym}(\mathcal{E}) \subset \mathrm{sym}(\pi)$ of higher symmetries of $\mathcal{C}(\pi)$ that are tangent to $\mathcal{E}^{(\infty)}$.

In coordinates, one can see that any $X \in \mathrm{Sym}(\pi)$ has the form

$$X = \sum_i a_i D_i + \sum_{|\sigma|\geq 0} \sum_{j=1}^{m} D_\sigma \varphi^j \, \partial_{u_\sigma^j}, \tag{11}$$

where $\varphi = (\varphi^j) = (X \lrcorner \omega_0^j)$ and a_i are some differentiable functions. Thus, any higher symmetry $[X]$ can be identified with its vertical (or evolutionary) representative

$$\ni_\varphi = \sum_{|\sigma|\geq 0} \sum_{j=1}^{m} D_\sigma \varphi^j \, \partial_{u_\sigma^j}, \tag{12}$$

with generating function $\varphi = (\varphi^a)$. Moreover, $\ni_\varphi \in \mathrm{sym}(\mathcal{E})$ whenever $\ni_\varphi$ is tangent to $\mathcal{E}^{(\infty)}$, i.e., φ satisfies the system of linear differential equations

$$\sum_{|\sigma|\geq 0} \sum_{j=1}^{m} D_\sigma(\varphi^j) \, \partial_{u_\sigma^j} \mathbf{F} = 0 \mod \mathcal{E}^{(\infty)}. \tag{13}$$

For any φ of fixed order, this is typically an overdetermined linear system for φ that involves only a finite number of differential consequences of $\mathcal{E}$ and can be explicitly solved by considering its integrability conditions. As with the symmetry condition for classical symmetries, the analysis of the linear system (13) is generally more feasible if one uses some symbolic manipulation package of the type developed in a computer algebra system like Maple.

For further details on the theory of classical and generalised symmetries, as well as for examples and explicit computations, see [8,38].

2.3.2. Pseudosymmetries of differential equations and factorisation

Pseudosymmetries were introduced by Sokolov in [42] as a generalization of infinitesimal symmetries.

In general, given a smooth distribution $\mathcal{D}$, a pseudosymmetry of $\mathcal{D}$ can be understood as a smooth vector field Y such that $L_Y\mathcal{D} \subseteq < Y > +\mathcal{D}$. For instance, for a distribution $\mathcal{D} =< X_1, ..., X_n >$, the pseudosymmetry condition is equivalent to say that for any $i \in \{1, ..., n\}$ one has $[Y, X_i] = \alpha_i Y + \sum_s \beta_i^s X_s$, for some smooth functions α_i and β_i^s.

Sokolov introduced also the more general notion of an r-pseudosymmetry of $\mathcal{D}$, that can be seen as a system $\{Y_1, ..., Y_r\}$ of vector fields such that $L_{Y_h}\mathcal{D} \subseteq < Y_1, ..., Y_r > +\mathcal{D}$, for any $h = 1, ..., r$. Hence, for a distribution $\mathcal{D} =< X_1, ..., X_n >$, one has an r-pseudosymmetry $\{Y_1, ..., Y_r\}$ if for any $i \in \{1, ..., n\}$ the $r \times 1$ matrix $\mathbb{Y} = [Y_1, ..., Y_r]^T$ is such that

$$[\mathbb{Y}, X_i] = \mathbb{U}_i \mathbb{Y} + \mathbb{V}_i \mathbb{X}, \tag{14}$$

where $[\mathbb{Y}, X_i] := [[Y_1, X_i], ..., [Y_r, X_i]]^T$ and $\mathbb{X} = [X_1, ..., X_n]^T$, whereas $\mathbb{U}_i$ and $\mathbb{V}_i$ are some $r \times r$ and $r \times n$ matrix-valued smooth functions, respectively.

It is well known that the flow of an infinitesimal symmetry always transforms an integral manifolds of $\mathcal{D}$ to another (possibly different) integral manifolds. Also, in view of $[Y, X_i] = \sum_s \alpha_i^s X_s$, it is readily seen that an infinitesimal symmetry of $\mathcal{D}$ sends an invariant (or first integral) I of $\mathcal{D}$ to another invariant $Y(I)$. These properties are in general not true for pseudosymmetries. However, pseudosymmetries share with symmetries another important property described by the following

Proposition 9. *Let $\left\{\xi^1, ..., \xi^n, \nu^1, ..., \nu^p\right\}$ be invariants of a pseudosymmetry Y (or, an r-pseudosymmetry $\{Y_1, ..., Y_r\}$) of $\mathcal{D} =< X_1, ..., X_n >$, such that*

$$\det \begin{pmatrix} X_1\left(\xi^1\right) & \cdots & X_1\left(\xi^n\right) \\ \vdots & & \vdots \\ X_n\left(\xi^1\right) & \cdots & X_n\left(\xi^n\right) \end{pmatrix} \neq 0.$$

Then the functions ν_i^j, defined by

$$\begin{pmatrix} \nu_1^1 & \cdots & \nu_1^p \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^p \end{pmatrix} = \begin{pmatrix} X_1\left(\xi^1\right) & \cdots & X_1\left(\xi^n\right) \\ \vdots & & \vdots \\ X_n\left(\xi^1\right) & \cdots & X_n\left(\xi^n\right) \end{pmatrix}^{-1} \begin{pmatrix} X_1(\nu^1) & \cdots & X_1(\nu^p) \\ \vdots & & \vdots \\ X_n(\nu^1) & \cdots & X_n(\nu^p) \end{pmatrix}, \tag{15}$$

are invariants of Y (resp., $\{Y_1, ..., Y_r\}$).

Proof. Consider the matrix identity

$$\begin{pmatrix} X_1\left(\xi^1\right) & \cdots & X_1\left(\xi^n\right) \\ \vdots & & \vdots \\ X_n\left(\xi^1\right) & \cdots & X_n\left(\xi^n\right) \end{pmatrix} \begin{pmatrix} \nu_1^1 & \cdots & \nu_1^p \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^p \end{pmatrix} = \begin{pmatrix} X_1(\nu^1) & \cdots & X_1(\nu^p) \\ \vdots & & \vdots \\ X_n(\nu^1) & \cdots & X_n(\nu^p) \end{pmatrix}.$$

In order to save space and avoid ambiguity, since we have the parentheses due to the derivatives along the field X, we will rewrite last identity as $\left[X_i(\xi^h)\right]\left[\nu_h^j\right] = \left[X_i(\nu^j)\right]$, instead of

$\left(X_i(\xi^h)\right)\left(\nu^j_h\right) = \left(X_i(\nu^j)\right)$. The same will be done for the other matrix identities that follow. Thus, by taking the Lie derivative of this identity with respect to Y on gets

$$\left[Y\left(X_i(\xi^h)\right)\right]\left[\nu^j_h\right] + \left[X_i(\xi^h)\right]\left[Y(\nu^j_h)\right] = \left[Y\left(X_i(\nu^j)\right)\right].$$

On the other hand, in view of $[Y, X_i] = \sum_s \alpha^s_i X_s + \beta_i Y$ and the fact that $Y(\xi^h) = Y(\nu^j) = 0$, last identity reduces to

$$\left[\sum_s \alpha^s_i X_s(\xi^h)\right]\left[\nu^j_h\right] + \left[X_i(\xi^h)\right]\left[Y(\nu^j_h)\right] = \left[\sum_s \alpha^s_i X_s(\nu^j)\right],$$

or equivalently

$$\left[\alpha^s_i\right]\left[X_s(\xi^h)\right]\left[\nu^j_h\right] + \left[X_i(\xi^h)\right]\left[Y(\nu^j_h)\right] = \left[\alpha^s_i\right]\left[X_s(\nu^j)\right].$$

Then by $\left[X_s(\xi^h)\right]\left[\nu^j_h\right] = \left[X_s(\nu^j)\right]$ one readily gets $\left[X_i(\xi^h)\right]\left[Y(\nu^j_h)\right] = 0$. The result follows from non-degeneracy condition $\det\left[X_i(\xi^h)\right] \neq 0$. An analogous proof holds in the case of an r-pseudosymmetry $\{Y_1, ..., Y_r\}$, since for any X_i and Y_h one has $[Y_h, X_i] = \sum_s \alpha^s_{hi} X_s + \sum_k \beta^k_{hi} Y_k$, for some functions a^s_{hi} and β^k_{hi}. $\square$

Concerning the pseudosymmetries or r-pseudosymmetries of $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}^{(\infty)}$, it is convenient to adopt the following

Definition 10. Let $\mathcal{E} = \{\mathbf{F} = \mathbf{0}\} \subset J^k(\pi)$ be a formally integrable differential equation. A *pseudosymmetry* of $\mathcal{E}$ is a vector field $\bar{Y}$ on $\mathcal{E}^{(\infty)}$ that is a pseudosymmetry of $\mathcal{C}(\mathcal{E})$. Analogously, an *r-pseudosymmetry* of $\mathcal{E}$ is a system of vector fields $\{\bar{Y}_1, ..., \bar{Y}_r\}$ on $\mathcal{E}^{(\infty)}$ that is an r-pseudosymmetry of $\mathcal{C}(\mathcal{E})$.

In his paper [42], Sokolov discussed a method for calculating pseudosymmetries of $\mathcal{E}$. Here we will describe that method by means of the following proposition, along with a short proof for the convenience of our readers.

Proposition 11. *Let $\mathcal{E} = \{\mathbf{F} = \mathbf{0}\} \subset J^k(\pi)$, with $\mathbf{F} = (F^1, ..., F^p)$, be a formally integrable differential equation and $\mu = U_1 dx^1 + ... + U_n dx^n$ an horizontally closed 1-form on $\mathcal{E}^{(\infty)}$, i.e., such that*

$$D_j\, U_i - D_i\, U_j = 0 \mod \mathcal{E}^{(\infty)}. \tag{16}$$

Then, the vector field on $J^\infty(\pi)$

$$Y = \sum_i a_i \partial_{x_i} + \sum_{|\sigma|\geq 0}\sum_j b^j_\sigma \partial_{u^j_\sigma} = \sum_i a_i D_i + \sum_{|\sigma|\geq 0}\sum_j (D+U)_\sigma\,(\varphi^j)\partial_{u^j_\sigma}, \tag{17}$$

where

$$\varphi^j := Y \lrcorner \omega_0^j, \qquad \omega_0^j = du^j - \sum_i u_i^j dx_i,$$

$$b_\sigma^j := \sum_i a_i u_{\sigma+1_i}^j + (D+U)_\sigma (\varphi^j),$$

$$(D+U)_\sigma := (D_1+U_1)^{\sigma_1} \circ \ldots \circ (D_n+U_n)^{\sigma_n},$$

restricts on $\mathcal{E}^{(\infty)}$ to a pseudosymmetry $\bar{Y}$ of $\mathcal{E}$ whenever $Y(F^h) = 0 \mod \mathcal{E}^{(\infty)}$, i.e.,

$$\sum_{|\sigma|\geq 0} \sum_j \partial_{u_\sigma^j} \left(F^h\right) (D+U)_\sigma (\varphi^j) = 0 \mod \mathcal{E}^{(\infty)}, \tag{18}$$

for any $h = 1, \ldots, p$, which is equivalent to the tangency condition of Y to $\mathcal{E}^{(\infty)}$.

Proof. In view of (17) one gets (for ease of notation, we use here the Einstein's summation convention)

$$\begin{aligned}
[D_s, Y] &= [D_s, a_i D_i] + \left[D_s, (D+U)_\sigma (\varphi^j)\partial_{u_\sigma^j}\right] \\
&= D_s(a_i)\ D_i + \left[D_s + U_s, (D+U)_\sigma (\varphi^j)\partial_{u_\sigma^j}\right] - \left[U_s, (D+U)_\sigma (\varphi^j)\partial_{u_\sigma^j}\right].
\end{aligned}$$

On the other hand, by (16) one has $[D_i + U_i, D_j + U_j] = 0 \mod \mathcal{E}^{(\infty)}$ and

$$(D_s + U_s) \circ (D+U)_\sigma = (D+U)_{\sigma+1_s} \mod \mathcal{E}^{(\infty)}.$$

Hence, above identity can be rewritten as

$$\begin{aligned}
[D_s, Y] = D_s(a_i)D_i &+ (D+U)_{\sigma+1_s} (\varphi^j)\partial_{u_\sigma^j} - (D+U)_\sigma (\varphi^j)\partial_{u_{\sigma-1_s}^j} \\
&- (D+U)_\sigma (\varphi^j)\partial_{u_\sigma^j} \circ U_s + (D+U)_\sigma (\varphi^j)\partial_{u_\sigma^j} \circ U_s \\
&- U_s \circ (Y - a_i D_i) \mod \mathcal{E}^{(\infty)},
\end{aligned}$$

which reduces to

$$[D_s, Y] = (D_s a_i + a_i U_s)\, D_i - U_s Y \mod \mathcal{E}^{(\infty)}, \tag{19}$$

in view of arbitrariness of σ. On the other hand (19) entails that

$$\begin{aligned}
D_s\left(Y(D_\sigma F^h)\right) - Y\left(D_s(D_\sigma F^h)\right) &= (D_s a_i + a_i U_s)\ D_i \left(D_\sigma F^h\right) - U_s\, Y(D_\sigma F^h) \\
&= -U_s\, Y(D_\sigma F^h) \mod \mathcal{E}^{(\infty)},
\end{aligned}$$

i.e.,

$$Y\left(D_{\sigma+1_s} F^h\right) = U_s\, Y(D_\sigma F^h) + D_s\left(Y(D_\sigma F^h)\right) \mod \mathcal{E}^{(\infty)}, \tag{20}$$

for any σ and for any h. Thus, by repeatedly using (20) with increasing $|\sigma|$, conditions $Y(F^h)=0 \mod \mathcal{E}^{(\infty)}$, $h=1,\ldots,p$, guarantee that $Y\left(D_\sigma F^h\right)=0 \mod \mathcal{E}^{(\infty)}$, i.e., that Y is tangent to $\mathcal{E}^{(\infty)}$. Therefore, in view of (19), one gets that Y restricts on $\mathcal{E}^{(\infty)}$ to a pseudosymmetry $\bar{Y}$ of $\mathcal{C}(\mathcal{E})$. □

Some remarks are in order here. First, in view of (17), one has recurrence formulas for the components of a pseudosymmetry that allow one to obtain higher order components from lower order ones. To distinguish the case of pseudosymmetries from that of symmetries, we will say that (17) is the *pseudoprolongation*, relatively to the 1-form μ, of the (possibly relative) vector field $\sum_i a_i\partial_{x_i}+\sum_j b^j\partial_{u^j}$ on $J^0(\pi)$. In particular, when $\mu=0$ the pseudoprolongation reduces to standard prolongation (11) and Y is a symmetry of $\mathcal{C}(\pi)$; thus $\varphi=(\varphi^j)$ is the generating function of a (classical or generalised) symmetry of $\mathcal{E}$ whenever $\mu=0$ and Y is tangent to $\mathcal{E}^{(\infty)}$. Also, we remark that for any given 1-form μ, the system (18) is a linear system of PDEs for $\varphi=(\varphi^j)$ that involves only a finite number of differential consequences of $\mathcal{E}$, whenever the order of φ is finite. As with the standard symmetry condition, the analysis of such a system of linear PDEs is generally more feasible if one uses some symbolic manipulation package of the type developed in a computer algebra system like Maple. Finally, we note that, as with generalised symmetries, also pseudosymmetries could be put in an evolutionary form (see [16,42]). In this article, however, we will not limit ourselves to considering pseudosymmetries in evolutionary form, because in general the space of invariants of a pseudosymmetry of $\mathcal{E}$ is not the same as that of its evolutionary form, and this is a non-negligible fact when, as in this paper, one is interested in studying factorisation by pseudosymmetries.

For r-pseudosymmetries there is an analogous result illustrated by the following proposition, which can be found in either the original paper [42] or the more recent paper [16].

Proposition 12. *Let $\mathcal{E}=\{\mathbf{F}=\mathbf{0}\}\subset J^k(\pi)$, with $\mathbf{F}=(F^1,\ldots,F^p)$, be a formally integrable differential equation and $\gamma=\mathbb{U}_1dx^1+\ldots+\mathbb{U}_ndx^n$ a horizontal 1-form with $r\times r$ matrix-valued components satisfying $d_H\gamma+\gamma\wedge\gamma=0 \mod \mathcal{E}^{(\infty)}$, i.e.,*

$$D_j\,\mathbb{U}_i-D_i\,\mathbb{U}_j-[\mathbb{U}_i,\mathbb{U}_j]=0 \mod \mathcal{E}^{(\infty)}. \tag{21}$$

Then, the $r\times 1$ matrix $\mathbb{Y}=[Y_1,\ldots,Y_r]^T$

$$\mathbb{Y}=\mathbb{A}\mathbb{D}+\sum_{|\sigma|\geq 0}(D+\mathbb{U})_\sigma\,(\Phi)\partial_{\mathbf{u}_\sigma}, \tag{22}$$

where $\mathbb{D}:=(D_1,\ldots,D_n)^T$, $\partial_{\mathbf{u}_\sigma}=(\partial_{u^1_\sigma},\ldots,\partial_{u^m_\sigma})^T$, $\mathbb{A}$ is an $r\times n$ matrix function, $\Phi=(\varphi^j_i)$ is an $r\times m$ matrix function with $\varphi^j_i:=Y_i\lrcorner\,\omega^j_0$ and

$$(D+\mathbb{U})_\sigma:=(D_1+\mathbb{U}_1)^{\sigma_1}\circ\ldots\circ(D_n+\mathbb{U}_n)^{\sigma_n},$$

defines on $\mathcal{E}^{(\infty)}$ an r-pseudosymmetry $\{\bar{Y}_1=Y_1|_{\mathcal{E}^{(\infty)}},\ldots,\bar{Y}_r=Y_r|_{\mathcal{E}^{(\infty)}}\}$ of $\mathcal{E}$ whenever $\mathbb{Y}(F^h)=0 \mod \mathcal{E}^{(\infty)}$, i.e.,

$$\sum_{|\sigma|\geq 0}\partial_{\mathbf{u}_\sigma}\left(F^h\right)(D+\mathbb{U})_\sigma\,(\Phi)=0 \mod \mathcal{E}^{(\infty)}, \tag{23}$$

for any $h=1,\ldots,p$, which is equivalent to the tangency condition of each $Y_1,\ldots,Y_r$ to $\mathcal{E}^{(\infty)}$.

For r-pseudosymmetries apply same considerations made above for pseudosymmetries. In particular, one can say that the system (22) is the *pseudoprolongation*, relatively to γ, of the (possibly relative) vector fields $\sum_i a_{hi}\partial_{x_i} + \sum_j b_h^j \partial_{u^j}$, $h = 1, ..., r$ (with $\varphi_h^j = b_h^j - \sum_i a_{hi} u_i^j$) on $J^0(\pi)$.

In general one has no guarantee that the system of "derived" (or prolonged) invariants $\{\xi^i, \nu^j, \nu_i^j\}$ (provided by Proposition 9) is functionally independent, even when $\{\xi^i, \nu^j\}$ is a functionally independent system. The scenario, however, changes in the case of the Cartan distribution $\mathcal{C}(\pi)$ on the infinite jet space, for an m-dimensional bundle $\pi : E \to M$ with $n = \dim M$. Indeed, in such a case the generators X_i of $\mathcal{C}(\pi)$ are the total derivatives, hence the jet-order of $\left[\nu_h^j\right] = \left[D_i \xi^h\right]^{-1} \left[D_i \nu^j\right]$ (that is equation (15)) is higher than that of $\{\xi^i, \nu^j\}$. Thus, if $\left\{\xi^i, \nu^j\right\}$ are functionally independent invariants of a pseudosymmetry Y (or, an r-pseudosymmetry $\{Y_1, ..., Y_r\}$) of $\mathcal{C}(\pi) =< D_1, ..., D_n >$ satisfying $\det\left(D_s \xi^i\right) \neq 0$, the prolongation formulas (15) provide an infinite sequence $\{x_i' = \xi^i,\ u'^j_\mu = \nu_\mu^j\}$ of invariants of Y (resp., $\{Y_1, ..., Y_r\}$); when $p = m$ this sequence defines a regular $\mathcal{C}$-morphism in $J^\infty(\pi)$, in view of Proposition 3.

On the other hand, given a formally integrable equation $\mathcal{E} = \{\mathbf{F} = 0\} \subset J^k(\pi)$, if $\left\{\xi^1, ..., \xi^n, \nu^1, ..., \nu^p\right\}$, $p \geq m$, are functionally independent invariants of a pseudosymmetry $\bar{Y}$ (resp., r-pseudosymmetry $\{\bar{Y}_1, ..., \bar{Y}_r\}$) of $\mathcal{C}(\mathcal{E}) =< \bar{D}_1, ..., \bar{D}_n >$ on $\mathcal{E}^{(\infty)}$ satisfying $\det\left(\bar{D}_s \xi^i\right) \neq 0$, then prolongation formulas (15) allow one to define a sequence $\{\xi^i,\ \nu_\mu^j\}$ of invariants of $\bar{Y}$ (resp., $\{\bar{Y}_1, ..., \bar{Y}_r\}$) that are functionally dependent, due to the constraints imposed by $\mathbf{F} = 0$ and its differential consequences. Indeed, assuming for instance that $\{\xi^i,\ \nu_\mu^j\}$ are ℓ-th order invariants of $\bar{Y}$, the cardinality of $\{\xi^i,\ \nu_\mu^j\}$ grows with $|\mu|$ faster than the dimension of the prolongation $\mathcal{E}^{(\ell+|\mu|-k)}$. Therefore, for sufficiently large $|\mu|$ the system $\{\xi^i,\ \nu_\mu^j\}$ is necessarily functionally dependent. In particular, a nontrivial $\bar{Y}$ has at most $\dim \mathcal{E}^{(\ell+|\mu|-k)} - 1$ functionally independent invariants of order $\ell + |\mu| \geq k$, with the maximum reached only when $\bar{Y}$ projects (or restricts) to a nontrivial vector field (i.e., a derivation) on $\mathcal{E}^{(\ell+|\mu|-k)}$. Indeed, when $\bar{Y}$ projects to a relative vector field on $\mathcal{E}^{(\ell+|\mu|-k)}$, the invariants of the projection are not necessarily also invariants of $\bar{Y}$. Analogous considerations hold when $\{\xi^i,\ \nu_\mu^j\}$ are invariants of an r-pseudosymmetry.

Therefore, since $\mathbf{F} = 0$ and its differential consequences entail that for sufficiently large $|\mu|$ the system $\{\xi^i,\ \nu_\mu^j\}$ is functionally dependent on $\mathcal{E}^{(\infty)}$, it follows that the invariants $\{\nu_\sigma^j\}$ satisfy an associated system of partial differential equations $\mathbf{Q} = 0$ (together with their prolongations), modulo $\mathbf{F} = 0$ and its differential consequences. This way one can look at $\{x_i' = \xi^i,\ u'^j_\mu = \nu_\mu^j\}$ as the components of a $\mathcal{C}$-morphism from $\mathcal{E} = \{\mathbf{F} = \mathbf{0}\}$ to $\mathcal{Q} = \{\mathbf{Q} = \mathbf{0}\}$. Of course, if one chooses another system of invariants $\{\hat{x}_i' = \hat{\xi}^i,\ \hat{u}'^j = \hat{\nu}^j\}$ equivalent to $\{\xi^i,\ \nu^j\}$, i.e., such that $\{\hat{\xi}^i = A^i(\xi^i, \nu^j),\ \hat{\nu}^j = B^j(\xi^i, \nu^j)\}$ and $\det\left(\bar{D}_s \hat{\xi}^i\right) \neq 0$ (where $\bar{D}_s$ are the total derivatives on $\mathcal{E}$), one would obtain a system $\hat{\mathcal{Q}}$ which is point equivalent to $\mathcal{Q}$, through the point transformation $\{\hat{x}_i' = A^i(x', u'),\ \hat{u}'^j = B^j(x', u')\}$. In particular, one can choose $\left\{\xi^i, \nu^j\right\}$ in such a way that none of its element is the prolongation of another one and, in addition to that, $\{\xi^i,\ \nu_\mu^j\}$ is a complete system of invariants, i.e., a system that generates invariants of all possible orders. In such a case, according to the common terminology used in the particular case of symmetries (see for instance [25,42,43]), one can look at $\mathcal{Q}$ as the factorisation, or the quotient, of $\mathcal{E}$ by $\bar{Y}$ (resp., $\{\bar{Y}_1, ..., \bar{Y}_r\}$).

Next two examples will provide a simple illustration of above factorisation procedure. Other examples, describing Bäcklund transformations, will be discussed in Section 4.

Example 13. The heat equation $\mathcal{E} = \{u_t - u_{xx} = 0\}$, where $u = u(x,t)$, admits the classical infinitesimal symmetry $Y = u\,\partial_u + u_x\partial_{u_x} + u_t\partial_{u_t} + \ldots$ (which defines a pseudosymmetry $\bar{Y}$ of $\mathcal{E}$ with $\mu = 0$) provided by the prolongation of $u\,\partial_u$; indeed it is readily seen that $Y(u_t - u_{xx}) = u_t - u_{xx} = 0 \mod \mathcal{E}$. In this case, since Y is a classical symmetry of the ambient Cartan distribution, it projects to a vector field on any finite order jet space $J^k(\pi)$ (where $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $(x,t,u) \mapsto (x,t)$). On the other hand, being also tangent to $\mathcal{E}$ (hence a symmetry of $\mathcal{E}$), its restriction $\bar{Y} := Y|_{\mathcal{E}^{(\infty)}}$ projects to a vector field on $\mathcal{E}$ as well as any of its prolongations $\mathcal{E}^{(\ell)}$. From now one we will limit ourself to domains where u and its derivatives do not vanish, hence above projections of Y are always nontrivial. Thus, being $\mathcal{E}$ a 7-dimensional submanifold of $J^2(\pi)$, $\bar{Y}$ admits $6 = \dim\mathcal{E} - 1$ functionally independent invariants of order 2, $8 = \dim\mathcal{E}^{(1)} - 1$ functionally independent invariants of order 3, and so on. These invariants can be obtained by means of prolongations from the following basic system of invariants $\{\xi^1 = x,\ \xi^2 = t,\ \nu^1 = u_x/u,\ \nu^2 = u_t/u\}$ of $\bar{Y}$, that define a $\mathcal{C}$-morphism $\mathcal{B} : J^\infty(\pi) \to J^\infty(\pi')$, where $\pi' : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $(x,t,u^{1\prime},u^{2\prime}) \mapsto (x,t)$. Indeed, non-degeneracy condition (4) is satisfied and a first prolongation provides

$$u^{1\prime}_x = \frac{u_{xx}}{u} - \left(\frac{u_x}{u}\right)^2, \qquad u^{1\prime}_t = u^{2\prime}_x = \frac{u_{xt}}{u} - \left(\frac{u_x}{u}\right)\left(\frac{u_t}{u}\right), \qquad u^{2\prime}_t = \frac{u_{tt}}{u} - \left(\frac{u_t}{u}\right)^2.$$

Thus, since the other two invariants $u^{1\prime}_x, u^{1\prime}_t$ satisfy the relations

$$u^{1\prime}_x = u^{2\prime} - \left(u^{1\prime}\right)^2, \qquad u^{1\prime}_t = u^{2\prime}_x \qquad \mod \{u_{xx} = u_t\},$$

a basic system of second order functionally independent invariants of $\bar{Y}$ is given by $\{x,\ t,\ u^{1\prime},\ u^{2\prime},\ u^{2\prime}_x,\ u^{2\prime}_t\}$. This means that $\mathcal{B}$ is a $\mathcal{C}$-morphism from the heat equation $\mathcal{E} = \{u_t - u_{xx} = 0\}$ to the system $\mathcal{Q} = \{u^{1\prime}_x - u^{2\prime} + \left(u^{1\prime}\right)^2 = 0,\ u^{1\prime}_t - u^{2\prime}_x = 0\}$, that one can interpret as the factorisation of $\mathcal{E}$ by $\bar{Y}$. Hence, $\left\{u^{1\prime} = u_x/u,\ u^{2\prime} = u_t/u\right\}$ transforms solutions of $\mathcal{E}$ to solutions of $\mathcal{Q}$, as well as of any of its differential consequences. In particular, since the Burgers equation $\mathcal{E}' = \{u'_t - u'_{xx} - 2u'u'_x = 0\}$ with $u' := u^{1\prime}$ is a differential consequence of $\mathcal{Q}$, one obtains the Cole-Hopf transformation $u' = u_x/u$ as a by-product of above transformation.

Example 14. The mKdV equation $\mathcal{E} = \{u_t - u_{xxx} + 6u^2u_x = 0\}$, can be written in conservation law form as $D_t u + D_x\left(-u_{xx} + 2u^3\right) = 0$. Thus any $\mu = c\left(u dx + (u_{xx} - 2u^3)dt\right)$, with $c \in \mathbb{R}$, is a conservation law of $\mathcal{E}$ and one can search for pseudosymmetries of the form (17), where $x_1 = x$ and $x_2 = t$. Here we consider the pseudosymmetry $\bar{Y}$ found by Sokolov in [42], obtained from the vector field ∂_u by a pseudoprolongation relative to 2μ:

$$\begin{aligned} Y \ &= \partial_u + 2u\partial_{u_x} + 2(u_{xx} - 2u^3)\partial_{u_t} + 2(u_x + 2u^2)\partial_{u_{xx}} + 2(u_t + 2u(u_{xx} - 2u^3))\partial_{u_{xt}} \\ &\quad + 2(u_{xxt} - 6u^2u_t + 2(u_{xx} - 2u^3)^2)\partial_{u_{tt}} + \ldots . \end{aligned}$$

In this case, the projections of Y to $J^k(\pi)$ (where $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $(x,t,u) \mapsto (x,t)$) are relative vector fields. The same occurs for projections of Y to $\mathcal{E}$ and prolongations $\mathcal{E}^{(\ell)}$. For instance, the projection of Y on $J^1(\pi)$ is the relative vector field $\partial_u + 2u\partial_{u_x} + 2(u_{xx} - 2u^3)\partial_{u_t}$. As a consequence, $\bar{Y}$ has only the 3 functionally independent invariants of first order $\{\xi^1 = x,\ \xi^2 = t,\ \nu^1 = u_x - u^2\}$, 3 further functionally independent invariants of second order $\{\nu^1_x =$

$u_{xx} - 2uu_x,\ v_t^1 = u_{xt} - 2uu_t,\ v^2 = u_t - 2u\, v_x^1 - 2u^2 v^1\}$ and only 3 of the derived invariants of third order $\{v_x^2, v_y^2, v_{xx}^1, v_{xy}^1, v_{yy}^1\}$ are functionally independent since $2v^1 v_x^1 + v_x^2 - v_t^1 = 0$, $2\left(v^1\right)^2 + v_{xx}^1 - v^2 = 0$. Then all higher order invariants are generated by $\{\xi^i,\ v_\mu^j\}$ and the factorisation of $\mathcal{E}$ by $\bar{Y}$ is described by the system of differential equations $\mathcal{Q} = \{2v^1 v_x^1 + v_x^2 - v_t^1 = 0,\ 2\left(v^1\right)^2 + v_{xx}^1 - v^2 = 0\}$, that describes the lower order functional dependencies between the invariants $\{\xi^i,\ v_\mu^j\}$. Hence, the map $\left\{v^1 = u_x - u^2,\ v^2 = u_t - 2u\ (u_{xx} - 2uu_x) - 2u^2\left(u_x - u^2\right)\right\}$ transforms solutions of $\mathcal{E}$ to solutions of $\mathcal{Q}$, as well as of any of its differential consequences. In particular, since the KdV equation $\mathcal{E}' = \{u_t' - u_{xxx}' - 6u'u_x' = 0\}$ with $u' := v^1$ is a differential consequence of $\mathcal{Q}$, one obtains the Miura transformation $u' = u_x - u^2$ as a by-product of above transformation.

2.4. *Differentiable coverings and Bäcklund transformations as nonlocal $\mathcal{C}$-morphisms*

Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation, with $\pi : E \to M$, $\dim M = n$. Throughout the paper it will be adopted the following

Definition 15. A *differentiable covering* of a formally integrable equation $\mathcal{E}$ is a fiber bundle $\tau : \tilde{\mathcal{E}} \longrightarrow \mathcal{E}^{(\infty)}$ with the following properties:

1. $\tilde{\mathcal{E}}$ is equipped with a n-dimensional involutive distribution $\tilde{\mathcal{C}}$;
2. for any $\theta \in \tilde{\mathcal{E}}$ the push-forward $\tau_{*\theta} : \tilde{\mathcal{C}}_\theta \longrightarrow \mathcal{C}_{\tau(\theta)}(\mathcal{E})$ is an isomorphism.

According to above definition (see [8]), when τ has finite fiber dimension q, $\tilde{\mathcal{E}}$ is locally diffeomorphic to the product $\mathcal{E}^{(\infty)} \times W$, where $W \subseteq \mathbb{R}^q$ is an open set. Thus, locally identifying $\tilde{\mathcal{E}}$ with $\mathcal{E}^{(\infty)} \times W$, τ can be locally realized as the natural projection $\mathcal{E}^{(\infty)} \times W \to \mathcal{E}^{(\infty)}$. Then, using standard coordinates $v^1, ..., v^q$ in $\mathbb{R}^q$, the generators $\tilde{D}_i$ of the involutive distribution $\tilde{\mathcal{C}}$ on $\mathcal{E}^{(\infty)} \times W$ can be locally written as $\tilde{D}_i = \bar{D}_i + \sum_s X_i^s \partial_{v^s}$ where $\bar{D}_i$ are the total derivatives on $\mathcal{E}^{(\infty)}$ and X_i^s are smooth functions such that $\left[\tilde{D}_i, \tilde{D}_j\right] = 0$, i.e., $\tilde{D}_i X_j^s = \tilde{D}_j X_i^s$, in view of $[\bar{D}_i, \bar{D}_j] = 0$.

In this paper we will consider only the case when $\tilde{\mathcal{E}}$ is the infinite prolongation of a formally integrable system $\mathcal{V} \subset J^l(\pi)$. Thus, in view of above considerations, the fiber coordinates $\{u^1, ..., u^m\}$ of π will be separated in two subsets $\{z^1, ..., z^{m-q}\}$ and $\{v^1, ..., v^q\}$, such that $\mathcal{E} = \{\mathbf{F}(x, z_\sigma) = 0\}$ and $\mathcal{V}^{(\infty)}$ is the infinite prolongation of a system

$$\begin{cases} F^h(x, z_\sigma) = 0, \\ v_i^s = X_i^s(x, z_\sigma, v). \end{cases} \tag{24}$$

It is noteworthy to observe that, in view of the definition of a differentiable covering $\tau : \mathcal{V}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$, any solution of $\mathcal{V}$ can be projected by τ to a solution of $\mathcal{E}$.

A system $\mathcal{V}$ like (24) will be called a q-dimensional *differentiable extension* of $\mathcal{E} = \{\mathbf{F} = 0\}$. Moreover, given a differentiable extension $\mathcal{V}$ of $\mathcal{E}$, it is customary to call $v^1, ..., v^q$ the nonlocal variables of the given extension (or the corresponding differentiable covering). Also, non-local symmetries and nonlocal conservation laws of $\mathcal{E}$ are the symmetries and conservation laws of a differentiable extension $\mathcal{Y}$ of $\mathcal{E}$.

Examples of differential extensions, hence of coverings, can be easily obtained by using in (24) the first order equations for the potentials of conservation laws, when $n = 2$. For instance, the equations $v_x = u$, $v_t = u_{xx} - 2u^3$ for the potential v of the conservation law $u\,dx + (u_{xx} - 2u^3)\,dt$ of mKdV (see Example 14), provide a differentiable extension of mKdV.

Other examples, that are particularly relevant in the study of Bäcklund transformations, are related to ZCRs. Indeed condition (1), for an $\ell \times \ell$ matrix-valued ZCR $\alpha = X\,dx + T\,dt$, is equivalent to the integrability condition of a linear system

$$\begin{pmatrix} v^1_x \\ \vdots \\ v^\ell_x \end{pmatrix} = X \begin{pmatrix} v^1 \\ \vdots \\ v^\ell \end{pmatrix}, \qquad \begin{pmatrix} v^1_t \\ \vdots \\ v^\ell_t \end{pmatrix} = T \begin{pmatrix} v^1 \\ \vdots \\ v^\ell \end{pmatrix}.$$

Example 16. The KdV equation $\mathcal{E} = \{z_t = -6zz_x - z_{xxx}\}$, with $z = z(x,t)$, admits the ZCR $\alpha = X\,dx + T\,dt$ given by

$$X = \begin{pmatrix} 0 & 1 \\ -z - \lambda & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} z_x & 4\lambda - 2z \\ z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2 & -z_x \end{pmatrix},$$

with λ a real parameter usually referred to as the spectral parameter. Thus, by introducing the fiber bundle $\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$, $(x, t, z, v^1, v^2) \mapsto (x, t)$, the infinite prolongation of the linear system

$$\mathcal{V}: \quad \begin{cases} v^1_x = v^2, \\ v^2_x = -(z + \lambda)v^1, \\ v^1_t = z_x v^1 + (4\lambda - 2z)\,v^2, \\ v^2_t = \left(z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2\right) v^1 - z_x v^2, \\ z_t = -6zz_x - z_{xxx}, \end{cases} \tag{25}$$

defines a 2-dimensional differentiable covering $\mathcal{V}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$ of $\mathcal{E}$. It is noteworthy to remark that, by setting $v^1 := \phi$ and $v^2 := \phi_x$, the above first order linear system can also be rewritten in the following equivalent form (Lax pair)

$$\begin{cases} \phi_{xx} = -z\phi - \lambda\phi, \\ \phi_t = 2(2\lambda - z)\phi_x + z_x\phi. \end{cases} \tag{26}$$

Indeed, it is well known [34] (see also [14,29,33,45] and references therein) that $\mathcal{E}$ is the integrability condition of (26).

Also, it is interesting to notice here that the considered 2-dimensional differentiable covering, defined by (25), is an extension of a 1-dimensional covering, by means of the equations for the potential of a conservation law. Indeed, by introducing $\rho := v^2/v^1$ one can readily see that (25) reduces to

$$\begin{cases} D_x\left(\ln v^1\right) = \rho, \\ D_t\left(\ln v^1\right) = z_x + (4\lambda - 2z)\,\rho, \\ \rho_x = -\rho^2 - z - \lambda, \\ \rho_t = -2(2\lambda - z)\rho^2 - 2z_x\rho + z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2, \\ z_t = -6zz_x - z_{xxx}. \end{cases}$$

Thus (25) describes an extension of the 1-dimensional differentiable covering $\tau : \mathcal{Y}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$, defined by the system

$$\mathcal{Y}: \quad \begin{cases} \rho_x = -\rho^2 - z - \lambda, \\ \rho_t = -2(2\lambda - z)\rho^2 - 2z_x\rho + z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2, \\ z_t = -6zz_x - z_{xxx}, \end{cases} \tag{27}$$

by means of the two differential equations $\left(\ln v^1\right)_x = \rho$ and $\left(\ln v^1\right)_t = z_x + (4\lambda - 2z)\,\rho$, for the potential $\ln v^1$ of the conservation law $\mu := \rho\,dx + (z_x + (4\lambda - 2z)\,\rho)\,dt$, defined in τ.

Example 17. The Tzitzeica equation $\mathcal{E} = \{z_{xt} = -\mathrm{e}^{-2z} + \mathrm{e}^z\}$, with $z = z(x,t)$, admits the ZCR $\alpha = X\,dx + T\,dt$ given by [9]

$$X = \begin{pmatrix} -z_x & 0 & \lambda \\ \lambda & z_x & 0 \\ 0 & \lambda & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & \frac{1}{\lambda}\mathrm{e}^{-2z} & 0 \\ 0 & 0 & \frac{1}{\lambda}\mathrm{e}^{z} \\ \frac{1}{\lambda}\mathrm{e}^{z} & 0 & 0 \end{pmatrix},$$

with λ a real parameter. Thus, by introducing the fiber bundle $\mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2$, $(x, t, z, v^1, v^2, v^3) \mapsto (x, t)$, the infinite prolongation of the linear system

$$\mathcal{V}: \quad \begin{cases} v^1_x = -z_x v^1 + \lambda v^3, & v^1_t = \dfrac{1}{\lambda} e^{-2z} v^2, \\ v^2_x = \lambda v^1 + z_x v^2, & v^2_t = \dfrac{1}{\lambda} e^{z} v^3, \\ v^3_x = \lambda v^2, & v^3_t = \dfrac{1}{\lambda} e^{z} v^1, \end{cases} \tag{28}$$

defines a 3-dimensional differentiable covering $\mathcal{V}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$ of $\mathcal{E}$. Again, similarly to what was observed for KdV in the previous example, by means of $v^2 = v^3_x/\lambda$ and $v^1 = \lambda e^{-z} v^3_t$ the first-order linear system can be reduced to a higher-order linear one

$$\begin{cases} \phi_{xx} = z_x\phi_x + \lambda^3 e^{-z}\phi_t, \\ \phi_{xt} = e^{z}\phi, \\ \phi_{tt} = z_t\phi_t + \dfrac{1}{\lambda^3} e^{-z}\phi_x, \end{cases} \tag{29}$$

where $\phi := v^3$. Indeed, one can readily see that $\mathcal{E}$ is also the integrability condition of (29). Moreover, similarly to the previous example, the 3-dimensional covering defined by (28) is the extension of a 2-dimensional covering, by means of the equations for the potential of a conservation law. Indeed, by introducing $\rho^1 = v^1/v^3$ and $\rho^2 = v^2/v^3$ one can readily see that (28) reduces to

$$\begin{cases} D_x\left(\ln v^3\right) = \lambda\rho^2, \\ D_t\left(\ln v^3\right) = \dfrac{1}{\lambda}e^z\rho^1, \\ \rho^1_x = -\lambda\,\rho^1\rho^2 - z_x\,\rho^1 + \lambda, \\ \rho^1_t = -\dfrac{1}{\lambda}\mathrm{e}^z\left(\rho^1\right)^2 + \dfrac{1}{\lambda}e^{-2z}\rho^2, \\ \rho^2_x = -\lambda\left(\rho^2\right)^2 + z_x\rho^2 + \lambda\rho^1, \\ \rho^2_t = -\dfrac{1}{\lambda}\mathrm{e}^z\rho^1\rho^2 + \dfrac{1}{\lambda}e^z, \\ z_{xt} = -\mathrm{e}^{-2z} + \mathrm{e}^z. \end{cases}$$

Thus (28) describes an extension of the 2-dimensional differentiable covering $\tau : \mathcal{Y}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$, defined by the system

$$\mathcal{Y}: \quad \begin{cases} \rho^1_x = -\lambda\,\rho^1\rho^2 - z_x\,\rho^1 + \lambda, \\ \rho^1_t = -\dfrac{1}{\lambda}\mathrm{e}^z\left(\rho^1\right)^2 + \dfrac{1}{\lambda}e^{-2z}\rho^2, \\ \rho^2_x = -\lambda\left(\rho^2\right)^2 + z_x\rho^2 + \lambda\rho^1, \\ \rho^2_t = -\dfrac{1}{\lambda}\mathrm{e}^z\rho^1\rho^2 + \dfrac{1}{\lambda}e^z, \\ z_{xt} = -\mathrm{e}^{-2z} + \mathrm{e}^z, \end{cases} \tag{30}$$

by means of the two differential equations $\left(\ln v^3\right)_x = \lambda\rho^2$ and $\left(\ln v^3\right)_t = e^z\rho^1/\lambda$, for the potential $\ln v^3$ of the conservation law $\mu := \lambda\rho^2\,dx + \frac{1}{\lambda}e^z\rho^1\,dt$, defined in τ.

For further details on the theory of differential coverings, as well as for examples and explicit computations, see [8] and references therein.

For Bäcklund transformations one can adopt the following

Definition 18. A Bäcklund transformation from a formally integrable equation $\mathcal{E}$ of order k to another formally integrable equation $\mathcal{E}' \subset J^{k'}(\pi')$, both with the same number n of independent variables, is a regular $\mathcal{C}$-morphism $\mathcal{B}: J^\infty(\pi) \to J^\infty(\pi')$ from $\mathcal{V}$ to $\mathcal{E}'$, for some differentiable covering $\tau : \mathcal{V}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$ defined by a differentiable extension $\mathcal{V} \subset J^l(\pi)$ of $\mathcal{E}$. In particular $\mathcal{B}$ is an auto-Bäcklund transformation whenever $\mathcal{E}'$ is a copy of $\mathcal{E}$.

According to this definition, a Bäcklund transformation can be seen as a pair of $\mathcal{C}$-morphisms

$$\begin{array}{ccc} & \mathcal{Y}^{(\infty)} & \\ \tau \swarrow & & \searrow \mathcal{B} \\ \mathcal{E}^{(\infty)} & & \mathcal{E}'^{(\infty)} \end{array}$$

which is analogous to the point of view followed by Krasil'shchik and Vinogradov in [26]. However, in this paper, we will not limit ourselves to considering Bäcklund transformations, which, like in [26], keep the independent variables unchanged.

Example 19. Continuing Example 16 we can provide an example of auto-Bäcklund transformation for KdV. Indeed, by considering the trivial bundle $\pi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $(x, t, z, \rho) \mapsto (x, t)$ and the 1-dimensional covering $\tau : \mathcal{Y}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$, it can be checked that the $\mathcal{C}$-morphism $\mathcal{B} : J^\infty(\pi) \to J^\infty(\pi')$ defined by

$$z' = -z - 2\rho^2 - 2\lambda, \tag{31}$$

through the prolongation formulas (5), defines a $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}' = \{z'_t + 6z'z'_x + z'_{xxx} = 0\}$, which is a copy of $\mathcal{E}$. Hence (31) provides an auto–Bäcklund transformation of KdV. In practice, for any known solution f_0 of the KdV, such a transformation associates a new solution

$$f_1 = -f_0 - 2g_0^2 - 2\lambda_0,$$

where g_0 is the solution of the Riccati-type system corresponding to f_0 for some λ_0, i.e.,

$$\begin{cases} g_{0x} = -g_0^2 - f_0 - \lambda_0, \\ g_{0t} = -2(2\lambda_0 - f_0)g_0^2 - 2f_{0x}g_0 + f_{0xx} + 2f_0^2 - 2\lambda_0 f_0 - 4\lambda_0^2, \end{cases}$$

Further iterations with pairwise different parameters $\lambda_1, \lambda_2, \ldots$ leads to a sequence of new solutions

$$f_i = -f_{i-1} - 2g_{i-1}^2 - 2\lambda_{i-1},$$

where each g_{i-1} is the solution of the Riccati-type system corresponding to f_{i-1} and λ_{i-1}.

Remark 20. Since $\rho = \phi_x/\phi$, one can readily check that in terms of the Lax potential above auto-Bäcklund transformation can be written as $z' = -z - 2(\phi_x/\phi)^2 - 2\lambda$, or equivalently as $z' = z + 2(\ln\phi)_{xx}$, in view of (26). Thus, for any known solution f_0 of the KdV, the transformation associates the new solution $f_1 = f_0 + 2(\ln g_0)_{xx}$ where g_0 is the corresponding solution of Lax system

$$\begin{cases} g_{0xx} = -f_0 g_0 - \lambda_0 g_0, \\ g_{0t} = 2(2\lambda_0 - f_0)g_{0x} + f_{0x}g_0. \end{cases}$$

Example 21. If in the previous example we augment the space by passing to the bundle $\hat{\pi} : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$, $(x, t, z, \rho, \hat{\rho}) \mapsto (x, t)$, and instead of (27) consider the system

$$\hat{\mathcal{Y}}: \quad \begin{cases} \hat{\rho}_x = -\hat{\rho}^2 - z - \hat{\lambda}, \\ \hat{\rho}_t = -2(2\hat{\lambda} - z)\hat{\rho}^2 - 2z_x\hat{\rho} + z_{xx} + 2z^2 - 2\hat{\lambda}z - 4\hat{\lambda}^2, \\ \rho_x = -\rho^2 - z - \lambda, \\ \rho_t = -2(2\lambda - z)\rho^2 - 2z_x\rho + z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2, \\ z_t = -6zz_x - z_{xxx}, \end{cases} \tag{32}$$

obtained by doubling the Riccati-type system, i.e., by augmenting another copy of the Riccati-type system with spectral parameter $\hat{\lambda} \neq \lambda$, then the natural projection $\hat{\tau} : \hat{\mathcal{Y}}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$ is another covering of KdV and the $\mathcal{C}$-morphism $J^\infty(\hat{\pi}) \to J^\infty(\pi')$ defined by

$$z' = -z - 2\hat{\rho}^2 - 2\hat{\lambda}, \qquad \rho' = -\frac{\hat{\lambda} - \lambda}{\hat{\rho} - \rho} - \hat{\rho}, \tag{33}$$

is a $\mathcal{C}$-morphism from $\hat{\mathcal{Y}}$ to the system

$$\mathcal{E}' : \quad \begin{cases} \rho'_x = -\rho'^2 - z' - \lambda, \\ \rho'_t = -2(2\lambda - z')\rho'^2 - 2z'_x\rho' + z'_{xx} + 2z'^2 - 2\lambda z' - 4\lambda^2, \\ z'_t = -6z'z'_x - z'_{xxx}. \end{cases}$$

In this case, to obtain a new solution z' of KdV out of a given one $z = f_0$, the first step is to compute the corresponding solution $\rho = g_0$ of the Riccati-type system, with arbitrary λ. Then, since g_0 depends on λ, one gets another instance $\hat{g}_0$ of that function, by substituting λ with $\hat{\lambda} \neq \lambda$. This way (33) provides the new pair of functions

$$z' = -f_0 - 2\hat{g}_0^2 - 2\hat{\lambda}, \qquad \rho' = -\frac{\hat{\lambda} - \lambda}{\hat{g}_0 - g} - \hat{g}_0,$$

that satisfy

$$\begin{cases} \rho_x = -\rho^2 - z - \lambda, \\ \rho_t = -2(2\lambda - z)\rho^2 - 2z_x\rho + z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2, \\ z_t = -6zz_x - z_{xxx}. \end{cases}$$

The peculiarity of this transformation, in comparison with that of previous example, is that it provides the new solution z' together with the corresponding solution ρ' of the Riccati-type system, with arbitrary λ. Thus, further applications of the transformations only require derivations and algebraic operations.

It is noteworthy to remark here that, passing to the Lax potential ϕ, the transformation (33) takes the form (with ϕ and $\hat{\phi}$ solutions of Lax equations with parameters λ and $\hat{\lambda}$, respectively)

$$z' = z + 2\left(\ln\hat{\phi}\right)_{xx}, \qquad \phi' = \phi_x - \phi\left(\ln\hat{\phi}\right)_x,$$

which is known in literature as the Darboux transformation of KdV (see for instance [14,33]). Thus, the notion of Bäcklund transformation described by Definition 18 allows one to regard also Darboux transformations as examples of Bäcklund transformations.

Example 22. Consider the covering $\tau : \mathcal{Y}^{(\infty)} \longrightarrow \mathcal{E}^{(\infty)}$ of the short-pulse equation $\mathcal{E} = \{z_{xt} = \frac{1}{2}z^2 z_{xx} + z z_x^2 + z\} \subset J^3(\pi)$, defined by the system

$$\mathcal{Y}: \quad \begin{cases} \rho_x = -\dfrac{z_x}{4\eta}\rho^2 + \dfrac{1}{2\eta}\rho + \dfrac{z_x}{4\eta}, \\[2ex] \rho_t = -\dfrac{z\,(z z_x + 4\,\eta)}{8\eta}\rho^2 + \dfrac{(8\,\eta^2 + z^2)}{4\eta}\rho - \dfrac{z\,(-z z_x + 4\,\eta)}{8\eta}, \\[2ex] z_{xt} = \dfrac{1}{2}z^2 z_{xx} + z z_x^2 + z, \end{cases} \tag{34}$$

where $\pi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $(x, t, z, \rho) \mapsto (x, t)$. One can check that the transformation

$$x' = x + 8\frac{\eta}{\rho^2 + 1}, \qquad t' = t, \qquad z' = -z + 8\eta\frac{\rho}{\rho^2 + 1}, \tag{35}$$

determines an auto-Bäcklund transformation, i.e., a Bäcklund transformation from $\mathcal{E} = \{z_{xt} = \frac{1}{2}z^2 z_{xx} + z z_x^2 + z\}$ to $\mathcal{E}' = \{z'_{xt} = \frac{1}{2}z'^2 z'_{xx} + z' z'^2_x + z'\}$.

3. Riccati-type differentiable coverings determined by ZCRs

In this section, starting from a ZCR of an equation $\mathcal{E}$, we will show how to determine a first-order Riccati-type system defining a differentiable covering together with a corresponding nonlocal conservation law. We will see in Section 4 that this type of covering and the corresponding nonlocal conservation law are fundamental for the computation of pseudosymmetries that allow one to determine Bäcklund transformations.

Let $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ be a ZCR of the form $\alpha = X\,dx + T\,dt$, for a formally integrable equation $\mathcal{E} := \{\mathbf{F} = \mathbf{0}\}$ with two independent variables (x, t). We assume here that X and T are $\ell \times \ell$ matrices, with $\ell \geq 2$.

One can readily check that (1) is equivalent to the integrability condition

$$\bar{D}_t X - \bar{D}_x T + [X, T] = 0 \tag{36}$$

of a linear system of the form

$$V_x = XV, \qquad V_t = TV, \tag{37}$$

where $V = (v^1, ..., v^\ell)^T$ is an auxiliary vector valued function of (x, t).

This means that, by suitably enlarging the space of dependent variables, α defines a differentiable covering $\mathcal{V}^{(\infty)} \to \mathcal{E}^{(\infty)}$, with $\mathcal{V} := \left\{\mathbf{F} = \mathbf{0},\ \bar{D}_x V = XV,\ \bar{D}_t V = TV\right\}$. In such an enlarged space, (36) can be also understood as the integrability condition of an equation of the form

$$\bar{d}_H V = X V\, dx + T V\, dt, \tag{38}$$

where $\bar{d}_H V = \bar{D}_x V\, dx + \bar{D}_t V\, dt$. Notice that $V_x = \tilde{D}_x V$ and $V_t = \tilde{D}_t V$, in terms of the total derivatives $\tilde{D}_x$ and $\tilde{D}_t$ in $\mathcal{V}^{(\infty)}$.

Now, it is noteworthy to observe that linear transformations $V \mapsto V^S := SV$, by a nonsingular matrix valued function S on $\mathcal{E}^{(\infty)}$, naturally determine a $\mathcal{C}$-morphism of $\mathcal{V}$. Under such transformations, (37) transforms as

$$V_x^S = X^S V^S, \qquad V_t^S = T^S V^S,$$

with

$$X^S := \bar{D}_x(S)\, S^{-1} + S X S^{-1}, \qquad T^S := \bar{D}_t(S)\, S^{-1} + S T S^{-1}.$$

Accordingly, (38) transforms as $\bar{d}_H V^S = X^S V^S\, dx + T^S V^S\, dt$. These transformations are usually referred to as *gauge transformations*.

We have the following

Lemma 23. *For any fixed $h \in \{1, \dots, \ell\}$, under the gauge transformation $V \mapsto V' = \frac{1}{v^h} V$ equation (38) rewrites as*

$$\begin{cases} \bar{d}_H\left(\ln v^h\right) = [\alpha]^h\, V', \\ \bar{d}_H V' = \alpha V' - \left([\alpha]^h\, V'\right) V', \end{cases} \tag{39}$$

where $[\alpha]^h$ denotes the h-th row of α. Thus, the first order system (37) is equivalent to the following one

$$\begin{cases} \dfrac{v_x^h}{v^h} = [X]^h\, V', \\[2ex] \dfrac{v_t^h}{v^h} = [T]^h\, V', \\[2ex] V'_x = X V' - \left([X]^h\, V'\right) V', \\ V'_t = T V' - \left([T]^h\, V'\right) V', \end{cases} \tag{40}$$

where $[X]^h$ and $[T]^h$ denote the h-th rows of X and T, respectively.

Proof. The result follows by substituting $V = v^h V'$ (with h fixed) in (38). $\square$

Since $V' = \frac{1}{v^h} V$, it is natural to write V' as

$$V' = (\rho^1, \dots, \rho^{h-1}, 1, \rho^h, \dots, \rho^{\ell-1})^T,$$

where

$$\rho^j := \begin{cases} v^j/v^h, & j < h, \\ v^{j+1}/v^h, & j > h. \end{cases}$$

It follows that the differentiable covering $\mathcal{V}^{(\infty)} \to \mathcal{E}^{(\infty)}$ is the extension, by means of the potential of a nonlocal conservation law, of a differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by a system of $\ell - 1$ Riccati-type equations. This generalises a fact already discovered by Chern and Tenenblat in [15], in the case of $\mathfrak{sl}_2(\mathbb{R})$-valued ZCRs for equations describing pseudospherical surfaces.

For instance, by taking $h = \ell$ for ease of notations, one has the following

Proposition 24. *To any zero-curvature representation $\alpha = X\,dx + T\,dt$ of $\mathcal{E} := \{\mathbf{F} = \mathbf{0}\}$, with $\ell \times \ell$ matrices X and T, it is associated the $(\ell-1)$-dimensional differentiable covering defined by the system*

$$\begin{cases} \rho^j_x = \displaystyle\sum_{s=1}^{\ell-1} \left(-X_{\ell s}\rho^s\rho^j + X_{js}\rho^s \right) - X_{\ell\ell}\rho^j + X_{j\ell}, & j = 1, ..., \ell-1 \\ \rho^j_t = \displaystyle\sum_{s=1}^{\ell-1} \left(-T_{\ell s}\rho^s\rho^j + T_{js}\rho^s \right) - T_{\ell\ell}\rho^j + T_{j\ell}, & j = 1, ..., \ell-1 \\ \mathbf{F} = \mathbf{0}. \end{cases}$$

In particular, with respect to this covering, $\mathcal{E}$ admits the following nonlocal conservation law

$$\mu = \left(\sum_{s=1}^{\ell-1} X_{\ell s}\rho^s + X_{\ell\ell} \right) dx + \left(\sum_{s=1}^{\ell-1} T_{\ell s}\rho^s + T_{\ell\ell} \right) dt.$$

Analogous results hold for any other $h = 1, 2, ..., \ell-1$, with the corresponding nonlocal variables $\bar{\rho}_1, \bar{\rho}_2, ..., \bar{\rho}_{\ell-1}$ reciprocally related, for any choice of h, to the $\rho_1, \rho_2, ..., \rho_{\ell-1}$ above.

Example 25. We consider here KdV equation $\mathcal{E} = \{z_{xxx} = -z_t - 6zz_x\}$ with the ZCR

$$\alpha = \begin{pmatrix} 0 & 1 \\ -\lambda - z & 0 \end{pmatrix} dx + \begin{pmatrix} z_x & 4\lambda - 2z \\ -4\lambda^2 - 2\lambda z + 2z^2 + z_{xx} & -z_x \end{pmatrix} dt, \qquad \lambda \in \mathbb{R}.$$

In this case, the procedure described above, with $\rho = v^2/v^1$, provides the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by the system (27) of Example 16, together with the nonlocal conservation law $\mu = \rho dx + (z_x + (4\lambda - 2z)\,\rho)\,dt$. According to first equation of (39) $\mu = \bar{d}_H(\ln v^1)$, thus $\rho = v^1_x/v^1$.

Example 26. We consider here Tzitzeica equation $\mathcal{E} = \{z_{xt} = -\mathrm{e}^{-2z} + \mathrm{e}^{z}\}$ with the ZCR

$$\alpha = \begin{pmatrix} -z_x & 0 & \lambda \\ \lambda & z_x & 0 \\ 0 & \lambda & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & \dfrac{1}{\lambda}\mathrm{e}^{-2z} & 0 \\ 0 & 0 & \dfrac{1}{\lambda}\mathrm{e}^{z} \\ \dfrac{1}{\lambda}\mathrm{e}^{z} & 0 & 0 \end{pmatrix} dt, \qquad \lambda \in \mathbb{R}.$$

In this case, the procedure described above, with $\rho^1 = v^1/v^3$ and $\rho^2 = v^2/v^3$, provides the 2-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by the system (30) of Example 17, together with the nonlocal conservation law $\mu = \lambda \rho^2\, dx + e^{z}\rho^1/\lambda\, dt$. In this case, according to first equation of (39) $\mu = \bar{d}_H(\ln v^3)$, thus $\rho^1 = \lambda e^{-z} v^3_t/v^3$ and $\rho^2 = \lambda^{-1} v^3_x/v^3$.

4. Nonlocal pseudosymmetries and Bäcklund transformations

As discussed in Subsection 2.4, a Bäcklund transformation from $\mathcal{E}$ to $\mathcal{E}'$ can be seen as a nonlocal $\mathcal{C}$-morphism, from a differentiable extension $\mathcal{V}$ of the equation $\mathcal{E}$ to the equation $\mathcal{E}'$. Our aim here is to show how such transformations can result from factorisations of differentiable extensions $\mathcal{V}$ with respect to nonlocal pseudosymmetries. In these cases one specialises to $\mathcal{V}$ the factorisation procedure described in the end of Subsection 2.3.2.

The basic assumption is that $\mathcal{E}$ admits a differentiable extension $\mathcal{V}$ which has a pseudosymmetry $\bar{Y}$ (or an r-pseudosymmetry $\{\bar{Y}_1, ..., \bar{Y}_r\}$) which leads to a quotient system $\mathcal{Q}$ that, among its equations, has a copy of equation $\mathcal{E}'$. In such a case, the Bäcklund transformation is determined by a suitable choice of some basic system of functionally independent invariants $S = \{x'_i = \xi^i,\, z'^j = \nu^j\}$ of the exploited pseudosymmetry (resp., r-pseudosymmetry). Indeed, since the concrete form of $\mathcal{Q}$ depends on the particular choice of S, if one chooses another basic system of functionally independent invariants $\hat{S} = \{\hat{x}'_i = \hat{\xi}^i,\, \hat{z}'^j = \hat{\nu}^j\}$ equivalent to $\{\xi^i,\, \nu^j\}$, i.e., such that $\{\hat{\xi}^i = A^i(\xi^i, \nu^j),\ \hat{\nu}^j = B^j(\xi^i, \nu^j)\}$ and $\det\left(\tilde{D}_s \hat{\xi}^i\right) \neq 0$ (where $\tilde{D}_s$ are the total derivatives on $\mathcal{V}^{(\infty)}$), one would obtain a system $\hat{\mathcal{Q}}$ which is point equivalent to $\mathcal{Q}$, through the point transformation $\{\hat{x}'_i = A^i(x', z'),\ \hat{z}'^j = B^j(x', z')\}$.

On the other hand, in view of Proposition 11 and Proposition 12, the infinitesimal conditions (18) and (23) for pseudosymmetries and r-pseudosymmetries are linear only with respect to φ and Φ. Thus, in general the determination of pseudosymmetries and r-pseudosymmetries of $\mathcal{V}$ is feasible if one already knows some μ and γ. It follows that, for the practical application of the procedure described above, it may be necessary to already have available the possible differential coverings $\mathcal{V}$ together with some corresponding 1-forms μ and γ, defined in $\mathcal{V}$.

Based on the results of the previous section, it is therefore clear that in the case of an equation with two independent variables a particularly advantageous situation is that of equations admitting ZCRs: indeed, starting from these representations, the Riccati-type differentiable coverings together with associated horizontally closed 1-forms μ remain automatically determined and one can directly pass to search for corresponding pseudosymmetries.

Unfortunately, one does not have yet an analogous result for r-pseudosymmetries. Hence, calculating the possible r-pseudosymmetries admitted by the Riccati-type differentiable coverings, associated with ZCRs, remains rather complicated; as is already the case without considering any differentiable covering so much so that in the literature it is not easy to find significant examples of r-pseudosymmetries of differential equations. Yet we believe that r-pseudosymmetries should play an important role when the Riccati-type differentiable covering has dimension greater than 1, because the larger number of covering equations in this case may lead to greater restrictions on the existence of pseudosymmetries; therefore, in the case of $\ell \times \ell$ matrix-valued ZCR, with $\ell > 2$, the more general r-pseudosymmetries should play a key role. For instance, in Example 36, we show that the auto-Bäcklund transformation of the Tzitzeica equation found in [9] can be obtained by using a 2-pseudosymmetry of the covering obtained from an $\mathfrak{sl}_3(\mathbb{R})$-valued ZCR. Thus, the computation of r-pseudosymmetries deserves further detailed study. The same is true for the more general question on the role that pseudosymmetries and r-pseudosymmetries can have in the integrability property of an equation; in particular, we believe that they can play a fundamental role in studying Bäcklund transformations in higher dimensions. These and other questions will be addressed in future works.

4.1. Examples

In this subsection, we will analyze several examples of equations with two independent variables for which, starting from a ZCR, it is possible to determine pseudosymmetries and subsequently Bäcklund transformations, following the procedure described above. Moreover, in the last example, we derive an auto-Bäcklund transformation of the Tzitzeica equation by using a 2-pseudosymmetry. Taken together, these examples describe not only the degree of generality of the method, but demonstrate also its practical applicability. Throughout this section $x_1 = x$ and $x_2 = t$.

Example 27. We consider here KdV $\mathcal{E} = \{z_{xxx} = -z_t - 6zz_x\}$ and show how the invariants of a nonlocal pseudosymmetry $\bar{Y}$ can be used to determine an auto-Bäcklund transformation of this equation. As already shown in Example 25, $\mathcal{E}$ admits the nonlocal conservation law $\mu = \rho dx + (z_x + (4\lambda - 2z)\,\rho)\,dt$ in the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y}: \begin{cases} \rho_x = -\rho^2 - z - \lambda, \\ \rho_t = -2(2\lambda - z)\rho^2 - 2z_x\rho + z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2, \\ z_{xxx} = -z_t - 6zz_x. \end{cases}$$

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x' = \xi^1,\ t' = \xi^2,\ z' = \nu\}$ such that ξ^1, ξ^2, ν are invariants of a pseudosymmetry Y of $\mathcal{Y}$.

First, by searching for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$a_1(x,t,z,\rho)\partial_x + a_2(x,t,z,\rho)\partial_t + b^1(x,t,z,\rho)\partial_z + b^2(x,t,z,\rho)\partial_\rho,$$

relatively to conservation laws $c\,\mu$, with $c \in \mathbb{R} - \{0\}$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar{Y} = \rho\partial_z - \frac{1}{4}\partial_\rho + \dots,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being $\{x,\ t,\ z+2\rho^2\}$ a basic system of zeroth order invariants of $\bar{Y}$, one has that the auto-Bäcklund transformation of KdV discussed in Example 19

$$x' = x, \qquad t' = t, \qquad z' = -z - 2\rho^2 - 2\lambda,$$

is of the type described above, i.e., it is provided by the invariants of the pseudosymmetry $\bar{Y}$.

Example 28. We show here that also the Darboux transformation discussed in Example 21 is provided by the invariants of a nonlocal pseudosymmetry $\bar{Y}$. To this end one can consider the 2-dimensional differentiable covering of KdV $\mathcal{E} = \{z_{xxx} = -z_t - 6zz_x\}$ defined by

$$\mathcal{Y}: \begin{cases} \hat{\rho}_x = -\hat{\rho}^2 - z - \hat{\lambda}, \\ \hat{\rho}_t = -2(2\hat{\lambda} - z)\hat{\rho}^2 - 2z_x\hat{\rho} + z_{xx} + 2z^2 - 2\hat{\lambda}z - 4\hat{\lambda}^2, \\ \rho_x = -\rho^2 - z - \lambda, \\ \rho_t = -2(2\lambda - z)\rho^2 - 2z_x\rho + z_{xx} + 2z^2 - 2\lambda z - 4\lambda^2, \\ z_{xxx} = -z_t - 6zz_x, \end{cases}$$

which coincides with (32) and is obtained by doubling the Riccati-type system of previous example. Hence, being $\hat{\mu} = \hat{\rho}dx + \left(z_x + \left(4\hat{\lambda} - 2z\right)\hat{\rho}\right)dt$ still a conservation law for $\mathcal{Y}$, one can search for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$\begin{aligned} &a_1(x,t,z,\rho,\hat{\rho})\partial_x + a_2(x,t,z,\rho,\hat{\rho})\partial_t + b^1(x,t,z,\rho,\hat{\rho})\partial_z + b^2(x,t,z,\rho,\hat{\rho})\partial_\rho \\ &\quad + b^3(x,t,z,\rho,\hat{\rho})\partial_{\hat{\rho}}\,, \end{aligned}$$

relatively to $2\,\hat{\mu}$. In this case, when $\hat{\lambda} \neq \lambda$ and $\hat{\rho} \neq \rho$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar{Y} = \hat{\rho}\partial_z - \frac{1}{4}\left(1 + \frac{(\rho - \hat{\rho})^2}{\lambda - \hat{\lambda}}\right)\partial_\rho - \frac{1}{4}\partial_{\hat{\rho}} + \dots .$$

Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being $\{x,\ t,\ z+2\hat{\rho}^2, \hat{\rho} + (\hat{\lambda} - \lambda)/(\hat{\rho} - \rho)\}$ a basic system of zeroth order invariants of $\bar{Y}$, one has that the Darboux transformation discussed in Example 21

$$x' = x, \qquad t' = t, \qquad z' = -z - 2\hat{\rho}^2 - 2\hat{\lambda}, \qquad \rho' = -\frac{\hat{\lambda} - \lambda}{\hat{\rho} - \rho} - \hat{\rho},$$

is provided by the invariants of the pseudosymmetry $\bar{Y}$. In this case the transformation describes a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, where by $\mathcal{E}'$ now we mean the system

$$\mathcal{E}' : \begin{cases} \rho'_x = -\rho'^2 - z' - \lambda, \\ \rho'_t = -2(2\lambda - z')\rho'^2 - 2z'_x\rho' + z'_{xx} + 2z'^2 - 2\lambda z' - 4\lambda^2, \\ z'_{xxx} = -6z'z'_x - z'_t. \end{cases}$$

Example 29. We consider here sine-Gordon equation $\mathcal{E} = \{z_{xt} = \sin z\}$ and show how the invariants of a nonlocal pseudosymmetry $\bar{Y}$ can be used to determine an auto-Bäcklund transformation of this equation. To this end we consider the following ZCR

$$\alpha = \frac{1}{2}\begin{pmatrix} \eta & -z_x \\ z_x & -\eta \end{pmatrix} dx + \frac{1}{2\eta}\begin{pmatrix} \cos z & \sin z \\ \sin z & -\cos z \end{pmatrix} dt, \qquad \eta \in \mathbb{R} - \{0\}.$$

In this case, the procedure described in Section 3, with $\rho = v^1/v^2$, provides the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y} : \begin{cases} \rho_x = -\frac{1}{2} z_x \rho^2 + \eta\rho - \frac{1}{2} z_x, \\ \rho_t = \frac{1}{2\eta} \sin z \left(1 - \rho^2\right) + \frac{1}{\eta} \cos z\, \rho, \\ z_{xt} = \sin z, \end{cases}$$

together with the nonlocal conservation law

$$\mu = \frac{1}{2}\left(\rho z_x - \eta\right) dx + \frac{1}{2\eta}\left(\rho \sin z - \cos z\right) dt.$$

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x' = \xi^1,\ t' = \xi^2,\ z' = \nu\}$ such that ξ^1, ξ^2, ν are invariants of a pseudosymmetry Y of $\mathcal{Y}$.

First, by searching for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$a_1(x,t,z,\rho)\partial_x + a_2(x,t,z,\rho)\partial_t + b^1(x,t,z,\rho)\partial_z + b^2(x,t,z,\rho)\partial_\rho,$$

relatively to conservation laws $c\,\mu$, with $c \in \mathbb{R} - \{0\}$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar{Y} = \left(\rho^2 + 1\right)\partial_z - \frac{\left(\rho^2 + 1\right)^2}{4}\partial_\rho + \dots,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being $\{x,\ t,\ z + 4 \arctan\rho\}$ a basic system of zeroth order invariants of $\bar{Y}$, one is naturally lead to consider $\xi^1 = x$, $\xi^2 = t$ and by taking $\nu = z + 4\arctan\rho$ one obtains $z'_{xt} - \sin z' = 0$, whenever $\{z, \rho\}$ is a solution of $\mathcal{Y}$. Hence, one has the auto-Bäcklund transformation of $\mathcal{E}$

$$x' = x, \qquad t' = t, \qquad z' = z + 4\arctan\rho,$$

provided by the invariants of the pseudosymmetry $\bar{Y}$.

Example 30. We consider here the short-pulse equation $\mathcal{E} = \{z_{xt} = \frac{1}{2}z^2 z_{xx} + z z_x^2 + z\}$ and show how the invariants of a nonlocal pseudosymmetry $\bar{Y}$ can be used to determine an auto-Bäcklund transformation of this equation. To this end we consider the following ZCR

$$\alpha = \frac{1}{4\eta}\begin{pmatrix} 1 & z_x \\ z_x & -1 \end{pmatrix} dx + \frac{1}{8\eta}\begin{pmatrix} 8\eta^2 + z^2 & z\,(z z_x - 4\,\eta) \\ z\,(z z_x + 4\,\eta) & -8\eta^2 - z^2 \end{pmatrix} dt, \qquad \eta \in \mathbb{R} - \{0\}.$$

In this case, the procedure described in Section 3, with $\rho = v^1/v^2$, provides the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y}\colon \begin{cases} \rho_x = -\dfrac{z_x}{4\eta}\rho^2 + \dfrac{1}{2\eta}\rho + \dfrac{z_x}{4\eta}, \\[2ex] \rho_t = -\dfrac{z\,(z z_x + 4\,\eta)}{8\eta}\rho^2 + \dfrac{\left(8\,\eta^2 + z^2\right)}{4\eta}\rho - \dfrac{z\,(-z z_x + 4\,\eta)}{8\eta}, \\[2ex] z_{xt} = \dfrac{1}{2}z^2 z_{xx} + z z_x^2 + z, \end{cases}$$

together with the nonlocal conservation law

$$\mu = \frac{1}{4\eta}\,(z_x\rho - 1)\,dx + \frac{1}{8\eta}\left(z^2 z_x\,\rho + 4\eta\,z\,\rho - z^2 - 8\,\eta^2\right)dt.$$

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x' = \xi^1,\ t' = \xi^2,\ z' = \nu\}$ such that ξ^1, ξ^2, ν are invariants of a pseudosymmetry Y of $\mathcal{Y}$.

First, by searching for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$a_1(x,t,z,\rho)\partial_x + a_2(x,t,z,\rho)\partial_t + b^1(x,t,z,\rho)\partial_z + b^2(x,t,z,\rho)\partial_\rho,$$

relatively to conservation laws $c\,\mu$, with $c \in \mathbb{R} - \{0\}$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar{Y} = -2\rho\partial_x + \left(\rho^2 - 1\right)\partial_z - \frac{1}{8\eta}\left(\rho^2 + 1\right)^2\partial_\rho + \dots,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being $\{x + 8\eta/(\rho^2+1),\ t, z - 8\eta\rho/(\rho^2+1)\}$ a basic system of zeroth order invariants of $\bar{Y}$, one is naturally lead to consider $\xi^1 = x + 8\eta/(\rho^2+1)$, $\xi^2 = t$ and by taking $\nu = -z + 8\eta\rho/(\rho^2+1)$ one obtains $z'_{x't'} - \frac{1}{2}z'^2 z'_{x'x'} - z' z'^2_{x'} - z' = 0$, whenever $\{z, \rho\}$ is a solution of $\mathcal{Y}$. Hence, one has the auto-Bäcklund transformation of $\mathcal{E}$

$$x' = x + 8\frac{\eta}{\rho^2+1}, \qquad t' = t, \qquad z' = -z + 8\eta\frac{\rho}{\rho^2+1},$$

provided by the invariants of the pseudosymmetry $\bar{Y}$.

Example 31. We consider here Camassa-Holm equation $\mathcal{E} = \{z_{txx} = 3\,z\,z_x - z\,z_{xxx} - 2\,z_x\,z_{xx} + z_t\}$ and show how the invariants of a nonlocal pseudosymmetry $\bar{Y}$ can be used to determine an auto-Bäcklund transformation of this equation. To this end we consider the following ZCR

$$\alpha = \frac{1}{2}\begin{pmatrix} 0 & -\dfrac{1}{\eta} \\ -z + z_{xx} - \eta & 0 \end{pmatrix} dx + \frac{1}{2}\begin{pmatrix} z_x & \dfrac{1}{\eta}(z - 2\,\eta) \\ z^2 - z\,z_{xx} - \eta\,z - 2\eta^2 & -z_x \end{pmatrix} dt, \quad \eta \in \mathbb{R} - \{0\}.$$

In this case, the procedure described in Section 3, with $\rho = v^2/v^1$, provides the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y}: \begin{cases} \rho_x = \dfrac{1}{2\eta}\rho^2 + \dfrac{z_{xx}}{2} - \dfrac{\eta + z}{2}, \\ \rho_t = -\dfrac{(-2\,\eta + z)}{2\,\eta}\rho^2 - z_x\,\rho - \dfrac{z\,z_{xx}}{2} - \dfrac{2\,\eta^2 + \eta\,z - z^2}{2}, \\ z_{txx} = 3\,z\,z_x - z\,z_{xxx} - 2\,z_x\,z_{xx} + z_t, \end{cases} \tag{41}$$

together with the nonlocal conservation law

$$\mu = -\frac{\rho}{2\eta}\,dx + \left(\frac{z_x}{2} - \left(1 - \frac{z}{2\eta}\right)\rho\right) dt. \tag{42}$$

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x' = \xi^1,\ t' = \xi^2,\ z' = \nu\}$ such that ξ^1, ξ^2, ν are invariants of a pseudosymmetry Y of $\mathcal{Y}$.

In this case, if one searches for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of (relative) vector fields of the form

$$a_1(x,t,z,\rho)\partial_x + a_2(x,t,z,\rho)\partial_t + b^1(x,t,z,z_x,\rho)\partial_z + b^2(x,t,z,z_x,\rho)\partial_\rho,$$

relatively to conservation laws $c\,\mu$, with $c \in \mathbb{R} - \{0\}$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar{Y} = -\partial_x + (2\rho - z_x)\,\partial_z + \frac{(\eta^2 - \rho^2)}{2\eta}\partial_\rho + \dots,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being

$$\xi^1 = x - \ln\left(\frac{\rho - \eta}{\rho + \eta}\right), \quad \xi^2 = t, \quad \nu^1 = \left(\frac{\rho^2 + \eta^2}{\rho^2 - \eta^2}\right) z + \frac{2\,\eta\,\rho}{\rho^2 - \eta^2}\, z_x,$$
$$\nu^2 = \frac{2\,\eta\,\rho}{\rho^2 - \eta^2} z + \left(\frac{\rho^2 + \eta^2}{\rho^2 - \eta^2}\right) z_x - 2\,\rho,$$

a basic system of first-order order invariants of $\bar{Y}$, one finds that in this case the system $\mathcal{Q}$ (the factorisation of $\mathcal{E}$ by $\bar{Y}$) reads

$$\mathcal{Q}: \quad \begin{cases} \nu^1_{\xi^1} = \nu^2, \\ \nu^2_{\xi^1\xi^2} = \left(-6\,\eta + 3\,\nu^1 - 2\,\nu^2_{\xi^1}\right)\nu^2 + \left(2\,\eta - \nu^1\right)\nu^2_{\xi^1\xi^1} + \nu^1_{\xi^2}. \end{cases}$$

Therefore, by taking

$$x' = x - \ln\left(\frac{\rho - \eta}{\rho + \eta}\right), \qquad t' = t, \qquad z' = \nu^1 - 2\eta = \left(\frac{\rho^2 + \eta^2}{\rho^2 - \eta^2}\right) z + \frac{2\,\eta\,\rho}{\rho^2 - \eta^2}\, z_x - 2\eta, \quad (43)$$

one gets that $z'_{t'x'x'} - 3\,z'\,z'_{x'} + z'\,z'_{x'x'x'} + 2\,z'_{x'}\,z'_{x'x'} - z'_{t'} = 0$, whenever $\{z, \rho\}$ is a solution of $\mathcal{Y}$. Hence, by means of (43), the invariants of $\bar{Y}$ determine an auto-Bäcklund transformation of $\mathcal{E}$. This auto-Bäcklund transformation coincides with that found in [40], by means of a different approach.

Example 32. We show here that, by following an approach similar to that used in Example 28, one can also find a Darboux transformation for the Camassa-Holm $\mathcal{E} = \{z_{txx} = 3\,z\,z_x - z\,z_{xxx} - 2\,z_x\,z_{xx} + z_t\}$. To this end one can consider the 2-dimensional differentiable covering of $\mathcal{E}$ defined now by

$$\mathcal{Y}: \quad \begin{cases} \hat{\rho}_x = \dfrac{1}{2\hat{\eta}}\hat{\rho}^2 + \dfrac{z_{xx}}{2} - \dfrac{\hat{\eta} + z}{2}, \\ \hat{\rho}_t = -\dfrac{\left(-2\,\hat{\eta} + z\right)}{2\,\hat{\eta}}\hat{\rho}^2 - z_x\,\hat{\rho} - \dfrac{z\,z_{xx}}{2} - \dfrac{2\,\hat{\eta}^2 + \hat{\eta}\,z - z^2}{2}, \\ \rho_x = \dfrac{1}{2\eta}\rho^2 + \dfrac{z_{xx}}{2} - \dfrac{\eta + z}{2}, \\ \rho_t = -\dfrac{(-2\,\eta + z)}{2\,\eta}\rho^2 - z_x\,\rho - \dfrac{z\,z_{xx}}{2} - \dfrac{2\,\eta^2 + \eta\,z - z^2}{2}, \\ z_{txx} = 3\,z\,z_x - z\,z_{xxx} - 2\,z_x\,z_{xx} + z_t, \end{cases}$$

which is obtained by doubling the Riccati-type system (41). Hence, being (42) still a conservation law for the new system $\mathcal{Y}$, one can search for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of (relative) vector fields of the form

$$a_1(x,t,z,\rho,\hat\rho)\partial_x + a_2(x,t,z,\rho,\hat\rho)\partial_t + b^1(x,t,z,z_x,\rho,\hat\rho)\partial_z + b^2(x,t,z,z_x,\rho,\hat\rho)\partial_\rho,$$
$$+ b^3(x,t,z,z_x,\rho,\hat\rho)\partial_{\hat\rho}\,,$$

relatively to $2\,\mu$. In this case, when $\hat\eta \neq \eta$ and $\hat\rho \neq \rho$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar Y = -\partial_x + (2\rho - z_x)\,\partial_z + \frac{\left(\eta^2-\rho^2\right)}{2\eta}\partial_\rho + \frac{\eta^2\hat\eta - \eta\,\hat\eta^2 - 2\eta\,\rho\,\hat\rho + \hat\eta\,\rho^2 + \eta\,\hat\rho^2}{2\,\eta\,\left(\eta-\hat\eta\right)}\partial_{\hat\rho} + \dots .$$

Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being

$$\xi^1 = x - \ln\left(\frac{\rho-\eta}{\rho+\eta}\right), \quad \xi^2 = t, \quad \nu^1 = \left(\frac{\rho^2+\eta^2}{\rho^2-\eta^2}\right)z + \frac{2\,\eta\,\rho}{\rho^2-\eta^2}\,z_x,$$
$$\nu^2 = \frac{2\,\eta\,\rho}{\rho^2-\eta^2}z + \left(\frac{\rho^2+\eta^2}{\rho^2-\eta^2}\right)z_x - 2\,\rho, \quad \nu^3 = \frac{\hat\eta\,\rho^2 - \hat\eta\,\rho\hat\rho - \eta\,\hat\eta\,\left(\eta-\hat\eta\right)}{\eta\,\hat\rho - \hat\eta\,\rho},$$

a basic system of first-order invariants of $\bar Y$, one finds that in this case

$$x' = x - \ln\left(\frac{\rho-\eta}{\rho+\eta}\right), \qquad t' = t, \qquad z' = \left(\frac{\rho^2+\eta^2}{\rho^2-\eta^2}\right)z + \frac{2\,\eta\,\rho}{\rho^2-\eta^2}\,z_x - 2\eta,$$
$$\rho' = \frac{\hat\eta\,\rho^2 - \hat\eta\,\rho\hat\rho - \eta\,\hat\eta\,\left(\eta-\hat\eta\right)}{\eta\,\hat\rho - \hat\eta\,\rho}, \tag{44}$$

defines a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to the system

$$\mathcal{Y}' : \begin{cases} \rho'_{x'} = \dfrac{1}{2\hat\eta}\rho'^2 + \dfrac{z'_{x'x'}}{2} - \dfrac{\hat\eta + z'}{2}, \\[2ex] \rho'_{t'} = -\dfrac{\left(-2\,\hat\eta + z'\right)}{2\,\hat\eta}\rho'^2 - z'_{x'}\,\rho' - \dfrac{z'\,z'_{x'x'}}{2} - \dfrac{2\,\hat\eta^2 + \hat\eta z' - z'^2}{2}, \\[2ex] z'_{t'x'x'} = 3\,z'\,z'_{x'} - z'\,z'_{x'x'x'} - 2\,z'_{x'}\,z'_{x'x'} + z'_{t'}. \end{cases}$$

Example 33. We consider here the Harry-Dym equation $\mathcal{E} = \{z_t = z^3 z_{xxx}\}$ and show how the invariants of a nonlocal pseudosymmetry $\bar Y$ can be used to determine an auto-Bäcklund transformation of this equation. To this end we consider the following ZCR

$$\alpha = \begin{pmatrix} 0 & 4\eta \\ -\frac{\eta}{z^2} & 0 \end{pmatrix} dx + \begin{pmatrix} 8\eta^2 z_x & -64\eta^3 z \\ 2\eta z_{xx} + \frac{16\eta^3}{z} & -8\eta^2 z_x \end{pmatrix} dt, \qquad \eta \in \mathbb{R} - \{0\}.$$

In this case, the procedure described in Section 3, with $\rho = v^2/v^1$, provides the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y}: \begin{cases} \rho_x = -4\eta\rho^2 - \dfrac{\eta}{z^2}, \\ \rho_t = 64\eta^3\, z\, \rho^2 - 16\eta^2 z_x \rho + 2\eta z_{xx} + \dfrac{16\,\eta^3}{z}, \\ z_t = z^3 z_{xxx}, \end{cases}$$

together with the nonlocal conservation law

$$\mu = 4\,\eta\,\rho\, dx + \left(-64\eta^3\,\rho\, z + 8\eta^2\, z_x\right) dt.$$

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x' = \xi^1,\ t' = \xi^2,\ z' = \nu\}$ such that ξ^1, ξ^2, ν are invariants of a pseudosymmetry Y of $\mathcal{Y}$.

First, by searching for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$a_1(x,t,z,\rho)\partial_x + a_2(x,t,z,\rho)\partial_t + b^1(x,t,z,\rho)\partial_z + b^2(x,t,z,\rho)\partial_\rho,$$

relatively to conservation laws $c\,\mu$, with $c \in \mathbb{R} - \{0\}$, one finds that the most general pseudosymmetry of this type with $b^1 \neq 0$ is a constant multiple of

$$\bar{Y} = \frac{1}{8\eta}\partial_x + z\,\rho\,\partial_z - \frac{\rho^2}{2}\,\partial_\rho + \dots,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being $\{x - 1/(4\eta\rho),\ t,\ z\,\rho^2\}$ a basic system of zeroth order invariants of $\bar{Y}$, one is naturally lead to consider $\xi^1 = x - 1/(4\eta\rho)$, $\xi^2 = t$ and taking $\nu = -1/(4z\rho^2)$ one obtains $z'_{t'} - z'^3 z'_{x'x'x'} = 0$, whenever $\{z, \rho\}$ is a solution of $\mathcal{Y}$. Hence, one has the auto-Bäcklund transformation of $\mathcal{E}$

$$x' = x - \frac{1}{4\eta\rho}, \qquad t' = t, \qquad z' = -\frac{1}{4z\rho^2},$$

provided by the invariants of the pseudosymmetry $\bar{Y}$.

Example 34. We consider here the following modification of Harry-Dym (mHD) equation

$$\mathcal{E} = \left\{ z_t = z^3 z_{xxx} + \left(-4\,k^2 z^3 - \frac{12a^2}{z} + b\right) z_x \right\} \tag{45}$$

where $a, b, k \in \mathbb{R}$. This is a new integrable equation that includes, as a particular instance, the equation recently studied in [22,46]. Here we show how the invariants of a nonlocal pseudosymmetry $\bar{Y}$ can be used to determine an auto-Bäcklund transformation of (45). To this end we consider the following ZCR of (45)

$$\alpha=\begin{pmatrix}\frac{P}{z^2} & 4\eta\\ -\frac{\eta}{z^2} & -\frac{P}{z^2}\end{pmatrix}dx+\begin{pmatrix}-2az_{xx}+8\eta^2z_x+\frac{PQ}{z^3} & \frac{4\eta\,(4azz_x+Q)}{z}\\ 2\eta z_{xx}-4\eta kz_x-\frac{\eta Q}{z^3} & 2az_{xx}-8\eta^2z_x-\frac{PQ}{z^3}\end{pmatrix}dt,$$

where $\eta\in\mathbb{R}-\{0\}$ and

$$P:=k\,z^2+a,\qquad Q:=8ak\,z^2-16\eta^2z^2-8a^2+bz.$$

In this case, the procedure described in Section 3, with $\rho=v^2/v^1$, provides the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)}\to\mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y}:\begin{cases}\rho_x=-4\eta\rho^2+\left(-2k-\frac{2a}{z^2}\right)\rho-\frac{\eta}{z^2},\\ \rho_t=\left(\left(-32\eta ak+64\eta^3\right)z-16\eta a\,z_x-4\eta b+\frac{32\eta\,a^2}{z}\right)\rho^2\\ \qquad+\left(\left(-16a\,k^2+32\eta^2k\right)z-16\eta^2z_x+4az_{xx}-2bk+\frac{32a\,\eta^2}{z}-\frac{2ab}{z^2}+\frac{16a^3}{z^3}\right)\rho\\ \qquad-4\eta kz_x+2\eta z_{xx}+\frac{-8a\eta k+16\eta^3}{z}-\frac{b\eta}{z^2}+\frac{8\eta\,a^2}{z^3}\\ z_t=z^3z_{xxx}+\left(-4\,k^2z^3-\frac{12a^2}{z}+b\right)z_x,\end{cases}$$

together with the nonlocal conservation law

$$\begin{aligned}\mu=\left(4\eta\rho+k+\frac{a}{z^2}\right)dx+\Bigg(&\left(-32\left(-ak+2\eta^2\right)\eta z+16\eta az_x+4\eta b-\frac{32\eta\,a^2}{z}\right)\rho\\ &-8\left(-ak+2\eta^2\right)kz+8\eta^2z_x-2az_{xx}+bk+\frac{-8a^2k-8\left(-ak+2\eta^2\right)a}{z}+\frac{ab}{z^2}-\frac{8a^3}{z^3}\Bigg)\,dt.\end{aligned}$$

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x'=\xi^1,\ t'=\xi^2,\ z'=\nu\}$ such that ξ^1,ξ^2,ν are invariants of a pseudosymmetry Y of $\mathcal{Y}$.

First, by searching for pseudosymmetries $\bar{Y}:=Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$a_1(x,t,z,\rho)\partial_x+a_2(x,t,z,\rho)\partial_t+b^1(x,t,z,\rho)\partial_z+b^2(x,t,z,\rho)\partial_\rho,$$

relatively to conservation laws $c\,\mu$, with $c\in\mathbb{R}-\{0\}$, one finds that the most general pseudosymmetry of this type with $b^1\neq 0$ is a constant multiple of

$$\bar{Y}=\left(\frac{a\rho}{4\eta^2}+\frac{1}{8\eta}\right)\partial_x+\left(\frac{a\rho^2}{\eta}+\rho+\frac{k}{4\eta}\right)z\partial_z+\frac{(-2a\rho-\eta)\,(2\eta\rho+k)\,\rho}{4\eta^2}\partial_\rho+\dots,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$.

From now on, it is convenient to distinguish the case where k is zero from that where k is non-zero. In fact, up to rescaling z and x, one can always reduce oneself to the cases: **(i)** $k = 0$; **(ii)** $k = 1$. In particular, case (i) could be further subdivided into subcases $b = 0$ and $b = 1$, although we will not do so here.

Case (i). In this case, being $\{x - 1/(4\eta\rho),\, t,\, \eta z\,\rho^2/(2a\rho + \eta)\}$ a basic system of zeroth order invariants of $\bar{Y}$, one is naturally lead to consider $\xi^1 = x - 1/(4\eta\rho)$, $\xi^2 = t$ and taking $\nu = -(2a\rho + \eta)/(4\eta z\,\rho^2)$ one obtains $z'_{t'} - z'^3 z'_{x'x'x'} + \left(12a^2/z' - b\right) z'_{x'} = 0$, whenever $\{z, \rho\}$ is a solution of $\mathcal{Y}$. Hence, one has the auto-Bäcklund transformation of $\mathcal{E}$

$$x' = x - \frac{1}{4\eta\rho}, \qquad t' = t, \qquad z' = -\frac{2a\rho + \eta}{4\eta z\,\rho^2},$$

provided by the invariants of the pseudosymmetry $\bar{Y}$, when $k = 0$.

Case (ii). In this case, being $\{x - \ln((2\eta\rho + 1)/\rho)/2,\, t,\, z(2\eta\rho^2 + \rho)/(2a\rho + \eta)\}$ a basic system of zeroth order invariants of $\bar{Y}$, one is naturally lead to consider $\xi^1 = x - \ln((2\eta\rho + 1)/\rho)/2$, $\xi^2 = t$ and taking $\nu = -(2a\rho + \eta)/(2z(2\eta\rho^2 + \rho))$ one obtains $z'_{t'} - z'^3 z'_{x'x'x'} + \left(4z'^3 + 12a^2/z' - b\right) z'_{x'} = 0$, whenever $\{z, \rho\}$ is a solution of $\mathcal{Y}$. Hence, one has the auto-Bäcklund transformation of $\mathcal{E}$

$$x' = x - \frac{1}{2}\ln\left(\frac{2\eta\rho + 1}{\rho}\right), \qquad t' = t, \qquad z' = -\frac{2a\rho + \eta}{z(4\eta\rho^2 + 2\rho)},$$

provided by the invariants of the pseudosymmetry $\bar{Y}$, when $k = 1$.

Example 35. We consider here the coupled Schrödinger system $\mathcal{E} = \{z^1_t = 2\left(z^1\right)^2 z^2 + z^1_{xx},\, z^2_t = -2z^1\left(z^2\right)^2 - z^2_{xx}\}$ and show how the invariants of a nonlocal pseudosymmetry $\bar{Y}$ can be used to determine an auto-Bäcklund transformation of $\mathcal{E}$. To this end we consider the following ZCR

$$\alpha = \begin{pmatrix} \eta & -z^2 \\ z^1 & -\eta \end{pmatrix} dx + \begin{pmatrix} -2\eta^2 - z^1 z^2 & 2\eta z^2 + z^2_x \\ -2\eta z^1 + z^1_x & 2\eta^2 + z^1 z^2 \end{pmatrix} dt, \qquad \eta \in \mathbb{R}.$$

In this case, the procedure described in Section 3, with $\rho = v^2/v^1$, provides the 1-dimensional differentiable covering defined by the system

$$\begin{cases} \rho_x = z^2\rho^2 - 2\eta\rho + z^1, \\ \rho_t = \left(-2\eta\, z^2 - z^2_x\right)\rho^2 + \left(2z^1 z^2 + 4\eta^2\right)\rho - 2\eta z^1 + z^1_x, \\ z^1_t = 2\left(z^1\right)^2 z^2 + z^1_{xx}, \\ z^2_t = -2z^1\left(z^2\right)^2 - z^2_{xx}, \end{cases}$$

together with the nonlocal conservation law

$$\mu = \left(-z^2\rho + \eta\right) dx + \left(2\eta z^2\rho - z^1 z^2 + z^2_x\rho - 2\eta^2\right) dt.$$

Here we double above covering, by considering the covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by the system

$$\mathcal{Y}: \begin{cases} \hat{\rho}_x = z^2\hat{\rho}^2 - 2\hat{\eta}\hat{\rho} + z^1, \\ \hat{\rho}_t = \left(-2\hat{\eta}\, z^2 - z^2_x\right)\hat{\rho}^2 + \left(2z^1z^2 + 4\hat{\eta}^2\right)\hat{\rho} - 2\hat{\eta}z^1 + z^1_x, \\ \rho_x = z^2\rho^2 - 2\eta\rho + z^1, \\ \rho_t = \left(-2\eta\, z^2 - z^2_x\right)\rho^2 + \left(2z^1z^2 + 4\eta^2\right)\rho - 2\eta z^1 + z^1_x, \\ z^1_t = 2\left(z^1\right)^2 z^2 + z^1_{xx}, \\ z^2_t = -2z^1\left(z^2\right)^2 - z^2_{xx}. \end{cases}$$

Hence, being μ still a conservation law for $\mathcal{Y}$, one can search for pseudosymmetries $\bar{Y} := Y|_{\mathcal{Y}^{(\infty)}}$ of $\mathcal{Y}$ determined by the pseudoprolongations Y of vector fields of the form

$$\begin{aligned} &a_1(x,t,z^1,z^2,\rho,\hat{\rho})\partial_x + a_2(x,t,z^1,z^2,\rho,\hat{\rho})\partial_t + b^{11}(x,t,z^1,z^2,\rho,\hat{\rho})\partial_{z^1} \\ &\quad + b^{12}(x,t,z^1,z^2,\rho,\hat{\rho})\partial_{z^2} + b^{21}(x,t,z^1,z^2,\rho,\hat{\rho})\partial_\rho, + b^{22}(x,t,z^1,z^2,\rho,\hat{\rho})\partial_{\hat{\rho}}\ , \end{aligned}$$

relatively to $c\,\mu$, with $c \in \mathbb{R} - \{0\}$. In this case, when $\hat{\eta} \neq \eta$ and $\hat{\rho} \neq \rho$, one finds that the most general pseudosymmetry of this type with $b^{11}b^{12} \neq 0$ is a constant multiple of

$$\bar{Y} = \rho^2\,\partial_{z^1} - \partial_{z^2} - \frac{\left(\rho - \hat{\rho}\right)^2}{2(\eta - \hat{\eta})}\partial_{\hat{\rho}} + \dots\,,$$

with $c = 2$. Such a pseudosymmetry of $\mathcal{Y}$ can be seen as a nonlocal pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$. Thus, being

$$\xi^1 = x, \quad \xi^2 = t, \quad \nu^1 = \frac{((2\eta - 2\hat{\eta})\rho\hat{\rho} + z^1\left(\rho - \hat{\rho}\right)}{\rho - \hat{\rho}}, \quad \nu^2 = \frac{(\rho - \hat{\rho})z^2 - 2\eta + 2\hat{\eta}}{\rho - \hat{\rho}}, \quad \nu^3 = \rho,$$

a basic system of zeroth-order invariants of $\bar{Y}$, one finds that in this case

$$x' = x, \qquad t' = t, \qquad z^{1\prime} = \frac{((2\eta - 2\hat{\eta})\rho\,\hat{\rho} + z^1\left(\rho - \hat{\rho}\right)}{\rho - \hat{\rho}}, \qquad z^{2\prime} = \frac{\left(\rho - \hat{\rho}\right)z^2 - 2\eta + 2\hat{\eta}}{\rho - \hat{\rho}}, \tag{46}$$

defines a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}$.

Example 36. We consider here Tzitzeica equation $\mathcal{E} = \{z_{xt} = -\mathrm{e}^{-2z} + \mathrm{e}^z\}$. As already shown in Example 26, $\mathcal{E}$ admits the nonlocal conservation law $\mu = \lambda\rho^2\,dx + \frac{1}{\lambda}e^z\rho^1\,dt$ in the 1-dimensional differentiable covering $\mathcal{Y}^{(\infty)} \to \mathcal{E}^{(\infty)}$ defined by

$$\mathcal{Y}: \begin{cases} \rho^1_x = -\lambda\,\rho^1\rho^2 - z_x\,\rho^1 + \lambda, \\ \rho^1_t = -\dfrac{1}{\lambda}\mathrm{e}^z\left(\rho^1\right)^2 + \dfrac{1}{\lambda}e^{-2z}\rho^2, \\ \rho^2_x = -\lambda\left(\rho^2\right)^2 + z_x\rho^2 + \lambda\rho^1, \\ \rho^2_t = -\dfrac{1}{\lambda}\mathrm{e}^z\rho^1\rho^2 + \dfrac{1}{\lambda}e^z, \\ z_{xt} = -\mathrm{e}^{-2z} + \mathrm{e}^z. \end{cases}$$

Contrary to previous examples, in this case, we did not find any nontrivial pseudosymmetry, which would be a pseudoprolongation of some vector field relative to a multiple of μ. However, we found that $\mathcal{Y}$ admits a 2-pseudosymmetry given by

$$\bar{\mathbb{Y}} = \mathbb{Y}|_{\mathcal{Y}^{(\infty)}} = \left[\partial_z - \frac{2\rho^1\rho^2-1}{2\rho^1}\partial_{\rho^2} + \dots\,, \quad \partial_{\rho^1} - \frac{\rho^2}{\rho^1}\partial_{\rho^2} + \dots\right]^T,$$

induced on $\mathcal{Y}^{(\infty)}$ by a pseudoprolongation $\mathbb{Y}$ relative to the horizontal 1-form

$$\gamma = \begin{pmatrix} -\dfrac{4\lambda\,(\rho^1\rho^2)^2+2\lambda\,(\rho^1)^3-4\lambda\,\rho^1\rho^2+\lambda}{2\rho^2(\rho^1)^2-\rho^1} & \dfrac{12\lambda\,(\rho^1\rho^2)^2+4\lambda\,(\rho^1)^3-12\lambda\,\rho^1\rho^2+3\lambda}{4\rho^1\rho^2-2} \\ -\dfrac{4\lambda\,\rho^1(\rho^2)^2+4\lambda\,(\rho^1)^2-2\lambda\,\rho^2}{2\rho^2(\rho^1)^2-\rho^1} & \dfrac{4\lambda(\rho^1)^2+4\lambda\rho^1(\rho^2)^2-2z_x\rho^1\rho^2-2\lambda\rho^2+z_x}{2\rho^1\rho^2-1} \end{pmatrix} dx$$

$$+ \begin{pmatrix} \dfrac{4\mathrm{e}^z(\rho^1)^3\rho^2-2\mathrm{e}^z(\rho^1)^2+6\rho^1\,(\rho^2)^2\mathrm{e}^{-2z}-2\rho^2\mathrm{e}^{-2z}}{\lambda\,\left(2\rho^2(\rho^1)^2-\rho^1\right)} & -\dfrac{\mathrm{e}^z(\rho^1)^2}{\lambda}-3\dfrac{\rho^2\mathrm{e}^{-2z}}{\lambda}+\dfrac{\mathrm{e}^{-2z}}{2\lambda\rho^1} \\ \dfrac{4\mathrm{e}^z(\rho^1)^2\rho^2-2\,\mathrm{e}^z\rho^1+4(\rho^2)^2\mathrm{e}^{-2z}}{\lambda\,\left(2\rho^2(\rho^1)^2-\rho^1\right)} & -2\dfrac{\mathrm{e}^z\rho^1}{\lambda}-\dfrac{\rho^2\,\mathrm{e}^{-2z}}{\lambda\rho^1} \end{pmatrix} dt,$$

satisfying $d_H\gamma + \gamma\wedge\gamma = 0 \mod \mathcal{Y}^{(\infty)}$. Such a 2-pseudosymmetry of $\mathcal{Y}$ can be seen as a non-local 2-pseudosymmetry of $\mathcal{E}$, with respect to the differentiable covering defined by $\mathcal{Y}$.

We are searching for a regular $\mathcal{C}$-morphism from $\mathcal{Y}$ to $\mathcal{E}'$, with $\mathcal{E}'$ being a copy of $\mathcal{E}$, defined by a system $\{x' = \xi^1,\ t' = \xi^2,\ z' = \nu\}$ such that ξ^1, ξ^2, ν are invariants of the 2-pseudosymmetry given by $\bar{\mathbb{Y}}$. Thus, being $\{x,\ t,\ z + \ln(2\rho^1\rho^2 - 1)\}$ a basic system of zeroth order invariants of $\bar{\mathbb{Y}}$, one is naturally lead to consider $\xi^1 = x$, $\xi^2 = t$ and taking $\nu = z + \ln(2\rho^1\rho^2 - 1)$ one obtains $z'_{x't'} + \mathrm{e}^{-2z'} - \mathrm{e}^{z'} = 0$, whenever $\{z, \rho^1, \rho^2\}$ is a solution of $\mathcal{Y}$. Hence, one has the auto-Bäcklund transformation of $\mathcal{E}$

$$x' = x, \qquad t' = t, \qquad z' = z + \ln(2\rho^1\rho^2 - 1),$$

provided by the invariants of the 2-pseudosymmetry given by $\bar{\mathbb{Y}}$. This auto-Bäcklund transformation coincides with that found in [9], by means of a different approach.

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No data was used for the research described in the article.

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