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Master’s Dissertation

# Random quasi-stability and long-time dynamics of stochastic piezoelectric systems with magnetic and thermal effects

by

**Santiago Alexander Sumire Benites**

Brasília, December 19, 2025

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Advisor: Prof. Dr. Mirelson Martins Freitas

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Dedicated to God, and to my parents for
their unconditional support.

# Resumo

Quase-estabilidade aleatória e dinâmica de longo prazo de sistemas piezoelétricos estocásticos com efeitos magnéticos e térmicos.

Neste trabalho, investigamos a quase-estabilidade aleatória e o comportamento assintótico de longo tempo de uma classe de sistemas piezoelétricos estocásticos não lineares que incorporam efeitos magnéticos e dissipação térmica descrita pela lei de Fourier. O estudo insere-se no contexto das equações diferenciais parciais estocásticas do tipo hiperbólico, as quais modelam a interação entre campos mecânicos, elétricos e térmicos sob a influência de ruído aleatório. Inicialmente, estabelecemos a boa-colocação global dos sistemas determinísticos obtidos por transformação caminho a caminho das equações estocásticas originais e demonstramos que o sistema dinâmico aleatório associado possui um único atrator *pullback* aleatório. Mesmo na presença de não linearidades com crescimento polinomial arbitrário e comportamento não globalmente Lipschitz, provamos que o atrator aleatório possui dimensão fractal finita e uniforme. Esses resultados são obtidos por meio da adaptação do método de quase-estabilidade ao contexto aleatório, originalmente desenvolvido por Chueshov e Schmalfuß [15]. Além disso, com base em estimativas uniformes em relação à intensidade do ruído, demonstramos a semicontinuidade superior dos atratores quando a intensidade do ruído tende a zero.

**Palavras-chave**: Sistemas piezoelétricos estocásticos; atratores aleatórios; dimensão fractal; quase-estabilidade; semicontinuidade superior.

# Abstract

In this work, we investigate the random quasi-stability and the long-time asymptotic behavior of a class of nonlinear stochastic piezoelectric systems that incorporate magnetic effects and thermal dissipation governed by Fourier's law. The study lies within the framework of stochastic partial differential equations of hyperbolic type, which model the interaction among mechanical, electrical, and thermal fields under the influence of random noise. We first establish the global well-posedness of the deterministic systems obtained through a pathwise transformation of the original stochastic equations and prove that the associated random dynamical system has a unique *pullback* random attractor. Even in the presence of nonlinearities with arbitrary polynomial growth and non-globally Lipschitz behavior, we show that the random attractor possesses a finite and uniform fractal dimension. These results are obtained by adapting the quasi-stability method to the random setting, originally developed by Chueshov and Schmalfuß [15]. Furthermore, based on uniform estimates with respect to the noise intensity, we establish the upper semicontinuity of the attractors as the noise intensity tends to zero.
**Keywords:** Stochastic piezoelectric systems; random attractors; fractal dimension; quasi-stability; upper semicontinuity

# Contents

# Introduction

### Motivation and context

The mathematical analysis of complex physical systems subject to random perturbations represents one of the most challenging and rapidly evolving areas in modern applied mathematics. Such systems arise naturally across various scientific disciplines, from materials science and engineering to climate modeling and biological systems, where uncertainty and stochastic effects play fundamental roles in the long-term dynamical behavior. Within this broad context, the study of piezoelectric materials, which are smart materials capable of converting mechanical energy into electrical energy and vice versa, has gained considerable attention due to their wide-ranging applications in energy harvesting, structural health monitoring, and precision actuation [41].

The interplay between mechanical deformation, electrical polarization, and thermal effects in piezoelectric materials gives rise to intricate coupled field equations that exhibit rich mathematical structure. When further subjected to random environmental fluctuations, these systems become stochastic partial differential equations of hyperbolic-parabolic type, presenting substantial challenges for both analytical treatment and numerical simulation. Understanding the long-time behavior of such systems, particularly the existence and structure of their random attractors, is not only mathematically profound but also crucial for the reliable design and operation of piezoelectric-based devices in real-world applications where noise and uncertainty are inevitable.

### The quasi-stability method and an open question of Chueshov and Schmalfuß

The quasi-stability method, originally developed by Chueshov and Lasiecka [11, 13, 14] in the context of deterministic infinite-dimensional dynamical systems, has emerged as a powerful and versatile technique for demonstrating the existence and finiteness of the fractal dimension of global attractors. This method provides a unified framework for analyzing long-time dynamics of dissipative evolutionary equations, particularly those with nonlinear damping and coupled mechanical-thermal interactions.

The fundamental idea behind quasi-stability is to establish a stabilizability estimate, which states that the difference between two trajectories emanating from a bounded set can be decomposed into a part with exponential decay and another part consisting of compact terms. This elegant approach not only guarantees the asymptotic compactness of the dynamical system but also yields crucial information about the finite-dimensional character of the attractor, despite the infinite-dimensional nature of the underlying phase space.

Recently, Chueshov and Schmalfuß [15] extended the quasi-stability method to the stochastic setting, proving the existence of a finite-dimensional random attractor for a stochastic fluid-plate interaction model with globally Lipschitz nonlinearities. This pioneering work opened new avenues for applying quasi-stability techniques to random dynamical systems, but it left a significant gap in the theory. As the authors explicitly noted in their concluding remarks, the case of non-globally Lipschitz nonlinearities remained an open challenge:

> "In conclusion we note that in the case when the force $f(u)$ is not globally Lipschitz the fluid-plate interaction model on the dimension of the attractor is still open. The reason is related to the fact that the corresponding analog of quasi-stability estimate in (19) implied by (15) contains a random coefficient $C_T(w)$ with uncontrolled growth behavior of the samples $t \mapsto C_T(\vartheta_t w)$ as $|t| \to \infty$."

This open question highlights a fundamental difficulty in stochastic dynamics: the potential uncontrolled growth of random coefficients in stability estimates, which can destroy the compactness properties essential for finite-dimensional reduction. The challenge is particularly acute for systems with polynomial growth nonlinearities, which arise naturally in physical models but fall outside the scope of globally Lipschitz theory.

Our contribution to this open problem is to provide a positive and constructive answer by developing an enhanced quasi-stability methodology that successfully handles non-globally Lipschitz nonlinearities with arbitrary polynomial growth. Our approach combines the random quasi-stability framework with ergodic theory and careful energy estimates to control the potentially unbounded random coefficients that thwart previous attempts.

The key innovation lies in exploiting Birkhoff's ergodic theorem to establish uniform control over the random coefficients appearing in the quasi-stability estimates, thereby overcoming the uncontrolled growth behavior identified by Chueshov and Schmalfuß. This enables us to prove the existence, finite-dimensionality, and robustness of random attractors for a broad class of stochastic systems that were previously inaccessible to quasi-stability methods.

Moreover, we demonstrate that our enhanced quasi-stability methodology possesses considerable generality and can be applied to various hyperbolic-type partial differential equations beyond the specific piezoelectric system studied here, including the original fluid-plate interaction model that motivated the open question.

**Piezoelectric systems: mathematical discussions**

Piezoelectric materials occupy a unique position at the intersection of mechanics, electromagnetism, and materials science. Their ability to convert mechanical energy into electrical energy (direct piezoelectric effect) and electrical energy into mechanical deformation (inverse piezoelectric effect) makes them invaluable for smart structure applications, energy harvesting devices, and precision sensors. A schematic representation of a clamped-free piezoelectric beam with electrodes is shown in Figure 1.

The mathematical modeling of piezoelectric beams began with simplified one-dimensional formulations that captured the essential coupling between mechanical displacement and electrical potential. The linear system proposed by Morris and Özer [32, 33] for a clamped-free piezoelectric beam of length $L$ represents a cornerstone in this development:

$$\begin{cases} \rho\, v_{tt} - \alpha\, v_{xx} + \gamma\beta\, p_{xx} = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\ \mu\, p_{tt} - \beta\, p_{xx} + \gamma\beta\, v_{xx} = 0 \quad \text{in } (0, L) \times \mathbb{R}^+. \end{cases} \tag{0.0.1}$$

with boundary and initial conditions

$$\begin{cases} v(0,t) = 0, \quad \alpha\, v_x(L,t) - \gamma\beta\, p_x(L,t) = \dfrac{G(t)}{h}, & t \in \mathbb{R}^+, \\ p(0,t) = 0, \quad \beta\, p_x(L,t) - \gamma\beta\, v_x(L,t) = -\dfrac{V(t)}{h}, & t \in \mathbb{R}^+, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in (0,L), \\ p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), & x \in (0,L). \end{cases} \tag{0.0.2}$$

Here, the variables $v$ and $p$ represent the longitudinal displacement and the electric charge distribution along the beam, respectively. The parameters $\rho$, $\alpha$, $\gamma$, $\mu$, and $\beta$ denote, respectively, the mass density per unit volume, elastic stiffness, piezoelectric coupling coefficient, magnetic permeability, and the reciprocal of the permittivity of the material. The functions $G(t)$ and $V(t)$ correspond to the mechanical strain and voltage controls applied at the free end of the beam. The parameter $\alpha$ is given by $\alpha = \alpha_1 + \gamma^2\beta > 0$, accounting for the coupling between the mechanical and electrical subsystems.

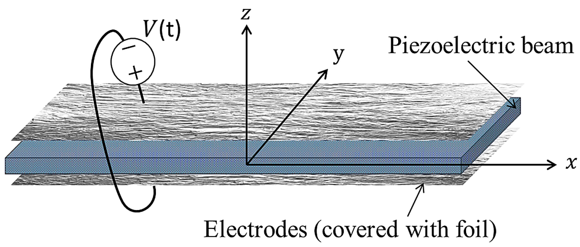


**Figure 1:** Schematic illustration of a voltage-actuated piezoelectric beam with surface electrodes. A voltage input $V(t)$ applied to the electrodes generates an electric field through the beam thickness, producing contraction or extension via the piezoelectric effect (adapted from [33]).

The stability analysis of piezoelectric systems has revealed a challenging mathematical problem. Early investigations [33, 35, 34] demonstrated that even with strong mechanical-electrical coupling, these systems typically lack exponential stability in the energy space under single boundary feedback control. This fundamental limitation motivated the development of enhanced control strategies, including multilayer configurations [37] and distributed damping mechanisms [36, 42], which can restore exponential stability.

The study of long-time dynamics through global attractors for nonlinear piezoelectric systems represents a more recent development. Works by [21, 22, 23, 24] have established the existence, regularity, finite fractal dimension, and upper semicontinuity of global/pullback attractors under various dissipation mechanisms. These contributions build upon the rich tradition of infinite-dimensional dynamical systems theory pioneered by [8, 9, 16, 20, 43, 45, 55, 56, 57], among others.

**The stochastic piezoelectric system**

Motivated by the need to incorporate environmental uncertainty and random fluctuations into piezoelectric modeling, we introduce a comprehensive stochastic framework that extends previous deterministic models. Our system incorporates three coupled physical effects: mechanical vibration, electrical polarization, and thermal diffusion, all subject to additive stochastic perturbations:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + v_t + f_1(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + p_t + f_2(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ d\theta - \kappa\theta_{xx}dt + \delta v_{xt}dt = \varepsilon\psi(x)dW_t & \text{in } (0,L)\times\mathbb{R}^+, \end{cases} \tag{0.0.3}$$

with boundary and initial conditions

$$\begin{cases} v(0,t) = 0, \quad \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t\in\mathbb{R}^+, \\ p(0,t) = 0, \quad p_x(L,t) - \gamma v_x(L,t) = 0, & t\in\mathbb{R}^+, \\ \theta(0,t) = \theta(L,t) = 0, & t\in\mathbb{R}^+, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x\in(0,L), \\ p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), & x\in(0,L), \\ \theta(x,0) = \theta_0(x), & x\in(0,L). \end{cases} \tag{0.0.4}$$

Here, the effective elastic coefficient $\alpha$ is given by $\alpha = \alpha_1 + \gamma^2\beta > 0$, and the physical constants $\rho, \alpha_1, \gamma, \mu, \beta, \kappa, \delta$ are positive, denoting the mass density, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermittivity, thermal diffusivity, and thermoelastic coupling strength, respectively. The functions $f_1(v,p)$ and $f_2(v,p)$ represent nonlinear internal forces. The stochastic thermal forcing is driven by a one-dimensional two-sided Wiener process $W_t$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\varepsilon > 0$ characterizes the noise intensity and $\psi$ is a prescribed spatial function.

We analyze the stochastic piezoelectric system within the framework of random dynamical systems (RDS), whose foundations were established in the pioneering works [2, 4, 17, 44]. This theory provides the mathematical machinery to describe the evolution of stochastic systems through cocycle formulations and pathwise transformations. Applications of this theory to physically relevant systems have been explored in various contexts, including [3, 15, 19, 26, 27, 28, 31, 48, 49, 52, 53, 54]. We also refer to [5, 7, 6, 10, 30, 47, 50] for key developments on random attractors in the abstract theory of RDS, which provide essential tools for the analysis carried out in this dissertation.

### Main contributions and original results

This dissertation contributes to the study of the asymptotic behavior of stochastic piezoelectric systems, within the framework of stochastic partial differential equations and infinite-dimensional dynamical systems, through the following main results:

1. *Global well-posedness:* We establish the well-posedness of both mild and strong global solutions for the stochastic piezoelectric system (0.0.3)-(0.0.4) in the natural energy space $\mathcal{H}$. This is achieved through a pathwise transformation that converts the original stochastic system into a family of deterministic problems parameterized by sample paths. The resulting deterministic systems are then analyzed using the theory of $C_0$-semigroups, following the framework developed by Pazy [39].

2. *Existence of random attractors:* We investigate the long-time behavior of the RDS $\Phi_\varepsilon$ generated by solutions to (2.0.1)-(2.0.2). For each noise intensity $\varepsilon \in (0, 1]$, we prove the existence of a $\mathcal{D}$-random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$ in $\mathcal{H}$, where $\mathcal{D}$ is the collection of tempered random bounded sets. This is done by proving the existence of a pullback absorbing set and the asymptotic compactness of $\Phi_\varepsilon$. The absorbing set is constructed via the energy method with appropriate pathwise multipliers. In contrast to the deterministic case studied in [22], the presence of additive noise requires the use of the frictional

damping term $v_t$ to suppress noise-induced instabilities. The asymptotic compactness is established through a stochastic version of the quasi-stability method introduced by Chueshov and Lasiecka [11, 14], and extended to RDS by Chueshov and Schmalfuß [15]. Our approach offers a more direct construction of the quasi-stability inequality tailored to the random setting.

3. *Finite fractal dimension:* We prove the finiteness of the fractal dimension of the random attractor $\mathcal{A}_\varepsilon(\omega)$ associated with a stochastic system whose nonlinearities $f_i$, $i = 1, 2$, exhibit arbitrary polynomial growth and are not globally Lipschitz. *This result provides a positive answer to the open question posed by Chueshov and Schmalfuß [10]*, where the authors emphasized the challenge in controlling the random coefficient appearing in the quasi-stability estimate due to the potential *uncontrolled* growth behavior of the sample paths $t \mapsto C_T(\vartheta_t\omega)$ as $|t| \to \infty$ (note that if we consider $f$ globally Lipschitz, in the random stabilizability estimate the tempered random variable disappears). In our setting, we overcome this difficulty by exploiting Birkhoff's ergodic theorem to control the random coefficient uniformly in time. This enables us to establish the finiteness of the fractal dimension of the attractor even in the presence of non-Lipschitz nonlinearities under stochastic forcing.

4. *Upper semicontinuity:* When $\varepsilon = 0$, the stochastic piezoelectric system with magnetic effects and Fourier's law (2.0.1)-(2.0.2) reduces to a deterministic one. In [22], the existence of a compact global attractor for this deterministic system was established. Here, we prove the robustness of the family of random attractors $\mathcal{A}_\varepsilon(\omega)$ as $\varepsilon \to 0$. A key step in our analysis is to show that

$$\overline{\bigcup_{\varepsilon\in(0,1]} \mathcal{A}_\varepsilon(\omega)}^{\mathcal{H}} \quad \text{is compact in } \mathcal{H}. \tag{0.0.5}$$

In general, the proof of (0.0.5) requires the existence of a compact attracting set independent of $\varepsilon$, which is not available in our context. We are only able to construct an $\varepsilon$-dependent compact attracting set, which by itself does not ensure the compactness of the union. Moreover, the high dimensionality of the system (involving three coupled components), the hyperbolic nature of the equations, and the nowhere differentiability of Brownian motion in time make the task of obtaining additional regularity in spaces stronger than $\mathcal{H}$ particularly challenging. To overcome these challenges, we combine the random quasi-stability method with a uniform continuity property of the solution map $\varepsilon \mapsto \Phi_\varepsilon$, which ultimately allows us to conclude the compactness of the union. To the best of our knowledge, such a result has not been previously addressed even in deterministic frameworks.

It is also worth highlighting that the present study gave rise to a scientific article entitled "Random Quasi-Stability and Long-Time Dynamics of Stochastic Piezoelectric Systems with Magnetic and Thermal Effects", developed in collaboration with Professors Renhai Wang, Mirelson Freitas, and Marcio Jorge Silva, which has been submitted for publication in a specialized journal in the field.

**Organization of the dissertation**

This dissertation is organized as follows:

- Chapter 1 provides the mathematical background on stochastic analysis and random dynamical systems theory necessary for understanding the subsequent developments.
- Chapter 2 establishes the global well-posedness of the stochastic piezoelectric system through pathwise transformations and semigroup methods.
- Chapter 3 is devoted to proving the existence of random attractors, constructing absorbing sets, and establishing asymptotic compactness.
- Chapter 4 contains our main result on the finite fractal dimension of random attractors for systems with non-globally Lipschitz nonlinearities.
- Chapter 5 addresses the upper semicontinuity of attractors with respect to noise intensity, establishing the robustness of our results.
- Conclusion summarizes our contributions, discusses the significance of our results, and outlines promising directions for future research.

# Chapter 1

# Preliminaries

In this chapter, we present the mathematical foundations necessary to understand the remainder of this work.

### 1.1 Stochastic differential equations

In this section, we introduce some basic concepts with the purpose of providing the reader with a clearer understanding of differential equations that involve stochastic noise. For further details on this topic, we refer the reader to the references by Arnold [1], Oksendal [38], Evans [18] and Langa [29].

#### 1.1.1 Stochastic processes

**Definition 1.1.1.** *Let $\Omega$ be a given set. A* $\sigma$-algebra $\mathcal{F}$ *on* $\Omega$ *is a family of subsets of* $\Omega$ *satisfying:*

*(i)* $\emptyset \in \mathcal{F}$*;*

*(ii) If* $F \in \mathcal{F}$*, then its complement* $F^C = \Omega \setminus F$ *also belongs to* $\mathcal{F}$*;*

*(iii) If* $A_1, A_2, \ldots \in \mathcal{F}$*, then*

$$A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

**Definition 1.1.2.** *The pair* $(\Omega, \mathcal{F})$ *is called a* measurable space*. A* probability measure $\mathbb{P}$ *on a measurable space* $(\Omega, \mathcal{F})$ *is a function*

$$\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$$

*such that*

*(i)* $\mathbb{P}(\emptyset) = 0$ *and* $\mathbb{P}(\Omega) = 1$*;*

*(ii) if* $A_1, A_2, \ldots \in \mathcal{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *are disjoint (i.e.,* $A_i \cap A_j = \emptyset$ *for* $i \neq j$*), then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

*The triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a* probability space*. It is said to be a* complete probability space *if* $\mathcal{F}$ *contains all subsets* $G \subset \Omega$ *with* $\mathbb{P}$*-outer measure zero, i.e.,*

$$\mathbb{P}^*(G) := \inf\{\mathbb{P}(F) : F \in \mathcal{F},\, G \subset F\} = 0.$$

**Remark 1.1.3.** *Any probability space can be made complete simply by adding to* $\mathcal{F}$ *all sets of outer measure zero and by extending* $\mathbb{P}$ *accordingly. From now on, we will assume that all our probability spaces are complete. The subsets* $F$ *of* $\Omega$ *which belong to* $\mathcal{F}$ *are called* $\mathcal{F}$-measurable sets*.*

**Definition 1.1.4.** *Let* $X$ *be a topological space, that is, a set endowed with a topology consisting of open subsets of* $X$*. The* Borel $\sigma$-algebra *on* $X$*, denoted by* $\mathcal{B}(X)$*, is the* $\sigma$*-algebra generated by all open subsets of* $X$*; equivalently,* $\mathcal{B}(X)$ *is the smallest* $\sigma$*-algebra on* $X$ *that contains every open subset of* $X$*.*

We now introduce the mathematical model corresponding to the random quantity, namely the *random variable*. In what follows, let $X$ denote a Banach space.

**Definition 1.1.5.** *A* random variable $\zeta : \Omega \to X$ *is a* $(\mathcal{F}, \mathcal{B}(X))$*-measurable function; that is,*

$$\zeta^{-1}(A) \in \mathcal{F}, \quad \forall A \in \mathcal{B}(X).$$

**Remark 1.1.6.** *Each random variable* $\zeta$ *induces a probability measure* $\eta_\zeta$ *on* $X$*, defined by*

$$\eta_\zeta(B) := \mathbb{P}(\zeta^{-1}(B)), \quad B \in \mathcal{B}(X).$$

*The measure* $\eta_\zeta$ *is called the* distribution *(or* law*) of* $\zeta$*.*

*If* $\int_\Omega |\zeta(\omega)|\, d\mathbb{P}(\omega) < \infty$*, then the quantity*

$$\mathbb{E}[\zeta] := \int_\Omega \zeta(\omega)\, d\mathbb{P}(\omega)$$

*is called the* expectation *(or* mean*) of* $\zeta$ *with respect to* $\mathbb{P}$*.*

**Definition 1.1.7.** *Let $\zeta : \Omega \to X$ be a square-integrable random variable, that is,*

$$\int_\Omega \|\zeta(\omega)\|_X^2 \, d\mathbb{P}(\omega) < \infty.$$

*The* variance *of $\zeta$, denoted by* $\mathrm{Var}(\zeta)$*, is defined by*

$$\mathrm{Var}(\zeta) := \mathbb{E}\big[\|\zeta - \mathbb{E}[\zeta]\|_X^2\big] .$$

**Definition 1.1.8.** *Two subsets $A, B \in \mathcal{F}$ are said to be* independent *if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

*A collection $\{\mathcal{F}_i\}_{i \in \mathcal{T}}$ of sub-$\sigma$-algebras $\mathcal{F}_i \subset \mathcal{F}$ is said to be* independent *if*

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k})$$

*for all choices of $A_{i_1} \in \mathcal{F}_{i_1}, \ldots, A_{i_k} \in \mathcal{F}_{i_k}$ with distinct indices $i_1, \ldots, i_k \in \mathcal{T}$.*

**Definition 1.1.9.** *Let $\zeta : \Omega \to X$ be a random variable. The* $\sigma$-algebra generated by $\zeta$, *denoted by $\mathcal{F}_\zeta$, is defined as*

$$\mathcal{F}_\zeta := \{\, \zeta^{-1}(A) : A \in \mathcal{B}(X) \,\}.$$

*That is, $\mathcal{F}_\zeta$ is the smallest $\sigma$-algebra on $\Omega$ with respect to which $\zeta$ is measurable.*

**Definition 1.1.10.** *A collection of random variables $\{\zeta_i\}_{i \in \mathcal{T}}$ is said to be* independent *if the collection of their generated $\sigma$-algebras $\{\mathcal{F}_{\zeta_i}\}_{i \in \mathcal{T}}$ is independent.*

**Definition 1.1.11.** *A* stochastic process *with values in $X$ is a collection of random variables $\{\zeta_t\}_{t \in \mathcal{T}}$, where each $\zeta_t : \Omega \to X$ for every $t \in \mathcal{T}$.*

The parameter space $\mathcal{T}$ is typically the half-line $[0, \infty)$, but it may also be the real line $\mathbb{R}$, an interval $[a, b]$, the set of integers $\mathbb{Z}$, the set of natural numbers $\mathbb{N}$, or more generally, a subset of $\mathbb{R}^n$ for some $n \geq 1$.

For each fixed $t \in \mathcal{T}$, we obtain a random variable

$$\omega \mapsto \zeta_t(\omega), \quad \omega \in \Omega.$$

Conversely, for each fixed $\omega \in \Omega$, we can view

$$t \mapsto \zeta_t(\omega), \quad t \in \mathcal{T},$$

as a realization of the process, called a *path* (or *trajectory*) of $\{\zeta_t\}_{t \in \mathcal{T}}$.

**Remark 1.1.12.** *It may be helpful, for intuition, to think of $t$ as "time" and each $\omega$ as an elementary outcome of the experiment. Under this interpretation, $\zeta_t(\omega)$ represents the state (or result) of the experiment $\omega$ at time $t$. In this sense, we can identify each $\omega$ with the function $t \mapsto \zeta_t(\omega)$ from $\mathcal{T}$ into $X$. Consequently, we may regard $\Omega$ as a subset of the space*

$$\tilde{\Omega} = X^{\mathcal{T}},$$

*which consists of all functions from $\mathcal{T}$ into $X$. The $\sigma$-algebra $\mathcal{F}$ then contains the $\sigma$-algebra $\mathcal{B}$ generated by sets of the form*

$$\{\, \omega : \omega(t_1) \in F_1, \, \ldots, \, \omega(t_k) \in F_k \,\}, \qquad F_i \subset X \text{ Borel sets,}$$

*where $\omega(t_i) = \zeta_{t_i}(\omega)$. Hence, one may also adopt the point of view that a stochastic process is a probability measure $\mathbb{P}$ on the measurable space $(X^{\mathcal{T}}, \mathcal{B})$.*

**Definition 1.1.13.** *A stochastic process $\{\zeta_t\}_{t\in\mathcal{T}}$ is called* stationary *if it has the same joint distribution as $\{\zeta_{t+h}\}$ for any $h > 0$.*

**Definition 1.1.14.** *Let $\{\zeta_t\}_{t\in\mathcal{T}}$ and $\{\xi_t\}_{t\in\mathcal{T}}$ be two stochastic processes. We say that $\{\zeta_t\}$ is a* version *of $\{\xi_t\}$ if*

$$\mathbb{P}\big(\{\omega \in \Omega : \zeta_t(\omega) = \xi_t(\omega)\}\big) = 1, \qquad \forall\, t \in \mathcal{T}.$$

*In other words, for each fixed $t \in \mathcal{T}$, the random variables $\zeta_t$ and $\xi_t$ coincide almost surely.*

We conclude this subsection by recalling a classical result from ergodic theory, whose proof can be found, for instance, in Walters [51].

**Theorem 1.1.15** (Poincaré recurrence theorem)**.** *Let $T : \Omega \to \Omega$ be a measurable transformation preserving the probability measure, that is,*

$$\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A), \quad \forall A \in \mathcal{F}.$$

*Then, for any measurable set $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, we have*

$$\mathbb{P}\Big(\big\{\omega \in A : T^n(\omega) \in A \text{ for infinitely many } n \in \mathbb{N}\big\}\Big) = \mathbb{P}(A).$$

*In particular, almost every point of $A$ returns to $A$ infinitely often.*

### 1.1.2 Brownian motion

In the early nineteenth century, the Scottish botanist Robert Brown observed that pollen grains suspended in water exhibited a continuous and erratic motion. Although previously noticed, the cause of this phenomenon was unclear. Brown's meticulous experiments showed that the movement was not biological but physical in nature, arising from the incessant collisions of fluid molecules with the suspended particles.

From a mathematical standpoint, Brownian motion is modeled by a stochastic process $\{W_t\}_{t\geq 0}$, known as the *Wiener process*.

**Definition 1.1.16.** *A* Wiener process *(or* standard Brownian motion*)* $\{W_t\}_{t\geq 0}$ *with values in* $\mathbb{R}$ *is a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *that satisfies* $\mathbb{P}$*-a.s.:*

*(i)* $W_0 = 0$*;*

*(ii) the trajectories* $t \mapsto W_t(\omega)$ *are continuous;*

*(iii) the increments are independent, that is, for any* $t_0 < t_1 < \cdots < t_n$ *and* $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$*, we have*

$$\mathbb{P}\big(W_{t_i} - W_{t_{i-1}} \in A_i,\ i = 1, \ldots, n\big) = \prod_{i=1}^{n} \mathbb{P}\big(W_{t_i} - W_{t_{i-1}} \in A_i\big);$$

*(iv) the process* $\{W_t\}_{t\geq 0}$ *it has stationary increments, which follow a Gaussian distribution;*

*(v) for all* $0 \leq s < t$*,* $W_t - W_s$ *is a Gaussian random variable with mean* $0$ *and variance* $t - s$*, i.e.*

$$\mathbb{P}(W_t - W_s \in A) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}\, dx, \quad A \in \mathcal{B}(\mathbb{R}).$$

The Wiener process possesses a number of important analytical and probabilistic properties, which will play a central role in the development of stochastic analysis. We summarize the most relevant ones below (see Arnold [1]).

**Proposition 1.1.17.** *Let* $\{W_t\}_{t\geq 0}$ *be a Wiener process in* $\mathbb{R}$*. Then the following properties hold:*

*(i)* $\mathbb{E}[W_t] = 0$ *and*

$$\mathbb{E}[W_t W_s] = \min(t, s),$$

*(ii)* *if $c \neq 0$ and $s \geq 0$, then the processes $-W_t$, $c^{-1}W_{c^2 t}$, and $W_{t+s} - W_s$ are also Wiener processes,*

*(iii)* ***(Strong law of large numbers)***

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \quad \mathbb{P}\text{-a.s.},$$

*(iv)* ***(Law of the iterated logarithm)***

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = 1, \qquad \liminf_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = -1, \quad \mathbb{P}\text{-a.s.}$$

*Consequently, for every $\varepsilon > 0$, there exists a random time $t_0(\omega)$ such that, for all $t > t_0(\omega)$,*

$$-(1+\varepsilon)\sqrt{2t\log\log t} < W_t(\omega) < (1+\varepsilon)\sqrt{2t\log\log t},$$

*(v)* ***(Local behavior)*** *almost every sample path satisfies*

$$\limsup_{t\to 0^+} \frac{W_t}{\sqrt{2t\log\log(1/t)}} = 1, \qquad \liminf_{t\to 0^+} \frac{W_t}{\sqrt{2t\log\log(1/t)}} = -1.$$

*Hence, with probability one, each path has infinitely many zeros in any interval $(0,\varepsilon)$, which accumulate at $t = 0$.*

**Remark 1.1.18.** *The local law of the iterated logarithm provides a sharper understanding of the oscillatory nature of Wiener paths. For almost every trajectory and any $0 < \varepsilon < 1$, as $h \to 0$,*

$$\frac{W_{t+h} - W_t}{\sqrt{2h\log\log(1/h)}} > (1-\varepsilon) \quad \textit{and} \quad \frac{W_{t+h} - W_t}{\sqrt{2h\log\log(1/h)}} < -(1-\varepsilon)$$

*infinitely often. As these bounds diverge to $+\infty$ and $-\infty$, respectively, the difference quotient*

$$\frac{W_{t+h} - W_t}{h}$$

*has, with probability one, the entire real line $\mathbb{R}$ as its set of cluster points for each fixed $t$. Consequently, almost all sample paths of $W_t$ are continuous but nowhere differentiable, crossing the time axis infinitely often in every neighborhood of any $t > 0$. This irregularity explains why the classical Riemann–Stieltjes integral cannot be used with $W_t$, motivating the development of the Itô integral and the framework of stochastic differential equations.*

### 1.1.3 Integration with respect to a Wiener process

As established in the previous discussion, almost all trajectories of a Wiener process are continuous but nowhere differentiable, and in particular, they possess unbounded variation on every finite interval. This fundamental irregularity prevents the interpretation of integrals with respect to $W_t$ in the classical Riemann–Stieltjes sense.

In the study of stochastic systems, however, one frequently encounters differential equations of the form

$$\frac{dy}{dt} = f(t, y(t)) + g(t, y(t))\dot{W}_t, \qquad \dot{W}_t = \frac{dW_t}{dt},$$

which are often written equivalently as

$$y(t) = y_0 + \int_0^t f(s, y(s))ds + \int_0^t g(s, y(s))\dot{W}_s ds.$$

Writing $\dot{W}_s ds = dW_s$, the equation becomes

$$y(t) = y_0 + \int_0^t f(s, y(s))ds + \int_0^t g(s, y(s))dW_s.$$

Since the trajectories of $W_t$ are of infinite total variation, the second integral in this expression cannot, in general, be defined through the classical calculus of Riemann or Riemann–Stieltjes type. Therefore, a new form of integration, the *Itô integral*, must be introduced. From now on, we refer to $\dot{W}$ as *white noise*.

**Definition 1.1.19.** *Given a Wiener process $\{W_t\}_{t\geqslant 0}$, we define for each $t \geqslant 0$ the $\sigma$-algebra*

$$\mathcal{F}_t = \sigma(W_s : 0 \leqslant s \leqslant t),$$

*that is, the smallest $\sigma$-algebra making $W_s$ measurable for all $s \leqslant t$. It is clear that $\{\mathcal{F}_t\}_{t\geqslant 0}$ forms an increasing family of sub-$\sigma$-algebras. We assume that $\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero. A stochastic process $\{\zeta_t\}_{t\geqslant 0}$ is said to be $\mathcal{F}_t$-*adapted *if, for every $t \geqslant 0$, the random variable $\zeta_t$ is $\mathcal{F}_t$-measurable.*

The construction of the stochastic integral with respect to a Wiener process is motivated by first defining it for a simple class of processes and then extending it by approximation. We begin with simple (or step) processes $\zeta : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ of the form

$$\zeta(t, \omega) = \sum_{j=0}^{n-1} e_j(\omega)\, \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\{e_j\}_{j=0}^{n-1}$ are bounded and $\mathcal{F}_{t_j}$-measurable random variables, and

$$S = t_0 < t_1 < \cdots < t_n = T$$

is a partition of the interval $[S, T]$. Note that, in particular, $\zeta$ is $\mathcal{F}_t$-adapted.

For such functions, we define

$$\left(\int_S^T \zeta(t)\, dW_t\right)(\omega) := \sum_{j=0}^{n-1} e_j(\omega)\big(W_{t_{j+1}} - W_{t_j}\big)(\omega).$$

The real-valued random variable defined above is called the *stochastic integral* of $\zeta$ on $[S, T]$ with respect to the Wiener process $W_t$.

In general, for an $\mathcal{F}_t$-adapted process $\zeta : \mathbb{R}^+ \times \Omega \to \mathbb{R}$, it is natural to approximate $\phi$ by step functions $\widetilde{\zeta}$ of the form

$$\widetilde{\zeta}(t, \omega) = \sum_{j=0}^{n-1} \zeta(t_j^*, \omega)\, \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where each evaluation point $t_j^*$ belongs to the interval $[t_j, t_{j+1}]$. The stochastic integral

$$\int_S^T \zeta(t, \omega)\, dW_t$$

is then defined as the limit (in a sense to be specified later) of the corresponding integrals

$$\left(\int_S^T \widetilde{\zeta}(t)\, dW_t\right)(\omega) = \sum_{j=0}^{n-1} \zeta(t_j^*, \omega)\big(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\big),$$

as the mesh of the partition tends to zero.

The choice of the evaluation points $t_j^*$ leads to different interpretations of the stochastic integral (see, e.g., Oksendal [38]). We highlight the two most important cases:

- if $t_j^* = t_j$, we obtain the *Itô integral*, denoted by

$$\int_S^T \zeta(t)\, dW_t,$$

- if $t_j^* = \frac{t_j + t_{j+1}}{2}$, we obtain the *Stratonovich integral*, denoted by

$$\int_S^T \zeta(t) \circ dW_t.$$

To rigorously define the Itô integral, we first restrict ourselves to a suitable class of functions for which the integral is well defined.

**Definition 1.1.20.** *We denote by $\mathfrak{L}^2(S,T)$ the class of functions*

$$\zeta : [S,T] \times \Omega \to \mathbb{R}$$

*satisfying the following properties:*

*(i) the mapping $(t,\omega) \mapsto \zeta(t,\omega)$ is $\mathcal{B}([S,T]) \otimes \mathcal{F}$-measurable;*

*(ii) $\zeta$ is $\mathcal{F}_t$-adapted;*

*(iii) $\zeta$ is square-integrable, that is,*

$$\mathbb{E}\left[\int_S^T |\zeta(t,\omega)|^2 \, dt\right] < \infty.$$

We can now define, for every $\zeta \in \mathfrak{L}^2(S,T)$, its stochastic integral with respect to $W_t$ in the sense of Itô. The construction proceeds in several steps.

**Step 1.** For an *elementary* function $\phi \in \mathfrak{L}^2(S,T)$, that is,

$$\phi(t,\omega) = \sum_{j=0}^{n-1} e_j(\omega) \, \mathbf{1}_{[t_j,t_{j+1})}(t),$$

where $e_j : \Omega \to \mathbb{R}$ are $\mathcal{F}_{t_j}$-measurable random variables and

$$S = t_0 < t_1 < \cdots < t_n = T,$$

we define

$$\left(\int_S^T \phi(t) \, dW_t\right)(\omega) := \sum_{j=0}^{n-1} e_j(\omega)\big(W_{t_{j+1}} - W_{t_j}\big)(\omega).$$

Note that $\int_S^T \phi(t) \, dW_t \in L^2(\Omega, \mathbb{P}; \mathbb{R})$.

**Step 2.** Let $g \in \mathfrak{L}^2(S,T)$ be bounded, and assume that $g(\cdot,\omega)$ is continuous for each $\omega \in \Omega$. Then there exists a sequence of elementary functions $\{\phi_n\} \subset \mathfrak{L}^2(S,T)$ such that

$$\mathbb{E}\left[\int_S^T (g - \phi_n)^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

It can be shown that the limit of the corresponding stochastic integrals exists in $L^2(\Omega, \mathbb{P}; \mathbb{R})$.

Where $L^2(\Omega, \mathbb{P}; \mathbb{R})$ is the space of square-integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathbb{R}$

We then define

$$\int_S^T g(t) \, dW_t := \lim_{n\to\infty} \int_S^T \phi_n(t) \, dW_t.$$

This definition does not depend on the choice of the approximating sequence $\{\phi_n\}$.

**Step 3.** Let $h \in \mathfrak{L}^2(S,T)$ be bounded. Then there exists a sequence of bounded functions $\{g_n\} \subset \mathfrak{L}^2(S,T)$ such that each $g_n(\cdot,\omega)$ is continuous for all $\omega \in \Omega$ and $n \in \mathbb{N}$, and

$$\mathbb{E}\left[\int_S^T (h-g_n)^2\,dt\right] \to 0 \quad \text{as } n \to \infty.$$

Again, the limit of the corresponding stochastic integrals exists in $L^2(\Omega,\mathbb{P};\mathbb{R})$ and does not depend on the particular choice of $\{g_n\}$. Hence, we define

$$\int_S^T h(t)\,dW_t := \lim_{n\to\infty}\int_S^T g_n(t)\,dW_t.$$

**Step 4.** Let $\zeta \in \mathfrak{L}^2(S,T)$. Then there exists a sequence $\{h_n\} \subset \mathfrak{L}^2(S,T)$ such that each $h_n$ is bounded and

$$\mathbb{E}\left[\int_S^T (\zeta-h_n)^2\,dt\right] \to 0 \quad \text{as } n \to \infty.$$

Once more, the limit of the corresponding stochastic integrals exists in $L^2(\Omega,\mathbb{P};\mathbb{R})$, and we define

$$\int_S^T \zeta(t)\,dW_t := \lim_{n\to\infty}\int_S^T h_n(t)\,dW_t.$$

The following result summarizes the basic properties of the Itô integral.

**Proposition 1.1.21.** *Let $f,g \in \mathfrak{L}^2(0,T)$, $0 \leqslant S < U < T$ and $c \in \mathbb{R}$. Then, $\mathbb{P}$-a.s.,*

*(i)* $\displaystyle\int_S^T f(t)\,dW_t = \int_S^U f(t)\,dW_t + \int_U^T f(t)\,dW_t,$

*(ii)* $\displaystyle\int_S^T (cf(t)+g(t))\,dW_t = c\int_S^T f(t)\,dW_t + \int_S^T g(t)\,dW_t,$

*(iii)* $\displaystyle\mathbb{E}\left[\int_S^T f(t)\,dW_t\right] = 0,$

*(iv)* $\displaystyle\int_S^T f(t)\,dW_t$ *is $\mathcal{F}_T$-measurable.*

One of the most fundamental properties of the Itô integral is that it defines a martingale.

**Definition 1.1.22.** *A stochastic process $M = \{M_t\}_{t\in[0,T]}$ is called an $\{\mathcal{F}_t\}$-*martingale *if:*

*(i) for each $t \in [0,T]$, $M_t$ is $\mathcal{F}_t$-measurable and integrable (i.e., $\mathbb{E}[|M_t|] < \infty$),*

*(ii) for all $0 \leqslant s \leqslant t \leqslant T$,*

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s \quad \mathbb{P}\text{-a.s.}$$

*Here,* $\mathbb{E}[M_t \mid \mathcal{F}_s]$ *denotes the conditional expectation with respect to the* $\sigma$*-algebra* $\mathcal{F}_s$*, that is, the (a.s. unique)* $\mathcal{F}_s$*-measurable random variable satisfying*

$$\int_A M_t \, d\mathbb{P} = \int_A \mathbb{E}[M_t \mid \mathcal{F}_s] \, d\mathbb{P}, \quad \forall A \in \mathcal{F}_s.$$

We now state an important result concerning the martingale property of the Itô integral (see, Oksendal [38]).

**Theorem 1.1.23.** *Let* $\zeta \in \mathfrak{L}^2(0,T)$*. Then there exists a continuous version of*

$$\int_0^t \zeta(s) \, dW_s, \qquad 0 \leqslant t \leqslant T,$$

*that is, a continuous stochastic process* $M : [0,T] \times \Omega \to \mathbb{R}$ *such that* $\mathbb{P}$*-a.s.,*

$$M_t = \int_0^t \zeta(s) \, dW_s, \quad \forall t \in [0,T].$$

*Moreover,* $\{M_t\}$ *is a square-integrable* $\{\mathcal{F}_t\}$*-martingale and satisfies Doob's inequality: for all* $\alpha > 0$*,*

$$\mathbb{P}\left[\sup_{0 \leqslant s \leqslant T} |M_s| > \alpha\right] \leqslant \frac{1}{\alpha^2} \, \mathbb{E}[M_T^2].$$

The basic definition of the Itô integral, although mathematically rigorous, is not particularly convenient for explicit computations. As in classical calculus, explicit evaluations rely not on the original definition but on higher-level results such as the *Itô formula*, which plays a role analogous to the fundamental theorem of calculus combined with the chain rule.

The essential difference between the Itô and Stratonovich integrals lies in their interpretation and analytical properties. The Itô integral is *nonanticipative*: the integrand depends only on past information, making it suitable for modeling causal systems such as those arising in finance, control theory, and population dynamics. It also preserves the martingale property, which provides strong analytical and computational advantages. In contrast, the Stratonovich integral preserves the classical chain rule and is therefore more natural in physical and engineering systems, where stochastic perturbations often appear as continuous limits of deterministic effects.

In what follows, we present the *Itô formula*, the stochastic analogue of the classical chain rule, which serves as a fundamental tool for transforming and evaluating Itô integrals.

**Theorem 1.1.24** (Itô formula)**.** *Let $y : [0,T] \times \Omega \to \mathbb{R}$ be a stochastic process defined by*

$$y(t) = y_0 + \int_0^t f(s)\, ds + \int_0^t g(s)\, dW_s, \tag{1.1.1}$$

*where $g \in \mathfrak{L}^2(0,T)$ and $f$ is an integrable $\mathcal{F}_t$-adapted process, $\mathbb{P}$-a.s. Let $v \in C^2([0,T] \times \mathbb{R})$. Then the process $Y(t) = v(t, y(t))$ satisfies*

$$\begin{aligned} Y(t) \overset{\mathbb{P}}{=} Y_0 &+ \int_0^t \frac{\partial v}{\partial t}(s, y(s))\, ds + \int_0^t \frac{\partial v}{\partial x}(s, y(s)) f(s)\, ds \\ &+ \int_0^t \frac{\partial v}{\partial x}(s, y(s)) g(s)\, dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 v}{\partial x^2}(s, y(s))\, (g(s))^2\, ds. \end{aligned} \tag{1.1.2}$$

**Remark 1.1.25.** *Equation (1.1.1) can be formally written in differential form as*

$$dy(t) = f(t)\, dt + g(t)\, dW_t.$$

*Consequently, the Itô formula (1.1.2) can also be expressed in differential form as*

$$dY(t) = \frac{\partial v}{\partial t}(t, y(t))\, dt + \frac{\partial v}{\partial x}(t, y(t))\, dy(t) + \tfrac{1}{2} \frac{\partial^2 v}{\partial x^2}(t, y(t))\, (dy(t))^2.$$

*Substituting $dy(t) = f(t)\, dt + g(t)\, dW_t$ and using the Itô rule $(dW_t)^2 = dt$, we obtain*

$$dY(t) = \left[ \frac{\partial v}{\partial t}(t, y(t)) + f(t) \frac{\partial v}{\partial x}(t, y(t)) + \tfrac{1}{2} (g(t))^2 \frac{\partial^2 v}{\partial x^2}(t, y(t)) \right] dt + g(t) \frac{\partial v}{\partial x}(t, y(t))\, dW_t.$$

We conclude this section with an example illustrating the application of Itô's formula to the explicit solution of stochastic differential equations.

**Example 1.1.26** (Ornstein–Uhlenbeck equation)**.** *Consider the stochastic differential equation*

$$dy(t) = \sigma\, y(t)\, dt + \lambda\, dW_t, \qquad y(0) = y_0 \in \mathbb{R},$$

*where $\sigma, \lambda \in \mathbb{R}$ are constants and $W_t$ is a one-dimensional Wiener process.*

*To solve the equation, consider the change of variable*

$$v(t, x) = e^{-\sigma t} x, \qquad Y(t) := v(t, y(t)) = e^{-\sigma t} y(t).$$

*Applying the Itô formula (cf. Remark 1.1.25), we obtain*

$$dY(t) = v_t(t, y(t))\, dt + v_x(t, y(t))\, dy(t) + \tfrac{1}{2} v_{xx}(t, y(t))\, (dy(t))^2.$$

*Since $v_t = -\sigma e^{-\sigma t} x$, $v_x = e^{-\sigma t}$, and $v_{xx} = 0$, we deduce*

$$dY(t) = -\sigma e^{-\sigma t} y(t)\, dt + e^{-\sigma t} \big( \sigma y(t)\, dt + \lambda\, dW_t \big) = \lambda e^{-\sigma t}\, dW_t.$$

*Integrating from* $0$ *to* $t$ *in the Itô sense yields*

$$Y(t) = Y(0) + \lambda \int_0^t e^{-\sigma s}\, dW_s.$$

*Since* $Y(0) = y_0$, *we obtain*

$$e^{-\sigma t} y(t) = y_0 + \lambda \int_0^t e^{-\sigma s}\, dW_s,$$

*and therefore,*

$$y(t) = e^{\sigma t} y_0 + \lambda \int_0^t e^{\sigma(t-s)}\, dW_s.$$

*This is the unique strong solution of the Ornstein–Uhlenbeck equation.*

## 1.2 Random attractors for random dynamical systems

This section aims to introduce some basic concepts and results concerning the theory of attractors for random dynamical systems. Subsequently, we present a result on the upper semicontinuity of random attractors with respect to a certain parameter $\varepsilon$, which appears in the stochastic perturbation of the differential equation.

### 1.2.1 Random attractor

The notion of random attractors was first introduced in the pioneering work of Crauel and Flandoli [17], as a generalization of the concept of global attractors for deterministic dynamical systems, as studied, for instance, by Hale [25], Temam [45], and Robinson [43]. Our aim in this subsection is to define this concept and recall some fundamental results concerning random attractors for random dynamical systems. The presentation follows closely the approach proposed by Wang [47].

We begin by recalling the notion of a random dynamical system that will be used throughout this work.

**Definition 1.2.1.** *Let* $\{\vartheta_t\}_{t\in\mathbb{R}}$ *be a family of mappings on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying the following properties:*

*(i) The mapping* $(t, \omega) \mapsto \vartheta_t \omega$ *is* $(\mathcal{B} \otimes \mathcal{F}, \mathcal{F})$*-measurable.*

*(ii)* *For each* $t \in \mathbb{R}$, $\vartheta_t : \Omega \to \Omega$ *preserves the probability measure, that is,*

$$\mathbb{P}(\vartheta_t^{-1} A) = \mathbb{P}(A), \quad \forall A \in \mathcal{F}, \ \forall t \in \mathbb{R}.$$

*(iii)* $\{\vartheta_t\}_{t\in\mathbb{R}}$ *has a group structure on* $\Omega$*, that is,*

*(*$i_a$*)* $\vartheta_0 = I$ *(the identity on* $\Omega$*),*

*(*$ii_a$*)* $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$*, for all* $t, s \in \mathbb{R}$.

*Endowing* $\Omega$ *with a metric and considering the Borel* $\sigma$*-algebra generated by the metric topology. Then* $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ *is called a* metric dynamical system.

**Example 1.2.2** (Wiener process)**.** *Let* $\{W_t\}_{t\in\mathbb{R}}$ *be a one-dimensional Wiener process. We define the underlying probability space as follows*

$$\Omega = \{\omega \in C(\mathbb{R}^+, \mathbb{R}) : \omega(0) = 0\},$$

*that is,* $\Omega$ *is the space of all real-valued continuous functions on* $\mathbb{R}$ *vanishing at the origin, endowed with the compact-open topology induced by the complete metric*

$$d(\omega, \omega') := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|\omega - \omega'\|_m}{1 + \|\omega - \omega'\|_m}, \quad \textit{with } \|\omega - \omega'\|_m := \sup_{-m \leqslant t \leqslant m} |\omega(t) - \omega'(t)|.$$

*Let* $\mathcal{F}$ *denote the Borel* $\sigma$*-algebra on* $\Omega$ *generated by the compact-open topology, and let* $\mathbb{P}$ *be the Wiener measure on* $(\Omega, \mathcal{F})$*, under which the canonical process*

$$W_t(\omega) := \omega(t), \qquad t \in \mathbb{R}, \ \omega \in \Omega,$$

*is a standard Wiener process.*

*Next, for each* $t \in \mathbb{R}$*, we define the family of mappings*

$$\vartheta_t : \Omega \to \Omega, \qquad (\vartheta_t \omega)(s) := \omega(s + t) - \omega(t), \quad \forall\, s \in \mathbb{R}.$$

*Intuitively, the mapping* $\vartheta_t$ *shifts the sample path* $\omega$ *by* $t$ *units and re-centers it so that the shifted path again starts at zero.*

*We now verify that* $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ *defines a metric dynamical system:*

*(i)* Measurability: *The mapping* $(t, \omega) \mapsto \vartheta_t \omega$ *is* $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$*-measurable since the shift operator is continuous on* $C(\mathbb{R}, \mathbb{R})$ *equipped with the compact-open topology.*

*(ii)* Measure preservation: *For every $t \in \mathbb{R}$ and $A \in \mathcal{F}$,*

$$\mathbb{P}(\vartheta_t^{-1} A) = \mathbb{P}(A),$$

*because the Wiener process has stationary increments, and hence the Wiener measure is invariant under time shifts.*

*(iii)* Group property: *The family $\{\vartheta_t\}_{t\in\mathbb{R}}$ forms a one-parameter group of transformations on $\Omega$:*

$$\vartheta_0 = I, \qquad \vartheta_{t+s} = \vartheta_t \circ \vartheta_s, \quad \forall\, t, s \in \mathbb{R}.$$

*Thus, the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system associated with the Wiener process. This construction provides the canonical framework for defining random dynamical systems driven by Brownian motion.*

In what follows, let $X$ be a Banach space and $2^X$ denote the collection of all subsets of $X$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system.

**Definition 1.2.3.** *A mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ is called a* random dynamical system *(RDS) on $X$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ if the following conditions hold:*

*(i)* $\Phi$ *is $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable,*

*(ii)* $\Phi(0, \omega, x) = x$ *for all $\omega \in \Omega$ and $x \in X$ (identity property),*

*(iii)* $\Phi$ *satisfies the* cocycle property*:*

$$\Phi(t + s, \omega, x) = \Phi\big(t, \vartheta_s\omega, \Phi(s, \omega, x)\big), \quad \forall\, t, s \in \mathbb{R}^+,\, \omega \in \Omega,\, x \in X.$$

*The RDS is said to be* continuous *if, for every fixed $t \in \mathbb{R}^+$ and $\omega \in \Omega$, the map $x \mapsto \Phi(t, \omega, x)$ is continuous on $X$.*

A fundamental notion in the theory of random dynamical systems is that of a *random set*.

**Definition 1.2.4.** *A set-valued mapping $K : \Omega \to 2^X$ is said to be* measurable *if:*

*(i) for every $\omega \in \Omega$, the value $K(\omega)$ is a nonempty closed subset of $X$;*

*(ii) for each fixed $x \in X$, the mapping $\omega \mapsto d\big(x, K(\omega)\big)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, where $d(x, K(\omega)) := \inf_{y \in K(\omega)} \|x - y\|_X$.*

**Definition 1.2.5.** *The* Hausdorff semi-distance *between two subsets $A, B \subset X$ is defined by*

$$\operatorname{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X.$$

**Definition 1.2.6.** *Let $\mathcal{D}$ denote a collection of families of nonempty subsets of $X$ indexed by $\omega \in \Omega$. Each element $D \in \mathcal{D}$ is a family $D = \{D(\omega)\}_{\omega \in \Omega}$, where each $D(\omega)$ is a nonempty subset of $X$. We write*

$$\mathcal{D} = \big\{ \, D = \{D(\omega)\}_{\omega \in \Omega} : D \textit{ satisfies some conditions} \, \big\}.$$

**Remark 1.2.7.** *The collection $\mathcal{D}$ typically describes the class of random sets from which attraction toward the random attractor is required. Depending on the context, $\mathcal{D}$ may represent, for instance, all tempered random bounded sets or all closed random subsets of $X$ possessing specific decay properties.*

**Definition 1.2.8.** *A family of random sets $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega \in \Omega} \subset \mathcal{D}$ is called a $\mathcal{D}$-*random attractor *for the RDS $\Phi$ if:*

*(i) $\mathcal{A}$ is measurable and each $\mathcal{A}(\omega)$ is compact in $X$,*

*(ii) $\mathcal{A}$ is invariant, that is,*

$$\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\vartheta_t \omega), \quad \forall \, t \in \mathbb{R}^+, \, \omega \in \Omega,$$

*(iii) $\mathcal{A}$ attracts every element of $\mathcal{D}$ in the pullback sense, that is, for all $D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$,*

$$\lim_{t \to \infty} \operatorname{dist}_X \big( \Phi(t, \vartheta_{-t}\omega, D(\vartheta_{-t}\omega)), \, \mathcal{A}(\omega) \big) = 0.$$

To establish the existence of a random attractor, we first introduce two key notions: a *$\mathcal{D}$-pullback absorbing set* and *$\mathcal{D}$-pullback asymptotic compactness*.

**Definition 1.2.9.** *A random set $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ is called a $\mathcal{D}$-*pullback absorbing set *for $\Phi$ if, for every $D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and every $\omega \in \Omega$, there exists a time $T_D(\omega) > 0$ such that*

$$\Phi(t, \vartheta_{-t}\omega, D(\vartheta_{-t}\omega)) \subset B(\omega), \quad \forall \, t \geqslant T_D(\omega).$$

*Moreover, $B$ is said to be* positively invariant *if*

$$\Phi(t, \omega, B(\omega)) \subseteq B(\vartheta_t \omega), \quad \forall \, t \geqslant 0.$$

**Definition 1.2.10.** *The random dynamical system $\Phi$ is said to be $\mathcal{D}$-*pullback asymptotically compact *in $X$ if, for every $\omega \in \Omega$, every $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, and any sequence $t_n \to \infty$ with $x_n \in D(\vartheta_{-t_n}\omega)$, the sequence $\{\Phi(t_n, \vartheta_{-t_n}\omega, x_n)\}$ has a convergent subsequence in $X$.*

The following result, adapted from Wang [47], provides sufficient conditions for the existence of a random attractor.

**Theorem 1.2.11.** *Let $\Phi$ be a continuous RDS on $X$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$. Suppose that there exists a closed $\mathcal{D}$-pullback absorbing set $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and that $\Phi$ is $\mathcal{D}$-pullback asymptotically compact in $X$. Then $\Phi$ admits a unique $\mathcal{D}$-random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$, given by*

$$\mathcal{A}(\omega) = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \Phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}, \quad \omega \in \Omega.$$

### 1.2.2 Upper semicontinuity of random attractors

We consider an abstract stochastic evolution equation in a Hilbert space $X$ of the form

$$du_\varepsilon(t) + \big(Au_\varepsilon(t) + F(u_\varepsilon(t))\big)\,dt = \varepsilon h\,dW_t, \qquad t > 0, \tag{1.2.1}$$

where $\varepsilon > 0$ is a small parameter representing the intensity of the stochastic perturbation, $A : D(A) \subset X \to X$ is a linear operator generating a strongly continuous semigroup, $F : X \to X$ is a nonlinear mapping satisfying suitable continuity and dissipativity assumptions, $h \in X$ is a fixed nonzero vector describing the spatial structure of the additive noise, and $W_t$ is a real-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Under appropriate conditions, equation (1.2.1) generates a random dynamical system

$$\Phi_\varepsilon : \mathbb{R}^+ \times \Omega \times X \to X, \qquad \Phi_\varepsilon(t, \omega, x_0) = u_\varepsilon(t, \omega; x_0),$$

whose trajectories describe the evolution of the system under additive stochastic perturbations. For each fixed $\varepsilon > 0$, we assume that $\Phi_\varepsilon$ possesses a random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega}$.

In the limiting case $\varepsilon = 0$, the stochastic term vanishes and (1.2.1) reduces to the deterministic evolution equation

$$\frac{du_0}{dt} + Au_0 + F(u_0) = 0, \tag{1.2.2}$$

which generates a deterministic semiflow $\Phi_0$ on $X$ possessing a global attractor $\mathcal{A}_0 \subset X$. That is, $\mathcal{A}_0$ is compact, invariant under $\Phi_0$ (i.e., $\Phi_0(t)\mathcal{A}_0 = \mathcal{A}_0$ for all $t \geqslant 0$), and uniformly attracting: for every bounded set $B \subset X$,

$$\lim_{t \to \infty} \operatorname{dist}_X(\Phi_0(t)B, \mathcal{A}_0) = 0.$$

A fundamental question concerns the relationship between the random attractors $\mathcal{A}_\varepsilon$ of the perturbed systems and the deterministic attractor $\mathcal{A}_0$ of the limiting system. Specifically, we seek to determine whether the random attractors converge to $\mathcal{A}_0$ as $\varepsilon \to 0$. This property, known as the *upper semicontinuity of random attractors*, requires that for each $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \operatorname{dist}_X\big(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0\big) = 0.$$

This convergence is of significant theoretical and practical interest, as it describes the robustness of the asymptotic dynamics with respect to small random perturbations.

The upper semicontinuity property ensures that the long-term behavior of the stochastic system (1.2.1) remains stable under vanishing noise. In particular, as the noise intensity $\varepsilon$ tends to zero, the random attractor $\mathcal{A}_\varepsilon$ continuously approaches the deterministic attractor $\mathcal{A}_0$, thus establishing the persistence of the asymptotic structure of the deterministic system.

To establish the upper semicontinuity of random attractors as $\varepsilon \to 0$, we employ a general criterion due to Wang [46].

**Proposition 1.2.12.** *Let $\Phi_\varepsilon$ be a continuous random dynamical system (RDS) on a Banach space $X$, defined over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and let $\mathcal{D}$ denote a collection of families of nonempty subsets of $X$. Assume that the following conditions hold:*

(A1) *For each $\varepsilon > 0$, the RDS $\Phi_\varepsilon$ admits a $\mathcal{D}$-random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, and a $\mathcal{D}$-pullback absorbing set $B_\varepsilon = \{B_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that, for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\limsup_{\varepsilon \to 0} \|B_\varepsilon(\omega)\|_X \leqslant c,$$

*for some deterministic constant $c > 0$, where $\|B\|_X := \sup_{x\in B} \|x\|_X$.*

(A2) *There exists $\varepsilon_0 > 0$ such that, for $\mathbb{P}$-almost every $\omega \in \Omega$, the set*

$$\overline{\bigcup_{\varepsilon\in(0,\varepsilon_0]} \mathcal{A}_\varepsilon(\omega)}^{X}$$

*is compact in $X$.*

(A3) *For* $\mathbb{P}$*-almost every* $\omega \in \Omega$*, for all* $t \geqslant 0$*, and for any sequences* $\varepsilon_n \to 0$ *and* $x_n \to x$ *in* $X$*, we have*

$$\lim_{n\to\infty} \|\Phi_{\varepsilon_n}(t,\omega,x_n) - \Phi_0(t)x\|_X = 0.$$

(A4) *The deterministic dynamical system* $\Phi_0$ *has a unique compact global attractor* $\mathcal{A}_0 \subset X$.

*Then the family of random attractors* $\{\mathcal{A}_\varepsilon\}_{\varepsilon>0}$ *is upper semicontinuous at* $\varepsilon = 0$*, that is, for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*,*

$$\lim_{\varepsilon\to 0} \operatorname{dist}_X\big(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0\big) = 0.$$

# Chapter 2

# Stochastic piezoelectric system

In this chapter, we investigate the well-posedness of the stochastic piezoelectric system (0.0.3)-(0.0.4). For convenience, we rewrite the system below:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + v_t + f_1(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + p_t + f_2(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ d\theta - \kappa\theta_{xx}dt + \delta v_{xt}dt = \varepsilon\psi(x)dW_t & \text{in } (0,L)\times\mathbb{R}^+, \end{cases} \tag{2.0.1}$$

where $f_1$ and $f_2$ denote nonlinear internal forces, $\varepsilon > 0$ measures the intensity of the stochastic perturbation, and $W_t$ is a one-dimensional two-sided Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The boundary and initial conditions are given by

$$\begin{cases} v(0,t) = 0, \quad \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t > 0, \\ p(0,t) = 0, \quad p_x(L,t) - \gamma v_x(L,t) = 0, & t > 0, \\ \theta(0,t) = \theta(L,t) = 0, & t > 0, \\ v(x,0) = v_0, \quad v_t(x,0) = v_1, & x \in (0,L), \\ p(x,0) = p_0, \quad p_t(x,0) = p_1, & x \in (0,L), \\ \theta(x,0) = \theta_0(x). & x \in (0,L). \end{cases} \tag{2.0.2}$$

The main objective of this chapter is to establish the well-posedness of (2.0.1)-(2.0.2). To this end, we first introduce the appropriate functional setting and structural assumptions on the nonlinear terms. Then, we apply a pathwise transformation to convert the stochastic system into a family of deterministic problems depending on $\omega \in \Omega$. Finally, we prove existence and uniqueness of solutions to the resulting deterministic system using semigroup methods.

## 2.1 Well-posedness

We now formulate the problem in an appropriate Hilbert space and introduce structural assumptions on the nonlinear terms.

### 2.1.1 Phase space and assumptions

For each $p \geq 1$ and $m > 0$, let $L^p(0, L)$ and $H^m(0, L)$ denote the standard Lebesgue and Sobolev spaces, respectively. The norm in $L^p(0, L)$ will be denoted by $\| \cdot \|_p$. Define

$$H^1_*(0, L) = \{u \in H^1(0, L) : u(0) = 0\}.$$

Due to the boundary condition $u(0) = 0$, the Poincaré inequality holds:

$$\lambda_0 \|u\|_2^2 \leq \|u_x\|_2^2, \quad \forall\, u \in H^1_*(0, L), \tag{2.1.1}$$

which implies that $\|u\|_{H^1_*(0,L)} := \|u_x\|_2$ is an equivalent norm on $H^1_*(0, L)$.

The phase space of the system is given by

$$\mathcal{H} = H^1_*(0, L) \times H^1_*(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2(0, L), \tag{2.1.2}$$

equipped with the norm

$$\|(v, p, \tilde{v}, \tilde{p}, \varphi)\|_{\mathcal{H}}^2 = \rho \, \|\tilde{v}\|_2^2 + \mu \, \|\tilde{p}\|_2^2 + \|\varphi\|_2^2 + \alpha_1 \, \|v_x\|_2^2 + \beta \, \|\gamma v_x - p_x\|_2^2. \tag{2.1.3}$$

We now impose structural hypotheses on the nonlinear terms and on the spatial structure of the noise in (2.0.1):

**(H1)** There exists a $C^1$-potential $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\nabla F(v, p) = \big(f_1(v, p), \; f_2(v, p)\big). \tag{2.1.4}$$

**(H2)** There exist $r \geq 1$ and $c_f > 0$ such that, for $i = 1, 2$,

$$\big|\nabla f_i(v, p)\big| \leq c_f \left(1 + |v|^{r-1} + |p|^{r-1}\right), \qquad \forall (v, p) \in \mathbb{R}^2. \tag{2.1.5}$$

**(H3)** There exists $m_f > 0$ such that

$$-m_f \leq F(v,p) \leq \nabla F(v,p) \cdot (v,p) + m_f, \qquad \forall (v,p) \in \mathbb{R}^2. \tag{2.1.6}$$

**(H4)** The deterministic function $\psi$ belongs to the space $H^2(0,L) \cap H_0^1(0,L)$.

### 2.1.2 Random Cauchy problem

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ be the metric dynamical system generated by the Wiener process $W_t$ (see Example 1.2.2). For $\sigma > 0$ and $\omega \in \Omega$, consider the stationary solution of the one-dimensional Ornstein-Uhlenbeck equation (see Example 1.1.26), given by

$$z(\vartheta_t\omega) := -\int_{-\infty}^{t} e^{-\sigma(t-\tau)}\, dW_\tau, \quad t \in \mathbb{R}. \tag{2.1.7}$$

Then $z(\vartheta_t\omega)$ satisfies

$$dz(\vartheta_t\omega) + \sigma\, z(\vartheta_t\omega)\, dt = dW_t. \tag{2.1.8}$$

The following lemma summarizes some standard properties of the stationary Ornstein-Uhlenbeck process. A proof can be found in Arnold [1].

**Lemma 2.1.1.** *There exists a full-measure invariant set $\widetilde{\Omega} \subset \Omega$ such that for every $\omega \in \widetilde{\Omega}$, the map $t \mapsto z(\vartheta_t\omega)$ is continuous and the following limits hold:*

$$\lim_{t\to\pm\infty} \frac{z(\vartheta_t\omega)}{t} = 0, \qquad \lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^{0} z(\vartheta_s\omega)\, ds = 0, \tag{2.1.9}$$

*and for every $\xi > 0$,*

$$\mathbb{E}\big(|z(\vartheta_t\omega)|^\xi\big) = \frac{\Gamma\left(\frac{1+\xi}{2}\right)}{\sqrt{\pi}\,\sigma^{\xi/2}}, \qquad t \in \mathbb{R}, \tag{2.1.10}$$

*where $\Gamma(\cdot)$ denotes the Gamma function.*

**Remark 2.1.2.** *In what follows, we do not distinguish between $\widetilde{\Omega}$ and $\Omega$.*

Let us define

$$\varphi := \theta - \varepsilon\, \psi\, z, \qquad z := z(\vartheta_t\omega), \tag{2.1.11}$$

where $\theta$ solves (2.0.1) and $z$ is the stationary Ornstein-Uhlenbeck process solving (2.1.8). A straightforward (formal) computation shows that the original system (2.0.1)-(2.0.2) can be rewritten as

$$
\begin{cases}
\rho\, v_{tt} - \alpha v_{xx} + \gamma\beta\, p_{xx} + \delta\, \varphi_x + v_t + f_1(v,p) = -\,\delta\,\varepsilon\,\psi_x\, z & \text{in } (0,L)\times\mathbb{R}^+,\\[2ex]
\mu\, p_{tt} - \beta p_{xx} + \gamma\beta\, v_{xx} + p_t + f_2(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+,\\[2ex]
\varphi_t - \kappa\,\varphi_{xx} + \delta\, v_{xt} = \sigma\varepsilon\,\psi\, z + \kappa\varepsilon\,\psi_{xx}\, z & \text{in } (0,L)\times\mathbb{R}^+.
\end{cases}
\tag{2.1.12}
$$

subject to the boundary and initial conditions

$$
\begin{cases}
v(0,t)=0, \quad \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t>0,\\[2ex]
p(0,t)=0, \quad p_x(L,t) - \gamma v_x(L,t) = 0, & t>0,\\[2ex]
\varphi(0,t) = \varphi(L,t) = 0, & t>0,\\[2ex]
v(x,0)=v_0(x), \quad v_t(x,0) = v_1(x), & x\in(0,L),\\[2ex]
p(x,0)=p_0(x), \quad p_t(x,0) = p_1(x), & x\in(0,L),\\[2ex]
\varphi(x,0) = \varphi_0(x) = \theta_0(x) - \varepsilon\,\psi(x)\, z(\omega), & x\in(0,L).
\end{cases}
\tag{2.1.13}
$$

Now, set the state variable $U = (v, p, \tilde{v}, \tilde{p}, \varphi) \in \mathcal{H}$, with $\tilde{v} := v_t$ and $\tilde{p} := p_t$. Then (2.1.12)-(2.1.13) can be written as the random abstract Cauchy problem

$$
\frac{dU}{dt} = \mathrm{A}U + \mathrm{F}(\vartheta_t\omega, U), \qquad U(0) = U_0 := (v_0, p_0, v_1, p_1, \varphi_0) \in \mathcal{H}, \tag{2.1.14}
$$

where $\mathcal{H}$ is the phase space defined in (2.1.2). The linear operator $\mathrm{A} : D(\mathrm{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$
\mathrm{A}U = \begin{bmatrix}
\tilde{v}\\[1ex]
\tilde{p}\\[1ex]
\dfrac{\alpha}{\rho} v_{xx} - \dfrac{\gamma\beta}{\rho} p_{xx} - \dfrac{\delta}{\rho}\varphi_x - \dfrac{1}{\rho}\tilde{v}\\
\dfrac{\beta}{\mu} p_{xx} - \dfrac{\gamma\beta}{\mu} v_{xx} - \dfrac{1}{\mu}\tilde{p}\\[1ex]
\kappa\,\varphi_{xx} - \delta\,\tilde{v}_x
\end{bmatrix}, \tag{2.1.15}
$$

with domain

$$
\begin{aligned}
D(\mathrm{A}) = \big\{ U = (v, p, \tilde{v}, \tilde{p}, \varphi) \in \mathcal{V} :\ & v(0) = p(0) = 0,\\
& \alpha v_x(L) - \gamma\beta p_x(L) = 0,\ p_x(L) - \gamma v_x(L) = 0 \big\},
\end{aligned}
$$

where

$$
\mathcal{V} = \big(H^2(0,L)\cap H^1_*(0,L)\big)^2 \times \big(H^1_*(0,L)\big)^2 \times \big(H^2(0,L)\cap H^1_0(0,L)\big).
$$

The nonlinear term $\mathrm{F} : \Omega \times \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathrm{F}(\omega, U) = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{1}{\rho} f_1(v,p) - \dfrac{\delta}{\rho} \varepsilon \psi_x z(\omega) \\ -\dfrac{1}{\mu} f_2(v,p) \\ \sigma\varepsilon \psi z(\omega) + \kappa\varepsilon \psi_{xx} z(\omega) \end{bmatrix}. \tag{2.1.16}$$

### 2.1.3 Global solutions

In this subsection, we establish the global well-posedness of the random Cauchy problem (2.1.14). The analysis relies on the semigroup theory (see, e.g., Pazy [39]). We begin with a technical lemma concerning the nonlinearity F, which will play a central role in the subsequent arguments.

**Lemma 2.1.3.** *Suppose that hypotheses* **(H1)**-**(H4)** *are satisfied. Then, for each $\omega \in \Omega$, the mapping $\mathrm{F}(\omega, \cdot) : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz. More precisely, there exists a constant $C_0 > 0$, independent of $U^1, U^2 \in \mathcal{H}$ and $\omega$, such that*

$$\|\mathrm{F}(\omega, U^1) - \mathrm{F}(\omega, U^2)\|_{\mathcal{H}} \leqslant C_0 \left(1 + \|U^1\|_{\mathcal{H}}^{r-1} + \|U^2\|_{\mathcal{H}}^{r-1}\right) \|U^1 - U^2\|_{\mathcal{H}}. \tag{2.1.17}$$

*Proof.* Let $U^i = (v^i, p^i, \tilde{v}^i, \tilde{p}^i, \varphi^i) \in \mathcal{H}$, $i = 1, 2$. From the definition of F in (2.1.16), we have

$$\mathrm{F}(\omega, U^1) - \mathrm{F}(\omega, U^2) = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\rho}\left(f_1(v^1,p^1) - f_1(v^2,p^2)\right) \\ -\frac{1}{\mu}\left(f_2(v^1,p^1) - f_2(v^2,p^2)\right) \end{bmatrix}.$$

Using the norm defined in (2.1.3), we obtain

$$\begin{aligned} \|\mathrm{F}(\vartheta_t\omega, U^1) - \mathrm{F}(\vartheta_t\omega, U^2)\|_{\mathcal{H}}^2 = {} & \frac{1}{\rho} \int_0^L |f_1(v^1,p^1) - f_1(v^2,p^2)|^2 \, dx \\ & + \frac{1}{\mu} \int_0^L |f_2(v^1,p^1) - f_2(v^2,p^2)|^2 \, dx. \end{aligned} \tag{2.1.18}$$

Using the mean value theorem together with hypothesis (2.1.5), we deduce for $i = 1, 2$,

$$\begin{aligned} |f_i(v^1,p^1) - f_i(v^2,p^2)|^2 \leqslant {} & C\left(1 + |v^1|^{2(r-1)} + |v^2|^{2(r-1)} + |p^1|^{2(r-1)} + |p^2|^{2(r-1)}\right) \\ & \times \left(|v^1 - v^2|^2 + |p^1 - p^2|^2\right). \end{aligned}$$

Integrating over $(0, L)$ and using the Sobolev embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$, we obtain

$$\begin{aligned}\int_0^L |f_i(v^1,p^1) - f_i(v^2,p^2)|^2\,dx \leqslant C\Big(1+\|v_x^1\|_2^{2(r-1)}+\|v_x^2\|_2^{2(r-1)}+\|p_x^1\|_2^{2(r-1)}+\|p_x^2\|_2^{2(r-1)}\Big)\\ \times\left(\|v^1-v^2\|_2^2+\|p^1-p^2\|_2^2\right). \qquad (2.1.19)\end{aligned}$$

Combining (2.1.18) and (2.1.19), we conclude that

$$\|\mathrm{F}(\omega,U^1)-\mathrm{F}(\omega,U^2)\|_{\mathcal{H}}^2 \leqslant C_0\big(1+\|U^1\|_{\mathcal{H}}^{2(r-1)}+\|U^2\|_{\mathcal{H}}^{2(r-1)}\big)\|U^1-U^2\|_{\mathcal{H}}^2,$$

which yields (2.1.17). The proof is complete. □

We now prove the existence and uniqueness of solutions to the random Cauchy problem. We start with the following notion:

**Definition 2.1.4.** *A random process $U(t) = U(t,\omega,U_0)$ is called a* mild solution *to problem* (2.1.14) *if, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $U \in C([0,T);\mathcal{H})$ for some $T > 0$, and $U$ satisfies the integral equation*

$$U(t) = e^{\mathrm{A}t}U_0 + \int_0^t e^{\mathrm{A}(t-s)}\mathrm{F}(\vartheta_s\omega, U(s))\,ds, \qquad t \in [0,T). \qquad (2.1.20)$$

*The mild solution is called a* strong solution *on $[0,T)$ if, in addition,*

$$U \in C([0,T);D(\mathrm{A})) \cap C^1([0,T);\mathcal{H}).$$

*A mild (resp. strong) solution is said to be* global *if $T = +\infty$.*

**Definition 2.1.5.** *A random process $U(t) = U(t,\omega,U_0)$ is called a* mild solution *to problem* (2.1.14) *if, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $U \in C([0,T);\mathcal{H})$ for some $T > 0$, and $U$ satisfies the integral equation*

$$U(t) = e^{\mathrm{A}t}U_0 + \int_0^t e^{\mathrm{A}(t-s)}\mathrm{F}(\vartheta_s\omega, U(s))\,ds, \qquad t \in [0,T). \qquad (2.1.21)$$

*The mild solution is called a* strong solution *on $[0,T)$ if, in addition,*

$$U \in C([0,T);D(\mathrm{A})) \cap C^1([0,T);\mathcal{H}).$$

*A mild (resp. strong) solution is said to be* global *if $T = +\infty$.*

**Theorem 2.1.6** (Well-posedness)**.** *Suppose that hypotheses* **(H1)**-**(H4)** *are satisfied. Then we have:*

(i) *For any $U_0 \in D(\mathrm{A})$, problem (2.1.14) has a unique global strong solution $U$.*

(ii) *For any $U_0 \in \mathcal{H}$, there exists a unique global mild solution $U$ to (2.1.14).*

(iii) *The solution map $U_0 \mapsto U(t, \omega, U_0)$ is continuous from $\mathcal{H}$ to $\mathcal{H}$, for each fixed $t \geqslant 0$ and $\omega \in \Omega$.*

*Proof.* We divide our argument into two steps:

*Step 1: Global existence.* Our goal is to apply the Lumer-Phillips Theorem [39, Theorem 4.3] to show that $\mathrm{A}$ is the infinitesimal generator of a $C_0$-semigroup of contractions. We first note that $D(\mathrm{A})$ is dense in $\mathcal{H}$. Moreover, using standard integration by parts over $(0, L)$, one easily verifies that for every $U = (v, p, \tilde{v}, \tilde{p}, \varphi) \in D(\mathrm{A})$,

$$\langle \mathrm{A}U, U\rangle_{\mathcal{H}} = -\|\tilde{v}\|_2^2 - \|\tilde{p}\|_2^2 - \kappa\|\varphi_x\|_2^2 \leqslant 0,$$

and therefore $\mathrm{A}$ is dissipative. To conclude, we prove that $R(I - \mathrm{A}) = \mathcal{H}$. This is equivalent to proving that given $h = (h_1, h_2, h_3, h_4, h_5) \in \mathcal{H}$, there exists a unique $U = (v, p, \tilde{v}, \tilde{p}, \varphi) \in D(\mathrm{A})$ such that $(I - \mathrm{A})U = h$. Writing the equivalent system, we obtain

$$\begin{aligned}
v - \tilde{v} &= h_1 \in H_*^1(0, L),\\
p - \tilde{p} &= h_2 \in H_*^1(0, L),\\
\rho\tilde{v} - (\alpha v - \gamma\beta p)_{xx} + \delta\varphi_x + \tilde{v} &= \rho h_3 \in L^2(0, L),\\
\mu\tilde{p} - (\beta p - \gamma\beta v)_{xx} + \tilde{p} &= \mu h_4 \in L^2(0, L),\\
\varphi - \kappa\varphi_{xx} + \delta\tilde{v}_x &= h_5 \in L^2(0, L).
\end{aligned}$$

Since $\tilde{v} = v - h_1$ and $\tilde{p} = p - h_2$, it follows that

$$(\rho + 1)v - (\alpha v - \gamma\beta p)_{xx} + \delta\varphi_x = \rho h_3 + (\rho + 1)h_1 \in L^2(0, L), \tag{2.1.22}$$

$$(\mu + 1)p - (\beta p - \gamma\beta v)_{xx} = \mu h_4 + (\mu + 1)h_2 \in L^2(0, L), \tag{2.1.23}$$

$$\varphi - \kappa\varphi_{xx} + \delta v_x = h_5 + \delta h_{1,x} \in L^2(0, L). \tag{2.1.24}$$

The objective is to use the Lax-Milgram theorem. To this end, let us first derive the variational formulation of the problem. That is, multiplying equations (2.1.22)-(2.1.24) by $(\hat{v}, \hat{p}, \hat{\varphi})$

respectively, and then integrate over $(0, L)$ to obtain:

$$\begin{aligned}
&\int_0^L [(\rho+1)v\hat{v} + (\mu+1)p\hat{p} + \varphi\hat{\varphi}]\,dx + \delta\int_0^L (\varphi_x\hat{v} + v_x\hat{\varphi})\,dx \\
&\quad + \alpha_1\int_0^L v_x\hat{v}_x\,dx + \beta\int_0^L (\gamma v - p)_x(\gamma\hat{v} - \hat{p})_x\,dx + \kappa\int_0^L \varphi_x\hat{\varphi}_x\,dx \\
&= \rho\int_0^L (h_3 + h_1)\hat{v}dx + \int_0^L h_1\hat{v}dx + \mu\int_0^L (h_4 + h_2)\hat{p}dx + \int_0^L h_2\hat{p}dx \\
&\quad + \int_0^L (h_5 + \delta h_{1,x})\hat{\varphi}dx.
\end{aligned} \tag{2.1.25}$$

Consider the Hilbert space $\hat{H} = (H_*^1(0, L) \times H_*^1(0, L)) \times H_0^1(0, L)$ with the equivalent norm

$$\|u\|_{\hat{H}}^2 = \alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2 + \kappa\|\varphi_x\|_2^2, \quad u = (v, p, \varphi) \in \hat{H}.$$

Then we can write (2.1.25) in the form

$$B[u, \hat{u}] = T(\hat{u}), \quad \forall \hat{u} = (\hat{v}, \hat{p}, \hat{\varphi}) \in \hat{H},$$

where $B : \hat{H} \times \hat{H} \to \mathbb{R}$ is a bilinear form defined by

$$\begin{aligned}
B[u, \hat{u}] = &\int_0^L [(\rho+1)v\hat{v} + (\mu+1)p\hat{p} + \varphi\hat{\varphi}]\,dx + \delta\int_0^L (\varphi_x\hat{v} + v_x\hat{\varphi})\,dx \\
&+ \alpha_1\int_0^L v_x\hat{v}_x\,dx + \beta\int_0^L (\gamma v - p)_x(\gamma\hat{v} - \hat{p})_x\,dx + \kappa\int_0^L \varphi_x\hat{\varphi}_x\,dx,
\end{aligned}$$

and $T : \hat{H} \to \mathbb{R}$ is a linear functional given by

$$\begin{aligned}
T(\hat{u}) = &\rho\int_0^L (h_3 + h_1)\hat{v}dx + \int_0^L h_1\hat{v}dx + \mu\int_0^L (h_4 + h_2)\hat{p}dx \\
&+ \int_0^L h_2\hat{p}dx + \int_0^L (h_5 + \delta h_{1,x})\hat{\varphi}dx.
\end{aligned}$$

Let us now verify that the conditions of the Lax-Milgram theorem are satisfied. We first note that $B$ is coercive, since

$$\begin{aligned}
B[u, u] &= (\rho+1)\|v\|_2^2 + (\mu+1)\|p\|_2^2 + \|\varphi\|_2^2 + \alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2 + \kappa\|\varphi_x\|_2^2 \\
&\geqslant \alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2 + \kappa\|\varphi_x\|_2^2 = \|u\|_{\hat{H}}^2.
\end{aligned}$$

It remains to show that $T$ is continuous. Indeed, using Hölder's inequality and Poincaré's inequality, it is easy to see that

$$\begin{aligned}
|T(\hat{u})| \leqslant &\rho\|h_3 + h_1\|_2\|\hat{v}\|_2 + \|h_1\|_2\|\hat{v}\|_2 + \mu\|h_4 + h_2\|_2\|\hat{p}\|_2 \\
&+ \|h_2\|_2\|\hat{p}\|_2 + \|h_5 + \delta h_{1,x}\|_2\|\hat{\varphi}\|_2 \leqslant C\|\hat{u}\|_{\hat{H}}.
\end{aligned}$$

This shows that $T$ is continuous. Thus, by the Lax-Milgram theorem, there exists a unique $u = (v, p, \varphi) \in \hat{H}$ satisfying (2.1.22)–(2.1.24). On the other hand, from equations (2.1.22) and (2.1.23), we obtain:

$$\alpha v_{xx} - \gamma\beta p_{xx} = (\rho + 1)v + \delta\varphi_x - \rho h_3 - (\rho + 1)h_1 =: S_1 \in L^2(0, L),$$
$$-\gamma\beta v_{xx} + \beta p_{xx} = (\mu + 1)p - \mu h_4 - (\mu + 1)h_2 =: S_2 \in L^2(0, L).$$

Solving the above linear system and using $\alpha = \alpha_1 + \gamma^2\beta$, we have

$$v_{xx} = \frac{S_1 + \gamma S_2}{\alpha_1} \in L^2(0, L),$$
$$p_{xx} = \frac{1}{\beta}S_2 + \frac{\gamma}{\alpha_1}(S_1 + \gamma S_2) \in L^2(0, L).$$

Consequently, $(v, p) \in \left(H^2(0, L) \cap H_*^1(0, L)\right)^2$. Now using equation (2.1.24), we find that

$$\kappa\varphi_{xx} = \varphi + \delta v_x - h_5 - \delta h_{1,x} \in L^2(0, L),$$

and thus $\varphi \in H^2(0, L) \cap H_0^1(0, L)$. Therefore, we conclude that there exists $U \in D(\mathrm{A})$ such that $(I - \mathrm{A})U = h$. It follows from the Lumer-Phillips theorem that $\mathrm{A}$ is the infinitesimal generator of a $C_0$-semigroup of contractions $T(t) = e^{\mathrm{A}t}$ on $\mathcal{H}$.

An immediate consequence of Lemma 2.1.3 and classical semigroup theory (see, e.g., Pazy [39]) is that there exists a unique mild solution on $[0, t_{\max})$, and Lemma 2.2.1 (below) allows us to extend it to a global solution, that is, $t_{\max} = \infty$. Moreover, if $U \in D(\mathrm{A})$, there exists a strong solution on $[0, t_{\max})$ that can be similarly extended. Therefore, items (i) and (ii) are proved.

*Step 2: Continuous dependence on initial data.* Let $U^1, U^2$ be two mild solutions with initial data $U_0^1, U_0^2 \in \mathcal{H}$ respectively. Denote $U := U^1 - U^2$ and $U_0 := U_0^1 - U_0^2$. By definition, it follows that

$$U^i(t) = e^{\mathrm{A}t}U_0^i + \int_0^t e^{\mathrm{A}(t-s)}\mathrm{F}(\vartheta_s\omega, U^i(s))\, ds, \quad i = 1, 2.$$

Considering the difference of the solutions, we obtain

$$U(t) = e^{\mathrm{A}t}U_0 + \int_0^t e^{\mathrm{A}(t-s)}\left[\mathrm{F}(\vartheta_s\omega, U^1(s)) - \mathrm{F}(\vartheta_s\omega, U^2(s))\right] ds.$$

Taking norms, and recalling that $e^{\mathrm{A}t}$ is a semigroup of contractions ($\|e^{\mathrm{A}t}\| \leq 1$), and applying

Lemma 2.1.3, we find

$$\begin{aligned}\|U(t)\|_{\mathcal{H}} &\leqslant \|U_0\|_{\mathcal{H}} + \int_0^t \|\mathrm{F}(\vartheta_s\omega, U^1(s)) - \mathrm{F}(\vartheta_s\omega, U^2(s))\|_{\mathcal{H}}\, ds \\ &\leqslant \|U_0\|_{\mathcal{H}} + \int_0^t \Upsilon(s)\|U(s)\|_{\mathcal{H}} ds,\end{aligned}$$

where

$$\Upsilon(s) = C_0\left(1 + \|U^1(s)\|_{\mathcal{H}}^{r-1} + \|U^2(s)\|_{\mathcal{H}}^{r-1}\right). \tag{2.1.26}$$

By Gronwall's inequality (integral form), it follows that

$$\|U(t)\|_{\mathcal{H}} \leqslant e^{\int_0^t \Upsilon(s)\, ds}\|U_0\|_{\mathcal{H}}, \quad \forall t \in [0, T). \tag{2.1.27}$$

By the regularity of the mild solution, we see that $\Upsilon$ is integrable on bounded intervals; the right-hand side of (2.1.27) remains finite for $t < T$, and the continuous dependence on initial data is established. This proves item (iii) and thus completes the proof. □

**Remark 2.1.7** (RDS)**.** *Thanks to Theorem 2.1.6, we are able to define the mapping*

$$\Phi_\varepsilon : \mathbb{R}^+ \times \Omega \times \mathcal{H} \to \mathcal{H}, \quad \Phi_\varepsilon(t, \omega, U_0) := U(t, \omega, U_0) = (v(t), p(t), v_t(t), p_t(t), \varphi(t)), \tag{2.1.28}$$

*where $U$ denotes the unique mild solution of* (2.1.14) *corresponding to the initial condition $U_0 \in \mathcal{H}$. Moreover, since mild solutions can be obtained as uniform limits of strong solutions on compact time intervals (via density arguments), the mapping $(t, \omega, U_0) \mapsto \Phi_\varepsilon(t, \omega, U_0)$ is measurable with respect to the product $\sigma$-algebra $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(\mathcal{H})$. In addition, for each fixed $(t, \omega)$, the mapping $U_0 \mapsto \Phi_\varepsilon(t, \omega, U_0)$ is continuous in $\mathcal{H}$. Therefore, $\Phi_\varepsilon$ defines a continuous random dynamical system (RDS) on $\mathcal{H}$ over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ in the sense of Definition 1.2.3.*

## 2.2 Energy estimates and tempered sets

In this section, we derive energy estimates for the random system and introduce a class $\mathcal{D}$ of tempered sets.

### 2.2.1 Energy functional

The energy functional associated with the random system (2.1.12) is defined by

$$\mathcal{E}(t) = E(t) + \int_0^L F(v(t), p(t))\, dx, \tag{2.2.1}$$

where

$$E(t) = \frac{1}{2} \|(v(t), p(t), v_t(t), p_t(t), \varphi(t))\|_{\mathcal{H}}^2. \tag{2.2.2}$$

We begin with the following stability estimate.

**Lemma 2.2.1.** *Suppose that hypotheses* **(H1)–(H4)** *are satisfied. Then, there exist constants* $\sigma_0, M, C_f > 0$, *independent of* $\varepsilon$, *such that*

$$\mathcal{E}(t) \leqslant M\mathcal{E}(0)e^{-\sigma_0 t} + C_f \int_0^t e^{\sigma_0(s-t)} \left(1 + \varepsilon^2 \left|z(\vartheta_s\omega)\right|^2\right) ds, \quad \forall\, t \geqslant 0. \tag{2.2.3}$$

To prove Lemma 2.2.1, we first establish several preliminary estimates.

**Lemma 2.2.2.** *There exists a constant* $C > 0$ *such that*

$$\frac{d}{dt}\mathcal{E}(t) \leqslant -\frac{1}{2}\|v_t\|_2^2 - \|p_t\|_2^2 - \frac{\kappa}{2}\|\varphi_x\|_2^2 + C\,\varepsilon^2|z|^2. \tag{2.2.4}$$

*Proof.* Multiplying the first equation in (2.1.12) by $v_t$, integrating over $[0, L]$, and using $\alpha = \alpha_1 + \gamma^2\beta$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\rho\|v_t\|_2^2 + \alpha_1\|v_x\|_2^2\right) + \gamma\beta \int_0^L (\gamma v_x - p_x)v_{tx}\, dx + \delta \int_0^L \varphi_x v_t\, dx + \|v_t\|_2^2 \\ + \int_0^L f_1(v,p)v_t\, dx = -\delta\varepsilon z \int_0^L \psi_x v_t\, dx. \end{aligned} \tag{2.2.5}$$

Multiplying the second equation by $p_t$ and integrating yields

$$\frac{\mu}{2}\frac{d}{dt}\|p_t\|_2^2 - \beta \int_0^L (\gamma v_x - p_x)p_{tx}\, dx + \|p_t\|_2^2 + \int_0^L f_2(v,p)p_t\, dx = 0. \tag{2.2.6}$$

Multiplying the third equation by $\varphi$ and integrating gives

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_2^2 + \kappa\|\varphi_x\|_2^2 - \delta \int_0^L v_t\varphi_x\, dx = \int_0^L (\sigma\varepsilon\psi z + \kappa\varepsilon\psi_{xx}z)\varphi\, dx. \tag{2.2.7}$$

Summing (2.2.5)–(2.2.7), and using hypothesis (2.1.4), we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\|v_t\|_2^2 - \|p_t\|_2^2 - \kappa\|\varphi_x\|_2^2 - \delta\varepsilon z \int_0^L \psi_x v_t\, dx + \int_0^L (\sigma\varepsilon\psi z + \kappa\varepsilon\psi_{xx}z)\varphi\, dx. \tag{2.2.8}$$

By Hölder's and Young's inequalities,

$$-\delta\varepsilon \int_0^L \psi_x z v_t dx \leqslant \delta\varepsilon|z|\|\psi_x\|_2\|v_t\|_2 \leqslant C\varepsilon^2|z|^2\|\psi_x\|_2^2 + \frac{1}{2}\|v_t\|_2^2.$$

Next, by Poincaré's inequality, we have

$$\begin{aligned}\int_0^L (\sigma\varepsilon\psi z + \kappa\varepsilon\psi_{xx} z)\varphi dx &\leqslant C\varepsilon|z|(\|\psi\|_2 + \|\psi_{xx}\|_2)\|\varphi\|_2 \\ &\leqslant C\varepsilon|z|(\|\psi\|_2 + \|\psi_{xx}\|_2)\|\varphi_x\|_2 \\ &\leqslant \frac{\kappa}{2}\|\varphi_x\|_2^2 + C|z|^2\varepsilon^2(\|\psi\|_2^2 + \|\psi_{xx}\|_2^2).\end{aligned}$$

Substituting the above estimates into (2.2.8) yields (2.2.4). □

Let us consider the functional

$$\chi(t) = \rho\int_0^L v_t v\, dx + \mu\int_0^L p_t p\, dx + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|p\|_2^2. \tag{2.2.9}$$

In the following lemma, we will construct this functional.

**Lemma 2.2.3.** *There exist constants $C, C_1 > 0$ such that*

$$\begin{aligned}\frac{d}{dt}\chi(t) \leqslant \rho\|v_t\|_2^2 + \mu\|p_t\|_2^2 - \frac{\alpha_1}{2}\|v_x\|_2^2 - \beta\|\gamma v_x - p_x\|_2^2 + C\|\varphi_x\|_2^2 \\ - \int_0^L F(v,p)\, dx + C\varepsilon^2|z|^2 + C_1.\end{aligned} \tag{2.2.10}$$

*Proof.* Multiplying the first equation of (2.1.12) by $v$ and the second by $p$, and integrating over $(0, L)$, we obtain

$$\begin{aligned}\frac{d}{dt}\chi(t) = \rho\|v_t\|_2^2 + \mu\|p_t\|_2^2 - \alpha_1\|v_x\|_2^2 - \beta\|\gamma v_x - p_x\|_2^2 - \int_0^L \nabla F(v,p)\cdot(v,p)\, dx \\ + \delta\int_0^L \varphi v_x\, dx - \delta\varepsilon\int_0^L \psi_x z\, v\, dx.\end{aligned}$$

Applying Young's and Poincaré's inequalities yields

$$\begin{aligned}\frac{d}{dt}\chi(t) \leqslant \rho\|v_t\|_2^2 + \mu\|p_t\|_2^2 - \frac{\alpha_1}{2}\|v_x\|_2^2 - \beta\|\gamma v_x - p_x\|_2^2 + C\|\varphi_x\|_2^2 \\ - \int_0^L \nabla F(v,p)\cdot(v,p)\, dx + C\varepsilon^2|z|^2.\end{aligned} \tag{2.2.11}$$

From assumption (2.1.6), we have

$$-\nabla F(v,p)\cdot(v,p) \leqslant -F(v,p) + m_f.$$

Substituting this estimate into (2.2.11) yields (2.2.10) with $C_1 = m_f L$, which completes the proof. □

Finally, we introduce the functional

$$\mathcal{L}(t) := N\,\mathcal{E}(t) + \chi(t), \tag{2.2.12}$$

for some constant $N > 0$ to be chosen later.

**Lemma 2.2.4.** *There exist constants $N_0 > 0$ and $K_f > 0$ such that, for any $N > N_0$, the following equivalence holds:*

$$\lambda_1\,\mathcal{E}(t) - K_f \leqslant \mathcal{L}(t) \leqslant \lambda_2\,\mathcal{E}(t) + K_f, \qquad \forall\, t \geqslant 0, \tag{2.2.13}$$

*where $\lambda_1 = N - N_0 > 0$ and $\lambda_2 = N + N_0 > 0$.*

*Proof.* From the definition of $\mathcal{L}(t)$ in (2.2.12) and of $\chi(t)$ in (2.2.9), and applying Hölder's and Poincaré's inequalities, we obtain

$$|\mathcal{L}(t) - N\,\mathcal{E}(t)| = |\chi(t)| = \left|\rho \int_0^L v_t v\,dx + \mu \int_0^L p_t p\,dx + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|p\|_2^2\right| \leqslant N_0 E(t),$$

for some constant $N_0 > 0$.

On the other hand, using (2.2.1), (2.2.2) and assumption (2.1.6), we deduce that

$$\mathcal{E}(t) \geqslant E(t) - m_f L, \tag{2.2.14}$$

and therefore

$$|\mathcal{L}(t) - N\,\mathcal{E}(t)| \leqslant N_0\big(\mathcal{E}(t) + m_f L\big).$$

This implies that

$$(N - N_0)\mathcal{E}(t) - N_0 m_f L \leqslant \mathcal{L}(t) \leqslant (N + N_0)\mathcal{E}(t) + N_0 m_f L,$$

that is, (2.2.13) holds with $K_f = N_0 m_f L$. Since $N > N_0$, it follows that $\lambda_1,\ \lambda_2 > 0$, and the proof is complete. $\square$

We are now in a position to complete the proof of Lemma 2.2.1.

*Proof.* **Proof of Lemma 2.2.1 (completion).** From Lemmas 2.2.2–2.2.4, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leqslant &-\left(\frac{N}{2} - \rho\right)\|v_t\|_2^2 - (N - \mu)\,\|p_t\|_2^2 - \left(\frac{N\kappa}{2} - C\right)\|\varphi_x\|_2^2 \\ &- \alpha_1\|v_x\|_2^2 - \beta\|\gamma v_x - p_x\|_2^2 - \int_0^L F(v,p)\,dx + C\varepsilon^2|z|^2 + C_1.\end{aligned}$$

By choosing $N > 0$ sufficiently large and using the fact that

$$F(v,p) + m_f \geqslant 0,$$

we deduce that there exist constants $\varkappa > 0$ and $C_f > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\varkappa\,\mathcal{E}(t) + C\varepsilon^2|z|^2 + C_f. \tag{2.2.15}$$

Using the equivalence estimate (2.2.13), it follows that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\frac{\varkappa}{\lambda_2}\mathcal{L}(t) + C\varepsilon^2|z|^2 + C_f.$$

Applying Gronwall's inequality gives

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)e^{-\sigma_0 t} + C_f\int_0^t e^{-\sigma_0(t-s)}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds,$$

where $\sigma_0 = \frac{\varkappa}{\lambda_2}$. Finally, applying once again the equivalence relation (2.2.13), we obtain the desired estimate (2.2.3). This completes the proof of Lemma 2.2.1. □

**Remark 2.2.5.** *Observe that, using* (2.2.14) *together with* (2.2.3)*, we obtain*

$$E(t) - m_f L \leqslant M\,\mathcal{E}(0)e^{-\sigma_0 t} + C_f\int_0^t e^{\sigma_0(s-t)}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds, \quad \forall\, t \geqslant 0. \tag{2.2.16}$$

*On the other hand, from assumption* (2.1.5) *and* (2.2.1)*, we deduce that there exists a constant* $C > 0$ *such that*

$$\mathcal{E}(0) \leqslant C\big(1 + (E(0))^{r+1}\big). \tag{2.2.17}$$

*Since, by definition,*

$$E(t) = \tfrac{1}{2}\left\|\Phi_\varepsilon(t,\omega,U_0)\right\|_{\mathcal{H}}^2,$$

*combining* (2.2.16) *and* (2.2.17)*, we obtain that, for all* $t \geqslant 0$*,* $\omega \in \Omega$*, and* $U_0 \in \mathcal{H}$*,*

$$\begin{aligned}\left\|\Phi_\varepsilon(t,\omega,U_0)\right\|_{\mathcal{H}}^2 &\leqslant C\big(1 + \|U_0\|_{\mathcal{H}}^{r+1}\big)e^{-\sigma_0 t}\\ &\quad + C_f\int_0^t e^{\sigma_0(s-t)}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds + m_f L,\end{aligned} \tag{2.2.18}$$

*for some constant* $C_f > 0$ *is independent of* $\varepsilon$*.*

### 2.2.2 Class of tempered random sets

Let $D = \{D(\omega)\}_{\omega\in\Omega}$ be a family of nonempty bounded subsets of $\mathcal{H}$. We say that $D$ is a *tempered random set* in $\mathcal{H}$ if

$$\lim_{t\to\infty} e^{-\eta t}\,\|D(\vartheta_{-t}\omega)\|_{\mathcal{H}} = 0, \quad \forall\,\eta > 0,\ \omega \in \Omega. \tag{2.2.19}$$

For any subset $A \subset \mathcal{H}$, we define

$$\|A\|_{\mathcal{H}} := \sup_{u\in A} \|u\|_{\mathcal{H}}.$$

We denote by $\mathcal{D}$ the collection of all tempered random sets in $\mathcal{H}$:

$$\mathcal{D} := \{\, D = \{D(\omega)\}_{\omega\in\Omega} : D \text{ satisfies (2.2.19)} \,\}. \tag{2.2.20}$$

In particular, we say that a random variable $\zeta : \Omega \to \mathbb{R}$ is *tempered* if

$$\lim_{t\to\infty} e^{-\eta t}\,|\zeta(\vartheta_{-t}\omega)| = 0, \quad \forall\,\eta > 0,\ \omega \in \Omega.$$

# Chapter 3

# Random attractors for the stochastic piezoelectric system

In this chapter, we establish the existence of $\mathcal{D}$-random attractors for the random dynamical system $\Phi_\varepsilon$ generated by the stochastic piezoelectric system (2.0.1)–(2.0.2).

### 3.1 Existence of absorbing sets

In this section, we construct a $\mathcal{D}$-pullback absorbing set for $\Phi_\varepsilon$. This analysis is justified, as the shift operator absorbs the explicit time dependence of the stochastic forcing.

**Lemma 3.1.1.** *Suppose that hypotheses* **(H1)–(H4)** *are satisfied. Then there exists a $\mathcal{D}$-pullback absorbing set*

$$\mathcal{B}_\varepsilon = \{\mathcal{B}_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D},$$

*given by*

$$\mathcal{B}_\varepsilon(\omega) = \left\{U \in \mathcal{H} : \|U\|_{\mathcal{H}}^2 \leqslant 2R_\varepsilon(\omega)\right\}, \qquad \forall\, \omega \in \Omega, \tag{3.1.1}$$

*where $R_\varepsilon(\omega)$ is defined as*

$$R_\varepsilon(\omega) = C_f \int_{-\infty}^{0} e^{\sigma_0 s} \left(1 + \varepsilon^2 |z(\vartheta_s\omega)|^2\right) ds + m_f L, \tag{3.1.2}$$

*for some constant $C_f > 0$ appearing in (2.2.18), which is independent of $\varepsilon$.*

*Proof.* Given $t \geqslant 0$, substituting $\omega$ by $\vartheta_{-t}\omega$ in (2.2.18), we obtain

$$\begin{aligned}\big\|\Phi_\varepsilon(t,\vartheta_{-t}\omega,U_0)\big\|_{\mathcal{H}}^2 &\leqslant C\big(1+\|U_0\|_{\mathcal{H}}^{r+1}\big)e^{-\sigma_0 t}\\ &\qquad + C_f\int_0^t e^{\sigma_0(s-t)}\left(1+\varepsilon^2|z(\vartheta_{s-t}\omega)|^2\right)\,ds + m_f L\\ &= C\big(1+\|U_0\|_{\mathcal{H}}^{r+1}\big)e^{-\sigma_0 t}\\ &\qquad + C_f\int_{-t}^0 e^{\sigma_0 s}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds + m_f L.\end{aligned}$$

Consequently,

$$\begin{aligned}\big\|\Phi_\varepsilon(t,\vartheta_{-t}\omega,U_0)\big\|_{\mathcal{H}}^2 &\leqslant C\big(1+\|U_0\|_{\mathcal{H}}^{r+1}\big)e^{-\sigma_0 t}\\ &\qquad + C_f\int_{-\infty}^0 e^{\sigma_0 s}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds + m_f L.\end{aligned} \tag{3.1.3}$$

Since (2.1.9) ensures that

$$\int_{-\infty}^0 e^{\sigma_0 s}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)ds < \infty,$$

it follows that $R_\varepsilon(\omega)$ is well-defined.

Now, given $\omega \in \Omega$ and $U_0 \in D(\vartheta_{-t}\omega)$ with $D \in \mathcal{D}$, it follows from the temperedness condition (2.2.19) that

$$Ce^{-\sigma_0 t}(1+\|U_0\|_{\mathcal{H}}^{r+1}) \leqslant Ce^{-\sigma_0 t} + C\left(e^{-\frac{\sigma_0}{r+1}t}\,\|D(\vartheta_{-t}\omega)\|_{\mathcal{H}}\right)^{r+1} \to 0, \quad \text{as } t\to\infty. \tag{3.1.4}$$

Combining (3.1.3) and (3.1.4), we conclude that for each $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T_D(\omega) > 0$ such that

$$\Phi_\varepsilon(t,\vartheta_{-t}\omega,D(\vartheta_{-t}\omega)) \subset \mathcal{B}_\varepsilon(\omega), \qquad \forall\, t \geqslant T_D(\omega). \tag{3.1.5}$$

This shows that $\mathcal{B}_\varepsilon$ satisfies the pullback absorption property.

To prove that $\mathcal{B}_\varepsilon \in \mathcal{D}$, let $\eta > 0$ and define $\ell = \min\{\eta,\sigma_0\}$. From the definition (3.1.1) of $\mathcal{B}_\varepsilon$, we have

$$e^{-\eta t}\|\mathcal{B}_\varepsilon(\vartheta_{-t}\omega)\|_{\mathcal{H}} \leqslant \left(2e^{-2\eta t}R_\varepsilon(\vartheta_{-t}\omega)\right)^{1/2}. \tag{3.1.6}$$

From the definition (3.1.2) of $R_\varepsilon$, we compute

$$\begin{aligned}e^{-2\eta t}R_\varepsilon(\vartheta_{-t}\omega) &= C_f e^{-2\eta t}\int_{-\infty}^0 e^{\sigma_0 s}\left(1+\varepsilon^2|z(\vartheta_{s-t}\omega)|^2\right)\,ds + m_f L e^{-2\eta t}\\ &= C_f e^{-2\eta t}\int_{-\infty}^{-t} e^{\sigma_0(s+t)}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds + C_f e^{-2\eta t}\\ &\leqslant C_f e^{-(2\eta-\ell)t}\int_{-\infty}^0 e^{\ell s}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)\,ds + C_f e^{-2\eta t} \to 0, \quad \text{as } t\to\infty.\end{aligned} \tag{3.1.7}$$

Combining (3.1.6) and (3.1.7), we obtain

$$e^{-\eta t}\|\mathcal{B}_\varepsilon(\vartheta_{-t}\omega)\|_{\mathcal{H}} \to 0, \quad \text{as } t \to \infty,$$

which proves that $\mathcal{B}_\varepsilon \in \mathcal{D}$. Moreover, since the mapping $\omega \mapsto R_\varepsilon(\omega)$ is $\mathcal{F}$-measurable, it follows that $\mathcal{B}_\varepsilon$ is a random set. Therefore, by (3.1.5), $\mathcal{B}_\varepsilon$ is a $\mathcal{D}$-pullback absorbing set. The proof is complete. □

**Remark 3.1.2.** *Before proceeding to the next section, we introduce the random set* $\widetilde{\mathcal{B}}_\varepsilon = \{\widetilde{\mathcal{B}}_\varepsilon(\omega)\}_{\omega\in\Omega}$ *defined by*

$$\widetilde{\mathcal{B}}_\varepsilon(\omega) = \bigcup_{s\geqslant T_{\mathcal{B}_\varepsilon}(\omega)} \Phi_\varepsilon\big(s, \vartheta_{-s}\omega, \mathcal{B}_\varepsilon(\vartheta_{-s}\omega)\big), \qquad \forall\, \omega \in \Omega.$$

*It follows from* (3.1.5) *that*

$$\widetilde{\mathcal{B}}_\varepsilon(\omega) \subset \mathcal{B}_\varepsilon(\omega), \qquad \forall\, \omega \in \Omega.$$

*Since* $\mathcal{B}_\varepsilon \in \mathcal{D}$*, we have*

$$e^{-\eta t}\,\|\widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega)\|_{\mathcal{H}} \leqslant e^{-\eta t}\,\|\mathcal{B}_\varepsilon(\vartheta_{-t}\omega)\|_{\mathcal{H}} \to 0 \quad \textit{as } t \to \infty,$$

*for all* $\eta > 0$*. Hence,* $\widetilde{\mathcal{B}}_\varepsilon \in \mathcal{D}$*. Moreover, for every* $t \geqslant 0$*,*

$$\begin{aligned}
\Phi_\varepsilon(t, \omega, \widetilde{\mathcal{B}}_\varepsilon(\omega)) &= \Phi_\varepsilon\Big(t, \omega, \bigcup_{s\geqslant T_{\mathcal{B}_\varepsilon}(\omega)} \Phi_\varepsilon(s, \vartheta_{-s}\omega, \mathcal{B}_\varepsilon(\vartheta_{-s}\omega))\Big)\\
&= \bigcup_{s\geqslant T_{\mathcal{B}_\varepsilon}(\omega)} \Phi_\varepsilon\big(t, \omega, \Phi_\varepsilon(s, \vartheta_{-s}\omega, \mathcal{B}_\varepsilon(\vartheta_{-s}\omega))\big)\\
&= \bigcup_{s\geqslant T_{\mathcal{B}_\varepsilon}(\omega)} \Phi_\varepsilon\big(t+s, \vartheta_{-s}\omega, \mathcal{B}_\varepsilon(\vartheta_{-s}\omega)\big)\\
&\subseteq \widetilde{\mathcal{B}}_\varepsilon(\vartheta_t\omega).
\end{aligned}$$

*Therefore,* $\widetilde{\mathcal{B}}_\varepsilon$ *is a positively invariant* $\mathcal{D}$*-pullback absorbing set.*

### 3.2 Random quasi-stability

The quasi-stability approach, originally developed for deterministic systems by Chueshov and Lasiecka [11, 14] and further refined in [12], provides a powerful framework for the analysis of long-time dynamics of nonlinear evolution equations. Motivated by its extension to the random setting by Chueshov and Schmalfuß [15], we adapt and further develop the quasi-stability strategy for the stochastic piezoelectric system considered in this work.

### 3.2.1 Random stabilizability estimate

In the deterministic context, a key ingredient in establishing quasi-stability is the so-called *stabilizability estimate* (see Chueshov and Lasiecka [11, Section 7.9]). In the stochastic framework, we employ its natural random counterpart, which we refer to as the *random stabilizability estimate* (see the estimate (3.2.1) below).

**Lemma 3.2.1.** *Suppose that hypotheses* **(H1)-(H4)** *are satisfied. Let $B = \{B(\omega)\}_{\omega\in\Omega}$ be a positively invariant random set in the class $\mathcal{D}$. Consider two solutions $\Phi_\varepsilon(t,\omega,U^i) = (v^i, p^i, v^i_t, p^i_t, \varphi^i)$, $i = 1, 2$, of the system (2.1.12) with initial conditions $U^1, U^2 \in B(\omega)$. Then there exist constants $\sigma_1, \mathrm{M} > 0$ and $C > 0$ such that, for all $t \geqslant 0$,*

$$\begin{aligned} \|\Phi_\varepsilon(t,\omega,U^1) - \Phi_\varepsilon(t,\omega,U^2)\|^2_{\mathcal{H}} \leqslant{}& \mathrm{M}e^{-\sigma_1 t}\|U^1 - U^2\|^2_{\mathcal{H}} \\ &+ C\int_0^t e^{-\sigma_1(t-s)}\zeta_B(\vartheta_s\omega)\big(\|v(s)\|_2^2 + \|p(s)\|_2^2\big)ds, \end{aligned} \tag{3.2.1}$$

*where $v = v^1 - v^2$, $p = p^1 - p^2$, and $\zeta_B(\omega) = 1 + \|B(\omega)\|^{2(r-1)}_{\mathcal{H}}$ is a tempered random variable.*

*Proof.* Let $v = v^1 - v^2$, $p = p^1 - p^2$, and $\varphi = \varphi^1 - \varphi^2$ denote the differences between two solutions of system (2.1.12). Taking the difference between the two systems, we obtain that $(v, p, v_t, p_t, \varphi)$ satisfies the following system

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\varphi_x + v_t + G_1(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + p_t + G_2(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ \varphi_t - \kappa\varphi_{xx} + \delta v_{xt} = 0 & \text{in } (0,L)\times\mathbb{R}^+, \end{cases} \tag{3.2.2}$$

where

$$G_i(v,p) := f_i(v^1,p^1) - f_i(v^2,p^2), \quad \text{for } i = 1, 2.$$

where the initial and boundary conditions associated with (3.2.2) are similar to (2.1.13).

The energy functional associated with system (3.2.2) is defined as

$$\widetilde{E}(t) := \frac{1}{2}\|(v(t), p(t), v_t(t), p_t(t), \varphi(t))\|^2_{\mathcal{H}} = \frac{1}{2}\left\|\Phi_\varepsilon(t,\omega,U^1) - \Phi_\varepsilon(t,\omega,U^2)\right\|^2_{\mathcal{H}}.$$

To estimate this energy, we divide the proof into a sequence of steps.

**Step 1.** We proceed in a manner analogous to the calculations used to derive (2.2.1), let us find the energy functional for (3.2.2). Multiplying the first equation in (3.2.2) by $v_t$ and integrating over $[0, L]$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\rho\|v_t\|_2^2+\alpha_1\|v_x\|_2^2\right)+\gamma\beta\int_0^L(\gamma v_x-p_x)v_{tx}dx+\delta\int_0^L\varphi_x v_t dx\\+\|v_t\|_2^2+\int_0^L G_1(v,\,p)v_t dx=0\end{aligned}\tag{3.2.3}$$

Multiplying the second equation in (3.2.2) by $p_t$ and integrating over $[0, L]$, we obtain

$$\frac{\mu}{2}\frac{d}{dt}\|p_t\|^2-\beta\int_0^L(\gamma v_x-p_x)p_{tx}dx+\|p_t\|_2^2+\int_0^L G_2(v,\,p)p_t dx=0\tag{3.2.4}$$

Multiplying the third equation in (3.2.2) by $\varphi$ and integrating over $[0, L]$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_2^2+\kappa\|\varphi_x\|_2^2+\delta\int_0^L v_{xt}\varphi dx=0\tag{3.2.5}$$

Adding (3.2.3)-(3.2.5), it follows that

$$\frac{d}{dt}\widetilde{E}(t)=-\|v_t\|_2^2-\|p_t\|_2^2-\kappa\|\varphi_x\|_2^2-\int_0^L G_1(v,\,p)v_t+G_2(v,\,p)p_t\,dx.\tag{3.2.6}$$

Using assumption (2.1.5), the embedding $H^1(0, L)\hookrightarrow L^\infty(0, L)$, and Young's inequality, we deduce

$$\begin{aligned}-\int_0^L G_1(v,p)v_t dx&\leqslant C\left(1+\|v^1\|_\infty^{r-1}+\|v^2\|_\infty^{r-1}+\|p^1\|_\infty^{r-1}+\|p^2\|_\infty^{r-1}\right)(\|v\|_2+\|p\|_2)\|v_t\|_2\\&\leqslant C\left(1+\|v_x^1\|_2^{r-1}+\|v_x^2\|_2^{r-1}+\|p_x^1\|_2^{r-1}+\|p_x^2\|_2^{r-1}\right)(\|v\|_2+\|p\|_2)\|v_t\|_2\\&\leqslant C(1+\|B(\vartheta_t\omega)\|_{\mathcal{H}}^{r-1})(\|v\|_2+\|p\|_2)\|v_t\|_2\\&\leqslant C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2)+\frac{1}{2}\|v_t\|_2^2,\end{aligned}\tag{3.2.7}$$

where $\zeta_B(\omega)=1+\|B(\omega)\|_{\mathcal{H}}^{2(r-1)}$. Similarly, we have

$$-\int_0^L G_2(v,p)p_t dx\leqslant C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2)+\frac{1}{2}\|p_t\|_2^2.\tag{3.2.8}$$

Inserting (3.2.7) and (3.2.8) into (3.2.6), we obtain

$$\frac{d}{dt}\widetilde{E}(t)\leqslant-\frac{1}{2}\|v_t\|_2^2-\frac{1}{2}\|p_t\|_2^2-\kappa\|\varphi_x\|_2^2+C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2).\tag{3.2.9}$$

**Step 2**. Multiplying the first and second equations in (3.2.2) by $v$ and $p$, respectively, and integrating over $[0, L]$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t)=\rho\|v_t\|_2^2+\mu\|p_t\|_2^2-\alpha_1\|v_x\|_2^2-\beta\|\gamma v_x-p_x\|_2^2\\+\delta\int_0^L\varphi v_x\,dx-\int_0^L(G_1(v,p)v+G_2(v,p)p)dx,\end{aligned}\tag{3.2.10}$$

where

$$\mathcal{G}(t)=\rho\int_0^L v_t v dx+\mu\int_0^L p_t p dx+\frac{1}{2}(\|v\|_2^2+\|p\|_2^2).$$

Using the Poincaré and Young's inequalities, we obtain

$$\delta\int_0^L \varphi v_x dx\leqslant \delta\|\varphi\|_2\|v_x\|_2\leqslant C\|\varphi_x\|_2\|v_x\|_2\leqslant C\|\varphi_x\|_2^2+\frac{\alpha_1}{2}\|v_x\|_2^2.$$

Analogously to (3.2.7) and (3.2.8), we deduce that

$$-\int_0^L (G_1(v,p)v+G_2(v,p)p)dx\leqslant C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2).$$

Inserting the last two estimates into (3.2.10), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t)\leqslant{}&\rho\|v_t\|_2^2+\mu\|p_t\|_2^2-\frac{\alpha_1}{2}\|v_x\|_2^2-\beta\|\gamma v_x-p_x\|_2^2\\&+C\|\varphi_x\|_2^2+C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2).\end{aligned}\tag{3.2.11}$$

**Step 3**. We consider the functional

$$\widetilde{\mathcal{L}}(t):=N\widetilde{E}(t)+\mathcal{G}(t),$$

for some $N>0$ to be fixed later. Similarly to Lemma 2.2.4, it is easy to prove that there exist constants $\lambda_1,\lambda_2>0$ such that

$$\lambda_1\widetilde{E}(t)\leqslant\widetilde{\mathcal{L}}(t)\leqslant\lambda_2\widetilde{E}(t),\quad\forall t\geqslant 0.\tag{3.2.12}$$

Combining (3.2.9) and (3.2.11), we obtain

$$\begin{aligned}\frac{d}{dt}\widetilde{\mathcal{L}}(t)\leqslant{}&-\left(\frac{N}{2}-\rho\right)\|v_t\|_2^2-\left(\frac{N}{2}-\mu\right)\|p_t\|_2^2-(N\kappa-C)\|\varphi_x\|_2^2\\&-\alpha_1\|v_x\|_2^2-\beta\|\gamma v_x-p_x\|_2^2+C(N+1)\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2).\end{aligned}\tag{3.2.13}$$

Choosing $N>0$ sufficiently large, we conclude that there exists $\varsigma>0$ such that

$$\frac{d}{dt}\widetilde{\mathcal{L}}(t)\leqslant-\varsigma\widetilde{E}(t)+C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2).\tag{3.2.14}$$

Combining (3.2.12) and (3.2.14), we find

$$\frac{d}{dt}\widetilde{\mathcal{L}}(t)\leqslant-\frac{\varsigma}{\lambda_2}\widetilde{\mathcal{L}}(t)+C\zeta_B(\vartheta_t\omega)(\|v\|_2^2+\|p\|_2^2).$$

Applying Gronwall's lemma, we obtain

$$\widetilde{\mathcal{L}}(t)\leqslant\widetilde{\mathcal{L}}(0)e^{-\sigma_1 t}+C\int_0^t e^{-\sigma_1(t-s)}\zeta_B(\vartheta_s\omega)(\|v(s)\|_2^2+\|p(s)\|_2^2)ds,$$

where $\sigma_1=\frac{\varsigma}{\lambda_2}$. Using (3.2.12) again, we obtain the desired estimate (3.2.1). The proof is now complete. □

### 3.2.2 Asymptotic compactness

In the deterministic setting, the quasi-stability property implies the asymptotic compactness of the associated dynamical system; see [11, Proposition 7.9.4]. In the random framework, we show that the random quasi-stability estimate established above ensures the $\mathcal{D}$-asymptotic compactness of the RDS $\Phi_\varepsilon$, by means of the Kuratowski measure of noncompactness (cf. [25] and [12, p. 54]).

**Definition 3.2.2.** *Let $X$ be a metric space and $C \subset X$. The* Kuratowski measure of noncompactness *of $C$ is defined by*

$$\kappa_X(C) = \inf\left\{ d > 0 : C \subset \bigcup_{i=1}^{n} D_i,\ \operatorname{diam}(D_i) < d,\ n \in \mathbb{N} \right\}.$$

It is well known that $\kappa_X(C) = 0$ if and only if the closure $\overline{C}$ is compact (see, e.g., [12]). The following lemma, which we will use in the proof of asymptotic compactness, is classical:

**Lemma 3.2.3.** *Let $X$ be a complete metric space, and let $\{C_t\}_{t\geqslant 0}$ be a decreasing family of nonempty closed subsets of $X$ (i.e., $C_s \subset C_t$ for $s > t$). If $\kappa_X(C_t) \to 0$ as $t \to \infty$, then $C := \bigcap_{t\geqslant 0} C_t$ is nonempty and compact.*

We now introduce the notion of a suitable pseudometric, which will play a key role in constructing compact attracting sets.

**Definition 3.2.4.** *Let $X$ be a Banach space, $B \subset X$ a bounded set, and let $\varrho$ be a pseudometric on $X$. We say that $\varrho$ is a* compact pseudometric on $B$ *if every sequence $\{x_n\} \subset B$ has a subsequence $\{x_{n_k}\}$ that is Cauchy with respect to $\varrho$, that is, $\varrho(x^{n_k}, x^{n_l}) \to 0$ as $k, l \to \infty$.*

The next result establishes the construction of a compact attracting set for the random dynamical system. The method follows the compactness argument of Ceron–Lopes in [25] (see also [13, 14]).

**Lemma 3.2.5.** *Suppose that hypotheses* **(H1)-(H4)** *hold. Then, for any $\varepsilon > 0$, there exists a compact $\mathcal{D}$-attracting set $\mathcal{C}_\varepsilon = \{\mathcal{C}_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ defined as*

$$\mathcal{C}_\varepsilon(\omega) := \bigcap_{t\geqslant 0} \overline{\Phi_\varepsilon(t, \vartheta_{-t}\omega, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega))}, \quad \forall \omega \in \Omega, \tag{3.2.15}$$

*where $\widetilde{\mathcal{B}}_\varepsilon \in \mathcal{D}$ is the positively invariant $\mathcal{D}$-pullback absorbing set given in Lemma 3.1.1.*

*Proof.* We first prove the existence of the compact set $\mathcal{C}_\varepsilon$, and subsequently show that it is a $\mathcal{D}$-attracting set. We organize the proof into some steps.

**Step 1. Compactness of $\mathcal{C}_\varepsilon(\omega)$.** Our strategy is to apply Lemma 3.2.3; thus, we begin by constructing a decreasing family of nonempty closed sets in $\mathcal{H}$. For each $\omega \in \Omega$ and $\varepsilon > 0$, define

$$C_t^\varepsilon(\omega) := \overline{\Phi_\varepsilon\Big(t, \vartheta_{-t}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega)\Big)}, \qquad t \geq 0.$$

By definition, we then have

$$\mathcal{C}_\varepsilon(\omega) = \bigcap_{t\geq 0} C_t^\varepsilon(\omega).$$

To show that $\{C_t^\varepsilon(\omega)\}_{t\geq 0}$ is decreasing, let $t > s \geq 0$. Using the cocycle property of $\Phi_\varepsilon$, we deduce

$$\begin{aligned}\Phi_\varepsilon(s, \vartheta_{-s}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-s}\omega)) &= \Phi_\varepsilon\big(t + (s-t),\, \vartheta_{-s}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-s}\omega)\big) \\ &= \Phi_\varepsilon\Big(t,\, \vartheta_{s-t}\vartheta_{-s}\omega,\, \Phi_\varepsilon(s-t, \vartheta_{-s}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-s}\omega))\Big) \\ &= \Phi_\varepsilon\Big(t,\, \vartheta_{-t}\omega,\, \Phi_\varepsilon(s-t, \vartheta_{-s}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-s}\omega))\Big). \end{aligned} \tag{3.2.16}$$

Since $\widetilde{\mathcal{B}}_\varepsilon$ is positively invariant, we have

$$\Phi_\varepsilon(s-t,\, \vartheta_{-s}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-s}\omega)) \subset \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{s-t}\vartheta_{-s}\omega) = \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega).$$

Substituting this into (3.2.16), we obtain

$$\Phi_\varepsilon(s, \vartheta_{-s}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-s}\omega)) \subset \Phi_\varepsilon(t, \vartheta_{-t}\omega,\, \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega)).$$

Taking closures on both sides yields

$$C_t^\varepsilon(\omega) \subset C_s^\varepsilon(\omega), \qquad \forall\, t > s \geq 0.$$

Thus, $\{C_t^\varepsilon(\omega)\}_{t\geq 0}$ is a decreasing family of nonempty closed subsets of $\mathcal{H}$. Therefore, by Lemma 3.2.3, the compactness of $\mathcal{C}_\varepsilon(\omega)$ will follow once we show that $\kappa_{\mathcal{H}}(C_t^\varepsilon(\omega)) \to 0$ as $t \to \infty$.

Given $t > 0$, we introduce the pseudometric $\varrho^t : \widetilde{\mathcal{B}}_\varepsilon(\omega) \times \widetilde{\mathcal{B}}_\varepsilon(\omega) \to \mathbb{R}$ defined by

$$\varrho^t(U^1, U^2) = \left(C \int_0^t e^{-\sigma_1(t-s)}\, \zeta_{\widetilde{\mathcal{B}}_\varepsilon}(\vartheta_s\omega)\, \big(\|v(s)\|_2^2 + \|p(s)\|_2^2\big)\, ds\right)^{\frac{1}{2}}, \quad U^1, U^2 \in \widetilde{\mathcal{B}}_\varepsilon(\omega),$$

where $\zeta_{\widetilde{\mathcal{B}}_\varepsilon}(\omega) = 1 + \|\widetilde{\mathcal{B}}_\varepsilon(\omega)\|_{\mathcal{H}}^{2(r-1)}$.

**Claim 1.** The pseudometric $\varrho^t$ is compact on $\widetilde{\mathcal{B}}_\varepsilon(\omega)$.

Indeed, let $\{U^n\} \subset \widetilde{\mathcal{B}}_\varepsilon(\omega)$. By the estimate (2.2.18), there exists a constant $C_{t,\omega} > 0$ such that

$$\|\Phi_\varepsilon(s, \omega, U^n)\|_{\mathcal{H}}^2 \leqslant C_{t,\omega}, \qquad \forall\, s \in [0, t],\ \forall\, n \in \mathbb{N}.$$

Writing $\Phi_\varepsilon(s, \omega, U^n) = (v^n(s), p^n(s), v_t^n(s), p_t^n(s), \varphi^n(s))$, the above estimate yields

$$\|\Phi_\varepsilon(s, \omega, U^n)\|_{\mathcal{H}}^2 = \rho\|v_t^n\|_2^2 + \mu\|p_t^n\|_2^2 + \|\varphi^n\|_2^2 + \alpha_1\|v_x^n\|_2^2 + \beta\|\gamma v_x^n - p_x^n\|_2^2 \leqslant C_{t,\omega},$$

for all $s \in [0, t]$. Consequently,

$$\{(v^n, p^n)\} \text{ is bounded in } L^\infty(0, t; (H_*^1(0, L))^2),$$

$$\{(v_t^n, p_t^n)\} \text{ is bounded in } L^\infty(0, t; (L^2(0, L))^2).$$

By the Aubin–Lions compactness lemma, we infer that

$$\{(v^n, p^n)\} \text{ is precompact in } C([0, t]; (L^2(0, L))^2).$$

Therefore, there exists a subsequence $\{(v^{n_k}, p^{n_k})\}$ such that

$$\lim_{k,l\to\infty} \max_{s\in[0,t]} \big(\|v^{n_k}(s) - v^{n_l}(s)\|_2^2 + \|p^{n_k}(s) - p^{n_l}(s)\|_2^2\big) = 0.$$

Using the definition of $\varrho^t$, we obtain

$$\begin{aligned}\lim_{k,l\to\infty} \varrho^t(U^{n_k}, U^{n_l}) \leqslant{}& \left(C\int_0^t e^{-\sigma_1(t-s)}\zeta_{\widetilde{\mathcal{B}}_\varepsilon}(\vartheta_s\omega)\, ds\right)^{1/2} \\ &\times \left(\lim_{k,l\to\infty} \max_{s\in[0,t]} \big(\|v^{n_k}(s) - v^{n_l}(s)\|_2^2 + \|p^{n_k}(s) - p^{n_l}(s)\|_2^2\big)\right)^{1/2} \\ ={}& 0.\end{aligned}$$

Thus,

$$\lim_{k,l\to\infty} \varrho^t(U^{n_k}, U^{n_l}) = 0,$$

which shows that $\varrho^t$ is a compact pseudometric on $\widetilde{\mathcal{B}}_\varepsilon(\omega)$. $\square$

By the definition of the Kuratowski measure of noncompactness, for any $\eta > 0$, there exist sets $D_1, \ldots, D_n \subset \mathcal{H}$ such that

$$\widetilde{\mathcal{B}}_\varepsilon(\omega) \subset \bigcup_{i=1}^n D_i, \qquad \operatorname{diam}(D_i) < \kappa_{\mathcal{H}}\big(\widetilde{\mathcal{B}}_\varepsilon(\omega)\big) + \eta. \tag{3.2.17}$$

**Claim 2.** There exists a finite set $N = \{y_j\}_{j=1}^m \subset \widetilde{\mathcal{B}}_\varepsilon(\omega)$ such that for every $y \in \widetilde{\mathcal{B}}_\varepsilon(\omega)$ there exists $j \in \{1, \ldots, m\}$ such that $\varrho^t(y,\, y_j) < \eta$.

Assume, towards a contradiction, that this claim is false. Then there exists a sequence $\{Z^n\} \subset \widetilde{\mathcal{B}}_\varepsilon(\omega)$ such that

$$\varrho^t(Z^n,\, Z^m) \geqslant \eta \quad \forall n \neq m. \tag{3.2.18}$$

Since $\varrho^t$ is compact (Claim 1), there exists a subsequence $\{Z^{n_k}\}$ of $\{Z^n\}$ such that

$$\lim_{k,\, l \to \infty} \varrho^t(Z^{n_k},\, Z^{n_l}) = 0.$$

This contradicts (3.2.18). Therefore, Claim 2 holds.

**Claim 3.** Given $y_j \in N = \{y_j\}_{j=1}^m$, we have

$$\widetilde{\mathcal{B}}_\varepsilon(\omega) \subset \bigcup_{j=1}^m C_j \quad \text{with} \quad C_j = \{y \in \mathcal{H} : \varrho^t(y, y_j) < \eta, j = 1, \ldots, m\}. \tag{3.2.19}$$

Indeed, let $y \in \widetilde{\mathcal{B}}_\varepsilon(\omega)$. By Claim 2, there exists $y_j \in N$ such that $\varrho^t(y,\, y_j) < \eta$. Therefore,

$$y \in C_j \subset \bigcup_{j=1}^m C_j.$$

Hence, Claim 3 is valid.

By Lemma 3.2.1, for any $U^1, U^2 \in \widetilde{\mathcal{B}}_\varepsilon(\omega)$ and $t \geqslant 0$, we have

$$\|\Phi_\varepsilon(t, \omega, U^1) - \Phi_\varepsilon(t, \omega, U^2)\|_{\mathcal{H}} \leqslant \sqrt{\mathrm{M}} e^{-\frac{\sigma_1}{2}t} \|U^1 - U^2\|_{\mathcal{H}} + \varrho^t(U^1, U^2). \tag{3.2.20}$$

Taking into account (3.2.17) and (3.2.20), and applying the triangle inequality for the pseudo-metric $\varrho^t$, we obtain, for all $U^1, U^2 \in D_i \cap C_j$,

$$\|\Phi_\varepsilon(t, \omega, U^1) - \Phi_\varepsilon(t, \omega, U^2)\|_{\mathcal{H}} \leqslant \sqrt{\mathrm{M}} e^{-\frac{\sigma_1}{2}t}\, \kappa_{\mathcal{H}}(\widetilde{\mathcal{B}}_\varepsilon(\omega)) + \big(\sqrt{\mathrm{M}} + 2\big)\eta.$$

Since $\eta > 0$ is arbitrary, we conclude that

$$\|\Phi_\varepsilon(t, \omega, U^1) - \Phi_\varepsilon(t, \omega, U^2)\|_{\mathcal{H}} \leqslant \sqrt{\mathrm{M}} e^{-\frac{\sigma_1}{2}t}\, \kappa_{\mathcal{H}}(\widetilde{\mathcal{B}}_\varepsilon(\omega)), \quad \forall\, U^1, U^2 \in D_i \cap C_j.$$

This implies that

$$\operatorname{diam}\big(\Phi_\varepsilon(t, \omega, D_i \cap C_j)\big) \leqslant \sqrt{\mathrm{M}} e^{-\frac{\sigma_1}{2}t} \operatorname{diam}\big(\widetilde{\mathcal{B}}_\varepsilon(\omega)\big).$$

Now, combining (3.2.17) and (3.2.19), we obtain

$$\Phi_\varepsilon(t, \omega, \widetilde{\mathcal{B}}_\varepsilon(\omega)) \subset \bigcup_{i,j} \Phi_\varepsilon(t, \omega, D_i \cap C_j).$$

Thus, by the definition of the measure of noncompactness, we deduce

$$\kappa_{\mathcal{H}}\big(\Phi_\varepsilon(t,\omega,\widetilde{\mathcal{B}}_\varepsilon(\omega))\big) \leqslant \sqrt{\mathrm{M}}e^{-\frac{\sigma_1}{2}t}\,\mathrm{diam}\big(\widetilde{\mathcal{B}}_\varepsilon(\omega)\big).$$

Replacing $\omega$ by $\vartheta_{-t}\omega$ and using the fact that $\widetilde{\mathcal{B}}_\varepsilon \in \mathcal{D}$, we obtain

$$\kappa_{\mathcal{H}}\big(C_t^\varepsilon(\omega)\big) \leqslant \sqrt{\mathrm{M}}e^{-\frac{\sigma_1}{2}t}\,\mathrm{diam}\big(\widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega)\big) \to 0 \quad \text{as } t\to\infty. \tag{3.2.21}$$

Combining this with the discussion at the beginning of Step 1 and Lemma 3.2.3, we conclude that $\mathcal{C}_\varepsilon(\omega)$ is compact.

**Step 2. Attracting property.** Suppose, by contradiction, that $\mathcal{C}_\varepsilon$ does not satisfy the $\mathcal{D}$-attracting property. Then there exist $\eta > 0$, $D \in \mathcal{D}$, and sequences $t_n \to \infty$ and $U_0^n \in D(\vartheta_{-t_n}\omega)$ such that

$$\mathrm{dist}_{\mathcal{H}}(\Phi_\varepsilon(t_n,\vartheta_{-t_n}\omega,U_0^n),\mathcal{C}_\varepsilon(\omega)) > \eta, \quad \forall n\in\mathbb{N}. \tag{3.2.22}$$

For an arbitrary $t \geqslant 0$, there exists $n_t \in \mathbb{N}$ such that $t_n \geqslant t$ for all $n \geqslant n_t$. Thus, by the cocycle property, we can write

$$\begin{aligned}\Phi_\varepsilon(t_n,\vartheta_{-t_n}\omega,U_0^n) &= \Phi_\varepsilon(t+(t_n-t),\vartheta_{-t_n}\omega,U_0^n)\\ &= \Phi_\varepsilon(t,\vartheta_{-t}\omega,\Phi_\varepsilon(t_n-t,\vartheta_{-t_n}\omega,U_0^n)), \quad \forall n\geqslant n_t.\end{aligned} \tag{3.2.23}$$

On the other hand, since $t_n - t \to \infty$ and $\widetilde{\mathcal{B}}_\varepsilon$ is $\mathcal{D}$-pullback absorbing, there exists $n_1 \geqslant n_t$ such that

$$\Phi_\varepsilon(t_n-t,\vartheta_{-t_n}\omega,U_0^n) = \Phi_\varepsilon(t_n-t,\vartheta_{-(t_n-t)}\vartheta_{-t}\omega,U_0^n) \in \widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega), \quad \forall n \geqslant n_1. \tag{3.2.24}$$

Combining (3.2.23) and (3.2.24), we obtain that

$$\Phi_\varepsilon(t_n,\vartheta_{-t_n}\omega,U_0^n) \in \Phi_\varepsilon(t,\vartheta_{-t}\omega,\widetilde{\mathcal{B}}_\varepsilon(\vartheta_{-t}\omega)) \subset C_t^\varepsilon(\omega), \quad \forall n\geqslant n_1.$$

Since $t \geqslant 0$ is arbitrary, we conclude that

$$\Phi_\varepsilon(t_n,\vartheta_{-t_n}\omega,U_0^n) \in \bigcap_{t\geqslant 0} C_t^\varepsilon(\omega) = \mathcal{C}_\varepsilon(\omega), \quad \forall n \geqslant n_1.$$

which contradicts (3.2.22). Therefore the proof is complete. □

### 3.2.3 Existence of the random attractor

Having now verified all the required assumptions, we are in a position to apply the abstract existence theorem for random attractors (cf. Teorema 1.2.11). We therefore state the main result of this chapter.

**Theorem 3.2.6** (Random attractor). *Suppose that hypotheses* **(H1)**–**(H4)** *hold. Then, for every* $\varepsilon \in (0,1]$*, the random dynamical system* $\Phi_\varepsilon$ *generated by the stochastic piezoelectric system* (2.0.1)-(2.0.2) *possesses a unique* $\mathcal{D}$*-random attractor* $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ *in the phase space* $\mathcal{H}$*, where*

$$\mathcal{A}_\varepsilon(\omega) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \Phi_\varepsilon(t, \vartheta_{-t}\omega, \mathcal{B}_\varepsilon(\vartheta_{-t}\omega))}.$$

*Proof.* Lemma 3.1.1 ensures that $\Phi_\varepsilon$ has a closed measurable $\mathcal{D}$-pullback absorbing set $\mathcal{B}_\varepsilon \in \mathcal{D}$. Moreover, Lemma 3.2.5 establishes that $\Phi_\varepsilon$ is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{H}$. Therefore, by Theorem 1.2.11, $\Phi_\varepsilon$ possesses a unique $\mathcal{D}$-random attractor $\mathcal{A}_\varepsilon$ in $\mathcal{H}$. This completes the proof. $\square$

# Chapter 4

# Finite-dimensionality of random attractors

In this chapter, we investigate the finite-dimensionality of the $\mathcal{D}$-random attractors obtained in Chapter 3. Establishing that a random attractor has finite fractal (or Hausdorff) dimension is a central issue in the qualitative theory of infinite-dimensional dynamical systems, since it shows that, despite the infinite-dimensional nature of the phase space, the long-time dynamics is effectively governed by only finitely many degrees of freedom.

A key novelty of our analysis, compared to the work of Chueshov and Schmalfuß [15], lies in the treatment of the nonlinear terms. In [15], the nonlinearities are assumed to be globally Lipschitz, which substantially simplifies the analysis of the fractal dimension. In contrast, the nonlinearities considered here may exhibit *arbitrary polynomial growth* and are *not globally Lipschitz*. This lack of global Lipschitz continuity leads to significant technical difficulties, particularly in the asymptotic control of the time-dependent coefficient $t \mapsto C(\vartheta_t\omega)$, as $|t| \to \infty$ which arises in the random quasi-stability estimate.

To overcome these difficulties, we employ the *random quasi-stability method*, following the framework developed by Chueshov and Lasiecka for deterministic systems [11, 14], together with its random extension introduced in [15], and complemented by ergodic-theoretic arguments to control the long-time behavior of the random coefficients.

### 4.1 Technical lemmas

In this section, we establish some technical lemmas that will play an important role in the proof of the finite-dimensionality of the random attractor.

**Lemma 4.1.1.** *Let $R_\varepsilon : \Omega \to \mathbb{R}$ be the random variable defined in* (3.1.2)*. Then,*

$$\mathbb{E}\big[R_\varepsilon^{r-1}(\omega)\big] < \infty.$$

*Proof.* We divide the argument into two cases.

**Case 1.** $1 < r \leqslant 2$**.** In this case, the inequality $a^{r-1} \leqslant 1 + a$ holds for all $a \geqslant 0$. Thus, using (2.1.10), we obtain

$$\begin{aligned}
\mathbb{E}\big[R_\varepsilon^{r-1}(\omega)\big] &= \mathbb{E}\left[\left(C_f \int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds + m_f L\right)^{r-1}\right]\\
&\leqslant \mathbb{E}\left[1 + m_f L + C_f \int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds\right]\\
&= 1 + m_f L + \frac{C_f}{\sigma_0} + C_f\varepsilon^2 \int_{-\infty}^{0} e^{\sigma_0 s}\,\mathbb{E}\big[|z(\vartheta_s\omega)|^2\big]\;ds\\
&= 1 + m_f L + \frac{C_f}{\sigma_0} + \frac{C_f\varepsilon^2\,\Gamma\big(\frac{3}{2}\big)}{\sigma_0\sqrt{\pi\sigma^2}}.
\end{aligned}$$

Consequently, $\mathbb{E}\big[R_\varepsilon^{r-1}(\omega)\big] < \infty$.

**Case 2.** $r > 2$**.** In this case, the inequality $(a+b)^{r-1} \leqslant 2^{r-2}\big(a^{r-1} + b^{r-1}\big)$, holds for all $a, b \geqslant 0$. Hence,

$$\begin{aligned}
\mathbb{E}\big[R_\varepsilon^{r-1}(\omega)\big] &= \mathbb{E}\left[\left(C_f \int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds + m_f L\right)^{r-1}\right]\\
&\leqslant \mathbb{E}\left[2^{r-2}(m_f L)^{r-1} + 2^{r-2}\left(C_f \int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds\right)^{r-1}\right]\\
&= 2^{r-2}(m_f L)^{r-1} + 2^{r-2}C^{r-1}\mathbb{E}\left[\left(\int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds\right)^{r-1}\right].
\end{aligned}$$

Therefore,

$$\mathbb{E}\big[R_\varepsilon^{r-1}(\omega)\big] \leqslant C_r + C_r\,\mathbb{E}\left[\left(\int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds\right)^{r-1}\right], \tag{4.1.1}$$

where $C_r = \max\{\,2^{r-2}(m_f L)^{r-1},\ 2^{r-2}C^{r-1}\,\}$.

Now we estimate the last term in (4.1.1). By Hölder's inequality, we have

$$\begin{aligned}
&\int_{-\infty}^{0} e^{\sigma_0 s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds = \int_{-\infty}^{0} e^{\frac{\sigma_0}{2}s}\,e^{\frac{\sigma_0}{2}s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)\,ds\\
&\leqslant \left(\int_{-\infty}^{0} e^{\frac{\sigma_0(r-1)}{2(r-2)}s}\,ds\right)^{\frac{r-2}{r-1}} \left(\int_{-\infty}^{0} e^{\frac{\sigma_0(r-1)}{2}s}\big(1+\varepsilon^2|z(\vartheta_s\omega)|^2\big)^{r-1}ds\right)^{\frac{1}{r-1}}.
\end{aligned}$$

Raising both sides to the power $(r-1)$ yields

$$
\begin{aligned}
&\left(\int_{-\infty}^{0} e^{\sigma_0 s}(1+\varepsilon^2|z(\vartheta_s\omega)|^2)\,ds\right)^{r-1}\\
&\leqslant \left(\int_{-\infty}^{0} e^{\sigma_0 \frac{s}{2}\frac{(r-1)}{(r-2)}}ds\right)^{r-2}\left(\int_{-\infty}^{0} e^{\sigma_0\frac{s}{2}(r-1)}(1+\varepsilon^2|z(\vartheta_s\omega)|^2)^{r-1}ds\right)\\
&= \left(\frac{(r-2)2}{(r-1)\sigma_0}\right)^{r-2}\left(\int_{-\infty}^{0} e^{\sigma_0\frac{s}{2}(r-1)}(1+\varepsilon^2|z(\vartheta_s\omega)|^2)^{r-1}ds\right).
\end{aligned}
$$

Taking expectation and using (2.1.10) gives

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_{-\infty}^{0} e^{\sigma_0 s}\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)ds\right)^{r-1}\right]\\
&\leqslant \left(\frac{(r-2)2}{(r-1)\sigma_0}\right)^{r-2}\mathbb{E}\left[\int_{-\infty}^{0} e^{\sigma_0\frac{s}{2}(r-1)}(1+\varepsilon^2|z(\vartheta_s\omega)|^2)^{r-1}ds\right]\\
&= \left(\frac{(r-2)2}{(r-1)\sigma_0}\right)^{r-2}\int_{-\infty}^{0} e^{(r-1)\frac{\sigma_0}{2}s}\mathbb{E}\left[\left(1+\varepsilon^2|z(\vartheta_s\omega)|^2\right)^{r-1}\right]ds\\
&\leqslant \left(\frac{(r-2)2}{(r-1)\sigma_0}\right)^{r-2}\int_{-\infty}^{0} e^{(r-1)\frac{\sigma_0}{2}s}\mathbb{E}\left[2^{r-2}\left(1+\varepsilon^{2(r-1)}|z(\vartheta_s\omega)|^{2(r-1)}\right)\right]ds\\
&= 2^{r-2}\left(\frac{(r-2)2}{(r-1)\sigma_0}\right)^{r-2}\left(\frac{2}{(r-1)\sigma_0}+\varepsilon^{2(r-1)}\int_{-\infty}^{0} e^{\frac{\sigma_0}{2}s(r-1)}\mathbb{E}\left[|z(\vartheta_s\omega)|^{2(r-1)}\right]ds\right)\\
&= 2^{r-2}\left(\frac{(r-2)2}{(r-1)\sigma_0}\right)^{r-2}\left(\frac{2}{(r-1)\sigma_0}+\frac{2\varepsilon^{2(r-1)}\Gamma\left(\frac{2r-1}{2}\right)}{(r-1)\sigma_0\sqrt{\pi\sigma^{2(r-1)}}}\right)<\infty.
\end{aligned}
\tag{4.1.2}
$$

Combining (4.1.1) and (4.1.2), we conclude that $\mathbb{E}[R_\varepsilon^{r-1}(\omega)]<\infty$. This completes the proof.

□

The following remark will be used in the sequel.

**Remark 4.1.2.** *Let $\zeta:\Omega\to\mathbb{R}$ be a random variable with finite expectation. Since $\vartheta_t$ preserves the measure $\mathbb{P}$, we have*

$$\mathbb{E}[\zeta(\vartheta_t\omega)]=\mathbb{E}[\zeta(\omega)],\quad \forall t\in\mathbb{R}.$$

*Note that this identity is independent of both $t$ and $\omega$.*

The following lemma provides an estimate for the difference between two solutions of (2.1.12)–(2.1.13) lying on the random attractor.

**Lemma 4.1.3.** *Let $T>0$, $\omega\in\Omega$, and $U_0^1,U_0^2\in\mathcal{A}_\varepsilon(\omega)$. Consider the corresponding solutions*

$$\Phi_\varepsilon(t,\omega,U_0^i)=\big(v^i(t),\,p^i(t),\,v_t^i(t),\,p_t^i(t),\,\varphi^i(t)\big),\qquad i=1,2,$$

*and define $(v,p) := (v^1 - v^2,\, p^1 - p^2)$. Then there exist a nonnegative scalar function $\tilde{b}(T)$ on $\mathbb{R}^+$ and a positive random variable $\tilde{C}_T(\omega)$ such that*

$$\|\Phi_\varepsilon(T,\omega,U_0^1) - \Phi_\varepsilon(T,\omega,U_0^2)\|_{\mathcal{H}}^2 + \int_T^{2T} \|\Phi_\varepsilon(t,\omega,U_0^1) - \Phi_\varepsilon(t,\omega,U_0^2)\|_{\mathcal{H}}^2\, dt$$
$$\leqslant \tilde{b}(T)\, \|U_0^1 - U_0^2\|_{\mathcal{H}}^2 + \tilde{C}_T(\omega) \sup_{s\in[0,2T]} \left(\|v(s)\|_2^2 + \|p(s)\|_2^2\right). \tag{4.1.3}$$

*Moreover,*

$$\lim_{T\to\infty} \tilde{b}(T) = 0.$$

*Proof.* Integrating estimate (3.2.1) from $T$ to $2T$ with respect to $t$, we obtain

$$\begin{aligned}\int_T^{2T} \|\Phi_\varepsilon(t,\omega,U_0^1) - \Phi_\varepsilon(t,\omega,U_0^2)\|_{\mathcal{H}}^2 dt &\leqslant \mathrm{M}\left(\int_T^{2T} e^{-\sigma_1 t}\, dt\right) \|U_0^1 - U_0^2\|_{\mathcal{H}}^2 \\ &+ C\int_T^{2T}\int_0^t e^{-\sigma_1(t-s)}\vartheta_{\mathcal{A}_\varepsilon}(\vartheta_s\omega)(\|v(s)\|_2^2 + \|p(s)\|_2^2)\, ds\, dt.\end{aligned} \tag{4.1.4}$$

Consequently,

$$\begin{aligned}\int_T^{2T} \|\Phi_\varepsilon(t,\omega,U^1) - \Phi_\varepsilon(t,\omega,U^2)\|_{\mathcal{H}}^2 dt &\leqslant b_T\|U^1 - U^2\|_{\mathcal{H}}^2 \\ &+ c_T(\omega) \sup_{s\in[0,2T]} \left(\|v(s)\|_2^2 + \|p(s)\|_2^2\right),\end{aligned} \tag{4.1.5}$$

where

$$b(T) = \frac{\mathrm{M}e^{-\sigma_1 T}}{\sigma_1}(1 - e^{-\sigma_1 T}), \quad c_T(\omega) = C\int_T^{2T}\int_0^t e^{-\sigma_1(t-s)}\zeta_{\mathcal{A}_\varepsilon}(\vartheta_s\omega) ds dt.$$

Moreover, it also follows from (3.2.1) that

$$\|\Phi_\varepsilon(T,\omega,U^1) - \Phi_\varepsilon(T,\omega,U^2)\|_{\mathcal{H}}^2 \leqslant \mathrm{M}e^{-\sigma_1 T}\|U^1 - U^2\|_{\mathcal{H}}^2 + \varkappa_T(\omega) \sup_{s\in[0,2T]} \left(\|v(s)\|_2^2 + \|p(s)\|_2^2\right), \tag{4.1.6}$$

where

$$\varkappa_T(\omega) = \int_0^T e^{-\sigma_1(T-s)}\zeta_{\mathcal{A}_\varepsilon}(\vartheta_s\omega) ds.$$

Adding (4.1.5) and (4.1.6) yields

$$\|\Phi_\varepsilon(T,\omega,U^1) - \Phi_\varepsilon(T,\omega,U^2)\|_{\mathcal{H}}^2 + \int_T^{2T} \|\Phi_\varepsilon(t,\omega,U^1) - \Phi_\varepsilon(t,\omega,U^2)\|_{\mathcal{H}}^2 dt$$
$$\leqslant \tilde{b}(T)\|U^1 - U^2\|_{\mathcal{H}}^2 + \tilde{C}_T(\omega) \sup_{s\in[0,2T]} \left(\|v(s)\|_2^2 + \|p(s)\|_2^2\right),$$

where

$$\tilde{b}(T) = \mathrm{M}e^{-\sigma_1 T} + \frac{\mathrm{M}e^{-\sigma_1 T}}{\sigma_1}(1 - e^{-\sigma_1 T}), \quad \tilde{C}_T(\omega) = c_T(\omega) + \varkappa_T(\omega).$$

Clearly, $\lim_{T\to\infty} \tilde{b}(T) = 0$. The proof is complete. □

### 4.2 Finite fractal dimension

In this section, we establish an upper bound for the fractal dimension of the random attractor. We begin by recalling the notion of fractal dimension in a metric space.

**Definition 4.2.1.** *Let $\mathbb{M}$ be a compact set in a metric space $X$. The fractal dimension (or say box-counting, metric or entropy) $\dim_f^X \mathbb{M}$ of $\mathbb{M}$ is defined as*

$$\dim_f^X \mathbb{M} = \limsup_{\nu \to 0} \frac{\ln N(\mathbb{M},\nu)}{-\ln \nu},$$

*where $N(\mathbb{M},\nu)$ is the Kolmogorov's $\nu$-entropy of $\mathbb{M}$, that is, the minimal number of closed balls in $\mathbb{X}$ of radius $\nu$ which cover the set $\mathbb{M}$. Since $\mathbb{M}$ is compact in $X$, we find that $N(\mathbb{M},\nu)$ is finite for any $\nu > 0$.*

**Theorem 4.2.2** (Finite fractal dimension)**.** *Assume that hypotheses* **(H1)–(H4)** *are satisfied. Then the random attractor $\mathcal{A}_\varepsilon$ associated with the random dynamical system generated by* (2.0.1)-(2.0.2) *possesses finite fractal dimension in $\mathcal{H}$. More precisely, for every $\varepsilon \in (0,1]$, we have*

$$\dim_f^{\mathcal{H}} \big(\mathcal{A}_\varepsilon(\omega)\big) < \infty, \qquad \mathbb{P}\text{-a.e.} \tag{4.2.1}$$

*Proof.* The proof is rather long, and in order to make it more comprehensible, we divide it into several steps and auxiliary claims.

**Step 1.** **$T$-trajectory space.** Let $T > 0$ be sufficiently large and define the space $\mathcal{H}_T = \mathcal{H} \times W(0,T)$, where

$$W(0,T) = \big\{(v,p) \in L^\infty\big(0,T;(H^1_*(0,L))^2\big) : \|(v,p)\|_{W(0,T)} = \|(v,p,v_t,p_t)\|_{L^2(0,T;(H^*_1)^2\times(L^2)^2)} < \infty\big\}.$$

Next, define

$$\mathcal{A}^\varepsilon_T(\omega) := \{(U_0, v(t), p(t)) : U_0 \in \mathcal{A}_\varepsilon(\omega),\ t \in [0,T]\} \subset \mathcal{H}_T, \tag{4.2.2}$$

where $(v(t),p(t))$ denote the first two components of

$$\Phi_\varepsilon(t,\omega,U_0) = (v(t),p(t),v_t(t),p_t(t),\varphi(t)), \quad t \geqslant 0.$$

Consider the mapping $V : \mathcal{A}^\varepsilon_T(\omega) \to \mathcal{H}_T$ given by

$$V(U_0,v(t),p(t)) = \big(\Phi_\varepsilon(T,\omega,U_0),\ v(t+T),\ p(t+T)\big).$$

**Claim 1.** The map $V$ is Lipschitz on $\mathcal{A}_T^\varepsilon(\omega)$; more precisely, there exists a positive random variable $L_T(\omega)$ such that

$$\|VZ^1 - VZ^2\|_{\mathcal{H}_T} \leqslant L_T(\omega)\, \|Z^1 - Z^2\|_{\mathcal{H}_T}, \quad \forall Z^1, Z^2 \in \mathcal{A}_T^\varepsilon(\omega). \tag{4.2.3}$$

Moreover,

$$\mathbb{E}\big[\ln L_T(\omega)\big] < \infty. \tag{4.2.4}$$

Indeed, let $Z^1 = (U_0^1, v^1(t), p^1(t))$ and $Z^2 = (U_0^2, v^2(t), p^2(t)) \in \mathcal{A}_T^\varepsilon(\omega)$, and set $v = v^1 - v^2$ and $p = p^1 - p^2$. Then

$$\begin{aligned}
\|VZ^1 - VZ^2\|_{\mathcal{H}_T}^2 &= \|\Phi_\varepsilon(T,\omega,U_0^1) - \Phi_\varepsilon(T,\omega,U_0^2)\|_{\mathcal{H}}^2 + \|(v(t+T), p(t+T))\|_{W(0,T)}^2 \\
&= \|\Phi_\varepsilon(T,\omega,U_0^1) - \Phi_\varepsilon(T,\omega,U_0^2)\|_{\mathcal{H}}^2 \\
&\quad + \int_0^T \|\Phi_\varepsilon(t+T,\omega,U_0^1) - \Phi_\varepsilon(t+T,\omega,U_0^2)\|_{\mathcal{H}}^2\, dt \\
&= \|\Phi_\varepsilon(T,\omega,U_0^1) - \Phi_\varepsilon(T,\omega,U_0^2)\|_{\mathcal{H}}^2 \\
&\quad + \int_T^{2T} \|\Phi_\varepsilon(t,\omega,U_0^1) - \Phi_\varepsilon(t,\omega,U_0^2)\|_{\mathcal{H}}^2\, dt.
\end{aligned}$$

Using (2.1.27), we obtain

$$\begin{aligned}
\|VZ^1 - VZ^2\|_{\mathcal{H}_T}^2 &\leqslant e^{2\int_0^T \Upsilon(s)\,ds}\, \|U_0^1 - U_0^2\|_{\mathcal{H}}^2 + \int_T^{2T} e^{2\int_0^t \Upsilon(s)\,ds}\, \|U_0^1 - U_0^2\|_{\mathcal{H}}^2\, dt \\
&= \tilde{G}_T(\omega)\, \|U_0^1 - U_0^2\|_{\mathcal{H}}^2 \leqslant \tilde{G}_T(\omega)\, \|Z^1 - Z^2\|_{\mathcal{H}_T}^2,
\end{aligned}$$

where $\Upsilon$ is given in (2.1.26) and

$$\tilde{G}_T(\omega) = e^{2\int_0^T \Upsilon(s)\,ds} + \int_T^{2T} e^{2\int_0^t \Upsilon(s)\,ds}\, dt.$$

Therefore,

$$\|VZ^1 - VZ^2\|_{\mathcal{H}_T} \leqslant L_T(\omega)\, \|Z^1 - Z^2\|_{\mathcal{H}_T},$$

where $L_T(\omega) = \sqrt{\tilde{G}_T(\omega)}$.

On the other hand, by the minimality of the attractor and the definition of $\Upsilon$, we obtain

$$\begin{aligned}
L_T(\omega) &\leqslant e^{C\int_0^T (1+R_\varepsilon(\vartheta_s(\omega)))\,ds} + \int_0^T e^{C\int_0^t (1+R_\varepsilon(\vartheta_s(\omega)))\,ds}\, dt \\
&\leqslant (1+T)\, e^{C\int_0^T (1+R_\varepsilon(\vartheta_s(\omega)))\,ds}.
\end{aligned}$$

Taking logarithms yields

$$\ln(L_T(\omega)) \leqslant \ln(1+T) + C\int_0^T (1+R_\varepsilon(\vartheta_s(\omega)))\,ds.$$

Since $\mathbb{E}[R_\varepsilon(\omega)] < \infty$ by Lemma 4.1.1, it follows from Remark 4.1.2 that

$$\begin{aligned}\mathbb{E}[\ln(L_T(\omega))] &\leqslant \ln(1+T) + C\int_0^T \left(1+\mathbb{E}[R_\varepsilon(\vartheta_s(\omega))]\right)\,ds \\ &= \ln(1+T) + C\int_0^T \left(1+\mathbb{E}[R_\varepsilon(\omega)]\right)\,ds \\ &= \ln(1+T) + CT\left(1+\mathbb{E}[R_\varepsilon(\omega)]\right) < \infty.\end{aligned}$$

This completes the proof of Claim 1.

**Claim 2.** $\mathcal{A}_T^\varepsilon(\omega)$ is invariant under $V$, that is,

$$V(\mathcal{A}_T^\varepsilon(\omega)) = \mathcal{A}_T^\varepsilon(\vartheta_T\omega), \quad \forall\omega\in\Omega. \tag{4.2.5}$$

Indeed, note that

$$V(\mathcal{A}_T^\varepsilon(\omega)) = \left(\Phi_\varepsilon(T,\omega,\mathcal{A}_\varepsilon(\omega)),\ v(t+T,\omega,\mathcal{A}_\varepsilon(\omega)),\ p(t+T,\omega,\mathcal{A}_\varepsilon(\omega))\right).$$

By the invariance property of the attractor $\mathcal{A}_\varepsilon$, we have

$$\Phi_\varepsilon(T,\omega,\mathcal{A}_\varepsilon(\omega)) = \mathcal{A}_\varepsilon(\vartheta_T\omega).$$

Therefore, by the cocycle property,

$$\Phi_\varepsilon(t+T,\omega,\mathcal{A}_\varepsilon(\omega)) = \Phi_\varepsilon\big(t,\vartheta_T\omega,\Phi_\varepsilon(T,\omega,\mathcal{A}_\varepsilon(\omega))\big) = \Phi_\varepsilon\big(t,\vartheta_T\omega,\mathcal{A}_\varepsilon(\vartheta_T\omega)\big).$$

Consequently,

$$V(\mathcal{A}_T^\varepsilon(\omega)) = \big(\mathcal{A}_\varepsilon(\vartheta_T\omega),\ v(t,\vartheta_T\omega,\mathcal{A}_\varepsilon(\vartheta_T\omega)),\ p(t,\vartheta_T\omega,\mathcal{A}_\varepsilon(\vartheta_T\omega))\big) = \mathcal{A}_T^\varepsilon(\vartheta_T\omega).$$

This completes the proof of Claim 2.

**Claim 3.** For any $Z^1, Z^2 \in \mathcal{A}_T^\varepsilon(\omega)$ there exists a compact seminorm $n_T$ on $\mathcal{H}_T$ such that

$$\|VZ^1 - VZ^2\|_{\mathcal{H}_T} \leqslant \eta_T\,\|Z^1 - Z^2\|_{\mathcal{H}_T} + K_T(\omega)\big[n_T(Z^1-Z^2) + n_T(VZ^1 - VZ^2)\big], \tag{4.2.6}$$

where $\eta_T > 0$ is a deterministic constant with $\eta_T \to 0$ as $T\to\infty$, and $\mathbb{E}[K_T(\omega)] < \infty$.

Indeed, given $Z^1 = (U_0^1, v^1, p^1)$ and $Z^2 = (U_0^2, v^2, p^2) \in \mathcal{A}_T^\varepsilon(\omega)$, we have

$$\|VZ^1 - VZ^2\|_{\mathcal{H}_T}^2 = \|\Phi_\varepsilon(T,\omega,U_0^1) - \Phi_\varepsilon(T,\omega,U_0^2)\|_{\mathcal{H}}^2 + \int_T^{2T} \|\Phi_\varepsilon(t,\omega,U_0^1) - \Phi_\varepsilon(t,\omega,U_0^2)\|_{\mathcal{H}}^2\, dt.$$

Let $(v,p) := (v^1 - v^2,\, p^1 - p^2)$. Then, by Lemma 4.1.3,

$$\begin{aligned}
\|VZ^1 - VZ^2\|_{\mathcal{H}_T}^2 &\leqslant \tilde{b}(T)\|U^1 - U^2\|_{\mathcal{H}}^2 + \tilde{C}_T(\omega) \sup_{s\in[0,2T]} \big(\|v(s)\|_2^2 + \|p(s)\|_2^2\big)\\
&\leqslant \tilde{b}(T)\|Z^1 - Z^2\|_{\mathcal{H}}^2 + \tilde{C}_T(\omega) \sup_{s\in[0,2T]} \big(\|v(s)\|_2^2 + \|p(s)\|_2^2\big)\\
&\leqslant \tilde{b}(T)\|Z^1 - Z^2\|_{\mathcal{H}}^2\\
&\quad + \tilde{C}_T(\omega)\left(\sup_{s\in[0,T]} \big(\|v(s)\|_2^2 + \|p(s)\|_2^2\big) + \sup_{s\in[0,T]} \big(\|v(s+T)\|_2^2 + \|p(s+T)\|_2^2\big)\right),
\end{aligned}$$

Define

$$\eta_T = \sqrt{\tilde{b}(T)}, \qquad K_T(\omega) = \sqrt{\tilde{C}_T(\omega)}, \qquad n_T(Z) = \sup_{s\in[0,T]} \sqrt{\|v(s)\|_2^2 + \|p(s)\|_2^2}.$$

Then,

$$\|VZ^1 - VZ^2\|_{\mathcal{H}_T} \leqslant \eta_T\|Z^1 - Z^2\|_{\mathcal{H}_T} + K_T(\omega)\big[n_T(Z^1 - Z^2) + n_T(VZ^1 - VZ^2)\big].$$

By Lemma 4.1.3, we have $\eta_T \to 0$ as $T \to \infty$ and $\mathbb{E}[K_T(\omega)] < \infty$. Finally, $n_T$ is a compact seminorm on $\mathcal{H}_T$, since

$$W(0,T) \hookrightarrow\hookrightarrow C([0,T], (L^2(0,L))^2).$$

This completes the proof of Claim 3.

**Claim 4.** $\mathcal{A}_T^\varepsilon(\omega)$ is compact for $\mathbb{P}$-almost every $\omega \in \Omega$.

First, note that by Claim 3 we may choose $T > 0$ sufficiently large such that $\eta_T < 1$. Then, by [11, Lemma 7.3.5], it follows that

$$\kappa_{\mathcal{H}_T}(V(\mathcal{A}_T^\varepsilon(\omega))) \leqslant \eta_T\, \kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\omega)), \quad \forall\, \omega \in \Omega,$$

where $\kappa_{\mathcal{H}_T}$ denotes the Kuratowski measure of noncompactness (cf. Definition 3.2.2). Iterating this estimate, we obtain for each $n \in \mathbb{N}$ that

$$\kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\vartheta_{Tn}\omega)) \leqslant \eta_T^n\, \kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\omega)). \tag{4.2.7}$$

Since $\mathcal{A}_T^\varepsilon(\omega)$ is bounded for every $\omega \in \Omega$, it follows from (4.2.7) that

$$\kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\vartheta_{nT}\omega)) \to 0 \quad \text{as } n \to \infty. \tag{4.2.8}$$

Now, for $k \in \mathbb{N}$ define the sets

$$B_k = \left\{\omega \in \Omega : \kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\omega)) > \frac{1}{k}\right\},$$

and observe that

$$B = \{\omega \in \Omega : \kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\omega)) > 0\} = \bigcup_{k=1}^{\infty} B_k.$$

Assume by contradiction that $\mathbb{P}(B) > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $\mathbb{P}(B_{k_0}) > 0$. Since $\vartheta_T : \Omega \to \Omega$ preserves the probability measure $\mathbb{P}$, we may apply the Poincaré recurrence theorem (cf. Theorem 1.1.15) to the set $B_{k_0}$. Therefore, for $\mathbb{P}$-almost every $\omega \in B_{k_0}$, there exist infinitely many $n \in \mathbb{N}$ such that $\vartheta_{nT}\omega \in B_{k_0}$. However, by (4.2.8), there exists $N(\omega) \in \mathbb{N}$ such that

$$\kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\vartheta_{nT}\omega)) \leqslant \frac{1}{k_0}, \quad \forall n \geqslant N(\omega).$$

But if $\vartheta_{nT}\omega \in B_{k_0}$, then by definition

$$\kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\vartheta_{nT}\omega)) > \frac{1}{k_0},$$

which yields a contradiction for all $n \geqslant N(\omega)$ sufficiently large. Therefore, $\mathbb{P}(B) = 0$. Consequently, $\kappa_{\mathcal{H}_T}(\mathcal{A}_T^\varepsilon(\omega)) = 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Since $\mathcal{A}_T^\varepsilon(\omega)$ is closed by definition, it follows that $\mathcal{A}_T^\varepsilon(\omega)$ is compact for $\mathbb{P}$-almost every $\omega \in \Omega$.

**Step 2. Fractal dimension of $\mathcal{A}_T^\varepsilon(\omega)$.** By the same reasoning as in [11, p. 354], it follows that there exists a random variable

$$\widetilde{L}_T(\omega) = \frac{4(1 + L_T(\omega))}{\alpha},$$

where $\alpha > 0$ is a deterministic parameter and $L_T(\omega)$ is the Lipschitz constant given in (4.2.3), such that

$$N(\mathcal{A}_T^\varepsilon(\vartheta_T\omega), \tilde{\eta}_T\nu) = N(V(\mathcal{A}_T^\varepsilon(\omega)), \tilde{\eta}_T\nu) \leqslant (1 + \widetilde{L}_T(\omega))^{d_T} N(\mathcal{A}_T^\varepsilon(\omega), \nu),$$

where $0 < \tilde{\eta}_T < 1$ and $d_T > 0$ are deterministic constants. Setting $\omega \mapsto \vartheta_{-T}\omega$, we obtain

$$N(\mathcal{A}_T^\varepsilon(\omega), \tilde{\eta}_T\nu) \leqslant (1 + \widetilde{L}_T(\vartheta_{-T}\omega))^{d_T} N(\mathcal{A}_T^\varepsilon(\vartheta_{-T}\omega), \nu), \quad \forall\, \nu > 0. \tag{4.2.9}$$

Let $\nu_n = \tilde{\eta}_T^n \nu_0$, for some fixed $\nu_0 \in (0,1]$. Then, applying (4.2.9) iteratively, we obtain

$$N(\mathcal{A}_T^\varepsilon(\omega), \nu_n) \leqslant \left( \prod_{k=1}^{n} (1 + \widetilde{L}_T(\vartheta_{-kT}\omega))^{d_T} \right) N(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0).$$

Taking logarithms, we obtain

$$\ln N(\mathcal{A}_T^\varepsilon(\omega), \nu_n) \leqslant d_T \sum_{k=1}^{n} \ln\left(1 + \widetilde{L}_T(\vartheta_{-kT}\omega)\right) + \ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0).$$

Therefore,

$$\begin{aligned} \frac{\ln N(\mathcal{A}_T^\varepsilon(\omega), \nu_n)}{-\ln \nu_n} &\leqslant \frac{d_T \sum_{k=1}^{n} \ln(1 + \widetilde{L}_T(\vartheta_{-kT}\omega))}{-\ln \nu_n} + \frac{\ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0)}{-\ln \nu_n} \\ &= \frac{d_T \frac{1}{n} \sum_{k=1}^{n} \ln(1 + \widetilde{L}_T(\vartheta_{-kT}\omega))}{-\ln \tilde{\eta}_T - \frac{\ln \nu_0}{n}} + \frac{\frac{1}{n} \ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0)}{-\ln \tilde{\eta}_T - \frac{\ln \nu_0}{n}}. \end{aligned} \tag{4.2.10}$$

Since $\mathbb{E}[\ln(L_T(\omega))] < \infty$ by (4.2.4), and $\widetilde{L}_T(\omega) = \frac{4(1+L_T(\omega))}{\alpha}$ with $\alpha > 0$ deterministic, it follows that

$$\mathbb{E}[\ln(1 + \widetilde{L}_T(\omega))] < \infty.$$

Thus, by ergodicity of the shift $\vartheta_t$ and the Birkhoff ergodic theorem [51], we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 + \widetilde{L}_T(\vartheta_{-kT}\omega)\right) = \mathbb{E}\big[\ln(1 + \widetilde{L}_T(\omega))\big] < \infty, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.2.11}$$

We now adapt the ideas from [15] to estimate the second term in (4.2.10). To this end, we first show that

$$\limsup_{n\to\infty} \frac{1}{n} \ln N\left(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0\right) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.2.12}$$

Fix $\nu_0 > 0$ and define, for $M \in \mathbb{N}$,

$$A_M := \{\omega \in \Omega : N(\mathcal{A}_T^\varepsilon(\omega), \nu_0) \leqslant M\}.$$

Since each $\mathcal{A}_T^\varepsilon(\omega)$ is compact in $\mathcal{H}_T$, its covering number $N(\mathcal{A}_T^\varepsilon(\omega), \nu_0)$ is finite for every $\omega \in \Omega$. Hence, the sets $(A_M)_{M\in\mathbb{N}}$ satisfy

$$A_1 \subset A_2 \subset \cdots \subset A_M \subset \cdots, \quad \text{and} \quad \bigcup_{M=1}^{\infty} A_M = \Omega,$$

which implies

$$\lim_{M\to\infty} \mathbb{P}(A_M) = \mathbb{P}\left(\bigcup_{M=1}^{\infty} A_M\right) = 1. \tag{4.2.13}$$

Fix $M$ sufficiently large so that $\mathbb{P}(A_M) > 0$. Since $\vartheta_T$ is measure-preserving with respect to $\mathbb{P}$, the Poincaré recurrence theorem ensures that for $\mathbb{P}$-almost every $\omega \in A_M$, the backward orbit $\{\vartheta_{-nT}\omega : n \in \mathbb{N}\}$ returns to $A_M$ infinitely many times.

Define the *first return time* of $\omega$ to $A_M$ along the backward orbit by

$$k_1^M(\omega) := \inf\{n \in \mathbb{N} : \vartheta_{-nT}\omega \in A_M\},$$

and recursively,

$$k_{j+1}^M(\omega) := k_j^M(\omega) + k_1^M(\vartheta_{-k_j^M(\omega)T}\omega), \qquad j \geqslant 1.$$

This yields an increasing sequence of return times

$$0 < k_1^M(\omega) < k_2^M(\omega) < \cdots \longrightarrow \infty, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in A_M.$$

By Kac's theorem (see Petersen [40, Theorem 2.4.6]), we have

$$\lim_{j\to\infty} \frac{k_j^M(\omega)}{j} = \frac{1}{\mathbb{P}(A_M)}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in A_M. \tag{4.2.14}$$

In particular, $k_j^M(\omega) \to \infty$ linearly as $j \to \infty$.

For each $n \in \mathbb{N}$, let $j(n)$ denote the unique integer such that

$$k_{j(n)}^M(\omega) \leqslant n < k_{j(n)+1}^M(\omega).$$

Then we can write

$$n = k_{j(n)}^M(\omega) + r(n), \qquad 0 \leq r(n) < k_{j(n)+1}^M(\omega) - k_{j(n)}^M(\omega).$$

Now, by applying the estimate (4.2.9) iteratively along these $r(n)$ backward time steps, we obtain the following recursive bound:

$$\begin{aligned}
&\ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0) \\
&= \ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-(k_{j(n)}^M(\omega)+r(n))T}\omega), \nu_0) \\
&\leqslant \ln\left[\left(\prod_{s=1}^{r(n)}(1+\widetilde{L}_T(\vartheta_{-(k_{j(n)}^M(\omega)+s)T}\omega))^{d_T}\right) N(\mathcal{A}_T^\varepsilon(\vartheta_{-k_{j(n)}^M(\omega)T}\omega), \tilde{\eta}_T^{-r(n)}\nu_0)\right] \\
&\leqslant \ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-k_{j(n)}^M(\omega)T}\omega), \nu_0) \\
&\quad + d_T\sum_{s=1}^{r(n)} \ln\big(1+\widetilde{L}_T(\vartheta_{-(k_{j(n)}^M(\omega)+s)T}\omega)\big),
\end{aligned} \tag{4.2.15}$$

where the first inequality follows from applying (4.2.9) $r(n)$ times, and the second inequality uses that $\tilde{\eta}_T^{-r(n)} \geqslant 1$ implies $N(X, \tilde{\eta}_T^{-r(n)}\nu_0) \leqslant N(X, \nu_0)$ for any compact set $X$. Therefore,

$$\begin{aligned} \ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0) \leqslant \ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-k^M_{j(n)}(\omega)T}\omega), \nu_0) \\ + d_T \sum_{s=1}^{r(n)} \ln\big(1 + \widetilde{L}_T(\vartheta_{-(k^M_{j(n)}(\omega)+s)T}\omega)\big). \end{aligned} \tag{4.2.16}$$

Since $\vartheta_{-k^M_{j(n)}(\omega)T}\omega \in A_M$ by the definition of the return time,

$$\ln N(\mathcal{A}_T^\varepsilon(\vartheta_{-k^M_{j(n)}(\omega)T}\omega), \nu_0) \leqslant \ln M.$$

Next, from (4.2.14), we also have (see also Chueshov-Schmalfuß [15, p. 849])

$$\lim_{j\to\infty} \frac{k^M_{j+1}(\omega)}{k^M_j(\omega)} = 1, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in A_M, \tag{4.2.17}$$

which implies

$$k^M_{j+1}(\omega) - k^M_j(\omega) = o\big(k^M_j(\omega)\big).$$

Thus,

$$0 \leqslant r(n) < k^M_{j(n)+1}(\omega) - k^M_{j(n)}(\omega) = o\big(k^M_{j(n)}(\omega)\big).$$

Using (4.2.14), we have $k^M_{j(n)}(\omega) \sim n$ as $n \to \infty$, and therefore

$$\frac{r(n)}{n} \longrightarrow 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in A_M. \tag{4.2.18}$$

Now dividing (4.2.16) by $n$, we obtain

$$\frac{1}{n}\ln N\big(\mathcal{A}_T^\varepsilon(\vartheta_{-nT}\omega), \nu_0\big) \leqslant \frac{\ln M}{n} + d_T\frac{r(n)}{n}\frac{1}{r(n)}\sum_{s=1}^{r(n)} \ln\big(1 + \widetilde{L}_T(\vartheta_{-(k^M_{j(n)}(\omega)+s)T}\omega)\big). \tag{4.2.19}$$

Since $\ln\big(1 + \widetilde{L}_T\big) \in L^1(\Omega)$, the Birkhoff ergodic theorem implies that for $\mathbb{P}$-almost every $\omega$, there exists a constant $C$ such that

$$\frac{1}{m}\sum_{s=1}^{m} \ln\big(1 + \widetilde{L}_T(\vartheta_{-sT}\omega)\big) \leqslant C \quad \forall m \in \mathbb{N}. \tag{4.2.20}$$

If $r(n) = 0$, the second term in (4.2.19) is interpreted as zero. If $r(n) > 0$, (4.2.20) implies that

$$\frac{1}{r(n)}\sum_{s=1}^{r(n)} \ln\big(1 + \widetilde{L}_T(\vartheta_{-(k^M_{j(n)}(\omega)+s)T}\omega)\big)$$

is bounded for $\mathbb{P}$-almost every $\omega$. Together with (4.2.18), we conclude that the second term in (4.2.19) tends to $0$ as $n \to \infty$. The first term also converges to $0$. Thus,

$$\limsup_{n\to\infty} \frac{1}{n} \ln N\big(\mathcal{A}_T^{\varepsilon}(\vartheta_{-nT}\omega), \nu_0\big) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in A_M.$$

Finally, since $\mathbb{P}(A_M) \to 1$ by (4.2.13), we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \ln N\big(\mathcal{A}_T^{\varepsilon}(\vartheta_{-nT}\omega), \nu_0\big) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.2.21}$$

As a consequence of (4.2.10), (4.2.11) and (4.2.21), we obtain

$$\begin{aligned}\dim_f^{\mathcal{H}_T}\big(\mathcal{A}_T^{\varepsilon}(\omega)\big) &\leqslant \limsup_{n\to\infty} \frac{\ln N(\mathcal{A}_T^{\varepsilon}(\omega), \nu_n)}{-\ln \nu_n} \\ &\leqslant \frac{d_T\, \mathbb{E}\big[\ln(1+\widetilde{L}_T(\omega))\big]}{-\ln \tilde{\eta}_T} < \infty,\end{aligned}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$.

**Step 3. (Conclusion) Fractal dimension of $\mathcal{A}_\varepsilon(\omega)$.** Finally, let $\mathcal{P} : \mathcal{H}_T \to \mathcal{H}$ be the projection defined by

$$\mathcal{P}(U_0, v(t), p(t)) = U_0.$$

Since $\mathcal{A}_\varepsilon(\omega) = \mathcal{P}(\mathcal{A}_T^{\varepsilon}(\omega))$ and $\mathcal{P}$ is Lipschitz continuous with Lipschitz constant $1$, the fractal dimension cannot increase under this projection. Hence,

$$\dim_f^{\mathcal{H}}\big(\mathcal{A}_\varepsilon(\omega)\big) \leqslant \dim_f^{\mathcal{H}_T}\big(\mathcal{A}_T^{\varepsilon}(\omega)\big) < \infty, \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

This completes the proof of Theorem 4.2.2. $\square$

# Chapter 5

# Upper semicontinuity of the random attractor

In this chapter, we establish the convergence of the family of random attractors associated with the stochastic piezoelectric system (2.0.1)–(2.0.2) as the noise intensity $\varepsilon \to 0$. In the limit, the stochastic perturbation vanishes and the system reduces to the deterministic piezoelectric model:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + v_t + f_1(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + p_t + f_2(v,p) = 0 & \text{in } (0,L)\times\mathbb{R}^+, \\ \theta_t - \kappa\theta_{xx} + \delta v_{xt} = 0 & \text{in } (0,L)\times\mathbb{R}^+, \end{cases} \tag{5.0.1}$$

supplemented with the same boundary and initial conditions as in (2.0.2).

**Remark 5.0.1.** *Assuming hypotheses* **(H1)–(H3)** *and setting $\varepsilon = 0$, the same arguments used in the proof of Theorem 2.1.6 guarantee that the deterministic problem (5.0.1) is well-posed in the phase space $\mathcal{H}$. Therefore, the associated solution operator $\Phi$ defines a nonlinear semigroup on $\mathcal{H}$. Moreover, the existence of a compact global attractor $\mathcal{A}_0$ for the semigroup $\Phi$ was established in [22].*

### 5.1 Preliminary lemmas

Our goal in this section is to verify conditions (A1)–(A2) of Proposition 1.2.12. To this end, we first establish some preliminary estimates.

Consider the random variable

$$R(\omega) = C_f \int_{-\infty}^{0} e^{\sigma_0 s}\big(1 + |z(\vartheta_s\omega)|^2\big)\, ds + m_f L,$$

where $C_f$ and $\sigma_0$ are the constants appearing in (3.1.2). Define the random set

$$\mathcal{B}(\omega) = \big\{U \in \mathcal{H} : \|U\|_{\mathcal{H}}^2 \leq 2R(\omega)\big\}.$$

By the minimality of the random attractor $\mathcal{A}_\varepsilon$, we have

$$\bigcup_{\varepsilon\in(0,1]} \mathcal{A}_\varepsilon(\omega) \subset \bigcup_{\varepsilon\in(0,1]} \mathcal{B}_\varepsilon(\omega) \subset \mathcal{B}(\omega), \quad \forall\omega \in \Omega. \tag{5.1.1}$$

We begin by establishing a result concerning the Lipschitz continuity of $\Phi_\varepsilon$ with respect to the parameter $\varepsilon \in [0, 1]$.

**Lemma 5.1.1.** *Assume that hypotheses* **(H1)–(H4)** *hold. Then, for every $t \geqslant 0$ and $\omega \in \Omega$, there exists a constant $C_{t,\omega} > 0$ such that*

$$\sup_{U\in\mathcal{B}(\omega)} \|\Phi_{\varepsilon_1}(t, \omega, U) - \Phi_{\varepsilon_2}(t, \omega, U)\|_{\mathcal{H}} \leqslant C_{t,\omega}\, |\varepsilon_1 - \varepsilon_2|, \quad \forall\varepsilon_1, \varepsilon_2 \in [0, 1]. \tag{5.1.2}$$

*Proof.* Let $U \in \mathcal{B}(\omega)$ and $\varepsilon_1, \varepsilon_2 \in [0, 1]$. Denote

$$\Phi_{\varepsilon_1}(t, \omega, U) = \big(v^{\varepsilon_1}(t),\, p^{\varepsilon_1}(t),\, v_t^{\varepsilon_1}(t),\, p_t^{\varepsilon_1}(t),\, \varphi^{\varepsilon_1}(t)\big),$$
$$\Phi_{\varepsilon_2}(t, \omega, U) = \big(v^{\varepsilon_2}(t),\, p^{\varepsilon_2}(t),\, v_t^{\varepsilon_2}(t),\, p_t^{\varepsilon_2}(t),\, \varphi^{\varepsilon_2}(t)\big).$$

Setting $v = v^{\varepsilon_1} - v^{\varepsilon_2}$, $p = p^{\varepsilon_1} - p^{\varepsilon_2}$, and $\varphi = \varphi^{\varepsilon_1} - \varphi^{\varepsilon_2}$, we see that $(v, p, \varphi)$ satisfies the system

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\varphi_x + v_t + \mathbf{F}_1(v, p) = -\delta(\varepsilon_1 - \varepsilon_2)\psi_x z, & \text{in } (0, L) \times \mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + p_t + \mathbf{F}_2(v, p) = 0, & \text{in } (0, L) \times \mathbb{R}^+, \\ \varphi_t - \kappa\varphi_{xx} + \delta v_{xt} = \sigma(\varepsilon_1 - \varepsilon_2)\psi z + \kappa(\varepsilon_1 - \varepsilon_2)\psi_{xx} z, & \text{in } (0, L) \times \mathbb{R}^+, \end{cases} \tag{5.1.3}$$

with initial conditions

$$(v(0), p(0), v_t(0), p_t(0), \varphi(0)) = 0,$$

where

$$z = z(\vartheta_t\omega), \qquad \mathbf{F}_i(v, p) := f_i(v^{\varepsilon_1}, p^{\varepsilon_1}) - f_i(v^{\varepsilon_2}, p^{\varepsilon_2}), \quad i = 1, 2.$$

Multiplying the three equations in (5.1.3) by $v_t$, $p_t$, and $\varphi$, respectively, and integrating over $(0, L)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(v,p,v_t,p_t,\varphi)\|_{\mathcal{H}}^2 = -\|v_t\|_2^2 - \|p_t\|_2^2 - \kappa\|\varphi_x\|_2^2 - \delta(\varepsilon_1-\varepsilon_2)\int_0^L \psi_x z v_t dx \\ + \int_0^L \left[\sigma(\varepsilon_1-\varepsilon_2)\psi z + \kappa(\varepsilon_1-\varepsilon_2)\psi_{xx}z\right]\varphi dx \\ - \int_0^L \left[\mathbf{F}_1(v,p)v_t + \mathbf{F}_2(v,p)p_t\right]dx.\end{aligned} \tag{5.1.4}$$

Using Hölder's and Young's inequalities, we deduce

$$-\delta(\varepsilon_1-\varepsilon_2)\int_0^L \psi_x z v_t dx \leqslant \delta|\varepsilon_1-\varepsilon_2||z|\|\psi_x\|_2\|v_t\|_2 \leqslant C|\varepsilon_1-\varepsilon_2|^2|z|^2\|\psi_x\|_2^2 + \frac{1}{2}\|v_t\|_2^2.$$

Similarly, applying Poincaré's inequality yields

$$\begin{aligned}\int_0^L \left[\sigma(\varepsilon_1-\varepsilon_2)\psi z + \kappa(\varepsilon_1-\varepsilon_2)\psi_{xx}z\right]\varphi\, dx \leqslant C|\varepsilon_1-\varepsilon_2|^2|z|^2\left(\|\psi\|_2^2 + \|\psi_{xx}\|_2^2\right) \\ + \frac{\kappa}{2}\|\varphi_x\|_2^2.\end{aligned}$$

Moreover, by estimates similar to those in (3.2.7) and (3.2.8), we obtain

$$-\int_0^L \left[\mathbf{F}_1(v,p)v_t + \mathbf{F}_2(v,p)p_t\right]dx \leqslant C\zeta_{\mathcal{B}}(\vartheta_t\omega)\|(v,p,v_t,p_t,\varphi)\|_{\mathcal{H}}^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|p_t\|_2^2.$$

Inserting the three estimates above into (5.1.4), we conclude that

$$\frac{d}{dt}\|(v,p,v_t,p_t,\varphi)\|_{\mathcal{H}}^2 \leqslant C\zeta_{\mathcal{B}}(\vartheta_t\omega)\|(v,p,v_t,p_t,\varphi)\|_{\mathcal{H}}^2 + C|\varepsilon_1-\varepsilon_2|^2|z|^2.$$

Applying Gronwall's inequality and using the fact that $\|(v(0),p(0),v_t(0),p_t(0),\varphi(0))\|_{\mathcal{H}}^2 = 0$, we obtain

$$\|(v(t),p(t),v_t(t),p_t(t),\varphi(t))\|_{\mathcal{H}}^2 \leqslant |\varepsilon_1-\varepsilon_2|^2\int_0^t e^{C\int_s^t \zeta_{\mathcal{B}}(\vartheta_r\omega)dr}|z(\vartheta_s\omega)|^2 ds.$$

This implies that

$$\|\Phi_{\varepsilon_1}(t,\omega,U) - \Phi_{\varepsilon_2}(t,\omega,U)\|_{\mathcal{H}}^2 \leqslant |\varepsilon_1-\varepsilon_2|^2\int_0^t e^{C\int_s^t \zeta_{\mathcal{B}}(\vartheta_r\omega)\,dr}|z(\vartheta_s\omega)|^2\,ds, \qquad t>0.$$

Therefore, (5.1.2) holds with $C_{t,\omega} = \left(\int_0^t e^{C\int_s^t \zeta_{\mathcal{B}}(\vartheta_r\omega)\,dr}|z(\vartheta_s\omega)|^2\,ds\right)^{1/2}$. $\square$

**Lemma 5.1.2.** *The random attractor is uniformly compact, that is,* $\overline{\bigcup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon(\omega)}^{\mathcal{H}}$ *is compact in* $\mathcal{H}$ *for* $\mathbb{P}$*-almost every* $\omega \in \Omega$.

*Proof.* Arguing similarly to Step 1 of Lemma 3.2.3, we can show that the pseudo-metric $\varrho^t : \mathcal{B}(\omega) \times \mathcal{B}(\omega) \to \mathbb{R}$ defined by

$$\varrho^t(U^1, U^2) = \left( C \int_0^t e^{-\sigma_1(t-s)} \zeta_{\mathcal{B}}(\vartheta_s \omega) \left( \|v(s)\|_2^2 + \|p(s)\|_2^2 \right) ds \right)^{1/2},$$

is compact.

Now let $t \in \mathbb{R}^+$ and $\omega \in \Omega$, and consider a sequence $\{U^n\} \subset \bigcup_{\varepsilon \in (0,1]} \mathcal{A}_\varepsilon(\omega)$. For each $n \in \mathbb{N}$, there exists $\varepsilon_n \in (0, 1]$ such that $U^n \in \mathcal{A}_{\varepsilon_n}(\omega)$. Since $\varepsilon_n \in (0, 1]$, then, $\varepsilon_n$ it is a bounded sequence, it follows from the Bolzano-Weierstrass theorem that there exists a subsequence $\{\varepsilon_{n_k}\}$ such that $\varepsilon_{n_k} \to \varepsilon_0 \in [0, 1]$ as $k \to \infty$. By the invariance of the attractors, for each $k \in \mathbb{N}$ and for all $t \in \mathbb{R}^+$, we have that

$$\mathcal{A}_{\varepsilon_{n_k}}(\omega) = \Phi_{\varepsilon_{n_k}}(t, \vartheta_{-t}\omega, \mathcal{A}_{\varepsilon_{n_k}}(\vartheta_{-t}\omega)),$$

then there exists $U_0^{n_k} \in \mathcal{A}_{\varepsilon_{n_k}}(\vartheta_{-t}\omega)$ such that

$$U^{n_k} = \Phi_{\varepsilon_{n_k}}(t, \vartheta_{-t}\omega, U_0^{n_k}). \tag{5.1.5}$$

By the triangle inequality, (5.1.1), Lemma 3.2.5, and (5.1.5), we obtain

$$\begin{aligned}\|U^{n_k} - U^{n_l}\|_{\mathcal{H}} \leqslant e^{-\frac{\sigma_1}{2}t} \|U_0^{n_k} - U_0^{n_l}\|_{\mathcal{H}} + \|\Phi_{\varepsilon_{n_k}}(t, \vartheta_{-t}\omega, U_0^{n_l}) - \Phi_{\varepsilon_{n_l}}(t, \vartheta_{-t}\omega, U_0^{n_l})\|_{\mathcal{H}} \\ + \varrho^t(U_0^{n_k}, U_0^{n_l}).\end{aligned}$$

Using (5.1.1), we get

$$e^{-\frac{\sigma_1}{2}t} \|U_0^{n_k} - U_0^{n_l}\|_{\mathcal{H}} \leqslant e^{-\frac{\sigma_1}{2}t} \operatorname{diam}\left( \bigcup_{\varepsilon \in (0,1]} \mathcal{A}_\varepsilon(\vartheta_{-t}\omega) \right) \leqslant e^{-\frac{\sigma_1}{2}t} \operatorname{diam}(\mathcal{B}(\vartheta_{-t}\omega)).$$

Therefore,

$$\begin{aligned}\|U^{n_k} - U^{n_l}\|_{\mathcal{H}} \leqslant e^{-\frac{\sigma_1}{2}t} \operatorname{diam}(\mathcal{B}(\vartheta_{-t}\omega)) + \|\Phi_{\varepsilon_{n_k}}(t, \vartheta_{-t}\omega, U_0^{n_l}) - \Phi_{\varepsilon_{n_l}}(t, \vartheta_{-t}\omega, U_0^{n_l})\|_{\mathcal{H}} \\ + \varrho^t(U_0^{n_k}, U_0^{n_l}).\end{aligned}$$

Given $\eta > 0$, since $\mathcal{B} = \{\mathcal{B}(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$, we can choose $t > 0$ sufficiently large such that

$$e^{-\frac{\sigma_1}{2}t} \operatorname{diam}(\mathcal{B}(\vartheta_{-t}\omega)) \leqslant \frac{\eta}{3},$$

which implies

$$\|U^{n_k} - U^{n_l}\|_{\mathcal{H}} \leqslant \frac{\eta}{3} + \|\Phi_{\varepsilon_{n_k}}(t, \vartheta_{-t}\omega, U_0^{n_l}) - \Phi_{\varepsilon_{n_l}}(t, \vartheta_{-t}\omega, U_0^{n_l})\|_{\mathcal{H}} + \varrho^t(U_0^{n_k}, U_0^{n_l}). \tag{5.1.6}$$

Since $\varepsilon_k \to \varepsilon_0$, by Lemma 5.1.1, there exists $k_0 \in \mathbb{N}$ such that, for all $k, l \geqslant k_0$, we have

$$\|\Phi_{\varepsilon_{n_k}}(t, \vartheta_{-t}\omega, U_0^{n_l}) - \Phi_{\varepsilon_{n_l}}(t, \vartheta_{-t}\omega, U_0^{n_l})\|_{\mathcal{H}} \leqslant \frac{\eta}{3}. \tag{5.1.7}$$

On the other hand, since $U_0^{n_k} \in \mathcal{A}_{\varepsilon_{n_k}}(\vartheta_{-t}\omega) \subset \mathcal{B}(\vartheta_{-t}\omega)$ and $\varrho^t$ is compact on $\mathcal{B}(\vartheta_{-t}\omega)$, there exists $k_1 > k_0$ such that for all $k, l \geqslant k_1$,

$$\varrho^t(U_0^{n_k}, U_0^{n_l}) \leqslant \frac{\eta}{3}. \tag{5.1.8}$$

Combining (5.1.6), (5.1.7), and (5.1.8), we deduce that for all $k, l \geqslant k_1$,

$$\|U^{n_k} - U^{n_l}\|_{\mathcal{H}} \leqslant \eta.$$

This shows that the subsequence $\{U^{n_k}\}$ is Cauchy in $\mathcal{H}$, and therefore convergent in $\mathcal{H}$. Hence, $\overline{\bigcup_{\varepsilon \in (0,1]} \mathcal{A}_\varepsilon(\omega)}^{\mathcal{H}}$ is compact in $\mathcal{H}$. □

## 5.2 Upper-semicontinuity

The following result concerning the upper-semicontinuity of the random attractor is the main result of this section.

**Theorem 5.2.1** (Upper-semicontinuity)**.** *Suppose that assumptions* **(H1)–(H4)** *hold. Then, the family of random attractors $\mathcal{A}_\varepsilon(\omega)$ is upper-semicontinuous with respect to $\varepsilon$ at $\varepsilon = 0$, with limit given by the global attractor $\mathcal{A}_0$ of the deterministic system. More precisely, for $\mathbb{P}$-almost every $\omega \in \Omega$, we have*

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0) = 0. \tag{5.2.1}$$

*Proof.* We verify conditions (A1)–(A2) of Proposition 1.2.12.

To verify (A1), we first observe from Theorem 3.2.6 that, for each $\varepsilon > 0$, the cocycle $\Phi_\varepsilon$ possesses a $\mathcal{D}$-random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Let $\mathcal{B}_\varepsilon = \{\mathcal{B}_\varepsilon(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ be the $\mathcal{D}$-pullback absorbing set provided by Lemma 3.1.1. From (3.1.1) and (3.1.2), we deduce that

$$\limsup_{\varepsilon \to 0} \|\mathcal{B}_\varepsilon(\omega)\|_{\mathcal{H}}^2 \leqslant 2 \limsup_{\varepsilon \to 0} R_\varepsilon(\omega).$$

Using (3.1.2) and the dominated convergence theorem, we obtain

$$\begin{aligned}
\limsup_{\varepsilon \to 0} R_\varepsilon(\omega) &= \limsup_{\varepsilon \to 0} \left( C_f \int_{-\infty}^0 e^{\sigma_0 s} \big(1 + \varepsilon^2 |z(\vartheta_s\omega)|^2\big)\, ds + m_f L \right) \\
&= C_f \int_{-\infty}^0 e^{\sigma_0 s}\, ds + m_f L = \frac{C_f}{\sigma_0} + m_f L.
\end{aligned}$$

Consequently,

$$\limsup_{\varepsilon\to 0} \|\mathcal{B}_\varepsilon(\omega)\|_{\mathcal{H}}^2 \leqslant \frac{2C_f}{\sigma_0} + 2m_f L.$$

Therefore, condition (A1) is satisfied.

Condition (A2) follows from Lemma 5.1.2. Condition (A3) follows similarly to the proof of Lemma 5.1.1 and is therefore omitted. Finally, condition (A4) follows immediately from Remark 5.0.1. This completes the proof. □

# Conclusion and future perspectives

In this dissertation, we investigated the random quasi-stability and long-term asymptotic behavior of a class of nonlinear stochastic piezoelectric systems incorporating magnetic effects and thermal dissipation governed by Fourier's law. The study was conducted within the framework of stochastic partial differential equations of hyperbolic type, which model the interaction among mechanical, electrical, and thermal fields under the influence of additive white noise.

The main results achieved can be summarized as follows:

1. **Global well-posedness**: We established the existence and uniqueness of global (mild and strong) solutions for the original stochastic system through a *pathwise* transformation that converts the problem into a family of deterministic systems parameterized by the trajectories of the stochastic process. The approach based on the theory of linear operator semigroups proved effective in ensuring the well-posedness of the problem in the natural phase space $\mathcal{H}$.

2. **Pullback random attractor**: We demonstrated that the random dynamical system $\Phi_\varepsilon$ associated with the problem possesses a unique random attractor $\mathcal{A}_\varepsilon$ in the class $\mathcal{D}$ of tempered random sets. The existence of this attractor was guaranteed through the construction of an absorbing set $\mathcal{B}_\varepsilon \in \mathcal{D}$ and the asymptotic compactness of the random flow, the latter obtained via an adaptation of the quasi-stability method to the stochastic context.

3. **Finite fractal dimension**: One of the most significant results of this work is the proof that the random attractor $\mathcal{A}_\varepsilon(\omega)$ has finite fractal dimension, even in the presence of nonlinearities with arbitrary polynomial growth and non-globally Lipschitz behavior. This advance provides a positive answer to an open question posed by Chueshov and Schmalfuß [15],

overcoming the difficulty related to the asymptotic control of random coefficients through Birkhoff's ergodic theorem.

4. **Upper semicontinuity**: Based on uniform estimates with respect to the noise intensity $\varepsilon$, we showed that the family of random attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon>0}$ is upper semicontinuous in the deterministic limit $\varepsilon \to 0$, converging to the global attractor $\mathcal{A}_0$ of the deterministic piezoelectric system. This result reinforces the robustness of the system's asymptotic dynamics against small-amplitude stochastic perturbations.

From a methodological perspective, the combination of techniques from random dynamical systems, energy estimates, ergodic theory, and the random quasi-stability method proved powerful for treating coupled systems with hybrid (hyperbolic-parabolic) structure and polynomial growth nonlinearities.

In terms of future perspectives, this work paves the way for the study of other models with multi-physical couplings subject to random perturbations, such as Bresse beam systems, plate systems, systems with memory and delay, among others that have not yet been addressed in the literature.

Finally, the results obtained here not only expand the theoretical knowledge about the long-term dynamics of piezoelectric systems under random fluctuations but also provide subsidies for the design and control of intelligent devices subjected to uncertain operational conditions.

For the future, I intend to pursue a PhD and deepen my studies in the theory of infinite dimensional stochastic dynamical systems, aiming to make contributions in this area.

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