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Geometric inequalities for Electrostatic Systems with Boundary

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To my wife Karla,
to my parents José Eduardo and Sara,
and to my daughter Catarina.

"A mathematician has not perfectly understood his own work until he has made it so clear that he can explain it to the first person he meets on the street."

- Joseph-Louis Lagrange.

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Resumo

DESIGUALDADES GEOMÉTRICAS PARA SISTEMAS ELETROSTÁTICOS COM FRONTEIRA

Neste trabalho, investigamos sistemas eletrostáticos com uma constante cosmológica não nula em variedades compactas com fronteira. Estabelecemos novas propriedades geométricas para variedades eletrostáticas em dimensões superiores, estendendo resultados anteriores na literatura. Além disso, provamos estimativas de fronteira precisas e desigualdades do tipo isoperimétrico para variedades eletrostáticas, bem como desigualdades de volume e fronteira envolvendo as massas de Brown-York e Hawking.

Além disso, provamos limites inferiores precisos para a massa de Hawking carregada de superfícies estáveis em espaços-tempos eletrostáticos em vários contextos. Um limite superior para o gênero de superfícies estáveis no sistema eletrostático é fornecido. Também estudamos a positividade para a massa de Hawking carregada de uma superfície minima com índice um nos espaços-tempos eletrostáticos. Um critério para que uma superfície CMC no espaço de Reissner-Nordström de Sitter seja estável é apresentado

Abstract

GEOMETRIC INEQUALITIES FOR ELECTROSTATIC SYSTEMS WITH BOUNDARY

In this work, we investigate electrostatic systems with a nonzero cosmological constant on compact manifolds with boundary. We establish new geometric properties for electrostatic manifolds in higher dimensions, extending previous results in the literature. Moreover, we prove sharp boundary estimates and isoperimetric-type inequalities for electrostatic manifolds, as well as volume and boundary inequalities involving the Brown-York and Hawking masses.

In addition, we prove sharp lower bounds for the charged Hawking mass of stable surfaces in electrostatic space-times in various contexts. An upper bound for the genus of stable surfaces in the electrostatic system is provided. We also study the positivity for the charged Hawking mass of a minimal surface with index one in the electrostatic space-times. A criterion for a CMC surface in the Reissner-Nordström deSitter space to be stable is presented.

Table of contents

Introduction

This work is based on the papers [44] and [2]. The first was co-authored with my supervisor, Professor Benedito Leandro, and published in Mathematical Research Letters. The second, which is set to appear in Classical and Quantum Gravity, was also in collaboration with my supervisor and Professors Allan Freitas (UFPB) and Ernani Ribeiro (UFC).

The Einstein-Maxwell equations with cosmological constant Λ on a Lorentzian manifold $(\widehat{M}^{n+1},\, \hat{g})$ are given by the following system:

$$\begin{cases} Ric_{\hat{g}} - \dfrac{R_{\hat{g}}}{2}\hat{g} + \Lambda\hat{g} = 2\left(F \circ F - \dfrac{1}{4}|F|^2_{\hat{g}}\hat{g}\right); \\ \\ dF = 0 \quad \text{and} \quad div_{\hat{g}}F = 0, \end{cases}$$

where F stands for the (Faraday) electromagnetic $(0, 2)$-tensor and $(F \circ F)_{\alpha\beta} = \hat{g}^{\sigma\gamma}F_{\alpha\sigma}F_{\beta\gamma}$, with Greek indices ranging from 1 to $n+1$. A static space-time is a product manifold $\widehat{M}^{n+1} = \mathbb{R} \times M^n$, endowed with the metric $\hat{g} = -f^2dt^2 + g$, where $(M^n,\, g)$ is an oriented n-dimensional Riemannian manifold and f is a positive smooth function on M^n. In particular, by choosing the electromagnetic field in the form $F = fE^\flat \wedge dt$, where $E^\flat$ denotes the dual 1-form to the electric field E, we prove in Chapter 1 that the Einstein-Maxwell system reduces to a set of equations adapted to the electrostatic setting (cf. [21, 26, 43]).

Definition 1. *Let (M^n, g) be an oriented n-dimensional Riemannian manifold, $E \in \mathfrak{X}(M)$ a tangent vector field, and f a positive smooth function on M^n. The Einstein-Maxwell equations with cosmological constant Λ for the electrostatic space-time associated to $(M^n,\, g,\, f,\, E)$ are* *given by*

$$\begin{cases} \nabla^2 f = f\left(Ric - \dfrac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \dfrac{2}{n-1}|E|^2 g\right), \\ \\ \Delta f = \dfrac{2}{n-1}\left((n-2)|E|^2 - \Lambda\right) f, \\ \\ \quad 0 = div(E) \quad and \quad 0 = d(fE^\flat). \end{cases} \tag{1}$$

We say that (M^n, g, f, E) is an electrostatic system if the system of equations (1) is satisfied for some constant $\Lambda \in \mathbb{R}$. If (M^n, g) is complete, then the electrostatic system is also complete.

We remark that Ric, ∇^2, div, and Δ denote the Ricci tensor, the Hessian tensor, the divergence, and the Laplacian with respect to the metric g, respectively. The smooth function f is referred to as the *lapse function* (or electrostatic potential), and E denotes the electric field. If M has a horizon boundary ∂M, we additionally assume that $f^{-1}(0) = \partial M$ (cf. [16, 21, 26]). An electrostatic system with zero cosmological constant is referred to as an electrovacuum system. In the absence of an electric field E, the system reduces to a vacuum static system. Among the explicit examples of electrostatic spacetimes, we highlight the vacuum static models (namely, the de Sitter and Nariai systems, see Example 1) as well as the Reissner-Nordström-de Sitter (RNdS), charged Nariai, cold black hole and ultracold black hole solutions. In Chapter 1, we explicitly prove that those systems are indeed solutions to the electrostatic system (1) introduced herein. Further details are provided in Section 1.5.

To proceed, we recall that the charge Q of a hypersurface Σ in (M^n, g) is given by

$$Q(\Sigma) = \frac{1}{\omega_{n-1}}\sqrt{\frac{2}{(n-1)(n-2)}}\int_\Sigma \langle E, \nu\rangle \, dA_g, \tag{2}$$

where ω_{n-1} denotes the area of the standard unit $(n-1)$-sphere, and ν is the unit normal vector to Σ. Several important contributions to the classification of electro-vacuum spacetimes can be found in the literature; see, for example, [16, 20, 22, 40, 42, 48]. In particular, by the positive mass theorem [64], any asymptotically flat electro-vacuum spacetime is conformally flat. That is, the Cotton tensor C vanishes in three dimensions, and the Weyl tensor W vanishes in higher dimensions. This, in turn, implies that the only admissible solutions to the electro-vacuum system under such conditions are precisely the Reissner-Nordström and Majumdar–Papapetrou spacetimes (cf. [20, Theorem 3.6]).

In this work, we investigate whether the de Sitter system is the unique compact, simply connected electrostatic system with positive scalar curvature. This question, originally posed in the context of vacuum static spaces, has been extensively studied within general relativity and is closely related to the *Cosmic No-Hair Conjecture*, formulated by Boucher, Gibbons, and Horowitz [14] (see also [13]): "*The only compact vacuum static system* (M^n, g, f) *with positive scalar curvature and connected boundary is the de Sitter system, with static potential* f *given by the height function*". While several results have confirmed the conjecture under additional assumptions (see, for example, [3, 12, 14, 32, 37, 39, 41, 59]), counterexamples were constructed by Gibbons, Hartnoll and Pope [33] in dimensions $4 \leq n \leq 8$, and a simply connected counterexample in all dimensions $n \geq 4$ was later presented by Costa, Diógenes, Pinheiro e Ribeiro [25]. To the best of our knowledge, the conjecture is still open for dimension $n = 3$.

A fruitful approach to investigating rigidity phenomena is through the establishment of obstruction results, such as geometric inequalities, which serve to obtain classification results as well as rule out some potential new examples. In this context, a distinguished result, by Boucher, Gibbons and Horowitz [14] and Shen [63], asserts that the de Sitter system maximizes the boundary area among all compact vacuum static spaces with positive scalar curvature and connected boundary. More precisely, they proved the following:

> *Let* (M^3, g, f) *be a compact, oriented vacuum static space with connected boundary and scalar curvature equal to* 6*. Then the area of* ∂M *satisfies the inequality*
>
> $$|\partial M| \leq 4\pi.$$
>
> *Moreover, equality holds if and only if* (M^3, g) *is isometric to the de Sitter system.*

In recent years, also motivated by the classical isoperimetric inequality, the study of boundary and volume estimates for special classes of manifolds (or metrics) has made significant advances. Such estimates have been established for static spaces in, e.g., [3, 11, 25, 35], as well as for critical metrics of the volume functional in, e.g., [6–9, 24, 31, 67]; see also [27, 28]. In this spirit, by employing the recent generalized Reilly formula due to Qiu and Xia [57] (see also [45]), we establish a sharp boundary estimate for compact electrostatic manifolds. More precisely, we obtain the following result.

Theorem 1. *Let (M^n, g, f, E) be a compact electrostatic system with connected boundary ∂M satisfying $|E|^2 < \dfrac{\Lambda}{(n-2)}$. Then we have:*

$$\sqrt{\frac{n+2}{2n\alpha}}|\partial M| \leq Vol(M), \tag{3}$$

where $\alpha = \max_M \dfrac{2}{n-1}\left(\Lambda - (n-2)|E|^2\right)$. Moreover, equality holds in (3) if (M^n, g, f) is isometric to the de Sitter system.

Remark 1. *We remark that the conclusion of Theorem 1 remains valid even when the boundary is disconnected, provided that the surface gravities $\kappa_i := |\nabla f||_{\Sigma_i}$, where Σ_i denotes the connected components of ∂M, are all equal to the same constant (see Theorem 11).*

Our next result may be viewed as an analogue of a Chruściel-type inequality in the setting of electrostatic systems. For motivating examples and a broader context in which such inequalities arise, we refer the reader to [35, p. 4].

In order to establish our main result, we consider the electric field E to be parallel to the gradient of the lapse function f. This condition provides a natural framework that complements the analysis found in the three-dimensional case studied by Cruz, Lima, and Sousa in [26, Theorem E]. As discussed in Remark 6 and Lemma 3, this alignment allows us to further explore the underlying geometry of the system. In this sense, the following result extends their original argument by providing a sharp inequality under these broader conditions.

Theorem 2. *Let (M^n, g, f, E) be a compact electrostatic system such that E is parallel to ∇f and $\partial M = \cup_{i=1}^{l}\Sigma_i$, where Σ_i are the connected components of ∂M. Suppose that $|E|^2 < \dfrac{\Lambda}{(n-2)}$. Then we have:*

$$\frac{1}{2}(n-1)(n-2)\omega_{n-1}^2\sum_{i=1}^{l}\frac{\kappa_i Q(\Sigma_i)^2}{|\Sigma_i|} + \frac{n-2}{n}\Lambda\sum_{i=1}^{l}\kappa_i|\Sigma_i| \leq \sum_{i=1}^{l}\kappa_i\int_{\Sigma_i}\frac{R^{\Sigma_i}}{2}dA_g, \tag{4}$$

where $\kappa_i = |\nabla f|\big|_{\Sigma_i}$ and ω_{n-1} is the area of the standard $(n-1)$-sphere. Here, R^{Σ_i} stands for the scalar curvature of Σ_i. Moreover, equality holds in (4) if and only if (M^n, g, f) is isometric to the de Sitter system.

Remark 2. *We emphasize that Theorem 2 can be regarded as a higher-dimensional generalization of [26, Theorem E]. The condition that E is parallel to ∇f always holds along ∂M;*

see assertion (iv) *in Proposition 3. Moreover, this property is satisfied in all known explicit examples of electrostatic systems, as discussed in Section 1.1.*

Remark 3. *The case* $n = 3$ *is of particular interest. In this specific dimension, the boundary* $\partial M = \bigcup_{i=1}^{\ell} \Sigma_i$ *consists of closed surfaces, and the scalar curvature of each component satisfies* $R^{\Sigma_i} = 2K$*, where* K *denotes the Gauss curvature. Consequently, by the Gauss–Bonnet theorem, we obtain the following area estimate:*

$$\sum_{i=1}^{\ell} k_i \left(\frac{16\pi^2 Q(\Sigma_i)^2}{|\Sigma_i|} + \frac{\Lambda}{3}|\Sigma_i| \right) \leq 4\pi \sum_{i=1}^{\ell} k_i,$$

which coincides precisely with [26, Theorem E]. Furthermore, if the boundary is assumed to be Einstein, we derive a related boundary estimate; see Theorem 12. For a significant topological characterization in the five-dimensional setting, see Corollary 2.

In the sequel, we relax the assumption that E is parallel to ∇f, which was required in the previous theorem. To this end, however, we must impose a control on the electric field. Our approach is based on an integral identity for electrostatic systems, which is of independent interest; see Proposition 4. Although an analogous result holds in higher dimensions (see Theorem 13), we state the three-dimensional case separately to allow a direct comparison with [26, Theorem D].

Theorem 3. *Let* (M^3, g, f, E) *be a compact electrostatic system with connected boundary* ∂M *and positive cosmological constant. Suppose that*

$$|E|^2 < \frac{\sqrt{5}}{5}\Lambda.$$

Then ∂M *is a* 2*-sphere and there exists a positive constant* c *such that*

$$\kappa c |\partial M| \leq 4\pi\Lambda, \tag{5}$$

where $\kappa = |\nabla f|\big|_{\partial M}$*. Moreover, equality holds in (5) if and only if* (M^3, g, f) *is isometric to the de Sitter system with* $c = \Lambda$ *and* $\kappa = \dfrac{\Lambda}{3}$*.*

Remark 4. *This bound on the norm of* E *considered in Theorem 3 arises naturally, even in the case where the scalar curvature is constant. In particular, when* $n = 3$*, it is known that the electric field satisfies* $|E|^2 < \Lambda$*, and the scalar curvature satisfies* $R < 4\Lambda$*; see assertion* (v) *of Proposition 3 in Section 1.1.*

By a different approach, we derive a boundary estimate for electrostatic manifolds that depends only on the cosmological constant Λ, that is, the estimate is independent of the surface gravity $\kappa = |\nabla f||_{\partial M}$. As before, we present here the result in the three-dimensional setting, although it arises as a special case of a more general boundary estimate valid in higher dimensions (see Theorem 14).

Theorem 4. *Let (M^3, g, f, E) be a compact electrostatic system with connected boundary ∂M. Suppose that*

$$|E|^2 < \frac{\Lambda}{5}.$$

Then ∂M is a 2-sphere and

$$\Lambda|\partial M| \leq 12\pi, \tag{6}$$

Moreover, equality holds in (6) if and only if (M^3, g, f) is isometric to the de Sitter system.

In the last part of this work, we focus on the role of quasi-local masses in the setting of electrostatic systems. In Newtonian gravity, the mass of a region can be naturally defined by integrating a mass density function. However, in general relativity, the situation is considerably more subtle due to the Equivalence Principle, which precludes the existence of a local energy density for the gravitational field. The definition of mass thus becomes a central and subtle problem in general relativity. In 1982, Penrose [54] identified several major open questions in the field, placing at the forefront the challenge of formulating a suitable quasi-local definition of energy-momentum. Mathematically, the formulation of this problem presents inherent challenges, and as a result, several definitions of quasi-local mass have been proposed over the years in an attempt to capture a coherent and physically meaningful notion of mass in general relativity. For a detailed discussion of the challenges and desirable properties associated with such definitions, we refer the reader to [1, 19]. A good definition of quasi-local mass should satisfy certain properties, such as

1. The mass must be positive for a large class of surfaces.

2. It should vanish for surfaces in flat space-time.

3. It should converge to the ADM mass at flat infinity.

4. It should converge to the Bondi mass at null infinity.

5. It should satisfy Penrose inequality if the surface enclosed a horizon.

6. It should satisfy some stability property for the rigidity case, i.e., it is expected that if we have a sequence of a finite domain and if the defined quasi-local mass of such a sequence of domain converges to zero, this sequence of finite domain converges to a domain in Minkowski space or Euclidean space.

7. Monotonicity property under inclusion, i.e., $\forall\, \Omega_1 \Subset \Omega_2 \subset M$, with $\partial\Omega_1$ outer-minimizing in Ω_2 and $\partial\Omega_2$ outer-minimizing in M. Then, $\mathrm{mass}(\Omega_1) \leq \mathrm{mass}(\Omega_2)$.

The ADM mass, defined by Arnowitt, Deser, and Misner [4], represents the total energy of an system and is defined for an asymptotically flat manifold (M^n, g) is given by:

$$\mathfrak{M} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \sum_{i,j=1}^{n} (\partial_j g_{ij} - \partial_i g_{jj}) \nu^i \, dA_\delta,$$

where S_r is a coordinate sphere of radius r and ν is its outward unit normal.

In order to proceed, we recall the definition of the Brown-York mass. Let Σ be a connected hypersurface in (M^n, g) such that $(\Sigma, g|_\Sigma)$ can be embedded in $\mathbb{R}^n$ as a convex hypersurface. Then, the Riemannian Brown-York mass $\mathfrak{M}_{BY}$ of Σ with respect to g is given by

$$\mathfrak{M}_{BY}(\Sigma, g) = \int_\Sigma (H_0 - H)\, dA_g,$$

where H_0 and H denote the mean curvatures of Σ as a hypersurface of $\mathbb{R}^n$ and M^n, respectively, and dA_g is the volume element on Σ induced by g. In [66], Yuan proved a boundary estimate for vacuum static spaces in terms of the Riemannian Brown-York mass. Similar estimates were obtained in [25, 27]. Motivated by these results, we have the following sharp boundary estimate involving the Riemannian Brown-York for electrostatic manifolds.

Theorem 5. *Let $(M^n,\, g,\, f,\, E)$ be a compact electrostatic system with (possibly disconnected) boundary ∂M and $\Lambda + |E|^2 > 0$. Suppose that each boundary component $(\Sigma_i,\, g)$ can be isometrically embedded in $\mathbb{R}^n$ as a convex hypersurface. Then we have*

$$|\Sigma_i| \leq c\, \mathfrak{M}_{BY}(\Sigma_i, g), \tag{7}$$

where c is a positive constant. Moreover, equality holds for some component Σ_i if and only if $(M^n,\, g,\, f)$ is isometric to the de Sitter system.

We note that, by virtue of the solution to the Weyl problem, the isometric embedding condition in Theorem 5 can be replaced by appropriate curvature bounds, as for instance, by requiring positive Gaussian curvature when $n = 3$, see, e.g. [31, 66]. Another important notion of quasi-local mass, the Hawking mass, has received considerable attention due to its essential role in the proof of the Riemannian Penrose inequality [36]. The standard Hawking mass of a surface Σ^2 in a given Riemannian manifold (M^3, g) is defined by

$$\mathfrak{M}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, dA_g\right),$$

where $|\Sigma|$ and H stand for the area of the surface and its mean curvature with respect to the metric g, respectively. A straightforward computation shows that if Σ is a minimal surface, its Hawking mass is positive. However on the one hand, the Hawking mass of any surface in $\mathbb{R}^3$ is less than or equal to zero, with equality if and only if the surface is a round sphere (cf. [52]). These examples contradict the first item of the nice properties that a quasi-local mass must satisfy. On the other hand, Christodoulou and Yau [19] proved that the Hawking mass is non-negative for stable constant mean curvature spheres in 3-manifolds with non-negative scalar curvature. This shows that even the concept of Hawking's mass being positive can be a bitter issue to deal with. The third chapter of this work aims to provide sharp lower bounds for the charged Hawking mass of stable surfaces in electrostatic space-times in various contexts, guaranteeing that the charged Hawking mass is positive under certain conditions.

In [50, Proposition 3], the author proved a lower bound for the Hawking mass of a connected stable CMC surface boundary Σ of an asymptotically flat and static manifold such that the Gauss curvature K_Σ and the constant mean curvature H_0 of Σ satisfy the inequality $4K_\Sigma \geq H_0^2$. They proved that

$$\mathfrak{M}_H(\Sigma) \geq \frac{1}{8}\sqrt{\frac{H_0^2|\Sigma|}{4\pi}}\mathfrak{M}.$$

Therefore, the Hawking mass is non-negative, according to the positive mass theorem.

From a quasi-local mass point of view, it is desirable to draw information on the quasi-local mass of a surface Σ purely from knowledge of the geometric data (g_Σ, H), where g_Σ is the intrinsic metric on Σ and H is the mean curvature. It is natural to ask if the Hawking mass of such a surface is positive when the intrinsic metric is not far from being round. The positivity of the standard Hawking mass for CMC surfaces on compact 3-manifolds with non-negative scalar curvature was studied by [51] without assuming stability. However, additional conditions were imposed on the geometric data.

The Hawking mass can also help provide Item the positivity for other definitions of quasi-local mass. A consequence of the proof of the Riemannian Penrose inequality via inverse mean curvature flow by [36] gives us

$$\mathfrak{M}(M, g) \geq \mathfrak{M}_H(\partial\Omega), \tag{8}$$

where (M^3, g) is an asymptotically flat Riemannian manifold (possibly with horizon boundary) with non-negative scalar curvature and $\Omega \subset M$ is a bounded open set with a smooth topological outer-minimizing boundary $\partial\Omega$.

The Hawking mass is often used as a lower bound for the Bartnik mass (a more delicate concept of quasi-local mass). Computing the Bartnik mass of a subset Ω is not easy, so one way to study the Bartnik mass is to look at its upper and lower bounds (cf. [51, 52, 47]). We define the Bartnik mass $\mathfrak{M}_B(\Omega)$ of Ω as

$$\mathfrak{M}_B(\Omega) = \inf\{\mathfrak{M}(\widetilde{M}, \widetilde{g}) : (\widetilde{M}, \widetilde{g}) \in \mathcal{A}\}, \tag{9}$$

where $\mathcal{A}$ is the set of asymptotically flat manifolds (possibly with horizon boundary) with non-negative scalar curvature into which Ω isometrically embeds such that $\partial\Omega \subset \widetilde{M}$ is outer-minimizing. The boundary $\partial\Omega$ is said to be outer-minimizing if $Perim(\Omega) \leq Perim(\Omega')$ for any set $\Omega' \subset M$ of a finite perimeter (denoted by $Perim(\Omega')$) and finite volume such that $\Omega \subset \Omega'$. The Bartnik mass is an important quasi-local mass since it satisfies the monotonicity property. Note that the positive mass theorem (or the Riemannian Penrose inequality, in case all elements in $\mathcal{A}$ have non-empty horizon boundary) immediately yields the non-negativity of the Bartnik mass (see more in [52]). In fact, since every smooth extension $(\widetilde{M}, \widetilde{g}) \in \mathcal{A}$ induces the same mean curvature on $\partial\Omega \subset \widetilde{M}$, the inequality (8) combined with (9) implies

$$\mathfrak{M}_B(\Omega) \geq \mathfrak{M}_H(\partial\Omega).$$

In [52, Theorem 1.6], the authors assumed that the Hawking mass satisfies a certain local non-positivity condition to obtain a global rigidity theorem for Riemannian manifolds with non-negative scalar curvature. If the manifold is in addition asymptotically locally simply connected, the space must be isometric to $\mathbb{R}^3$. Moreover, to obtain a lower bound on the Bartnik mass [52, Theorem 1.7] out of the expansion of the Hawking mass obtained in [52, Equation 11], the authors proved that the optimally perturbed geodesic spheres are outer-minimizing.

We can see how useful it can be to prove lower bounds for the Hawking mass since computing the Bartnik mass of a subset is, in general, non-trivial. On the other hand, from the definition, it is conceivable to expect upper bounds for the Bartnik quasi-local mass by direct comparison with somewhat explicit competitors. However, the issue of finding explicit lower bounds is more delicate.

Inspired by the above discussion, we will focus on finding sharp lower bounds for the charged Hawking mass of surfaces possessing an index equal to zero or one in non-compact (or compact) manifolds, particularly stable minimal surfaces, stable CMC surfaces, and minimal surfaces of index one. Moreover, we will provide examples proving that such lower bounds are sharp.

Definition 2. *[5] Let (M^3, g) be a three-dimensional Riemannian manifold and Σ^2 a closed surface on M^3. The charged Hawking mass is defined by*

$$\mathfrak{M}_{CH}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(\frac{1}{2}\chi(\Sigma) - \frac{1}{16\pi}\int_{\Sigma}(H^2 + \frac{4}{3}\Lambda)d\sigma + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2\right),$$

where $|\Sigma|$ stands for the area of Σ. Here, $\chi(\Sigma) = 2(1 - g(\Sigma))$ and $g(\Sigma)$ stands for the genus of Σ. Moreover, Λ and $Q(\Sigma)$ stand for the cosmological constant and the electric charge of Σ given by (3.7), respectively.

Proposition 1. *Let (M^3, g, f, E) be a three-dimensional electrostatic system (compact or non-compact) and Σ a closed surface on M satisfying*

$$\int_{\Sigma}(Ric(\nu, \nu) + |A|^2)d\sigma \leq C. \tag{10}$$

Then,

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left[\frac{1}{16\pi}\int_{\Sigma}H^2 d\sigma + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right) - \frac{C}{8\pi}\right],$$

where $C \in \mathbb{R}$. Moreover, the genus of Σ is bounded from above, i.e.,

$$1 + \frac{C}{4\pi} \geq g(\Sigma).$$

This proposition is strongly used to demonstrate the main theorems of Chapter 3:

Theorem 6. *Let (M^3, g, f, E) be a three-dimensional electrostatic system (compact or non-compact) and Σ a closed surface on M with $Index(\Sigma) = 0$. Then, the genus of Σ is at most 1. Moreover,*

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2} \left[\frac{1}{16\pi}\int_\Sigma H^2 d\sigma + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right)\right].$$

The equality holds if and only if Σ is totally umbilical, E is orthogonal to Σ, and equality holds in (10) for $C = 0$.

Theorem 7. *Let (M^3, g, f, E) be a three-dimensional Electrostatic system with a non-null cosmological constant and $\Sigma \subset M$ an embedded stable minimal surface. Then, the genus of Σ is at most 1. Moreover,*

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2} \left[\frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right)\right].$$

In addition, if Σ is a stable minimal two-sphere, equality holds if Σ is the horizon boundary of the ultracold black hole system.

Theorem 8. *Let (M^3, g, f, E) be a three-dimensional Electrostatic system with Σ a closed stable CMC surface in M. Then, the genus of Σ is at most 3. Moreover,*

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2} \left[\frac{H^2}{16\pi}|\Sigma| + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right) - 1\right].$$

The equality holds if and only if Σ is totally umbilical, E is orthogonal to Σ, and equality holds in (10) for $C = 8\pi$.

Theorem 9. *Let (M^3, g, f, E) be a three-dimensional Electrostatic system with Σ a closed minimal surface of index one in M. Then,*

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2} \left(\frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right) - 1 - Int\left[\frac{1+g(\Sigma)}{2}\right]\right).$$

The equality holds if and only if Σ is totally umbilical, E is orthogonal to Σ, and the equality holds in (10) for $C = 8\pi\left(1 + Int\left[\frac{1+g(\Sigma)}{2}\right]\right)$.

More details about the above theorems will be seen in Chapter 3.

Chapter 1

Electrostatic system

1.1 Preliminary

First, we define some useful geometry concepts for a Riemannian manifold (M^n, g). Manfredo in [30, Chapter 0] defines $\mathbf{x}_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ such that $(U_\alpha, \mathbf{x}_\alpha)$ is the maximum differential structure of M. We will indicate by $(x_1^\alpha, \cdots, x_n^\alpha)$ the coordinates of U_α and by $(\partial_1^\alpha, \cdots, \partial_n^\alpha)$ The associated bases in the tangent spaces of $\mathbf{x}_\alpha(U_\alpha)$.

It is convenient to say that the functions Γ_{ij}^k defined in U by $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$ are the Christoffel Symbols of the conetion ∇ in U. Since ∇ is metric-compatible we have

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left\{\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right\} g^{km}. \tag{1.1}$$

As defined in [30, Chapter 2].

Also in [30, Chapter 4] the author expresses the components of the curvature of a Riemannian manifold in terms of the coefficients Γ_{ij}^k of the Riemannian connection by writing

$$R_{ijk}^s = \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \partial_j \Gamma_{ik}^s - \partial_i \Gamma_{jk}^s. \tag{1.2}$$

Some curvature combinations appear so frequently that they deserve to be remembered: the Ricci curvature and the scalar curvature. We call the contraction of the Riemann curvature the Ricci curvature tensor, and its coefficients are given by.

$$R_{ik} = \sum_j R_{ijk}^j. \tag{1.3}$$

The scalar curvature is obtained by taking the trace of the Ricci curvature.

$$R = \sum_{ik} R_{ik} g^{ik}. \tag{1.4}$$

At this point we will establish some operators in differential forms, that is, the divergent and the curl. As we see in [29, Chapter 1], let v be a vector field differentiable on M^n, if ω denotes the 1-differential form obtained from v by the canonical isomorphism induced by the metric g, and ν is the volume element of M^n, the divergence can be obtained as follows

$$d(\star\omega) = (\operatorname{div} v)\nu,$$

and the curl curl v of v is the $(n-2)$-form defined by

$$\star(d\omega) = \operatorname{curl} v,$$

where we use the operation $\star$, introduced in Definition 3 above. We further denote that $v^\flat = \omega$.

Definition 3. *Given a k-form ω in M^n, we define an $(n-k)$-form $\star\omega$ by setting*

$$\star\left(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = (-1)^\sigma \left(dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}\right)$$

and extending this definition by linearity, where $i_1 < \cdots < i_k, j_1 < \cdots < j_{n-k}$, $\left(i_1, \cdots, i_k, j_1, \cdots, j_{n-k}\right)$ is a permutation of $(1, 2, \cdots, n)$ and σ is 0 or 1, depending on whether the permutation is even or odd, respectively.

We will describe some properties using lemmas, and the first one that will be very useful is Poincaré's Lemma:

Lemma 1. *Let M be a contractible differentiable manifold, and let ω be a differentiable k-form on M, with $d\omega = 0$. Then, ω is exact i.e., there exists a $(k-1)$-form α on M, such that $d\alpha = \omega$.*

Another very useful equation is the relation between the Laplacian on the ambient manifold and the intrinsic Laplacian on a hypersurface, that is

Proposition 2. *Let (M^n, g) be a Riemannian manifold and $f \in C^\infty(M)$, the Laplacian of f can be decomposed as*

$$\Delta f = \Delta_\Sigma f + Hg(\nabla f, \eta) + \nabla^2 f(\eta, \eta), \tag{1.5}$$

where $\Sigma^{n-1} \subset M^n$ is a hypersurface with unit normal vector field η and mean curvature H and $\nabla^2 f$ is the Hessian of f on M.

Proof. Let $X, Y \in \mathfrak{X}(\Sigma)$ and notice that

$$\nabla^2 f(X,Y) = g(\nabla_X \nabla f, Y)$$

and

$$\nabla f = \nabla_\Sigma f + g(\nabla f, \eta)\eta.$$

Then,

$$\begin{array}{rcl}\nabla^2 f(X,Y) & = & g\{\nabla_X[\nabla_\Sigma f + g(\nabla f,\eta)\eta], Y\}; \\ & = & g(\nabla_X \nabla_\Sigma f, Y) + g[\nabla_X g(\nabla f,\eta)\eta, Y]; \\ & = & \nabla^2_\Sigma f(X,Y) + g(\nabla f,\eta) g(\nabla_X \eta, Y); \\ & = & \nabla^2_\Sigma f(X,Y) + A(X,Y) g(\nabla f,\eta).\end{array}$$

Finally, taking the trace

$$\Delta f - \nabla^2 f(\eta,\eta) = \Delta_\Sigma f + H g(\nabla f,\eta).$$

□

1.2 The electrostatic system

The Einstein–Maxwell equations it was the result of an attempt to connect the Maxwell's theory for electromagnetism and the Einstein's theory for gravitation, showing how the curvature of spacetime is influenced not only by mass but also by electromagnetic fields.

In [17, 21, 26] the authors consider a Lorentzian $n+1$-manifold $(\mathcal{M}^{n+1}, \widetilde{g})$ and a 2-form F on $\mathcal{M}$. The (source-free) Einstein Maxwell equations with cosmological constant $\Lambda \in \mathbb{R}$ for the triple $(\mathcal{M}, \widetilde{g}, F)$ are expressed by the following system

$$\begin{cases} \mathrm{Ric}_{\widetilde{g}} - \dfrac{\mathrm{R}_{\widetilde{g}}}{2}\mathrm{g} + \Lambda \widetilde{g} = 2\left(F \circ F - \dfrac{1}{4}|F|^2_{\widetilde{g}} \widetilde{g}\right); \\ \\ dF = 0 \quad \text{and} \quad \mathrm{div}_{\widetilde{g}} F = 0. \end{cases} \tag{1.6}$$

The solutions of this system are called electrovacuum spacetimes, where F stands for the (Faraday) electromagnetic (0, 2)-tensor, $(F \circ F)_{\alpha\beta} = \widetilde{g}^{\sigma\gamma} F_{\alpha\sigma} F_{\beta\gamma}$, stands for the Hadamard-Schur product and $|F|^2 = \widetilde{g}^{\alpha\beta} \widetilde{g}^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$ is the Hilbert-Schmidt norm of F.

Our goal in this section is construct the electrostatic system, for this is necessary to consider (M^n, g) an n dimensional Riemannian manifold with $n \geq 3$, and $f : M \to \mathbb{R}$ a positive function. Now, let

$$\mathcal{M}^{n+1} = M^n \times_f \mathbb{R}$$

be a warped product with metric

$$\widetilde{g} = g + \varepsilon f^2 dt^2,$$

where $\varepsilon = \pm 1$. Hence, suppose that $x_i = (x_1, \cdots, x_n)$, $x_\mu = (x_0, x_1, \cdots, x_n)$ and $x_0 = t$, and are ordinary vectors of $\mathcal{M}, M$ and $\mathbb{R}$, respectively, in such a way that a vector field $X \in \mathfrak{X}(\mathcal{M}^{n+1})$ can be written as

$$X = \left\langle X, \varepsilon f^{-1}\partial_t \right\rangle_t \partial_t + \sum_{i=1}^{n} g(X, \partial_i)\partial_i. \tag{1.7}$$

Therefore, observe that the Christoffel's symbols (1.1) for $(\mathcal{M}^{n+1}, \widetilde{g})$ are

$$\begin{aligned}
\widetilde{\Gamma}^0_{00} &= \frac{1}{2}\left\{\frac{\partial}{\partial t}\widetilde{g}_{00} + \frac{\partial}{\partial x_t}\widetilde{g}_{00} - \frac{\partial}{\partial t}\widetilde{g}_{00}\right\}\widetilde{g}^{00} = 0,\\
\widetilde{\Gamma}^0_{ij} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_i}\widetilde{g}_{j0} + \frac{\partial}{\partial x_j}\widetilde{g}_{0i} - \frac{\partial}{\partial t}\widetilde{g}_{ij}\right\}\widetilde{g}^{00} = 0,\\
\widetilde{\Gamma}^m_{i0} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_i}\widetilde{g}_{00} + \frac{\partial}{\partial t}\widetilde{g}_{0i} - \frac{\partial}{\partial t}\widetilde{g}_{i0}\right\}\widetilde{g}^{0m} = 0,\\
\widetilde{\Gamma}^0_{i0} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_i}\widetilde{g}_{00} + \frac{\partial}{\partial t}\widetilde{g}_{0i} - \frac{\partial}{\partial t}\widetilde{g}_{i0}\right\}\widetilde{g}^{00} = \frac{1}{2}\frac{2\varepsilon f \partial_i f}{\varepsilon f^2} = \frac{\partial_i f}{f},\\
\widetilde{\Gamma}^m_{00} &= \frac{1}{2}\sum_{k\neq 0}\left\{\frac{\partial}{\partial t}\widetilde{g}_{0k} + \frac{\partial}{\partial t}\widetilde{g}_{k0} - \frac{\partial}{\partial x_k}\widetilde{g}_{00}\right\}\widetilde{g}^{km} = -\sum_k \frac{1}{2}2\varepsilon f \partial_k f \widetilde{g}^{km};\\
&= -\varepsilon f \sum_k \partial_k f\, g^{km} = -\varepsilon f \nabla^m f,\\
\widetilde{\Gamma}^m_{ij} &= \Gamma^m_{ij}.
\end{aligned}$$

Thus the Riemannian curvature tensor (1.2) of $(\mathcal{M}^{n+1}, g)$ is

$$\begin{aligned}
\widetilde{R}^{s}_{0j0} &= \sum_{l} \widetilde{\Gamma}^{l}_{00}\widetilde{\Gamma}^{s}_{jl} - \sum_{l} \widetilde{\Gamma}^{l}_{j0}\widetilde{\Gamma}^{s}_{0l} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^{s}_{00} - \frac{\partial}{\partial t}\widetilde{\Gamma}^{s}_{j0};\\
&= -\sum_{l} \varepsilon f \nabla^{l} f \widetilde{\Gamma}^{s}_{jl} - \widetilde{\Gamma}^{0}_{j0}\widetilde{\Gamma}^{s}_{00} - \frac{\partial}{\partial x_j}\left(\varepsilon f \nabla^{s} f\right);\\
&= \varepsilon \partial_j f \nabla^{s} f - \frac{\partial}{\partial x_j}\left(\varepsilon f \nabla^{s} f\right);\\
&= \varepsilon f \nabla_j \nabla^{s} f,\\
\widetilde{R}^{0}_{i0k} &= \sum_{l} \widetilde{\Gamma}^{l}_{ik}\widetilde{\Gamma}^{0}_{0l} - \sum_{l} \widetilde{\Gamma}^{l}_{0k}\widetilde{\Gamma}^{0}_{il} + \frac{\partial}{\partial t}\widetilde{\Gamma}^{0}_{ik} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}^{0}_{0k};\\
&= \sum_{l} \widetilde{\Gamma}^{l}_{ik}\frac{\partial_l f}{f} - \frac{\partial_k f}{f}\frac{\partial_i f}{f} - \frac{\partial}{\partial x_i}\frac{\partial_k f}{f};\\
&= \frac{\partial_k f}{f}\frac{\partial_i f}{f} - \frac{f\partial_k\partial_i f - \partial_k f \partial_i f}{f^2};\\
&= \frac{\nabla_k \nabla_i f}{f},\\
\widetilde{R}^{0}_{000} &= \sum_{l} \widetilde{\Gamma}^{l}_{00}\widetilde{\Gamma}^{0}_{0l} - \sum_{l} \widetilde{\Gamma}^{l}_{00}\widetilde{\Gamma}^{0}_{0l} + \frac{\partial}{\partial t}\widetilde{\Gamma}^{0}_{00} - \frac{\partial}{\partial t}\widetilde{\Gamma}^{0}_{00} = 0,\\
\widetilde{R}^{s}_{ijk} &= R^{s}_{ijk}.
\end{aligned}$$

For the Ricci tensor (1.3) of $(\mathcal{M}, \widetilde{g})$ we have

$$\widetilde{R}_{ik} = \sum_{j\neq 0} \widetilde{R}^{j}_{ijk} + \widetilde{R}^{0}_{i0k} = R_{ik} - \frac{\nabla_k \nabla_i f}{f}, \tag{1.8}$$

$$\widetilde{R}_{00} = \sum_{j\neq 0} \widetilde{R}^{j}_{0j0} + \widetilde{R}^{0}_{000} = -\varepsilon f \nabla_j \nabla^{j} f = -\varepsilon f \Delta f.$$

Thus, the scalar curvature (1.4) of $\mathcal{M}$ is

$$\widetilde{R} = \sum_{ik} \widetilde{R}_{ik}\widetilde{g}^{ik} + \widetilde{R}_{00}\widetilde{g}^{00} = R - \frac{\Delta f}{f} - \varepsilon\frac{f\Delta f}{\varepsilon f^2},$$

$$\widetilde{R} = R - 2\frac{\Delta f}{f}. \tag{1.9}$$

We recovered from (1.6) that the Einstein field equation is given by

$$\mathrm{Ric}_{\widetilde{g}} - \frac{\mathrm{R}_{\widetilde{g}}}{2}\widetilde{g} + \Lambda\widetilde{g} = 2\left(F \circ F - \frac{1}{4}|F|^2_{\widetilde{g}}\widetilde{g}\right). \tag{1.10}$$

Moreover, we can observe that

$$\mathrm{Tr}_{\widetilde{g}}\left(2F \circ F - \frac{1}{2}|F|^2_{\widetilde{g}}\widetilde{g}\right) = \widetilde{g}^{\alpha\beta}\left(2F \circ F - \frac{1}{2}|F|^2_{\widetilde{g}}\widetilde{g}\right)_{\alpha\beta} = 2|F|^2 - \frac{n+1}{2}|F|^2 = -\frac{n-3}{2}|F|^2$$

and

$$\mathrm{Tr}_{g}\left(2F \circ F - \frac{1}{2}|F|^2_{\widetilde{g}}\widetilde{g}\right) = \widetilde{g}^{ij}\left(2F \circ F - \frac{1}{2}|F|^2_{\widetilde{g}}\widetilde{g}\right)_{ij} = |F|^2 - \frac{n}{2}|F|^2 = -\frac{n-2}{2}|F|^2.$$

Contracting (1.10) over $\widetilde{g}$ we obtain the scalar curvature of $(\mathcal{M}, \widetilde{g})$, given by

$$\begin{aligned}
&\widetilde{\mathrm{R}} - \frac{\widetilde{\mathrm{R}}}{2}(n+1) + \Lambda(n+1) = -\frac{n-3}{2}|F|^2,\\
&-\widetilde{\mathrm{R}}(n-1) + 2\Lambda(n+1) = -(n-3)|F|^2,\\
&\widetilde{\mathrm{R}} = \frac{1}{n-1}\left[(n-3)|F|^2 + 2\Lambda(n+1)\right].
\end{aligned} \tag{1.11}$$

Replacing the curvature (1.11) in (1.10) we have the following equation

$$\begin{aligned}
&\mathrm{Ric}_{\widetilde{g}} - \frac{1}{n-1}\left[\frac{n-3}{2}|F|^2 + \Lambda(n+1)\right]\widetilde{g} + \Lambda\widetilde{g} = 2\left(F \circ F - \frac{1}{4}|F|^2_{\widetilde{g}}\widetilde{g}\right),\\
&\mathrm{Ric}_{\widetilde{g}} - \frac{1}{n-1}\left[\frac{n-3}{2}|F|^2 - \frac{n-1}{2}|F|^2 + \Lambda(n+1) - \Lambda(n-1)\right]\widetilde{g} = 2F \circ F,\\
&\mathrm{Ric}_{\widetilde{g}} = 2F \circ F + \frac{1}{n-1}\left(2\Lambda - |F|^2\right)\widetilde{g}.
\end{aligned}$$

Now, replacing (1.8) in the previous equation we get

$$R_{ik} = \frac{\nabla_k \nabla_i f}{f} + 2(F \circ F)_{ij} + \frac{1}{n-1}\left(2\Lambda - |F|^2\right) g_{ij}. \tag{1.12}$$

Contracting this equation over metric g, we have the scalar curvature of M,

$$\begin{aligned} R &= \frac{\Delta f}{f} + |F|^2 + \frac{n}{n-1}\left(2\Lambda - |F|^2\right); \\ &= \frac{\Delta f}{f} + \frac{1}{n-1}\left(2n\Lambda + (n-1)|F|^2 - n|F|^2\right); \\ &= \frac{\Delta f}{f} + \frac{1}{n-1}\left(2n\Lambda - |F|^2\right). \end{aligned} \tag{1.13}$$

Also, we can combine (1.11) with (1.9) to obtain

$$R = 2\frac{\Delta f}{f} + \frac{1}{n-1}\left[(n-3)|F|^2 + 2\Lambda(n+1)\right],$$

then using (1.13) yields

$$\begin{aligned} 0 &= -\frac{\Delta f}{f} + \frac{1}{n-1}\left[-|F|^2 - (n-3)|F|^2 + 2n\Lambda - 2\Lambda(n+1)\right], \\ \frac{\Delta f}{f} &= \frac{1}{n-1}\left[-(n-2)|F|^2 - 2\Lambda\right]. \end{aligned} \tag{1.14}$$

We now assume that the Maxwell field takes the form

$$F = fE^\flat \wedge dt,$$

where $E = (E_1, \cdots, E_n)$ is a vector field and

$$F_{ij} = 0, \quad F_{i0} = -F_{0i} = fE_i, \quad F_{00} = 0, \quad \text{for } i, j = 1, \cdots, n.$$

In what follows we will consider $\varepsilon = -1$. Thus,

$$\begin{aligned} |F|^2 &= \widetilde{g}^{\alpha\beta}\widetilde{g}^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} \\ &= \widetilde{g}^{00}\widetilde{g}^{ij}F_{0i}F_{0j} + \widetilde{g}^{0j}\widetilde{g}^{i0}F_{0i}F_{j0} + \widetilde{g}^{i0}\widetilde{g}^{0j}F_{i0}F_{0j} + \widetilde{g}^{ij}\widetilde{g}^{00}F_{i0}F_{j0} \\ &= 2f^2\widetilde{g}^{00}\widetilde{g}^{ij}E_iE_j = -2|E|^2 \end{aligned}$$

and

$$(F \circ F)_{\alpha\beta} = \widetilde{g}^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} = \widetilde{g}^{00}F_{i0}F_{j0} = f^2\widetilde{g}^{00}E_iE_j = -E^\flat \otimes E^\flat.$$

Then by (1.12) and (1.14) we get, respectively,

$$\begin{aligned}\nabla^2 f &= f(\mathrm{Ric} - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g),\\ \Delta f &= \frac{2}{n-1}\left[(n-2)|E|^2 - \Lambda\right] f.\end{aligned}$$

For the other two equations of the electrostatic system, we may observe that

$$0 = dF = d\left(fE^\flat\right) \wedge dt + fE^\flat \wedge ddt,$$

and once $ddt = 0$, $d\left(fE^\flat\right)$, must be zero. Since $d\left(fE^\flat\right) = 0$, by Lemma 1 we can infer that $fE = \nabla\psi$ for some $\psi : M \longrightarrow \mathbb{R}$ smooth function. Now we can consider $\widetilde{\psi} : \mathcal{M} \longrightarrow \mathbb{R}$ such that $\widetilde{\psi}\Big|_M = \psi$ and $\partial_t \widetilde{\psi} = 0$ so, $d\widetilde{\psi} = d\psi$ and $\sum_{i=0}^{n} dd\widetilde{\psi}_{ii} = \widetilde{\Delta}\widetilde{\psi}$, to prove that $\mathrm{div}(E) = 0$, by the definition of divergent,

$$\mathrm{div}_{\widetilde{g}}(F)v = d\left[\star\left(fE^\flat \wedge dt\right)\right] = d\left[\star\left(d\psi \wedge dt\right)\right].$$

and by Definition 3

$$\begin{aligned} d\left[\star\left(d\psi \wedge dt\right)\right] &= d\left[\star\left(d\widetilde{\psi} \wedge dt\right)\right];\\ &= d\left[\sum_{i=0}^{n}(-1)^{i-1} d\widetilde{\psi}_i d\widehat{x}_i\right];\\ &= \sum_{i=0}^{n}(-1)^{i-1}(-1)^{i-1} dd\widetilde{\psi}_{ii} v.\end{aligned}$$

Then

$$0 = \mathrm{div}_{\widetilde{g}}(F) = \widetilde{\Delta}\widetilde{\psi}$$

and by Laplacian's definition

$$
\begin{aligned}
\widetilde{\Delta}\widetilde{\psi} &= \sum_{i,j=0}^{n} \widetilde{g}^{ij}\left(\frac{\partial^2\widetilde{\psi}}{\partial x_i \partial x_j} - \sum_{k=0}^{n} \widetilde{\Gamma_{ij}^{k}} \frac{\partial \psi}{\partial x_k}\right);\\
&= g^{00}\left(\frac{\partial^2\widetilde{\psi}}{\partial t^2} - \widetilde{\Gamma_{00}^{0}}\frac{\partial\widetilde{\psi}}{\partial t} - \sum_{k=1}^{n}\Gamma_{00}^{k}\frac{\partial\widetilde{\psi}}{\partial x_k}\right);\\
&+ \sum_{i,j=1}^{n} g^{ij}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j} - \widetilde{\Gamma_{ij}^{0}}\frac{\partial\widetilde{\psi}}{\partial t} - \sum_{k=1}^{n}\Gamma_{ij}^{k}\frac{\partial\psi}{\partial x_k}\right);\\
&= -g^{00}\sum_{k=1}^{n}\Gamma_{00}^{k}\frac{\partial\psi}{\partial x_k} + \sum_{i,j=1}^{n} g^{ij}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j} - \sum_{k=1}^{n}\Gamma_{ij}^{k}\frac{\partial\psi}{\partial x_k}\right);\\
&= \frac{1}{f^2}\varepsilon f\sum_{k}\frac{\partial\psi}{\partial x_k} g^{kk}\frac{\partial\psi}{\partial x_k} + \Delta\psi;\\
&= \Delta\psi - \frac{g(\nabla\psi,\nabla f)}{f}.
\end{aligned}
$$

So

$$
\operatorname{div}(E) = \operatorname{div}\left(\frac{\nabla\psi}{f}\right) = \frac{1}{f}\left(\Delta\psi - \frac{g(\nabla\psi,\nabla f)}{f}\right) = 0
$$

and now we finally got to the system we were looking for:

Definition 4. *Let (M^n, g) be a n-dimensional smooth Riemannian manifold with $n \geq 3$, and let E a tangent vector field on M and $f \in C^\infty(M)$ satisfying*

$$
\begin{aligned}
\nabla^2 f &= f(\mathrm{Ric} - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g),\\
\Delta f &= \frac{2}{n-1}\left[(n-2)|E|^2 - \Lambda\right] f,\\
0 &= \operatorname{div}(E) \quad \textit{and} \quad 0 = d(fE).
\end{aligned}
$$

Here, Ric, ∇^2, div *and* Δ *stand for the Ricci tensor, Hessian tensor, divergence, and Laplacian operator concerning the metric g, respectively. Moreover, $E^\flat$ is the one-form metrically dual to E. If M has an horizon boundary ∂M, we also assume that $f^{-1}(0) = \partial M$. We refer to the above equations as electrostatic system with cosmological constant Λ for the electrostatic space-time associated to (M^n, g, f, E).*

1.3 Properties

This section was entirely based on [2]. Now, we begin to collecting some properties of the electrostatic system Definition 4. Although some of these features are already known (see, for example, [26, Lemma 4]), we include them here for the sake of completeness.

Proposition 3. *Let (M^n, g, f, E) be an electrostatic system with a non-empty boundary. Then the following assertions hold:*

(i) The scalar curvature of (M^n, g) is given by

$$R = 2\Lambda + 2|E|^2; \tag{1.15}$$

(ii) The boundary ∂M is totally geodesic. In particular,

$$\mathring{Ric}(\nu, \nu) = \frac{n-2}{2n}R - \frac{1}{2}R^{\partial M}, \tag{1.16}$$

where $\mathring{Ric} = Ric - \frac{R}{n}g$ is the traceless Ricci tensor, and $R^{\partial M}$ is the scalar curvature of ∂M.

(iii) If $\partial M = \cup_{i=1}^{l}\Sigma_i$, where Σ_i are the connected components of ∂M, then $\kappa_i := |\nabla f|\Big|_{\Sigma_i}$ are non-null constant (they are called surface gravities);

(iv) ∇f and E are proportional along ∂M. In particular, $|\langle E, \nabla f\rangle| = |E||\nabla f|$ along ∂M;

(v) If M is compact and $|E|$ constant, then

$$\Lambda > (n-2)|E|^2.$$

In particular,

$$2\Lambda \leq R < \frac{2(n-1)}{n-2}\Lambda;$$

(vi) (M^n, g, f) is sub-static, i.e.,

$$fRic - \nabla^2 f + (\Delta f)g \geq 0. \tag{1.17}$$

Proof. The proof is structured according to each assertion:

(i) We take the trace of the first equation in (1) in order to infer

$$\Delta f = f\left(R - \frac{2n}{n-1}\Lambda - \frac{2}{n-1}|E|^2\right) = \left(\frac{2(n-2)}{n-1}|E|^2 - \frac{2}{n-1}\Lambda\right) f,$$

which combined with the second equation in (1) proves the stated identity.

(ii) First, we observe that $|\nabla f(x)| \neq 0$ for all $x \in \partial M$. Indeed, consider a unit-speed geodesic $\sigma : [0,1] \longrightarrow M^n$ such that $\sigma(0) = x \in \partial M$ and $\sigma(1) \in \mathrm{int}(M)$. Define the function $\theta : [0,1] \longrightarrow \mathbb{R}$ by $\theta(t) = f(\sigma(t))$. Then $\theta(0) = f(x) = 0$, and

$$\theta''(t) = \nabla^2 f(\sigma'(t), \sigma'(t)) = A(t)\theta(t),$$

for some function $A(t)$. If $\nabla f(x) = 0$, then $\theta(t)$ would satisfy the initial value problem $\theta''(t) = A(t)\theta(t)$ with $\theta(0) = \theta'(0) = 0$, implying that θ is identically zero near x, and thus, f would vanish along σ, leading to a contradiction.

Since $\partial M = f^{-1}(0)$, we may take $\nu = -\dfrac{\nabla f}{|\nabla f|}$ as a unit normal vector field on ∂M. Hence, for any vector fields $X, Y \in \mathfrak{X}(\partial M)$, the second fundamental form A of ∂M satisfies

$$A(X,Y) = -\langle \nabla_X \nu, Y\rangle = \frac{1}{|\nabla f|}\nabla^2 f(X,Y) = 0,$$

along ∂M. The Gauss equation then yields identity (1.16) directly.

(iii) For any $X \in \mathfrak{X}(\partial M)$, we compute

$$X(|\nabla f|^2) = 2\langle \nabla_X \nabla f, \nabla f\rangle = 2\nabla^2 f(X, \nabla f) = 0,$$

along ∂M.

(iv) Since

$$0 = d(fE^\flat) = df \wedge E^\flat + f\, dE^\flat,$$

and $f = 0$ on ∂M, it follows that $df \wedge E^\flat = 0$ on ∂M.

(v) Integrating the second equation in (1) and using the notation from item (iii), we obtain

$$-\sum_i \kappa_i |\Sigma_i| = \int_M \Delta f \, dV_g = \frac{2}{n-1}\left((n-2)|E|^2 - \Lambda\right)\int_M f\, dV_g.$$

Since $f > 0$ in M^n, the result follows.

(vi) A direct computation using system (1) gives

$$f\,Ric - \nabla^2 f + (\Delta f)g = 2f\left(|E|^2 g - E^\flat \otimes E^\flat\right),$$

and the desired conclusion follows from the Cauchy–Schwarz inequality.

This finishes the proof. $\square$

Remark 5. *We also highlight that the condition (vi) established in Proposition 3 carries an important physical interpretation related to Einstein's field equations. Consider an $(n+1)$-dimensional static spacetime $(\widehat{M}, \hat{g})$, where*

$$\widehat{M} = \mathbb{R} \times M\,, \qquad \hat{g} = -f^2\,\mathrm{d}t \otimes \mathrm{d}t + g, \tag{1.18}$$

(M^n, g) is a Riemannian manifold and $f \in C^\infty(M)$ is a positive smooth function. Suppose this spacetime satisfies the Einstein field equations:

$$Ric_{\hat{g}} + \left(\Lambda - \frac{1}{2}R_{\hat{g}}\right)\hat{g} = \mathfrak{T}, \tag{1.19}$$

where $\mathfrak{T}$ stands for the stress-energy tensor and Λ is the cosmological constant.

The spacetime $(\widehat{M}, \hat{g})$ is said to satisfy the null energy condition (NEC) *for (1.19) if*

$$\mathfrak{T}(Y, Y) \geq 0$$

for all null vectors Y, that is, vectors satisfying $\hat{g}(Y,Y) = 0$. This condition is a natural geometric assumption and plays a central role in Penrose's singularity theorem [55]. It is also closely related to the sub-static condition (1.17) (see, e.g., [65, Lemma 3.8] and [23, Section 2]).

Indeed, up to a rescaling, a null vector Y can be expressed as $Y = \widehat{e_0} + X$, where $\widehat{e_0} = \dfrac{1}{f}\dfrac{\partial}{\partial t}$ and $X \in \mathfrak{X}(M)$ satisfies $g(X,X) = 1$. Given the form of the metric in (1.18) and the fact that $\hat{g}(\widehat{e_0}, \widehat{e_0}) = -1$, applying the (NEC) to Y yields:

$$\begin{aligned}
0 \leq \mathfrak{T}(Y,Y) &= \mathfrak{T}_{00} + \mathfrak{T}_{ij}X^iX^j \\
&= \left(-\Lambda + \frac{R}{2}\right) + \left[Ric - \frac{\nabla^2 f}{f} + \frac{\Delta f}{f} g\right](X,X) + \left(\Lambda - \frac{R}{2}\right) g(X,X) \\
&= \left[Ric - \frac{\nabla^2 f}{f} + \frac{\Delta f}{f} g\right](X,X),
\end{aligned}$$

where R denotes the scalar curvature of (M, g). This is precisely the condition stated in item (vi) of Proposition 3.

Proceeding, it follows from (1) and (1.15) that

$$\nabla^2 f = f\left(Ric + 2E^\flat \otimes E^\flat - \frac{R}{(n-1)}g\right) \tag{1.20}$$

and

$$\Delta f = \left(\frac{(n-2)}{(n-1)}R - 2\Lambda\right)f. \tag{1.21}$$

The next result is a divergence-type formula in the setting of electrostatic systems, primarily inspired by the classical Robinson-Shen identity [63]; see also [42, 43] for related formulas.

Lemma 2. *Let (M^n, g, f, E) be an electrostatic system. Then we have:*

$$\begin{aligned} div\left[\frac{1}{f}\left(\nabla|\nabla f|^2 - 4f\langle\nabla f, E\rangle E^\flat + \frac{2Rf}{n(n-1)}\nabla f\right)\right] &= 2f|\mathring{Ric}|^2 + 4f\mathring{Ric}(E, E) \\ &+\frac{n-2}{n}\langle\nabla R, \nabla f\rangle. \end{aligned}$$

Proof. One easily verifies that

$$div(\mathring{Ric}(\nabla f)) = div\mathring{Ric}(\nabla f) + \langle\mathring{Ric}, \nabla^2 f\rangle.$$

This jointly with the twice-contracted second Bianchi identity ($2div\,Ric = \nabla R$) and (1.20) yields

$$div(\mathring{Ric}(\nabla f)) = f|\mathring{Ric}|^2 + \frac{n-2}{2n}\langle\nabla R, \nabla f\rangle + 2f\mathring{Ric}(E, E). \tag{1.22}$$

On the other hand, it follows from (1.20) that

$$2\mathring{Ric}(\nabla f) = \frac{1}{f}\nabla|\nabla f|^2 - 4\langle\nabla f, E\rangle E^\flat + \frac{2R}{n(n-1)}\nabla f,$$

Plugging this into (1.22), one sees that

$$
\begin{aligned}
div\left[\frac{1}{f}\left(\nabla|\nabla f|^2 - 4f\langle\nabla f, E\rangle E^\flat + \frac{2Rf}{n(n-1)}\nabla f\right)\right]\\
= 2f|\mathring{Ric}|^2 + \frac{n-2}{n}\langle\nabla R, \nabla f\rangle + 4f\mathring{Ric}(E, E),
\end{aligned}
$$

as asserted. □

As a consequence of the previous result, we obtain the following integral identity.

Proposition 4. *Let (M^n, g, f, E) be a compact electrostatic system with boundary ∂M. Then we have:*

$$
\begin{aligned}
\int_M \left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right] dV_g &= \int_{\partial M} |\nabla f| R^{\partial M} dA_g \\
&+ \frac{1}{n}\int_M \left(4(n-1)f|E|^4 + (n-2)R\Delta f\right) dV_g. \quad (1.23)
\end{aligned}
$$

In particular, if $\Lambda \geq 0$, then

$$
\begin{aligned}
&\frac{4(n-1)}{n}\int_M f\left\{|E|^2 + \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right)} + R\right]\right\} \\
&\qquad\times\left\{\frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right)} - R\right] - |E|^2\right\} dV_g \qquad (1.24)\\
&\qquad\leq \int_{\partial M} |\nabla f| R^{\partial M}\, dA_g.
\end{aligned}
$$

Moreover, equality holds in (1.24) if and only if (M^n, g, f, E) is isometric to the de Sitter system.

Proof. On integrating (1.22) and using Stokes' theorem, we obtain

$$
\begin{aligned}
\int_M div\left(\mathring{Ric}(\nabla f)\right) dV_g &= \int_M \left[f|\mathring{Ric}|^2 + 2f\mathring{Ric}(E, E)\right] dV_g \\
&\quad + \frac{n-2}{2n}\int_M \langle \nabla R, \nabla f\rangle dV_g \\
&= \int_M \left[f|\mathring{Ric}|^2 + 2f\mathring{Ric}(E, E)\right] dV_g \\
&\quad + \frac{n-2}{2n}\left(\int_{\partial M} R\langle \nabla f, \nu\rangle dS_g - \int_M R\Delta f\, dV_g\right) \\
&= \int_M \left[f|\mathring{Ric}|^2 + 2f\mathring{Ric}(E, E)\right] dV_g \\
&\quad - \frac{n-2}{2n}\left(\int_{\partial M} R|\nabla f| dA_g + \int_M R\Delta f\, dV_g\right).
\end{aligned} \tag{1.25}
$$

On the other hand, it follows from (1.16) and Stokes' theorem that

$$
\begin{aligned}
\int_M div\left(\mathring{Ric}(\nabla f)\right) dV_g &= \int_{\partial M} \langle \mathring{Ric}(\nabla f), \nu\rangle dA_g \\
&= -\int_{\partial M} |\nabla f|\mathring{Ric}(\nu, \nu) dA_g \\
&= -\int_{\partial M} |\nabla f|\left(\frac{n-2}{2n}R - \frac{1}{2}R^{\partial M}\right) dA_g.
\end{aligned}
$$

This combined with (1.25) gives

$$
\begin{aligned}
\int_{\partial M} |\nabla f| R^{\partial M} dA_g &= 2\int_M \left(f|\mathring{Ric}|^2 + 2f\mathring{Ric}(E, E)\right) dV_g \\
&\quad - \frac{(n-2)}{n}\int_M R\Delta f\, dV_g.
\end{aligned} \tag{1.26}
$$

Next, since $\mathring{Ric} = Ric - \frac{R}{n}g$, one deduces from (1.20) that

$$
\nabla^2 f = f\left(\mathring{Ric} + 2E^\flat \otimes E^\flat - \frac{R}{n(n-1)}g\right),
$$

and consequently,

$$
|\nabla^2 f|^2 = f^2|\mathring{Ric}|^2 + 4f^2\mathring{Ric}(E, E) + 4f^2|E|^4 - \frac{4f^2R|E|^2}{n(n-1)} + \frac{f^2R^2}{n(n-1)^2},
$$

which can be rewrite as

$$|\mathring{\nabla}^2 f|^2 + \frac{(\Delta f)^2}{n} = f^2|\mathring{Ric}|^2 + 4f^2\mathring{Ric}(E,\,E) + 4f^2|E|^4 - \frac{4f^2R|E|^2}{n(n-1)} + \frac{f^2R^2}{n(n-1)^2},$$

where we have used that $\mathring{\nabla}^2 f = \nabla^2 f - \frac{\Delta f}{n}g$. By using (1.21), we get

$$\begin{aligned}2\mathring{Ric}(E,\,E) &= \frac{1}{2f^2}|\mathring{\nabla}^2 f|^2 - \frac{1}{2}|\mathring{Ric}|^2 + \frac{1}{2n}\left(\frac{(n-2)}{(n-1)}R - 2\Lambda\right)^2 - 2|E|^4\\ &\quad+\frac{2R|E|^2}{n(n-1)} - \frac{R^2}{2n(n-1)^2}.\end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned}|\mathring{Ric}|^2 + 2\mathring{Ric}(E,\,E) &= \frac{1}{2f^2}|\mathring{\nabla}^2 f|^2 + \frac{1}{2}|\mathring{Ric}|^2\\ &\quad+\frac{(n-3)}{2n(n-1)}R^2 - \frac{2(n-2)}{n(n-1)}\Lambda R + \frac{2\Lambda^2}{n} - 2|E|^4\\ &\quad+\frac{2R|E|^2}{n(n-1)}.\end{aligned}$$

Now, we use (1.15) to infer

$$\begin{aligned}|\mathring{Ric}|^2 + 2\mathring{Ric}(E,\,E) &= \frac{1}{2f^2}|\mathring{\nabla}^2 f|^2 + \frac{1}{2}|\mathring{Ric}|^2\\ &\quad+\frac{(n-3)}{2n(n-1)}(2R|E|^2 + 2\Lambda R) - \frac{2(n-2)}{n(n-1)}\Lambda R + \frac{1}{n}(\Lambda R - 2\Lambda|E|^2)\\ &\quad-2|E|^4 + \frac{2R|E|^2}{n(n-1)}\\ &= \frac{1}{2f^2}|\mathring{\nabla}^2 f|^2 + \frac{1}{2}|\mathring{Ric}|^2 - \frac{2(n-1)}{n}|E|^4,\end{aligned}$$

so that

$$2(f|\mathring{Ric}|^2 + 2f\mathring{Ric}(E,\,E)) = \frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2 - \frac{4(n-1)}{n}f|E|^4.$$

Besides, upon integrating the above expression, we use (1.26) to deduce

$$\begin{aligned}&\int_M \left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{R}ic|^2\right] dV_g\\ = &\int_{\partial M} |\nabla f| R^{\partial M} dA_g + \frac{1}{n}\int_M \left[4(n-1)f|E|^4 + (n-2)R\Delta f\right] dV_g,\end{aligned}$$

which proves (1.23).

We now address the inequality (1.24). By Definition 1, one sees that

$$4(n-1)f|E|^4 + (n-2)R\Delta f = 4(n-1)f|E|^4 + \frac{2(n-2)R}{n-1}\left[(n-2)|E|^2 - \Lambda\right] f.$$

In particular, we consider

$$F(|E|^2) := 4(n-1)|E|^4 + \frac{2(n-2)^2 R}{(n-1)}|E|^2 - \frac{2(n-2)R}{(n-1)}\Lambda,$$

where F is a quadratic function of $|E|^2$. Computing its discriminant, one deduces that

$$\triangle = \frac{4(n-2)^4R^2}{(n-1)^2} + 32(n-2)R\Lambda = \frac{4(n-2)^4R}{(n-1)^2}\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right),$$

which is nonnegative because $\Lambda \geq 0$. Consequently, $F(|E|^2) = 0$ if and only if

$$\begin{aligned}|E|^2 &= -\frac{(n-2)^2R}{4(n-1)^2} \pm \frac{(n-2)^2}{4(n-1)^2}\sqrt{R\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}\\ &= \frac{(n-2)^2}{4(n-1)^2}\left[\pm\sqrt{R\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right)} - R\right].\end{aligned}$$

From this, it follows that

$$\begin{aligned}F(|E|^2) &= 4(n-1)\left\{|E|^2 + \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right)} + R\right]\right\}\\ &\times\left\{|E|^2 - \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R + \frac{8(n-1)^2}{(n-2)^3}\Lambda\right)} - R\right]\right\}.\end{aligned}$$

Plugging this into (1.23) yields

$$
\begin{aligned}
&\int_M \left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right] dV_g \\
= &\int_{\partial M} |\nabla f| R^{\partial M} dA_g \\
&+\frac{4(n-1)}{n}\int_M f\left\{|E|^2 + \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}+R\right]\right\} \\
&\times\left\{|E|^2 - \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-R\right]\right\} dV_g,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\frac{4(n-1)}{n}\int_M f\left\{|E|^2 + \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}+R\right]\right\} \\
&\times\left\{\frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-R\right]-|E|^2\right\} dV_g \leq \int_{\partial M} |\nabla f| R^{\partial M} dA_g
\end{aligned}
\tag{1.27}
$$

as asserted.

Finally, equality holds in (1.27) if and only if $\mathring{\nabla}^2 f = 0$. Hence, the rigidity conclusion follows from Reilly's generalization of Obata's theorem to compact manifolds with boundary (see [58, Lemma 3]).

□

The next lemma establishes a key equivalence within our framework by using the condition that the gradient of the potential ∇f is parallel to the electric field E, a condition that holds in all known examples (see Section 1.5). In particular, it will be used to establish a divergence-free tensor, which provides a suitable setting for applying integral identities such as in Theorem 10.

Remark 6. *As previously observed, even in the three-dimensional case this condition that the gradient of the potential ∇f is parallel to the electric field E appears to be necessary. Indeed, as in the proof of [26, Proposition 21], by setting*

$$
T = Ric - \frac{R}{2}g + 2E^\flat \otimes E^\flat - |E|^2 g, \tag{1.28}
$$

one obtains that

$$\begin{aligned} div\, T &= g^{jk}\nabla_k\left(R_{ij} - \frac{R}{2}g_{ij} + 2E_iE_j - |E|^2 g_{ij}\right) \\ &= g^{jk}\nabla_k R_{ij} - \frac{1}{2}\nabla_i R + 2g^{jk}(\nabla_k E_i)E_j + 2g^{jk}(\nabla_k E_j)E_i - \nabla_i|E|^2 \\ &= 2(\nabla_j E_i)E_j - 2(\nabla_i E_j)E_j \\ &= 2(\nabla_j E_i - \nabla_i E_j)E_j, \end{aligned}$$

where we have used the twice-contracted second Bianchi identity and the fact that $div(E) = 0$. In particular, $div\, T = 0$ if and only if $\nabla_j E_i = \nabla_i E_j$. This fact motivates our condition that the gradient of the potential ∇f is parallel to the electric field E in the next lemma.

Lemma 3. *Let (M^n, g, f, E), $n \geq 3$, be an electrostatic system. Then E is parallel to ∇f if and only if $div\, T = 0$.*

Proof. By using (1.20), in tensorial notation[1], we notice that

$$\begin{aligned} \nabla_i\nabla_i\nabla_j f &= \nabla_i f R_{ij} + 2\nabla_i f E_i E_j - \frac{R}{(n-1)}\nabla_j f + f\nabla_i R_{ij} \\ &\quad +2fE_i\nabla_i E_j - \frac{f}{(n-1)}\nabla_j R, \end{aligned}$$

where we used that $div(E) = 0$. Moreover, by the twice contracted second Bianchi identity, we get

$$\begin{aligned} \nabla_i\nabla_i\nabla_j f &= \nabla_i f R_{ij} + 2\nabla_i f E_i E_j - \frac{R}{(n-1)}\nabla_j f + \frac{1}{2}f\nabla_j R \\ &\quad +2fE_i\nabla_i E_j - \frac{f}{(n-1)}\nabla_j R \\ &= \nabla_i f R_{ij} + 2\langle\nabla f, E\rangle E_j - \frac{R}{(n-1)}\nabla_j f + \frac{(n-3)}{2(n-1)}f\nabla_j R \\ &\quad +2fE_i\nabla_i E_j. \end{aligned} \tag{1.29}$$

On the other hand, the Ricci identity:

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = R_{ijkl}\nabla_l f$$

[1]The Einstein convention of summing over the repeated indices is adopted.

yields

$$g^{ik}\nabla_i\nabla_j\nabla_k f - g^{ik}\nabla_j\nabla_i\nabla_k f = g^{ik}R_{ijkl}\nabla_l f,$$

so that

$$\nabla_k\nabla_k\nabla_j f - \nabla_j\Delta f = R_{jl}\nabla_l f.$$

Rearranging the indices and using (1.29), one obtains that

$$\nabla_j\Delta f = 2\langle\nabla f, E\rangle E_j - \frac{R}{(n-1)}\nabla_j f + \frac{(n-3)}{2(n-1)}f\nabla_j R + 2fE_i\nabla_i E_j. \tag{1.30}$$

Now, substituting (1.21) into (1.30) gives

$$\begin{aligned}\nabla_j\left[\left(\frac{(n-2)}{(n-1)}R - 2\Lambda\right)f\right] \quad = \quad & 2\langle\nabla f, E\rangle E_j - \frac{R\nabla_j f}{(n-1)} + \frac{(n-3)}{2(n-1)}f\nabla_j R \\ & + 2fE_i\nabla_i E_j,\end{aligned}$$

which can be written as

$$\frac{1}{2}f\nabla_j R + (R - 2\Lambda)\nabla_j f = 2\langle\nabla f, E\rangle E_j + 2fE_i\nabla_i E_j,$$

From this, it follows that

$$\nabla_j\left[(R-2\Lambda)f^2\right] = 4f\langle\nabla f, E\rangle E_j + 4f^2E_i\nabla_i E_j = 4f\,div(fE^\flat\otimes E^\flat).$$

Now, we use (1.15) to infer

$$\nabla_j(f^2|E|^2) = \nabla_i(f^2|E|^2g_{ij}) = 2fg^{ik}\nabla_k(fE_iE_j).$$

In particular, we have

$$\nabla_i(f^2|E|^2g_{ij}) = \nabla_i f(f|E|^2g_{ij}) + f\nabla_i(f|E|^2g_{ij}) = fg^{ik}\nabla_k(2fE_iE_j).$$

Since $f > 0$ in the interior of M^n, one deduces that

$$|E|^2\nabla_j f = \nabla_i(2fE_iE_j - f|E|^2g_{ij}).$$

Therefore,

$$|E|^2\nabla_j f = (2E_iE_j - |E|^2 g_{ij})\nabla_i f + f\nabla_i(2E_iE_j - |E|^2 g_{ij}),$$

which can be reformulated as

$$0 = 2(E_iE_j - |E|^2 g_{ij})\nabla_i f + f\nabla_i(2E_iE_j - |E|^2 g_{ij}). \tag{1.31}$$

If E is parallel to ∇f, there exists a function h on M^n such that $E = h\nabla f$. Thus, one easily verifies that

$$\begin{aligned}(E_iE_j - |E|^2 g_{ij})\nabla_i f &= \langle E, \nabla f\rangle E_j - |E|^2\nabla_j f \\ &= h^2\langle\nabla f, \nabla f\rangle\nabla_j f - h^2|\nabla f|^2\nabla_j f = 0.\end{aligned}$$

Then, it follows from (1.31) that

$$\nabla_i(2E_iE_j - |E|^2 g_{ij}) = div\left(2E^\flat \otimes E^\flat - |E|^2 g\right) = div\, T = 0.$$

Conversely, if $\nabla_i(2E_iE_j - |E|^2 g_{ij}) = 0$, one obtains from (1.31) that

$$0 = 2(E_iE_j - |E|^2 g_{ij})\nabla_i f,$$

and hence,

$$0 = 2(E_iE_j - |E|^2 g_{ij})\nabla_i f\nabla_j f,$$

which is equivalent to

$$|E|^2|\nabla f|^2 = \langle E, \nabla f\rangle^2.$$

Then, E is parallel to ∇f. So, the proof is completed. □

Remark 7. *An advantage of the proof of Lemma 3 is that it does not rely on the condition $d(fE^\flat) = 0$. In other words, the result holds in a more general setting.*

Next, we establish the following proposition, which holds in a general context, allowing M^n to be non-compact or without boundary.

Proposition 5. *Let (M^n, g, f, E) be an electrostatic system such that E is parallel to ∇f. If $\nabla^2 f = \dfrac{\Delta f}{n} g$, then $|E|$ and the scalar curvature R are constants.*

Proof. Initially, by using $\nabla^2 f = \dfrac{\Delta f}{n} g$ into the first equation of Definition 1, we obtain

$$\frac{\Delta f}{n} g = f\left(Ric - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g\right),$$

so that

$$\frac{2}{n(n-1)}\left[(n-2)|E|^2 - \Lambda\right] g = (Ric - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g),$$

which can be rewrite as

$$Ric - \frac{2\Lambda}{n} g = 2\left(\frac{2}{n}|E|^2 g - E^\flat \otimes E^\flat\right).$$

So, one easily verifies from (1.15) that

$$Ric - \frac{R}{n} g = \frac{2}{n}|E|^2 g - 2E^\flat \otimes E^\flat. \tag{1.32}$$

In particular, (M, g) is Einstein if and only if

$$E^\flat \otimes E^\flat = \frac{|E|^2}{n} g.$$

Next, by taking the divergence of (1.32) and using Lemma 3, we arrive at

$$div\left(Ric - \frac{R}{n} g\right) = 2\left(\frac{2-n}{2n}\right)\nabla|E|^2.$$

By applying the twice-contracted second Bianchi identity together with the Equation 1.15, we obtain

$$\frac{2(n-2)}{n}\nabla|E|^2 = 0,$$

which implies that $|E|^2$ is constant. Consequently, the scalar curvature must also be constant.

□

To conclude this section, we recall two important integral identities that will be useful in our context. The first one is a very important generalization of the classical Reilly formula [58, Theorem 1], recently established by Qiu and Xia [57] (see also [45]).

Proposition 6 (Generalized Reilly Identity [57]). *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. Given two functions f and u on M^n and a constant k, we have*

$$\begin{aligned}\int_M f\left[(\Delta u + knu)^2 - |\nabla^2 u + kug|^2\right] dV_g &= (n-1)k \int_M \left(\Delta f + nkf\right) u^2 dV_g \\ &+ \int_M \left(\nabla^2 f - (\Delta f)g - 2(n-1)kfg + fRic\right)(\nabla u, \nabla u) dV_g \\ &+ \int_{\partial M} f\left[2\Delta_{\partial M} u \frac{\partial u}{\partial \nu} + H\left(\frac{\partial u}{\partial \nu}\right)^2 + A(\nabla_{\partial M} u, \nabla_{\partial M} u) + 2(n-1)ku\frac{\partial u}{\partial \nu}\right] dA_g \\ &+ \int_{\partial M} \frac{\partial f}{\partial \nu}\left[|\nabla_{\partial M} u|^2 - (n-1)ku^2\right] dA_g,\end{aligned}$$

where H and A stand for the mean curvature and second fundamental form of ∂M, respectively.

The second is a Pohozaev-type identity, originally observed by Schoen in [62] in the context of the Einstein tensor, and subsequently extended to its present form for $(0,2)$-tensors B that are locally conserved, that is, $div\, B = 0$ (divergence-free).

Theorem 10 (Generalized Pohozaev-Schoen Identity [34]). *Let (M^n, g) be a compact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field. If B is a divergence-free symmetric $(0,2)$-tensor, then*

$$\int_M X(\operatorname{tr} B)\, dV_g = \frac{n}{2}\int_M \langle \mathring{B}, \mathcal{L}_X g\rangle\, dV_g - n\int_{\partial M} \mathring{B}(X,\nu)\, dA_g, \tag{1.33}$$

where $\operatorname{tr} B$ denotes the trace of B with respect to g, $\mathring{B} = B - \frac{\operatorname{tr} B}{n} g$ is the traceless part of B, $\mathcal{L}_X g$ is the Lie derivative of g in the direction of X, and ν is the outward unit normal vector field along ∂M.

1.4 Examples

1.4.1 Reissner-Nordström-de Sitter space

In this section we will explore the Reissner-Nordström-de Sitter (RNdS) space as an example of elec electrostatic space. Consider $(M^n, \widetilde{g}, f, E)$, where

$$M = I \times \mathbb{S}^{n-1}$$

for some $I \subset \mathbb{R}$ which depends on the roots of the static potential $f(r)$. With the metric given by

$$\widetilde{g} = f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}},$$

Also, the charge Q of a hypersurface Σ of $(M^n,\, g)$ is given by

$$Q(\Sigma) = \frac{1}{\omega_{n-1}}\sqrt{\frac{2}{(n-1)(n-2)}}\int_{\Sigma}\langle E,\, \nu\rangle dA_g, \tag{1.34}$$

where ω_{n-1} stand for the area of the standard unit $(n-1)$-sphere and ν the normal vector of Σ.

Here we want to find the potential functions f that are indeed a solution for the electrostatic system. At First, we will calculate the Christoffel symbols, followed by the Riemann and Ricci curvatures.

$$\begin{aligned}
\widetilde{\Gamma}^0_{ij} &= \frac{1}{2}\sum_k\left\{\frac{\partial}{\partial x_i}\widetilde{g}_{jk} + \frac{\partial}{\partial x_j}\widetilde{g}_{ki} - \frac{\partial}{\partial x_k}\widetilde{g}_{ij}\right\}\widetilde{g}^{k0} = \frac{1}{2}\left\{\frac{\partial}{\partial x_i}\widetilde{g}_{j0} + \frac{\partial}{\partial x_j}\widetilde{g}_{0i} - \frac{\partial}{\partial r}\widetilde{g}_{ij}\right\}\widetilde{g}^{00}\\
&= -f^2 r g_{ij},\\
\widetilde{\Gamma}^m_{0j} &= \frac{1}{2}\sum_k\left\{\frac{\partial}{\partial r}\widetilde{g}_{jk} + \frac{\partial}{\partial x_j}\widetilde{g}_{k0} - \frac{\partial}{\partial x_k}\widetilde{g}_{0j}\right\}\widetilde{g}^{km} = \frac{1}{2}\sum_{k\neq 0}\frac{\partial}{\partial r}\widetilde{g}_{jk}\widetilde{g}^{km} = r^{-1}\delta^m_j,\\
\widetilde{\Gamma}^0_{0j} &= \frac{1}{2}\sum_k\left\{\frac{\partial}{\partial r}\widetilde{g}_{jk} + \frac{\partial}{\partial x_j}\widetilde{g}_{k0} - \frac{\partial}{\partial x_k}\widetilde{g}_{0j}\right\}\widetilde{g}^{k0} = \frac{1}{2}\left\{\frac{\partial}{\partial r}\widetilde{g}_{j0} + \frac{\partial}{\partial x_j}\widetilde{g}_{00} - \frac{\partial}{\partial r}\widetilde{g}_{0j}\right\}\widetilde{g}^{00} = 0,\\
\widetilde{\Gamma}^m_{00} &= \frac{1}{2}\sum_k\left\{\frac{\partial}{\partial r}\widetilde{g}_{0k} + \frac{\partial}{\partial r}\widetilde{g}_{k0} - \frac{\partial}{\partial x_k}\widetilde{g}_{00}\right\}\widetilde{g}^{km} = 0,\\
\widetilde{\Gamma}^0_{00} &= \frac{1}{2}\sum_k\left\{\frac{\partial}{\partial r}\widetilde{g}_{0k} + \frac{\partial}{\partial r}\widetilde{g}_{k0} - \frac{\partial}{\partial x_k}\widetilde{g}_{00}\right\}\widetilde{g}^{k0} = \frac{1}{2}\left\{\frac{\partial}{\partial r}\widetilde{g}_{00} + \frac{\partial}{\partial r}\widetilde{g}_{00} - \frac{\partial}{\partial r}\widetilde{g}_{00}\right\}\widetilde{g}^{00} = -\frac{f'}{f}.
\end{aligned}$$

$$
\begin{aligned}
\widetilde{R}^s_{ijk} &= \widetilde{\Gamma}^0_{ik}\widetilde{\Gamma}^s_{j0} - \widetilde{\Gamma}^0_{jk}\widetilde{\Gamma}^s_{i0} + \sum_{l\neq 0}\widetilde{\Gamma}^l_{ik}\widetilde{\Gamma}^s_{jl} - \sum_{l\neq 0}\widetilde{\Gamma}^l_{jk}\widetilde{\Gamma}^s_{il} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^s_{ik} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}^s_{jk} \\
&= -f^2(\widetilde{g}_{ik}\delta^s_j - \widetilde{g}_{jk}\delta^s_i) + R^s_{ijk} = (1-f^2)(\widetilde{g}_{ik}\delta^s_j - \widetilde{g}_{jk}\delta^s_i), \\
\widetilde{R}^s_{0jk} &= \sum_l \widetilde{\Gamma}^l_{0k}\widetilde{\Gamma}^s_{jl} - \sum_l \widetilde{\Gamma}^l_{jk}\widetilde{\Gamma}^s_{0l} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^s_{0k} - \frac{\partial}{\partial r}\widetilde{\Gamma}^s_{jk} = 0, \\
\widetilde{R}^0_{i0k} &= \sum_l \widetilde{\Gamma}^l_{ik}\widetilde{\Gamma}^0_{0l} - \sum_l \widetilde{\Gamma}^l_{0k}\widetilde{\Gamma}^0_{il} + \frac{\partial}{\partial r}\widetilde{\Gamma}^0_{ik} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}^0_{0k} = \widetilde{\Gamma}^0_{ik}\widetilde{\Gamma}^0_{00} - \sum_{l\neq 0}\widetilde{\Gamma}^l_{0k}\widetilde{\Gamma}^0_{il} - \frac{\partial}{\partial r}f^2 r \\
&= \left(f' f r + f^2 - 2 f' f r - f^2\right)\widetilde{g}_{ik} = -f' f r \widetilde{g}_{ik}, \\
\widetilde{R}^s_{ij0} &= \sum_l \widetilde{\Gamma}^l_{i0}\widetilde{\Gamma}^s_{jl} - \sum_l \widetilde{\Gamma}^l_{j0}\widetilde{\Gamma}^s_{il} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^s_{i0} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}^s_{j0} = 0, \\
\widetilde{R}^0_{00k} &= \sum_l \widetilde{\Gamma}^l_{0k}\widetilde{\Gamma}^0_{0l} - \sum_l \widetilde{\Gamma}^l_{0k}\widetilde{\Gamma}^0_{0l} + \frac{\partial}{\partial r}\widetilde{\Gamma}^0_{0k} - \frac{\partial}{\partial r}\widetilde{\Gamma}^0_{0k} = 0, \\
\widetilde{R}^s_{0j0} &= \sum_l \widetilde{\Gamma}^l_{00}\widetilde{\Gamma}^s_{jl} - \sum_l \widetilde{\Gamma}^l_{j0}\widetilde{\Gamma}^s_{0l} + \frac{\partial}{\partial x_j}\widetilde{\Gamma}^s_{00} - \frac{\partial}{\partial r}\widetilde{\Gamma}^s_{j0} = \widetilde{\Gamma}^0_{00}\widetilde{\Gamma}^s_{j0} - \sum_{l\neq 0}\widetilde{\Gamma}^l_{j0}\widetilde{\Gamma}^s_{0l} - \frac{\partial}{\partial r}r^{-1} \\
&= -\frac{f'}{fr}\delta^s_j - r^{-2}\delta^s_j + r^{-2}\delta^s_j = -\frac{f'}{fr}\delta^s_j, \\
\widetilde{R}^0_{i00} &= \sum_l \widetilde{\Gamma}^l_{i0}\widetilde{\Gamma}^0_{0l} - \sum_l \widetilde{\Gamma}^l_{00}\widetilde{\Gamma}^0_{il} + \frac{\partial}{\partial r}\widetilde{\Gamma}^0_{i0} - \frac{\partial}{\partial x_i}\widetilde{\Gamma}^0_{00} = 0, \\
\widetilde{R}^0_{000} &= \sum_l \widetilde{\Gamma}^l_{00}\widetilde{\Gamma}^0_{0l} - \sum_l \widetilde{\Gamma}^l_{00}\widetilde{\Gamma}^0_{0l} + \frac{\partial}{\partial r}\widetilde{\Gamma}^0_{00} - \frac{\partial}{\partial r}\widetilde{\Gamma}^0_{00} = 0.
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}_{ik} &= \widetilde{R}^0_{i0k} + \sum_{j\neq 0}\widetilde{R}^j_{ijk} = -f' f r\widetilde{g}_{ik} + \sum_{j\neq 0} R^j_{ijk} = -f' f r \widetilde{g}_{ik} - f^2(\widetilde{g}_{ik}\delta^j_j - \widetilde{g}_{jk}\delta^j_i) + \widetilde{R}^j_{ijk} \\
&= -f' f r\widetilde{g}_{ik} - f^2(\widetilde{g}_{ik}(n-1) - \widetilde{g}_{ik}) + (n-2)\widetilde{g}_{ik} = \left(-f' f r + (1-f^2)(n-2)\right)\widetilde{g}_{ik}, \\
\widetilde{R}_{0k} &= \widetilde{R}^0_{00k} + \sum_{j\neq 0}\widetilde{R}^j_{0jk} = 0, \\
\widetilde{R}_{i0} &= \widetilde{R}^0_{i00} + \sum_{j\neq 0}\widetilde{R}^j_{ij0} = 0, \\
\widetilde{R}_{00} &= \widetilde{R}^0_{000} + \sum_{j\neq 0}\widetilde{R}^j_{0j0} = -(n-1)\frac{f'}{rf}.
\end{aligned} \tag{1.35}
$$

Now, we evaluate the Hessian and the Laplacian of f by using the Lie derivative, Hence.

$$
\begin{aligned}
2\nabla^2 f &= \mathcal{L}_{\nabla f} g_{RNdS} = \mathcal{L}_{\nabla f}(f^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}}) = \mathcal{L}_{\nabla f} f^{-2}dr^2 + \mathcal{L}_{\nabla f} r^2 g_{\mathbb{S}^{n-1}} \\
&= \mathcal{L}_{\nabla f} f^{-2}dr^2 + f^{-2}\left(\mathcal{L}_{\nabla f}(dr)\otimes dr\right) + f^{-2}\left(dr \otimes \mathcal{L}_{\nabla f}(dr)\right) + \mathcal{L}_{\nabla f} r^2 g_{\mathbb{S}^{n-1}}.
\end{aligned}
$$

Since $\nabla f = f' g^{rr}\partial_r$ and by the definition of Lie derivative,

$$
\mathcal{L}_{\nabla f}(dr) = d(i_{\nabla f}dr) + i_{\nabla f}(ddr) = d(dr(\nabla f)) = d(f'g^{rr}) = f''f^2 + 2(f')^2 f\, dr
$$

and then

$$
\begin{aligned}
2\nabla^2 f &= f'g^{rr}\partial_r f^{-2}dr^2 + 2f^{-2}f''f^2dr^2 + 4f^{-2}(f')^2 f\,dr^2 + f'g^{rr}\partial_r r^2 g_{\mathbb{S}^{n-1}}; \\
&= -2f'f^2f^{-3}f'dr^2 + 2f''dr^2 + 4(f')^2f^{-1}dr^2 + 2f'f^2 r g_{\mathbb{S}^{n-1}}; \\
\nabla^2 f &= f''dr^2 + (f')^2 f^{-1}dr^2 + f'f^2 r g_{\mathbb{S}^{n-1}}. \qquad (1.36)
\end{aligned}
$$

Now, taking the trace

$$
\Delta f = g^{rr}\left(f'' + (f')^2 f^{-1}\right)dr^2 + g^{ij} f'f^2 r g_{\mathbb{S}^{n-1}} = f''f^2 + (f')^2 f + (n-1)f'f^2 r^{-1}.
$$

Using the second equation of the electrostatic system

$$
\begin{aligned}
\Delta f &= 2\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right)f; \\
ff'' + (f')^2 + (n-1)r^{-1}ff' &= 2\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right); \\
ff'' + (f')^2 + (n-1)r^{-1}ff' - 2\frac{n-2}{n-1}|E|^2 &= -\frac{2}{n-1}\Lambda.
\end{aligned}
$$

Now, substituting the Hessian (1.36) into the first equation of the electrostatic system

$$f''dr^2 + (f')^2 f^{-1} dr^2 + f' f^2 r g_{\mathbb{S}^{n-1}} = f\Bigg(\mathrm{Ric} - \frac{2}{n-1}\Lambda\left(f(r)^{-2}dr^2 + r^2 g_\Omega\right)$$
$$+ 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2\left(f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}}\right)\Bigg);$$

$$f'' f^{-1} dr^2 + (f')^2 f^{-2} dr^2 + f' f r g_{\mathbb{S}^{n-1}} = \mathrm{Ric}$$
$$+ \left(f f'' + (f')^2 + (n-1) r^{-1} f f' - 2\frac{n-2}{n-1}|E|^2\right)\left(f(r)^{-2}dr^2 + r^2 g_\Omega\right)$$
$$+ 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2\left(f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}}\right);$$

$$f'' f^{-1} dr^2 + (f')^2 f^{-2} dr^2 + f' f r g_{\mathbb{S}^{n-1}} = \mathrm{Ric}$$
$$+ \left(f f'' + (f')^2 + (n-1) r^{-1} f f'\right)\left(f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}}\right)$$
$$+ 2E^\flat \otimes E^\flat - 2|E|^2\left(f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}}\right);$$

$$-(n-1) r^{-1} f' f^{-1} dr^2 - f f'' r^2 g_{\mathbb{S}^{n-1}} - (f')^2 r^2 g_{\mathbb{S}^{n-1}} + (n-2) f' f r g_{\mathbb{S}^{n-1}}$$
$$= \mathrm{Ric} + 2E^\flat \otimes E^\flat - 2|E|^2\left(f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^{n-1}}\right).$$

Here, we applying (∂_r, ∂_r) in the above equation, and from (1.35)
$\mathrm{Ric}(\partial_r, \partial_r) = -(n-1) r^{-1} f' f^{-1} dr^2$,

$$0 = 2\widetilde{g}(E, \partial_r)^2 - 2|E|^2 f(r)^{-2};$$

$$\widetilde{g}(E, \partial_r)^2 = |E|^2 f(r)^{-2}. \tag{1.37}$$

Once, by (1.7), $E = \left\langle E, f\partial_r\right\rangle_r \partial_r + g_{\mathbb{S}^{n-1}}(E, r\partial_i)\partial_i$ and $|E|^2 = \left\langle E, f\partial_r\right\rangle_r^2 + g_{\mathbb{S}^{n-1}}(E, r\partial_i)^2$. The Equation 1.37 gives us that E is parallel to ∂_r. Thus, since the electric field has only radial coordinates, by (1.34), we get

$$Q(\mathbb{S}^{n-1}) = \frac{1}{\omega_{n-1}}\sqrt{\frac{2}{(n-1)(n-2)}}\int_{\mathbb{S}^{n-1}} \langle E, \nu\rangle\, dA_g = \frac{1}{\omega_{n-1}}\sqrt{\frac{2}{(n-1)(n-2)}}\,\omega_{n-1} r^{n-1}|E|,$$

$$E = \sqrt{\frac{(n-1)(n-2)}{2}}\,\frac{Q}{r^{n-1}} f\partial_r,$$

and so

$$|E|^2 = \frac{(n-1)(n-2)}{2}\frac{Q^2}{r^{2(n-1)}}.$$

Thus, we use the second equation of the electrostatic system and solve the ODE with initial values: $f(\rho) = f'(\rho) = 0$ such that

$$\mathfrak{M} = \rho^{n-2}\left(1 - \frac{2\Lambda\rho^2}{n(n-2)}\right) \tag{1.38}$$

and

$$Q^2 = \rho^{2(n-2)}\left(1 - \frac{2\Lambda\rho^2}{(n-1)(n-2)}\right). \tag{1.39}$$

Then,

$$\begin{aligned}
\Delta f &= 2\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right)f;\\
ff'' + f'f' + (n-1)r^{-1}ff' &= (n-2)^2\frac{Q^2}{r^{2(n-1)}} - \frac{2\Lambda}{n-1};\\
\left(ff'\right)' + (n-1)r^{-1}ff' &= (n-2)^2\frac{Q^2}{r^{2(n-1)}} - \frac{2\Lambda}{n-1};\\
r^{n-1}\left(ff'\right)' + (n-1)r^{n-2}ff' &= (n-2)^2\frac{Q^2}{r^{n-1}} - \frac{2\Lambda r^{n-1}}{n-1};\\
r^{n-1}ff' &= C - (n-2)\frac{Q^2}{r^{n-2}} - \frac{2\Lambda r^n}{n(n-1)};\\
ff' &= \frac{C}{r^{n-1}} - (n-2)\frac{Q^2}{r^{2(n-2)+1}} - \frac{2\Lambda r}{n(n-1)};\\
\frac{f^2}{2} &= D - \frac{C}{(n-2)r^{n-2}} + \frac{Q^2}{2r^{2(n-2)}} - \frac{\Lambda r^2}{n(n-1)};
\end{aligned} \tag{1.40}$$

i.e.,

$$f^2 = 2D - \frac{2C}{(n-2)r^{n-2}} + \frac{Q^2}{r^{2(n-2)}} - \frac{2\Lambda r^2}{n(n-1)}. \tag{1.41}$$

Making $r = \rho$ in (1.40) and using the Equation 1.39,

$$
\begin{aligned}
0 &= \frac{C}{\rho^{n-1}} - (n-2)\frac{Q^2}{\rho^{2(n-2)+1}} - \frac{2\Lambda\rho}{n(n-1)};\\
0 &= \frac{C}{\rho^{n-1}} - (n-2)\frac{1-\dfrac{2\Lambda\rho^2}{(n-1)(n-2)}}{\rho} - \frac{2\Lambda\rho}{n(n-1)};\\
0 &= \frac{C}{\rho^{n-2}} - (n-2) + \frac{2\Lambda\rho^2}{(n-1)} - \frac{2\Lambda\rho^2}{n(n-1)};\\
0 &= \frac{C}{\rho^{n-2}} - (n-2) + \frac{2\Lambda\rho^2}{n};\\
\frac{C}{\rho^{n-2}} &= (n-2)\left(1 - \frac{2\Lambda\rho^2}{n(n-2)}\right);\\
\frac{C}{(n-2)} &= \rho^{n-2}\left(1 - \frac{2\Lambda\rho^2}{n(n-2)}\right) = \mathfrak{M}.
\end{aligned}
$$

Then, we have from the Equation 1.41,

$$
f^2 = 2D - \frac{2\mathfrak{M}}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}} - \frac{2\Lambda r^2}{n(n-1)},
$$

by making $r = \rho$ and using the equations (1.38) and (1.39), we obtain that

$$
\begin{aligned}
0 &= 2D - \frac{2\mathfrak{M}}{\rho^{n-2}} + \frac{Q^2}{\rho^{2(n-2)}} - \frac{2\Lambda\rho^2}{n(n-1)};\\
0 &= 2D - 2 + 2\frac{2\Lambda\rho^2}{n(n-2)} + 1 - \frac{2\Lambda\rho^2}{(n-1)(n-2)} - \frac{2\Lambda\rho^2}{n(n-1)};\\
-2D &= -1 + \frac{4(n-1)\Lambda\rho^2}{n(n-1)(n-2)} - \frac{2n\Lambda\rho^2}{n(n-1)(n-2)} - \frac{2(n-2)\Lambda\rho^2}{n(n-1)(n-2)};\\
2D &= 1.
\end{aligned}
$$

Therefore,

$$
f^2 = 1 - \frac{2\mathfrak{M}}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}} - \frac{2\Lambda r^2}{n(n-1)}.
$$

1.4.2 Charged Nariai space

In this section, we will explore more examples of electrostatic space (Definition 4). by making a small change in the metric of the space $(M^n, \widetilde{g}, f, E)$. So, let

$$M^n = I \times \mathbb{S}^{n-1}$$

for some $I \subset \mathbb{R}$ which depends on the roots of the static potential $f(s)$. With the metric metric given by

$$\widetilde{g} = ds^2 + \rho^2 g_{\mathbb{S}^{n-1}},$$

for ρ such that satisfies the equations (1.38) and (1.39).

There we want to find the potential functions f that are indeed a solution for the electrostatic system. At first, we evaluate the Hessian and the Laplacian of f by using the Lie derivative:

$$\begin{aligned} 2\nabla^2 f &= \mathcal{L}_{\nabla f} g = \mathcal{L}_{\nabla f}(ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}) = \mathcal{L}_{\nabla f} ds^2 + \mathcal{L}_{\nabla f} \rho^2 g_{\mathbb{S}^{n-1}}; \\ &= \left(\mathcal{L}_{\nabla f}(ds) \otimes ds\right) + \left(ds \otimes \mathcal{L}_{\nabla f}(ds)\right). \end{aligned}$$

Since $\nabla f = f' g^{ss} \partial_s$ and by the definition of Lie derivative

$$\mathcal{L}_{\nabla f}(ds) = d(i_{\nabla f} ds) + i_{\nabla f}(dds) = d(ds(\nabla f)) = d(f') = f'' ds.$$

So

$$\nabla^2 f = f'' ds^2.$$

Taking the trace

$$\Delta f = f''$$

and now we use the equations of the electrostatic system and solve the ODE with initial values $f(0) = 0$ and $f'(0) = \gamma$, where γ will be defined later. Then, from the second equation of the electrostatic space

$$\begin{aligned} \Delta f &= 2\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right) f; \\ f'' &= 2\frac{n-2}{n-1}|E|^2 f - \frac{2}{n-1}\Lambda f; \\ \frac{f''}{f} - \frac{2(n-2)}{n-1}|E|^2 &= -\frac{2}{n-1}\Lambda. \end{aligned} \tag{1.42}$$

Now, combining (1.42) and the first equation of the electrostatic space

$$
\begin{aligned}
\nabla^2 f &= f\left(\mathrm{Ric} - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g\right);\\
f'' ds^2 &= f\left[\mathrm{Ric} + \left(\frac{f''}{f} - \frac{2(n-2)}{n-1}|E|^2\right)\left(ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}\right);\right.\\
&\left. + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2\left(ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}\right)\right];\\
f'' ds^2 &= f\left[\mathrm{Ric} + \left(\frac{f''}{f} - 2|E|^2\right)\left(ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}\right) + 2E^\flat \otimes E^\flat\right].
\end{aligned}
$$

Here, we applying (∂_s, ∂_s) in the above equation, and observe that $\mathrm{Ric}(\partial_s, \partial_s) = 0$,

$$
\begin{aligned}
f'' &= f\left[\frac{f''}{f} - 2|E|^2 + 2g(E, \partial_s)^2\right];\\
0 &= -2|E|^2 + 2g(E, \partial_s)^2;\\
|E|^2 &= g(E, \partial_s)^2.
\end{aligned}
$$

Concluding that E is parallel to ∂_s. Then, assuming that the electrostatic field is constant and using (1.34)

$$
Q(\mathbb{S}^{n-1}) = \frac{1}{\omega_{n-1}}\sqrt{\frac{2}{(n-1)(n-2)}}\int_{\mathbb{S}^{n-1}} \langle E, \nu\rangle \, dA_g = \frac{1}{\omega_{n-1}}\sqrt{\frac{2}{(n-1)(n-2)}}\omega_{n-1}\rho^{n-1}|E|,
$$

$$
E = \sqrt{\frac{(n-1)(n-2)}{2}}\frac{Q}{\rho^{n-1}}\partial_s,
$$

and so,

$$
|E|^2 = \frac{(n-1)(n-2)}{2}\frac{Q^2}{\rho^{2(n-1)}}.
$$

Therefore, from the second equation of the electrostatic space,

$$
\begin{aligned}
\Delta f &= 2\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right) f;\\
f'' &= \left((n-2)^2\frac{Q^2}{\rho^{2(n-1)}} - \frac{2\Lambda}{n-1}\right) f. \qquad (1.43)
\end{aligned}
$$

From now, we will study tree cases, are they:

- Charged Nariai space, if $|Q| < \sqrt{\frac{2\Lambda}{n-1}}\frac{\rho^{n-1}}{n-2}$.
- Cold black hole space if $|Q| > \sqrt{\frac{2\Lambda}{n-1}}\frac{\rho^{n-1}}{n-2}$.
- Utracold black hole space, if $|Q| = \sqrt{\frac{2\Lambda}{n-1}}\frac{\rho^{n-1}}{n-2}$.

To solve de ODE (1.43), we first observe that the roots of the auxiliary equation

$$x^2 + \left(\frac{2\Lambda}{n-1} - (n-2)^2\frac{Q^2}{\rho^{2(n-1)}}\right) = 0$$

are

$$x = \pm\sqrt{(n-2)^2\frac{Q^2}{\rho^{2(n-1)}} - \frac{2\Lambda}{n-1}}.$$

Using the Equation 1.39,

$$\begin{aligned}
x &= \pm\sqrt{(n-2)^2\rho^{-2}\left(1 - \frac{2\Lambda\rho^2}{(n-1)(n-2)}\right) - \frac{2\Lambda}{n-1}};\\
x &= \pm\sqrt{(n-2)^2\rho^{-2} - \frac{2\Lambda n - 4\Lambda}{n-1} - \frac{2\Lambda}{n-1}};\\
x &= \pm\sqrt{(n-2)^2\rho^{-2} - 2\Lambda}.
\end{aligned}$$

Then, if $\rho^2 > \frac{(n-2)^2}{2\Lambda}$ we have the charged Nariai space and

$$\begin{aligned}
f(s) &= c_1\cos s\alpha + c_2\sin s\alpha,\\
f'(s) &= -\alpha c_1\sin s\alpha + \alpha c_2\cos s\alpha,
\end{aligned}$$

where

$$\alpha = \sqrt{\frac{2\Lambda}{n-1} - (n-2)^2\frac{Q^2}{\rho^{2(n-1)}}}.$$

Taking $s = 0$ and considering the initial value $\gamma = \alpha$,

$$f(0) = c_1 \cos 0\alpha + c_2 \sin 0\alpha;$$
$$0 = c_1$$

and

$$f'(0) = -\alpha c_1 \sin 0\alpha + \alpha c_2 \cos 0\alpha;$$
$$\alpha = \alpha c_2,$$

obtaining

$$f(s) = \sin \alpha s.$$

1.4.3 Cold black hole space

If $\rho^2 < \dfrac{(n-2)^2}{2\Lambda}$ we have the cold black hole space and

$$f(s) = c_1 e^{s\beta} + c_2 e^{-s\beta},$$
$$f'(s) = \beta c_1 e^{s\beta} - \beta c_2 e^{-s\beta},$$

for

$$\beta = \sqrt{(n-2)^2 \frac{Q^2}{\rho^{2(n-1)}} - \frac{2\Lambda}{n-1}}.$$

Taking $s = 0$ and considering the initial value $\gamma = \beta$,

$$f(0) = c_1 e^{0\beta} + c_2 e^{-0\beta}$$
$$0 = c_1 + c_2$$

and

$$f'(0) = \beta c_1 e^{0\beta} - \beta c_2 e^{0\beta}$$
$$\beta = \beta c_1 - \beta c_2$$
$$1 = 2c_1,$$

obtaining

$$f(s) = \frac{e^{s\beta} - e^{-s\beta}}{2} = \sinh \beta s.$$

1.4.4 Ultracold black hole space

If $\rho^2 = \dfrac{(n-2)^2}{2\Lambda}$ we have the ultracold black hole space and

$$\begin{aligned} f(s) &= c_1 + c_2 s, \\ f'(s) &= c_2. \end{aligned}$$

Taking $s = 0$ and considering the initial value $\gamma = 1$,

$$\begin{aligned} f(0) &= c_1 + c_2 0 \\ 0 &= c_1 \end{aligned}$$

and

$$\begin{aligned} f'(0) &= c_2 \\ 1 &= c_2, \end{aligned}$$

obtaining

$$f(s) = s.$$

Regarding the ultra-cold black hole, we can obtain additional information about the parameters $\mathfrak{M}, Q$ and E. Remembering that in the ultracold black hole $|Q| = \sqrt{\dfrac{2\Lambda}{n-1}}\dfrac{\rho^{n-1}}{n-2}$ and so,

$$\begin{aligned} |E| &= \sqrt{\frac{(n-1)(n-2)}{2}}\frac{|Q|}{\rho^{n-1}} \\ &= \sqrt{\frac{(n-1)(n-2)}{2}}\sqrt{\frac{2\Lambda}{n-1}}\frac{1}{n-2} \\ &= \sqrt{\frac{\Lambda}{n-2}}. \end{aligned}$$

Here remembering that in the ultracold black hole $\rho^2 = \dfrac{(n-2)^2}{2\Lambda}$ and rescaling the equations (1.38) and (1.39) we get that

$$
\begin{aligned}
\mathfrak{M} &= \rho^{n-2}\left(1-\frac{2\Lambda\rho^2}{n(n-2)}\right)\\
&= \left(\frac{n-2}{\sqrt{2\Lambda}}\right)^{n-2}\left(1-\frac{n-2}{n}\right)\\
&= \left(\frac{n-2}{\sqrt{2\Lambda}}\right)^{n-2}\frac{2}{n}
\end{aligned}
$$

and

$$
\begin{aligned}
Q^2 &= \rho^{2(n-2)}\left(1-\frac{2\Lambda\rho^2}{(n-1)(n-2)}\right)\\
&= \left(\frac{(n-2)^2}{2\Lambda}\right)^{n-2}\left(1-\frac{n-2}{n-1}\right)\\
&= \left(\frac{(n-2)^2}{2\Lambda}\right)^{n-2}\frac{1}{n-1}.
\end{aligned}
$$

1.4.5 de Sitter space

A very important and useful solution of the electrostatic space for this work is the de Sitter space. Consider $(M^n, \widetilde{g}, f, E)$, where

$$M^n = \mathbb{S}^n_+,$$

with the metric metric given by

$$\widetilde{g} = \frac{n(n-1)}{2\Lambda}\left(dr^2+\sin^2 r g_{\mathbb{S}^{n-1}}\right),$$

Here we want to find the potential functions f that are indeed a solution for the electrostatic system. At First, we evaluate the Hessian and the Laplacian of f by using the Lie derivative.

$$
\begin{aligned}
2\nabla^2 f &= \mathcal{L}_{\nabla f}\widetilde{g} = \frac{n(n-1)}{2\Lambda}\mathcal{L}_{\nabla f}\left(dr^2+\sin^2 r g_{\mathbb{S}^{n-1}}\right);\\
&= \frac{n(n-1)}{2\Lambda}\mathcal{L}_{\nabla f} dr^2 + \frac{n(n-1)}{2\Lambda}\mathcal{L}_{\nabla f}\sin^2 r g_{\mathbb{S}^{n-1}};\\
&= \frac{n(n-1)}{2\Lambda}\left[\left(\mathcal{L}_{\nabla f}(dr)\otimes dr\right)+\left(dr\otimes\mathcal{L}_{\nabla f}(dr)\right)\right]+\frac{n(n-1)}{2\Lambda}\mathcal{L}_{\nabla f}\sin^2 r g_{\mathbb{S}^{n-1}}.
\end{aligned}
$$

Since $\nabla f = f' g^{rr}\partial_r$ and by the definition of Lie derivative,

$$\mathcal{L}_{\nabla f}(dr) = d(i_{\nabla f}dr) + i_{\nabla f}(ddr) = d(dr(\nabla f)) = d(f'g^{rr}) = \frac{2\Lambda}{n(n-1)}f''dr,$$

then

$$\begin{aligned} 2\nabla^2 f &= 2f''dr^2 + \frac{n(n-1)}{\Lambda}f'g^{rr}\partial_r \sin^2 r g_{\mathbb{S}^{n-1}}; \\ &= 2f''dr^2 + 2f'\sin r\cos r g_{\mathbb{S}^{n-1}}. \end{aligned}$$

Now, taking the trace

$$\Delta f = g^{rr}\left(f''\right)dr^2 + g^{ij}f'\sin r\cos r g_{\mathbb{S}^{n-1}} = \frac{2\Lambda}{n(n-1)}\left[f'' + (n-1)f'\cot(r)\right].$$

Using the second equation of the electrostatic system and assuming that the electric field is null

$$\begin{aligned} \Delta f &= 2\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right)f; \\ \frac{2\Lambda}{n(n-1)}\left[f'' + (n-1)f'\cot r\right] &= -\frac{2}{n-1}\Lambda f; \\ f'' + (n-1)f'\cot r &= -nf; \\ f''\sin r + (n-1)f'\cos r + nf\sin r &= 0. \end{aligned} \tag{1.44}$$

Since the ODE depends only on trigonometric functions, let's assume the solution is of the type $f(r) = A\sin r + B\cos r$, then

$$f'(r) = A\cos r - B\sin r,$$

$$f''(r) = -A\sin r - B\cos r.$$

Replacing f and its derivatives in (1.44)

$$\begin{aligned} &-A\sin^2 r - B\sin r\cos r + (n-1)\left(A\cos^2 r - B\sin r\cos r\right) \\ &+ n\left(A\sin^2 r + B\sin r\cos r\right) = 0; \\ &-A\sin^2 r + (n-1)A\cos^2 r + nA\sin^2 r = 0; \\ &-A + A = 0. \end{aligned}$$

Concluding that $f(r) = A\sin r + B\cos r$ is a solution for any values of A and B but then, considering the initial value $f(0) = 1$. We finally get that

$$f(r) = \cos r.$$

1.5 Summary of examples

We now will present a bunched of the examples previously demonstrated of the electrostatic manifolds with boundary for arbitrary dimensions $n \geq 3$.

Example 1 (Vacuum static systems). *The vacuum static systems (those for which $E \equiv 0$) are trivial examples of electrostatic systems. Among them, the* **de Sitter system** *corresponds to the standard hemisphere $(\mathbb{S}^n_+, g)$ endowed with the metric $g = \frac{n(n-1)}{2\Lambda}\left(dr^2 + \sin^2 r\, g_{\mathbb{S}^{n-1}}\right)$ and the lapse function by $f(r) = \cos(r)$ with $r \leq \pi/2$ representing the height function.*

Another distinguished vacuum static example is the **Nariai system**, *given by $M^n = [0,\pi] \times \mathbb{S}^{n-1}$, with metric $g = ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}$ and lapse function $f(s) = \sin(s)$, which yields a compact, oriented electrostatic manifold whose boundary consists of two disconnected components.*

Example 2 (Reissner-Nordström-de Sitter (RNdS)). *The $(n+1)$-dimensional RNdS spacetime is a three-parameter family (characterized by the ADM mass $\mathfrak{M}$, the charge Q, and the cosmological constant Λ) of static, electrically charged solutions to the Einstein equations. Consider the system $(M^n, g_{\mathrm{RNdS}}, f, E)$, where*

$$M^n = I \times \mathbb{S}^{n-1},$$

for some interval $I \subset \mathbb{R}$ determined by the roots of the static potential

$$f(r)^2 = 1 - \frac{2\mathfrak{M}}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}} - \frac{2\Lambda r^2}{n(n-1)},$$

and

$$E = \sqrt{\frac{(n-1)(n-2)}{2}}\,\frac{Q}{r^{n-1}} f(r)\,\partial_r.$$

The metric is given by

$$g_{\mathrm{RNdS}} = f(r)^{-2}\,dr^2 + r^2 g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ denotes the standard metric on the unit sphere $\mathbb{S}^{n-1}$ and r represents the radial coordinate. When the cosmological constant vanishes identically, the space is referred to as the Reissner–Nordström (RN) space.

Example 3 (Cold Black Hole). *The Cold Black Hole system is defined on $[0, +\infty) \times \mathbb{S}^{n-1}$ endowed with the metric $g = ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}$ and lapse function $f(s) = \sinh(\tau s)$, where*

$$\tau = \sqrt{\left(\frac{(n-2)^2 Q^2}{\rho^{2(n-1)}} - \frac{2\Lambda}{n-1}\right)}.$$

The electric field is given by

$$E = \sqrt{\frac{(n-1)(n-2)}{2}}\frac{Q}{\rho^{n-1}}\partial_s.$$

Example 4 (Ultracold Black Hole). *The ultracold black hole system is defined on $[0, +\infty) \times \mathbb{S}^{n-1}$ with metric $g = ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}$ and lapse function $f(s) = s$. The electric field is given by*

$$E = \sqrt{\frac{\Lambda}{(n-2)}}\partial_s,$$

where

$$Q^2 = \frac{1}{n-1}\left(\frac{(n-2)^2}{2\Lambda}\right)^{n-2} = \frac{\rho^{2(n-2)}}{n-1}.$$

Example 5 (Charged Nariai). *Based on the Nariai system, one can construct the Charged Nariai system defined on $\left[0, \frac{\pi}{\gamma}\right] \times \mathbb{S}^{n-1}$ with metric $g = ds^2 + \rho^2 g_{\mathbb{S}^{n-1}}$ and lapse function $f(s) = \sin(\gamma s)$, where $\gamma = \sqrt{\frac{2\Lambda}{n-1} - \frac{(n-2)^2 Q^2}{\rho^{2(n-1)}}}$. Moreover, the electric field is given by*

$$E = \sqrt{\frac{(n-1)(n-2)}{2}}\frac{Q}{\rho^{n-1}}\partial_s.$$

Chapter 2

Geometric inequalities for Electrostatic Systems

2.1 Boundary area estimates and rigidity results

In this section, we establish results concerning boundary area estimates and their implications. We begin with a slightly more general result, from which Theorem 1 follows as a particular case, assuming additionally that the boundary is connected.

Theorem 11. *Let* (M^n, g, f, E) *be a compact electrostatic system with boundary* $\partial M = \cup_{i=1}^{l} \Sigma_i$*, where* Σ_i *are the connected components of* ∂M*. If* $|E|^2 < \dfrac{\Lambda}{(n-2)}$*, then*

$$\sqrt{\frac{n+2}{2n\alpha}} \left(\frac{(\sum_i \kappa_i |\Sigma_i|)^3}{\sum_i \kappa_i^3 |\Sigma_i|} \right)^{\frac{1}{2}} \leq Vol(M),$$

where $\alpha = \max\limits_{M} \dfrac{2}{n-1} \left[\Lambda - (n-2)|E|^2 \right]$ *and* $\kappa_i := |\nabla f|\Big|_{\Sigma_i}$ *are the surface gravities. Moreover, equality holds if* (M^n, g, f) *is isometric to the de Sitter system.*

Proof. By taking f as the potential of the electrostatic system in Proposition 6, and noting that $f = 0$ on ∂M, we deduce

$$\begin{aligned} &\int_M f \left[(\Delta u + knu)^2 - |\nabla^2 u + kug|^2 \right] dV_g = (n-1)k \int_M \left(\Delta f + nkf \right) u^2 dV_g \\ &+ \int_M \left(\nabla^2 f - (\Delta f) g - 2(n-1)kfg + fRic \right) (\nabla u, \nabla u) dV_g \\ &+ \int_{\partial M} \frac{\partial f}{\partial \nu} \left[|\nabla_{\partial M} u|^2 - (n-1)ku^2 \right] dA_g. \end{aligned}$$

One sees from Definition 1 that

$$\nabla^2 f(\nabla u, \nabla u) = f\left(Ric - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g\right)(\nabla u, \nabla u),$$

so that

$$\begin{aligned}
&\int_M f\left[(\Delta u + knu)^2 - |\nabla^2 u + kug|^2\right] dV_g = (n-1)k\int_M (\Delta f + nkf)\, u^2 dV_g \\
&+ \int_M \left(2fRic - 2\left(\frac{\Lambda}{n-1} + (n-1)k\right) fg + 2fE^\flat \otimes E^\flat - \frac{2}{n-1} f|E|^2 g - (\Delta f)g\right)(\nabla u, \nabla u) dV_g \\
&+ \int_{\partial M} \frac{\partial f}{\partial \nu}\left[|\nabla_{\partial M} u|^2 - (n-1)ku^2\right] dA_g.
\end{aligned}$$

Taking into account that

$$\Delta f = \frac{2}{n-1}\left((n-2)|E|^2 - \Lambda\right) f,$$

one obtains that

$$\begin{aligned}
&\int_M f\left[(\Delta u + knu)^2 - |\nabla^2 u + kug|^2\right] dV_g = (n-1)k\int_M (\Delta f + nkf)\, u^2 dV_g \\
&+ \int_M 2f\left[Ric(\nabla u, \nabla u) - (n-1)k|\nabla u|^2\right] dV_g + \int_M 2f\left(\langle E, \nabla u\rangle^2 - |E|^2|\nabla u|^2\right) dV_g \\
&+ \int_{\partial M} \frac{\partial f}{\partial \nu}\left[|\nabla_{\partial M} u|^2 - (n-1)ku^2\right] dA_g.
\end{aligned}$$

Now, we take $u = f$ and use the fact that $f = 0$ on ∂M to infer

$$\begin{aligned}
&\int_M f\left[\left(\Delta f + knf\right)^2 - |\nabla^2 f + kfg|^2\right] dV_g = (n-1)k\int_M (\Delta f + nkf)\, f^2 dV_g \\
&+ \int_M 2f\left[Ric(\nabla f, \nabla f) - (n-1)k|\nabla f|^2\right] dV_g + \int_M 2f\left(\langle E, \nabla f\rangle^2 - |E|^2|\nabla f|^2\right) dV_g.
\end{aligned}$$

We then use the classical Bochner's formula:

$$\Delta|\nabla f|^2 = 2Ric(\nabla f, \nabla f) + 2|\nabla^2 f|^2 + 2\langle \nabla\Delta f, \nabla f\rangle$$

to compute the second term on the right-hand side of (3.9), observing that

$$\begin{aligned}
&\int_M 2f\left[Ric(\nabla f,\nabla f)-(n-1)k|\nabla f|^2\right]dV_g\\
&=\int_M f\left[\Delta|\nabla f|^2-2|\nabla^2 f|^2-2\langle\nabla\Delta f,\nabla f\rangle-2(n-1)k|\nabla f|^2\right]dV_g\\
&=\int_M f\left[\Delta|\nabla f|^2-2|\nabla^2 f|^2-2(n-1)k|\nabla f|^2\right]dV_g+2\int_M \Delta f(f\Delta f+|\nabla f|^2)dV_g,
\end{aligned}$$

where we have used that $\Delta f=0$ at ∂M. Also by Green's identity and again the fact that $f=0$ in ∂M, one deduces that

$$\int_M \left(f\Delta|\nabla f|^2-|\nabla f|^2\Delta f\right)dV_g=\int_{\partial M}\left(f\frac{\partial|\nabla f|^2}{\partial\nu}-|\nabla f|^2\frac{\partial f}{\partial\nu}\right)dA_g=\sum_i\kappa_i^3|\Sigma_i|,$$

where $\partial M=\cup_{i=1}^l\Sigma_i$, Σ_i are the connected components of ∂M and $\kappa_i=|\nabla f|\Big|_{\Sigma_i}$ (see Proposition 3). Consequently,

$$\int_M f\Delta|\nabla f|^2 dV_g=\sum_i\kappa_i^3|\Sigma_i|+\int_M|\nabla f|^2\Delta f\,dV_g.$$

Therefore,

$$\begin{aligned}
&\int_M 2f\left[Ric(\nabla f,\nabla f)-(n-1)k|\nabla f|^2\right]dV_g\\
&=\sum_i\kappa_i^3|\Sigma_i|-\int_M f\left[2|\nabla^2 f|^2+2(n-1)k|\nabla f|^2\right]dv+2\int_M f(\Delta f)^2dV_g\\
&+3\int_M|\nabla f|^2\Delta f\,dV_g.
\end{aligned}$$

Now, substituting the above identity into (3.9), we obtain

$$\begin{aligned}
&\int_M f\left[(\Delta f+knf)^2-|\nabla^2 f+kfg|^2\right]dV_g=(n-1)k\int_M(\Delta f+nkf)\,f^2dV_g\\
&+\sum_i\kappa_i^3|\Sigma_i|-\int_M f\left[2|\nabla^2 f|^2+2(n-1)k|\nabla f|^2\right]dV_g+2\int_M f(\Delta f)^2dV_g\\
&+3\int_M|\nabla f|^2\Delta f\,dV_g+\int_M 2f\left(\langle E,\nabla f\rangle^2-|E|^2|\nabla f|^2\right)dV_g,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_M 2f\left(|E|^2|\nabla f|^2 - \langle E, \nabla f\rangle^2\right) dV_g + \int_M f|\nabla^2 f|^2 dV_g \\
&= \sum_i \kappa_i^3 |\Sigma_i| + 3\int_M |\nabla f|^2 \Delta f\, dV_g + \int_M f\Delta f(\Delta f - k(n-1)f) dV_g \\
&-2(n-1)k\int_M f|\nabla f|^2 dV_g.
\end{aligned}$$

Taking into account that $|\nabla^2 f|^2 = |\mathring{\nabla}^2 f|^2 + \dfrac{(\Delta f)^2}{n}$, we infer

$$\begin{aligned}
0 &\leq \int_M 2f\left(|E|^2|\nabla f|^2 - \langle E, \nabla f\rangle^2\right) dV_g + \int_M f|\mathring{\nabla}^2 f|^2 dV_g \\
&= \sum_i \kappa_i^3 |\Sigma_i| + 3\int_M |\nabla f|^2 \Delta f\, dV_g + (n-1)\int_M f\Delta f\left(\frac{\Delta f}{n} - kf\right) dV_g \\
&\quad -2(n-1)k\int_M f|\nabla f|^2 dV_g. \qquad (2.1)
\end{aligned}$$

We now consider

$$\frac{2}{n-1}\left[\Lambda - (n-2)|E|^2\right] \leq \frac{2}{n-1}\max_M \left[\Lambda - (n-2)|E|^2\right] = \alpha,$$

and by , one sees that $\Delta f \geq -\alpha f$, which implies that

$$\int_M f(\Delta f)^2 dV_g \leq -\alpha\int_M f^2 \Delta f\, dV_g = 2\alpha\int_M f|\nabla f|^2 dV_g, \qquad (2.2)$$

where in the last identity we used integration by parts and the fact that $f = 0$ on ∂M.

Substituting (2.2) into (2.1) yields

$$\begin{aligned}
0 &\leq \int_M 2f\left(|E|^2|\nabla f|^2 - \langle E, \nabla f\rangle^2\right) dV_g + \int_M f|\mathring{\nabla}^2 f|^2 dV_g \\
&\leq \sum_i \kappa_i^3|\Sigma_i| - 3\alpha \int_M f|\nabla f|^2 dV_g + \frac{2\alpha(n-1)}{n}\int_M f|\nabla f|^2 dV_g \\
&\quad -(n-1)k\int_M f^2\Delta f\, dV_g - 2(n-1)k\int_M f|\nabla f|^2 dV_g \\
&= \sum_i \kappa_i^3|\Sigma_i| - \frac{\alpha(n+2)}{n}\int_M f|\nabla f|^2 dV_g. \qquad (2.3)
\end{aligned}$$

Now, we need to estimate the term $\int_M f|\nabla f|^2 dV_g$. To do so, we first observe that our assumption jointly with (1.15) imply $-\Delta f \geq 0$ and hence, by by Hölder's inequality and integration by parts, one sees that

$$\begin{aligned}
\left(\int_M f\Delta f\, dV_g\right)^2 &\leq \int_M (-\Delta f) dV_g \int_M f^2(-\Delta f) dV_g \\
&= \sum_i \kappa_i|\Sigma_i| \int_M f^2(-\Delta f) dV_g \\
&= 2\sum_i \kappa_i|\Sigma_i| \int_M f|\nabla f|^2 dV_g. \qquad (2.4)
\end{aligned}$$

Again, by Hölder's inequality, one obtains that

$$\left(\int_M \Delta f\, dV_g\right)^2 \leq Vol(M)\int_M (\Delta f)^2\, dV_g \leq -\alpha Vol(M)\int_M f\Delta f\, dV_g.$$

From this, it follows that

$$\left(\sum_i \kappa_i|\Sigma_i|\right)^4 = \left(\int_M \Delta f\, dV_g\right)^4 \leq \alpha^2 Vol(M)^2\left(\int_M f\Delta f\, dV_g\right)^2.$$

This jointly with (2.4) yields

$$\frac{(\sum_i \kappa_i|\Sigma_i|)^3}{2\alpha^2 Vol(M)^2} \leq \int_M f|\nabla f|^2 dV_g.$$

Now, it suffices to use (2.3) to infer

$$\frac{\alpha(n+2)}{n}\int_M f|\nabla f|^2 dV_g \leq \sum_i \kappa_i^3|\Sigma_i|,$$

consequently,

$$\frac{\alpha(n+2)}{2n\alpha^2 Vol(M)^2}\left(\sum_i \kappa_i|\Sigma_i|\right)^3 \leq \sum_i \kappa_i^3|\Sigma_i|,$$

which proves the asserted inequality.

We now consider the case of equality. In this situation, it follows from (2.3) that E is parallel to ∇f, and $\mathring{\nabla}^2 f = 0$. Hence, we use Proposition 5 to deduce that M^n has constant scalar curvature. Consequently, since the boundary ∂M is totally geodesic, we may then invoke Reilly's generalization of Obata's theorem (see [58, Lemma 3]) to conclude that (M^n, g, f) is isometric to the de Sitter system. This finishes the proof of the theorem. □

Proof of Theorem 1. It suffices to simplify the (unique) constant κ appearing in Theorem 11. □

Proceeding, we present the proof of Theorem 2.

Proof of Theorem 2. We first consider the tensor T given by

$$T := Ric - \frac{R}{2}g + 2E^\flat \otimes E^\flat - |E|^2 g.$$

It follows from Lemma 3 that T satisfies $div\, T = 0$. Moreover, taking the trace of T, we have

$$\begin{aligned} tr\, T &= \frac{2-n}{2}R + (2-n)|E|^2 \\ &= \frac{2-n}{2}\left(2\Lambda + 2|E|^2\right) + (2-n)|E|^2 \\ &= -(n-2)\Lambda + 2(2-n)|E|^2, \end{aligned}$$

where we have used (1.15). Besides, one sees that

$$\begin{aligned}\mathring{T} &= T - \frac{tr\,T}{n}g \\ &= Ric - \frac{R}{n}g + 2E^\flat \otimes E^\flat - \frac{2}{n}|E|^2 g.\end{aligned}$$

By choosing $X = \nabla f$ in (1.33), using integration by parts and the fact that $\nu = -\frac{\nabla f}{|\nabla f|}$ along ∂M, one obtains that

$$\begin{aligned}\int_M \nabla f(tr\,T)dV_g &= \int_M \left\langle \nabla f, \nabla\left(-(n-2)\Lambda + (4-2n)|E|^2\right)\right\rangle dV_g \\ &= 2(n-2)\int_M |E|^2 \Delta f\, dV_g - 2(n-2)\int_{\partial M} |E|^2 \langle \nabla f, \nu\rangle\, dA_g \\ &= 2(n-2)\int_M f|E|^2\left(2\left(\frac{n-2}{n-1}\right)|E|^2 - \frac{2}{n-1}\Lambda\right) dV_g \\ &\quad +2(n-2)\int_{\partial M} |E|^2|\nabla f|dA_g. \end{aligned} \tag{2.5}$$

Next, since $\mathcal{L}_{\nabla f}\, g = 2\nabla^2 f$, we use the first equation in Definition 1 to deduce

$$\int_M \langle \mathring{T}, \mathcal{L}_{\nabla f}\, g\rangle dV_g = 2\int_M f|\mathring{T}|^2 dV_g. \tag{2.6}$$

At the same time, by the Gauss equation and Proposition 3, one sees that

$$\begin{aligned}\int_{\partial M} \mathring{T}(\nabla f, \nu)dA_g &= \int_{\partial M} \Bigg[Ric(\nabla f, \nu) - \frac{R}{n}\langle \nabla f, \nu\rangle + 2\langle E, \nabla f\rangle\langle E, \nu\rangle \\ &\quad - \frac{2}{n}|E|^2\langle \nabla f, \nu\rangle\Bigg] dA_g; \\ &= \int_{\partial M} \Bigg[-|\nabla f|\left(Ric(\nu,\nu) - \frac{R}{n}\right) - \frac{2}{|\nabla f|}\langle E, \nabla f\rangle^2 \\ &\quad + \frac{2}{n}|E|^2|\nabla f|\Bigg] dA_g; \\ &= \int_{\partial M} \left[-|\nabla f|\left(-\frac{R^{\partial M}}{2} + \frac{n-2}{2n}R\right) + \frac{2-2n}{n}|E|^2|\nabla f|\right] dA_g. \end{aligned} \tag{2.7}$$

Substituting (2.5), (2.6) and (2.7) into (1.33) of Theorem 10, we get

$$
\begin{aligned}
& 2(n-2)\int_M f|E|^2\left[2\left(\frac{n-2}{n-1}\right)|E|^2-\frac{2}{n-1}\Lambda\right]dV_g+2(n-2)\int_{\partial M}|E|^2|\nabla f|dA_g \\
= \;& n\int_M f|\mathring{T}|^2 dV_g+n\int_{\partial M}|\nabla f|\left[-\frac{R^{\partial M}}{2}+\frac{n-2}{2n}R+\frac{2(n-1)}{n}|E|^2\right]dA_g,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \int_M\left[f|\mathring{T}|^2+\frac{2(n-2)}{n}|E|^2\left(\frac{2}{n-1}\Lambda-\frac{2(n-2)}{n-1}|E|^2\right)f\right]dV_g \\
= \;& \int_{\partial M}|\nabla f|\left[\frac{R^{\partial M}}{2}-\frac{(n-2)}{2n}R-\frac{2}{n}|E|^2\right]dA_g. \qquad (2.8)
\end{aligned}
$$

Hence, taking into account that $\kappa_i=|\nabla f|\big|_{\Sigma_i}$, where Σ_i are the connected components of ∂M, we use the assumption and Eq. (2) to infer

$$
\begin{aligned}
0 \;\leq\; & \int_M\left[f|\mathring{T}|^2+\frac{2(n-2)}{n}|E|^2\left(\frac{2}{n-1}\Lambda-\frac{2(n-2)}{n-1}|E|^2\right)f\right]dV_g \\
= \;& \sum_{i=1}^{l}\kappa_i\int_{\Sigma_i}\left[\frac{R^{\Sigma_i}}{2}-\frac{n-2}{2n}R-\frac{2}{n}|E|^2\right]dA_g \qquad (2.9)\\
= \;& \sum_{i=1}^{l}\kappa_i\int_{\Sigma_i}\left[\frac{R^{\Sigma_i}}{2}-\frac{n-2}{2n}(2|E|^2+2\Lambda)-\frac{2}{n}|E|^2\right]dA_g \\
= \;& \sum_{i=1}^{l}\kappa_i\int_{\Sigma_i}\left[\frac{R^{\Sigma_i}}{2}-|E|^2\right]dA_g-\frac{n-2}{n}\Lambda\sum_{i=1}^{l}\kappa_i|\Sigma_i| \\
\leq \;& \sum_{i=1}^{l}\kappa_i\int_{\Sigma_i}\frac{R^{\Sigma_i}}{2}dA_g-\frac{1}{2}(n-1)(n-2)\omega_{n-1}^2\sum_{i=1}^{l}\frac{\kappa_i Q(\Sigma_i)^2}{|\Sigma_i|}-\frac{n-2}{n}\Lambda\sum_{i=1}^{l}\kappa_i|\Sigma_i|,
\end{aligned}
$$

which proves the stated inequality (4).

Finally, observe that, since both terms in the integral over M^n are nonnegative, equality holds if and only if

$$
\frac{2}{n-1}|E|^2\left(\Lambda-(n-2)|E|^2\right)=0 \quad \text{and} \quad \mathring{T}=0.
$$

From this, and using once again the assumption $\Lambda - (n-2)|E|^2 > 0$, we conclude that $E \equiv 0$ and, consequently, $\mathring{Ric} = 0$, that is, M^n is an Einstein manifold. By applying the classical Reilly's theorem (see [58, Lemma 3]), we obtain the rigidity result.

□

As a direct consequence of Theorem 2, Theorem E of [26] can be recovered by using the Gauss-Bonnet formula. To be precise, we have the following corollary.

Corollary 1. *([26, Theorem E]) Let (M^3, g, f, E) be a compact electrostatic system such that E is parallel to ∇f and connected boundary ∂M. Suppose that $|E|^2 < \Lambda$. Then we have:*

$$16\pi^2 \frac{Q(\partial M)^2}{|\partial M|} + \frac{1}{3}\Lambda|\partial M| \leq 4\pi, \tag{2.10}$$

Moreover, equality occurs in (2.10) if and only if (M^3, g, f, E) is isometric to the de Sitter system.

In the higher-dimensional setting, the additional assumption that the boundary is Einstein yields the following boundary estimate.

Theorem 12. *Let (M^n, g, f, E) be a compact electrostatic system with connected Einstein boundary ∂M and such that E is parallel to ∇f. Suppose that $|E|^2 < \dfrac{\Lambda}{(n-2)}$. Then we have:*

$$|\partial M| \left[\min_{\partial M}\left(R + \frac{4}{n-2}|E|^2\right)\right]^{\frac{n-1}{2}} \leq [n(n-1)]^{\frac{n-1}{2}}\omega_{n-1}, \tag{2.11}$$

where ω_{n-1} is the area of the standard unitary sphere $\mathbb{S}^{n-1}$. Moreover, equality holds in (2.11) if and only if (M^n, g, f, E) is isometric to the de Sitter system.

Proof. Since ∂M is Einstein, we may consider $Ric^{\partial M} = \delta(n-2)g^{\partial M}$, where δ is constant. By Bishop's theorem, one deduces that

$$|\partial M| \leq \delta^{-\frac{1}{a}}\omega_{n-1}. \tag{2.12}$$

On the other hand, since $R^{\partial M} = (n-1)(n-2)\delta$, Theorem 2 implies

$$\delta(n-1)(n-2) \geq \min_{\partial M}\left(\frac{n-2}{n}R + \frac{4}{n}|E|^2\right),$$

so that

$$\delta n(n-1) \geq \min_{\partial M}\left(R + \frac{4}{n-2}|E|^2\right). \tag{2.13}$$

Next, by assumption, we have

$$\min_{\partial M}\left(R+\frac{4}{n-2}|E|^2\right)>0.$$

Thereby, substituting (2.13) into (2.12), we arrive at (2.11). Moreover, the equality in (2.13) guarantees the equality in (4) of Theorem 2 and therefore, the rigidity result follows. $\square$

When $n=5$, so that the boundary ∂M is four-dimensional, the Gauss–Bonnet–Chern theorem reveals a deep connection between the topology and geometry of ∂M. In particular, it expresses the Euler characteristic $\chi(\partial M)$ through the following identity:

$$8\pi^2\chi(\partial M)=\frac{1}{4}\int_{\partial M}|W|^2\,dA_g+\frac{1}{24}\int_{\partial M}\left(R^{\partial M}\right)^2\,dA_g-\frac{1}{2}\int_{\partial M}|\mathring{Ric}_{\partial M}|^2\,dA_g, \tag{2.14}$$

where W, $R^{\partial M}$ and $\mathring{Ric}_{\partial M}$ denote the Weyl tensor, the scalar curvature and the traceless part of its Ricci tensor of ∂M, respectively. This result originates from the seminal contributions of S.-S. Chern [18] in the 1940s, wherein he generalized the classical Gauss–Bonnet theorem to higher even dimensions via the framework of characteristic classes and differential forms.

Corollary 2. *Let $(M^5,\,g,\,f,\,E)$ be a compact electrostatic system with connected Einstein boundary ∂M such that E parallel to ∇f. Then we have:*

$$\left[\min_{\partial M}\left(\frac{3}{5}R+\frac{4}{5}|E|^2\right)\right]^2|\partial M|\leq 192\pi^2\chi(\partial M). \tag{2.15}$$

Moreover, equality holds in (2.15) if and only if $(M^5,\,g,\,f)$ is isometric to the de Sitter system.

Proof. Considering $n=5$ in Theorem 2 (see (2.9)), one obtains that

$$\int_{\partial M}\left[R^{\partial M}-\frac{3}{5}R-\frac{4}{5}|E|^2\right]dA_g\geq 0,$$

which then implies

$$\min_{\partial M}\left(\frac{3}{5}R+\frac{4}{5}|E|^2\right)|\partial M|\leq\int_{\partial M}R^{\partial M}dA_g.$$

Now, we use the Holder's inequality to infer

$$\left[\min_{\partial M}\left(\frac{3}{5}R+\frac{4}{5}|E|^2\right)\right]^2|\partial M|\leq\int_{\partial M}\left(R^{\partial M}\right)^2 dA_g.$$

To conclude, it suffices to apply the Gauss–Bonnet–Chern formula (2.14). Furthermore, the equality case follows from Theorem 2. □

In the sequel, assuming a suitable upper bound for the scalar curvature, we obtain a boundary estimate for compact electrostatic systems with connected boundary. This estimate will be used in the proof of Theorem 3.

To this end, we first establish a lower bound for the scalar curvature in electrostatic systems, which will also play a key role in the proof of Theorem 13.

Lemma 4. *Let (M^n, g, f, E) be a compact electrostatic system with positive cosmological constant Λ. Then the scalar curvature satisfies the following lower bound:*

$$R\geq\frac{4(n-1)}{3n-4+\sqrt{n^2-4}}\Lambda.$$

Proof. Consider the function

$$\ell(n)=\frac{4(n-1)}{3n-4+\sqrt{n^2-4}}.$$

A straightforward computation shows that $\ell(n)$ is a decreasing function of n. Moreover,

$$\ell(2)=2,\ \ \ell(3)=2\left(1-\frac{\sqrt{5}}{5}\right)\approx 1.10,\quad\text{and}\quad\lim_{n\to+\infty}\ell(n)=1.$$

In particular, for all $n\geq 2$, we have

$$\ell(n)\Lambda\leq 2\Lambda.$$

Next, it follows from (1.15) that $R\geq 2\Lambda$ and hence, the claimed inequality follows. □

We are now in a position to discuss our next result. Essentially, it provides a condition under which the left-hand side of inequality (1.24) in Proposition 4 is nonnegative. See also Remark 8 for a discussion on the meaning of the hypothesis.

Theorem 13. *Let* (M^n, g, f, E) *be a compact electrostatic system with connected boundary* ∂M *and positive cosmological constant satisfying*

$$R \leq \frac{4(n-1)}{3n-4-\sqrt{n^2-4}}\Lambda. \tag{2.16}$$

Then we have:

$$\kappa C|\partial M| \leq \frac{2\Lambda}{(n-1)}\int_{\partial M} R^{\partial M} dA_g, \tag{2.17}$$

where $\kappa = |\nabla f|\Big|_{\partial M}$ *and* $C \geq 0$ *is a constant. Moreover, equality holds in (2.17) if and only if* (M^n, g, f, E) *is isometric to the de Sitter system with* $C = \dfrac{4(n-2)\Lambda^2}{\kappa n(n-1)}$ *and* $\kappa = \dfrac{\Lambda}{n}$*.*

Proof. We start by defining

$$\begin{aligned} \overline{F}(|E|^2) &= \frac{4(n-1)}{n\kappa}\left\{|E|^2 + \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}+R\right]\right\} \\ &\quad \times\left\{\frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-R\right]-|E|^2\right\}, \end{aligned} \tag{2.18}$$

where $\kappa = \big|\nabla f\big|\big|_{\partial M}$. Note that $\overline{F}(|E|^2)$ is a quadratic polynomial in the variable $|E|^2$. From now on, the proof is divided into some steps.

We first need to obtain a geometric condition to guarantee that $\overline{F}(|E|^2) \geq 0$. Indeed, since $R \geq 0$, it suffices to control the sign of the second factor in the product in (2.18). Using (1.15), we rewrite this factor in terms of the scalar curvature R as

$$\begin{aligned} P(R) &= \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-R\right]-|E|^2 \\ &= \frac{(n-2)^2}{4(n-1)^2}\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-\frac{(n-2)^2}{4(n-1)^2}R-\frac{R}{2}+\Lambda. \end{aligned}$$

By squaring this expression, we get

$$\begin{aligned}\left[\frac{(n-2)^2}{4(n-1)^2}\right]^2 R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right) &= P(R)^2+2P(R)\left(\frac{(n-2)^2}{4(n-1)^2}R+\frac{R}{2}-\Lambda\right)\\ &\quad+\left(\frac{(n-2)^2}{4(n-1)^2}R+\frac{R}{2}-\Lambda\right)^2.\end{aligned}$$

Observe from (1.15) that $R \geq 2|E|^2$ and therefore,

$$\left[\frac{(n-2)^2}{4(n-1)^2}R+\frac{R}{2}-\Lambda\right] \geq \left[\frac{(n-2)^2}{2(n-1)^2}|E|^2+|E|^2\right] \geq 0.$$

Furthermore, to guarantee that $P(R) \geq 0$ we only need to show that

$$Q(R) := -\frac{(n-1)^2+(n-2)^2}{4(n-1)^2}R^2+\frac{3n-4}{2(n-1)}\Lambda R-\Lambda^2 \geq 0.$$

The roots of $Q(R)$ are

$$R_\pm = \frac{(n-1)\left(3n-4\pm\sqrt{(n+2)(n-2)}\right)}{(n-1)^2+(n-2)^2}\Lambda.$$

By a simple simplification

$$\begin{aligned}R_\pm &= \frac{3n-4\mp\sqrt{n^2-4}}{3n-4\mp\sqrt{n^2-4}}\frac{(n-1)\left(3n-4\pm\sqrt{n^2-4}\right)}{2n^2-6n+5}\Lambda\\ &= \frac{(n-1)\left(8n^2-24n+20\right)}{(2n^2-6n+5)\left(3n-4\mp\sqrt{n^2-4}\right)}\Lambda\\ &= \frac{4(n-1)}{3n-4\mp\sqrt{n^2-4}}\Lambda.\end{aligned}$$

Consequently, by Lemma 4 and the assumption (2.16), one sees that

$$R_- \leq R \leq R_+,$$

and therefore $Q(R)$, $P(R)$ and $\overline{F}(|E|^2)$ are nonnegative.

Proceeding, we return to inequality (1.24) in Proposition 4 which asserts that

$$\begin{aligned}
&\frac{1}{\kappa}\int_M \left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right] dV_g \\
&+\frac{4(n-1)}{n\kappa}\int_M f\left\{|E|^2 + \frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}+R\right]\right\} \\
&\times\left\{\frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{R\left(R+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-R\right]-|E|^2\right\} dV_g \\
=& \int_{\partial M} R^{\partial M} dA_g,
\end{aligned}$$

so that

$$\frac{1}{\kappa}\int_M \left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right] dV_g + \int_M f\overline{F}(|E|^2) dV_g = \int_{\partial M} R^{\partial M} dA_g.$$

By letting $C := \min_M \overline{F}(|E|^2) \geq 0$, one obtains that

$$C\int_M f dV_g \leq \int_{\partial M} R^{\partial M} dA_g. \tag{2.19}$$

Moreover, equality in (2.19) occurs if and only if $\mathring{Ric} = 0$, and $\mathring{\nabla}^2 f = 0$.

We now need to remove the dependence on f. To that end, on integrating the equation in (1):

$$\Delta f = \frac{2}{n-1}\left[(n-2)|E|^2 - \Lambda\right] f$$

and applying Proposition 3, we obtain

$$-\kappa|\partial M| \geq \frac{2}{n-1}\left((n-2)\min_M |E|^2 - \Lambda\right)\int_M f\, dV_g.$$

Since $C \geq 0$, one deduces that

$$\begin{aligned}
\kappa C|\partial M| &\leq \frac{2C}{n-1}\left(\Lambda - (n-2)\min_M |E|^2\right)\int_M f\, dV_g \\
&\leq \frac{2C\Lambda}{n-1}\int_M f\, dV_g \\
&\leq \frac{2\Lambda}{n-1}\int_{\partial M} R^{\partial M}\, dA_g,
\end{aligned}$$

so that

$$\kappa C|\partial M| \leq \frac{2\Lambda}{n-1}\int_{\partial M} R^{\partial M}\, dA_g, \tag{2.20}$$

which proves the asserted inequality.

Finally, we address the equality case. If equality is achieved, then (2.19) yields $\mathring{\mathrm{Ric}} = 0$ and $\mathring{\nabla}^2 f = 0$. By Reilly's theorem [58], it follows that the manifold is the de Sitter space and then $\kappa = \dfrac{\Lambda}{n}$. Furthermore, we can explicitly compute the constant C by evaluating

$$\begin{aligned}
\overline{F}(0) &= \frac{4(n-1)}{n\kappa}\left\{\frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{2\Lambda\left(2\Lambda+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}+2\Lambda\right]\right\}\\
&\quad\times\left\{\frac{(n-2)^2}{4(n-1)^2}\left[\sqrt{2\Lambda\left(2\Lambda+\frac{8(n-1)^2}{(n-2)^3}\Lambda\right)}-2\Lambda\right]\right\}\\
&= \frac{(n-2)^4\Lambda^2}{\kappa n(n-1)^3}\left[\sqrt{1+\frac{4(n-1)^2}{(n-2)^3}}+1\right]\left[\sqrt{1+\frac{4(n-1)^2}{(n-2)^3}}-1\right]\\
&= \frac{4(n-2)\Lambda^2}{\kappa n(n-1)}.
\end{aligned}$$

Thus, we must have $C = \dfrac{4(n-2)\Lambda^2}{\kappa n(n-1)}$. So, the proof is finished. □

Remark 8. *Regarding the scalar curvature restriction in the assumption of Theorem 13, notice that*

$$\lim_{n\to+\infty}\frac{4(n-1)}{3n-4-\sqrt{n^2-4}} = 2.$$

Moreover, for $n = 3$, we have the explicit bound

$$1,10\Lambda \approx 2\left(1-\frac{\sqrt{5}}{5}\right)\Lambda \leq R \leq 2\left(1+\frac{\sqrt{5}}{5}\right)\Lambda \approx 2,89\Lambda.$$

Note that this hypothesis is consistent with the restriction observed in Proposition 3.

We proceed to prove Theorem 3, which follows as a direct consequence of Theorem 13.

Proof of Theorem 3. We begin by using (1.15) to verify that the assumption in Theorem 13 is satisfied. Therefore, it suffices to apply the Gauss-Bonnet formula to the right-hand side of The Equation 2.20 in order to obtain the stated inequality. □

It is also relevant to consider the case where the surface gravity does not appear explicitly. In this context, we have the following theorem.

Theorem 14. *Let (M^n, g, f, E) be a compact electrostatic system and $\partial M = \cup_{i=1}^{l}\Sigma_i$, where Σ_i are the connected components of ∂M. Suppose that*

$$[(n-1)^2 + (n-2)^2]|E|^2 \leq (n-2)\Lambda.$$

Then we have:

$$\frac{n-2}{n}\Lambda\sum_{i=1}^{l}\kappa_i|\Sigma_i| \leq \sum_{i=1}^{l}\kappa_i\int_{\Sigma_i}\frac{R^{\Sigma_i}}{2}dA_g,$$

where $\kappa_i = |\nabla f|\Big|_{\Sigma_i}$ and ω_{n-1} is the area of the standard $(n-1)$-sphere. Moreover, equality holds if and only if (M^n, g, f, E) is isometric to the de Sitter system.

Proof. By Proposition 4, we have

$$\begin{aligned}\int_M\left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right]dV_g &= \int_{\partial M}|\nabla f|R^{\partial M}dA_g\\ &+\frac{1}{n}\int_M\left(4(n-1)f|E|^4 + (n-2)R\Delta f\right)dV_g,\end{aligned}$$

by using (1.15), one obtains that

$$\begin{aligned}n\int_M\left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right]dV_g &= n\kappa\int_{\partial M}R^{\partial M}dA_g\\ &+\int_M\left(4(n-1)f|E|^4 + 2(n-2)(|E|^2+\Lambda)\Delta f\right)dV_g\\ &= n\kappa\int_{\partial M}R^{\partial M}dA_g\\ &+\int_M 2|E|^2\left(2(n-1)f|E|^2 + (n-2)\Delta f\right)dV_g\\ &-2(n-2)\Lambda\kappa|\partial M|\end{aligned}$$

where $\kappa = |\nabla f|\Big|_{\partial M}$. Then, by Definition 1, we get

$$n\int_M \left[\frac{1}{f}|\mathring{\nabla}^2 f|^2 + f|\mathring{Ric}|^2\right] dV_g + 2(n-2)\Lambda\kappa|\partial M| = n\kappa\int_{\partial M} R^{\partial M} dA_g$$
$$+\frac{4}{(n-1)}\int_M f|E|^2\left[((n-1)^2+(n-2)^2)|E|^2 - (n-2)\Lambda\right] dV_g.$$

Therefore, by assuming that $[(n-1)^2+(n-2)^2]|E|^2 \le (n-2)\Lambda$, one concludes that

$$\Lambda|\partial M| \le \frac{n}{2(n-2)}\int_{\partial M} R^{\partial M} dA_g.$$

Furthermore, equality holds if and only if $|\mathring{\nabla}^2 f|^2 = 0$, $|\mathring{Ric}|^2 = 0$ and $E = 0$. By applying the classical Reilly's theorem [58], we obtain the rigidity result. □

Proof of Theorem 4. The proof follows immediately from Theorem 14, in combination with the Gauss–Bonnet formula. □

In the remainder of this section, we present an additional boundary area estimate in higher dimensions under the assumption of constant scalar curvature. In this case, the technique differs slightly from the previous ones and is based on a volume comparison argument involving the Bishop–Gromov inequality. We note that it would be of particular interest to replace the constant scalar curvature assumption by a condition involving a bound on $|E|$.

Theorem 15. *Let (M^n, g, f, E) be a compact electrostatic system with connected boundary ∂M and constant scalar curvature. Then we have:*

$$|\partial M| \le c\,\Omega_{n-1}, \tag{2.21}$$

for a specific constant $c > 0$, where Ω_{n-1} denotes the volume of the unit $(n-1)$-dimensional sphere. Moreover, equality holds in (2.21) if and only if (M^n, g, f, E) is isometric to the de Sitter system.

Remark 9. *The constant c in Theorem 15 is given by*

$$c := \frac{1}{\kappa}\cdot\frac{n^{\frac{n-1}{2}}}{\zeta^{\frac{n+1}{2}}},$$

where $\zeta = \dfrac{2}{n-1}\left[\Lambda - (n-2)|E|^2\right]$.

Proof. Initially, it follows from the system (1) that

$$-\frac{\Delta f}{f} = \frac{2}{n-1}\left[\Lambda - (n-2)|E|^2\right] =: \zeta,$$

where $\zeta > 0$ is a constant by the constant scalar curvature assumption and (1.15). Next, consider the warped product $N^{n+1} = \mathbb{S}^1 \times_f M$, and denote geometric tensors on M and N with the subscripts M and N, respectively. Using standard curvature formulas for warped products (see [53, Chapter 7]), one sees that

$$\begin{aligned}
Ric_N(X,Y) &= Ric_M(X,Y) - \frac{1}{f}\nabla^2 f(X,Y),\\
Ric_N(X,V) &= 0,\\
Ric_N(U,V) &= -\frac{\Delta f}{f}\, g_{\mathbb{S}^1}(U,V),
\end{aligned}$$

for vector fields $X, Y \in \mathfrak{X}(M)$ and $U, V \in \mathfrak{X}(\mathbb{S}^1)$. Hence, one deduce from (1.17) in Proposition 3 that

$$Ric_M - \frac{1}{f}\nabla^2 f \geq -\frac{\Delta f}{f}\, g_M, \tag{2.22}$$

and consequently,

$$Ric_N \geq \zeta\, g_N.$$

Applying the Bishop-Gromov comparison theorem, we obtain

$$Vol(N) \leq \Omega_{n+1}\left(\frac{n}{\zeta}\right)^{\frac{n+1}{2}}. \tag{2.23}$$

On the other hand, by Fubini's theorem and integration by parts, we have

$$\begin{aligned}
Vol(N) &= \int_{\mathbb{S}^1}\left[\int_M f\, dV_g\right] d\theta = 2\pi\int_M f\, dV_g\\
&= -\frac{2\pi}{\zeta}\int_M \Delta f\, dV_g = \frac{2\pi}{\zeta}\int_{\partial M}|\nabla f|\, dA_g\\
&= \frac{2\pi}{\zeta}\kappa\,|\partial M|,
\end{aligned} \tag{2.24}$$

where $\kappa = |\nabla f|_{\partial M}$ is constant. Plugging (2.24) into (2.23), we get

$$|\partial M| \leq \frac{\Omega_{n+1}}{2\pi\kappa}\left(\frac{n}{\zeta}\right)^{\frac{n+1}{2}}.$$

Using the recurrence formula $\Omega_{n-1} = \frac{n}{2\pi}\Omega_{n+1}$, one deduces that

$$|\partial M| \leq \frac{1}{\kappa} \cdot \frac{n^{\frac{n-1}{2}}}{\zeta^{\frac{n+1}{2}}}\Omega_{n-1}.$$

In the case of equality, Bishop's theorem implies that (N^{n+1}, h) must be isometric to the round sphere. The equality in (2.22) then yields $Ric_M - \frac{1}{f}\nabla^2 f = -\frac{\Delta f}{f} g_M$, which holds if and only if $E = 0$, meaning the system is vacuum static. Furthermore, by the classical characterization of Einstein manifolds associated with vacuum static spaces (see, for instance, [3, Section 5]), we conclude that (M, g) is isometric to the de Sitter system, which finishes the proof. □

Chapter 3

Charged Hawking mass in the electrostatic space

3.1 The Brown-York and charged Hawking masses

In this section, we present the proof of Theorem 5. It is worth noting that the technique adopted here is inspired by recent works in various contexts of Einstein-type spaces, including the vacuum static case [31, 35, 56, 66], perfect fluid space-times [25] and quasi-Einstein manifolds [27]. We begin by establishing the bound for the Brown–York mass in electrostatic systems. To do so, we consider the following notation

$$\beta^{-1} = \max_{M}\left(f^2 + \frac{n(n-1)}{R}|\nabla f|^2\right)^{\frac{1}{2}}.$$

In addition, we introduce the function v along with a conformal change of the metric $\overline{g}$ by

$$v = (1+\beta f)^{-\frac{n-2}{2}} \quad \text{and} \quad \overline{g} = v^{\frac{4}{n-2}}g.$$

With this notation, we have the following lemma.

Lemma 5. *Let $(M^n,\, g,\, f,\, E)$ be a compact electrostatic system with $\Lambda + |E|^2 > 0$. Then $R_{\bar{g}}$ is non-negative. Furthermore, $\overline{R} = 0$ if and only if $E \equiv 0$, $\Delta f = -\dfrac{R}{n-1}f$ and $f^2+\dfrac{n(n-1)}{R}|\nabla f|^2$ is constant in M^n. Here, $\overline{R}$ stands for the scalar curvature of $\overline{g}$.*

Proof. First, observe that

$$
\begin{aligned}
\Delta f &= \left(2\left(\frac{n-2}{n-1}\right)|E|^2 - \frac{2}{(n-1)}\Lambda\right) f \\
&= \left(2\left(\frac{n-2}{n-1}\right)|E|^2 + \frac{2}{n-1}|E|^2 - \frac{R}{n-1}\right) f \\
&= \left(2|E|^2 - \frac{R}{n-1}\right) f \\
&\geq -\frac{R}{n-1} f,
\end{aligned}
\tag{3.1}
$$

where equality holds if and only if $E = 0$. From this, it follows that

$$
\begin{aligned}
\Delta v &= \Delta\left((1+\beta f)^{-\frac{n-2}{2}}\right) \\
&= -\frac{(n-2)\beta}{2} div\left((1+\beta f)^{-\frac{n}{2}}\nabla f\right) \\
&= -\frac{(n-2)\beta}{2}(1+\beta f)^{-\frac{n}{2}}\Delta f + \frac{n(n-2)}{4}\beta^2(1+\beta f)^{-\frac{n+2}{2}}|\nabla f|^2 \\
&\leq \frac{n(n-2)}{4}\beta(1+\beta f)^{-\frac{n+2}{2}}\left(\frac{2R}{n(n-1)}(1+\beta f) f + \beta|\nabla f|^2\right).
\end{aligned}
\tag{3.2}
$$

On the other hand, we use the following formula for the scalar curvature as we see in [10]

$$
\overline{R} = v^{-\frac{n+2}{n-2}}\left(Rv - 4\frac{(n-1)}{(n-2)}\Delta v\right),
$$

in order to infer

$$
\begin{aligned}
\Delta v &= \frac{(n-2)}{4(n-1)}Rv - \frac{(n-2)}{4(n-1)}v^{\frac{n+2}{n-2}}\overline{R} \\
&= \frac{(n-2)}{4(n-1)}R(1+\beta f)^{-\frac{n-2}{2}} - \frac{(n-2)}{4(n-1)}(1+\beta f)^{-\frac{n+2}{2}}\overline{R} \\
&= \frac{(n-2)}{4(n-1)}(1+\beta f)^{-\frac{n+2}{2}}\left[(1+\beta f)^2 R - \overline{R}\right].
\end{aligned}
$$

Substituting this information into (3.2) yields

$$
R(1+\beta f)^2 - \overline{R} \leq n(n-1)\beta\left(\frac{2R}{n(n-1)}(1+\beta f) f + \beta|\nabla f|^2\right).
$$

Consequently,

$$\begin{aligned}\overline{R} &\geq R(1+\beta f)^2 - n(n-1)\beta\left(\frac{2R}{n(n-1)}(1+\beta f)f + \beta|\nabla f|^2\right)\\ &= R\left[(1+\beta f)^2 - 2\beta(1+\beta f)f - \frac{n(n-1)\beta^2}{R}|\nabla f|^2\right]\\ &= R\left[1-\beta^2\left(f^2 + \frac{n(n-1)}{R}|\nabla f|^2\right)\right].\end{aligned}$$

Since $R > 0$ (whenever $\Lambda + |E|^2 > 0$), and using the chosen value of β, we conclude that $\overline{R} \geq 0$.

Finally, it follows from (3.1) that $\overline{R} = 0$ if and only if $E \equiv 0$ and $\Delta f = -\dfrac{R}{n-1}f$. Moreover, from the characterization of β, we deduce that the quantity $f^2 + \dfrac{n(n-1)}{R}|\nabla f|^2$ is constant on M^n, which finishes the proof. □

We are now ready to present the proof of Theorem 5.

Proof of Theorem 5. Following the notation introduced above, it is well known that the mean curvature $\overline{H}^i$ of Σ_i with respect to the conformal metric $\overline{g} = v^{\frac{4}{n-2}}g$ is strictly positive. Indeed, since $f|_{\partial M} = 0$, it follows that $v\ |_{\partial M} = 1$. Hence, $\overline{g} = g$ over the boundary ∂M and $(\Sigma_i, \overline{g})$ is isometric to $(\Sigma_i,\ g)$, which, by assumption, can be isometrically embedded in $\mathbb{R}^n$ as a convex hypersurface with mean curvature H_0^i, induced by the Euclidean metric. Moreover, taking into account that the mean curvature of Σ_i with respect to the conformal metric $\overline{g}$ is given by

$$\overline{H}^i = v^{-\frac{2}{n-2}}\left(H^i + 2\frac{n-1}{n-2}\partial_\nu(\log(v))\right),$$

as we see in [10], so that

$$\begin{aligned}\overline{H}^i &= \frac{2(n-1)}{(n-2)}\left\langle\nabla(1+\beta f)^{-\frac{n-2}{2}}, \nu\right\rangle\\ &= (n-1)\beta|\nabla f|\Big|_{\Sigma_i},\end{aligned} \tag{3.3}$$

where we have used that $H^i = 0$ and $\nu = -\dfrac{\nabla f}{|\nabla f|}$. This proves that $\overline{H}^i > 0$, as claimed (cf. Lemma 4 in [26]).

Proceeding, one sees from (3.3) that

$$\begin{aligned}\mathfrak{M}_{BY}(\Sigma_i,\overline{g}) &= \int_{\Sigma_i}(H_0^i-\overline{H}^i)dA_g\\ &= \mathfrak{M}_{BY}(\Sigma_i,g)-(n-1)\beta|\nabla f|\Big|_{\Sigma_i}|\Sigma_i|.\end{aligned}$$

Since Lemma 5 implies that $\overline{R}\geq 0$, and $\overline{H}^i>0$, we apply the Positive Mass Theorem for the Brown-York mass [64] to conclude that $\mathfrak{M}_{BY}(\Sigma_i,\overline{g})\geq 0$. Consequently,

$$\begin{aligned}|\Sigma_i| &\leq \frac{1}{(n-1)\beta|\nabla f|\big|_{\Sigma_i}}\mathfrak{M}_{BY}(\Sigma_i,g)\\ &= \frac{1}{(n-1)\beta|\nabla f|\big|_{\Sigma_i}}\int_{\Sigma_i}H_0^i dA_g,\end{aligned}\tag{3.4}$$

which proves the stated inequality (7).

We now consider the case of equality. If equality holds in (3.4) for some component Σ_{i_0}, then

$$\mathfrak{M}_{BY}(\Sigma_{i_0},\overline{g})=0.$$

By invoking the equality case of the Positive Mass Theorem for the Brown–York mass, it follows that the conformal metric $\overline{g}$ is flat, and consequently, $(M^n,\,\overline{g})$ is isometric to a bounded domain in $\mathbb{R}^n$. Moreover, taking into account that $\overline{R}=0$, Lemma 5 implies that $E\equiv 0$, $\Delta f=-\dfrac{R}{n-1}f$ and the quantity $\dfrac{n(n-1)}{R}|\nabla f|^2+f^2$ is constant on M^n. Thus, since $(M^n,\,g)$ is vacuum static, the scalar curvature R is constant. Consequently,

$$0=\nabla\left[\frac{n(n-1)}{R}|\nabla f|^2+f^2\right]=\frac{2n(n-1)}{R}\nabla^2 f(\nabla f)+2f\nabla f$$

so that

$$\nabla|\nabla f|^2-2\frac{\Delta f}{n}\nabla f=0.$$

Now, it suffices to use the Robinson-Shen-type identity in Lemma 2 to conclude that $|\mathring{Ric}|^2=0$. Finally, we use Reilly's theorem [58] linked with the fact that ∂M is totally geodesic in order to conclude that $(M^n,\,g)$ is isometric to $\mathbb{S}^n_+$.

On the other hand, if (M^n, g) is isometric to $\mathbb{S}^n_+$, one deduces that $\partial M = \mathbb{S}^{n-1}$. Thereby, the Brown-York mass of $\mathbb{S}^{n-1}$ is given by

$$\mathfrak{M}_{BY}(\mathbb{S}^{n-1}) = \int_{\mathbb{S}^{n-1}} (n-1)dA_{g_{\mathbb{S}^{n-1}}} = (n-1)\omega_{n-1},$$

where ω_{n-1} is the volume of the standard $(n-1)$-sphere. At the same time, since $\mathbb{S}^n_+$ has constant scalar curvature $R = n(n-1)$, it follows that the height function h satisfies $\Delta h = -\dfrac{R}{n-1}h = -nh$. Therefore, $(\mathbb{S}^n_+, g, h)$ is a static vacuum space with non-zero cosmological constant. Consequently, $h^2 + \dfrac{n(n-1)}{R}|\nabla h|^2$ is constant. Indeed, a direct computation yields

$$\begin{aligned}
\nabla\left(h^2 + \frac{n(n-1)}{R}|\nabla h|^2\right) &= 2h\nabla h + 2\nabla^2 h(\nabla h) \\
&= -\frac{2\Delta h}{n}\nabla h + 2\nabla^2 h(\nabla h) \\
&= 2\mathring{\nabla}^2 h(\nabla h) = 2h\mathring{Ric}(\nabla h) = 0.
\end{aligned}$$

From this, it follows that

$$\beta^{-2} = \left(h^2 + \frac{n(n-1)}{R}|\nabla h|^2\right)\Bigg|_{\partial M} = |\nabla h|^2_{|\partial M},$$

so that

$$\beta = \frac{1}{|\nabla h|_{|\partial M}}.$$

Hence,

$$\frac{\mathfrak{M}_{BY}(\partial M, g)}{\alpha(n-1)|\nabla h|_{|\partial M}} = \omega_{n-1} = |\partial M|,$$

which is the equality in (3.4). So, the proof is completed. □

Remark 10. *We note that Theorem 5 can also be formulated within a broader class of sub-static systems (see Remark 5). The core of its proof relies on inequality (3.1), which remains valid in this more general setting. This extension has already been observed, for example, in the case where the energy–momentum tensor corresponds to that of a perfect fluid, under a sub-static condition related to the dominant energy condition, specifically, the inequality $\rho + P \geq 0$, where ρ denotes the energy density and P is the pressure; see [25, Theorem 2]. This broader* *perspective allows the result to be extended to other classes of energy–momentum tensors, such as those associated with wave maps, including Klein–Gordon-type matter fields (cf. [23, Section 2]).*

This chapter also focuses on the study of surfaces embedded in three-dimensional Riemannian manifolds. To establish the necessary structure, it is appropriate to begin by recalling the foundational concepts of the electrostatic space for the three-dimensional case ($n = 3$). Hence, consider a static space-time is a product manifold $\mathrm{M}^4 = \mathbb{R}\times M^3$ with metric $\mathrm{g} = -f^2dt^2+g$, for a smooth, positive function $f : M \to (0, +\infty)$ and Riemannian metric g. Here, (M^3, g) is an oriented three-dimensional Riemannian manifold. We will explore the Reissner-Nordström-de Sitter (RNdS) space as an example of electrostatic space. The 3 + 1-dimensional RNdS space-time is a 3-parameter family (labeled with a mass $\mathfrak{M}$, a charge Q and cosmological constant Λ) of static, electrically charged, solutions to the Einstein-Maxwell equations. Consider (M^3, g_{RNdS}, f, E), where

$$M^3 = I \times \mathbb{S}^2$$

for some $I \subset \mathbb{R}$ which depends on the roots of the lapse function given by

$$f(r)^2 = 1 - \frac{2\mathfrak{M}}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}.$$

This polynomial equation has four distinct solutions. There is a negative root with no physical relevance. We denote by $r_c > r_+ > r_-$ the positive roots. The metric and the electric field for RNdS space are given by

$$g_{RNdS} = f(r)^{-2}dr^2 + r^2 g_{\mathbb{S}^2} \quad \text{and} \quad E = \frac{Q}{r^2}f(r)\partial r.$$

In the above, Q, $\mathfrak{M}$, and $g_{\mathbb{S}^2}$ stand for the charge, mass, and the standard metric of the unit sphere $\mathbb{S}^2$, respectively. Here, r represents the radial coordinate. The set $\{r = r_-\}$ is the inner (Cauchy) black hole horizon, $\{r = r_+\}$ is the outer (Killing) black hole horizon, and $\{r = r_c\}$ is the cosmological horizon. We can have solutions possessing some extreme conditions: the cold black hole $r_- = r_+$, the Nariai black hole $r_+ = r_c$, and the ultracold black hole $r_- = r_+ = r_c$, see more details in [26]. These solutions for the electrostatic system (Definition 1) can be seen as a static black hole model with a given Hawking temperature (the reason behind the term "cold").

Theorem 16. *Let (M^3, g, f, E) be a compact electrostatic system with connected boundary ∂M. Then the following assertions hold:*

(i) If E is parallel to ∇f and $|E|^2 < \Lambda$, then

$$\mathfrak{M}_{CH}(\partial M) \geq \left(\frac{8\pi}{|\partial M|}Q(\partial M)^2\right)\sqrt{\frac{|\partial M|}{16\pi}}. \tag{3.5}$$

(ii) If $|E|^2 < \frac{\Lambda}{5}$, then

$$\mathfrak{M}_{CH}(\partial M) \geq \left(\frac{4\pi}{|\partial M|}Q(\partial M)^2\right)\sqrt{\frac{|\partial M|}{16\pi}}. \tag{3.6}$$

Moreover, equality holds in (3.5) or (3.6) if and only if $E = 0$ and (M^3, g, f) is isometric to the de Sitter system.

Proof. To begin with, we notice that a direct computation by using the definition of charged Hawking mass and Eq. (2.10) yields

$$\begin{aligned}
\mathfrak{M}_{CH}(\partial M) &= \sqrt{\frac{|\partial M|}{16\pi}}\left(\frac{1}{2}\chi(\partial M) - \frac{1}{16\pi}\int_{\partial M}(H^2 + \frac{4}{3}\Lambda)dA_g + \frac{4\pi}{|\partial M|}Q(\partial M)^2\right) \\
&= \sqrt{\frac{|\partial M|}{16\pi}}\left(1 - \frac{\Lambda}{12\pi}|\partial M| + \frac{4\pi}{|\partial M|}Q(\partial M)^2\right) \\
&\geq \left(\frac{8\pi}{|\partial M|}Q(\partial M)^2\right)\sqrt{\frac{|\partial M|}{16\pi}},
\end{aligned}$$

which proves the first assertion. Moreover, the rigidity statement follows from Corollary 1.

Proceeding, we shall prove the second inequality. Indeed, by combining the definition of charged Hawking mass and Theorem 4, one obtains that

$$\begin{aligned}
\mathfrak{M}_{CH}(\partial M) &= \sqrt{\frac{|\partial M|}{16\pi}}\left(\frac{1}{2}\chi(\partial M) - \frac{1}{16\pi}\int_{\partial M}(H^2 + \frac{4}{3}\Lambda)dA_g + \frac{4\pi}{|\partial M|}Q(\partial M)^2\right) \\
&= \sqrt{\frac{|\partial M|}{16\pi}}\left(1 - \frac{\Lambda}{12\pi}|\partial M| + \frac{4\pi}{|\partial M|}Q(\partial M)^2\right) \\
&\geq \left(\frac{4\pi}{|\partial M|}Q(\partial M)^2\right)\sqrt{\frac{|\partial M|}{16\pi}},
\end{aligned}$$

which gives the stated inequality. Furthermore, the rigidity statement follows from Theorem 4. This finishes the proof of the theorem. $\square$

Remember the total electric charge contained within an orientable closed surface Σ with unit normal ν, i.e.,

$$Q(\Sigma) = \frac{1}{4\pi}\int_{\Sigma}\langle E, \nu\rangle d\sigma, \tag{3.7}$$

and the ADM mass of a Riemannian manifold (M^3, g), which is given by

$$\mathfrak{M} = \mathfrak{M}(M, g) = \frac{1}{8\pi}\lim_{r\to+\infty}\int_{\mathbb{S}(r)}\sum_{i,j=1}^{3}(\partial_i g_{ij} - \partial_j g_{ii})\nu^j,$$

where $\mathbb{S}(r)$ is the standard sphere. Here, ν^j represents one component of the normal vector field of $\mathbb{S}(r)$. Moreover, a straightforward computation shows that the charged Hawking mass of a sphere in the Reissner-Nordström-de Sitter space satisfies $\mathfrak{M}_{CH}(\mathbb{S}(r)) = \mathfrak{M}_{ADM}$ (see [5, p. 6]).

A relevant concept in the electrostatic theory is the photon sphere. A time-like embedded orientable surface Σ in $(\mathbb{R}\times M^3, -f^2dt^2 + g)$ is called a photon sphere if it is totally umbilical and f is constant on every connected component of Σ. In the three-dimensional Reissner–Nordström space (i.e., $\Lambda = 0$) with mass parameter $\mathfrak{M} > 0$, there is a photon sphere located at the radius

$$\frac{3\mathfrak{M}}{2}\left(1 + \sqrt{1 - \frac{8Q^2}{9\mathfrak{M}^2}}\right), \tag{3.8}$$

whenever $9\mathfrak{M}^2 \geq 8Q^2$ (e.g., subextremal Reissner–Nordström space), see [16, 38]. A Reissner–Nordström space is called subextremal (extremal, superextremal) if $\mathfrak{M}^2 > Q^2$ (if $\mathfrak{M}^2 = Q^2$, $\mathfrak{M}^2 < Q^2$). The photon sphere appears naturally in our study and plays an important role in our main results.

The Reissner-Nordström photon sphere models an embedded submanifold ruled by photons spiraling around the central black hole "at a fixed distance". The Reissner–Nordström photon spheres are related to the existence of relativistic images in the context of gravitational lensing [16, 38]. Such a photon sphere is a stable constant mean curvature (CMC) surface (see Proposition 7 below). In fact, any sphere $\mathbb{S}^2(r)$ of radius r in the RNdS is stable if

$$\frac{3\mathfrak{M}}{2}\left(1 - \sqrt{1 - \frac{8Q^2}{9\mathfrak{M}^2}}\right) \leq r \leq \frac{3\mathfrak{M}}{2}\left(1 + \sqrt{1 - \frac{8Q^2}{9\mathfrak{M}^2}}\right).$$

The concept of stability of a surface is related to its Morse index. The Morse index of a closed surface Σ is given by the number of negative eigenvalues of the Jacobi operator J, where

$$J = -\Delta^{\Sigma} - Ric(\nu, \nu) - |A|^2.$$

A constant mean curvature (CMC) surface Σ in a Riemannian manifold is stable if the Jacobi operator is non-negative, i.e., the Morse index is zero ($Index(\Sigma) = 0$). Here, Δ^{Σ} and A stand for the Laplacian and the second fundamental form of Σ. Moreover, ν represents the normal vector field of Σ. In the proof of Theorem 17, we do not impose any information about the mean curvature H of a closed surface in the electrostatic space (M^3, g, f, E). It can be either non-constant or constant, and it can change signs. In Theorem 17, we will only impose that J must be non-negative, that is, $Index(\Sigma) = 0$. Moreover, the photon sphere will be an important example of this theorem. Stable CMC surfaces are a fundamental tool in studying general relativity. For example, the theory of stable minimal surfaces was essential in proving the positive mass theorem [61].

Note that, even for minimal surfaces, the positivity for the charged Hawking mass is not trivially obtained. It is a straightforward computation to show that the charged Hawking mass of a sphere in the Reissner-Nordström-de Sitter space satisfies $\mathfrak{M}_{CH}(\mathbb{S}(r)) = \mathfrak{M}$ (cf. [5]). The result we will provide does not impose (M^3, g) to be compact, and we do not ask for any additional condition over the mean curvature or Gauss curvature of a surface Σ in the electrostatic system. It is well-known that if a surface Σ has $Index(\Sigma) = 0$, then for any $\phi \in C^{\infty}(\Sigma)$ we have

$$0 \leq \int_{\Sigma} |\nabla_{\Sigma}\phi|^2 d\sigma - \int_{\Sigma} (Ric(\nu, \nu) + |A|^2)\phi^2 d\sigma.$$

Taking $\phi = 1$ we get

$$\int_{\Sigma} (Ric(\nu, \nu) + |A|^2) d\sigma \leq 0. \tag{3.9}$$

Theorem 17. *Let (M^3, g, f, E) be a three-dimensional electrostatic system (compact or non-compact) and Σ a closed surface on M satisfying (3.9). Then, the genus of Σ is at most 1. Moreover,*

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2} \left[\frac{1}{16\pi}\int_{\Sigma} H^2 d\sigma + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right)\right]. \tag{3.10}$$

The equality holds if and only if Σ is totally umbilical, E is orthogonal to Σ, and equality holds in (3.9).

Remark 11. *In Proposition 7 (below), we will prove that any CMC surface in the RNdS space must be a sphere. The equality in Theorem 17 holds for a sphere of radius (3.8) in the Reissner-Nordström-de Sitter space. Moreover, we can conclude that the charged Hawking mass is non-negative if $\Lambda \geq 0$.*

We will apply the above theorem to a stable minimal surface to get rigidity (Theorem 18). Nonetheless, we will also provide the lower bound for the charged Hawking mass considering a stable CMC surface with non-zero mean curvature and a minimal surface of index one (cf. Theorem 19 and Theorem 20).

Some important results use the Hawking mass to obtain the rigidity of a given space [5, 49]. In [5], the authors proved the rigidity of 3-manifolds satisfying Einstein's constraints, assuming the existence of a strictly stable minimal two-sphere that locally maximizes the charged Hawking mass, concluding that the ambient space must be locally a copy of RNdS, see also [49] for the analogous result without charge. If we consider just stability, the rigidity for area-minimizing two-sphere holds if the two-sphere has a fixed area. In this case, the sphere is a global maximum of the Hawking mass [49, Remark 4.4]. We will assume that [5, Theorem 2] holds for a stable area-minimizing two-sphere with a fixed area (see the details in the proof of Theorem 18).

In the following theorem, we will use the results of [5, 49] to obtain a sharp lower bound for the charged Hawking mass of minimal surfaces.

Theorem 18. *Let (M^3, g, f, E) be a three-dimensional electrostatic system with a non-null cosmological constant and $\Sigma \subset M$ an embedded stable minimal surface. Then, the genus of Σ is at most 1. Moreover,*

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left[\frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right)\right]. \tag{3.11}$$

In addition, if Σ is a stable minimal two-sphere, equality holds if Σ is the horizon boundary of the ultracold black hole system.

We want to obtain a lower bound for the charged Hawking mass for a stable CMC surface ($H \neq 0$), i.e., a constant mean curvature surface Σ satisfying the stability condition (3.9) for any function $\phi \in C^\infty(\Sigma)$ such that $\displaystyle\int_\Sigma \phi d\sigma = 0$. However, we can not provide the rigidity since, as far as we know, we do not possess a result like the main theorem in [5] for stable CMC surfaces. It is well-known that if a surface Σ is a stable

CMC surface, then
$$\int_\Sigma (Ric(\nu,\nu)+|A|^2)d\sigma \le 8\pi, \tag{3.12}$$
see the details in [19, 46].

Theorem 19. *Let (M^3, g, f, E) be a three-dimensional electrostatic system with Σ a closed stable CMC surface in M. Then, the genus of Σ is at most 3. Moreover,*
$$\mathfrak{M}_{CH}(\Sigma) \ge \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left[\frac{H^2}{16\pi}|\Sigma| + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right) - 1\right]. \tag{3.13}$$
The equality holds if and only if Σ is totally umbilical, E is orthogonal to Σ, and equality holds in (3.12).

Remark 12. *Considering $\Lambda > 0$, the charged Hawking mass is non-negative for any stable CMC surface in the RNdS space such that $|\Sigma| \ge \dfrac{12\pi}{\Lambda}$. The Nariai system is $M^3 = [0, \pi/\sqrt{\Lambda}] \times \mathbb{S}^2$ with metric tensor $g = ds^2 + \dfrac{1}{\Lambda}g_{\mathbb{S}^2}$, where $f(s) = \sin(\sqrt{\Lambda}s)$, $\mathfrak{M} = \dfrac{1}{3\sqrt{\Lambda}}$, and $Q = 0$. Therefore, the area of a two-sphere Σ in the Nariai system is $|\Sigma| = \dfrac{4\pi}{\Lambda}$. The equality holds in Theorem 19 for stable CMC sphere such that $s = \dfrac{\pi}{2\sqrt{\Lambda}}$. This will be discussed in more detail in the proof of Theorem 19.*

The last theorem above (Theorem 19) inspired us to pursue a similar result for minimal surfaces of index one. It is well-known that such a surface must satisfy the following inequality:
$$\int_\Sigma (Ric(\nu,\nu)+|A|^2)d\sigma \le 8\pi\left(1 + Int\left[\frac{1+g(\Sigma)}{2}\right]\right). \tag{3.14}$$
Here, $Int[x]$ denotes the integer part of x. See details in [26, 60].

Theorem 20. *Let (M^3, g, f, E) be a three-dimensional electrostatic system with Σ a closed minimal surface of index one in M. Then,*
$$\mathfrak{M}_{CH}(\Sigma) \ge \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left(\frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right) - 1 - Int\left[\frac{1+g(\Sigma)}{2}\right]\right). \tag{3.15}$$
The equality holds if and only if Σ is totally umbilical, E is orthogonal to Σ, and equality holds in (3.14).

Remark 13. *Considering $\Lambda > 0$, the charged Hawking mass is non-negative for any minimal surface of index one in the RNdS space such that* $|\Sigma| \geq \dfrac{12\pi}{\Lambda}\left(1 + Int\left[\dfrac{1+g(\Sigma)}{2}\right]\right)$. *Consider the deSitter system which is* $M^3 = \mathbb{S}^3_+$ *with metric* $g = \dfrac{3}{\Lambda} g_{\mathbb{S}^3}$, $\mathfrak{M} = Q = 0$ *and* $f(x) = x_4$, *with Cartesian coordinates* $x = (x_1, x_2, x_3, x_4)$. *We can see that the equator is a minimal sphere* Σ *of index one with area* $|\Sigma| = \dfrac{12\pi}{\Lambda}$. *Therefore, the equality holds in Theorem 20.*

3.2 Proof of the main results

In this section, we present the proof of the main results. Before we prove our main results concerning a lower bound for the charged Hawking mass, we will prove a criterion for CMC surfaces in the RNdS space to be stable (Proposition 7). This result will be important for a better understanding of Theorem 17. Then, we will prove the sharp lower bounds for the charged Hawking mass of stable surfaces in the electrostatic space-time (Proposition 8).

Proposition 7. *Any CMC surface in the three-dimensional Reissner-Nordström-de Sitter space must be a sphere* $\mathbb{S}^2(r)$ *with mean curvature* H *given by*

$$H = \frac{2}{r}\left(1 - \frac{2\mathfrak{M}}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}\right)^{1/2}.$$

Moreover, if

$$\frac{3\mathfrak{M}}{2}\left(1 - \sqrt{1 - \frac{8Q^2}{9\mathfrak{M}^2}}\right) \leq r \leq \frac{3\mathfrak{M}}{2}\left(1 + \sqrt{1 - \frac{8Q^2}{9\mathfrak{M}^2}}\right)$$

such CMC surface must be stable.

Proof of Proposition 7. We start with a result due to Brendle proving that any CMC surface must be $\mathbb{S}^2(r)$ in the Reissner-Nordström-de Sitter space (cf. [15, Section 5] to see that Λ plays no role in the proof of Corollary 1.3). For every spherical slice, $\Sigma = \{r\} \times \mathbb{S}^2$ in the RNdS space, we have $H(r) = \dfrac{2}{r} f(r)$ and $|\Sigma| = 4\pi r^2$. By the spherical symmetry, Σ is umbilical. Moreover, $E = \dfrac{Q}{r^2} f(r)\partial r$ is parallel to the normal vector field $\nu = f(r)\partial_r$.

Remember by 1.5 that

$$\Delta f = \Delta^{\Sigma} f + \nabla^2 f(\nu, \nu) + H\langle \nabla f, \nu \rangle.$$

By Definition 1 we have

$$\begin{aligned}
H\langle \nabla f, \nu\rangle - f|A|^2 &= \Delta f - \Delta^{\Sigma} f - \nabla^2 f(\nu,\nu) - f|A|^2 \\
&= (|E|^2 - \Lambda) f - \Delta^{\Sigma} f \\
&\quad - f[Ric(\nu,\nu) + 2\langle E, \nu\rangle^2 - (|E|^2 + \Lambda)] - f|A|^2 \\
&= 2f(|E|^2 - \langle E, \nu\rangle^2) - \Delta^{\Sigma} f - f Ric(\nu,\nu) - f|A|^2. \qquad (3.16)
\end{aligned}$$

Since E is parallel to ν (unitary), i.e., $|E|^2 = \langle E, \nu\rangle^2$, we obtain

$$\left(\frac{H}{f}\langle \nabla f, \nu\rangle - |A|^2\right) f = (-\Delta^{\Sigma} - Ric(\nu,\nu) - |A|^2) f.$$

i.e.,

$$\frac{H}{f}\langle \nabla f, \nu\rangle - |A|^2 = -Ric(\nu,\nu) - |A|^2, \qquad (3.17)$$

where we use that $\Delta^{\Sigma} f = 0$, since f is a non-null constant at $\Sigma = \{r\} \times \mathbb{S}^2$.

In the Reissner-Nordström-de Sitter space, we have

$$|E|^2 = \frac{Q^2}{r^4}$$

and

$$\nabla f = f(r)^2 f'(r)\partial_r = f(r)^2 \frac{1}{2f(r)}\left(\frac{2\mathfrak{M}}{r^2} - \frac{2Q^2}{r^3} - \frac{2\Lambda r}{3}\right)\partial_r.$$

Thus,

$$\frac{H}{f}\langle \nabla f, \nu\rangle = \frac{2}{r}\left(\frac{\mathfrak{M}}{r^2} - \frac{Q^2}{r^3} - \frac{\Lambda r}{3}\right).$$

Moreover,

$$\begin{aligned}
\frac{H}{f}\langle \nabla f, \nu\rangle - |A|^2 &= \frac{2}{r}\left(\frac{\mathfrak{M}}{r^2} - \frac{Q^2}{r^3} - \frac{\Lambda r}{3}\right) - \frac{H^2}{2} \\
&= \frac{2}{r}\left(\frac{\mathfrak{M}}{r^2} - \frac{Q^2}{r^3} - \frac{\Lambda r}{3}\right) - \frac{1}{2}\frac{4}{r^2} f(r)^2 \\
&= \frac{2}{r}\left[\frac{\mathfrak{M}}{r^2} - \frac{Q^2}{r^3} - \frac{\Lambda r}{3} - \frac{1}{r}\left(1 - \frac{2\mathfrak{M}}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}\right)\right] \\
&= \frac{2}{r}\left[\frac{3\mathfrak{M}}{r^2} - \frac{2Q^2}{r^3} - \frac{1}{r}\right] = -\frac{2}{r^4}\left[2Q^2 - 3\mathfrak{M}r + r^2\right]. \qquad (3.18)
\end{aligned}$$

Note that if $\mathfrak{M} = Q = 0$ we always have an unstable sphere (see the deSitter space). We can conclude that if

$$\frac{3\mathfrak{M}}{2}\left(1-\sqrt{1-\frac{8Q^2}{9\mathfrak{M}^2}}\right) \leq r \leq \frac{3\mathfrak{M}}{2}\left(1+\sqrt{1-\frac{8Q^2}{9\mathfrak{M}^2}}\right) \tag{3.19}$$

the CMC surface must be stable. Moreover, we can see that the photon sphere of radius $\frac{3\mathfrak{M}}{2}\left(1+\sqrt{1-\frac{8Q^2}{9\mathfrak{M}^2}}\right)$ must be a stable CMC surface in the RNdS space.

In fact, from (3.18) and (3.19) for any $\phi \in C^\infty(\Sigma)$ such that $\int_\Sigma \phi d\sigma = 0$ we have

$$\int_\Sigma |\nabla_\Sigma \phi|^2 d\sigma - \int_\Sigma (Ric(\nu,\nu) + |A|^2)\phi^2 d\sigma$$
$$= \int_\Sigma |\nabla_\Sigma \phi|^2 d\sigma + \int_\Sigma \left(\frac{H}{f}\langle \nabla f, \nu\rangle - |A|^2\right)\phi^2 d\sigma \geq 0.$$

□

Proposition 8. *Let (M^3, g, f, E) be a three-dimensional electrostatic system (compact or non-compact) and Σ a closed surface on M satisfying*

$$\int_\Sigma (Ric(\nu,\nu) + |A|^2) d\sigma \leq C.$$

Then,

$$\mathfrak{M}_{CH}(\Sigma) \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left[\frac{1}{16\pi}\int_\Sigma H^2 d\sigma + \frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right) - \frac{C}{8\pi}\right],$$

where $C \in \mathbb{R}$. Moreover, the genus of Σ is bounded from above, i.e.,

$$1 + \frac{C}{4\pi} \geq g(\Sigma).$$

Proof of Proposition 8. From Definition 1 and the Gauss equation we have, respectively,

$$\nabla^2 f(\nu,\nu) = f[Ric(\nu,\nu) + 2\langle E, \nu\rangle^2 - (|E|^2 + \Lambda)]$$

and

$$\frac{R}{2} = K + Ric(\nu,\nu) + \frac{1}{2}(|A|^2 - H^2),$$

where K is the Gauss curvature of Σ. Combining the above equations and using (1.15), we obtain

$$\nabla^2 f(\nu,\nu) = f[\frac{1}{2}(H^2 - |A|^2) - K + 2\langle E,\nu\rangle^2].$$

Now, from

$$\Delta f = \Delta^{\Sigma} f + \nabla^2 f(\nu,\nu) + H\langle \nabla f, \nu\rangle$$

and the last equation we deduce

$$\frac{\Delta f}{f} = \frac{\Delta^{\Sigma} f}{f} + [\frac{1}{2}(H^2 - |A|^2) - K + 2\langle E,\nu\rangle^2] + \frac{H}{f}\langle \nabla f,\nu\rangle.$$

By integrating the above identity, we have

$$\begin{aligned}\int_{\Sigma}(|E|^2 - \Lambda)d\sigma &= \int_{\Sigma}\frac{H}{f}\langle \nabla f,\nu\rangle d\sigma + \int_{\Sigma}\frac{|\nabla_{\Sigma} f|^2}{f^2}d\sigma \\ &+ \int_{\Sigma}\left[\frac{1}{2}(H^2 - |A|^2) - K + 2\langle E,\nu\rangle^2\right]d\sigma. \qquad (3.20)\end{aligned}$$

We consider that

$$\int_{\Sigma}(Ric(\nu,\nu) + |A|^2)d\sigma \leq C, \qquad (3.21)$$

where $C \in \mathbb{R}$. Dividing (3.16) by f and by using (3.21) we get

$$\begin{aligned}&\int_{\Sigma}\left(\frac{H}{f}\langle \nabla f,\nu\rangle + \frac{1}{f^2}|\nabla_{\Sigma} f|^2\right)d\sigma \\ &\geq \int_{\Sigma}|A|^2 d\sigma + 2\int_{\Sigma}(|E|^2 - \langle E,\nu\rangle^2)d\sigma - C. \qquad (3.22)\end{aligned}$$

Thus, from (3.20), (3.22) and the Gauss-Bonnet theorem we have

$$4\pi(1 - g(\Sigma)) \geq \int_{\Sigma}[\frac{1}{2}(H^2 + |A|^2)]d\sigma + \int_{\Sigma}(|E|^2 + \Lambda)d\sigma - C,$$

and since $2|A|^2 \geq H^2$,

$$4\pi(1 - g(\Sigma)) \geq \frac{3}{4}\int_{\Sigma}H^2 d\sigma + \int_{\Sigma}(|E|^2 + \Lambda)d\sigma - C. \qquad (3.23)$$

With this inequality, we can analyze the topology of Σ. Indeed, $4\pi(1-g(\Sigma))+C \geq 0$, i.e.,

$$1+\frac{C}{4\pi} \geq g(\Sigma).$$

Combine the above inequality (3.23) with Definition 2, i.e.,

$$2\pi\chi(\Sigma) = 4\pi\sqrt{\frac{16\pi}{|\Sigma|}}\mathfrak{M}_{CH}(\Sigma) + \frac{1}{4}\int_{\Sigma}(H^2+\frac{4}{3}\Lambda)d\sigma - \frac{16\pi^2}{|\Sigma|}Q(\Sigma)^2,$$

to obtain

$$4\pi\sqrt{\frac{16\pi}{|\Sigma|}}\mathfrak{M}_{CH}(\Sigma) \geq \frac{16\pi^2}{|\Sigma|}Q(\Sigma)^2 + \frac{1}{2}\int_{\Sigma}H^2 d\sigma + \int_{\Sigma}|E|^2 d\sigma + \frac{2}{3}\Lambda|\Sigma| - C.$$

Moreover, applying Hölder's inequality to the definition of charge given by (3.7) yields to

$$16\pi^2 Q(\Sigma)^2 = \left(\int_{\Sigma}\langle E, \nu\rangle d\sigma\right)^2 \leq |\Sigma|\int_{\Sigma}\langle E, \nu\rangle^2 d\sigma \leq |\Sigma|\int_{\Sigma}|E|^2 d\sigma.$$

Finally,

$$4\pi\sqrt{\frac{16\pi}{|\Sigma|}}\mathfrak{M}_{CH}(\Sigma) \geq \frac{32\pi^2}{|\Sigma|}Q(\Sigma)^2 + \frac{1}{2}\int_{\Sigma}H^2 d\sigma + \frac{2}{3}\Lambda|\Sigma| - C.$$

The inequality above proves the theorem. □

Proof of Theorem 17. By hypothesis, we have

$$0 \geq \int_{\Sigma}(Ric(\nu, \nu) + |A|^2)d\sigma. \tag{3.24}$$

Therefore, Proposition 8 holds for $C = 0$, and from (3.23) we have $g(\Sigma) \leq 1$.

Let us apply (3.10) to a sphere in the Reissner-Nordström-de Sitter space. We know that in this case $\mathfrak{M}_{CH}(\mathbb{S}(r)) = \mathfrak{M}$, see [5]. Since

$$\frac{1}{16\pi}\int_{\mathbb{S}(r)}H^2 d\sigma = f(r)^2$$

our inequality (3.10) is reduced to

$$\mathfrak{M} \geq r - 2\mathfrak{M} + 2\frac{Q^2}{r},$$

where the equality holds for

$$r = \frac{3\mathfrak{M}}{2}\left(1 \pm \sqrt{1 - \frac{8Q^2}{9\mathfrak{M}^2}}\right).$$

This is a totally umbilical stable constant mean curvature surface in the RNdS space in which equality holds in (3.24), and E is parallel to ν. See Proposition 7, equations (3.17) and (3.18). □

Proof of Theorem 18. Consider a stable minimal sphere of radius r in the RNdS space, and applying the inequality presented by Theorem 17, we get

$$\mathfrak{M} \geq \frac{Q^2}{r} + \frac{\Lambda}{3}r^3,$$

where equality holds if and only if

$$\frac{\Lambda}{3}r^4 - \mathfrak{M}r + Q^2 = 0. \tag{3.25}$$

The mean curvature of a two-sphere in the RNdS space is given by $H(r) = \frac{2}{r}f(r)$. Moreover, the ADM mass is

$$\mathfrak{M} = \frac{1}{2}\left(r + \frac{Q^2}{r} - \frac{\Lambda}{3}r^3\right). \tag{3.26}$$

Hence, combining (3.25) and (3.26) we get

$$\Lambda r^4 - r^2 + Q^2 = 0,$$

i.e.,

$$r^2 = \frac{1 \pm \sqrt{1 - 4\Lambda Q^2}}{2\Lambda}.$$

We can consult [26, Section 3.8] to conclude that a sphere in the RNdS space is stable when the radius r is bounded, i.e.,

$$\frac{1 - \sqrt{1 - 4\Lambda Q^2}}{2\Lambda} \leq r^2 \leq \frac{1 + \sqrt{1 - 4\Lambda Q^2}}{2\Lambda},$$

and it is strictly stable if

$$\frac{1-\sqrt{1-4\Lambda Q^2}}{2\Lambda} < r^2 < \frac{1+\sqrt{1-4\Lambda Q^2}}{2\Lambda}.$$

In fact, the Jacobi operator for the RNdS space is given by

$$J = -\Delta^{\Sigma} - \frac{1}{r^4}(\Lambda r^4 - r^2 + Q^2).$$

On the other hand, consider the equality holds in (3.11) for a stable minimal sphere of radius r in the electrostatic system. Thus,

$$\mathfrak{M}_{CH}(\Sigma) = \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left[\frac{4\pi}{|\Sigma|}Q(\Sigma)^2 + \frac{\Lambda}{3}\left(\frac{|\Sigma|}{4\pi}\right)\right] = \frac{Q^2}{r} + \frac{\Lambda}{3}r^3.$$

We must have a priori

$$r^2 = \frac{1 \pm \sqrt{1-4\Lambda Q^2}}{2\Lambda}, \quad \text{i.e.,} \quad Q^2 = \frac{1}{4\Lambda}[1-(1-2\Lambda r^2)^2]$$

Otherwise, the Equation 3.25 does not hold.

Hence,

$$\mathfrak{M}_{CH}(\Sigma) = \frac{Q^2}{r} + \frac{\Lambda}{3}r^3 = \frac{1}{4\Lambda r}[1-(1-2\Lambda r^2)^2] + \frac{\Lambda}{3}r^3 = r - \frac{2}{3}\Lambda r^3.$$

Therefore, the sphere $\Sigma = \mathbb{S}(r)$ of radius $r^2 = \dfrac{1}{2\Lambda}$ maximizes the charged Hawking mass, which is strictly stable. Hence, from Theorem 2 in [5], there is a neighborhood of such sphere in (M, g) that is isometric to the RNdS space $((-\varepsilon, \varepsilon) \times \Sigma, g_{RNdS})$ for some $\varepsilon > 0$. Since in the RNdS, Σ is minimal if and only if $f = 0$, such a surface must be the horizon boundary of M, i.e., $\Sigma = \partial M$.

Considering that the roots of f in the RNdS space are $r_+ = r_- = r_c$, we can perform a change of variable such that $g_{RNdS} = ds^2 + \rho^2 g_{\mathbb{S}^2}$ with $f(s) = s$, $\rho^2 = \dfrac{1}{2\Lambda}$, $Q^2 = \dfrac{1}{4\Lambda}$ and $\mathfrak{M}^2 = \dfrac{2}{9\Lambda}$, which correspond to the ultracold black hole system, where $M = [0, +\infty) \times \mathbb{S}^2$ (cf. [5, Equation 1.12] and [26, Section 3.7]), with a stable horizon at $s = 0$. □

Proof of Theorem 19. A stable CMC surface satisfies

$$8\pi \geq \int_\Sigma \left(|A|^2 + Ric(\nu,\nu)\right)\, d\sigma.$$

See more details about the above inequality in [19]. So, applying Proposition 8 and considering $C = 8\pi$ we obtain inequality (3.13). Assuming $\Lambda \geq 0$ from (3.23) we get $g(\Sigma) \leq 3$.

Let us apply (3.13) to a sphere in the RNdS. We know that in this case $\mathfrak{M}_{CH}(\mathbb{S}(r)) = \mathfrak{M}$, see [5]. Moreover, from (3.18) if $Q = 0$, $\mathfrak{M} = \dfrac{1}{3\sqrt{\Lambda}}$ and $r = \dfrac{1}{\sqrt{\Lambda}}$, then such sphere must be stable. Furthermore,

$$\frac{1}{16\pi}\int_{\mathbb{S}(r)} H^2 d\sigma = f(r)^2.$$

So,

$$\mathfrak{M} \geq \left(\frac{|\Sigma|}{4\pi}\right)^{1/2}\left[f^2 + \frac{4\pi}{|\Sigma|}Q^2 + \frac{\Lambda}{3}\frac{|\Sigma|}{4\pi} - 1\right]. \tag{3.27}$$

The Nariai system is $M^3 = [0, \pi/\sqrt{\Lambda}] \times \mathbb{S}^2$ with metric tensor $g = ds^2 + \dfrac{1}{\Lambda}g_{\mathbb{S}^2}$, where $f(s) = \sin(\sqrt{\Lambda}s)$, $\mathfrak{M} = \dfrac{1}{3\sqrt{\Lambda}}$ and $Q = 0$. The equality in (3.27) holds for $s = \dfrac{\pi}{2\sqrt{\Lambda}}$. □

Proof of Theorem 20. It is well-known that a closed minimal surface of index one satisfies the inequality:

$$8\pi\left(1 + Int\left[\frac{1+g(\Sigma)}{2}\right]\right) \geq \int_\Sigma \left(|A|^2 + Ric(\nu,\nu)\right)\, d\sigma,$$

where $Int[x]$ denotes the integer part of x. See for instance [26, Proposition 17] and [60]. Considering $C = 8\pi\left(1 + Int\left[\dfrac{1+g(\Sigma)}{2}\right]\right)$ in Proposition 8 will give us the desired result.

Consider the deSitter system which is $M^3 = \mathbb{S}^3_+$ with metric $g = \dfrac{3}{\Lambda}g_{\mathbb{S}^3}$, $\mathfrak{M} = Q = 0$ and $f(x) = x_4$, where $x = (x_1, x_2, x_3, x_4)$. We can see that the equator is a minimal sphere Σ of index one with area $|\Sigma| = \dfrac{12\pi}{\Lambda}$. Therefore, the equality holds in (3.15). □

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