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Multiplicity of Solutions for Elliptic Problems via multiple Applications of the Rayleigh Quotient

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Resumo

Multiplicidade de Soluções para Problemas Elípticos por meio de Múltiplas Aplicações do Quociente de Rayleigh.

O Quociente de Rayleigh e a análise de seus valores extremais permitem examinar e caracterizar a estrutura da variedade de Nehari associada a um funcional de energia. Nesta dissertação, apresentamos diferentes formulações desses valores extremais o que possibilita determinar se a variedade de Nehari, ou alguns de seus subconjuntos, são vazios ou não. Essa informação é fundamental, pois os candidatos a soluções dos problemas elípticos considerados pertencem exatamente a um desses subconjuntos.

Uma vez estabelecida essa caracterização, aborda-se a existência de soluções de equações diferenciais parciais não lineares por meio de métodos variacionais, por exemplo, através de procedimentos de minimização sobre as "subvariedades" de Nehari. Além disso, são apresentadas aplicações específicas que ilustram os resultados teóricos desenvolvidos e evidenciam seu alcance ao demonstrar a multiplicidade de soluções para diferentes classes de problemas com um ou mais parâmetros.
Palavras-chave: Quociente de Rayleigh, multiplicidade de soluções, subvariedades da variedade de Nehari, técnicas de minimização.

Abstract

The Rayleigh Quotient and the analysis of its extremal values make it possible to examine and characterize the structure of the Nehari manifold associated with an energy functional. In this dissertation, we present different formulations of these extremal values, which make it possible to determine whether the Nehari manifold, or some of its subsets, are empty or not. This information is fundamental, as the candidate solutions of the elliptic problems considered belong precisely to one of these subsets.

Once this characterisation has been established, the existence of solutions to nonlinear partial differential equations is addressed through variational methods, for example, by means of minimisation procedures on the Nehari “submanifolds”. In addition, specific applications are presented that illustrate the theoretical results developed and demonstrate their scope by showing the multiplicity of solutions for different classes of problems with one or more parameters.
Keywords: Rayleigh quotient, multiplicity of solutions, submanifolds of the Nehari manifold, minimization techniques.

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- We denote by $W = W_1 \times W_2 \times \cdots \times W_n$ the product of Banach spaces W_i, and $\|\cdot\|_{W_i},\ i = 1,2,\ldots,n$, is the norm. The norm on W is given by $\|\cdot\| = \|\cdot\|_{W_1} + \|\cdot\|_{W_2} + \cdots + \|\cdot\|_{W_n}$;
- $\dot{W} = (W_1 \setminus \{0\}) \times (W_2 \setminus \{0\}) \times \cdots \times (W_n \setminus \{0\}) \qquad \dot{\mathbb{R}}^+ = \mathbb{R}^+ \setminus \{0\}$;
- $u_i e_i = (0,..,0,u_i,0,..0)$;
- $\nabla_u F(u) := (D_{u_1}F(u),\ldots,D_{u_n}F(u))^T$; where $D_{u_i}F(u)$ are applications;
- $\nabla_u F(u)(v) := (D_{u_1}F(u)(v_1),\ldots,D_{u_n}F(u)(v_n))^T$;
- $D_u F(u)(v) := \sum_{i=1}^{n} D_{u_i}F(u)(v_i)$.

If we consider the functional $\Phi(u) = T(u) - \lambda G(u)$, then we introduce the following notation.

- $\mathcal{J}_1(u) = D_u T(u)(u)$;
- $\mathcal{J}_2(u) = D_u G(u)(u)$;
- $0_n = (0,0,0,...,0)$;
- $\mathcal{J}_1,\, \mathcal{J}_2 \in C^1$;
- $\Phi'(u) = \mathcal{J}_1(u) - \lambda\, \mathcal{J}_2(u)$;
- $\varphi(t) := \Phi(tu)$, called the fiber map of the functional Φ;
- $\mathcal{J}_2(u) > 0$ for all $u \in W \setminus \{0\}$.

Introduction

The Rayleigh Quotient has its roots in classical mathematical physics, being introduced and popularized by Lord Rayleigh (John William Strutt). His monumental work, The Theory of Sound, first published in 1877.

Advancing almost a century, the Israeli-American mathematician Zeev Nehari introduced a revolutionary method in 1963 to address nonlinear eigenvalue problems. Nehari observed that the nontrivial solutions of his nonlinear problem could be characterized by the stationarity condition of an energy functional, or, equivalently, through a nonlinear generalization of the Rayleigh Quotient.

On reading these pages, the reader might think that the thesis focuses exclusively on a single method for guaranteeing the existence of solutions. However, the scope of this work goes far beyond that: it not only presents results associated with the Nehari manifold, but also explores modern tools that establish a deeper connection between this manifold and elliptic partial differential equations.

In this study, we combine the Nehari set with the nonlinear Rayleigh quotient, two fundamental tools which, when used together, allow us to identify and understand the location of possible solutions within the subsets of the Nehari manifold. This methodological integration provides a clearer and more robust perspective on the problem, showing how modern theory can facilitate both the existence and the characterisation of solutions.
In this work, we shall generally consider an energy functional of the form

$$\Phi(u) = T(u) - \lambda G(u), \text{ where } \Phi : W \to \mathbb{R},\ \Phi \in C^1(W \setminus \{0_n\}, \mathbb{R}).$$

Here W denotes a Banach space or, in the particular case of Chapter 1, the Cartesian product of Banach spaces, λ a real parameter. Thus, the the definition of the Nehari set is as follows

$$\mathcal{N} = \Big\{ u \in W \setminus \{0\} : \nabla_u \Phi(u)(u) \equiv \frac{d}{dt}\Phi(tu)\Big|_{t=1} = 0 \Big\}, \tag{0.0.1}$$

where, $\nabla_u \Phi(u)\,(u) := (D_{u_1}\Phi(u)(u_1), \ldots, D_{u_n}\Phi(u)(u_n))^T$ besides $W = W_1 \times \ldots \times W_n$, W_i is Banach space, and $D_{u_i}\Phi(u)(u_i)$ this is the Fréchet derivative at point u and in the $u_i e_i$ direction. Note that Φ depends on λ; therefore, the Nehari set $\mathcal{N}$ will also depend on the parameter λ. For the sake of simplicity in our notation, we shall omit writing Φ_λ e $\mathcal{N}_\lambda$. Note that when u belongs to a single Banach space, the definition will be the same, but without using the gradient, employing only the usual classical Fréchet derivative.
Let H be a real Hilbert space and let

$$A : D(A) \subset H \to H$$

be a self-adjoint linear operator. For $u \in D(A) \setminus \{0\}$, the *Rayleigh quotient* is defined by

$$R(u) = \frac{A(u)(u)}{||u||^2},$$

where $A(u)(u)$ is the inner product of $A(u)$ with u. On the other hand, for nonlinear operators, let W be a real reflexive Banach space and let

$$A, B : W \to W^*$$

be (possibly nonlinear) operators satisfying

$$B(u)(u) > 0 \quad \text{for all } u \in W \setminus \{0\},$$

the *Rayleigh quotient associated with the pair* (A,B) is defined by

$$R(u) = \frac{A(u)(u)}{B(u)(u)}, \qquad u \in W \setminus \{0\}.$$

For our energy functional, we obtain the Rayleigh quotient from the Fréchet derivative, that is,

$$D_u\Phi(u)(u) = D_uT(u)(u) - \lambda D_uG(u)(u) = 0 \Longleftrightarrow R(u) := \frac{D_uT(u)(u)}{D_uG(u)(u)},$$

where $D_uG(u)(u) \neq 0$ forall $u \in \dot{W} = (W_1 \setminus \{0\}) \times (W_2 \setminus \{0\}) \times \cdots \times (W_n \setminus \{0\})$. Thus R represents the Rayleigh quotient from the Fréchet derivative (see e.g. [19], [28]). Note that u belongs to $\mathcal{N}$ if and only if it lies on the level set $R(u) = \lambda$, which will allow us to analyse in greater depth the geometry of the functional and, in particular, to determine conditions under which the associated Nehari set is non-empty. This analysis is essential because, if the Nehari set is not empty, this implies that it contains an element that constitutes a strong candidate to be a solution of the elliptic partial differential equations under study. So, to enrich our analysis, let we shall begin by us examining how the Nehari set can be subdivided; to this end, we will introduce and refine the extremal values of the Rayleigh quotient.

In the first chapter, we analyze the critical points of the energy functional restricted to the Nehari set, which under suitable assumptions is generally a Nehari manifold. In fact, the Fréchet derivative of the energy functional Φ satisfies the corresponding weak formulation, and by applying the Implicit Function Theorem-or alternatively, as in Chapter 3, using the method of Lagrange multipliers-we conclude that the critical values of the functional restricted to the Nehari set correspond precisely to weak solutions of the problem. For this purpose, we work in the vectorial setting. This result and its consequences are of great importance, since later we shall focus solely on proving that the energy functionals attain their minima on the Nehari C^1-manifold and, therefore, that critical points exist. This will serve as a basis for concluding that a specific problem indeed admits solutions.
Furthermore, in this chapter we shall undertake an extensive and in-depth study of the Rayleigh quotient from the Fréchet derivative, together with its naturally defined extremal values, from which various fundamental properties will be derived. We will also show that, by defining the extremal values based on the derivatives of the fibering map associated with the Rayleigh quotient of the Fréchet derivative, one obtains additional information that is essential for the main objective: the subdivision of the Nehari set into three disjoint subsets

$$\mathcal{N}^+ := \left\{ u \in \mathcal{N} : D^2_{uu}\Phi(u)(u,u) > 0 \right\},$$
$$\mathcal{N}^0 := \left\{ u \in \mathcal{N} : D^2_{uu}\Phi(u)(u,u) = 0 \right\},$$
$$\mathcal{N}^- := \left\{ u \in \mathcal{N} : D^2_{uu}\Phi(u)(u,u) < 0 \right\}.$$

The definitions of these subsets may be also expressed either in terms of the fiber map of the energy functional or in terms of the fiber map of the Rayleigh quotient from the Fréchet derivative (see (1.1.1), and (1.7.1) in Chapter 1).
We also formulate hypotheses concerning the fiber map the Rayleigh quotient from the Fréchet derivative and show that, under certain conditions, they provide crucial information for achieving our main results. Finally, we study the Rayleigh quotient from the energy functional and

show how it is closely related to the Rayleigh quotient from the Fréchet derivative, inheriting many of its properties at various stages of the analysis.

In the second chapter, we present a complement to the previous one. Once the naturally defined extremal values are established, and since they allow us to subdivide the Nehari set, the main objective is to prove—by means of the minimization method—that the energy functional Φ attains its minimum when restricted to the Nehari subsets. In other words, the minimization method consists in minimizing the energy functional Φ on these Nehari subsets. Moreover, we show that such a minimum corresponds to a solution of the elliptic partial differential equations associated with Φ. Finally, we determine to which subset of the Nehari manifold this solution belongs.

To obtain this result, it will be necessary to impose additional hypotheses that constrain the energy functional. In general terms, if the functional satisfies all these hypotheses, we shall be able to guarantee the existence of a solution to the corresponding problem. This fact will be stated in the first theorem of this chapter, which we shall develop in the vectorial case, that is, for $u \in W$, where W is a product of Banach spaces.
After this, and in order to simplify the calculations, we shall work in the scalar case; that is, we shall consider functions u belonging to a single Banach space. Consequently, all the results presented from this point onwards will be developed under this framework.
As an application of this theorem, we shall show that the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-2}u + |u|^{\gamma-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{0.0.2}$$

admits at least two weak solutions in $W_0^{1,2}(\Omega)$, where $1 < \alpha < p < \gamma \leq p^* = \dfrac{PN}{N-p}$, if $p < N$, and $p^* = \infty$ if $p \geq N$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and λ is a real parameter (see [30]).
It is worth mentioning that, later on, in Chapter 3, this equation will be extended to the case of two parameters.

Since in Chapter 1 we refined the definition of the extremal values associated with the standard Rayleigh quotient, it is natural that in Chapter 2 more elaborate theorems, propositions, and lemmas will appear. We shall introduce new hypotheses, which are typically satisfied in

problems that arise naturally in the study of the energy functional.
The minimization method will allow us to approach the search for solutions to the problem in a clearer and more accessible manner. Throughout this chapter, relevant and carefully structured results are presented, so that the reader may appreciate how the developments carried out in Chapter 1 now acquire a deeper and more coherent meaning. Moreover, the use of the refined extremal values simplifies the formulation of the hypotheses and allows for a more precise and effective manipulation of the different subsets of the Nehari manifold, which greatly enriches the analysis. Finally, we shall study the extremal values of the Rayleigh quotient arising from the energy functional, showing how these values allow for a refinement of fundamental results and, in particular, how they enable us to determine whether the critical point of the energy functional has positive, zero, or negative energy.

To conclude, we shall present an application of the theory developed by studying a Kirchhoff-type problem (see [7] and [35]), which will serve to illustrate how the results obtained in this chapter can be employed in a concrete and relevant case within the analysis of nonlinear elliptic problem

$$\begin{cases} -(a+\lambda \int |\nabla u|^2)\Delta u = |u|^{\gamma-2}u, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.3}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $a>0$ and $\gamma \in (2,4)$.
Finally, in Chapter 3 we shall study a problem involving two real parameters (see [28]). Following the same line as in the previous chapters, we shall work extensively with all the tools developed beforehand. Although the presence of two parameters introduces an additional level of complexity into the problem, the analysis essentially becomes manageable thanks to the systematic and repeated use of the Rayleigh quotients—both the from Fréchet derivative and the from the energy functional, and their respective extreme values.

In this chapter, the energy functional will take the appropriate form to incorporate both parameters, and its structure will once again allow the application of methods based on the Nehari manifold and on fibering theory. The combined use of the Rayleigh quotients, evaluated strategically at different points, will be crucial for understanding the geometry of the problem and for identifying the location of the solutions within the subsets of the Nehari manifold.

The problem to be studied in this chapter will be the following:

$$\begin{cases} -\Delta_p u = |u|^{\gamma-2}u + \lambda |u|^{\alpha-2}u - \mu |u|^{q-2}u := f(\lambda,\mu,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.4}$$

where $1 < q < \alpha < p < \gamma < p^*$, $p^* = \dfrac{pN}{N-p}$ if $p < N$ and $p^* = +\infty$ if $p \geq N$, and $\lambda, \mu \in \mathbb{R}$, and Ω is a smooth bounded domain.

Once this relationship has been established, we introduce the Rayleigh quotient from the Fréchet derivative and the Rayleigh quotient from the energy functional in the following way:

$$R(u) := \frac{\int |\nabla u|^p + \mu \int |u|^q - \int |u|^\gamma}{\int |u|^\alpha}, \quad u \in W_0^{1,p} \setminus \{0\},\ \mu \in \mathbb{R},$$

and

$$R^e(u) := \frac{\frac{1}{p}\int |\nabla u|^p + \frac{\mu}{q}\int |u|^q - \frac{1}{\gamma}\int |u|^\gamma}{\frac{1}{\alpha}\int |u|^\alpha}, \quad u \in W_0^{1,p} \setminus \{0\},\ \mu \in \mathbb{R}.$$

In fact, we can observe that this problem constitutes an extension of the first problem presented in Chapter 2. It will be shown that by incorporating two parameters, the Rayleigh quotients allow for a clear and comprehensible analysis that leads to the proof of the existence of three distinct solutions for this problem.

Analogously to the applications in Chapter 2, the defined extremal values will enable us to understand and accurately describe the structure of the equation. We shall show that, under appropriate hypotheses, the Nehari manifold can once again be decomposed into suitable subsets, and that the refined extremal values of the Rayleigh quotients allow for a precise characterization of this decomposition. This structure will be essential for proving the existence of three weak solutions to the problem. Two of these solutions will be obtained through minimization techniques applied to different parts of the Nehari set; one of them is denoted by $u^2 \in \mathcal{N}^+$, and the other by $u^3 \in \mathcal{N}^-$.

The third solution, u^1, which is deduced to belong to $\mathcal{N}^-$, will instead be obtained using the Mountain Pass method. This approach identifies a variational geometry that cannot be captured solely through minimization techniques.

It will also be shown that these three solutions are distinct from one another, which is a very important result. For this purpose, we will use Theorem (3.3.4). Clearly, the solutions u^2 and

u^3 are different, as are u^2 and u^1, since they belong to disjoint sets. To show that u^3 and u^1 are different, we will rely on the sign of the energy functional Φ associated with our problem. In this way, it will be shown that $\Phi(u^3) \leq 0$ and $\Phi(u^1) > 0$. Moreover, it will be proven that u^1 is linearly stable, whereas u^3 is linearly unstable.

Chapter 1

Rayleigh quotient

In this chapter, we shall address the development of the Rayleigh quotient derived from the Fréchet derivative, when we refer to this quotient, we will denote it by RQ. To simplify the exposition, whenever we mention the Rayleigh quotient we shall be referring specifically to this quotient.

From this quotient, we shall be able to study the various extremal values of the fiber of the Rayleigh quotient (see [19]), which will allow us to establish a direct connection with the Nehari set. More precisely, these extremal values will help us to understand when the Nehari set is empty or not.

In addition, we shall refine these extremal values and define, from the derivative of the fiber of the Rayleigh quotient, new characterisations which, together with the equivalences in the definition of the Nehari set and its subsets, will enable us to determine in which cases these subsets are empty or not, and how they are closely related to one another.

We shall also present the geometry of the fiber of the Rayleigh quotient, showing situations in which this fiber possesses a critical point and others in which it does not. These results will be fundamental for the subsequent chapters.

Finally, we shall introduce the Rayleigh quotient associated with the energy functional, and we shall show how it relates to the previous Rayleigh quotient, demonstrating that their properties are closely interconnected

In this chapter, we shall work in the space $W = \prod_{i=1}^{n} W_i$ the product of Banach spaces W_i, adopting a vectorial perspective on the problem. Within this framework, the Nehari manifold and the corresponding energy functional will also be defined from a vectorial approach.

In this chapter, we shall study the following equation

$$\Phi(u) = T(u) - \lambda G(u),$$

where $\lambda \in \mathbb{R}$, $T, G \in C^1(W \setminus \{0\}, \mathbb{R})$. Taking into account that weak solutions are defined by differentiating Φ in the Fréchet sense and setting it equal to zero, we have

$$\nabla_u \Phi(u) := \nabla_u T(u) - \lambda \nabla_u G(u) = 0, \quad u \in W \tag{1.0.1}$$

Let us define the Nehari vector manifold associated with (1.0.1):

Definition 1.0.1. *The Nehari manifold is defined as*

$$\mathcal{N} = \left\{ u \in W \setminus \{0_n\} : \nabla_u \Phi(u)(u) \equiv \frac{d}{dt}\Phi(tu)|_{t=1} = 0 \right\} \tag{1.0.2}$$

where $\dot{W} = \prod_{i=1}^{n} (W_i \setminus \{0\})$.

The problem of the Nehari manifold is:

$$\begin{cases} \Phi(u) \to \text{critical}, \\ u \in \mathcal{N}. \end{cases} \tag{1.0.3}$$

Remark 1.0.1. *The notation (1.0.3) means that* $\nabla_u \Phi(u) = 0$ *for all direction* $v \in T_u\mathcal{N}$*, that is,* $\nabla_u \Phi(u)(v) = 0$ *for all* $v \in T_u\mathcal{N}$*.*

Proposition 1.0.1. *Let* $\Phi \in C^1(W, \mathbb{R})$*. Assume that for some* $u \in \mathcal{N}$ *one has*

$$W = T_u\mathcal{N} \oplus \mathbb{R}u,$$

and moreover,

$$\nabla_u \Phi(u)(v) = 0, \quad \textit{for every } v \in T_u\mathcal{N}.$$

Then $\nabla_u \Phi(u) = 0$*, that is,* u *is a critical point of* Φ *in the whole space* W*.*

Proof. Since $W = T_u\mathcal{N} \oplus \mathbb{R}u$, any $w \in W$ can be written uniquely as $w = v + \alpha u$ with $v \in T_u\mathcal{N}$ and $\alpha \in \mathbb{R}$. By the linearity of the derivative in its second argument,

$$\nabla_u \Phi(u)(w) = \nabla_u \Phi(u)(v) + \alpha \nabla_u \Phi(u)(u)$$

by hypothesis, $\nabla_u\Phi(u)(v) = 0$ for all $v \in T_u\mathcal{N}$, hence $\nabla_u\Phi(u)(w) = \alpha\nabla_u\Phi(u)(u)$, since u belongs to the Nehari, one has $\nabla_u\Phi(u)(u) = 0$. Therefore,

$$\nabla_u\Phi(u)(w) = 0, \quad \text{for all } w \in W,$$

which means that $\nabla_u\Phi(u) = 0$. Consequently, u is a critical point of Φ in the whole space W. □

Firstly, we shall show that the critical points of the energy functional (1.0.3) satisfy equation (1.0.1). That is, by demonstrating this, we are establishing that the critical points of the functional correspond to solutions of equation (1.0.3). Consequently, these critical points represent solutions of a partial elliptic differential equation associated with the considered energy functional.
For this reason, the following theorem is of particular importance throughout this chapter. Once the hypotheses of this theorem are verified, we will be able to guarantee the existence of solutions. This analysis not only underpins the validity of the variational methods employed but also provides a deeper understanding of the relationship between the geometry of the energy functional and the structure of the problem's solutions.

Theorem 1.0.1. *Assume that $\Phi \in C^1(\dot{W}, \mathbb{R})$ and that the function*

$$F(u,t) := D_u\Phi(tu)(u)$$

is of class C^1 on $\dot{W} \times \dot{\mathbb{R}}^+$. Suppose that $\mathcal{N} \neq \emptyset$ and that

$$\frac{d}{dt}F(u,t)\,|_{t=1} \neq 0, \quad \text{for all } u \in \mathcal{N} \tag{1.0.4}$$

Then $\mathcal{N}$ is a C^1-manifold of codimension 1, $W = T_u(\mathcal{N}) \oplus \mathbb{R}u$ for every $u \in \mathcal{N}$, and any solution of (1.0.3) satisfies (1.0.1).

Proof. To prove that $\mathcal{N}$ is a C^1 submanifold of codimension 1, we apply the Implicit Function Theorem (see Appendix B.0.2). Indeed recall that the Nehari set is defined by

$$\mathcal{N} = \left\{u \in W \setminus \{0_n\} : \nabla_u\Phi(u)(u) \equiv \frac{d}{dt}\Phi(tu)|_{t=1} = 0\right\}.$$

Let us introduce the auxiliary function

$$F(u,t) := D_u\Phi(tu)(u), \qquad (u,t) \in \dot{W} \times \dot{\mathbb{R}}^+,$$

then we have that

$$\mathcal{N} = \{u \in \dot{W} : F(u,1) = 0\}.$$

Fix $u_0 \in \mathcal{N}$. Then

$$F(u_0,1) = D_u\Phi(u_0)(u_0) = 0.$$

By hypothesis, $F \in C^1(\dot{W} \times \dot{\mathbb{R}}^+, \mathbb{R})$ and

$$\partial_t F(u_0,1) = \frac{d}{dt} F(u_0,t)\Big|_{t=1} \neq 0.$$

Since t is a real scalar variable, the partial derivative

$$\partial_t F(u_0,1) \colon \mathbb{R} \to \mathbb{R}, \qquad s \to F_t(u_0,1)\, s,$$

is a linear isomorphism. Hence, all the hypotheses of the Implicit Function Theorem (see Appendix B.0.2) are satisfied at the point $(u_0,1)$. Therefore, there exist an open neighborhood $U_0 \subset \dot{W}$ of u_0, an open interval $I \subset \mathbb{R}^+$ containing 1, and a unique function

$$t \in C^1(U_0, \mathbb{R}^+)$$

such that

$$F(u,t(u)) = 0 \quad \text{for all } u \in U_0, \qquad \text{and} \qquad t(u_0) = 1.$$

From the definition of F, the identity $F(u,t(u)) = 0$ is equivalent to

$$D_u\Phi(t(u)u)(u) = 0, \qquad \forall u \in U_0.$$

Multiplying by the positive scalar $t(u)$, we obtain

$$D_u\Phi(t(u)u)(t(u)u) = 0, \qquad \forall u \in U_0,$$

which shows that

$$t(u)u \in \mathcal{N}, \qquad \forall u \in U_0.$$

Since $u_0 \in \mathcal{N}$ was arbitrary, we conclude that $\mathcal{N}$ is a C^1 submanifold of W. On the other hand, we define the following function

$$G(u) := F(u,1) = D_u\Phi(u)(u), \qquad u \in \dot{W}.$$

Then $G \in C^1(\dot{W},\mathbb{R})$ and

$$\mathcal{N} = G^{-1}(0).$$

Moreover, since

$$D_uG(u)(u) = \frac{d}{dt}F(u,t)\Big|_{t=1} = F_t(u,1) \neq 0,$$

the differential $D_uG(u)$ is a nonzero continuous linear functional on W. Therefore, by Theorem (B.0.1)

$$T_u(\mathcal{N}) = \ker D_uG(u), \qquad \forall u \in \mathcal{N}.$$

Thus, by Theorem (B.0.4), $\mathcal{N}$ has codimension 1. On the other hand, let us prove that

$$T_u(\mathcal{N}) \cap \mathbb{R}u = \{0\}. \tag{1.0.5}$$

Let $w \in T_u(\mathcal{N}) \cap \mathbb{R}u$. Then there exists $\alpha \in \mathbb{R}$ such that $w = \alpha u$. Since $w \in \ker D_uG(u)$, we have

$$0 = D_uG(u)(w) = \alpha D_uG(u)(u).$$

Since $D_uG(u)(u) = F_t(u,1) \neq 0$, it follows that $\alpha = 0$ and hence $w = 0$.

Next, let $v \in W$ be arbitrary and define

$$\alpha := \frac{D_uG(u)(v)}{D_uG(u)(u)}.$$

Set

$$v_0 := v - \alpha u.$$

By linearity of $D_uG(u)$, we obtain

$$D_uG(u)(v_0) = 0,$$

which implies $v_0 \in T_u(\mathcal{N})$. Consequently,

$$v = v_0 + \alpha u \in T_u(\mathcal{N}) + \mathbb{R}u. \tag{1.0.6}$$

Since $v \in W$ was arbitrary, combining (1.0.5) and (1.0.6), we conclude that

$$W = T_u(\mathcal{N}) \oplus \mathbb{R}u, \qquad \forall u \in \mathcal{N}.$$

On the other hand, to prove that any solution of (1.0.3) satisfies (1.0.1), it follows from the proof of Proposition (1.0.1). □

The previous Theorem implies that, from it, we can obtain a general and simple view of which critical values on the Nehari manifold satisfy the variational form of a given problem. Once this is understood, it will be sufficient to prove the existence of critical values on that manifold to guarantee the existence of solutions to the corresponding elliptic equations.

Remark 1.0.2. *We say that the vector SM−method is applicable in geral to problem (1.0.1) for a given $\lambda \in \mathbb{R}$ if condition (1.0.4) is satisfied for each $u \in \mathcal{N}$.*

1.1 Rayleigh quotient from the Fréchet derivative

The RQ allows us to understand the behavior of the fiber, which will facilitate the demonstration of the existence of solutions for different elliptic problems, as well as to gain a deeper insight into the properties of the RQ. By means of an argument based on infima and suprema, one aims to determine whether the Nehari manifold is empty or not; moreover, this provides fundamental information on whether the subsets of the Nehari manifold are empty or not. Once this is established, through other methods and theorems, the existence of a solution to a specific problem can be achieved.
Let Φ be the energy functional associated with equation

$$\Phi(u) = T(u) - \lambda G(u).$$

Then the derivative in the Fréchet sense in the direction u is

$$D_u\Phi(u)(u) = D_uT(u)(u) - \lambda D_uG(u)(u).$$

In the present thesis, we introduce the hypotheses necessary for the proper definition of the RQ. Based on these, we shall study the fiber function, which will help us to understand its nature and behaviour, and will later enable us to search for solutions.

(A$_1$) : $D_uG(u)(u) \neq 0$, for all $u \in \dot{W}$.

From this we define the Rayleigh quotient, that is,

$$r_u(t) := R(tu) = \frac{D_uT(tu)(tu)}{D_uG(tu)(tu)}, \quad t \in \mathbb{R}^+ \setminus 0,\ u \in \dot{W}, \tag{1.1.1}$$

where

$$R(u) = \frac{D_uT(u)(u)}{D_uG(u)(u)}.$$

(A$_2$) : The maps $\frac{d}{dt}T(tu)$ and $\frac{d}{dt}G(tu)$ are of class C^1 on $\dot{\mathbb{R}}^+ \times W$.

(A$_3$) : For every fixed $u \in \dot{W}$ there exists

$$\lim_{t\to 0} r_u(t) = \hat{r}_u(0), \qquad \text{where } \hat{r}(0) \in [-\infty, \infty]$$

Remark 1.1.1. *Observe, that (A_1) and (A_2) imply that $r_u(t)$ and $\frac{d}{dt}\Phi_\lambda(tu)$ are maps of class C^1 on $\mathbb{R}^+ \times W$.*

We note that the following statement is clear; therefore, we denote it by

(R) : For $u \in \mathcal{N}$, if $\frac{d}{dt}\Big(D_u\Phi(tu)(u)\Big)\Big|_{t=1} = 0$, then 1 is a critical point of r_u.

Let $u \in \dot{W}$, $t_0 \in \dot{\mathbb{R}}^+$. If $\frac{d}{dt}r_u(t)\Big|_{t=t_0} = 0$, then t_0 is said to be a critical point of $r_u(t)$ and $\lambda = r_u(t_0)$ is said to be a critical value, for $tu \in \mathcal{N}$, we observe that

$$\frac{d}{dt}r_u(t) = \frac{1}{D_uG(tu)(u)}\frac{d}{dt}(D_u\Phi(tu)(u)) \tag{1.1.2}$$

Indeed the derivative of the Rayleigh quotient (1.1.1) with respect to t

$$\frac{d}{dt}r_u(t) = \frac{\frac{d}{dt}\left(D_uT(tu)(tu)\right)D_uG(tu)(tu) - D_uT(tu)(tu)\frac{d}{dt}\left(D_uG(tu)(tu)\right)}{(D_uG(tu)(tu))^2}.$$

By applying the chain rule and taking into account that the derivative is linear in its second argument, we obtain that

$$\frac{d}{dt}r_u(t) = \frac{tD^2_{uu}G(tu)(u^2)D_uG(tu)(tu) - tD^2_{uu}G(tu)(u^2)D_uT(tu)(tu)}{(D_uG(tu)(tu))^2}. \tag{1}$$

O the other hand, since $tu \in \mathcal{N}$ then, the derivative of $\Phi(tu)$ in the Fréchet sense in the direction of u is $D_u\Phi(tu)(u) = D_uT(tu)(u) - \lambda D_uG(tu)(u) = 0$ multiplying both sides by t and using the linearity of the derivative in its second argument, we obtain $D_uT(tu)(tu) = \lambda D_uG(tu)(tu)$ Substituting this into (1), and using the fact that $\frac{d}{dt}D_u\Phi(tu)(u) = D^2_{uu}T(tu)(u^2) - \lambda D^2_{uu}G(tu)(u^2)$ we then obtain the result.

We call $t_0 \in \dot{\mathbb{R}}^+$ the extreme point of $r_u(t)$ if the function $r_u(t)$ attains at t_0 its local maximum or minimum on $\mathbb{R}^+$.

Proposition 1.1.1. *For $u \in \dot{W}$ and $t > 0$ there holds:*

(a) $tu \in \mathcal{N}$ *if and only if* $\lambda = r_u(t)$.

Furthermore, if $D_uG(tu)(tu) > 0$ (respectively, $D_uG(tu)(tu) < 0$) for $u \in \dot{W}$ and $t \in \dot{\mathbb{R}}^+$, then:

(b) $r_u(t) > \lambda$ *if and only if* $\frac{d}{dt}\Phi(tu) > 0 \quad \left(\frac{d}{dt}\Phi(tu) < 0\right)$;

(c) $r_u(t) < \lambda$ *if and only if* $\frac{d}{dt}\Phi(tu) < 0 \quad \left(\frac{d}{dt}\Phi(tu) > 0\right)$;

(d) $\frac{d}{dt}r_u(t) < 0$ *if and only if* $\frac{d^2}{dt^2}\Phi(tu) < 0$ *(respectively,* > 0*);*

(e) $\frac{d}{dt}r_u(t) > 0$ *if and only if* $\frac{d^2}{dt^2}\Phi(tu) > 0$ *(respectively,* < 0*).*

Proof. The proof of these items is essentially intuitive and relies primarily on straightforward derivative computations.

(a) Let $tu \in \mathcal{N}$. Then $D_u\Phi(tu)(tu) = 0$ (the Fréchet derivative in the direction tu) as $\Phi(tu) = T(tu) - \lambda G(tu)$ and by the linearity of the differential in its second argument, we have

$$\begin{aligned} &D_u\Phi(tu)(u)t = D_uT(tu)(u)t - \lambda D_uG(tu)(u)t = 0 \\ &\Longleftrightarrow \lambda = \frac{D_uT(tu)(tu)}{D_uG(tu)(tu)} \\ &\Longleftrightarrow r_u(t) = \lambda. \end{aligned}$$

(b) For this item, let us first consider $D_uG(tu)(tu) > 0$ then

$$\begin{aligned} r_u(t) = \frac{D_uT(tu)(tu)}{D_uG(tu)(tu)} > \lambda \\ \iff D_uT(tu)(tu) - \lambda D_uG(tu)(tu) > 0 \\ \iff \frac{d}{dt}\Phi(tu) > 0. \end{aligned}$$

Note that the last inequality follows from the application of the chain rule for derivatives. On the other hand, if we consider $D_uG(tu)(tu) < 0$, we have the following:

$$\begin{aligned} r_u(t) = \frac{D_uT(tu)(tu)}{D_uG(tu)(tu)} > \lambda \\ \iff D_uT(tu)(tu) - \lambda D_uG(tu)(tu) < 0 \\ \iff \frac{d}{dt}\Phi(tu) < 0. \end{aligned}$$

This proves what we wanted to show.

(c) The proof of this item is completely analogous to the previous one; therefore, let us proceed to the demonstration of the next item.

(d) Firstly, let us note that the second derivative of Φ with respect to t is obtained using the chain rule for the Fréchet derivative in the direction of u we have $\frac{d^2}{dt^2}\Phi(tu) = \frac{d}{dt}(D_u\Phi(tu)(u)$ by (1.1.2) Therefore, our inequality for the Rayleigh derivative is determined by the sign of $D_uG(tu)(u)$ then if $D_uG(tu)(u) > 0$

$$\begin{aligned} \frac{d}{dt}r_u(t) = \frac{1}{D_uG(tu)(u)}\frac{d}{dt}(D_u\Phi(tu)(u)) > 0 \\ \iff \frac{d^2}{dt^2}\Phi(tu) > 0. \end{aligned}$$

And it is clear that for $D_uG(tu)(u) < 0$ the inequality is reversed, yielding the desired result.

Note that the last inequality follows from the application of the chain rule for derivatives.

(e) The proof of this item is completely analogous to the previous one, so there is no need to repeat it.

□

1.2 Extremal values for the Rayleigh quotient from the Fréchet derivative

Once the RQ has been obtained, we can define certain parameters associated with it. These parameters are introduced strategically and conveniently, with the purpose of later linking them to our variational problem. In this way, we define the following parameters:

$$\lambda_i(u) := \inf_{t\in\mathbb{R}^+} r_u(t), \quad u \in \dot{W},$$
$$\lambda_s(u) := \sup_{t\in\dot{\mathbb{R}}^+} r_u(t), \quad u \in \dot{W},$$

and we restrict our main attention to the extremal values:

$$\lambda_{ii} = \inf_{u\in\dot{W}} \lambda_i(u), \qquad \lambda_{ss} = \sup_{u\in\dot{W}} \lambda_s(u), \tag{1.2.1}$$

$$\lambda_{si} = \sup_{u\in\dot{W}} \lambda_i(u), \qquad \lambda_{is} = \inf_{u\in\dot{W}} \lambda_s(u). \tag{1.2.2}$$

When defining them, we will adopt precise and consistent notations to facilitate both exposition and analysis. For example, if λ_{ii} denotes the infimum of the infimum, and λ_{ss} represents the supremum of the supremum, we will use this type of notation systematically throughout the text.

From these parameters, we will be able to deduce and explain in which cases the Nehari manifold is empty or not. It is important to understand this part, since in order to solve the proposed elliptic equations, we may already determine in which cases we have potential candidates for the existence of solutions.

Lemma 1.2.1. *It is fulfilled that*

(a) If $\lambda_{ii} > -\infty(\lambda_{ss} < +\infty)$, then $\mathcal{N} = \emptyset$, for any $\lambda < \lambda_{ii}(\lambda > \lambda_{ss})$;

(b) $\mathcal{N} \neq \emptyset$, for $\lambda \in (\lambda_{ii}, \lambda_{ss})$, and $\mathcal{N} = \emptyset$ for $\lambda \in \mathbb{R}\backslash[\lambda_{ii}, \lambda_{ss}]$.

Proof. (a) It is known that

$$\lambda_{ii} = \inf_{u\in\dot{W}} \inf_{t\in(\mathbb{R}^+)} r_u(t)$$

suppose that $\lambda < \lambda_{ii} < R(tu)$, $tu \in \dot{W}$ on the other hand, it is known that for $u \in \mathcal{N}$ we have $\lambda = R(u)$ then $\lambda = R(u) < \lambda_{ii}$ for $u \in \mathcal{N}$ contradiction, analogously where

$$\lambda_{ss} = \sup_{u\in\dot{W}} \sup_{t\in(\mathbb{R}^+)} r_u(t)$$

suppose that $\lambda > \lambda_{ss} > R(tu)$, $tu \in \dot{W}$on the other hand, it is known that for $u \in \mathcal{N}$ we have $\lambda = R(u)$ then $\lambda = R(u) > \lambda_{ss}$ contradiction, then (a) is satisfied.

(b) $u \in \mathcal{N}$ if and only if $R(u) = \lambda$, recall that

$$\mathcal{N} := \left\{u \in \dot{W} : R(u) = \lambda\right\} \text{ and } \quad R(u) := \frac{D_u T(u)(u)}{D_u G(u)(u)},$$

and that the scalar extremal values are defined by

$$\lambda_{ii} := \inf_{u\in\dot{W}} \inf_{t\in(\mathbb{R}^+)} r_u(t), \quad \lambda_{ss} := \sup_{u\in\dot{W}} \sup_{t\in(\mathbb{R}^+)} r_u(t).$$

Case 1: Assume that $\lambda \in (\lambda_{ii}, \lambda_{ss})$. Then, there exist vectors $u_1, u_2 \in \dot{W}$ and $t_1, t_2 \in \dot{\mathbb{R}}^+$ such that

$$r_{u_1}(t_1) < \lambda < r_{u_2}(t_2).$$

Let us define

$$\begin{aligned} \gamma : \quad & [0,1] \longrightarrow W \times (\dot{\mathbb{R}}^+) \\ & s \longmapsto \gamma(s) = ((1-s)u_1 + su_2,\ (1-s)t_1 + st_2). \end{aligned}$$

Note that $\dot{W}$ and $(\mathbb{R}^+)$ are convex sets, it is also continuous in $[0,1]$ now define

$$\begin{aligned} f : \quad & [0,1] \longrightarrow \mathbb{R} \\ & s \longmapsto f(s) = r_{u_s}(\gamma(s)), \end{aligned}$$

where $f(0) = r_{u_1}(t_1)$ and $f(1) = r_{u_2}(t_2)$ besides is continuous and differentiable because it is a composition of continuous and differentiable functions. By the intermediate value theorem exists s_0 such that $f(s_0) = \lambda$. Let us call $\gamma(s_0) = (u_0, t_0) \to f(s_0) = r_{u_0}(t_0)$
Case 2: Assume now that $\lambda \notin [\lambda_{ii}, \lambda_{ss}]$, i.e., either $\lambda < \lambda_{ii}$ or $\lambda > \lambda_{ss}$.

- If $\lambda < \lambda_{ii}$, then for every $u \in \dot{W}$, we have $R(u) > \lambda$, so there is no u such that $R(u) = \lambda$, contradiction. Hence, $\mathcal{N} = \emptyset$.
- Similarly, if $\lambda > \lambda_{ss}$, then for all $u \in \dot{W}$, we have $R(u) < \lambda$, and again $\mathcal{N} = \emptyset$.

Therefore, $\mathcal{N} = \emptyset$ for all $\lambda \in \mathbb{R} \setminus [\lambda_{ii}, \lambda_{ss}]$.

□

1.3 Applicability of the Nehari method

Once the fiber of the RQ is known, we will add some conditions and see how these allow us to obtain important results. In this work, we aim to study a particular case of the behaviour of $r_u(t)$, since in many problems discussed in the literature we encounter this type of behaviour. That is, in order to obtain useful results, we impose conditions on $r_u(t)$. Such a condition can be written as follows.

(S) For all $u \in \dot{W}$, $r_u(t)$ has no critical points in $\mathbb{R}^+$ except the points of global minimum or maximum of $r_u(t)$ on $\mathbb{R}^+$.

We will see in the forthcoming examples that condition (S) could easily be verified. Typical graphs of function $r_u(t)$ satisfying (S) are presented in Figure 1.1.

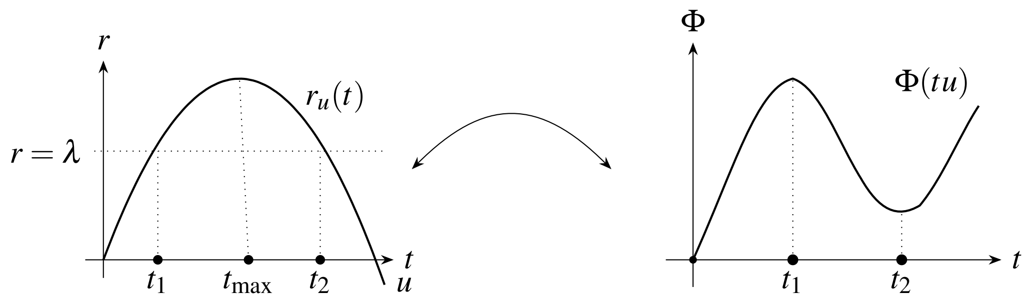


Fig. 1.1 $r_u(t)$ and a perfectly smoothed fibering function $\Phi(tu)$ with parabola-like transitions.

This will allow us to obtain the following important result.

Theorem 1.3.1. *Assume* $(A_1),(A_2)$ *and* (A_3) *hold. Suppose* $r_u(t)$ *satisfies* $(R),(S)$ *and* $\lambda_{si} < \lambda_{is}$. *Then for each* $\lambda \in (\lambda_{si}, \lambda_{is})$ *the vector Nehari manifold method is applicable to (1.0.1) so that if* $\mathcal{N} \neq \emptyset$, *then* $\mathcal{N}$ *is a* C^1*-manifold of codimension* 1 *and any solution of (1.0.3) satisfies (1.0.1).*

Proof. Let $\lambda \in (\lambda_{si}, \lambda_{is})$ and $u \in \mathcal{N}$. Suppose by contradiction that $\frac{d}{dt} D_u\Phi(tu)(u) \mid_{t=1} = 0$. Then by (1.2.2) and (R)

$$\left.\frac{d}{dt} r_u(t)\right|_{t=1} = \frac{1}{D_u G(tu)(u)} \frac{d}{dt}(D_u\Phi(tu)(u))\bigg|_{t=1} = 0, \tag{1.3.1}$$

then $t = 1$ is a critical point for $r_u(t)$, and by (S) the function $r_u(t)$ attains its global minimum or maximum at $t = 1$. Assume, for instance, that this is a global minimum point. Since $\lambda > \lambda_{si}$ and $\lambda = r_u(1)$ for $u \in \mathcal{N}$, (1.2.2) implies

$$r_u(1) = \min_{t\in\mathbb{R}^+} r_u(t) = \lambda > \lambda_{si} = \sup_{u\in W}\left(\inf_{t\in\mathbb{R}^+} r_u(t)\right) \geq \inf_{t\in\mathbb{R}^+} r_u(t) = r_u(1).$$

Thus we get a contradiction and the proof follows from Theorem (1.0.1). □

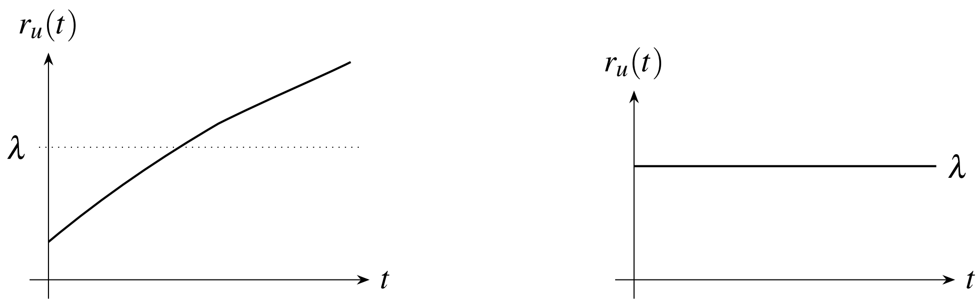
If condition (S) is strengthened by the introduction of additional constraints, one can expect more precise estimates of the extremal values of the SM method. Throughout this thesis, those extremals will be progressively refined. Let us now consider the following particular case of (S).

(S_0) For any $u \in \dot{W}$ one of the following holds:

(a) $r_u(t)$ has no critical point $t \in \mathbb{R}$ such that $tu \in \mathcal{N}$;

(b) $\frac{d}{dt} r_u(t) = 0$ for all $t \in \mathbb{R}$.

The typical graphs are the below



(a) No critical points (b) Constant function

Fig. 1.2 figures with the S_0 condition

Thus, if one ever works with an energy functional and its quotient satisfying these conditions, the following theorem would be very useful.

Theorem 1.3.2. *Assume* $(A_1),(A_2)$ *and* (A_3) *hold. Suppose* $r_u(t)$ *satisfies* $(R),(S_0)$ *and* $\lambda_{ii} < \lambda_{is}$ $(\lambda_{si} < \lambda_{ss})$. *Then for each* $\lambda \in (\lambda_{ii}, \lambda_{is})$ $(\lambda \in (\lambda_{si}, \lambda_{ss}))$ *the vector Nehari manifold method is applicable to (1.0.1) so that if* $\mathcal{N} \neq \emptyset$, *then* $\mathcal{N}$ *is a* C^1-*manifold of codimension* 1 *and any solution of (1.0.3) satisfies (1.0.1).*

Proof. We prove the statement for the case $\lambda_{ii} < \lambda_{is}$. The proof in the case $\lambda_{si} < \lambda_{ss}$ is similar, Given $u \in \mathcal{N}$, suppose by contradiction that $\frac{d}{dt} D_u\Phi(tu)(u) \mid_{t=1}= 0$. Then by (1.2.2) and (R)

$$\left.\frac{d}{dt} r_u(t)\right|_{t=1} = \frac{1}{D_uG(tu)(u)} \frac{d}{dt}\left(D_u\Phi(tu)(u)\right)\Bigg|_{t=1} = 0, \tag{1.3.2}$$

then $t = 1$ is a critical point of the function $r_u(t)$. Hence, (S_0) entails that the function $r_u(t)$ identically equals to the constant λ in $\mathbb{R}^+$ and attains its global minimum and maximum at any point $t \in \mathbb{R}^+$. However, the assumption $\lambda < \lambda_{is}$ yields that

$$\lambda < \lambda_{is} = \inf_{u\in W}\left(\sup_{t\in\mathbb{R}^+} r_u(t)\right) \leq \sup_{t\in\mathbb{R}^+} r_u(t) = \max_{t\in\mathbb{R}^+} r_u(t) = r_u(1) \equiv R(u) = \lambda.$$

□

Thus we get a contradiction and proof follows from (1.0.1).

1.4 Subsets of the Nehari manifold

In recent years, the Nehari manifold and its subsets have become a fundamental tool for proving the existence of solutions in variational problems. As a result, the concept of the Nehari manifold has acquired multiple formulations and equivalences—some depending solely on the energy functional, while others are expressed in terms of the Rayleigh quotient; both, however, are closely related.

Strategically, this relationship allows the Nehari manifold to be splited into different regions or components, facilitating the analysis of the critical points of the functional. Many authors and researchers refer to this process as the multi-Nehari manifold, since it precisely consists of considering several partitions or internal structures within the original manifold.

Definition 1.4.1. *The subsets of the Nehari manifold are defined as follows:*

$$\mathcal{N}^+ := \Big\{ u \in \mathcal{N} : D^2_{uu}\Phi(u)(u,u) > 0 \Big\},$$
$$\mathcal{N}^0 := \Big\{ u \in \mathcal{N} : D^2_{uu}\Phi(u)(u,u) = 0 \Big\},$$
$$\mathcal{N}^- := \Big\{ u \in \mathcal{N} : D^2_{uu}\Phi(u)(u,u) < 0 \Big\}.$$

Where $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \cup \mathcal{N}^0$, note that in this chapter we are working vectorly; we have $D^2_{uu}\Phi(u)(u,u) = \sum_{i,j=1}^{n} D^2_{u_i,u_j}\Phi(u)(u_i,u_j)$ and $\mathcal{H}\Phi(u) = \Big(D^2_{u_iu_j}\Phi(u)\Big)_{1\leq i,j\leq n}$, which represents the Hessian matrix with second derivatives in the Fréchet sense. Furthermore, each entry of the Hessian matrix is represented by $D^2_{u_iu_j}\Phi(u) := \dfrac{\partial^2\Phi(u)}{\partial u_i \partial u_j}$, that basically means the second derivative in the u_i and u_j directions.

Remark 1.4.1. *Note that* $\mathcal{N}^-$, $\mathcal{N}^+$ *and* $\mathcal{N}^0$ *are disjoint. In other words,*

$$\mathcal{N}^- \cap \mathcal{N}^+ = \emptyset.$$

From this definition, we can identify other sets equivalent to those previously mentioned, modifying only the definition in terms of the fibering function. This reformulation will allow us to gain a deeper understanding of the structure of the problem and will become a fundamental tool for determining more precisely the location of the solutions of a particular partial differential equation.

Lemma 1.4.1. *Definition (1.4.1) is equivalent to*

$$\mathcal{N}^+ := \Big\{ u \in \mathcal{N} : \frac{d}{dt}\Phi(tu)\Big|_{t=1} = 0,\ \frac{d^2}{dt^2}\Phi(tu)\Big|_{t=1} > 0 \Big\}, \tag{1.4.1}$$

$$\mathcal{N}^0 := \Big\{ u \in \mathcal{N} : \frac{d}{dt}\Phi(tu)\Big|_{t=1} = 0,\ \frac{d^2}{dt^2}\Phi(tu)\Big|_{t=1} = 0 \Big\}, \tag{1.4.2}$$

$$\mathcal{N}^- := \Big\{ u \in \mathcal{N} : \frac{d}{dt}\Phi(tu)\Big|_{t=1} = 0,\ \frac{d^2}{dt^2}\Phi(tu)\Big|_{t=1} < 0 \Big\}. \tag{1.4.3}$$

Proof. Let us prove that $\mathcal{N}^+$ is equivalent to (1.4.1). Indeed, as we know, the fibering function is denoted by $\varphi(t) = \Phi(tu)$, and since $\Phi(tu) = T(tu) - \lambda G(tu)$, then by differentiating and applying the chain rule, we easily obtain that

$$\varphi'(t) = D_u T(tu)(u) - \lambda D_u G(tu)(u).$$

Moreover, for $u \in \mathcal{N}$ we have that $r_u(1) = \lambda$. Replacing the value of λ and knowing that we are working under the condition $D_u G(tu)(u) \neq 0$, it is clear that $\varphi'(1) = 0$, that is,

$$\frac{d}{dt}\Phi(tu)\Big|_{t=1} = 0.$$

Now, to prove the second condition, we again apply the chain rule, but this time for the second derivative:

$$\varphi''(t) = D^2_{uu} T(tu)(u,u) + \lambda D^2_{uu} G(tu)(u,u).$$

Since $u \in \mathcal{N}$, then for $t = 1$ we have that $\varphi''(1) = D^2_{uu}\Phi(u)(u,u)$, which is bigger than 0 by hypothesis, and the proof is complete. In a completely analogous way, one can prove (1.4.2) and (1.4.3). □

Lemma 1.4.2. *If $D_u G(tu)(tu) > 0$ and $t_0 \in \dot{\mathbb{R}}^+$, then the subsets defined above are equivalent to*

$$\mathcal{N}^+ := \left\{u \in \mathcal{N} : \frac{d}{dt} r_u(t)\Big|_{t=1} > 0\right\}, \tag{1.4.4}$$

$$\mathcal{N}^0 := \left\{u \in \mathcal{N} : \frac{d}{dt} r_u(t)\Big|_{t=1} = 0\right\}, \tag{1.4.5}$$

$$\mathcal{N}^- := \left\{u \in \mathcal{N} : \frac{d}{dt} r_u(t)\Big|_{t=1} < 0\right\}. \tag{1.4.6}$$

Proof. For the proof, we use the previous lemma, thus by (1.1.2) the proof of (1.4.5) is immediate and by (1.1.1) items (d) and (e) the proof of (1.4.4) and (1.4.6) is immediate. □

Remark 1.4.2. *If $r_u(t) = \lambda$ for some $t > 0$.*

1. *If $r'_u(t) > 0$, then $tu \in \mathcal{N}^+$,*
2. *If $r'_u(t) < 0$, then $tu \in \mathcal{N}^-$,*
3. *If $r'_u(t) = 0$, then $tu \in \mathcal{N}^0$.*

Thus, from this point onward and throughout the dissertation, we will assume that $\mathcal{J}_1 = D_u G(u)(u) > 0$. Therefore, we will no longer mention it; if at any moment a negative value is assumed, we will state it explicitly.

We now show that there exists a relationship between the Nehari manifold and its subsets and the extremal values of the Rayleigh quotient from the Fréchet derivative.

1.5 Extremals values related to these Nehari subsets

Now, we introduce other extremal values from the derivative of the Rayleigh quotient. The idea behind working with additional extremals is to gain more detailed information about Nehari submanifolds and their interrelationships. We will see how the theory that will help us better understand a partial elliptic differential equation arises.
We define extreme values as:

$$\lambda_{+ii} = \inf_{u \in W \setminus 0} \inf_{\substack{t>0 \\ r_u'(t)>0}} r_u(t); \tag{1.5.1}$$

$$\lambda_{-ii} = \inf_{u \in W \setminus 0} \inf_{\substack{t>0 \\ r_u'(t)<0}} r_u(t); \tag{1.5.2}$$

$$\lambda_{0ii} = \inf_{u \in W \setminus 0} \inf_{\substack{t>0 \\ r_u'(t)=0}} r_u(t); \tag{1.5.3}$$

$$\lambda_{+is} = \inf_{u \in W \setminus 0} \sup_{\substack{t>0 \\ r_u'(t)=>0}} r_u(t); \tag{1.5.4}$$

$$\lambda_{+si} = \sup_{u \in W \setminus 0} \inf_{\substack{t>0 \\ r_u'(t)>0}} r_u(t). \tag{1.5.5}$$

Remark 1.5.1. *Note that the definition of these parameters is strategic, since the plus sign indicates that r' can be positive, negative, or zero, and the first i is for the first infimum, while the second i is for the second infimum. Thus, we can define other parameters such as λ_{-si}, λ_{-is}, λ_{+ss}, λ_{-ss}, λ_{0si}, λ_{0is}, λ_{0ss}.*

Remark 1.5.2. *We assume that* $\inf \emptyset = \infty$ *and* $\sup \emptyset = -\infty$.

From now on we call $\inf_{t>0: r_u'(t)>0} r_u(t)$ and their similar a nonlinear generalized Rayleigh's quotient (NGRQ for short)

Theorem 1.5.1. *There holds:*

1. *If* $\lambda_{+ii} > -\infty$, *then for each* $\lambda < \lambda_{+ii}$ *we have that* $\mathcal{N}^+ = \emptyset$;

2. *If* $\lambda_{+ss} < \infty$, *then for each* $\lambda > \lambda_{+ss}$ *we have that* $\mathcal{N}^+ = \emptyset$;

3. *If* $\lambda_{-ii} > -\infty$, *then for each* $\lambda < \lambda_{-ii}$ *we have that* $\mathcal{N}^- = \emptyset$;

4. *If* $\lambda_{-ss} < \infty$, *then for each* $\lambda > \lambda_{-ss}$ *we have that* $\mathcal{N}^- = \emptyset$;

5. *If* $\lambda_{0ii} > -\infty$, *then for each* $\lambda < \lambda_{0ii}$ *we have that* $\mathcal{N}^0 = \emptyset$;

6. *If* $\lambda_{0ss} < \infty$, *then for each* $\lambda > \lambda_{0ss}$ *we have that* $\mathcal{N}^0 = \emptyset$.

Similar to $\lambda_{-ii}, \lambda_{-ss}, \lambda_{0ii}, \lambda_{0ss}$ *and their corresponding Nehari sets* $\mathcal{N}^-,\ \mathcal{N}^0$.

Proof. Indeed

1. Let us suppose, by contradiction, $\mathcal{N}^+ \neq \emptyset$. Then there exists $u_0 \in W \setminus \{0\}$ e $t_0 > 0$ such that $r_{u_0}(t_0) = \lambda < \lambda_{+ii}$ and $r'_{u_0}(t_0) > 0$. On the other hand, we have by definition that $r_u(t) \geq \lambda_{+ii}$ for all $t > 0$ and $u \in W \setminus \{0\}$. This is a contradiction for u_0 and to above.

2. Let us suppose, by contradiction, $\mathcal{N}^+ \neq \emptyset$. Then there exists $u_0 \in W \setminus \{0\}$ e $t_0 > 0$ such that $r_{u_0}(t_0) = \lambda < \lambda_{+ss}$ and $r'_{u_0}(t_0) > 0$. On the other hand, we have by definition that $r_u(t) \geq \lambda_{+ss}$ for all $t > 0$ and $u \in W \setminus \{0\}$. This is a contradiction for u_0 and to above. A similar proof hold to (3)- (6).

□

The purpose of this chapter is to analyse a Rayleigh quotient with a unique critical point, as such a situation frequently arises in many elliptic equations.

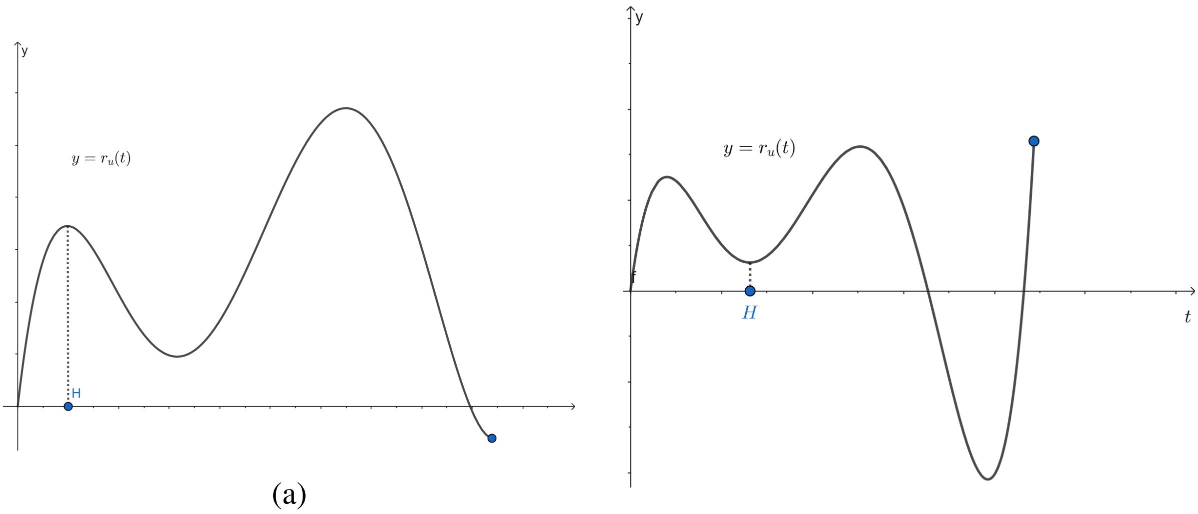
1.6 Fibering maps from the Rayleigh quotient with at most one critical point

Next, we present conditions under which the fiber has at most one critical value, and we show that in each case we obtain very important properties that will later allow us to search for the existence of solutions in some nonempty subset of the Nehari manifold. We develop these conditions because we frequently encounter problems whose Rayleigh quotients satisfy one of them.

1.6.1 Local maximum or minimum

Therefore, we introduce the following condition.

(Λ_1) : Fix $u \in W \setminus \{0\}$. Then every critical point of $r_u(t)$ is isolated and corresponds to a local maximum or a local minimum.



(a) (b)

Fig. 1.3 Typical graphs that satisfy the condition (Λ_1).

Theorem 1.6.1. *Assume (Λ_1), then:*

(1) If $\lambda_{+ii} < \lambda_{+is}$, then for each $\lambda \in (\lambda_{+ii}, \lambda_{+is})$ we have that $\mathcal{N}^+ \neq \emptyset$;

(2) If $\lambda_{-ii} < \lambda_{-is}$, then for each $\lambda \in (\lambda_{+ii}, \lambda_{+ss})$ we have that $\mathcal{N}^- \neq \emptyset$;

(3) If $\lambda_{+si} < \lambda_{+ss}$, then for each $\lambda \in (\lambda_{+si}, \lambda_{+ss})$ we have that $\mathcal{N}^+ \neq \emptyset$;

(1) If $\lambda_{-si} < \lambda_{-ss}$, then for each $\lambda \in (\lambda_{+si}, \lambda_{+ss})$ we have that $\mathcal{N}^- \neq \emptyset$.

Proof. (1) Fix $\lambda \in (\lambda_{+ii}, \lambda_{+is})$ and choose $u \in W \setminus \{0\}$ such that

$$\lambda_{+ii} \leq \inf_{\{t>0: r_u'(t)>0\}} r_u(t) < \lambda.$$

From the definition of λ_{+is} we know that $\sup_{\{t>0: r_u'(t)>0\}} r_u(t) \geq \lambda_{+is} > \lambda$, thus the set

$$A_\lambda = \{t > 0 : \lambda = r_u(t)\}$$

is non-empty.

We claim that there exists $t \in A_\lambda$ such that $r_u'(t) > 0$. Indeed take $t_0 > 0$ such that $r_u(t_0) < \lambda$ and set

$$B_1 = \{t > t_0 : r_u(t) < \lambda\}.$$

Since $A_\lambda \neq \emptyset$ we conclude that $t_1 = \sup B_1 < \infty$ and $r_u(t_1) = \lambda$. If $r_u'(t_1) > 0$ we are done, otherwise we have that $r_u'(t_1) = 0$ and from condition (Λ_1) we obtain that t_1 is an isolated local maximizer. Therefore by setting

$$B_2 = \{t > t_1 : r_u(t) < \lambda\}$$

and arguing as before we conclude that $t_2 = \sup B_2 < \infty$ and $r_u(t_2) = \lambda$. If $r_u'(t_2) > 0$ we are done, otherwise we can continue this process. Since $\sup_{\{t>0: r_u'(t)>0\}} r_u(t) \geq \lambda_{+is} > \lambda$ this process has to stop in finite time and then the proof is complete. A similar proof hold to (2), (3) and (4). □

Corollary 1.6.1. *Suppose* (Λ_1). *If* $\lambda_{+si} < \lambda_{+is}$, *then* $\mathcal{N}^+ \neq \emptyset$ *for each* $\lambda \in (\lambda_{+ii}, \lambda_{+ss})$. *Similar to* $\mathcal{N}^-$.

Remark 1.6.1. *Note that Theorem (1.6.1) and Corollary (1.6.1) still valid if we assume, instead of* (Λ_1), *whenever there exists* $t > 0$ *such that* $r_u'(t) = r_u''(t) = 0$, *then* r_u *is constant.*

Remark 1.6.2. *It is clear that if* $\lambda_{+si} = -\infty$ *or* $\lambda_{+is} = \infty$, *then* $\lambda_{+si} < \lambda_{+is}$. *In fact, this is the case in many applications. A similar result holds to the other extremal parameters.*

Next, we focus on the case $\mathcal{N}^0 \neq \emptyset$. The idea of studying this set is similar to the previous cases; however, this case constitutes the main difference between this chapter and the previous one. To study it, let us denote the following

$$D = \{u \in W \setminus \{0\} : \exists\, t > 0, r_u'(t) = 0\}$$

and

$$\mathcal{R}_i(u) = \inf_{\{t>0:\ r_u'(t)=0\}} r_u(t) \quad \forall u \in D$$

the NGRQ. Similarly define

$$\mathcal{R}_s(u) = \sup_{\{t>0:\ r_u'(t)=0\}} r_u(t) \quad \forall u \in D.$$

Theorem 1.6.2. *Suppose that D is conected and $\mathcal{R}_i$ is continuos. If $\lambda_{0ii} < \lambda_{0si}$, then for each $\lambda \in (\lambda_{0ii}, \lambda_{0si})$ we have that $\mathcal{N}^0 \neq \emptyset$.*

Proof. In fact, fix $\lambda \in (\lambda_{0ii}, \lambda_{0ss})$ and choose $u_1, u_2 \in D$ such that $\mathcal{R}_i(u_1) < \lambda < \mathcal{R}_i(u_2)$. Since $\mathcal{R}_i$ is continouos and D is connected, by the intermediate Value theorem, we obtain that there exists $u \in D$ such that $\mathcal{R}_i(u) = \lambda$ and the proof is complete. □

Remark 1.6.3. *It is clear that if λ_{0ii} is attained, that is, if there exits $u \in \dot{W} \setminus 0$ and $t > 0$ with $r_u'(t) = 0$ such that $\lambda_{0ii} = r_u(t)$, then $\mathcal{N}^0 \neq \emptyset$ for $\lambda = \lambda_{0ii}$. Similar to $\lambda_{0is}, \lambda_{0si}$ and λ_{0ss}.*

1.6.2 Global maximum for an unbounded fiber map

Consider a functional Φ such that each fibering map φ has at most two critical points.
We introduce the following condition

(Λ_{cc}) : For each $u \in W \setminus \{0\}$ the fibering r_u satisfies one of the following:

1) r_u has no critical points, it is increasing with $r_u(0) := \lim_{t\to 0^+} r_u(t) = 0$ and $r_u(t) \to \infty$ as $t \to \infty$;
2) $r_u(t)$ has a unique non-degenerate critical point at $t_u > 0$, which corresponds to a global maximum with positive energy and moreover $r_u(0) := \lim_{t\to 0^+} r_u(t) = 0$ and $r_u(t) \to -\infty$. as $t \to \infty$.

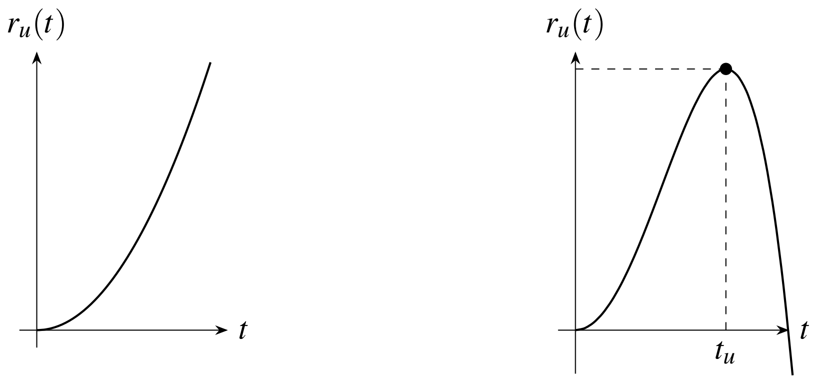


Fig. 1.4 Typical functions that fulfill the condition (Λ_{cc}).

Furthermore, there exists $u \in \dot{W}$ such that $2)$ is satisfied.

Let $D = \{u \in W \setminus \{0\} : t_u$ is well defined$\}$ and define $r(u) = r_u(t_u)$ for $u \in D$.

Lemma 1.6.1. *Suppose* (Λ_{cc}). *Then :*

i) $\lambda_{+ii} = 0 = \lambda_{+si}$,
$\lambda_{+is} = \inf_{u \in D} r(u)$,
$\lambda_{+ss} = \min\{\sup_{u \in D} r(u), \infty\}$.

ii) $\lambda_{-ii} = -\infty = \lambda_{-si}$,
$\lambda_{-is} = \inf_{u \in D} r(u)$,
$\lambda_{-ss} = \sup_{u \in D} r(u)$.

iii) $\lambda_{0ii} = \lambda_{0is} = \inf_{u \in D} r(u)$,
$\lambda_{0si} = \lambda_{0ss} = \sup_{u \in D} r(u)$.

Proof. The proof is straightforward. We just point out that the minimum appearing in item *i*) comes from the fact that $1)$ of (Λ_{cc}) may not be satisfied. □

Let us denote $\lambda^* := \lambda_{+is} = \lambda_{-is} = \lambda_{0ii} = \lambda_{0is}$.

Corollary 1.6.2. *Suppose* (Λ_{cc}) *and* $\lambda^* > 0$, *then*

i) $\mathcal{N}^+ \neq \emptyset$, *if and only if* $0 < \lambda < \lambda_{+ss}$;

ii) $\mathcal{N}^- \neq \emptyset$, *if only if,* $-\infty < \lambda < \lambda_{-ss}$;

iii) $\mathcal{N}^0 = \emptyset$ *if* $\lambda < \lambda^*$.

Since $\lambda^* \leq \min\{\lambda_{+ss}, \lambda_{-ss}\}$ it follows from Corollary (1.6.2) that

Theorem 1.6.3. *Suppose* (Λ_{cc}) *and* $\lambda^* > 0$, *then* $\mathcal{N}$ *is a manifold for all* $\lambda < \lambda^*$. *Moreover, if* $\lambda \leq 0$, *then* $\mathcal{N} = \mathcal{N}^-$ *and if* $\lambda \in (0, \lambda^*)$, *then* $\mathcal{N} = \mathcal{N}^- \cup \mathcal{N}^+$.

Proof. The idea of the proof is precisely an application of the Implicit Function Theorem, which shows= that $\mathcal{N}$ is C^1 manifold, known as Nehari manifolds. The second part consists in understanding the geometry of the fibering map $r_u(t)$ and using the corollary (1.6.2). □

Proposition 1.6.1. *Suppose* (Λ_{cc}) *and* $\lambda^* > 0$*, then for each* $u \in W \setminus \{0\}$ *and* $\lambda \in \mathbb{R}$ *the equation* $r_u(t) = \lambda, t > 0$ *has at most two solutions. Moreover:*

1) *If* $\lambda \leq 0$*, then* $r_u(t) = \lambda$ *has only one solution at* $t^-(u)$ *which corresponds to a non-degenerate global maximum of* φ*.*

2) *If* $\lambda > 0$*, then there are four possibilities:*

   a) *The equation* $r_u(t) = \lambda$ *has two solutions at* $t^+(u) < t_u < t^-(u)$ *. Moreover* $t^-(u)$ *corresponds to a non-degenerate local maximum of* φ *while* $t^+(u)$ *corresponds to a non-degenerate local minimum of* φ*;*

   b) *The equation* $r_u(t) = \lambda$ *has only one solutions at* t_u*, which corresponds to a degenerate critical point* φ*. Moreover* φ *is increasing;*

   c) *The equation* $r_u(t) = \lambda$ *has no solution and* φ *is increasing and has no critical points.*

   d) *The equation* $r_u(t) = \lambda$ *has only one solution at* $t^+(u)$*, which corresponds to a non-degenerate global minimum of* φ*.*

Proof. The fact that $r_u(t) = \lambda, t > 0$ has at most two solutions follows from (Λ_{cc}). If $\lambda \leq 0$, by (Λ_{cc}), the equation $r_u(t) = \lambda$ has only one solution at $t^-(u)$ and since $r_u'(t^-(u)) < 0$, it follows by Lemma(1.4.2) that $\varphi''(t^-(u)) < 0$ which prove $1)$. The proof of $2)$ is similar. □

Corollary 1.6.3. *Suppose* (Λ_{cc}) *and* $\lambda^* > 0$ *and fix* $u \in W \setminus \{0\}$*.*

1) *If* $\varphi'(t) > 0$*, then* $\lambda \leq 0$ *or item* $2) - a)$ *or item* $2) - (d)$ *of Proposition (1.6.1) are satisfied.*

2) *If* $\lambda < \lambda^*$*, then* $\lambda \leq 0$*, or item* $2) - (b)$ *and* $2) - (c)$ *does not happen.*

Proof. Item (1) is straightforward. To prove item (2) note that if items $2) - (b)$ or $2) - (c)$ holds true, then $\lambda \leq r(u)$ which implies that $\lambda \geq \lambda^*$. □

1.6.3 Global maximum for bounded fiber map

By adding certain hypotheses on the function r, we can derive properties of the Nehari submanifolds through the defined parameters.

(Λ^{cc}): For each $u \in W \setminus 0$ the fiber r_u satisfies :

r_u has a unique non-degenerate critical point at $t_u > 0$, which corresponds to a global maximun with $r_u(t) > 0$. Moreover $r_u(t) \to 0$ as $t \to \infty$.

We denote $r_u(0) := \lim_{t \to 0^+} r_u(t)$ and observe from (Λ^{cc}) that or $r_u(0) \in \mathbb{R}$ or $r_u(0) = -\infty$. We also write $r(u) := r_u(t_u)$ for each $u \in W \setminus 0$.

Lemma 1.6.2. *Suppose* (Λ^{cc})*, then*

i) $\lambda_{+ii} = \inf_{u \in W \setminus 0} r_u(0)$,
$\lambda_{+si} = \sup_{u \in W \setminus 0} r_u(0)$,
$\lambda_{+is} = \inf_{u \in W \setminus 0} r_u(t_u)$,
$\lambda_{+ss} = \sup_{u \in W \setminus 0} r_u(t_u)$.

ii) $\lambda_{-ii} = 0 = \lambda_{-si}$,
$\lambda_{-is} = \inf_{u \in W \setminus 0} r_u(t_u)$,
$\lambda_{-ss} = \sup_{u \in W \setminus 0} r(t_u)$

iii) $\lambda_{0ii} = \lambda_{0is} = \inf_{u \in W \setminus 0} r_u(t_u)$
$\lambda_{0si} = \lambda_{0ss} = \sup_{u \in W \setminus 0} r_u(t_u)$

Proof. The proof is straightforward. □

Let us denote $\lambda^* := \lambda_{+ss} = \lambda_{-ss} = \lambda_{0si} = \lambda_{0ss}$. It is clear that $\lambda^* > 0$.

Corollary 1.6.4. *Suppose* (Λ^{cc}) *and* $\lambda_{+si} < \lambda_{+is}$ *then*

i) $\mathcal{N}^+ \neq \emptyset$*, if and only if* $0 < \lambda < \lambda^*$.

ii) $\mathcal{N}^- \neq \emptyset$ *for all* $\lambda_{+ii} < \lambda < \lambda^*$.

iii) $\mathcal{N}^0 = \emptyset$ *is* $\lambda < \lambda_{0ii}$ *or* $\lambda > \lambda^*$.

There are cases where $\mathcal{N}^0 \neq \emptyset$ for all $\lambda \in (\lambda_{0ii}, \lambda^*]$ which, in turn, makes it more difficult to minimize over the Nehari $\mathcal{N}^-, \mathcal{N}^+$.
Let us introduce the following condition, which will allow us to understand the behaviour of the fiber.

(Λ_1^{cc}): For each $u \in W \setminus 0$ the fibering map r_u satisfies $r_u(0) = -\infty$.

Proposition 1.6.2. *Suppose* (Λ^{cc}) *and* (Λ_1^{cc})*, then for each* $u \in W \setminus 0$ *and* $\lambda \in \mathbb{R}$ *the equation* $r_u(t) = \lambda, t > 0$ *has at a most two solutions. Morever*

1) *If* $\lambda \leq 0$*, then* $r_u(t) = \lambda$ *has only one solution at* $t^-(u)$*, which corresponds to a non-degenerate global maximum of* $\varphi(t)$*.*

2) *If* $\lambda > 0$*, then there are three possibilities :*

   a) *The equation* $r_u(t) = \lambda$ *has two solutions at* $t^-(u) < t^+(u)$*. Morever* $t^-(u)$ *corresponds to a non-degenerate local maximum of* $\varphi(t)$*, while* $t^+(u)$ *corresponds to a non-degenerate local minimum of* $\varphi(t)$*;*

   b) *The equation* $r_u(t) = \lambda$ *has only one solution at* $t(u)$ *which corresponds to a degenerate critical point* $\varphi(t)$*. Morever* $\varphi(t)$ *is increasing ;*

   c) *The equations* $r_u(t)$ *has no critical points.*

Moreover, fixed $u \in W \setminus 0$ *there holds:*

Proof. That $r_u(t) = \lambda, t > 0$ has at most two solutions follows from (Λ^{cc}). Now observe that

$$r_u'(t) = \frac{\varphi''(t)}{t\mathcal{J}_2(tu)}, \quad t > 0. \tag{1.6.1}$$

If $\lambda < 0$, by (Λ^{cc}), the equation $r_u(t) = \lambda$ has only one solution at $t^-(u)$ and since $r_u'(t^-(u)) > 0$, it follows from (1.6.1) that $\varphi''(t^-(u)) < 0$ which prove 1). The proof of 2) is similar and we leave it to the reader. □

Corollary 1.6.5. *Suppose* (Λ^{cc}) *and* (Λ_1^{cc})*. If* $\varphi'(t) < 0$*, then* $\lambda \leq 0$ *or* $\lambda > 0$ *and* φ *satisfies item* $2 - b)$ *of Proposition (1.6.2)*

1.7 Rayleigh quotient from the energy functional

Now, another important concept is the Rayleigh quotient from the energy functional, since it will allow us to understand in which cases the minimum of the energy functional on a given energy subset is positive or negative. Moreover, we will show that it inherits the properties of the Rayleigh quotient. For each $u \in W \setminus 0$ define

$$R^e(u) = \frac{T(u)}{G(u)}.$$

Called the Rayleigh quotient from the energy functional.

Lemma 1.7.1. *If $\mathcal{J}_2(u) > 0$ for all $u \in W \setminus \{0\}$, then $G(u) > 0$, for all $u \in W \setminus \{0\}$.*

Proposition 1.7.1. *Fix $u \in W \setminus 0$, then $\Phi(u) = 0$ if, and only if, $\lambda = R^e(u)$.*

For each $u \in W \setminus 0$ define

$$R^e(tu) = \lambda(tu), \quad t > 0. \tag{1.7.1}$$

Proposition 1.7.2. *For each $u \in W \setminus 0$ there holds*

$$\frac{d}{dt}R^e(tu) = \frac{\mathcal{J}_2(tu)}{tG^2(tu)}[r_u(t) - R^e(tu)], \quad t > 0$$

and

$$\frac{d^2}{dt^2}R^e(tu) = \frac{\mathcal{J}_2(tu)}{tG^2(tu)}[r_u'(t) - \frac{d}{dt}R^e(tu)] + \frac{d}{dt}\left(\frac{\mathcal{J}_2(tu)}{tG^2(tu)}\right)[r_u(t) - R^e(tu)], \quad t > 0.$$

Proof. Indeed

$$\frac{d}{dt}R^e(tu) = \frac{G(tu)D_uT(tu)(u) - T(tu)D_uG(tu)u}{G^2(tu)} = \frac{\mathcal{J}_2(tu)}{tG^2(tu)}[r_u(t) - R^e(tu)], \quad t > 0.$$

The other equality is clear □

(Λ_2) Fix $u \in W \setminus 0$, then r_u has at most one critical point.

Proposition 1.7.3. *Suppose (Λ_1) and (Λ_2), then $R^e(tu)$ also satisfies (Λ_1) and (Λ_2).*

Proof. The proof will be divided in four cases:
Case 1: r_u is increasing.
First we claim that $R^e(tu) \le r_u(t)$ for all $t > 0$. Otherwise, if there exists $t_0 > 0$ such that $R^e(tu) > r_u(t_0)$, then since r_u is increasing, we must conclude that $r_u(t) < r_u(t_0)$ for all $t \in (0,t_0)$, which imply that φ has no critical points on the interval $(0,t_0)$, howeber $\varphi(0) = \varphi(t_0) = 0$, which imply that φ has a critical point on the interval $(0,t_0)$, a contradiction. Thus $R^e(tu) \le r_u(t)$

for all $t > 0$.
Now we clain that $\frac{d}{dt}R^e(tu)\Big|_{t=t_0} = 0$, then by Proposition (1.7.2) we obtain that $r_u(t_0) = R^e(t_0u)$, however, since $r_u'(t_0) > 0$, we can find $t_1 > 0$ in a neighborhood of t_0 such that $R^e(t_1u) < r_u(t_1)$, a contradiction, thus $R^e(tu)$ has no critical points and therefore it must be increasin or decreasing over $(0,\infty)$. Moreover it is clear that $R^e(tu)$ has to be increasing.

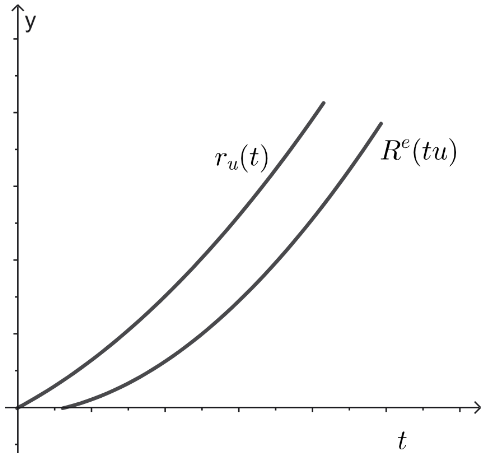


Fig. 1.5 Typical graph of Case 1

Case 2: r_u is decreasing. This case is similar to Case 1.
Case 3: r_u is has a global maximum at t_u.
Firs we claim that if $\frac{d}{dt}R^e(tu)\Big|_{t=t_0} = 0$, then $\frac{d^2}{dt^2}R^e(tu)\Big|_{t=t_0} \neq 0$. In fact, on the contrary, we infer by Proposition (1.7.2) that $t_0 = t_u$ and consequently φ has a unique critical point at t_u with zero energy and this is clearly a contradiction, therefore, each critical point of $R^e(tu)$ (if any) degenerate.
Now observe that a similar argument, as the one used in Case 1, inplies that there exists $\delta > 0$ such that $R^e(tu) < r_u(t)$ for all $t \in (0,\delta)$. Thus, by Proposition (1.7.2) we conclude that $R^e(tu)$ is increasing near the origin. If $R^e(tu)$ does not have any critical point, then it is increasing in $(0,\infty)$ and we are done. Therefore, let us assume that t_0 is the first critical point of $R^e(tu)$, that is, if $t \in (0,t_0)$, then $\frac{d}{dt}R^e(tu) > 0$ and $\frac{d}{dt}R^e(tu)\Big|_{t=t_0} = 0$. By Proposition (1.7.2) is clear that $t_0 > t_u$ and since t_0 is non-degenerate, it has to be a local maximum. We clain that $R^e(tu)$ has no other critical points. If fact, suppose on the contrary, that $t_1 > t_0$ is another critical point and the second one, that is, if $t \in (t_0,t_1)$, then $\frac{d}{dt}R^e(tu) < 0$ and $\frac{d}{dt}R^e(tu)\Big|_{t=t_1} = 0$. Since $t_1 > t_0 > t_u$ we know that $r_u'(t_1) < 0$. Moreover, since t_1 is no-degenerate, then t_1 is a local minimum to $R^e(tu)$ and thus there exists $t_0 < t_2 < t_1$ such that $R^e(t_2u) < r_u(t_2)$, however this is

a contradiction, since close to the local maximum t_0 there exists $t_3 > 0$ such that $t_0 < t_3 < t_2 < t_1$ and $R^e(t_3u) > r_u(t_3)$ and hence we can find $t_4 \in (t_0, t_2)$ such that $R^e(t_4u) = r_u(t_4)$, which implies by Proposition (1.7.2) that $r'_u(t_4) = 0$, and absurd, thus $R^e(tu)$ has only one non-degenerate critical point.

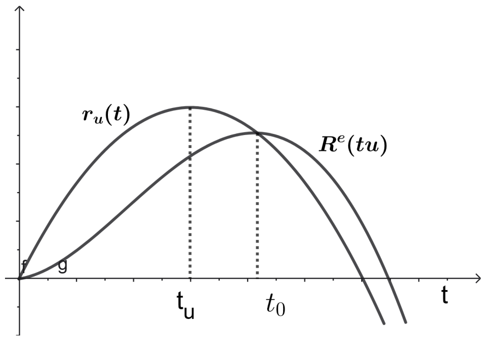


Fig. 1.6 Typical graph of Case 3

□

Let us denote $r_u(\infty) = \lim_{t\to\infty} r_u(t)$ and similarly to $R^e(tu)$. The next theorem is clear from the proof of Proposition (1.7.3).

Theorem 1.7.1. *Suppose* (Λ_1) *and* (Λ_2)*, then* $r_u(0) = R^e(0)$ *and* $r_u(\infty) = R^e(\infty)$ *. Moreover*

1. r_u *is increasing if, and only if,* $R^e(tu)$ *is increasing.*
2. r_u *is decreasing if, and only if,* $R^e(tu)$ *is decreasing.*
3. r_u *has a global maximum at* t_u *if, and only if,* $R^e(tu)$ *has a global maximum at* $t_{0,u} > t_u$*.*
4. r_u *has a global minimum at* t_u *if, and only if,* $R^e(tu)$ *has a global minimum at* $t_{0,u} > t_u$*.*

Let

$$D_0 = \{u \in W \setminus \{0\} : \exists t > 0, \text{ such that } \frac{d}{dt} R^e(tu) = 0\},$$

and define

$$\mathcal{R}_{0,i}(u) := \inf_{\{t>0:\, \frac{d}{dt} R^e(tu)=0\}} R^e(tu), \quad u \in D_0.$$

And similarly define $\mathcal{R}_{0,s}$

$$\mathcal{R}_{0,s}(u) := \sup_{\{t>0:\, \frac{d}{dt} R^e(tu)=0\}} R^e(tu), \quad u \in D_0.$$

Also define

$$\lambda_{ii0} = \inf_{u\in D_0} \mathcal{R}_{0,i}(u), \quad \lambda_{si0} = \sup_{u\in D_0} \mathcal{R}_{0,i}(u), \quad \lambda_{is0} = \inf_{u\in D_0} \mathcal{R}_{0,s}(u), \quad \lambda_{ss0} = \sup_{u\in D_0} \mathcal{R}_{0,s}(u).$$

Theorem 1.7.2. *Suppose* $(\Lambda)_{cc}$*, then* $\lambda_{0ii} = \lambda_{0is} := \mu_*$, $\lambda_{0si} = \lambda_{0ss} := \mu^*$, $\lambda_{ii0} = \lambda_{is0} := \mu_{0^*}$ *and* $\lambda_{si0} = \lambda_{ss0} := \mu^{0^*}$*. Moreover* $0 \leq \mu_{0^*} \leq \mu_*$ *and* $0 \leq \mu^{0^*} \leq \mu^*$*.*

Proof. Since r_u has at most one critical point, then the equalities $\lambda_{0ii} = \lambda_{0is}$ and $\lambda_{0si} = \lambda_{0ss}$ are obvious. By Theorem (1.7.1) we know that $R^e(tu)$ also satisfies (Λ_1) $(\Lambda)_{cc}$ and then, $\lambda_{ii0} = \lambda_{is0}$ and $\lambda_{si0} = \lambda_{ss0}$ are also clear. Once $r_u(t_u), R^e(tu)(t_{0,u}) > 0$ we also have that $0 \leq \mu_{0^*}$ and $0 \leq \mu^{0^*}$. To conclude, note by Theorem (1.7.1) that $R^e(tu)(t_{0,u}) > r_u(t_u)$, whenever $u \in D_0$. □

From now on we use the notation established in Theorem (1.7.2), so whenever (Λ_1), (Λ_{cc}) are satisfied, we write $\mu_* := \lambda_{0ii} = \lambda_{0is}$, $\mu^* := \lambda_{0si} = \lambda_{0ss}$, $\mu_{0^*} := \lambda_{ii0} = \lambda_{is0}$ and $\mu^{0^*} := \lambda_{si0} = \lambda_{ss0}$.

Theorem 1.7.3. *Suppose* (Λ_{cc})*. There holds:*

i) suppose $0 < \mu_{0^*}$ *and fix* $\lambda < \mu_{0^*}$*. If* $u \in D_0$*, then* φ *satisfies item* (a) *of Proposition (1.6.1) and* $\varphi(t(u)^-) > 0$;

ii) suppose that $\lambda > \mu_{0^*}$ *and there exists* $u \in \mathcal{N}^-$*, then* $\Phi_1 < 0$.

Proof. $i)$ By definition of infimum we know taht $\lambda < R^e(t_{0,u}u)$ and thus the equations $R^e(tu) = \lambda$ has two solutions at $t_1 < t_2$. By theorem (1.7.2) we have that $\mu_{0^*} \leq \mu_*$ and therefore the equation $r_u(t) = \lambda$ has two solutions at $t^+(u) < t^-(u)$. It is clear that $t^+(u) < t_1 < t^-(u) < t_2$ and thus $i)$ is proved.
$ii)$ Indeed, suppose on the contrary that $\Phi_1 \geq 0$, then it follows that for all $u \in D_0$ we have $\lambda = R^e(t^-(u)u) \leq R^e(t_{0,u}u)$, which implies the contradiction $\lambda \leq \mu_{0^*}$. □

Concerning μ_{0^*} we have the following.

Theorem 1.7.4. *Suppose* (Λ_{cc}) *and* $\mu_{0^*} > 0$*. If there exists* $u \in W \setminus \{0\}$ *such that* $\mu_{0^*} = \mathcal{R}_{0,i}(u)$*, then* $\Phi_{\mu_{0^*}}(t_u u) = 0$ *and* $\Phi'_{\mu_{0^*}}(t_u u) = 0$

Proof. The proof is similar to the proof of Proposition (1.7.2) □

Remark 1.7.1. *Theorem (1.7.4) says that, whenever* μ_{0^*} *is archived, we have a critical point to* $\Phi_{\mu_0^*}$ *with zero energy. Compare with Proposition (1.7.2).*

For a better understanding of what follows, we denote $\lambda_0^* := \mu_{0^*}$.

Proposition 1.7.4. *Suppose* (Λ^{cc}) *and* (Λ_1^{cc})*, then* Λ *also satisfies* (Λ^{cc}) *and* Λ_1^{cc} *for all* $u \in W \setminus \{0\}$.

Proof. The proof is similar to the proof of Proposition (1.7.3). □

From Proposition (1.7.4), for each $u \in W \setminus \{0\}$, there exists $t_{0,u} > 0$ that corresponds to the unique global maximizer of the fibering map R^e. Define $r_0(u) = R^e(t_{0,u}u)$ and
$\lambda_0^* = \sup\limits_{u \in W \setminus \{0\}} r_0(u) > 0$.

Proposition 1.7.5. *Suppose* (Λ^{cc}) *and* (Λ_1^{cc})*, then*

1) *for all* $u \in W \setminus \{0\}$ *we have that* $R^e(tu) < r_u(t)$ *for all* $t \in (0, t_{0,u})$*,* $R^e(t_{0,u}u) = r_u(t_{0,u})$ *and* $R^e(tu) > r_u(t)$ *for all* $t > t_{0,u}$.

2) $t_u < t_{0,u}$ *for all* $u \in \dot{W} \setminus \{0\}$.

3) $r_0(u) < r(u)$ *for all* $u \in \dot{W} \setminus \{0\}$ *and* $\lambda_0^* \leq \lambda^*$.

Proof. The proof is similar to the proof of Theorem (1.7.1). □

By the definition of $R^e(tu)$, the next theorem is clear:

Theorem 1.7.5. *There holds:*

1. *if* $\lambda > \lambda_0^*$*, then for each* $u \in W \setminus 0$ *we have that* $\varphi(t) > 0$ *for* $t > 0$*;*
2. *if* $\lambda < \lambda_0^*$*, then for each* $u \in W \setminus 0$ *we have that* $\inf\limits_{t>0} \varphi(t) < 0$;
3. *if* $\lambda = \lambda_0^*$*, then for each* $u \in \dot{W} \setminus \{0\}$ *we have that* $\varphi(t) \geq 0$ *for all* $t > 0$*. Morever* $\varphi_{\lambda_0^*,u}(t) = 0$ *for some* $t > 0$ *and some* $u \in W \setminus \{0\}$ *if, and only if,* u *is a maximizer of* λ_0^* *and* $t = t_{0,u}$.

Chapter 2

Multiplicity of Solutions for Elliptic Problems

2.1 Multiplicity of global minimun solutions

In this chapter, we focus on the existence and multiplicity of solutions through the analysis of the fibering function. For this purpose, we shall employ the minimisation method. Next, we introduce several important assumptions that will allow us to show that certain elliptic partial differential equations admit solutions.

We will study (1.0.1) using the scalar SM-method (see [19]). In what follows, we always assume that $DG(u)(u) > 0$ for all $u \in W \setminus \{0\}$ so that (A_1) is satisfied. Denote $S = \{v \in W : ||v||_W = 1\}$. We will suppose $r_u(t)$ satisfied the following conditions:

(H_1) for all $u \in W \setminus \{0\}$, $r_u(t)$ has a unique critical point $t_{\max}(u) \in \mathbb{R}^+$ which is a global maximun point;

(H_2) there exists $\delta_0 > 0$, such that $t_{\max}(v) > \delta_0$, for any $v \in S$;

(H_3) if $(v_m) \subset S$ is weakly separated from $0 \in W$, then the set of functions $(r_{v_m}(s))_{m=1}^{\infty}$ is bounded in $C^1[\sigma, T]$ for any $\sigma, T \in (0, \infty)$;

(H_4) for any $\lambda \in \mathbb{R}$, $0 < \sigma < T < \infty$, the set $\mathcal{N} \cap \{\sigma < ||u||_W < T\}$ is weakly separated from $0 \in W$.

Typical graph of the function $r_u(t)$ satisfying (H_1) is presented in Figure (1.1). The condition of type (H_2) is common in study of variatonial problems, where fibering functions are used.

Roughly speaking, this condition, as well as condition (H_3) and (H_4), ensures that the solutions of problem (1.0.3) are separated from zero and from each other. In the example below, we will see that conditions (H_3) and (H_4) will be verified by standard methods as a consequence of Sobolev's and Hölder's inequalities and the Rellich-Kondrachov theorem.

Evidently, (H_1) yields (R), (S) defined in Chapter one and that it satisfies $dr_u(t_{\max}(u))/dt = 0$, thus, by using basic calculus, we can obtain information about where the fiber of the function $r_u(t)$ increases and decreases, this is

$$\frac{d}{dt}r_u(t) > 0 \iff 0 < t < t_{\max}(u) \quad \text{and} \quad \frac{d}{dt}r_u(t) < 0 \iff t > t_{\max}(u). \tag{2.1.1}$$

Furthermore, it follows that for all $u \in W \setminus \{0\}$ there exist limits: $r_u(t) \to r_u(0)$ as $t \to 0$ and $r_u(t) \to r_u(\infty)$ as $t \to \infty$, where $-\infty \le r_u(0),\, r_u(\infty) < +\infty$. Introduce

$$\lambda_M = \sup_{u \in W\setminus\{0\}} \max\{r_u(0), r_u(\infty)\}.$$

Consider

$$\lambda_{is} = \inf_{u \in W\setminus\{0\}} \sup_{t>0} r_u(t).$$

Observe that (H_1) entails

$$\lambda_{si} = \sup_{u \in W\setminus\{0\}} \inf_{t>0} r_u(t) \le \lambda_M. \tag{2.1.2}$$

The proof is immediate by Lemma (1.4.2).

Remark 2.1.1. *In view of Lemma (1.2.1), $\mathcal{N}^- \neq \emptyset$, $\mathcal{N}^+ \neq \emptyset$ if $\lambda \in (\lambda_M, \lambda_{is})$.*

Thus, for $\lambda \in (\lambda_M, \lambda_{is})$, one may split the minimization problem (1.0.3) into the study of the subsets on which the energy functional can be minimized. To understand how to apply the theory developed so far, we first define the minima on each subset of the Nehari manifold as follows:

$$\Phi_1 = \min\{\Phi(u) : u \in \mathcal{N}^-\}, \tag{2.1.3}$$

$$\Phi_2 = \min\{\Phi(u) : u \in \mathcal{N}^+\}. \tag{2.1.4}$$

We now establish the existence of solutions in a general setting through the following theorem. When applying this theorem to a specific problem, it will be straightforward to verify whether its hypotheses are satisfied and, consequently, to conclude directly the existence of solutions. Moreover, the proof of this theorem will rely on the results previously established.

Theorem 2.1.1. *Suppose W is a reflexive Banach space, $\Phi \in C^1(W \setminus 0_n, \mathbb{R})$, $dT(tu)/dt, dG(tu)/dt$ are maps of class C^1 on $\mathbb{R}^+ \times (W \setminus 0_n)$, $D_uG(u)(u) > 0$, for all $u \in W \setminus 0_n$, $(H_1) - (H_4)$ hold and the following conditions are fulfilled:*

(a) for all $\lambda \in \mathbb{R}$, $\Phi(u) \to \infty$ as $||u||_W \to \infty$, $u \in \mathcal{N}$;

(b) $\Phi(u)$, for all $\lambda \in \mathbb{R}$ and $R(u)$ are sequentially weakly lower semicontinous functionals on W.

Assume $\lambda_M < \lambda_{is}$. Then for every $\lambda \in (\lambda_M, \lambda_{is})$ system of equations (1.0.1) has two distinct solutions $u^1_\lambda, u^2_\lambda \in W \setminus 0_n$ such that

$$d^2\Phi(tu^1_\lambda)/dt^2|_{t=1} < 0, \quad d^2\Phi(tu^2_\lambda)/dt^2|_{t=1} > 0, \quad \Phi(u^2_\lambda) \equiv \Phi^2 < \Phi(0).$$

Furthermore, for $\lambda \in (\lambda_M, \lambda_{is})$, u^2_λ is a ground state of (1.0.1) and $\mathcal{N}^-, \mathcal{N}^+$ are C^1-manifolds of codimension 1.

Proof. Since $\mathcal{N}^- \cap \mathcal{N}^+ = \emptyset$ and $\mathcal{N}^- \cup \mathcal{N}^+ = \mathcal{N}$ for $\lambda \in (\lambda_M, \lambda_{is})$, any solution of (2.1.3) or (2.1.4) is a critical point (minimizer) of Φ in $\mathcal{N}$. Thus, in view of Theorem (1.3.1), to prove the existence of two distinct solutions of (1.0.1), it is sufficient to show that (2.1.3) and (2.1.4) for $\lambda \in (\lambda_M, \lambda_{is})$ possess minimizers $u^1_\lambda \in \mathcal{N}^-$ and $u^2_\lambda \in \mathcal{N}^+$, respectively.

Let $\lambda \in (\lambda_M, \lambda_{is})$ and $(u^i_m), i = 1, 2$, be minimizing sequences of (2.1.3) and (2.1.4), respectively.

Proposition 2.1.1. *For $i = 1, 2$, the minimizing sequence (u^i_m) has a non-zero limit point $u_0 \in W$.*

Proof. Firstly, note that Φ is coercive and continuos; therefore, it is bounded from below. Consequently, there exists a sequence u^i_m such that

$$\Phi(u^i_m) \to \inf_{u \in W} \Phi(u) = c < \infty.$$

Afirmation 1. For $i = 1, 2$ then (u^i_m) is bounded in W. Indeed suppose by contradiction $|u^i_m| \to \infty$, then by hypothesis (a) $\Phi(u^i_m) \to \infty$, this leads to a contradiction with the previous argument.

Now write $u^i_m = t^i_m v^i_m, i = 1, 2$ where $t^i_m = ||u^i_m||_W$. By the Bolzano-Weierstrass theorem, there exists a subsequence such that $t^i_m \to t^i_0 \geq 0$. Besides, $v^i_m \in S$ is bounded in W, and W is reflexive,

then there exists a subsequence $v_m^i \rightharpoonup v_0^i$ weakly in W.
Let us show that $u_0^i := t_0^i v_0^i \neq 0_n$, $i = 1,2$.
Afirmation 2. One has $t_0^1 \neq 0$. Indeed, consider the first minimizing problem (2.1.3). Then (H_2) entails $t_m^1 > t_{\max}(v_m)$, and thus $\inf_m t_m^1 > \delta_0 > 0$ for any $m = 1,2,..$ then

$$s_0^1 = \lim_{m\to\infty} \inf_m t_m^1 > \delta_0 > 0$$

then $s_0^1 \neq 0$.
Afirmation 3. One has $\Phi_2 < 0$ for $\lambda \in (\lambda_M, \lambda_{is})$. Indeed, consider now the minimizing problem (2.1.4). Let $u \in \mathcal{N}^2$. Then (H_1) entails $r_u(t) < \lambda = r_u(1)$ for every $t \in (0,1)$. Consequently, by Proposition (1.1.1),

$$\frac{d}{dt}\Phi(tu) < 0 \quad \text{for all } t \in (0,1),$$

and therefore

$$0 = \Phi(0) > \Phi(u) \geq \Phi_2.$$

Afirmation 4. One has $t_0^2 \neq 0$. Indeed, assume $t_m^2 \to 0$. Then $||u_m^2||_W \to 0$, hence $u_m^2 \to 0$, and thus $\Phi(u_m^2) \to 0$. Now, as strong convergence implies weak convergence, we have $u_m^2 \rightharpoonup 0$. By hypothesis (b),

$$0 = \Phi(0) \leq \lim_{m\to\infty} \inf_m \Phi(u_m) = \inf_{u\in W} \Phi(u),$$

because the sequence is minimizing. Since the minimum exists, it coincides with the infimum, thus

$$\Phi_2 = \min\{\Phi(u) : u \in \mathcal{N}^+\} \geq 0,$$

which contradicts Affirmation 3. Therefore, $t_0^2 \neq 0$, hence $t_0^2 > 0$. In real numbers, no matter how close two numbers are, there is always a number between them; then $t_0^2 > \delta_1 > 0$. Taking $\delta = \min\{\delta_0, \delta_1\}$, and since u_m is bounded, we have

$$\delta < ||u_m^i||_W < K < \infty, \quad m = 1,2,\ldots,\ i = 1,2,$$

for some $\delta, K \in (0,\infty)$, and assumption (H_4) entails that $u_0^i \neq 0_n$, $i = 1,2$.

□

Proposition 2.1.2. *One has*

$$\frac{d}{dt} r_{u_0^1}(t)\Big|_{t=1} < 0, \tag{2.1.5}$$

$$\frac{d}{dt} r_{u_0^2}(t)\Big|_{t=1} > 0, \tag{2.1.6}$$

Proof. Let $i = 1,2$. Since (v_m^i) is weakly separated from $0_n \in W$, assumption (H_3) yields that the set of functions $(r_{v_m}(s))_{m=1}^{\infty}$ is bounded in $C^1[\sigma, T]$ for any $\sigma, T \in (0,\infty)$, that is, the set of functions is equicontinuous and equilimited. Consequently, by the Arzelá-Ascoli compactness criterion, we can assume that

$$r_{v_m^i}(s) \to \psi^i(s), \quad \text{in } C[\sigma, T], \quad \text{as } m \to \infty, \quad \text{for all } \sigma, T \in (0,\infty), \tag{2.1.7}$$

for some limit function $\psi^i \in C(0,\infty)$. Since $t_0^i > 0$,

$$r_{u_m^i}(s) = R(st_m^i v_m^i) \to \psi^i(st_0^i) =: \hat{\psi}^i(s), \quad \text{as } m \to \infty \tag{2.1.8}$$

for all $s \in (0,\infty)$. Observe that by the weak lower semi-continuity of R,

$$r_{u_0^i}(t) \equiv R(tu_0) \le \lim_{m\to\infty} R(tu_m^i), \quad \text{for all } t > 0. \tag{2.1.9}$$

This and (2.1.8) yield that for $t \ge 0$,

$$r_{u_0}(t) \le \hat{\psi}^i(t). \tag{2.1.10}$$

Let us show (2.1.5). Suppose, contrary to our claim, that $\frac{dr_{u_0^1}(t)}{dt}|_{t=1} \ge 0$. Then (2.1.1) entails $t_{\max}(u_0^1) \ge 0$. Since $r_{u_m^1}(t) \le \lambda$ for $t \in [1,\infty)$, $m = 1,2,\dots$, (2.1.8) implies $\hat{\psi}^1(t) \le \lambda$ for $t \in [1,\infty)$, and consequently, by (2.1.10), $r_{u_0^1}(t) \le \lambda$ for $t \in [1,\infty)$. Hence

$$\max_{t>0} r_{u_0^1}(t) = r_{u_0^1}(t_{\max}(u_0^1)) \le \lambda.$$

However, by the assumption $\lambda < \lambda_{is} \le \max_{t>0} r_{u_0^1}(t)$, we get a contradiction.

Assertion (2.1.6) can be handled in a similar way. Indeed, if $\frac{dr_{u_0^2}(t)}{dt}|_{t=1} \le 0$, then (2.1.1) entails $t_{\max}(u_0^2) \le 1$. Since $r_{u_m^2}(t) \le \lambda$ for $t \in (0,1]$, $m = 1,2,\dots$, (2.1.8) and (2.1.10) yield $r_{u_0^2}(t) \le \lambda$ for any $t \in (0,1]$. Hence

$$\lambda < \lambda_{is} \le \max_{t>0} r_{u_0^2}(t) = r_{u_0^2}(t_{\max}(u_0^2)) \le \lambda,$$

which is a contradiction. □

Proposition 2.1.3. *There exist u_λ^1 and u_λ^2 minimizers of (2.1.3) and (2.1.4), respectively.*

Proof. By the weak lower semi-continuity of Φ and R, we have

$$-\infty < \Phi(u_0^i) \leq \lim_{m\to\infty} \inf \Phi(u_m^i) = \Phi^i, \quad i = 1,2. \tag{2.1.11}$$

$$-\infty < R(u_0^i) \leq \lim_{m\to\infty} \inf R(u_m^i) = \lambda, \quad i = 1,2. \tag{2.1.12}$$

Note that if $R(u_0^i) = \lambda, i = 1,2$, then (2.1.5), (2.1.6) imply that $u_0^1 \in \mathcal{N}^-$, $u_0^2 \in \mathcal{N}^+$. Consequently, in (2.1.11) the only possible equalities are

$$\Phi(u_0^i) \leq \lim_{m\to\infty} \inf \Phi(u_m^i) = \inf_{u\in W} \Phi^i(u) \leq \Phi(u_0^i), \quad i = 1,2,$$

then $\Phi(u_0^i) = \Phi^i, i = 1,2$, and we take $u_\lambda^1 = u_0^1$ and $u_\lambda^2 = u_0^2$, which yields the proof of the proposition.

Now let us show that $r_{u_0^2}(1) = R(u_0^2) = \lambda$. Suppose by contradiction that $R(u_0^2) < \lambda$. Then, since $\lambda < \lambda_{is}$, we have

$$R(u_0^2) < \lambda < \lambda_{is} = \inf_{u\in W} \sup_{t>0} r_u(t) \leq \sup_{t>0} r_{u_0^2}(t) = \max_{t>0} r_{u_0^2}(t) \leq r_{u_0^2}(t_{\max}) = R(t_{\max}(u_0^2)u_0^2).$$

Note that since the maximum of $r_{u_0^2}$ exists, it coincides with the supremum. By (2.1.6), we have $dR(tu_0^2)/dt|_{t=1} > 0$. Hence, assumption (H_1) yields that there exists $t_1 \in (1, t_{\max})$ such that $r_{u_0^2}(t_1) = \lambda$. This is immediately deduced by the intermediate value theorem, besides $dr_{u_0^2}(t)/dt|_{t=t_1} > 0$, i.e., $t_1 u_0^2 \in \mathcal{N}^+ \subset W$. On the other hand, by Proposition (1.1.1), the inequality $r_{u_0^2}(t) < \lambda$, $t \in [1, t_1)$, implies $d\Phi(tu_0^2)/dt < 0$ for $t \in [1, t_1)$. Consequently, by (2.1.11) we have

$$\Phi(t_1 u_0^2) < \Phi(u_0^2) \leq \Phi_2 = \inf_{u\in W} \Phi(u) \leq \Phi(t_1 u_0^2),$$

which is a contradiction, proving our assertion.

Now suppose that $r_{u_0^1}(t) < \lambda$, Consider (2.1.5). Then, since $\lambda < \lambda_{is}$ and (H_1) entail that

$$\frac{dr_{u_0^1}(t)}{dt} < 0 \Longleftrightarrow t > t_{\max},$$

then $t_{\max} < 1$, $r_{u_0^1}(1) < r_{u_0^1}(t_{\max})$. By the intermediate value theorem, there exists $t_1 < 1$ such that

$$R(t_1 u_0^1) = r_{u_0^1}(t_1) = \lambda \quad \text{and} \quad \frac{dR(tu_0^1)}{dt}\Big|_{t=t_1} < 0,$$

then $t_1 u_0^1 \in \mathcal{N}^-$.

Note that $R(t_1 u_m^1) \geq R(t_1 u_0^1)$, otherwise $R(t_1 u_m^1) < R(t_1 u_0^1)$, then

$$\lim_{m\to\infty} \inf R(t_1 u_m^1) < R(t_1 u_0^1),$$

but by (2.1.12) this is an absurdity.

Now, as $u_m^1 \in \mathcal{N}^-$, then $R(u_m^1) = \lambda$ for $m = 1, 2, \ldots$. Assumption (H_1) entails $R(tu_m^1) > \lambda$ for all $m = 1, 2, \ldots$ and all $t \in (t_1, 1]$. Then, by Proposition (1.1.1),

$$\frac{d\Phi(tu_m^1)}{dt} > 0, \quad m = 1, 2, \ldots, \text{ for all } t \in (t_1, 1].$$

Consequently,

$$\Phi(t_1 u_m^1) < \Phi(u_m^1),$$

and by the weak lower semi-continuity of Φ we have

$$\Phi(t_1 u_0^1) \leq \lim_{m\to\infty} \inf \Phi(t_1 u_m^1) \leq \lim_{m\to\infty} \inf \Phi(u_m^1) = \Phi_1.$$

On the other hand, $\Phi_1 \leq \Phi(t_1 u_0^1)$ since $t_1 u_0^1 \in \mathcal{N}^-$. Then

$$\Phi_1 = \Phi(t_1 u_0^1),$$

and we take $u_\lambda^1 = t_1 u_0^1$. □

Now let us conclude the proof of the theorem. Since (2.1.2), Proposition (2.1.3) and Theorem (1.3.1) yield u_λ^1 and u_λ^2 satisfy (1.0.1). Since (2.1.5), (2.1.6), Proposition (1.1.1) implies that

$$\frac{d^2\Phi(tu_\lambda^1)}{dt^2}\Big|_{t=1} < 0 \quad \text{and} \quad \frac{d^2\Phi(tu_\lambda^2)}{dt^2}\Big|_{t=1} > 0.$$

Thus, it remains to show that u_λ^2 is a ground state of (1.0.1). Assumption (a) and $\lambda \in (\lambda_M, \lambda_{is})$ yield that the equation

$$\frac{d\Phi(\tau u_\lambda^1)}{d\tau} = 0$$

has precisely two solutions $\tau_{\min} < 1$ and $\tau_{\max} = 1$, such that

$$\frac{d^2\Phi(\tau u_\lambda^1)}{d\tau^2}\Big|_{\tau=1} < 0 \quad \text{and} \quad \frac{d^2\Phi(\tau u_\lambda^1)}{d\tau^2}\Big|_{\tau=\tau_{\min}} > 0.$$

This and Proposition (1.1.1) yield that $\tau_{\min} u_\lambda^1 \in \mathcal{N}^+$. Hence

$$\Phi(u_\lambda^1) > \Phi(\tau_{\min} u_\lambda^1) \geq \Phi_2 = \min\{\Phi(u) : u \in \mathcal{N}^+\} \equiv \Phi(u_\lambda^2).$$

Since $\mathcal{N}^- \cup \mathcal{N}^+ = \mathcal{N}$ for $\lambda \in (\lambda_M, \lambda_{is})$, this implies that u_λ^2 is a minimizer of the problem

$$\Phi = \min\{\Phi(u) : u \in \mathcal{N}\},$$

i.e., u_λ^2 is a ground state of (1.0.1). We emphasize that the value λ_M has been used above only in order to locate the values λ in $(\lambda_M, \lambda_{is})$ for which $\mathcal{N}^- \neq \emptyset$, $\mathcal{N}^+ \neq \emptyset$. In fact, the above proof of Theorem (2.1.1) can be easily adapted to other assumptions on the behaviour of $r_v(t)$ as $t \to 0$ and $t \to \infty$. In particular, let us assume that for all $v \in S$ there holds

$$r_v(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad r_v(t) \to -\infty \quad \text{as} \quad t \to 0. \tag{2.1.13}$$

Then $\lambda_M = 0$, $\lambda_{si} = -\infty$, and now $\mathcal{N}^- \neq \emptyset$ for all $\lambda < \lambda_{is}$. It is easily seen that the above proof of the existence of the minimizer u_λ^1 of (2.1.3) remains valid for all $\lambda < \lambda_{is}$, provided (2.1.13) is satisfied. Thus we have

Corollary 2.1.1. *Suppose the assumptions of Theorem (2.1.1) and (2.1.13) hold. Then for every $\lambda < \lambda_{is}$ there exists a minimizer u_λ^1 of (2.1.3) which satisfies (1.0.1). Furthermore, $d^2\Phi_\lambda(tu_\lambda^1)/dt^2|_{t=1} < 0$ and u_λ^1 is the ground state of (1.0.1) for $\lambda \leq 0$.*

Thus, having understood and established this theorem, we are now in a position to apply it to find solutions to a particular elliptic partial differential equations. In our case, we consider the following quasilinear elliptic problem and will show that it has at least two solutions.

2.1.1 Extremals for the p-Laplacian concave–convex problem (0.0.2)

Consider the problem with convex-concave nonlinearity studied in [34], it will be shown that this problem has at least two solutions, and through Theorem (2.1.1) above we will see that one solution lies in $\mathcal{N}^+$ and the other in $\mathcal{N}^-$

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-2}u + |u|^{\gamma-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1.14}$$

where $1 < \alpha < p < \gamma < p^*$, λ is a real parameter and by a solution of (2.1.14) we mean a weak solution $u \in W_0^{1,p}(\Omega)$ moreover, let us consider that Ω is a smooth bounded domain.
We say that u is a weak solution of (2.1.14) if the following holds:

i) $u \in W_0^{1,p}(\Omega)$;

ii) $\displaystyle\int |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int |u|^{\alpha-2}uvdx + \int |u|^{\gamma-2}uvdx, \forall v \in W_0^{1,p}(\Omega).$

Let us define the following energy functional $\Phi \in C^1(W_0^{1,p}, \mathbb{R})$, associated with our variational problem

$$\begin{aligned} \Phi : \quad & W_0^{1,p}(\Omega) \to \mathbb{R} \\ & u \to \Phi(u) = \frac{1}{p}\int |\nabla u|^p dx - \frac{\lambda}{\alpha}\int |u|^\alpha dx - \frac{1}{\gamma}\int |u|^\gamma dx. \end{aligned}$$

Following our notation we identify the working terms

$$\Phi(u) = T(u) - \lambda G(u) \quad \to \quad T(u) = \frac{1}{p}\int |\nabla u|^p dx - \frac{1}{\gamma}\int |u|^\gamma dx, \quad G(u) = \frac{1}{q}\int |u|^\alpha dx.$$

Note that,

$$D_u\Phi(u)(u) = \int |\nabla u|^p dx - \lambda \int |u|^\alpha dx - \int |u|^\gamma dx.$$

Therefore, Rayleigh's quotient is

$$R(u) := \frac{\displaystyle\int |\nabla u|^p dx - \int |u|^\gamma dx}{\displaystyle\int |u|^\alpha dx}, \quad u \in W_0^{1,p} \setminus \{0\}. \tag{2.1.15}$$

For $u \in W_0^{1,p} \setminus \{0\}$, $t > 0$, we have

$$R(tu) := r_u(t) = \frac{t^{p-\alpha} \displaystyle\int |\nabla u|^p dx - t^{\gamma-\alpha} \int |u|^\gamma dx}{\displaystyle\int |u|^\alpha dx}. \tag{2.1.16}$$

Thus, in this case we are working in a single Banach space, so derivatives in the vectorial sense are not relevant.

Lemma 2.1.1. *The fibre maps $r_u(t)$ of the Rayleigh quotient has a unique global maximum; moreover, its extremal values are characterized as follows:*

i) $\lambda_i = \lambda_{ii} = \lambda_{si} = -\infty$;

ii) $\lambda_s > 0$;

iii) $\lambda_{ss} > 0$;

iv) $\lambda_{is} > 0$.

Proof. One first

$$\frac{d}{dt} r_u(t) = \frac{(p-\alpha)t^{p-\alpha-1} \int |\nabla u|^p dx - (\gamma-\alpha)t^{\gamma-\alpha-1} \int |u|^\gamma dx}{\int |u|^\alpha dx}$$

critical point, $dr_u(t)/dt = 0$ if and only if

$$(p-\alpha)t^{p-\alpha-1} \int |\nabla u|^p dx = (\gamma-\alpha)t^{\gamma-\alpha-1} \int |u|^\gamma dx$$

then

$$t_{\max}(u) = \left(\frac{(p-\alpha) \int |\nabla u|^p dx}{(\gamma-\alpha) \int |u|^\gamma dx} \right)^{1/(\gamma-p)}. \tag{2.1.17}$$

Besides this we conclude that assumption (S) is satisfied. Substituting $t_{\max}(u)$ into $r_u(t)$ yields the following SG-Rayleigh quotient:

$$\lambda_i(u) = \inf \left\{ \frac{t^{p-\alpha} \int_\Omega |\nabla u|^p dx - t^{\gamma-\alpha} \int_\Omega |u|^\gamma dx}{\int_\Omega |u|^\alpha dx} : t \geq 0 \right\} = -\infty.$$

From (2.1.16), when t gets too big, the dominant term is the one with the highest exponent, that is, $(\gamma - \alpha) > 0$. Then $\lambda_i(u) = -\infty$. Moreover, when t is close to zero the dominant one is the term $(p-\alpha)$ but $r_u(t)\to 0$, so it would not be the lowest of the $r_u(t)$. Thus

$$\lambda_{si}=\sup_{u\in W\setminus\{0\}}\lambda_i(u)=-\infty,$$

now, following the same idea, let us calculate the other extremal values.

$$\lambda_s(u)=\sup_{t>0} r_u(t)=r_u(t_{\max})=c_{p,\alpha}\frac{(\int|\nabla u|^p)^{(\gamma-\alpha)/(\gamma-\alpha)}}{\int|u|^\alpha dx\,(\int|u|^\gamma dx)^{(p-\alpha)/(\gamma-p)}}$$

where

$$c_{p,\alpha}=\frac{\gamma-p}{p-\alpha}\left(\frac{p-\alpha}{\gamma-\alpha}\right)^{(\gamma-\alpha)/(\gamma-p)}$$

and

$$\lambda_{ss}=c_{p,\alpha}\sup\left\{\frac{(\int|\nabla u|^p)^{(\gamma-\alpha)/(\gamma-p)}}{\int|u|^\alpha dx\,(\int|u|^\gamma dx)^{(p-\alpha)/(\gamma-p)}}:u\in W\right\}.$$

Thus

$$\lambda_{is}=c_{p,\alpha}\inf\left\{\frac{(\int|\nabla u|^p)^{(\gamma-\alpha)/(\gamma-p)}}{\int|u|^\alpha dx\,(\int|u|^\gamma dx)^{(p-\alpha)/(\gamma-p)}}\ :\ u\in W\setminus\{0\}\right\}. \tag{2.1.18}$$

On the other hand

$$\lambda_{ii}=\inf_{u\in W}\lambda(u)=-\infty,$$

also

$$\lambda_{si}=\sup_{u\in W}\lambda(u)=-\infty.$$

Affirmation: $\lambda_{is}>0$. Indeed , as $W_0^{1,p}(\Omega)\hookrightarrow L^\alpha(\Omega)$ and $W_0^{1,p}(\Omega)\hookrightarrow L^\gamma(\Omega)$, then

$$||u||_{L^\alpha}\le C_1||u||_{W_0^{1,p}},\quad ||u||_{L^\gamma}\le C_2||u||_{W_0^{1,p}}$$

and

$$c_{p,\alpha}\frac{(\int|\nabla u|^p)^{(\gamma-\alpha)/(\gamma-p)}}{\int|u|^\alpha dx\,(\int|u|^\gamma dx)^{(p-\alpha)/(\gamma-p)}}=c_{p,\alpha}\frac{||u||_{W_0^{1,p}}^{p(\gamma-\alpha)/(\gamma-p)}}{||u||_{L^\alpha}^{\alpha}||u||_{L^\gamma}^{\gamma(p-\alpha)/(\gamma-p)}}\ge\frac{c_{p,\alpha}}{K}$$

where K is the constant of the embedding. Thus $\lambda_{is}>0$.
As in Hypothesis$-\infty=\lambda_{si}<\lambda_{is}$, let us return to Chapter 1 and apply Theorem (1.3.1), in

this way, we are directly approaching the demonstration of the existence of solutions for our problem. The following steps will allow us to consolidate the results and achieve our goal. $\square$

Proposition 2.1.4. *The functional Φ is coercive on the Nehari manifold $\mathcal{N}$.*

Proof. For $u \in \mathcal{N}$, we have

$$\Phi(u) = \Phi(u) - \frac{1}{\gamma} D_u \Phi(u)(u) = \left(\frac{1}{p} - \frac{1}{\gamma}\right) \int |\nabla u|^p \, dx - \lambda \left(\frac{1}{\alpha} - \frac{1}{\gamma}\right) \int |u|^\alpha \, dx.$$

Moreover, since $\lambda < |\lambda|$ and using the Sobolev embedding, we obtain

$$\Phi(u) \geq \left(\frac{1}{p} - \frac{1}{\gamma}\right) \|u\|^p_{W_0^{1,p}} - |\lambda| \left(\frac{1}{\alpha} - \frac{1}{\gamma}\right) C \|u\|^\alpha_{W_0^{1,p}}.$$

Then, since $\alpha < p$, as $\|u\|_{W_0^{1,p}} \to \infty$ the dominant term corresponds to the higher exponent. Therefore, $\Phi(u) \to +\infty$, which implies that Φ is coercive for every $u \in \mathcal{N}$. $\square$

Proposition 2.1.5. *The functional Φ is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$.*

Proof. Let (u_n) be a sequence such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. We shall prove that

$$\liminf_{n\to\infty} \Phi(u_n) \geq \Phi(u).$$

It is known that the norm in W is weakly semicontinuous

$$\frac{1}{p} \int |\nabla u|^p \, dx \leq \liminf_{n\to\infty} \frac{1}{p} \int |\nabla u_n|^p \, dx.$$

See (Appendix B (B.0.7)).
Since $\alpha < p < \gamma < p^*$, the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for every $r \in [1, p^*)$, in particular,

$$u_n \to u \quad \text{in } L^\alpha(\Omega) \quad \text{and} \quad L^\gamma(\Omega),$$

and therefore,

$$\int |u_n|^\alpha \, dx \to \int |u|^\alpha \, dx, \text{and} \qquad \int |u_n|^\gamma \, dx \to \int |u|^\gamma \, dx.$$

Combining these results we obtain that

$$\begin{aligned}
\liminf_{n\to\infty} \Phi(u_n) &= \liminf_{n\to\infty} \left(\frac{1}{p}\int |\nabla u_n|^p \, dx - \frac{\lambda}{\alpha}\int |u_n|^\alpha \, dx - \frac{1}{\gamma}\int |u_n|^\gamma \, dx \right) \\
&\geq \frac{1}{p}\int |\nabla u|^p \, dx - \frac{\lambda}{\alpha}\int |u|^\alpha \, dx - \frac{1}{\gamma}\int |u|^\gamma \, dx \\
&= \Phi(u).
\end{aligned}$$

□

Proposition 2.1.6. *R is weakly lower semicontinuous in $W_0^{1,p}(\Omega)\setminus\{0\}$.*

Proof. Let $(u_n) \subset W_0^{1,p}(\Omega)$ be such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u \neq 0$. We will use the same convergence arguments from the previous proposition. Define

$$A_n = \int |\nabla u_n|^p \, dx - \int |u_n|^\gamma \, dx \quad \text{and} \quad A = \int |\nabla u|^p \, dx - \int |u|^\gamma \, dx,$$

analogously

$$B_n = \int |u_n|^\alpha \, dx, \quad \text{and} \quad B = \int |u|^\alpha \, dx.$$

From the above, we have $B_n \to B > 0$ (since $u \neq 0$) and

$$\liminf_{n\to\infty} A_n \geq \left(\liminf_{n\to\infty} \int |\nabla u_n|^p \right) - \left(\lim_{n\to\infty} \int |u_n|^\gamma \right) \geq \int |\nabla u|^p \, dx - \int |u|^\gamma \, dx = A.$$

We wish to estimate $\liminf R(u_n) = \liminf (A_n/B_n)$. Since $B_n \to B$, it follows that

$$\liminf_{n\to\infty} \frac{A_n}{B_n} \geq \frac{\liminf_{n\to\infty} A_n}{\limsup_{n\to\infty} B_n} = \frac{\liminf_{n\to\infty} A_n}{B}.$$

See (Appendix B (B.0.8)). Therefore we have to

$$\liminf_{n\to\infty} \frac{A_n}{B_n} \geq \frac{A}{B} = R(u).$$

Therefore, R is weakly lower semicontinuous in $W_0^{1,p}(\Omega)\setminus\{0\}$. □

2.1.2 Multiplicity of solutions for the problem (0.0.2)

We will now show that there exist two minimizers of our functional: one corresponding to the subset of the Nehari manifold $\mathcal{N}^+$ and the other to $\mathcal{N}^-$, as stated in the following theorem.

Theorem 2.1.2. *Assume $1 < \alpha < p < \gamma \leq p^*$, Ω is a smooth bounded domain. Then the extremal value associated with the Rayleigh quotient of Φ, $\lambda_{is} > 0$ and for any $\lambda < \lambda_{is}$, problem (2.1.14) admits a weak solution $u^1_\lambda \neq 0$ in $\mathcal{N}^-$. Furthermore, when $\lambda \in (0, \lambda_{is})$, problem (2.1.14) has a second weak solution $u^2_\lambda \neq 0$ in $\mathcal{N}^+$.*

Moreover:

(a) $\displaystyle \frac{d^2}{ds^2}\Phi(su^1_\lambda)\Big|_{s=1} < 0, \quad \frac{d^2}{ds^2}\Phi(su^2_\lambda)\Big|_{s=1} > 0, \quad \Phi(u^2_\lambda) < 0;$

(b) if $\lambda \in (-\infty, 0]$, then u^1_λ is a ground state (1.0.1);

(c) if $\lambda \in (0, \lambda_{is})$, then u^2_λ is a ground state of (1.0.1).

Proof. For the proof of this theorem, we will verify all the hypotheses of Theorem (2.1.1). Indeed, let us first prove points (H_1) -(H_4) mentioned in the theorem.

(H_1) By (2.1.17), we have that $r_u(t)$ has a unique critical point $t_{\max}$, which corresponds to a global maximum. Hence, condition (H_1) is satisfied.

(H_2) We must prove that there exists $\delta_0 > 0$ such that $t_{\max}(v) > \delta_0$ for some $v \in S$. Indeed, suppose by contradiction that there exists a sequence $(v_m) \subset S$ such that $t_m := t_{\max}(v_m) \to 0$ as $m \to \infty$. Since $\dfrac{d}{dt} r_{v_m}(t) = 0$, we have

$$(p-\alpha)t_m^{p-\alpha-1} - (\gamma-\alpha)t_m^{\gamma-\alpha-1}\int |v_m|^\gamma = 0.$$

Now, by the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$, we know that

$$\|v_m\|^\gamma_{L^\gamma} \leq C\|v_m\|^\gamma_{W_0^{1,p}}$$

therefore, we obtain

$$(p-\alpha)t_m^{p-\alpha-1} - (\gamma-\alpha)t_m^{\gamma-\alpha-1}C\|v_m\|^\gamma_{W_0^{1,p}} \leq 0.$$

Since v_m belongs to the unit sphere S, it follows that

$$(p-\alpha)t_m^{p-\alpha-1} - C(\gamma-\alpha)t_m^{\gamma-\alpha-1} \leq 0,$$
$$t_m^{p-\alpha-1}[(p-\alpha) - C(\gamma-\alpha)t_m^{\gamma-p}] \leq 0.$$

Because $\alpha < p < \gamma$, when $t_m \to 0^+$, we have $t_m^{\gamma-p} \to 0$. Hence, there exists $t_0 > 0$ such that, for all $0 < t_m < t_0$,

$$t_m^{\gamma-p} \leq \frac{p-\alpha}{2C(\gamma-\alpha)}.$$

Thus,

$$(p-\alpha) - C(\gamma-\alpha)t_m^{\gamma-p} > \frac{(p-\alpha)}{2} > 0.$$

Since $t_m^{p-\alpha-1} > 0$, we arrive at a contradiction with the previous estimate. Therefore, condition (H_2) is satisfied.

(H_3) Assume that $(v_m) \subset S$ is weakly separated from 0_n in $W_0^{1,p}$. Since (v_m) is bounded in $W_0^{1,p}$ and $W_0^{1,p}$ is a reflexive Banach space, we may assume that $v_m \rightharpoonup v_0$ weakly in $W_0^{1,p}$ for some $v_0 \in W_0^{1,p}$. Furthermore, by the Rellich–Kondrachov theorem, $||v_m||_{L^d} \leq C||v_m||_{W_0^{1,p}} < \infty$, hence $||v_m||_{L^d} < C_1$, for $m = 1,2,\ldots$, $1 \leq d \leq p^*$ and $v_m \to v_0$ in $L^d(\Omega)$ up to a subsequence for $d < p^*$ (compact embedding).

Since $(v_m) \subset S$ is weakly separated from 0 in W, it follows that $v_0 \neq 0$, and consequently, there exists $\delta_1 > 0$ such that

$$||v_0||_{L^\alpha}^\alpha = \int |v_0|^\alpha dx > \delta_1^\alpha,$$

using the triangle inequality, $||v_m||_{L^\alpha} \geq ||v_0||_{L^\alpha} - ||v_m - v_0||_{L^\alpha} > \delta_1 - \varepsilon$, hence

$$\int |v_m|^\alpha dx \geq \delta_0 = \delta_1^\alpha \quad \text{for all } m = 1,2,\ldots.$$

on the other hand, note that since $v_m \subset S$, we have $||v_m||_{W_0^{1,p}} = 1$, and therefore we start from

$$r_u(t) = \frac{t^{p-\alpha}\int |\nabla u|^p dx - t^{\gamma-\alpha}\int |u|^\gamma dx}{\int |u|^\alpha dx}.$$

We have, for any $s \in [\sigma, T],\ \sigma, T \in (0,\infty)$,

$$\begin{aligned}
|r_{v_m}(t)| &\leq \frac{1}{\delta_0}\left[t^{p-\alpha}\left|\int |\nabla v_m|^p dx\right| + t^{\gamma-\alpha}\int |v_m|^\gamma dx\right] \\
&\leq \frac{1}{\delta_0}\left[t^{p-\alpha} + t^{\gamma-\alpha}\int |v_m|^\gamma dx\right] \\
&< \delta_0^{-1} t^{p-\alpha} - C t^{\gamma-\alpha},
\end{aligned}$$

where $C > 0$ comes from the Sobolev embedding. Therefore, since $t \leq T$, the sequence $(r_{v_m}(t))_{m=1}^{\infty}$ is bounded. Now we show that its derivative is also bounded; indeed, we have

$$\begin{aligned}
\frac{d}{dt} r_{v_m}(t) &= \frac{(p-\alpha)t^{p-\alpha-1}\int |\nabla v_m|^p dx - (\gamma-\alpha)t^{\gamma-\alpha-1}\int |v_m|^\gamma dx}{\int |v_m|^\alpha dx}, \\
\left|\frac{d}{dt} r_{v_m}(t)\right| &\leq \delta_0^{-1}\left[(p-\alpha)t^{p-\alpha-1} + c_1(\gamma-\alpha)t^{\gamma-\alpha-1}\right]
\end{aligned}$$

where $c_1 > 0$ comes from the Sobolev embedding. Therefore, for $s \in [\sigma, T]$, the family $\left\{\frac{d}{dt} r_{v_m}(t)\right\}_{m=1}^{\infty}$ is bounded. Thus, we obtain (H_3).

(H_4) Suppose, by contradiction, that there exists a sequence $(t_m v_m) \subset \mathcal{N}$ with $(v_m) \subset S$ (that is, $\|v_m\|_W = 1$ for all m), $\sigma < t_m < T$ for all m, and $v_m \rightharpoonup 0$ weakly in W.

Since (v_m) is bounded in $W_0^{1,p}$ and $\alpha, \gamma < p^*$, the Sobolev embeddings

$$W_0^{1,p}(\Omega) \hookrightarrow L^\alpha(\Omega) \quad \text{and} \quad W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$$

are compact. Hence, from $v_m \rightharpoonup 0$ in W, we deduce (up to a subsequence) that $v_m \to 0 \quad \text{in} \quad L^\alpha(\Omega) \quad \text{and in} \quad L^\gamma(\Omega)$. in particular

$$\int |v_m|^\alpha dx \to 0 \quad \text{and} \quad \int |v_m|^\gamma dx \to 0.$$

Since $(t_m v_m) \in \mathcal{N}$, we have $r_{v_m}(t_m) = \lambda$. Moreover, we know that $\|v_m\|_{W_0^{1,p}} = 1$, hence

$$r_{v_m}(t) = \frac{t^{p-\alpha}\|v_m\|_W^p - t^{\gamma-\alpha}\displaystyle\int |v_m|^\gamma dx}{\displaystyle\int |v_m|^\alpha dx} = \frac{t^{p-\alpha} - t^{\gamma-\alpha}\displaystyle\int |v_m|^\gamma dx}{\displaystyle\int |v_m|^\alpha dx}.$$

Evaluating at $t = t_m$, we obtain

$$t_m^p - t_m^\gamma \int |v_m|^\gamma dx = \lambda\, t_m^\alpha \int |v_m|^\alpha dx,$$

and therefore, after a suitable rearrangement,

$$t_m^p = t_m^\gamma \int |v_m|^\gamma dx + \lambda\, t_m^\alpha \int |v_m|^\alpha dx.$$

Since $v_m \to 0$ strongly in L^γ and L^α, we have

$$\int |v_m|^\gamma dx \to 0 \quad \text{and} \quad \int |v_m|^\alpha dx \to 0.$$

Because $\sigma < t_m < T$ and $\lambda \in \mathbb{R}$ is fixed, the right-hand side of the previous equality tends to 0, thus

$$t_m^p \to 0.$$

However, this contradicts the fact that $t_m \geq \sigma > 0$ for all m. Therefore, we obtain (4). Finally, let us verify conditions (a) and (b) of Theorem (2.1.1).

(a) Is satisfied by the proposition (2.1.4)

(b) The functional Φ and R are weakly lower semicontinuous by Propositions (2.1.5) and (2.1.6) respectively.

Besides by Lemma (2.1.1) item $iv)$ we already know that $\lambda_{is} > 0$. Thus, all assumptions of Theorem (2.1.1) and Corollary (2.1.1) are satisfied.

□

2.2 Multiplicity of solutions for functionals with $\mathcal{N}^0 \neq \emptyset$

In this subsection, we will improve the understanding of the subsets of the Nehari manifold and show that, by defining the extremal values as (1.5.1), we obtain much more precise information about potential candidates for solutions. Moreover, under some general hypotheses, we will obtain results that allow us to minimize the energy functional. We will also analyze the subset $\mathcal{N}^0$ (see [7]), which is not necessarily a Nehari manifold, and show that, in some cases, there exist elements in this set that minimize our functional.

In this context, W will be studied as a single normed Banach vector space, which will make the manipulation of derivatives and the presentation of results more convenient. Nevertheless, it is worth noting that the vectorial treatment would also be valid, provided that the partial derivatives associated with each component are properly considered.
Recalling the definition of the Nehari manifold and its subsets, introduced in the previous chapter for the vectorial case, in this chapter we will work in the scalar case, that is, for $N = 1$. Thus, by adjusting all definitions to the scalar case, we obtain.

Definition 2.2.1. *The Nehari manifold is defined as*

$$\mathcal{N} = \{u \in W \setminus \{0\} : D_u\Phi(u)(u) \equiv \frac{d}{dt}\Phi(tu)|_{t=1} = 0\} \tag{2.2.1}$$

where W is a normed vector space and $\Phi(u) = T(u) - \lambda G(u), \lambda \in \mathbb{R}$.

Remembering our notes, we will make heavy use of T, G are C^1 and denote:

- $\mathcal{J}_1(u) = T'(u)(u)$ fréchet derivative in the u direction;
- $\mathcal{J}_2(u) = G'(u)(u)$ fréchet derivative in the u direction;
- $\mathcal{J}_1; \mathcal{J}_2 \in C^1$;
- $\Phi'(u)(u) = \mathcal{J}_1(u) - \lambda \mathcal{J}_2(u)$ fréchet derivative in the u direction;
- $\varphi(t) := \Phi(tu)$ called the fiber of the function;
- $\mathcal{J}_2(u) > 0$ for all $u \in W \setminus \{0\}$.
- Since we are working in a single Banach space, when we write $D\Phi$ we will be referring to the derivative in the Fréchet sense.

When we differentiate φ, we will be referring to the derivative with respect to the variable t. From Chapter one and with these new notations, we already have the Rayleigh quotient.

$$R(tu) := r_u(t) = \frac{\mathcal{J}_1(tu)}{\mathcal{J}_2(tu)}.$$

With our new notations and using the equivalences from the definition of $\mathcal{N}^+, \mathcal{N}^0, \mathcal{N}^-$, as shown in Chapter 1, we have that

$$\mathcal{N}^+ := \Big\{ u \in \mathcal{N} : \varphi'(1) = 0,\ \varphi''(1) > 0 \Big\}, \tag{2.2.2}$$

$$\mathcal{N}^0 := \Big\{ u \in \mathcal{N} : \varphi'(1) = 0,\ \varphi''(1) = 0 \Big\}, \tag{2.2.3}$$

$$\mathcal{N}^- := \Big\{ u \in \mathcal{N} : \varphi'(1) = 0,\ \varphi''(1) < 0 \Big\}. \tag{2.2.4}$$

Then, recalling that if $\mathcal{J}_1(tu) > 0$, the subsets of the Nehari manifold are obtained in the following way

$$\mathcal{N}^+ := \Big\{ u \in \mathcal{N} : \frac{d}{dt} r_u(t)|_{t=1} > 0 \Big\}, \tag{2.2.5}$$

$$\mathcal{N}^0 := \Big\{ u \in \mathcal{N} : \frac{d}{dt} r_u(t)|_{t=1} = 0 \Big\}, \tag{2.2.6}$$

$$\mathcal{N}^- := \Big\{ u \in \mathcal{N} : \frac{d}{dt} r_u(t)|_{t=1} < 0 \Big\}. \tag{2.2.7}$$

Proposition 2.2.1. *There holds: $u \in \mathcal{N}$ if, and only if, $r_u(1) = \lambda$*

Proof. In fact, $u \in \mathcal{N}$ is and only if, $\mathcal{J}(u) = \mathcal{J}_1(u) - \lambda \mathcal{J}_2(u) = 0$. □

In this section, we focus on the existence and multiplicity of solutions through the analysis of the fiber function. For this purpose, we shall employ the minimisation method. From now on we denote

$$\Phi_1 = \inf\{\Phi(u) : u \in \mathcal{N}^-\},$$
$$\Phi_2 = \inf\{\Phi(u) : u \in \mathcal{N}^+\},$$
$$\Phi_0 = \inf\{\Phi(u) : u \in \mathcal{N}^0\}.$$

Note that each of these terms depends on λ; instead of writing each term explicitly as a function of λ, we will understand this dependence implicitly.
We often encounter problems that require certain conditions on the energy functional. Therefore, in order to facilitate the understanding of a specific problem and to develop a more systematic analysis, we strategically introduce the following assumptions.

(N_1) $\Phi, \mathcal{J}_1, \mathcal{J}_1'(.)$, are weakly lower semi-continuous and $\mathcal{J}_2, \mathcal{J}_2'(.)$ are weakly continuous.

(N_2) There holds:

1. whenever non-empty, there exists $C > 0$ such that $C \leq \|u\|$ for all $u \in \mathcal{N}^-$. Moreover, the functional Φ is coercive over $\mathcal{N}^-$;
2. whenever non-empty, there exists $C > 0$ such that $\|u\| \leq C$ for all $u \in \mathcal{N}^+$;

(N_3) if $u_n \rightharpoonup u$ in W and $\mathcal{J}_1(u) = \liminf_{n\to\infty} \mathcal{J}_1(u_n)$, then, up to a sub-equence, $u_n \to u$ in W.

Thus, this allows us to develop an understanding of when an energy functional can be minimized and what conditions must be satisfied for this to occur.

Theorem 2.2.1. *Suppose* (Λ_{cc})*,* $\lambda^* > 0, (N_1)$ *and* $(N_2) - (1)$ *and* (N_3)*. If* $\lambda < \lambda^*$*, then there exists* $u \in \mathcal{N}^-$ *such that* $\Phi(u) = \Phi_1$ *.*

Proof. It is clear by (N_1) and $(N_2) - (1)$ that $\Phi_1 > -\infty$. Let $u_n \in \mathcal{N}^-$ be a minimizing sequence to Φ_1. By $(N_2) - (1)$ we can assume that $u_n \rightharpoonup u$ in W. We clain that $u \neq 0$, otherwise

$$0 = \mathcal{J}_1(0) \leq \lim\inf_{n\to\infty} \mathcal{J}_1(u_n) = \lim\inf_{n\to\infty} (\mathcal{J}_1(u_n) - \lambda \mathcal{J}_2(u_n)) = 0,$$

which implies by (N_3) that, up to a subsequence, $u_n \to 0$ in W, which contradicts $(N_2) - (1)$, therefore $u \neq 0$. Now we have two cases:
Case 1: $u_n \to u$ in W.
This is clear, since $\lambda < \lambda^*$ implies, by Corollary (1.6.3) that $\mathcal{N}^0 = \emptyset$ and hence $u \in \mathcal{N}^-$, thus $\Phi(u) = \Phi_1$.
Case 2: $u_n \not\to u$ in W.
We can assume that u_n has no sub-sequence converging to u. Since, by $(N_1), r'_u(1) \leq \lim\inf_{n\to\infty} r'_{u_n}(1) \leq 0$ and $\lambda < \lambda^*$, by Proposition (1.6.1) there exists $t := t^-(u) > 0$ such that $tu \in \mathcal{N}^-$ and hence

$$0 = \mathcal{J}_1(tu) < \lim\inf_{n\to\infty} \mathcal{J}_1(tu_n).$$

Since equality would imply by (N_3) a contradiction. Thus for n sufficiently large conclude that $\varphi'(t) > 0$ and by Corollary (1.6.3) and Proposition (1.6.1) it follows that $\varphi(t) < \varphi(1)$. Therefore

$$\Phi(tu) = \varphi(t) \leq \lim_{n\to\infty}\inf \varphi(t) \leq \lim_{n\to\infty}\inf \varphi(1) = \Phi_1,$$

and thus the proof is complete. □

Remark 2.2.1. *It is clear from Proposition (1.6.2) that if* $(N_2)-(1)$ *holds, then* $||u|| \geq C$ *for all* $u \in \mathcal{N}^+$*. Also, if* $(N_2)-(2)$ *holds, then* $||u|| \leq C$ *for all* $u \in \mathcal{N}^-$*. Moreover, if* $\lambda > 0$*, then* $\mathcal{N}^- \neq \emptyset$ *if, and only if,* $\mathcal{N}^+ \neq \emptyset$*.*

Theorem 2.2.2. *Suppose* (Λ_{cc})*,* $\lambda^* > 0$*,* $(N_1), (N_2)-(2)$ *and* (N_3)*. If* $\lambda \in (0, \lambda^*)$*, then there exists* $u \in \mathcal{N}^+$ *such that* $\Phi(u) = \Phi_2 < 0$*.*

Proof. It is clear by (N_1), $(N_2)-(2)$ and Proposition (1.6.1) that $-\infty < \Phi_2 < 0$. Let $u_n \in \mathcal{N}^+$ be a minimizing sequence to Φ_2. By $(N_2)-(2)$ we can assume that $u_n \rightharpoonup u$ in W and since $\Phi_2 < 0$ it follows that $u \neq 0$.
We clain that, up to a sub-sequence, $u_n \to u$ in W. Suppose not, then we can assume u_n has no sub-sequence converging to u. Since $\lambda < \lambda^*$, by Corollary (1.6.3) and Proposition (1.6.1) there exists $t := t^+(u) > 0$ such that $tu \in \mathcal{N}^+$ and hence

$$0 = \mathcal{J}_1(tu) < \liminf_{n\to\infty} \mathcal{J}_1(tu_m).$$

Since equality, by (N_3), a contradiction. Thus $t > 1$ and

$$\Phi(tu) = \varphi(t) < \varphi(1) \leq \liminf_{n\to\infty} \varphi(1) = \Phi_2$$

and hence the proof is complete. □

Now, recalling the extreme values defined from the derivative of the Rayleigh quotient fiber map, $\lambda^* := \lambda_{+is} = \lambda_{-is} = \lambda_{0ii} = \lambda_{0is}$ we consider the case $\lambda = \lambda^*$. This case is particularly interesting and represents something new in this chapter. Although, intuitively, we can already anticipate the behaviour of our fibre and the parameters associated with the Rayleigh quotient, in order to simplify the analysis and obtain significant results, we introduce a certain degree of uniformity in conditions $(N_1)-(N_3)$. These conditions are essential for proving the existence of solutions.

$(N_1)_u$: $\mathcal{J}_2(u) = D_uG(u)(u)$ is strongly continuous

$(N_2)_u$: There holds:

1. Fix $[a,b] \subset (0,\infty)$, then there exists $C > 0$ such that $C \leq ||u||$ for all $u \in \mathcal{N}^-$ and $\lambda \in [a,b]$. Moreover, if $\lambda_n \to \lambda > 0$ and $u \in \mathcal{N}^-_{\lambda_n}$ satisfies $||u_n|| \to \infty$, then $\Phi_{\lambda n}(u_n) \to \infty$.

2. Fix $[a,b] \subset (0,\infty)$, then exists $C>0$ such that $||u|| \leq C$ for all $u \in \mathcal{N}^+$ and $\lambda \in [a,b]$

(S^+): If $u_n \rightharpoonup u$ in W and $\limsup_{n\to\infty} T'(u_n)(u_n - u) \leq 0$, then $u_n \to u$ in W.

Theorem 2.2.3. *Suppose* (Λ_{cc}), $\lambda^* > 0$, $(N_1),(N_3),(N_1)_u,(N_2)_u-(1)$ *and* (S^+). *Then, there exists* $u \in \mathcal{N}^-_{\lambda^*} \cup \mathcal{N}^0_{\lambda^*}$ *such that* $\Phi'_{\lambda^*}(u) = 0$.

Proof. It is clear that $(N_2)_u - (1)$ implies $(N_2)-(1)$ for $\lambda > 0$ thus, combined with the other hypothesis, we can use Theorem (2.2.1). Take a sequence $\lambda_n < \lambda^*$ such that $\lambda_n \to \lambda^*$ and a corresponding sequence $u_n \in \mathcal{N}^-_{\lambda_n}$ such that $\Phi_{\lambda_n}(u_n) = \Phi_{1,\lambda_n}$ (it's Φ_1, which depends on λ_n) and $\Phi'_{\lambda_n}(u_n) = 0$. By (N_1) and $(N_2)-(1)$ we can assume that $u_n \rightharpoonup u$ in W and $u \neq 0$. Therefore

$$T'(u_n)(u_n - u) = \lambda_n G'(u_n)(u_n - u), \forall n \in \mathbb{N}.$$

From $(N_1)_u$ we conclude that $\limsup_{n\to\infty} T'(u_n)(u_n-u) = 0$ and by (S^+) it follows that $u_n \to u$ and thus the proof is complete. □

Theorem 2.2.4. *Suppose* (Λ_{cc}), $\lambda^* > 0$, $(N_1),(N_3),(N_1)_u,(N_2)_u-(2)$ *and* (S^+). *Then, there exists* $u \in \mathcal{N}^-_{\lambda^*} \cup \mathcal{N}^0_{\lambda^*}$ *such that* $\Phi'_{\lambda^*}(u) = 0$.

Proof. It is clear that $(N_2)_u - (2)$ implies $(N_2)-(2)$ for $\lambda > 0$ thus, combined with the other hypothesis, we can use Theorem (2.2.2) take a sequence $\lambda_n < \lambda^*$ such that $\lambda_n \to \lambda^*$ and a corresponding sequence $u_n \in \mathcal{N}^+_{\lambda_n}$ such that $\Phi_{\lambda_n}(u_n) = \Phi_{2,\lambda_n}$ (it's Φ_2, which depends on λ_n) and $\Phi'_{\lambda_n}(u_n) = 0$. Since Φ is decreasing in λ, it follows that $\Phi_{2,\lambda_n} \to c < 0$. By (N_1) and $(N_2)-(2)$ we have that $c > -\infty$. We may also assume that $u_n \rightharpoonup u$ in W and since $c<0$ it follows that $u \neq 0$.
Therefore

$$T'(u_n)(u_n - u) = \lambda_n G(u_n)(u_n - u), \forall n \in \mathbb{N}.$$

From $(N_1)_u$ we conclude that $\limsup_{n\to\infty} T(u_n)(u_n-u) = 0$ and by (S^+) it follows that $u_n \to u$ and thus the proof is complete. □

Remark 2.2.2. *Since* $\mathcal{N}^0_{\lambda^*}$ *may be non-empty, it is not clear whether the solutions provided by Theorems (2.2.3) and (2.2.4) belong to* $\mathcal{N}^-_{\lambda^*}$ *or* $\mathcal{N}^+_{\lambda^*}$, *respectively.*

Proposition 2.2.2. *Suppose* (Λ_{cc}) *and* $\lambda^* > 0$. *If there exists* $u \in D$ *such that* $\lambda^* = r(u)$, *then* $\Phi'_{\lambda^*}(t_u u) = 0$.

Proof. It is clear by (Λ_{cc}) and $\lambda^* > 0$ that D is open and $D \ni u \to t_u$ is C^1. Given $v \in W$ denote $w = t_u v + t_u' v u$, where t_u' represents the derivative of the function $D \ni u \to t_u$ on the point u. Since u is a global minimizer, we have that

$$0 = r'(u)w = \lambda'(t_u u)(w) = \frac{\mathcal{J}_2(t_u u)\mathcal{J}_1'(t_u u)w - \mathcal{J}_1(t_u u)\mathcal{J}_2'(t_u u)w}{\mathcal{J}_2(t_u u)^2} = \frac{\Phi_{\lambda^*}'(t_u u)w}{\mathcal{J}_2(t_u u)^2}.$$

Therefore $\Phi'(t_u u) = 0$. □

Then we can conclude

Theorem 2.2.5. *Suppose (Λ_{cc}) and $\lambda^* > 0$. If $\mathcal{N}_{\lambda^*}^0 \neq \emptyset$, then every $u \in \mathcal{N}_{\lambda^*}^0$ satisfies $\Phi_{\lambda^*}'(u) = 0$.*

By Theorem (2.2.5), one sufficiently condition to ensure that u, given by Theorem (2.2.4), belongs to $\mathcal{N}^+$ is the following compatibility condition:

(CC) If $\Phi_{\lambda^*}'(u) = 0$ then $u = 0$.

Theorem 2.2.6. *Suppose (CC). If $u \in \mathcal{N}_{\lambda^*}$ is a critical point to Φ_{λ^*}, then $u \in \mathcal{N}_{\lambda^*}^+ \cup \mathcal{N}_{\lambda^*}^-$.*

Proof. In fact, if $u \in \mathcal{N}_{\lambda^*}^0$ is a critical point to Φ_{λ^*}, then by Theorem (2.2.5) we obtain that $\Phi_{\lambda^*}'(u) = 0$ which implies that $u = 0$, a condration, therefore $u \in \mathcal{N}_{\lambda^*}^+ \cup \mathcal{N}_{\lambda^*}^-$. □

Now we can prove multiplicity of solutions when $\lambda = \lambda^*$.

Theorem 2.2.7. *Suppose (Λ_{cc}), $\lambda^* > 0$, (N_1), (N_3), $(N_1)_u$, $(N_2)_u$, S^+ and (CC). Then, there exist $u \in \mathcal{N}_{\lambda^*}^+$ and $v \in \mathcal{N}_{\lambda^*}^-$ such that $\Phi_{\lambda^*}'(u) = \Phi_{\lambda^*}'(v) = 0$.*

Proof. By Theorems (2.2.3) and (2.2.4), there exist $u \in \mathcal{N}_{\lambda^*}^+ \cup \mathcal{N}_{\lambda^*}^0$ and $v \in \mathcal{N}_{\lambda^*}^- \cup \mathcal{N}_{\lambda^*}^0$ such that $\Phi_{\lambda^*}'(u) = \Phi_{\lambda^*}'(v) = 0$. By Theorem (2.2.6) it follows that $u \in \mathcal{N}_{\lambda^*}^+$ and $v \in \mathcal{N}_{\lambda^*}^-$. □

Remark 2.2.3. *It is clear that $u \neq v$.*

Taking into account Observation (2.2.1), let us introduce the following condition. This will allow us to prove important results concerning the existence of solutions. Moreover, whenever we encounter a problem satisfying these hypotheses, the existence of a solution will follow immediately.

$(N_2)^{cc}$: We assume that the following conditions hold.

1. If $\lambda \leq 0$, then there exists $c > 0$ such that $c \leq ||u||$ for all $u \in \mathcal{N}^-$. Moreover Φ is coercive over $\mathcal{N}^-$.
2. If $\lambda \in (0, \lambda^*)$, then there exists, c, C such that $c \leq ||u|| \leq C$ for all $u \in \mathcal{N}$.

Theorem 2.2.8. *Suppose* $(\Lambda^{cc}), (\Lambda_1^{cc}), (N_1)$, $(N_2)^{cc}$, (N_3). *If* $-\infty < \lambda < \lambda^*$, *then there exists* $u \in \mathcal{N}^- \cup \mathcal{N}^0$ *such that* $\Phi(u) = \Phi_1$.

Proof. It is clear by Proposition (1.6.2) that $\Phi_1 \geq 0$. By $(N_2)^{cc}$, if $u_n \in \mathcal{N}^-$ is a minimizing sequence to Φ_1, then we can assume that $u_n \rightharpoonup u$ in W. If $u = 0$, then by (N_1) we obtain $0 \leq \liminf \mathcal{J}_1(u_n) = \liminf(\mathcal{J}_1(u_n) - \lambda \mathcal{J}_2(u_n)) = 0$, which implies by $N3$ that $u_n \to 0$ in W, which contradicts (N_2). Therefore $u \neq 0$ and we have two cases :

Case 1: $u_n \to u$.

This case is clear

Case 2: $u_n \not\to u$.

By (N_3) it follows that

$$\varphi'(1) < \liminf \varphi'(1) = 0,$$

and by Corollary (1.6.5) and Proposition (1.6.2), there exists $t < 1$ such that $tu \in \mathcal{N}^-$. Therefore

$$0 = \varphi'(t) < \liminf \varphi'(t)$$

and since $t < 1$, it follows that for sufficiently large n, we have that $\varphi(t) < \varphi(1)$. So

$$\Phi(tu) \leq \liminf \varphi(t) \leq \liminf \varphi(1) = \Phi_1,$$

which implies that $tu \in \mathcal{N}^-$ satisfies $\Phi(tu) = \Phi_1$ and the proof is complete. $\square$

Theorem 2.2.9. *Suppose* $(\Lambda^{cc}), (\Lambda_1^{cc}), (N_1), (N_2)^{cc} - (2)$, (N_3). *If* $0 < \lambda < \lambda^*$, *then there exists* $u \in \mathcal{N}^+ \cup \mathcal{N}^0$ *such that* $\Phi(u) = \Phi_2$.

Proof. By $(N_1), (N_2)^{cc} - (2)$ we have that $\Phi_2 > -\infty$ and if $u_n \in \mathcal{N}^+$ is a minimizing sequence to Φ_2, then we can assume that $u_n \rightharpoonup u$ in W. If $u = 0$, then by (N_1) we obtain $0 \leq \liminf \mathcal{J}_1(u_n) = \liminf(\mathcal{J}_1(u_n) - \lambda \mathcal{J}_2(u_n)) = 0$, which implies by $N3$ that $u_n \to 0$ in W, which contradicts $(N_2)^{cc} - (2)$. Therefore $u \neq 0$ and we claim that $u_n \to u$. Otherwise by (N_3) it follows that

$$\varphi'(1) < \liminf \varphi'(1) = 0$$

and by Corollary (1.6.5) and Proposition (1.6.2), there exists $t<1$ such that $tu \in \mathcal{N}^+$ and $\varphi(t)<\varphi(1)$. Therefore

$$\Phi(tu)=\varphi(t)<\varphi(1)\leq \liminf \varphi(1)=\Phi_2$$

which is a contradiction and hence $u_m \to u$ and the proof is complete. □

2.3 A sharper range of the parameter for solutions with negative energy

Now that we have Theorems (2.2.9) and (2.2.8) in hands, we need to be able to say whether the minimizer belongs to $\mathcal{N}^0$ or not. The next result follows immediately from Theorems (2.2.9) and (2.2.8).

Theorem 2.3.1. *Suppose* $(\Lambda^{cc}),(\Lambda_1^{cc}),(N_1),(N_2)^{cc}$ *and* (N_3).

1. *If* $0<\lambda<\lambda^*$ *and* $\Phi_2<\Phi_0$*, then there exists* $u\in\mathcal{N}^+$ *such that* $\Phi(u)=\Phi_2$.
2. *If* $-\infty<\lambda<\lambda^*$ *and* $\Phi_1<\Phi_0$*, then there exists* $u\in\mathcal{N}^-$ *such that* $\Phi(u)=\Phi_1$

So in order to apply Theorem (2.3.1) we need to understand the inequalities $\Phi_2<\Phi_0$ and $\Phi_1<\Phi_0$. To this end, let us introduce the extremal parameter that controls the energy of the local minimun of φ. Given $u\in W\setminus\{0\}$ recall the definition of

$$R^e(tu)=\frac{T(tu)}{G(tu)},\quad t>0.$$

Remark 2.3.1. *Since* $G(0)=0$ *and* $\mathcal{J}_2(u)<0$ *for all* $u\in W\setminus\{0\}$*, it is clear from the mean value theorem that* $G(u)<0$ *for all* $u\in W\setminus\{0\}$.

From Proposition (1.7.4), for each $u\in W\setminus\{0\}$, there exists $t_{0,u}>0$ that corresponds to the unique global maximizer of the fiber map R^e. Define $r_0(u)=R^e(t_{0,u}u)$ and
$\lambda_0^*=\sup_{u\in W\setminus\{0\}} r_0(u)>0$, called the extreme value of zero energy.

Theorem 2.3.2. *Suppose* (Λ^{cc}) , (Λ_1^{cc}) *and* $(N_1),(N_2)^{cc}-(2)$ *and* N_3*, then for all* $\lambda\in(0,\lambda_0^*)$*, there exists* $u\in\mathcal{N}^+$ *such that* $\Phi(u)=\Phi_2$*. Moreover* $\Phi_2<0$ *for* $\lambda\in(0,\lambda_0^*)$.

Proof. It is clear from Proposition (1.6.2) that $\Phi(u) > 0$ for all $u \in \mathcal{N}^0$ and thus $\Phi_0 \geq 0$. By Theorem (1.7.5) we know that $\Phi_2 < 0$ and hence the existence of $u \in \mathcal{N}^+$ such that $\Phi(u) = \Phi_2$ follows from Theorem (2.3.1). □

Remark 2.3.2. *If λ_0^* is attained by some $u \in W \setminus \{0\}$, then $\varphi_{\lambda_0^*}(t_{0,u}) = \varphi'_{\lambda_0^*}(t_{0,u}) = 0$. By Theorem (1.7.5) it follows that $\Phi_{\lambda_0^*}(t_{0,u}u) = \Phi_{2,\lambda_0^*}$ (were Φ_{2,λ_0^*} denotes that Φ_2 depends on λ_0^*) and by Proposition (1.6.2) we also have that $t_{0,u}u \in \mathcal{N}^+_{\lambda_0^*}$.*

Now we show existence in $\mathcal{N}^-$.

Theorem 2.3.3. *Suppose (Λ^{cc}), (Λ_1^{cc}) and N_1, $(N_2)^{cc} - (1)$, (N_3), then for all $\lambda \in (-\infty, 0]$, there exists $v \in \mathcal{N}^-$ such that $\Phi(v) = \Phi_1 > 0$.*

Proof. In fact, by Proposition (1.6.2) we have that $\mathcal{N}^0 = \emptyset$ for all $\lambda \leq 0$, therefore, by Theorem (2.3.1) the proof is complete. □

2.4 Application to an elliptic Kirchhoff-type problem

We will apply the theory developed to a Kirchhoff-type problem studied in [7] and [35]. We will demonstrate the existence of solutions by analyzing the extremal values associated with the Rayleigh quotient and by precisely identifying the ranges of the parameters for which the existence of solutions is guaranteed. We now present the Kirchhoff-type problem that will serve as the main application of the theory developed in this chapter.

$$\begin{cases} -(a + \lambda \int |\nabla u|^2)\Delta u = |u|^{\gamma-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.4.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, $a > 0$ and $\gamma \in (2, 4)$.
We say that u is a weak solution of (2.4.1) if the following holds:

i) $u \in H_0^1(\Omega)$;

ii) $\left(a + \lambda \int |\nabla u|^2 dx\right) \int \nabla u \nabla v dx = \int |u|^{\gamma-2} uv dx, \forall v \in H_0^1(\Omega)$.

Let us define the following energy functional $\Phi \in C^1(H_0^1, \mathbb{R})$, associated with our variational problem

$$\begin{aligned}\Phi: \quad & H_0^1(\Omega) \to \mathbb{R} \\ & u \to \Phi(u) = \frac{a}{2}\int |\nabla u|^2 dx + \frac{\lambda}{4}\left(\int |\nabla u|^2 dx\right)^2 - \frac{1}{\gamma}\int |u|^\gamma dx.\end{aligned}$$

Using our notations we have the following terms

$$\Phi(u) = T(u) - \lambda G(u) \quad \to \quad T(u) = \frac{a}{2}\int |\nabla u|^2 dx - \frac{1}{\gamma}\int |u|^\gamma dx, \quad G(u) = -\frac{1}{4}\left(\int |\nabla u|^2 dx\right)^2.$$

Note that,

$$D\Phi(u)u = a\int |\nabla u|^2 dx + \lambda\left(\int |\nabla u|^2 dx\right)^2 - \int |u|^\gamma dx.$$

So, using our notations from this chapter, we have $\mathcal{J}_1(u) = DT(u)(u) = a\int |\nabla u|^2 dx - \int |u|^\gamma dx$ and $\mathcal{J}_2(u) = DG(u)(u) = -\left(\int |\nabla u|^2 dx\right)^2$.

Therefore, we define the Rayleigh quotient and zero-energy Rayleigh quotient as follows

$$R(u) := -\frac{a\int |\nabla u|^2 dx - \int |u|^\gamma dx}{\left(\int |\nabla u|^2 dx\right)^2}, \quad u \in H_0^1 \setminus \{0\}$$

and

$$R^e(u) = -\frac{\dfrac{a}{2}\displaystyle\int |\nabla u|^2 dx - \dfrac{1}{\gamma}\int |u|^\gamma dx}{\dfrac{1}{4}\left(\displaystyle\int |\nabla u|^2 dx\right)^2}, \quad u \in H_0^1 \setminus \{0\}$$

Thus, we have that its associated fiber functions, $R(tu)$ and $R^e(tu)$ for $u \in H_0^1 \setminus \{0\}$, $t > 0$, are defined as follows:

$$r_u(t) = -\frac{a}{t^2}\frac{1}{\int |\nabla u|^2 dx} + \frac{1}{t^{4-\gamma}}\frac{\int |u|^\gamma dx}{\left(\int |\nabla u|^2 dx\right)^2} \tag{2.4.2}$$

$$R^e(tu) = -\frac{\frac{a}{2}t^2\int |\nabla u|^2 dx - \frac{1}{\gamma}t^\gamma\int |u|^\gamma dx}{\frac{1}{4}t^4\left(\int |\nabla u|^2 dx\right)^2}, \quad u \in H_0^1 \setminus \{0\}.$$

and it is clear that r_u satisfies (Λ^{cc}).

Proposition 2.4.1. *The Rayleigh quotient $r_u(t)$ has a unique global maximum.*

Proof. Therefore it has a unique global minimizer. Indeed

$$\frac{dr_u(t)}{dt} = \frac{2a}{t^3}\frac{1}{\int|\nabla u|^2\,dx} - (4-\gamma)\frac{1}{t^{5-\gamma}}\frac{\int|u|^\gamma\,dx}{\left(\int|\nabla u|^2\,dx\right)^2} \tag{2.4.3}$$

$$t_u = \left(\frac{2a}{4-\gamma}\frac{\int|\nabla u|^2}{\int|u|^\gamma}\right)^{\dfrac{1}{\gamma-2}}$$

and

$$r(u) = r_u(t_u) = C(\gamma,a)\frac{\left(\displaystyle\int|u|^\gamma\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\displaystyle\int|\nabla u|^2\,dx\right)^{\frac{\gamma}{\gamma-2}}}$$

where

$$C(\gamma,a) = \left(\frac{\gamma-2}{2}\right)\left(\frac{4-\gamma}{2a}\right)^{\frac{4-\gamma}{\gamma-2}} > 0.$$

Note that $H_0^1(\Omega) \hookrightarrow L^\gamma(\Omega)$

$$\frac{(\displaystyle\int|u|^{,\gamma}\,dx)^{\frac{2}{\gamma-2}}}{(\displaystyle\int|\nabla u|^2\,dx)^{\frac{\gamma}{\gamma-2}}} \le C\frac{(\displaystyle\int|\nabla u|^2\,dx)^{\frac{2}{\gamma-2}}}{(\displaystyle\int|\nabla u|^2\,dx)^{\frac{\gamma}{\gamma-2}}} < \infty.$$

□

Besides (Λ_1^{cc}) is satisfied $\lim_{t\to 0} r_u(t) = -\infty$.

2.4.1 Extremals of the Rayleigh quotient from the Fréchet derivative

First, we compute all the parameters associated with our Rayleigh quotient. This step is essential, since it will allow us to determine whether the Nehari submanifolds are empty or not, and thus to clearly identify in which of these sets it is possible to find solutions for our equation.

Proposition 2.4.2. *The following extremes comply:*

i) $\lambda_{+ii} = \lambda_{+si} = -\infty$;

ii) $\lambda_{+is} = \lambda_{-ii} = \lambda_{-si} = \lambda_{-is} = \lambda_{0ii} = \lambda_{0is} = 0$;

iii) $\lambda_{-ss} = \lambda_{0ss} = \lambda_{0si} = \lambda_{+ss} < +\infty$.

Proof. We will calculate all the extremal values using fundamental properties of analysis.

1. $\lambda_{+ii} = \inf_{u\in H_0^1\setminus\{0\}} \inf_{\{t>0, r_u'(t)>0\}} \Big\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2 dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{(\int|\nabla u|^2 dx)^2}\Big\} = -\infty$ since, when $t\to 0^+$ $r_u(t)\to -\infty$ thus $r_u(t)$ is increasing for in $(0,t_u)$ besides t^{-2} it increases at a faster rate than $t^{\gamma-4}$ for $\gamma \in (2,4)$.

2. $\lambda_{+si} = \sup_{u\in H_0^1\setminus\{0\}} \inf_{\{t>0, r_u'(t)>0\}} \Big\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2 dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{(\int|\nabla u|^2 dx)^2}\Big\} = -\infty$ by the same justification as for 1.

3. $\lambda_{+is} = \inf_{u\in H_0^1\setminus\{0\}} \sup_{\{t>0, r_u'(t)>0\}} \Big\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2 dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{(\int|\nabla u|^2 dx)^2}\Big\}$ we fix u; moreover, since r_u is increasing on $(0,t_u)$, the supremum of r for $r_u'(t)>0$ is

$$\sup_{\{t>0, r_u'(t)>0\}} r_u(t) = \lim_{t\to t_u^-} r_u(t) = r(t_u)$$

therefore, it is enough to analyse

$$\inf_{u\in H_0^1\setminus\{0\}} r_u(t_u) = C(\gamma,a) \inf_{u\in H_0^1\setminus\{0\}} \frac{\left(\int |u|^\gamma dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^2 dx\right)^{\frac{2}{\gamma-2}}}$$

now, since we approximate the infimum by εu, then for $\varepsilon \to 0$

$$\frac{\left(\int |\varepsilon u|^\gamma dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\varepsilon\nabla u|^2 dx\right)^{\frac{2}{\gamma-2}}} = \varepsilon^{(2\gamma-4)/(\gamma-2)} \frac{\left(\int |u|^\gamma dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^2 dx\right)^{\frac{2}{\gamma-2}}} \to 0.$$

Therefore $\lambda_{+is} = 0$.

4. One has

$$\lambda_{+ss} = \sup_{u\in W\setminus\{0\}} \sup_{\{t>0:\, r_u'(t)>0\}} \left\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2\,dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma\,dx}{\left(\int|\nabla u|^2\,dx\right)^2} \right\}$$

$$= C(\gamma,a) \sup_{u\in W\setminus\{0\}} \frac{\left(\int|u|^\gamma\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\int|\nabla u|^2\,dx\right)^{\frac{2}{\gamma-2}}} < \infty$$

since $H_0^1(\Omega) \hookrightarrow L^\gamma(\Omega)$.

5. $\lambda_{-ii} = \inf_{u\in H_0^1\setminus\{0\}} \inf_{\{t>0, r_u'(t)<0\}} \left\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{\left(\int|\nabla u|^2dx\right)^2} \right\} = 0$ besides $t\to\infty$ then $r_u(t)\to 0$.

6. $\lambda_{-si} = \sup_{u\in H_0^1\setminus\{0\}} \inf_{\{t>0, r_u'(t)<0\}} \left\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{\left(\int|\nabla u|^2dx\right)^2} \right\} = 0$ besides $t\to\infty$ then $r_u(t)\to 0$.

7. $\lambda_{-is} = \inf_{u\in H_0^1\setminus\{0\}} \sup_{\{t>0, r_u'(t)<0\}} \left\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{\left(\int|\nabla u|^2dx\right)^2} \right\}$ we fix u; moreover, since r_u is decreasing on (t_u,∞), the supremum of r for $r_u'(t)<0$ is

$$\sup_{\{t>0, r_u'(t)<0\}} r_u(t) = \lim_{t\to t_u^+} r_u(t) = r(t_u)$$

therefore, it is enough to analyse

$$\inf_{u\in H_0^1\setminus\{0\}} r_u(t_u) = C(\gamma,a) \inf_{u\in H_0^1\setminus\{0\}} \frac{\left(\int|u|^\gamma\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\int|\nabla u|^2\,dx\right)^{\frac{2}{\gamma-2}}}$$

now, since we approximate the infimum by εu, then for $\varepsilon \to 0$

$$\frac{\left(\int |\varepsilon u|^{\gamma}\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\varepsilon \nabla u|^{2}\,dx\right)^{\frac{2}{\gamma-2}}} = \varepsilon^{(2\gamma-4)/(\gamma-2)} \frac{\left(\int |u|^{\gamma}\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^{2}\,dx\right)^{\frac{2}{\gamma-2}}} \to 0.$$

Therefore $\lambda_{-is} = 0$.

Analogously to item 4, we have

8. $\lambda_{-ss} = C(\gamma,a) \sup\limits_{u\in W\setminus\{0\}} \dfrac{\left(\int |u|^{\gamma}\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^{2}\,dx\right)^{\frac{2}{\gamma-2}}} < \infty$ by the same argument as in items 4.

9. $\lambda_{0ii} = \inf\limits_{u\in H_0^1\setminus\{0\}} \inf\limits_{\{t>0, r_u'(t)=0\}} \left\{ -\dfrac{a}{t^2}\dfrac{1}{\int |\nabla u|^2 dx} + \dfrac{1}{t^{4-\gamma}} \dfrac{\int |u|^{\gamma} dx}{\left(\int |\nabla u|^2 dx\right)^2} \right\} = \inf\limits_{u\in W\setminus\{0\}} r_u(t_u) =$ 0 by the same argument as in items 3 and 7.

10. One has

$$\begin{aligned}\lambda_{0si} &= \sup_{u\in H_0^1\setminus\{0\}} \inf_{\{t>0, r_u'(t)=0\}} \left\{ -\frac{a}{t^2}\frac{1}{\int |\nabla u|^2 dx} + \frac{1}{t^{4-\gamma}} \frac{\int |u|^{\gamma} dx}{\left(\int |\nabla u|^2 dx\right)^2} \right\} \\ &= C(\gamma,a) \sup_{u\in W\setminus\{0\}} \frac{\left(\int |u|^{\gamma}\,dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^{2}\,dx\right)^{\frac{2}{\gamma-2}}} < \infty.\end{aligned}$$

11. $\lambda_{0is} = \inf\limits_{u\in H_0^1\setminus\{0\}} \sup\limits_{\{t>0, r_u'(t)=0\}} \left\{ -\dfrac{a}{t^2}\dfrac{1}{\int |\nabla u|^2 dx} + \dfrac{1}{t^{4-\gamma}} \dfrac{\int |u|^{\gamma} dx}{\left(\int |\nabla u|^2 dx\right)^2} \right\} = 0$ by the same justification as for 3.

12. One has

$$\lambda_{0ss} = \sup_{u\in H_0^1\setminus\{0\}} \sup_{\{t>0, r_u'(t)=0\}} \Big\{ -\frac{a}{t^2}\frac{1}{\int|\nabla u|^2dx} + \frac{1}{t^{4-\gamma}}\frac{\int|u|^\gamma dx}{(\int|\nabla u|^2dx)^2}\Big\}$$

$$= C(\gamma,a) \sup_{u\in W\setminus\{0\}} \frac{\left(\int |u|^\gamma dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^2 dx\right)^{\frac{2}{\gamma-2}}} < \infty.$$

□

We denote$\lambda^* := \lambda_{+ss} = \lambda_{-ss} = \lambda_{0si} = \lambda_{0ss}$. It is clear that $\lambda^* > 0$.
$\lambda_0^* := \lambda_{ii0} = \lambda_{is0}$.

Proposition 2.4.3. *Let, $u \in H_0^1(\Omega)\setminus\{0\}$ then $R^e(tu)$ possesses a unique critical point $t_{0,u} > 0$, which is a maximum.*

Proof. Differentiate $R^e(tu)$ with respect to t. Writing the numerator and denominator explicitly and applying the quotient rule yields, after simplification, the critical equation

$$a\left(\int |\nabla u|^2 dx\right) t^{-3} = \frac{4-\gamma}{\gamma}\left(\int |u|^\gamma dx\right) t^{\gamma-5}.$$

Rearranging gives

$$t^{\gamma-2} = \frac{\gamma a \int |\nabla u|^2 dx}{(4-\gamma)\int |u|^\gamma dx}$$

and since the right-hand side is positive for $2 < \gamma < 4$, there is exactly one positive solution, namely the displayed $t_{0,u}$.

To determine the nature of this critical point, observe that $R^e(tu) \to -\infty$ as $t \to 0^+$ and $R^e(tu) \to 0^+$ as $t \to \infty$. Moreover, a sign check of $\frac{d}{dt}R^e(tu)$ shows that R^e increases for $t < t_{0,u}$ and decreases for $t > t_{0,u}$. Hence the unique critical point $t_{0,u}$ is a global maximum.
Then we evaluate at the fiber of the null function and obtain the global maximum point

$$t_{0,u} = \left(\frac{\gamma a \int |\nabla u|^2 dx}{(4-\gamma)\int |u|^\gamma dx}\right)^{1/(\gamma-2)}, \quad 2 < \gamma < 4,\ a > 0$$

$$R^e(t_{0,u}u) = K(\gamma,a)\,\frac{\left(\int |u|^\gamma dx\right)^{\frac{2}{\gamma-2}}}{\left(\int |\nabla u|^2 dx\right)^{\frac{\gamma}{\gamma-2}}}$$

where

$$K(\gamma,a) = \frac{2(\gamma-2)}{\gamma^{\frac{2}{\gamma-2}}}\,a^{\frac{\gamma-4}{\gamma-2}}\,(4-\gamma)^{\frac{4-\gamma}{\gamma-2}}. \tag{2.4.4}$$

□

Note that $R^e(t_{0,u}u) = r_0(u)$ is a multiple of $r(u)$.

2.4.2 Extremals of the Rayleigh quotient from the energy functional

Recalling what was developed on previous pages, we now require the following set with the aim of finding the extrema of the Rayleigh quotient associated with the energy functional.

$$D_0 = \{u \in W \setminus \{0\} : \exists t > 0 \text{ such that } \frac{d}{dt}R^e(tu) = 0\}.$$

As defined in this chapter, the extreme values of the Rayleigh quotient from the energy functional are

$$\mathcal{R}_{0,i}(u) := \inf_{\{t>0:\, \frac{d}{dt}R^e(tu)=0\}} R^e(tu),$$

$$\mathcal{R}_{0,s}(u) := \sup_{\{t>0:\, \frac{d}{dt}R^e(tu)=0\}} R^e(tu).$$

Let u $\in D_0$, Clearly, since it has only one critical value, we have to $\mathcal{R}_{0,i}(u) = \mathcal{R}_{0,s}(u) = R^e(t_{0,u}u)$.

Proposition 2.4.4. *Assume $2 < \gamma < 4$, $a > 0$, and $\gamma < 2^*$ so that the Sobolev embedding is subcritical on Ω. Define*

$$\lambda_{ii0} = \inf_{u\in D_0} \mathcal{R}_{0,i}(u), \qquad \lambda_{si0} = \sup_{u\in D_0} \mathcal{R}_{0,i}(u),$$

$$\lambda_{is0} = \inf_{u\in D_0} \mathcal{R}_{0,s}(u), \qquad \lambda_{ss0} = \sup_{u\in D_0} \mathcal{R}_{0,s}(u).$$

Then

$$\lambda_{ii0} = \lambda_{is0} = 0, \qquad \lambda_{si0} = \lambda_{ss0} = K(\gamma,a)\, C_S^{\frac{2\gamma}{\gamma-2}},$$

where $K(\gamma,a)$ *of* (2.4.4) *and* $C_S > 0$ *denotes the best constant in the embedding* $H_0^1(\Omega) \hookrightarrow L^\gamma(\Omega)$, *i.e.* $\|v\|_{L^\gamma} \leq C_S \|\nabla v\|_{L^2}$ *for all* $v \in H_0^1(\Omega)$.

Proof. From Proposition 2.4.3 we have, for every $u \in D_0$,

$$\mathcal{R}_{0,i}(u) = \mathcal{R}_{0,s}(u) = K(\gamma,a)J(u) \quad \text{and} \quad J(u) := \frac{(\int |u|^\gamma)^{2/(\gamma-2)}}{(\int |\nabla u|^2)^{\gamma/(\gamma-2)}}.$$

Thus each λ equals $K(\gamma,a)$ times the corresponding extremum of J on D_0.
(1)*Upper bounds and supremum.* By the Sobolev embedding $\int |u|^\gamma \leq C_S^\gamma (\int |\nabla u|^2)^{\gamma/2}$, hence

$$J(u) \leq C_S^{\frac{2\gamma}{\gamma-2}} \quad \text{for every } u \neq 0.$$

Therefore $\sup_{u \in D_0} J(u) \leq C_S^{2\gamma/(\gamma-2)}$. Conversely, by taking a (sub)sequence of functions that approximate the extremal of the Sobolev inequality (or a sequence that attains the Sobolev constant in the limit), one obtains $J(u_n) \to C_S^{2\gamma/(\gamma-2)}$. Hence

$$\sup_{u \in D_0} J(u) = C_S^{\frac{2\gamma}{\gamma-2}},$$

and consequently

$$\lambda_{si0} = \lambda_{ss0} = K(\gamma,a)\, C_S^{\frac{2\gamma}{\gamma-2}}.$$

(2) *Infimum equals zero.* Completely analogous to point 3, we have $\lambda_{ii0} = \lambda_{is0} = 0$. □

2.4.3 Multiplicity of solutions for the Kirchhoff-type problem

Note that $\Phi, \mathcal{J}_1, \mathcal{J}_1'(.)$, are weakly lower semi-continuous and $\mathcal{J}_2, \mathcal{J}_2'(.)$ are weakly continuous, therefore condition (N_1) is satisfied.

Proposition 2.4.5. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded smooth domain,* $a > 0$ *and* $\gamma \in (2,4)$. *Define, for* $u \in H_0^1(\Omega)$,

$$\mathcal{J}_1(u) = a \int |\nabla u|^2 \, dx - \int |u|^\gamma \, dx.$$

Assume that $(u_n) \subset H_0^1(\Omega)$ *satisfies*

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega) \qquad \text{and} \qquad \mathcal{J}_1(u) = \liminf_{n\to\infty} \mathcal{J}_1(u_n).$$

Then, up to a subsequence,

$$u_n \to u \quad \text{in } H_0^1(\Omega).$$

Proof. Since $\gamma \in (2,4) \subset (2,6)$ and $\Omega \subset \mathbb{R}^3$ is bounded, the embedding $H_0^1(\Omega) \hookrightarrow L^\gamma(\Omega)$ is compact. Hence, up to a subsequence, $u_n \to u$ strongly in $L^\gamma(\Omega)$, i.e $||u_n||_{l^\gamma}^\gamma \to ||u||_{l^\gamma}^\gamma$

Moreover, the weak convergence $u_n \rightharpoonup u$ yields the lower semicontinuity

$$\int |\nabla u|^2\, dx \leq \liminf_{n\to\infty} \int |\nabla u_n|^2\, dx.$$

By the definition of $\mathcal{J}_1$ and the hypothesis $\mathcal{J}_1(u) = \liminf_{n\to\infty} \mathcal{J}_1(u_n)$, we have

$$a \int |\nabla u|^2\, dx - \int |u|^\gamma\, dx = \liminf_{n\to\infty} \left(a \int |\nabla u_n|^2\, dx - \int |u_n|^\gamma\, dx \right).$$

Using the strong L^γ-convergence, the nonlinear term passes to the limit, and we obtain

$$\int |\nabla u|^2\, dx = \liminf_{n\to\infty} \int |\nabla u_n|^2\, dx.$$

Since $\int |\nabla u|^2 \leq \liminf \int |\nabla u_n|^2$ and the reverse inequality follows from the equality of the $\mathcal{J}_1$ limits, we conclude that

$$\int |\nabla u_n|^2\, dx \to \int |\nabla u|^2\, dx.$$

Hence, $\|u_n\|_{H_0^1} \to \|u\|_{H_0^1}$.

Lemma 2.4.1. *If* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$ *and* $\|u_n\|_{H_0^1} \to \|u\|_{H_0^1}$*, then* $u_n \to u$ *strongly in* $H_0^1(\Omega)$.

Proof of the lemma. In $H_0^1(\Omega)$ the norm is induced by the scalar product $\langle v, w\rangle = \int \nabla v \cdot \nabla w\, dx$. Hence

$$\|u_n - u\|_{H_0^1}^2 = \|u_n\|_{H_0^1}^2 + \|u\|_{H_0^1}^2 - 2\langle u_n, u\rangle.$$

The weak convergence implies $\langle u_n, u\rangle \to \langle u, u\rangle = \|u\|_{H_0^1}^2$ then $\lim_{n\to\infty} \|u_n - u\|_{H_0^1}^2 = 0$, and thus $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Therefore, we have that (N_3) of this chapter is satisfied.
Now we will show that condition $(N_2)^{cc}$ is met . Indeed From (2.4.2) we have, for every $u \in S$,

$$r_u(t) \leq -\frac{a}{t^2} + \frac{C}{t^{4-\gamma}}, \qquad t > 0,$$

where $C > 0$ is a constant independent of u. Let $f(t) = -\frac{a}{t^2} + \frac{C}{t^{4-\gamma}}$. Since $2 < \gamma < 4$, we have $f(t) \to -\infty$ as $t \to 0^+$ and $f(t) \to 0^+$ as $t \to +\infty$. Hence there exists a unique $t_0 > 0$ such that $f(t_0) = 0$, and $f(t) < 0$ for $t < t_0$, $f(t) > 0$ for $t > t_0$. Because the parameters a, C, γ are fixed, this t_0 does not depend on u.

Consequently, for each $u \in S$, the function $r_u(t)$, which satisfies $r_u(t) \leq f(t)$, changes sign in an interval (t_1, t_2) containing t_0. Therefore, there exists $t_u \in (t_1, t_2)$ such that $r_u(t_u) = 0$, i.e. $t_u u \in \mathcal{N}$.

Since $\|t_u u\| = t_u \|u\| = t_u$, we conclude that every element $v \in \mathcal{N}$ can be written as $v = t_u u$ with $t_u \in (t_1, t_2)$ and $u \in S$, so that

$$t_1 \leq \|v\| \leq t_2.$$

Hence there exist positive constants $c = t_1$ and $C = t_2$ such that

$$c \leq \|u\| \leq C, \qquad \forall u \in \mathcal{N}.$$

To prove coerciveness, for $\lambda \leq 0$, note that

$$\Phi(u) = \frac{\gamma - 2}{2\gamma} \int |\nabla u|^2 dx + \frac{\gamma - 4}{4\gamma} \lambda \left(\int |\nabla u|^2 dx \right)^2, \quad \forall u \in \mathcal{N}$$

Theorem 2.4.1. *There holds*

1) *If $0 < \lambda \leq \lambda_0^*$, then problem (2.4.1) has a solution $u_\lambda \in \mathcal{N}^+$. Moreover, if $\lambda < \lambda_0^*$, then u_λ is a global minimizer to Φ over $\mathcal{N}^+$ with negative energy, while $u_{\lambda_0^*}$ is a global minimizer to $\Phi_{\lambda_0^*}$ over $\mathcal{N}^+$ with zero energy.*

2) *If $\lambda \in (-\infty, 0]$, then problem (2.4.1) has a solution $v_\lambda \in \mathcal{N}^-$. Moreover v_λ is a global minimizer to Φ over $\mathcal{N}^-$ with posirive energy.*

3) *Equation (2.4.1) has no non-zero solutions for $\lambda > \lambda^*$.*

Proof. Therefore by Corollary (1.6.4), Theorems (2.3.2), (2.3.3) and Remark (2.3.2) the proof is concluded. □

Chapter 3

Fibering maps from the Rayleigh quotient with at least two critical points

In this section, we investigate an equation depending on two real parameters. Our objective is to show that the Rayleigh quotients from the Fréchet derivative and from the energy functional provide a precise framework for establishing the existence of solutions lying in the different components of the Nehari set. Furthermore, we demonstrate that the fibering maps associated with these Rayleigh quotients possesses two distinct critical points, revealing a richer structure than the classical case discussed in Chapter 1.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$. We consider the following boundary value problem studied in [28]

$$\begin{cases} -\Delta_p u = |u|^{\gamma-2}u + \lambda |u|^{\alpha-2}u - \mu |u|^{q-2}u := f(\lambda,\mu,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.0.1}$$

Here $1 < q < \alpha < p < \gamma < p^*$, $p^* = \dfrac{pN}{N-p}$ if $N < p$, and $p^* = +\infty$ if $p \geq N$, and $\mu, \lambda \in \mathbb{R}$. We are interesed on the existence of branches of solutions with S-shaped type bificurcation curve. The $S-$shaped bifurcation curve arises in the study of many problems such as the the Lioville-Bratu-Gerlfand equation [17], the Kolmogorov- petrovsky-Piscounov equation [22]. Most of the results on the existence of an-S shaped bifurcation curve deal with a nonlinearly f that satisfied $f(u) > 0$ on $(0,r)$ for some $r \in (0,+\infty]$, and $f \in C^2[0,+\infty)$[3]. Note that in (3.0.1) we are facing the opposite case, namely, for any $\mu > 0$, $\lambda > 0$, there exists $r_{\mu,\lambda} > 0$ such that $f(\lambda,\mu,s) < 0, s \in (0,r_{\lambda,\mu})$. Morever, the nonlinearity in (3.0.1) is non-Lipschitz at

$u = 0$ if $p \leq 2$ and $\mu \neq 0$ or $\lambda \neq 0$. An additional feature of (1), which implies our results is that the S−shaped bifurcation curve of (3.0.1) exibits the so-called dual cusp catastrophe [18]. Let us state our main results. The problem (3.0.1) has a variational structure with the energy functional $\Phi(u) \in C^1(W_0^{1,p}(\Omega))$ given by

$$\Phi(u) = \frac{1}{p}\int |\nabla u|^p + \frac{\mu}{q}\int |u|^q - \frac{\lambda}{\alpha}\int |u|^\alpha - \frac{1}{\gamma}\int |u|^\gamma dx.$$

Here $W_0^{1,p}(\Omega)$ is the Sobolev space endowed with the norm $||u|| := \left(\int |\nabla u|^p dx\right)^{1/p}$ and $\Phi(u) = \Phi_{\mu,\lambda}(u)$. By a weak solution of (3.0.1) we mean a critical point u of $\Phi(u)$ on $W_0^{1,p}(\Omega)$, i.e., $D\Phi(u) = 0$ this is the derivative in the Fréchet sense. and $D\Phi(u)(v)$ denote its the directional derivate in direction $v \in W_0^{1,p}$. Hereinafter, for $F \in C^1(W_0^{1,p}(\Omega))$ we use the abbreviated notations $F'(tu) := \frac{d}{dt}F(tu), t > 0, u \in W_0^{1,p}(\Omega)$. A nonzero weak solution u of (3.0.1) is said to be *ground state* if $\Phi(u) \leq \Phi(w)$, for any non-zero weak solution $w \in W_0^{1,p}(\Omega)$ of (3.0.1). We say that weak solution $\overline{u}$ of (3.0.1) is *linearly stable* is $\overline{u}$ is a local minimizer of $\Phi(u)$ in $W_0^{1,p}(\Omega)$.

3.1 Generalized nonlinear Rayleigh extremal values

From this point onwards, the defined parameters depend on μ and; however, to simplify our notation, we make a change of variable, for example, $\lambda_\mu^{e,+} := \lambda^{e,+}$.

$$\lambda^{e,+} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \left\{ R^e(u) : \frac{d}{dt}R^e(tu)|_{t=1} = 0,\ \frac{d^2}{dt^2}R^e(tu)|_{t=1} > 0 \right\}, \tag{3.1.1}$$

$$\lambda^{e,-} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \left\{ R^e(u) : \frac{d}{dt}R^e(tu)|_{t=1} = 0,\ \frac{d^2}{dt^2}R^e(tu)|_{t=1} < 0 \right\}, \tag{3.1.2}$$

$$\lambda^{n,+} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \left\{ R(u) : \frac{d}{dt}R(tu)|_{t=1} = 0,\ \frac{d^2}{dt^2}R(tu)|_{t=1} > 0 \right\}, \tag{3.1.3}$$

$$\lambda^{n,-} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \left\{ R(u) : \frac{d^2}{dt^2}R(tu)|_{t=1} = 0, \frac{d^2}{dt^2}R(tu)|_{t=1} < 0 \right\}. \tag{3.1.4}$$

Here $R, R^e : W_0^{1,p} \setminus \{0\} \to \mathbb{R}$ are the Rayleigh quotients given by

$$\begin{aligned} R(u) &:= \frac{\displaystyle\int |\nabla u|^p + \mu \int |u|^q - \int |u|^\gamma}{\displaystyle\int |u|^\alpha}, \qquad u \in W_0^{1,p} \setminus \{0\},\ \mu \in \mathbb{R}, \\ R^e(u) &:= \frac{\frac{1}{p}\displaystyle\int |\nabla u|^p + \frac{\mu}{q}\int |u|^q - \frac{1}{\gamma}\int |u|^\gamma}{\frac{1}{\alpha}\displaystyle\int |u|^\alpha}, \quad u \in W_0^{1,p} \setminus \{0\},\ \mu \in \mathbb{R}. \end{aligned} \tag{3.1.5}$$

We introduce also the so-called *NG-Rayleigh* $\mu-$ *extremal value:*

$$\overline{\mu} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \frac{(\displaystyle\int |\nabla u|^p)^{\frac{\gamma-q}{\gamma-p}}}{(\displaystyle\int |u|^q)(\displaystyle\int |u|^\gamma)^{\frac{p-q}{\gamma-p}}}$$

and define

$$\mu^e = c^e \overline{\mu}, \quad \mu^n = c^n \overline{\mu}, \tag{3.1.6}$$

$$\text{where} \quad c^e := c^e_{q,\gamma} = \frac{q\gamma^{\frac{p-q}{\gamma-p}}}{p^{\frac{\gamma-q}{\gamma-p}}} c^n, \qquad c^n := c^n_{q,\gamma} = \frac{(p-\alpha)^{\frac{\gamma-q}{\gamma-p}}(p-q)^{\frac{p-q}{\gamma-q}}(\gamma-p)}{(\alpha-q)(\gamma-\alpha)^{\frac{p-q}{\gamma-p}}(\gamma-q)^{\frac{\gamma-q}{\gamma-p}}}.$$

3.2 Fibering maps with at least two critical points

Recalling what was developed in Chapter 1, the critical values of the energy functional $\Phi(u)$, restricted to the Nehari manifold, correspond to the weak solutions of the associated problem.

In this chapter, we will present the specific formulation and the preliminary results that will serve as a basis for the existence of solutions to the proposed problem

Lemma 3.2.1. *Let $u \in \mathcal{N}$ be an extremal point of Φ on the Nehari manifold. Suppose that $\Phi''(u) = D\Phi'(u)(u) \neq 0$. Then $D\Phi(u) = 0$.*

Proof. Due to the assumption we may apply the Lagrange multipler rule, and thus, we have $D\Phi(u) + \nu D\Phi'(u) = 0$, for some $\nu \in \mathbb{R}$. Testing this equality by u we obtain $\nu D\Phi'(u)(u) = 0$. Since $D\Phi'(u)(u) \neq 0, \nu = 0$, and therefore, $D\Phi(u) = 0$. □

Observe that Φ is coercive on $\mathcal{N}$, $\forall\mu > 0, \lambda \in \mathbb{R}$. Indeed, by the Sobolev inequality,

$$\Phi(u) - \frac{1}{\gamma}\Phi'(u) \geq \frac{\gamma - p}{p\gamma}||u||^p - \lambda C||u||^\alpha, \quad \forall u \in \mathcal{N},$$

for some constant $C > 0$ independent of $u \in \mathcal{N}$. Since $\alpha < p$, we have $\Phi(u) \to +\infty$ for $||u|| \to +\infty$ and $u \in \mathcal{N}$.

It is easily seen that for $u \in W_0^{1,p} \setminus \{0\}$, the fibering function $\Phi(su), s > 0$, may have at most three nonzero critical points

$$0 < s_1(u) \leq s_2(u) \leq s_3(u) < \infty$$

such that $\Phi''(s_1(u)u) \leq 0$, $\Phi''(s_2(u)u) \geq 0, \Phi''(s_3(u)u) \leq 0$ (see Fig. 3.1).

To apply the Nehari manifold method, we need to find values μ, λ, where the strong inequalities $0 < s_1(u) < s_2(u) < s_3(u)$ hold. We solve this by the recursively application of the nonlinear generalized Rayleigh quotient method. In the first step of this recursive procedure, we

$$R(u) := \frac{\int |\nabla u|^p + \mu \int |u|^q - \int |u|^\gamma}{\int |u|^\alpha}, \quad u \in W_0^{1,p} \setminus \{0\}, \ \mu \in \mathbb{R}, \tag{3.2.1}$$

Notice that for $u \in W_0^{1,p} \setminus \{0\}$,

$$\begin{aligned} &R(u) = \lambda \Longleftrightarrow \Phi'(u) = 0, \\ &R(u) := \lambda, \quad R'(u) > 0(<)0 \Longleftrightarrow \Phi''(u) > 0(<)0. \end{aligned} \tag{3.2.2}$$

In particular,

$$\mathcal{N} = \{u \in W_0^{1,p} \setminus \{0\} : R(u) = \lambda\}.$$

Moreover, since for any $\mu > 0, R(tu) \to \infty$ at $t \to 0$, and $R(tu) \to -\infty$ as $t \to +\infty$,

$$\mathcal{N} \neq \emptyset, \quad \forall \lambda \in (-\infty, +\infty), \forall \mu \in (0, +\infty). \tag{3.2.3}$$

Using the Implicit Function Theorem we have

Proposition 3.2.1. *If $R'(s_i(u_0)u_0) \neq 0$ for $u_0 \in W \setminus \{0\}$, $i = 1, 2, 3$, then there exists a neighbourhood $U_{u_0} \subset W_0^{1,p} \setminus \{0\}$ of u_0 such that $s_i(\cdot) \in C^1(U_0)$.*

Simple analysis swhows that for any given $u \in W_0^{1,p} \setminus \{0\}$, the fibering function $R(tu)$ may have at most two-non-zero critical points $t^{n,+}(u), t^{n-}(u)$, where $t^{n,+}(u)$ is a local minimum, $t^{n,-}(u)$ is a local maximum point of $R(tu)$, and

$$0 < s_1(u) \le t^{n,+}(u) \le s_2(u) \le t^{n,+}(u) \le s_3(u) < \infty. \tag{3.2.4}$$

(see Fig.3.2).

To split the points $t^{n+}(u), t^{n,-}(u)$, in the second step of the recursive procedure, we apply the nonlinear generalized Rayleigh quotient method to the functional R with respect to the parameter μ, i.e., we consider

$$M^n(u) := \frac{(p-\alpha)\int|\nabla u|^p - (\gamma-\alpha)\int|u|^\gamma}{(\alpha-q)\int|u|^q}, \quad u \in W_0^{1,p} \setminus \{0\}. \tag{3.2.5}$$

Notice that for any $u \in W_0^{1,p} \setminus \{0\}$, $R'(tu) = 0 \iff M^n(tu) = \mu$. The only solution of $\frac{d}{dt} M^n(tu) = 0$ is a global maximum point $t^n(u)$ of the function $M^n(tu)$ which can be found

$$t^n(u) := \left(C_n \frac{\int |\nabla u|^p}{\int |u|^\gamma} \right)^{1/(\gamma-p)}, \quad \forall u \in W_0^{1,p} \setminus \{0\} \tag{3.2.6}$$

where

$$C_n = \frac{(p-\alpha)(p-q)}{(\gamma-\alpha)(\gamma-q)}.$$

This allows us to introduce the following *NG-Rayleigh μ-quotient*

$$M^n(t^n(u)u) = c^n \frac{(\int |\nabla u|^p)^{\frac{\gamma-q}{\gamma-p}}}{(\int |u|^q)(\int |u|^\gamma)^{\frac{p-q}{\gamma-p}}} \tag{3.2.7}$$

and the corresponding principal extremal value

$$\mu^n(u) := \inf_{u \in W_0^{1,p} \setminus \{0\}} \sup_{t>0} M^n(tu) = c^n \inf_{u \in W_0^{1,p} \setminus \{0\}} \frac{(\int |\nabla u|^p)^{\frac{\gamma-q}{\gamma-p}}}{(\int |u|^q)(\int |u|^\gamma)^{\frac{p-q}{\gamma-p}}}$$

where

$$c^n = \frac{(p-\alpha)^{\frac{\gamma-q}{\gamma-p}}(p-q)^{\frac{p-q}{\gamma-q}}(\gamma-p)}{(\alpha-q)(\gamma-\alpha)^{\frac{p-q}{\gamma-p}}(\gamma-q)^{\frac{\gamma-q}{\gamma-p}}}.$$

Nothe that this definition of μ^n coincides with (3.1.6).
It easily follows (cf. [29])

Proposition 3.2.2. *For any $\mu \in (0,\mu^n)$ and $u \in W_0^{1,p} \setminus \{0\}$, the function $R(tu)$ has precisely two distinct critical points such that $0 < t^{n,+}(u) < t^{n,-}(u)$, with $t^{n,+}(.), t^{n,-} \in C^1(W_0^{1,p} \setminus \{0\})$. Moreover,*

- $R''(t^{n,+}(u)u) > 0$, $R''(t^{n,-}(u)u) < 0$,
- $R'(tu) < 0 \iff t \in (0,t^{n,+}(u)) \cup (t^{n,-}(u),\infty)$,
- $R'(tu) > 0 \iff t \in (t^{n,+}(u),t^{n,-}(u))$.

Observe that this and (3.2.4) imply that $0 < s_1(u) < s_3(u) < \infty$ for any $\mu \in (0,\mu^n), u \in W \setminus \{0\}$. Thus, for $\mu \in (0,\mu^n)$, we are able to introduce the following *NG-Rayleigh $\lambda-$quotients*

$$\lambda^{n,+}(u) := R(t^{n,+}(u)u), \quad \lambda^{n,-} := R(t^{n,-}(u)u), \quad u \in W_0^{1,p} \setminus \{0\}.$$

By proposition (3.2.2) and regularity of R it follows that $\lambda^{n,+}$ and $\lambda^{n,-}$ are $C^1(W_0^{1,p} \setminus \{0\})$, $\mu \in (0,\mu^n)$. It is easily seen that the corresponding *principal extremal values.*

$$\lambda^{n,+} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \lambda^{n,+}(u), \tag{3.2.8}$$

$$\lambda^{n,-} = \inf_{u \in W_0^{1,p} \setminus \{0\}} \lambda^{n,-}(u), \tag{3.2.9}$$

coincide with (3.1.3) and (3.1.4), respectively. We also need the so-called zero-energy level Rayleigh quotient

$$R^e(u) := \frac{\frac{1}{p}\int |\nabla u|^p + \frac{\mu}{q}\int |u|^q - \frac{1}{\gamma}\int |u|^\gamma}{\frac{1}{\alpha}\int |u|^\alpha}, \quad u \in W_0^{1,p} \setminus \{0\},\ \mu \in \mathbb{R}$$

which is characterized by the fact that $R^e(u) = \lambda \iff \Phi(u) = 0$. It is easy to see that $R^e(u)$ possesses similar propiertis to that $R(u)$. In particular, the fibering function $R^e(tu)$ may have at most two non-zero fibering critical points $0 < t^{e,+}(u) \leq t^{e,-}(u) < +\infty$ so that $t^{e,+}(u)$ is a local minimum while $t^{e,-}(u)$ is a local maximum point of $R^e(tu)$. Moreover, the same conclusion as for $M^n(u)$ can be drawn for the Rayleigh quotient

$$M^e(u) := q\,\frac{\frac{p-\alpha}{p}\displaystyle\int |\nabla u|^p - \frac{\gamma-\alpha}{\gamma}\displaystyle\int |u|^\gamma}{(\alpha-q)\displaystyle\int |u|^q} \tag{3.2.10}$$

which is characterized by the fact that $(R^e)'(tu) = 0 \Longleftrightarrow M^e(tu) = 0$ for any $u \in W_0^{1,p} \setminus \{0\}$. The unique solution of $\dfrac{d}{dt}M^e(tu) = 0$ is a global maximum point of the function $M^e(tu)$ define by

$$t^e(u) := \left(C_e\frac{||u||_1^p}{||u||_{L^\gamma}^\gamma}\right)^{\dfrac{1}{(\gamma-p)}}, \quad \forall u \in W_0^{1,p} \setminus \{0\}, \tag{3.2.11}$$

where

$$C_e = \frac{\gamma(p-\alpha)(p-q)}{p(\gamma-\alpha)(\gamma-q)}.$$

Thus we have the following NG-Rayleigh quotient $\mu^e(u) := M^e(t^e(u)u), \quad u \in W_0^{1,p} \setminus \{0\}$ with the corresponding principal extremal value

$$\mu^e = \inf_{u\in W_0^{1,p}\setminus\{0\}} \sup_{t>0} M^e(tu) = c^e \inf_{u\in W_0^{1,p}\setminus\{0\}} \frac{\|u\|_1^{p\frac{\gamma-q}{\gamma-p}}}{\|u\|_{L^q}^q\,\|u\|_{L^\gamma}^{\gamma\frac{p-q}{\gamma-p}}}$$

where $c^e = c^n\, q\, \dfrac{\gamma^{\frac{p-q}{\gamma-p}}}{p^{\frac{\gamma-q}{\gamma-p}}}$. Note that this definition of μ^e coincides with (3.1.6) we thus have

Proposition 3.2.3. *For any $\mu \in (0,\mu^e)$ and $u \in W_0^{1,p} \setminus \{0\}$, the function $R^e(tu)$ has precisely two distinct critical points such that $0 < t^{e,+}(u) < t^{e,-}(u)$ with $t^{e,+}(.), t^{e,-}(.) \in C^1(W_0^{1,p} \setminus \{0\})$. Moreover,*

- $(R^e)''(t^{e,+}(u)u) > 0$, *and* $(R^e)''(t^{e,-}(u)u) < 0$,
- $(R^e)'(tu) < 0 \Longleftrightarrow t \in (0,t^{e,+}(u)) \cup (t^{e,-}(u),\infty)$,
- $(R^e)'(tu) > 0 \Longleftrightarrow t \in (t^{e,+}(u),t^{e,-}(u))$.

Hence, for $\mu \in (0,\mu^e)$, we are able to introduce the following zero-energy level *NG-Rayleigh λ-quotients*

$$\lambda^{e,+}(u) := R^e(t^{e,+}(u)u), \quad \lambda^{e,-}(u) := R(t^{e,-}(u)u), \quad u \in W_0^{1,p} \setminus \{0\}.$$

It is not difficult to see that the corresponding zero-energy level principal extremal values

$$\lambda^{e,+} = \inf_{u \in W_0^{1,p}\setminus\{0\}} \lambda^{e,+}(u), \quad \lambda^{e,-} = \inf_{u \in W_0^{1,p}\setminus\{0\}} \lambda^{e,-}(u) \tag{3.2.12}$$

coincide with (3.1.1) and (3.1.2), respectively. The relationships among the above introduced Rayleigh quotients are given by the following lemma (see Fig. 3.2).

Lemma 3.2.2. *Assume that* $1 < q < \alpha < p < \gamma, \quad u \in W_0^{1,p} \setminus \{0\}, t > 0.$

(i) $M^e(tu) = M^n(tu) \Longleftrightarrow t = t^e(u)$,

(ii) $R^e(tu) = R(tu) \Longleftrightarrow t = t^{e,+}(u)$ or $t = t^{e,-}(u), \quad \mu \in (0,\mu^e)$,

(iii) $t^{n,+}(u) < t^{e,+}(u) < t^e(u) < t^{n,-}(u) < t^{e,-}(u), \quad \forall \mu \in (0,\mu^e)$

(iv) $R(tu) < R^e(tu) \Longleftrightarrow t \in (0,t^{e,+}(u))$ or $t \in (t^{e,-}(u),\infty), \quad \forall \mu \in (0,\mu^e)$.

Proof. (i) $M^e(tu) = M^n(tu)$ is equivalent to

$$\begin{aligned}
&t^{p-q}||u||^p - \frac{(\gamma-\alpha)}{(p-\alpha)}t^{\gamma-q}||u||^\gamma_{L^\gamma} = q\left(t^{p-q}\frac{1}{p}||u||^p - t^{\gamma-q}\frac{(\gamma-\alpha)}{\gamma(p-\alpha)}||u||^\gamma_{L^\gamma}\right).\\
&\Longleftrightarrow ||u||^p\left(\frac{p-q}{p}\right) = t^{\gamma-p}\left(\frac{\gamma(\gamma-\alpha)-q(\gamma-\alpha)}{\gamma(p-\alpha)}\right)||u||^\gamma_{L^\gamma} = t^{\gamma-p}\frac{(\gamma-\alpha)(\gamma-q)}{\gamma(p-\alpha)}||u||^\gamma_{L^\gamma}\\
&\Longleftrightarrow t^{\gamma-p} = \frac{||u||^p}{||u||^\gamma_{L^\gamma}}\frac{\gamma(p-\alpha)(p-q)}{p(\gamma-\alpha)(\gamma-q)}.
\end{aligned}$$

Therefore $t = t^e(u)$.

(ii) Observe, $R^e(tu) = R(tu)$ for $t > 0$ if and only if

$$\begin{aligned}
&t^{p-\alpha}||u||^p + \mu t^{q-\alpha}||u||^q_{L^q} - t^{\gamma-\alpha}||u||^\gamma_{L^\gamma} = \frac{\alpha t^{p-\alpha}}{p}||u||^p + \frac{\mu\alpha t^{q-\alpha}}{q}||u||^q_{L^q} - \frac{\alpha t^{\gamma-\alpha}}{\gamma}||u||^\gamma_{L^\gamma},\\
&\Longleftrightarrow 0 = \frac{(p-\alpha)}{p}t^{p-\alpha}||u||^p - \frac{\gamma-\alpha}{\gamma}t^{\gamma-\alpha}||u||^\gamma_{L^\gamma} - \frac{\mu(\alpha-q)}{q}t^{q-\alpha}||u||^q_{L^q}\\
&\qquad = \frac{(\alpha-q)||u||^q_{L^q}t^{q-\alpha}}{q}(M^e(tu)-\mu).
\end{aligned}$$

Since $(R^e)'(tu) = 0 \Longleftrightarrow M^e(tu) = 0$, we get (ii). The proof of (iii) and (iv) follow from items $(i), (ii)$ and simple accounts.

□

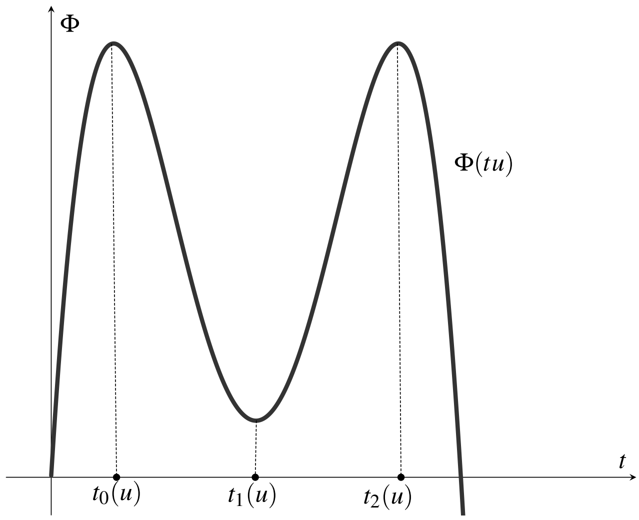


Fig. 3.1 Fibering function $\Phi(tu), t \geq 0$, $u \in W_0^{1,p}$

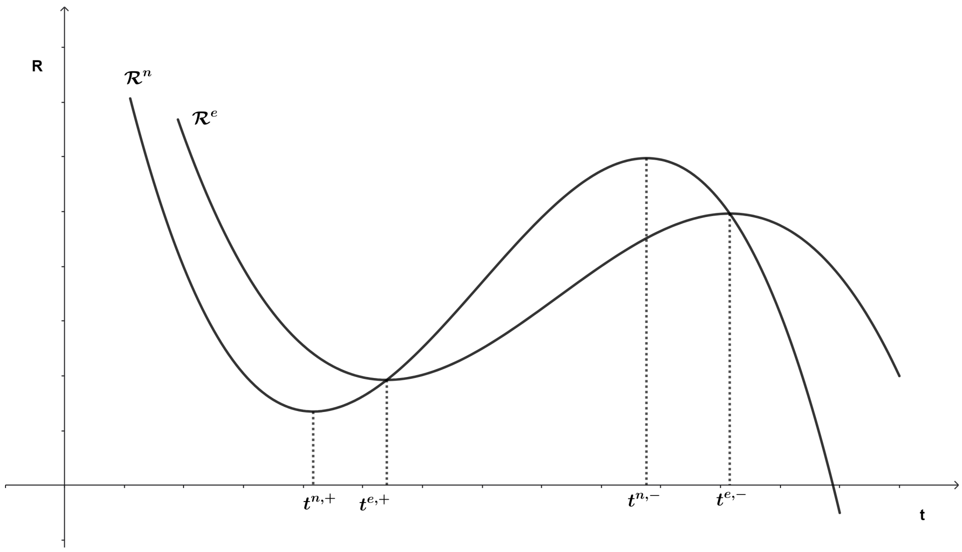


Fig. 3.2 The functions $\mathcal{R}^e(tu), \mathcal{R}^n(tu)$

We need also

Corollary 3.2.1. *The functionals* $\mu^e(u), \mu^n(u)$ *and* $\lambda^{n,\pm}(u)$ *for* $\mu < \mu^n$, $\lambda^{e,\pm}(u)$ *for* $\mu < \mu^e$ *are weakly lower semi-continuous on* $W_0^{1,p}$. *Futhermore,* $t^{n,-}(u), t^{e,-}(u)$ *are lower semicontinous, while* $t^{n,+}(u), t^{e,-}(u)$ *upper semi-continuous on* $W_0^{1,p}$.

Proof. The weakly lower semi-continuity of $\mu^e(u), \mu^n(u)$ on $W_0^{1,p}$, let (u_m) be a sequence in $W_0^{1,p}$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}$ then $||u|| \leq \lim_{n\to\infty}\inf ||u_m||$ an by Theorem (B.0.8) have

$$\mu^n(u) \leq \frac{\liminf_{m\to\infty}\left(\int |\nabla u_m|^p\right)^{\frac{\gamma-q}{\gamma-p}}}{\left(\int |u_m|^q\right)\left(\int |u_m|^\gamma\right)^{\frac{p-q}{\gamma-p}}}$$

$$= \frac{\liminf_{m\to\infty}\left(\int |\nabla u_m|^p\right)^{\frac{\gamma-q}{\gamma-p}}}{\limsup_{m\to\infty}\left(\int |u_m|^q\right)\left(\int |u_m|^\gamma\right)^{\frac{p-q}{\gamma-p}}}$$

$$\leq \liminf_{m\to\infty} \mu^n(u).$$

analogously for $\mu^e(u)$.

Let us prove as an example that $\lambda^{n,+}(u)$ is weakly lower semi-continuous on $W_0^{1,p}$. Assume that $\mu < \mu^n$. Let (u_m) be a sequence in $W_0^{1,p}$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}$ as $m \to \infty$ for some $u \neq 0$. Then $\mu < \mu^n < \mu^n(u)$ and by the weakly lower semi-continuity of $\mu^n(u)$ we have $\mu < \mu^n < \mu^n(u) \leq \liminf_{m\to+\infty} \mu^n(u_m)$. Hence by Proposition (3.2.2), there exist $t^{n,\pm}(u), t^{n,\pm}(u_m) \in (0,+\infty), m = 1,2,...$ Moreover, since $M^n(tu) \leq M^n(tu_m)$, for $t > 0$ and for sufficiently large $m = 1,2,..$, we have

$$t^{n,+}(u_m) \leq t^{n,+}(u) < t^{n,-}(u) \leq t^{n,-}(u_m). \tag{3.2.13}$$

On the other hand, we have that $R(tu) \leq \lim_{m\to\infty}\inf R(tu_m)$, for all $t > 0$. Hence and using the fact that $R'(tu) < 0$ for $t \in (0, t^{n,+}(u))$ and (3.2.13) we infer

$$\begin{aligned}\lambda^{n,+}(u) = R(t^{n,+}(u)u) &\leq \lim_{m\to\infty}\inf R(t^{n,+}(u_m)u) \\ &\leq \lim_{m\to\infty}\inf R(t^{n,+}(u_m)u_m) = \lim_{m\to\infty}\inf \lambda^{n,+}(u_m).\end{aligned}$$

The proof of the last part of corollary The proof of $t^{n,-}(u), t^{e,-}(u)$ are lower semicontinous, while $t^{n,+}(u), t^{e,-}(u)$ upper semi-continuous on W follows from (3.2.13). ☐

Lemma 3.2.3. *Assume* $1 < q < \alpha < p < \gamma < p^*$. *Then,*

(I) for any $\mu \in (0, \mu^e)$,

(i) *(3.1.1) has a minimizer* $u^{e,+} \in W_0^{1,p} \setminus \{0\}$ *such that* $0 < \lambda^{e,+} = \lambda^{e,+}(u^{e,+}), (R^e)''(u^{e,+}) > 0$;

(ii) *(3.1.2) has a minimizer* $u^{e,-} \in W_0^{1,p} \setminus 0$ *such that* $0 < \lambda^{e,-}(u^{e,-}), (R^e)''(u^{e,-}) < 0$;

(II) for any $\mu \in (0, \mu^n)$,

(i) *(3.1.3) has a minimizer* $u^{n,+} \in W_0^{1,p} \setminus \{0\}$ *such that* $0 < \lambda^{n,+} = \lambda^{n,+}(u^{n,+}), R''(u^{n,+}) > 0$;

(ii) *(3.1.4) has a minimizer* $u^{n,-} \in W_0^{1,p} \setminus 0$ *such that* $0 < \lambda^{n,-}(u^{n,-}), R''(u^{n,-}) < 0$.

Proof. The proofs of these assertions are similar. Let us prove as an example assertion (i), (I). Let $\mu > 0$. Define the set $\mathcal{Z} := \{u \in W_0^{1,p} \setminus \{0\} : (R^e)'(u) = 0\}$. Note that

$$R^e(u) = \alpha \frac{\left[\frac{(\gamma - p)}{p}||u||^p + \mu \frac{(\gamma - q)}{q}||u||^q_{L^q}\right]}{(\gamma - \alpha)||u||^\alpha_{L^\alpha}}, \quad \forall u \in \mathcal{Z}_\mu. \tag{3.2.14}$$

Hence by the Sobolev inequality we derive that $R^e(u) \geq \alpha \frac{(\gamma - p)}{p(\gamma - \alpha)}||u||^{p-\alpha} \to \infty$ if $u \in \mathcal{Z}$ and $||u|| \to +\infty$. Thus, R^e is coercive on $\mathcal{Z}$, $\forall \mu > 0$.
Let (u_m) be a minimizing sequence of (3.1.1), i.e., $\lambda^{e,+}(u_m) = R^e(u_m) \to \lambda^{e,+}$, where by the homogeneity of $\lambda^{e,+}(u)$ we may assume that $t_m = t^{e,+}(u_m) = 1$ for $m = 1, 2, \ldots$ The coerciveness of R^e on $\mathcal{Z}$ implies that the minimizing sequence (u_m) of (3.1.1) has a weak in $W_0^{1,p}$ and strong in $L^r, 1 < r < p^*$ limit point $u^e \in W_0^{1,p}$.
Let us show that $u^e \neq 0$. Observe,

$$R^e(u_m) = \mathcal{R}^0(u_m) - \frac{\alpha ||u_m||^\gamma_{L^\gamma}}{\gamma ||u_m||^\alpha_{L^\alpha}}, m = 1, \ldots \tag{3.2.15}$$

where

$$\mathcal{R}^0(u) := \frac{\frac{1}{p}||u||^p + \frac{\mu}{q}||u||^q_{L^q}}{\frac{1}{\alpha}||u||^\alpha_{L^\alpha}}, \quad u \in W_0^{1,p} \setminus \{0\}.$$

In can be shown (see, e.g., [10]) that

$$\min_{u \in W_0^{1,p} \setminus \{0\}} \min_{t > 0} \mathcal{R}^0(tu) = \lambda_0 > 0. \tag{3.2.16}$$

Denote $a_m := \frac{\alpha ||u_m||^{\gamma}_{L^{\gamma}}}{\gamma ||u_m||^{\alpha}_{L^{\alpha}}}$, $m = 1, 2, \ldots$ Then

$$\mathcal{R}^0(u_m) = a_m \gamma \frac{\frac{1}{p}||u_m||^p + \frac{\mu}{q}||u_m||^q_{L^q}}{||u_m||^{\gamma}_{L^{\gamma}}}.$$

We may assume that $a_0 := \lim_{m\to\infty} a_m \geq 0$. Since $\lambda_0 > 0$, $a_0 \neq 0$. Suppose, conversely to our claim, that $u^e = 0$. Then

$$R^e(u_m) \geq a_m \left(\gamma \frac{C}{||u_m||^{\gamma}_{L^{\gamma}}} - 1 \right) \to +\infty$$

which is a contradiction. Thus, $u^e \neq 0$ and $\lambda^{e,+} > 0$. Now by the weakly lower semi-continuity of $\lambda^{e,+}(u)$ it follows that $\lambda^{e,+}(u^{e,+}) \leq \lambda^{e,+}$, which obviously implies $\lambda^{e,+} = \lambda^{e,+}(u^{e,+})$. □

Corollary 3.2.2. *The following holds*

(i) if $\mu \in (0, \mu^e)$*, then* $0 < \lambda^{e,+} < \lambda^{e,-} < +\infty$*,*

(ii) if $\mu \in (0, \mu^n)$*, then* $0 < \lambda^{n,+} < \lambda^{n,-} < +\infty$*.*

Proof. By Lemma (3.2.3), we have

$$0 < \lambda^{e,+} \leq \lambda^{e,+}(u^{e,+}) < \lambda^{e,-}(u^{e,-}) = \lambda^{e,-} < \infty.$$

Thus, we get (i). The proof of (ii) is similar. □

Corollary 3.2.3. *If* $\mu \in (0, \mu^e)$*, then* (i) $\lambda^{e,-} < \lambda^{n,-}$*;* (ii) $\lambda^{n,+} < \lambda^{e,+}$*.*

Proof. By Lemma (3.2.3), there exists $u^{n,-}$ such that $\lambda^{n,-} = \lambda^{n,-}(u^{n,-})$. Lemma (3.2.2) entails that $R^e(t^{e,-}(u^{n,-})u^{n,-}) = R(t^{e,-}(u^{n,-})u^{n,-})$ and the function $t \to R(tu^{n,-})$ is decreasing on the interval $(t^{n,-}(u^{n,-}), t^{e,-}(u^{n,-}))$. Hence,

$$\lambda^{e,-} \leq R^e(t^{e,-}(u^{n,-})u^{n,-}) = R(t^{e,-}(u^{n,-})u^{n,-}) < R(t^{n,-}(u^{n,-})u^{n,-}) = \lambda^{n,-},$$

and we get (i). The proof of (ii) is similar. □

Lemma 3.2.4. *3.3.4 Assume that* $1 < q < \alpha < p < \gamma < p^*$*. Then*

(1) $0 < \mu^e < \mu^n < +\infty$*,*

(2) $0 < \lambda^{n,+} < \lambda^{e,-} < \lambda^{n,-} < +\infty$, $\forall \mu \in (0, \mu^e)$.

Proof. Follows from Corollaries (3.2.2) and (3.2.3) □

Corollary 3.2.4. *The following holds*

(i) The minimizer $u^{e,+}$ of (3.1.1) (perhaps, after a scaling) is a non-negative critical point of $\Phi_{\lambda^{e,+}}$, moreover $\Phi_{\lambda^{e,+}}(u^{e,+}) = 0$ and $\Phi''_{\lambda^{e,+}}(u^{e,+}) > 0$.

(ii) The minimizer $u^{e,-}$ of (3.1.2) (perhaps, after a scaling) is a non-negative critical point of $\Phi_{\lambda^{e,-}}$ moreover $\Phi_{\lambda^{e,-}}(u^{e,-}) = 0$ and $\Phi''_{\lambda^{e,-}}(u^{e,-}) < 0$.

Proof. (i) Let $u^{e,+}$ be a minimizer of (3.1.1). Then $u^{e,+}$ is also a minimizer of (3.2.12) with $t^{e,+}(u^{e,+}) = 1$. Hence, $0 = D\lambda^{e,+}(u^{e,+}) = DR^e(u^{e,+})$, and consequently, $D\Phi_{\lambda^{e,+}}(u^{e,+}) = 0$. Moreover, $R^e(u^{e,+}) = \lambda^{e,+}$, $(R^e)''(u^{e,+}) > 0$ yield $\Phi_{\lambda^{e,+}}(u^{e,+}) = 0$ and $\Phi''_{\lambda^{e,+}}(u^{e,+}) > 0$, respectively. Since $\lambda^{e,+}(|u^{e,+}|) = \lambda^{e,+}(u^{e,+})$ one may assume that $u^{e,+} \geq 0$. The proof of (ii) is similar. □

3.3 Existence of three solutions for a two-parameter problem

Once the Rayleigh quotient from the Fréchet derivative and the Rayleigh quotient from the energy functional have been introduced, and following the same strategy developed in Chapters 1 and 2, we shall show in this section the existence of at least three solutions to our problem with two parameters, μ and λ.
Two of these solutions will be obtained by minimization on the appropriate subsets of the Nehari manifold, while the third one will be established by means of the Mountain Pass Theorem.

3.3.1 A linearly stable ground state solution

Theorem 3.3.1. *Assume that $1 < q < \alpha < p < \gamma < p^*$, $\mu \in (0, \mu^e)$.
There exists $\hat{\lambda}^* \in (\lambda^{n,+}, \lambda^{e,+})$ such that for any $\lambda \in [\hat{\lambda}^*, \lambda^{n,-}]$, problem (3.0.1) possesses a positive solution $u^2 \in C^{1,k}(\overline{\Omega}), k \in (0,1)$ such that*

(i) u^2 is ground state;

(ii) u^2 is linearly stable and $\Phi''(u^2) > 0$;

(iii) The function $\lambda \to \Phi(u^2)$ is continuos and monotone decreasing on $(\hat{\lambda}^*, \lambda^{n,-})$;

(iv) $\Phi(u^2) > 0$ *for* $\lambda \in (\hat{\lambda}^*, \lambda^{e,+})$, $\Phi_{\lambda^{e,+}}(u^2_{\lambda^{e,+}}) = 0$, *and* $\Phi(u^2) < 0$ *for* $\lambda \in (\lambda^{e,+}, \lambda^{n,-})$.

Proof. We obtain the solution u^2 using the following Nehari minimization problem

$$\Phi_2 = \min\{\Phi(u) : u \in \mathcal{RN}^2\}, \tag{3.3.1}$$

where $\mathcal{RN}^2 := \{u \in \mathcal{N} : R'(u) \geq 0\}$. Observe, $\mathcal{RN}^2 \neq \emptyset$, for all $\mu \in (0, \mu^e)$ and $\lambda \in (\lambda^{n,+}, \lambda^{n,-})$. Indeed, if $\lambda \in (\lambda^{n,+}, \lambda^{n,-})$, then $\lambda < \lambda^{n,-} < \lambda^{n,-}(u), \forall u \in W_0^{1,p} \setminus \{0\}$, and therefore, there exists $\tilde{u} \in W_0^{1,p} \setminus \{0\}$ such that $\lambda^{n,+} < \lambda^{n,+}(\tilde{u}) < \lambda < \lambda^{n,-}(\tilde{u})$. Lemma (3.2.2) implies that there exists $s_2(\tilde{u}) \in (t^{n,+}(\tilde{u}), t^{n,-}(\tilde{u}))$, and thus, $s_2(\tilde{u})\tilde{u} \in \mathcal{RN}^2$.
Futhermore, the assumption $\mu \in (0, \mu^e), \lambda \in (\lambda^{n,+}, \lambda^{n,-})$ implies that

$$\lambda^{n,+}(u) \leq \lambda < \lambda^{n,-}(u), \quad \forall u \in \mathcal{RN}^2. \tag{3.3.2}$$

Indeed, $\lambda < \lambda^{n,-} \leq \lambda^{n,-}(u)$ for any $u \in W_0^{1,p} \setminus \{0\}$, whereas the conditions $R(u) = \lambda$, $R'(u) \geq 0$ for $u \in \mathcal{RN}^2$ yield $\lambda^{n,+}(u) \leq \lambda$.

In the following lemma, our u_2 depends on λ.

Lemma 3.3.1. *Let* $\mu \in (0, \mu^e)$ *and* $\lambda \in (\lambda^{n,+}, \lambda^{n,-})$. *There exists a minimizer* $u^2 \in W_0^{1,p} \setminus \{0\}$ *of (3.3.1) and*

$$\begin{cases} \Phi_2 = \Phi(u^2) > 0 & \text{if} \quad \lambda \in (\lambda^{n,+}, \lambda^{e,+}), \\ \Phi_2 = \Phi(u^2) = 0 & \text{if} \quad \lambda = \lambda^{e,+}, \\ \Phi_2 = \Phi(u^2) < 0 & \text{if} \quad \lambda \in (\lambda^{e,+}, \lambda^{n,-}). \end{cases} \tag{3.3.3}$$

Proof. Note that by Lemma (3.2.3), the minimum of (3.3.1) for $\lambda = \lambda^{e,+}$ attains (perphas, after a scaling) at the solution $u^{e,+} \in W_0^{1,p} \setminus \{0\}$ of (3.1.1) and $\Phi_2|_{\lambda=\lambda^{e,+}} = 0$. This easily implies the proof of the lemma in the case $\lambda = \lambda^{e,+}$.
Assume that $\lambda \in (\lambda^{n,+}, \lambda^{n,-})$. Let u_m be a minimizing sequence of (3.3.1). The coerciveness of Φ implies that the sequence (u_m) is bounded in $W_0^{1,p}$ and thus, up to subsequence,

$$u_m \to u^2 \quad \text{in } L^r, \text{ and } u_m \rightharpoonup u^2 \quad \text{in } W_0^{1,p},$$

for some $u^2 \in W_0^{1,p}$, where $r \in (1, p^*)$. Moreover, $\Phi(u^2) \leq \liminf_{m\to\infty} \Phi(u_m) = \Phi_2$.Moreover $\Phi(u^2) \leq \lim_{m\to\infty} \inf \Phi(u_m) = \Phi_2$.
If $\lambda \in (\lambda^{e,+}, \lambda^{n,-})$, then there exists $\tilde{u} \in \mathcal{RN}^2$ such that $1 = s_2(\tilde{u}) \in (t^{e,+}(\tilde{u}), t^{e,-}(\tilde{u}))$. Hence

by (iv), Lemma (3.2.2), $\lambda = R(s_2(\tilde{u})\tilde{u}) > R^e(s_2(\tilde{u})\tilde{u}) \equiv R^e(\tilde{u})$, with implies $0 > \Phi(\tilde{u}) \geq \Phi_2$. Hence, $\Phi(u^2) < 0$, and consequently, $u^2 \neq 0$ for $\lambda \in (\lambda^{e,+}, \lambda^{n,-})$.
If $\lambda \in (\lambda^{n,+}, \lambda^{e,+})$, then $\lambda < \lambda^{e,+} < \lambda^{e,+}(u) = R^e(t^{e,+}(u)u)$ for any $u \in \mathcal{RN}^2$, and therefore, $1 = s_2(u) \in (0, t^{e,+}(u))$. Hence by (iv), Lemma (3.2.2), $R^e(u) \equiv R^e(s_2(u)u) > \lambda$, and consequently, $\Phi(u) > 0$, $\forall \lambda \in (\lambda^{n,+}, \lambda^{e,+})$, $\forall u \in \mathcal{RN}^2$. Let us show that $u^2 \neq 0$ for $\lambda \in (\lambda^{n,+}, \lambda^{e,+})$, then $t^{n,+}(u_m) < s_2(u_m) = 1$, $m = 1,2,\ldots$ Consequently, $t^{n,+}(u_m)u_m \to 0$ as $m \to +\infty$ strongly in L^r, $r \in (1, p^*)$ and weakly in $W_0^{1,p}$. Analysis similar to that in the proof of Lemma (3.2.3) shows that this implies $R(t^{n,+}(u_m)u_m) \to +\infty$. However, this contradicts $R(t^{n,+}(u_m)u_m) \leq R(u_m) = \lambda, m = 1,2,\ldots$, and thus, $u^2 \neq 0$ for $\lambda \in (\lambda^{n,+}, \lambda^{e,+})$.
Let $\lambda \in (\lambda^{n,+}, \lambda^{e,+})$ and suppose, contrary to our claim, that $\Phi(u_m) \to 0$ as $m \to +\infty$. Then

$$\Phi(u_m) = ||u_m||^{\alpha}_{L^\alpha}(R^e(u_m) - \lambda) \to 0. \tag{3.3.4}$$

Since $\Phi(u_m) > 0, m = 1,2,\ldots$ and $u^2 \neq 0$, this implies $R^e(u_m) \downarrow \lambda$ as $m \to +\infty$. Since $\lambda < \lambda^{e,+} \leq \lambda^{e,+}(u_m) = R^e(t^{e,+}(u_m)u_m)$, $1 = s_2(u_m) \in (0, t^{e,+}(u_m))$, and therefore, $R^e(u_m) > R^e(t^{e,+}(u_m)u_m) = \lambda^{e,+}(u_m) > \lambda, m = 1,2,\ldots$ Consequently, $\lim_{m\to\infty} \lambda^{e,+}(u_m) = \lambda < \lambda^{e,+}$, which is a contradiction since $\lambda^{e,+}(u_m) \geq \lambda^{e,+}, m = 1,2,\ldots$ Thus, $\Phi_2 > 0$ if $\lambda \in (\lambda^{n,+}, \lambda^{e,+})$, and we have proved (3.3.1).
Let us show that u^2 is a minimizer of (3.3.1), i.e., $u^2 \in \mathcal{RN}^2$ and $\Phi(u^2) = \Phi_2$. By (3.3.2), $\lambda^{n,+}(u^2) \leq \lambda < \lambda^{n,-}(u^2)$, and therefore $\exists s_2(u^2) \in (s_1(u^2), s_3(u^2))$ such that

$$\lambda = R(s_2(u^2)u^2) \leq \lim_{m\to\infty} \inf R(s_2(u^2)u_m),$$

$$0 < R'(s_2(u^2)u^2) \leq \lim_{m\to\infty} \inf R'(s_2(u^2)u_m).$$

This means that $1 = s_2(u_m) \leq s_2(u^2) < s_3(u_m)$, $m = 1,\ldots$. Hence by

$$R(u^2) \leq \lim_{m\to\infty} \inf R(u_m) = \lambda,$$

we obtain $s_1(u^2) \leq 1 \leq s_2(u^2)$. Since $\Phi'(su^2) < 0$, for any $s \in (s_1(u^2), s_2(u^2))$, we derive

$$\Phi(s_2(u^2)u_2) \leq \Phi(u^2) \leq \lim_{m\to\infty} \inf(u_m) = \Phi_2,$$

which yields that $s_2(u^2)u^2 = u^2$ is a minimizer of (3.3.1) and thus, $u_m \to u^2$ strongly in W. $\square$

Consider the following subset of $\mathcal{M}$

$$\mathcal{M}_2 := \{u \in \mathcal{RN}^2 : \Phi_2 = \Phi(u)\}.$$

Lemma (3.3.1) yields that $\mathcal{M}_2 \neq \emptyset$ for any $\mu \in (0,\mu^e)$ and $\lambda \in (\lambda^{n,+},\lambda^{n,-})$. Note that the minimizer $u^2 \in \mathcal{M}_2$ of(3.3.1) does not necessarily provide a solution of (3.0.1). By Lemma (3.2.1) and (3.2.2), $u_2 \in \mathcal{M}_2$ corresponds to a solution of (3.0.1), if the strict inequality $R'(u^2) > 0$ holds. Note that by Proposition (3.2.2),

$$R'(u^2) > 0 \Longleftrightarrow \lambda^{n,+}(u^2) < \lambda < \lambda^{n,-}(u^2).$$

By (3.3.2), if $\mu \in (0,\mu^e)$ and $\lambda \in (\lambda^{n,+},\lambda^{n,-})$, then $\lambda < \lambda^{n,-}(u^2)$ for any $u^2 \in \mathcal{M}_2$. Thus, to obtain that $u^2 \in \mathcal{M}_2$ is a weak solution of (3.0.1) it is sufficient to show that $\lambda^{n,+}(u^2) < \lambda$.

Corollary 3.3.1. *Let $\mu \in (0,\mu^e)$. If $\lambda \in [\lambda^{e,+},\lambda^{n,-})$, then $u^2 \in \mathcal{M}_2$ is a weak solution of (3.0.1).*

Proof. Indeed, the inequality $\Phi(u^2) = \Phi_2 \leq 0$ implies $\lambda^{n,+}(u^2) < \lambda^{e,+}(u^2) \leq R^e(u^2) \leq \lambda$, and thus, $\lambda^{n,+}(u^2) < \lambda$. □

Lemma 3.3.2. *Let $\mu \in (0,\mu^e)$. There exists $\tilde{\lambda}^* \in (\lambda^{n,+},\lambda^{e,+})$ such that if $\lambda \in (\tilde{\lambda}^*,\lambda^{e,+})$, then $u^2 \in \mathcal{M}_2$ weakly satisfies (3.0.1), moreover $\Phi(u^2) = \Phi_2 > 0$, $\Phi''(u^2) > 0$.*

Proof. By the above, it is sufficient to show that there exists $\tilde{\lambda}^* \in (\lambda^{n,+},\lambda^{e,+})$ such that $\lambda^{n,+}(u) < \lambda, \forall u \in \mathcal{M}_2$, $\forall \lambda \in (\tilde{\lambda}^*,\lambda^{e,+})$. Suppose this is false. Then there exist sequences $\lambda_m \in (\lambda^{n,+},\lambda^{e,+})$ and $u_m \in \mathcal{M}_2, m = 1,2,...$ such that $\lambda_m \to \lambda^{e,+}$ as $m \to +\infty$ and $\lambda_m = \lambda^{n,+}(u_m), \forall m = 1,2,...$ By Proposition (3.3.3) , up to a subsequence, $u_m \to u_0$ strongly in $W_0^{1,p}$ as $m \to \infty$ for some $u_0 \in \mathcal{M}_{2,\lambda^{e,+}}$. Hence $\lambda^{n,+}(u_0) = \lambda^{e,+}$, and consequently $u_0 \in \mathcal{RN}^2_{\lambda^{e,+}}$. Futhermore, Corollary (3.3.2) implies that $\Phi_{\lambda^{e,+}}(u_0) = \lim_{m \to +\infty} \Phi_{\lambda_m}(u_m) = \Phi_{2,\lambda^{e,+}}$. Hence $u_0 \in \mathcal{M}^2_{\lambda^{e,+}}$ and $\lambda^{n,+}(u_0) = \lambda^{e,+}(u_0) = \lambda^{e,+}$, which contradicts (2^0), Lemma (3.3.4). □

Conclusion of the proof of (1^0), theorem (3.3.1)

Let $0 < \mu < \mu^e$. Introduce,

$$\lambda^* := \sup\{\lambda \in (\lambda^{n,+},\lambda^{n,-}) : \lambda = \lambda^{n,+}(u), \exists u \in \mathcal{M}_2\}. \tag{3.3.5}$$

Corollary (3.3.1) and Lemma (3.3.2) imply that $\lambda^{n,+} \leq \lambda^* < \lambda^{e,+}$. Hence for any $\lambda \in (\lambda^*,\lambda^{n,-})$, there holds $\lambda^{n,+}(u) < \lambda^{e,+}, \forall u \in \mathcal{M}^2_{\lambda^{e,+}}$, and therefore, each $u^2 \in \mathcal{M}_2$ is a weak

solution of (3.0.1). By the above, $\Phi''(u^2) > 0, \Phi(u^2) > 0$, for $\lambda \in (\lambda^*, \lambda^{e,+}), \Phi_{\lambda^{e,+}}(u^2_{\lambda^{e,+}})$, and $\Phi(u^2) < 0$ for $\lambda \in (\lambda^{e,+}, \lambda^{n,-})$. From the above, u^2 is a local minimizer of $\Phi(u)$ in the Nehari manifold $\mathcal{N}$. This by $\Phi''(u^2) > 0$ implies that u_2 is a local minimizer of $\Phi(u) \in W_0^{1,p}$. Thus, u^2 is a linearly stable solution. It is obvious that u^2 is a ground state.
Note that $\Phi(|u|^2) = \Phi(u^2)$ and $|u^2| \in \mathcal{RN}^2$. Hence, one may assume that $u^2 \geq 0$. The bootstrap argument and the Sobolev embedding theorem yield that $u^2 \in L^\infty$. Then $C^{1,k}$- regularity results of DiBenedetto [9] and Tolksdorf [31] (interior regularity) combined with Lieberman [25] (regularity up to the boundary) yield $u^2 \in C^{1,k}(\overline{\Omega})$ for $k \in (0,1)$. Furthermore, since $p < \gamma$, the Harnack inequality due to Trudinger [32] implies that $u^2 > 0$ in Ω.
From Corollary (3.3.2) it follows that the function $(\lambda^*, \lambda^{n,-}) \ni \lambda \longmapsto \Phi(u^2)$ is continuous and monotone decreasing. This concludes the proof of Theorem (3.3.1). □

3.3.2 Linearly unstable local solution

Theorem 3.3.2. *Assume that* $1 < q < \alpha < p < \gamma < p^*$, $\mu \in (0, \mu^e)$ *for any* $\lambda \in (-\infty, \lambda^{n,-})$, *problem (3.0.1) possesses a positive solution* $u^3 \in C^{1,k}(\overline{\Omega})$, $k \in (0,1)$ *such that*

(i) u^3 *is linearly unstable and* $\Phi''(u^3) < 0$,

(ii) *the function* $(-\infty, \lambda^{n,-}) \ni \lambda \longmapsto \Phi(u^3)$ *is continuous and monotone decreasing;*

(iii) $\Phi(u^3) > 0$ *if* $\lambda \in (-\infty, \lambda^{e,-})$, $\Phi_{\lambda^{e,-}}(u^3_{\lambda^{e,-}}) = 0$, *and* $\Phi(u^3) < 0$ *if* $\lambda \in (\lambda^{e,-}, \lambda^{n,-})$*;*

(iv) $\Phi(u^3) \to +\infty$, $||u^3|| \to \infty$ *as* $\lambda \to -\infty$.
Moreover, if $\lambda \in (-\infty, \lambda^{n,+})$, *then* u^3 *is a ground state of (3.0.1).*

Proof. Let $\mu \in (0, \mu^n)$ and $\lambda \in (-\infty, +\infty)$. Consider

$$\Phi_3 = \min\{\Phi(u) : u \in \mathcal{RN}^3\}, \tag{3.3.6}$$

where

$$\mathcal{RN}^3 := \{u \in \mathcal{N} : R'(u) \leq 0, (M^n)(u) < 0\}.$$

Notice that $\mathcal{RN}^3 = \{u \in \mathcal{N} : (R)'(u) \leq 0, t^n(u) < 1\}$, where $t^n(u)$ is defined by (3.2.6).

Lemma 3.3.3. *Let* $\mu \in (0, \mu^n)$ *and* $\lambda \in (-\infty, +\infty)$. *Then there exists a minimizer* $u^3 \in \mathcal{RN}^3$ *of (3.3.6) and* $\Phi''(u^3) \leq 0$.

Proof. Since $\sup_{u\in W_0^{1,p}\setminus 0} \lambda^{n,-}(u) = +\infty$, one can find $u \in W_0^{1,p}\setminus 0$ for any $\lambda \in (-\infty,+\infty)$ such that $\lambda < \lambda^{n,-}(u)$, and therefore, there exists $s_3(u) > t^n(u)$. Hence $s_3(u)(u) \in \mathcal{RN}^3$ and therefore, $\mathcal{RN}^3 \neq \emptyset$ for any $\mu \in (0,\mu^n), \lambda \in (-\infty,+\infty)$. Let (u_m) be a minimizing sequence of (3.3.6). Similar to the proof of (1^o), Theorem (3.3.1) one can deduce that there exists a subsequence, which we again denote by (u_m), and a limit point u^3 such that $u_m \to u^3$ strongly in $L^r, r \in (1,p^*)$ and weakly in $W_0^{1,p}$. Observe that if u^3, then by (3.2.6) we obtain a contradiction

$$1 > (t^n(u_m))^{\gamma-p} = C_n \frac{||u_m||^p}{||u_m||^{\gamma}_{L^\gamma}} \geq c\frac{1}{||u_m||^{\gamma-p}_{L^\gamma}} \to \infty \quad \text{as } m \to \infty,$$

where $c \in (0,+\infty)$ does not depend on m. Thus $u^3 \neq 0$, and therefore, there exists $s_3(u^3) > t^n(u^3)$ so that $s_3(u^3)u^3 \in \mathcal{RN}^3$.
By corollary , we have $t^{n,+}(u_m) \leq t^{n,+}(u^3) < t^{n,-}(u^3) \leq t^{n,-}(u_m)$ for sufficiently large m. From this and since $R(tu^3) \leq \lim_{m\to\infty} \inf R(tu_m)$, for any $t > 0$, it follows that for sufficiently large m there holds $s_3(u^3) \leq s_3(u_m) = 1$, and if $s_2(u_m)$ exists, $s_2(u_m) < s_2(u^3) \leq s_3(u^3)$. Hence by the weak lower semi-continuity of $\Phi(u)$ we have

$$\Phi(s_3(u^3)u^3) \leq \liminf_{m\to+\infty} \Phi(s_3(u^3)u_m) \leq \liminf_{m\to\infty} \Phi(u_m) = \Phi_3,$$

which implies that $s_3(u^3)u^3$ is a minimizer, and consequently, $s_3(u^3) = 1$ and $u_m \to u^3$ strongly in $W_0^{1,p}$. Since $\mu < \mu^n$, we have $(M^n)'(u^3) < 0$, and therefore, $R'(u^3) \leq 0$. □

Assume that $\lambda \in (-\infty, \lambda^{n,-})$. Then $\lambda < \lambda^{n,-} < \lambda^{n,-}(u^3)$, and therefore, $R'(u^3) < 0$. This by Lemma (3.2.1) and (3.2.2) implies that u^3 is a weak solution of (3.0.1). Moreover, since $R'(u^3) < 0$, we have $\Phi''(u^3) < 0$, and therefore, u^3 is a linearly unstable solution. Analysis similar to that in the proof of (1^o), Theorem (3.3.1) show that $u^3 \in C^{1,k}(\overline{\Omega})$ for $k \in (0,1)$ and $u^3 > 0$. As (3.3.1) it can by shown that $\Phi(u^3) > 0$ if $\lambda \in (-\infty, \lambda^{e,-})$, $\Phi_{\lambda^{e,-}}(u^3_{\lambda^{e,-}}) = 0$, and $\Phi(u^3) < 0$ if $\lambda \in (\lambda^{e,+}, \lambda^{n,-})$. Corollary (3.3.2) implies that the function
$(-\infty, \lambda^{n,-}) \ni \lambda \longmapsto \Phi(u^3)$is continuous and monotone decreasing.
Let us show (iv). From the monotonicity of $\Phi(u^3)$ it follows $\Phi(u^3) \to C$ as $\lambda \to -\infty$ for some $C \in (0,+\infty]$. Suppose, contrary to our claim, that $C < +\infty$. Since $u^3 \in \mathcal{N}$,

$$\frac{C}{2} < \Phi(u^3) = \frac{\gamma-p}{\gamma p}\|u^3\|^p + \mu\frac{\gamma-q}{\gamma q}\|u^3\|^q_{L^q} - \lambda\frac{\gamma-\alpha}{\gamma\alpha}\|u^3\|^\alpha_{L^\alpha} < \frac{3C}{2}. \tag{3.3.7}$$

for sufficiently large $|\lambda|$. This implies that $||u^3||^{\alpha}_{L^\alpha} \to 0$ as $\lambda \to -\infty$, and (u^3) is bounded. Thus there exists a subsequence $\lambda_j \to -\infty$ such that $u^3_{\lambda_j} \rightharpoonup \bar{u}$ in $W_0^{1,p}$ as $j \to \infty$ for some $\bar{u} \in W_0^{1,p}$. Since $||u^3||^{\alpha}_{L^\alpha} \to 0$ as $\lambda \to -\infty$, $||u^3_{\lambda_j}||^q_{L^q} \to 0$. Hence passing to the limit in $\Phi'(u^3_{\lambda_j}) = 0$ we obtain $\lim_{j\to\infty} ||u^3_{\lambda_j}||^p = 0$. this and (3.3.7) yield $0 < C/2 \leq \lim_{\lambda_j \to \infty} \Phi(u^3_{\lambda_j}) = 0$, which is a contradiction. Thus $\Phi(u^3) \to \infty$ and $||u^3|| \to +\infty$ as $\lambda \to -\infty$.
Let us show that u^3 is a ground state of (3.0.1) if $\lambda \in (-\infty, \lambda^{n,+})$. By Proposition (3.2.2) and (3.2.8), if $\lambda \in (-\infty, \lambda^{n,+})$, then for any $u \in W_0^{1,p} \setminus 0$, the fibering function $\Phi(su)$ has only critical point $s_3(u) > 0$. Hence if $\lambda \in (-\infty, \lambda^{n,+})$, then $\mathcal{RN}^3 = \mathcal{N}$ and $\inf_{u \in W_0^{1,p} \setminus 0} \max_{s>0} \Phi(su) = \Phi_3$ and we obtain the desired. This concludes the proof of Theorem (3.3.2). □

3.3.3 A ground state mountain-pass solution

To establish the existence of the third solution, we shall employ the method based on the Mountain Pass Theorem. We begin by observing that:
A weak solution $u \in W_0^1 \setminus 0$ of (3.0.1) is said to be mountain pass type if

$$\Phi(u_0) = \Phi_m := \inf_{\gamma \in P} \sup_{u \in \gamma} \Phi(u) \tag{3.3.8}$$

for the paths set $P := \{g \in C([0,1]; W_0^{1,p}) : g(0) = 0, g(1) = w_1\}$ with some $w_1 \in W_0^{1,p}$ such that $\Phi(w_1) < 0$.

Theorem 3.3.3. *Assume that $1 < q < \alpha < p < \gamma < p^*, \mu > 0, -\infty < \lambda < +\infty$. Then (3.0.1) admits a positive mountain pass type solution $u \in C^{1,k}(\overline{\Omega}), k \in (0,1)$ such that $\Phi(u) > 0$. Moreover, if $\mu \in (0, \mu^e), \lambda \in (-\infty, \lambda^{n,+})$, then:*

(i) u is a ground state of (3.0.1), i.e., $u \in \mathcal{M}$;

(ii) $\Phi(u) \to +\infty$, $||u|| \to +\infty$ as $\lambda \to -\infty$;

(iii) $\Phi(u) \to 0$, $||u|| \to 0$ as $\lambda \to \infty$.

The proof of this theorem begins by establishing several auxiliary results that will be essential for achieving an optimal argument.

Lemma 3.3.4. *Let $\mu > 0, -\infty < \lambda < +\infty$. Then (3.0.1) has a mountain pass type solution $u \in C^{1,k}(\overline{\Omega}), k \in (0,1)$ such that $u > 0$ in Ω and $\Phi(u) > 0$.*

Proof. The functional Φ satisfies the Palais-Smale condition. Indeed, suppose that $(u_n) \subset W_0^1 \setminus 0$ is a Palais-Smale sequence, i.e., $\Phi(u_n) \to c$, $D\Phi(u_n) \to 0$. By the Sobolev embedding theorem, we have

$$\begin{aligned} c + o(1)||u_m|| &= \frac{\gamma - p}{p\gamma}||u||^p + \mu\frac{\gamma - q}{q\gamma}||u_n||^q_{L^q} - \lambda\frac{\gamma - \alpha}{\alpha}||u_n||^\alpha_{L^\alpha} \\ &\geq \frac{\gamma - p}{p\gamma}||u_n||^p - |\lambda|\frac{\gamma - \alpha}{\alpha}||u_n||^\alpha, \quad \text{ as } n \to \infty \end{aligned}$$

This implies that $||u_n||$ is bounded, and hence, after choosing a subsequence if necessary, we have $u_n \rightharpoonup$ weakly in $W_0^{1,p}$, and $u_n \to u$ strongly in $L^r(\Omega), 1 \leq r \leq p^*$ to some $u \in W_0^{1,p}$. Hence the convergence $D\Phi(u_n)(u_n) \to 0$ as $n \to \infty$ implies

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u \rangle = 0.$$

Thus by S^+ property of the $p-$Laplacian operator (see [13]) it follows that $u_n \to u$ strongly in $W_0^{1,p}$, which means that Φ satisfies the Palis-Smale condition.
The functional Φ possesses a mountain pass type geometry for $\mu > 0, -\infty < \lambda < +\infty$. Indeed, for each $\mu > 0, -\infty < \lambda < +\infty$, there exists $c(\mu, \lambda) > 0$ such that $(\mu/q)s^q - (\lambda/\alpha)s^\alpha - (1/\gamma)s^\gamma \geq -c(\mu, \lambda)s^\gamma, \forall s > 0$. Therefore, by the Sobolev embedding theorem we have

$$\Phi(u) \geq \frac{1}{p}||u||^p - c(\mu, \lambda)||u||^\gamma_{L^\gamma} \geq (\frac{1}{p} - \tilde{c}(\mu, \lambda)||u||^{\gamma - p})||u||^p, \tag{3.3.9}$$

where $\tilde{c}(\mu, \lambda) > 0$ does not depend on $u \in W_0^{1,p}$. We thus can find a sufficiently small $\rho > 0$ such that $\Phi(u) > \delta$ for some $\delta > 0$ provided $||u|| = \rho$. Evidently, $\Phi(su) \to -\infty$ as $t \to +\infty$ for any $u \in W_0^{1,p} \setminus 0$, and thus, there is $w_1 \in W_0^{1,p}$, $||w_1|| > \rho$ such that $\Phi(w_1) < 0$. Since $\Phi(0) = 0$, Φ posses a mountain pass type geometry. It easily seen that the same conclusion holds if we replace the function $f(\lambda, \mu, u) := |u|^{\gamma-2}u + \lambda|u|^{\alpha-2}u - \mu|u|^{q-2}u$ by the truncation function: $f^+(\lambda, \mu, u) := f(\lambda, \mu, u)$ if $u \geq 0$, $f^+(u) := 0$, if $u < 0$. Thus, the mountain pass Theorem [2] provides us the critical point u of $\Phi(u)$ such that

$$\Phi(u) = \Phi_m := \inf_{\gamma \in P} \sup_{u \in \gamma} \Phi(u) > 0.$$

And $u \geq 0$. As in the proof of Theorem (3.3.1), it follows that $u \in C^{1,k}(\overline{\Omega})$ for $k \in (0, 1)$ and $u > 0$ in Ω. □

Proposition 3.3.1. *If $\mu \in (0,\mu^e), \lambda \in (-\infty, \lambda^{n,+})$, then u is a ground state of (3.0.1). Moreover, $\Phi(u) \to +\infty$, $||u|| \to +\infty$ as $\lambda \to -\infty$.*

Proof. Let $\mu \in (0,\mu^e), \lambda \in (-\infty, \lambda^{n,+})$ and u by a mountain pass solution. Then $u \in \mathcal{N}$, and in view of Theorem (3.3.2),

$$\Phi(u) = \inf_{g\in P} \max_{s\in[0,1]} \Phi(g(s)) \le \inf_{u\in W_0^{1,p}\setminus 0} \max_{s>0} \Phi(su) = \Phi_3 \le \Phi(u)$$

where $P := \{g \in C([0,1], W_0^{1,p}) : g(0), g(1) = w_1\}$ with $w_1 \in W_0^{1,p}$ such that $||w_1|| > \rho$ and $\Phi(w_1) < 0$. Hence $\Phi(u_3) = \Phi_3 = \Phi(u)$. Thus, for $\mu \in (0,\mu^e), \lambda \in (-\infty, \lambda^{n,+})$, any mountain pass type solution u is a ground state of (3.0.1), i.e., $u \in \mathcal{M}$. Futhermore, by (iv), (2^o), Theorem (3.3.1), it follows that $\Phi_3 = \Phi(u) \to +\infty, ||u|| \to +\infty$ as $\lambda \to -\infty$. □

Lemma 3.3.5. *Let $\mu \in (0,\mu^n)$. Then $\Phi(u) \to 0$ and $||u|| \to 0$ as $\lambda \to +\infty$.*

Proof. The proof is based on the use of the following auxiliary variational problem

$$\Phi_1 = \min\{\Phi(u) : u \in \mathcal{RN}^1\}, \tag{3.3.10}$$

where

$$\mathcal{RN}^1 := \{u \in \mathcal{N} : \lambda^{n,-}(u) \le \lambda\},$$

and $\mu \in (0,\mu^n), \lambda \in (\lambda^{n,-}, +\infty)$. □

Lemma 3.3.6. *Let $\mu \in (0,\mu^n)$ and $\lambda \in (\lambda^{n,-}, +\infty)$. There exists a minimizer $u^1 \in \mathcal{RN}^1$ of (3.3.10) such that $\Phi_1 = \Phi(u^1) > 0$.*

Proof. Since $\mu \in (0,\mu^n)$, the functional $\lambda^{n,-}(u)$ is well defined on $W_0^{1,p} \setminus 0$. This implies that $\mathcal{RN}^1 \neq \emptyset$ for $\mu \in (0,\mu^n), \lambda \in (\lambda^{n,-}, +\infty)$. By the proof of Lemma (3.3.4), there exists $\rho > 0$ such that $\inf_{\{u:||u||=\rho\}} \Phi(u) > 0$, and therefore $\Phi_1 > 0$ for $\mu \in (0,\mu^n), \lambda \in (\lambda^{n,-}, +\infty)$.
Let $(u_m)_{m=1}^{\infty}$ by a minimizing sequence of (3.3.10). The coerciveness of Φ on $\mathcal{N}$ implies that the sequence (u_m) is bounded in $W_0^{1,p}$ and thus, up to a subsequence,

$$u_m \to u^1 \text{ strongly in } L^r \text{ for } r \in (1,p^*) \text{ and weakly in } W_0^{1,p},$$

for some $u^1 \in W_0^{1,p}$. It is easily seen that if $u_m \to u^1$ strongly in $W_0^{1,p}$, then u_1 is a nonzero minimizer of (3.3.10).
To obtain a contradiction, suppose that the convergence $u_m \to u^1$ in $W_0^{1,p}$, is not strong.

Let us show that $u^1 \neq 0$. Observe $\lim_{m\to\infty} ||u_m||^p = \beta > 0$, since $\Phi_1 > 0$. Thus, if $u^1 = 0$, then $0 = \lim_{m\to+\infty} (\Phi)'(u_m) = (1/p)\beta > 0$ is a contradiction. By the weak lower-semicontinuity of $\lambda^{n,-}(u)$ we have $\lambda^{n,-}(u^1) \leq \liminf_{m\to\infty} \lambda^{n,-}(u_m) \leq \lambda$, and therefore, there exists $s_1(u^1) > 0$ such that

$$\lambda = R(s_1(u^1)u^1) < \liminf_{m\to\infty} R(s_1(u^1)u_m).$$

Hence $s_1(u^1) < s_1(u_m), m = 1, 2, \ldots$ In view of that $\Phi'(su_m) > 0$ for $s \in (0, s_1(u_m))$, this implies

$$\Phi(s_1(u^1)u^1) < \liminf_{m\to\infty} \Phi(s_1(u^1)u_m) \leq \liminf_{m\to\infty} \Phi(s_1(u_m)u_m) = \Phi_1$$

which is a contradiction since $s_1(u^1)u^1 \in \mathcal{RN}^1$. □

Proposition 3.3.2. *Let $\mu \in (0, \mu^n)$. Then $\Phi(u^1) \to 0$ as $\lambda \to +\infty$.*

Proof. Corollary 3.3.2 implies that $\Phi(u^1)$ is monotone decreasing on $(\lambda^{n,-}, +\infty)$, and therefore, $\Phi(u^1) \to \delta$ as $\lambda \to +\infty$ for some $\delta \in (0, +\infty)$. Assume by contradiction that $\delta > 0$. Then, since $\Phi'(u^1) = 0$, we have

$$\frac{\delta}{2} < \Phi(u^1) = \frac{\gamma - p}{\gamma p}||u^1||^p + \mu\frac{\gamma - q}{\gamma q}||u^1||^q_{L^q} - \lambda\frac{\gamma - \alpha}{\gamma\alpha}||u^1||^\alpha_{L^\alpha} < \frac{3\delta}{2}, \tag{3.3.11}$$

for sufficiently large λ, whence follows by the embedding $W_0^{1,p} \hookrightarrow L^\alpha(\Omega)$

$$\frac{\gamma - p}{\gamma p}||u^1|| - \lambda C\frac{\gamma - \alpha}{\gamma\alpha}||u^1||^\alpha < \frac{3\delta}{2}, \tag{3.3.12}$$

for some positive constant C. Hence (u^1) is bounded in $W_0^{1,p}$. Consequently, there exists a subsequence $u^1_{\lambda_j}$ such that $\lim_{j\to\infty} \lambda_j = +\infty$ and $u^1_{\lambda j} \rightharpoonup \overline{u}$ weakly in $W_0^{1,p}$ and $u^1_{\lambda j} \to \overline{u}$ strongly in $L^r(\Omega), 1 \leq r \leq p^*$ as $j \to \infty$ for some $\overline{u} \in W_0^{1,p}$. Observe that (3.3.11) implies $||u^1||^\alpha_{L^\alpha} \to 0$ as $\lambda \to +\infty$, which implies $\overline{u} = 0$. Passing to the limit in $\Phi(u^1_{\lambda_j}) = 0$ we obtain $\lim_{j\to+\infty} ||u^1_{\lambda_j}||^p = 0$, and consequently, (3.3.11) implies that $\lambda_j ||u^1||^\alpha_{L^\alpha} \to 0$. Hence

$$\frac{\delta}{2} \leq \Phi(u^1) = \lim_{\lambda\to+\infty} \Phi(u^1) = 0.$$

Thus $\delta = 0$ and we obtain $||u^1|| \to 0$ and $\Phi(u^1) \to 0$ as $\lambda \to +\infty$. □

Let us now conclude the proof of Lemma (3.3.5). Since $\lambda^{n,-}(u^1) \leq \lambda$, the function $\Phi(su^1)$ has a unique global maximun point $s = s^1(u^1) = 1$. Take a sufficiently large $s_0 > 1$ such that

$\Phi(s_0u^1) < 0$. Then by the above there exists a mountain pass solution u such that

$$\Phi(u) = \Phi_m := \inf_{\gamma \in P} \sup_{u \in \gamma} \Phi(u) > 0,$$

where $P := \{g \in C([0,1]; W_0^{1,p}) : g(0) = 0, g(1) = s_0u^1\}$. Note that $\tilde{g} \in P$, where $\tilde{g} = su^1$, $s \in [0, s_0]$. Hence

$$\Phi(u^1) = \sup_{s>0} \Phi(su^1) \geq \inf_{\gamma \in P} \sup_{u \in \gamma} \Phi(u) = \Phi(u),$$

for any $\mu \in (0, \mu^n)$ and $\lambda \in (\lambda^{n,-}, +\infty)$. Then by Proposition (3.3.2), $\Phi(u) \to 0$, and consequently, $||u|| \to 0$ as $\lambda \to +\infty$. □

This concludes the proof of Theorem (3.3.3).

Remark 3.3.1. *Theorems (3.3.1) and (3.3.3) yield the following result on the existence of three distinct branches of weak positive solutions of (3.0.1).*

Theorem 3.3.4. *Asume $\mu \in (0, \mu^e)$ and $\lambda \in [\lambda^{e,-}, \lambda^{n,-})$. Then (3.0.1) admits at least three distinct positive solutions: u^1, u^2, u^3 such that $\Phi(u^1) > 0$, $\Phi(u^2) < 0$, $\Phi(u^3) \leq 0$ and $\Phi''(u^2) > 0$, $\Phi''(u^3) < 0$. Futhermore, u^2 is linearly stable while u^1, u^3 are linearly unstable solutions.*

Proof. The existence of three solutions u^i, $i = 1,2,3$, for $\mu \in (0, \mu^e)$ and $\lambda \in [\lambda^{e,-}, \lambda^{n,-})$ follows from Theorems (3.3.1), (3.3.2) and (3.3.3), where we set $u^1 := u$, for $\mu \in (0, \mu^e)$, $\lambda \in [\lambda^{e,-}, \lambda^{n,-})$. They are distinct since $\Phi(u^1) > 0$ by Theorem (3.3.3), while $\Phi(u^2) < 0, \Phi(u^3) < 0$ and $\Phi''(u^2) > 0$, $\Phi''(u^3) < 0$ by Theorems (3.3.1), (3.3.2). By (3.3.1), (3.3.2), u^2 is linearly stable while u^3 is linearly unstable solutions. Consider

$$K_{\Phi_m} := \{u \in W_0^{1,p} : \Phi(u) = \Phi_m, \;\; D\Phi(u) = 0\}$$

where Φ is replaced by the truncation functional as in the proof of Lemma (3.3.4). Let us show that for $\mu \in (0, \mu^e)$ and $\lambda \in [\lambda^{e,-}, \lambda^{n,-})$, K_{Φ_m} contains a point u^1 which is a linearly unstable solutions. Indeed, by (3.3.9) it is easily seen that $0 \in W_0^{1,p}$ is a local minimizer of $\Phi(u)$ and $0 = \Phi(0) > \Phi(u^2)$. Hence, by the result of Holfer [36], Pucci, Serrin [37] it follows that the set K_{Φ_m} contains a critical point u^1 which is not local minimum of Φ, and therefore it is a linearly unstable solutions. □

Corollary 3.3.2. *The functions $\lambda \to \Phi(\tilde{u}^1)$ on $(\lambda^{n,-}, +\infty)$ for $\mu \in (0, \lambda^n)$; $\lambda \to \Phi(u^2)$ on $(\lambda^{n,+}, \lambda^{n,-})$ for $\mu \in (0, \mu^e)$, $\lambda \to \Phi(u^3)$ on $(-\infty, \infty)$ for $\mu \in (0, \mu^e)$ are continuous and monotone decreasing.*

Proposition 3.3.3. *Let $\lambda_0, \lambda_m \in (\lambda^{n,+}, \lambda^{n,-})$ ($\lambda_0, \lambda_m \in (-\infty, +\infty)$)), $m = 1, 2 \ldots$ such that $\lambda_m \to \lambda_0$ as $m \to +\infty$. Then there exist a subsequence, which we again denote by (λ_m), and a sequence $u^2_{\lambda_m} \in \mathcal{M}^2_{\lambda_m}$ ($u^2_{\lambda_m} \in \mathcal{M}^3_{\lambda_m}$) such that $u^2_{\lambda_m} \to u^2_{\lambda_0}$ strongly in $W_0^{1,p}$.*

Proof. As an example we prove the proposition in the case $i = 2$. By the above it follows that $\Phi_{\lambda_m}(v^2_{\lambda_m}) \to \Phi_{2,\lambda_0}$ as $m \to +\infty$, which easily implies that $(s^2_{\lambda_0}(v^2_{\lambda_m}) v^2_{\lambda_m})_{m=1}^{\infty}$ is a minimizing sequence of (3.3.1) for $\lambda = \lambda_0$. Then from the proof of Lemma (3.3.1) it follows that up to a subsequence, $u^2_{\lambda_m} \to u^2_{\lambda_0}$ strongly in W_0^1, for some $u^2_{\lambda_0} \in \mathcal{M}^2_{\lambda_0}$. □

Conclusion

As shown throughout this thesis, we worked with the Rayleigh quotient both in its form arising from the Fréchet derivative and in the form derived from the energy functional. From this approach, we obtained valuable information that contributes to a better understanding of the structure and behavior of various elliptic problems.

In the first place, it was established that, under certain hypotheses, the critical values of the energy functional restricted to the Nehari manifold satisfy the variational formulation of the problem. Secondly, the extremal values associated with the Rayleigh quotients mentioned above were identified, which allowed for a meaningful subdivision of the Nehari manifold.

Furthermore, we analyzed the cases in which the fiber map associated with the Rayleigh quotient derived from the Fréchet derivative exhibits a maximum, a minimum, or no critical point at all. In each situation, a key conclusion was obtained that ensures when the corresponding subsets of the Nehari manifold are empty or nonempty. In addition, through a general theorem (2.1.1), it was shown that two distinct solutions belonging to different subsets of the Nehari manifold $\mathcal{N}^+, \mathcal{N}^-$ respectively can be found for a quasilinear problem depending on one real parameter.

We also showed that, by means of the extremal values defined from the derivative of the fibering map of the Rayleigh quotient, classical hypotheses for obtaining solutions to certain PDEs can be improved or weakened, illustrating even that solutions may exist in the subset $\mathcal{N}^0$. In the last chapter, we considered the case in which the fiber map of the Rayleigh quotient derived from the Fréchet derivative has two critical points, making the problem more intricate.

Moreover, we related this behavior with that of the fiber map associated with the Rayleigh quotient derived from the energy functional, showing that their properties are closely connected.

This analysis allowed us to obtain at least three positive solutions: two obtained by minimization methods and a third one obtained via the mountain pass theorem, for a problem depending on two real parameters.

Among the research directions opened by this work are, for instance, the study of the case in which the corresponding fiber map possesses more than two critical points, the introduction of three real parameters into the problem analyzed in Chapter 3, and the investigation of vectorial systems in which multiple critical points of the associated fiber map may arise.

Appendix A

Rayleigh quotient for vectorial quotients

From this point onward, with the aim of working in the vectorial setting, we introduce the following notation, which will help us to clearly understand the subsequent developments. We denote by $W = W_1 \times W_2 ... \times W_n$ Banach Spaces product W_i and $||.||_{W_i}, i = 1, 2, ..., n$ is the norm $||.|| = ||.||_{W_1} + ||.||_{W_2} + ... + ||.||_{W_n}$. To simplify the notation, let's write

- $\dot{W} = (W_1 \setminus \{0\}) \times (W_2 \setminus \{0\}) \times \cdots \times (W_n \setminus \{0\}), \quad \dot{\mathbb{R}}^+ = \mathbb{R}^+ \setminus \{0\}$
- $t := (t_1, t_2, ..., t_n) \in \mathbb{R}^n$ *and* $\dot{\mathbb{R}}^+ = \mathbb{R}^+ \setminus 0$;
- $tu := (t_1 u_1, ..., t_n u_n)$, $\langle t, u \rangle = \sum_{i=1}^{n} t_i u_i$ pra $u \in W$, $t \in \mathbb{R}^n$;
- $1_n = (1, 1, ..., 1)^T$ é $0_n = (0, 0, ..., 0)^T$ denotes the vector $1 \times n$ en $\mathbb{R}^n$;
- $\nabla_u F(u) := (D_{u_1} F(u), \ldots, D_{u_n} F(u))$;
- $\nabla_u F(u) v := (D_{u_1} F(u)(v_1), \ldots, D_{u_n} F(u)(v_n))^T$;
- $D_u F(u)(v) := \sum_{i=1}^{n} D_{u_i} F(u)(v_i)$;
- $\nabla_t F(tu) := (\partial_{t_1} F(tu), \ldots, \partial_{t_n} F(tu))^T$;
- $\nabla_t F(tu) t := (\partial_{t_1} F(tu) t_1, \ldots, \partial_{t_n} F(tu) t_n)^T$;
- $\partial F(tu)/\partial t = \langle \nabla_t F(tu), t \rangle = \sum_{i=1}^{n} \partial_{t_i} F(tu) t_i$.

Where $D_{u_i}F(u)$ denotes the derivative in the Frechet sense with respect to $u_i \in W_i$ é $D_{u_i}F(u)(v_i)$ denotes $D_{u_i}F(u)$ applied in $v_i \in W_i$ for $i = 1, 2, \dots, n$ for Ω a bounded domain with $\Omega \subset \mathbb{R}^N$ with border $\partial\Omega$ bounded and $W := W_0^{1,p}(\Omega), 1 < p < +\infty$ sovoleb space with norm $||u||_W = \left(\int |\nabla u|^p dx \right)^{1/p}$, p^* denotes the critical exponent of sobolev space.

In this section, we shall study the following equation.

$$\Phi(u) = T(u) - \lambda G(u)$$

where $\lambda \in \mathbb{R}$, $T, G \in C^1(W \setminus \{0\}, \mathbb{R})$ Taking into account that weak solutions are defined by differentiating Φ in the Fréchet sense and setting it equal to zero, we have

$$\nabla_u \Phi(u) := \nabla_u T(u) - \lambda \nabla_u G(u) = 0, \quad u \in W \tag{A.0.1}$$

where $W = \prod_{i=1}^{n} W_i$ the product of Banach spaces W_i, $\lambda \in \mathbb{R}$, $T, G \in C^1(\dot{W}, \mathbb{R})$ é $\Phi(u) = T(u) - \lambda G(u)$ in this case $n = 1$ (A.0.1) note that this case was already studied in Chapter 1.
Thus, we consider the case $t \in \mathbb{R}^n$, which is the framework we shall work with throughout this section.
Let us define the Nehari vector manifold associated with (A.0.1):

$$\mathcal{N} = \{u \in \dot{W} : \nabla_u \Phi(u) u \equiv \nabla_t \Phi(tu)|_{t=1_n} = \bar{0}\} \tag{A.0.2}$$

where $\dot{W} = \prod_{i=1}^{n} (W_i \setminus 0)$.The problem of Nehari set is:

$$\begin{cases} \Phi(u) \to \text{critical}, \\ u \in \mathcal{N}. \end{cases} \tag{A.0.3}$$

Theorem A.0.1. *Let $\lambda \in \mathbb{R}$. Assume $\Phi \in C^1(\dot{W}, \mathbb{R})$, $F(t,u) = \nabla_t \Phi(tu)$ is a map of class C^1 on $(\dot{\mathbb{R}}^+)^n \times \dot{W}$. Suppose that $\mathcal{N} \neq \emptyset$ for all $u \in \mathcal{N}$*

$$\det \mathcal{H}(\Phi(u)) \neq 0. \tag{A.0.4}$$

Then $\mathcal{N}$ is a C^1 -manifold of codimension n, $W = T_u(\mathcal{N}) \oplus \mathbb{R}^n u$ for every $u \in \mathcal{N}$ and any solutions of (A.0.3) satisfies (A.0.1)

Proof. Let $u \in \dot{W}$ and define the following function

$$\Psi(u) := \nabla_u \Phi(u)(u) = (D_{u_1}\Phi(u)(u_1), \ldots, D_{u_n}\Phi(u)(u_n))^T .$$

Now, for each $(t,u) \in \dot{\mathbb{R}}^+ \times \dot{W}$, the function $F(t,u)$ is of class C^1. Now, fixing the point $(1_n, u)$, we have that in a neighborhood of this point the function F is continuous and has continuous derivative; hence, since $F(1_n, u) = \Psi(u)$, it follows that $\Psi(u)$ is of class C^1. Since the point is arbitrary, we conclude that Ψ is of class C^1 on $\dot{W}$. On the other hand, let us prove the following affirmations.

Affirmation 1. For $u \in \dot{W}$

$$J_u(\Psi(u))au = J_t(\Psi(ut))a|_{t=1_n}, \quad \forall a \in \mathbb{R}^n \tag{A.0.5}$$

indeed, note that $\Psi : \dot{W} \to \mathbb{R}^n$ is defined by:

$$\Psi(u) := \nabla_u \Phi(u)(u) = \begin{pmatrix} D_{u_1}\Phi(u)(u_1) \\ \vdots \\ D_{u_n}\Phi(u)(u_n) \end{pmatrix}$$

where $u = (u_1, \ldots, u_n) \in \dot{W}$. The Jacobian matrix of $\Psi(u)$ respect to u is:

$$J_u(\Psi(u)) = \begin{bmatrix} D_{u_1}\Psi_1(u) & D_{u_2}\Psi_1(u) & \cdots & D_{u_n}\Psi_1(u) \\ D_{u_1}\Psi_2(u) & D_{u_2}\Psi_2(u) & \cdots & D_{u_n}\Psi_2(u) \\ \vdots & \vdots & \ddots & \vdots \\ D_{u_1}\Psi_n(u) & D_{u_2}\Psi_n(u) & \cdots & D_{u_n}\Psi_n(u) \end{bmatrix}$$

each entry $D_{u_j}\Psi_i(u)$ is the directional derivative of Ψ_i in the direction $u_j \in W_j$, where

$$\Psi_i(u) = D_{u_i}\Phi(u)(u_i),$$

the product of this matrix by a vector $ua = (u_1a_1, u_2a_2, \ldots, u_na_n)^T$:

$$J_u(\Psi(u))au = \begin{bmatrix} \sum_{j=1}^{n} D_{u_j}\Psi_1(u)a_ju_j \\ \sum_{j=1}^{n} D_{u_j}\Psi_2(u)a_ju_j \\ \vdots \\ \sum_{j=1}^{n} D_{u_j}\Psi_n(u)a_ju_j \end{bmatrix}$$

in a similar way, the Jacobian matrix $\Psi(tu)$ respect to t is:

$$J_t(\Psi(tu)) = \begin{bmatrix} \frac{\partial}{\partial t_1}\Psi_1(tu) & \frac{\partial}{\partial t_2}\Psi_1(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_1(tu) \\ \frac{\partial}{\partial t_1}\Psi_2(tu) & \frac{\partial}{\partial t_2}\Psi_2(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_2(tu) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial t_1}\Psi_n(tu) & \frac{\partial}{\partial t_2}\Psi_n(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_n(tu) \end{bmatrix}$$

note that $\frac{\partial}{\partial t_1}\Psi_j(tu) = D_{u_1}\Psi_j(tu)(u_1)$ and the product with $a = (a_1, a_2, ..., a_n)$ is:

$$J_t(\Psi(tu))a = \begin{bmatrix} \sum_{j=1}^{n} D_{u_j}\Psi_1(tu)a_ju_j \\ \sum_{j=1}^{n} D_{u_j}\Psi_2(tu)a_ju_j \\ \vdots \\ \sum_{j=1}^{n} D_{u_j}\Psi_n(tu)a_ju_j \end{bmatrix}$$

evaluating $t = 1_n$, let us get, therefore, we conclude that:

$$J_u(\Psi(u))au = J_t(\Psi(tu))a|_{t=1_n}$$

note that for $u_0 \in \mathcal{N}$ we have $F(1, u_0) = 0$
Then, for a neighborhood of the arbitrary point $(1, u_0)$, we have the following statement.

Affirmation 3. For $u_0 \in \mathcal{N}$

$$J_t(\Psi(tu_0))|_{t=1_n} = \mathcal{H}(\Phi(u_0)) + \nabla_{u_0}\Phi(u_0)u_0 = \mathcal{H}(\Phi(u_0))$$

$$J_t(\Psi(tu)) = \begin{bmatrix} \frac{\partial}{\partial t_1}\Psi_1(tu) & \frac{\partial}{\partial t_2}\Psi_1(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_1(tu) \\ \frac{\partial}{\partial t_1}\Psi_2(tu) & \frac{\partial}{\partial t_2}\Psi_2(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_2(tu) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial t_1}\Psi_n(tu) & \frac{\partial}{\partial t_2}\Psi_n(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_n(tu) \end{bmatrix}$$

where $\Psi_i(tu) = D_{u_i}\Phi(tu)(t_iu_i) = t_iD_{u_i}\Phi(tu)(u_i)$.
for example

$$\frac{\partial\Psi(tu)}{\partial t_1} = t_1D^2_{u_1u_1}\Phi(tu)(u_1u_1) + D_{u_1}\Phi(tu)(u_1)$$

in general, it is feared

$$\frac{\partial}{\partial t_j}\Psi_i(tu) = \begin{cases} t_jD^2_{u_ju_j}\Phi(tu)u_ju_j + D_{u_j}\Phi(tu)u_j & \text{se } i = j \\ t_jD^2_{u_ju_i}\Phi(tu)u_ju_i & \text{se } i \neq j \end{cases}$$

then

$$J_t(\Psi(tu))|_{t=1_n} = \begin{bmatrix} D^2_{u_1u_1}\Phi(u)u_1u_1 & D^2_{u_1u_2}\Phi(u)u_1u_2 & \cdots & D^2_{u_1u_n}\Phi(u)u_1u_n \\ D^2_{u_2u_1}\Phi(u)u_2u_1 & D^2_{u_2u_2}\Phi(u)u_2u_2 & \cdots & D^2_{u_2u_n}\Phi(u)u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ D^2_{u_nu_1}\Phi(u)u_nu_1 & D^2_{u_nu_2}\Phi(u)u_nu_2 & \cdots & D^2_{u_nu_n}\Phi(u)u_nu_n \end{bmatrix}$$

$$+ \begin{bmatrix} D_{u_1}\Phi(u)u_1 & 0 & \cdots & 0 \\ 0 & D_{u_2}\Phi(u)u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{u_n}\Phi(u)u_n \end{bmatrix}$$

then for $u_0 \in \mathcal{N}$

$$J_t(\Psi(tu_0))|_{t=1_n} = \mathcal{H}(\Phi(u_0)) + \nabla_{u_0}\Phi(u_0)u_0 = \mathcal{H}(\Phi(u_0)).$$

On the other hand This last equality is (A.0.4), it implies that the function

$$J_t(\Psi(tu_0))\Big|_{t=1_n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is bijective.

Affirmation 4. $J_{u_0}(\Psi(u_0)) : W \longrightarrow \mathbb{R}^n$ is surjective.

Indeed, by (A.0.5) $J_{u_0}(\Psi(u_0))au_0 = J_t(\Psi(u_0t))a|_{t=1_n}$ as $J_t(\Psi(tu_0))\Big|_{t=1_n}$ is bijective in particular is surjective, then given $\overline{b} \in \mathbb{R}^n$ then $\exists a \in \mathbb{R}^n$ such that $J_t(\Psi(u_0t))a|_{t=1_n} = \overline{b}$ therefore$\exists au_0 \in W$ such that

$$J_{u_0}(\Psi(u_0))au_0 = J_t(\Psi(u_0t))a|_{t=1_n} = \overline{b}.$$

Then $\mathcal{N}$ is a C^1 -manifold of codimension n, $W = T_u(\mathcal{N}) \oplus \mathbb{R}^n u$ for every $u \in \mathcal{N}$.

Now let us show that solutions of (A.0.3) satisfy (A.0.1). Let $u \in \mathcal{N}$ be a critical point by (A.0.3). Indeed, since $W = T_u(\mathcal{N}) \oplus \mathbb{R}^n u$ then $w = v + \mu u$ for every $v \in T_u(\mathcal{N})$ and $\mu \in \mathbb{R}^n$. Using the linearity of the Fréchet derivative we have

$$\nabla_u \Phi(u)(w) = \nabla_u \Phi(u)(v) + \mu^T \nabla_u \Phi(u)(u).$$

Now, using the hypothesis that u belongs to the Nehari set we have $u \in T_u(\mathcal{N})$, hence both terms vanish. Therefore $\nabla_u \Phi(u)(w) = 0$ for all $w \in \dot{W}$. □

Definition A.0.1. *We say that the vetor NM-method is applicable in general (applicable for short) to problem (A.0.3) for a given $\lambda \in \mathbb{R}$ if condition (A.0.4) is satisfied for each $u \in \mathcal{N}$.*

Proposition A.0.1. *Let $\theta : (\dot{\mathbb{R}}^+)^n \longrightarrow (\dot{\mathbb{R}}^+)^n$ be C^1-map such that*

$$\theta(1_n), \quad \det\left(J_{\overline{\tau}}(\theta(\overline{\tau}))\right)|_{\overline{\tau}=1_n} \neq 0$$

where $J_{\overline{\tau}}(\theta(\overline{\tau}))$ is the Jacobian matrix of $\theta(\overline{\tau})$. Then

(a) $\nabla_{\overline{\tau}} \Phi(\theta(\overline{\tau})u)|_{\overline{\tau}=1} = 0$ *if and only if* $\nabla_t \Phi(tu)|_{t=1_n} = 0$*;*

(b) $\det \mathcal{H}(\Phi(\theta(\overline{\tau})u))|_{\overline{\tau}=1_n} \neq 0$ *if only if* $\det \mathcal{H}(\Phi(u)) \neq 0$.

Proof. Indeed, we have $\nabla_{\overline{\tau}} \Phi(\theta(\overline{\tau})u)|_{\overline{\tau}=1_n} = J_{\overline{\tau}}(\theta(\overline{\tau}))|_{\overline{\tau}=1_n} \nabla_t \Phi(tu)|_{t=1_n}$ and $\det \mathcal{H}(\Phi_{\lambda,u}(\theta(\overline{\tau})))|_{\overline{\tau}=1_n} = \det J_{\overline{\tau}}(\theta(1_n)) \det \mathcal{H}(\Phi(u))$. □

A.1 Nonlinear generalized Rayleigh quotient

In the sequel, we always assume:

(A_1): $\quad D_uG(u)(u) \neq 0 \quad$ forall $u \in W$

Analogously to the definition of the Rayleigh quotient from the Fréchet derivative given in Chapter 1, in this section we also introduce the corresponding definition, now taking into account that $t \in (\mathbb{R}^+)^n$. We define it as follows

$$r_u(t) := R(tu) = \frac{D_uT(tu)(tu)}{D_uG(tu)(tu)}. \quad u \in \dot{W}$$

where

$$R(u) = \frac{D_uT(u)(u)}{D_uG(u)(u)}.$$

It is the original Rayleigh quotient from the Fréchet derivative. Analogously to Chapter 1, let us now present some conditions that will allow us to carry out an appropriate study of the Rayleigh quotient in the vector case:

(A_2) : $\nabla_t T(tu)$, $\nabla_t G(tu)$ are maps of class C^1 on $(\dot{\mathbb{R}}^+)^n \times \dot{W}$

(A_3) : for every fixed $u \in \dot{W}$ and $a_n \in (\mathbb{R}^+)^n \backslash (\dot{\mathbb{R}}^+)$, there exists

$$\lim_{t \to a_n} r_u(t) = \hat{r}_u(a_n) \text{ where } \hat{r}(u) \in [-\infty, \infty].$$

Note, since (A_1), $r_u(t)$ in $(\dot{\mathbb{R}}^+)^n \times \dot{W}$ are well defined. Clearly, $T, G \in C^1(W, \mathbb{R})$ implies $R(.) \in C(\dot{W}, \mathbb{R})$ and $r_u(.) \in C((\dot{\mathbb{R}}^+)^n, \mathbb{R})$ for every $u \in \dot{W}$.
Observe, that (A_1) and (A_2) imply that $r_u(t)$ and $\nabla_t \Phi(tu)$ are maps of class C^1 on $(\dot{\mathbb{R}})^n \times \dot{W}$.
Note that (A_3) entails the existence of a continuation of the fibering Rayleigh quotient $r_u(t) := r_u(t)$ to $(\mathbb{R}^+) \times \dot{W}$ such that $r_u(a_n) := \hat{r}(a_n)$ for each $u \in \dot{W}$ and $a_n \in (\dot{\mathbb{R}^+})^n \backslash (\dot{\mathbb{R}^+})^n$.
Notice that in the scalar case of NM-method, (A_3) is represented as follows: for every fixed $u \in \dot{W}$, there exists $\lim_{s \to 0} r_u(s) = \hat{r}_u(0)$, where $|\hat{r}_u(0)| \leq \infty$.

Let $u \in \dot{W}$, $t_0 \in (\dot{\mathbb{R}}^+)^n$. If $\nabla_t r_u(t_0) = 0_n$, then t_0 is said to be a critical point of $r_u(t)$ and $\lambda = r_u(t_0)$ is said to be a critical value. We call $t_0 \in (\mathbb{R}^+)^n$ the extreme point of $r_u(t)$ if the function $r_u(t)$ attains at t_0 its local maximum or minimum on $(\mathbb{R}^+)^n$.

Proposition A.1.1. *Let $tu \in \mathcal{N}$, then*

$$\nabla_t r_u(t) = \frac{\mathcal{H}(\Phi(tu))1_n}{D_u G(tu)(tu)}. \tag{A.1.1}$$

Proof. Applying the coefficient rule and using our notations strongly, we have to

$$\nabla_t r_u(t) = \frac{\nabla_t(D_u T(tu)(tu)).D_u G(tu)(tu) - D_u T(tu)(tu).\nabla_t(D_u G(tu(tu)))}{(D_u G(tu)(tu))^2}$$

where $\Phi(tu) = T(tu) - \lambda G(tu) \longrightarrow D_u\Phi(tu)(tu) = D_u T(tu)(tu) - \lambda D_u G(tu)(tu)$ then

$$\nabla_t(D_u\Phi(tu)(tu)) = \nabla_t(D_u T(tu)(tu)) - \lambda\nabla_t(D_u G(tu)(tu)).$$

Since tu is in the Nehari manifold $D_u T(tu)(tu) = \lambda D_u G(tu)(tu)$ and $\nabla_t(D_u T(tu)(tu)) = \lambda\nabla_t(D_u G(tu)(tu))$

$$\begin{aligned}\nabla_t r_u(t) &= \frac{D_u G(tu)(tu)\,(\nabla_t D_u T(tu)(tu) - \lambda\nabla_t G(tu)(tu))}{(D_u G(tu)(tu))^2}\\ &= \frac{\nabla_t\Phi(tu)(tu)}{D_u G(tu)(tu)}.\end{aligned}$$

On the other hand, if $\Psi(tu) = \nabla_t\Phi(tu)(tu) = (D_{u_1}\Phi(tu)(t_i u_i), \ldots, D_{u_n}\Phi(tu)(t_n u_n))$

$$J_t(\Psi(tu)) = \begin{bmatrix} \frac{\partial}{\partial t_1}\Psi_1(tu) & \frac{\partial}{\partial t_2}\Psi_1(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_1(tu) \\ \frac{\partial}{\partial t_1}\Psi_2(tu) & \frac{\partial}{\partial t_2}\Psi_2(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_2(tu) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial t_1}\Psi_n(tu) & \frac{\partial}{\partial t_2}\Psi_n(tu) & \cdots & \frac{\partial}{\partial t_n}\Psi_n(tu) \end{bmatrix}$$

where $\Psi_i(tu) = D_{u_i}\Phi(tu)(t_i u_i) = t_i D_{u_i}\Phi(tu)(u_i)$.
In general, it is feared

$$\frac{\partial}{\partial t_j}\Psi_i(tu) = \begin{cases} t_j D^2_{u_j u_j}\Phi(tu)u_j u_j + D_{u_j}\Phi(tu)u_j, & \text{se } i = j, \\ t_j D^2_{u_j u_i}\Phi(tu)u_j u_i, & \text{se } i \neq j. \end{cases}$$

as well as $tu \in \mathcal{N}$

$$\frac{\partial}{\partial t_j}\Psi_i(tu) = \begin{cases} t_j D^2_{u_j u_j}\Phi(tu)u_j u_j & \text{if } i = j \\ t_j D^2_{u_j u_i}\Phi(tu)u_j u_i & \text{if } i \neq j \end{cases}$$

$$J_t(\Psi(tu)) = \begin{bmatrix} t_1 D^2_{u_1u_1}\Phi(tu)u_1u_1 & t_2 D^2_{u_2u_1}\Phi(tu)u_2u_1 & \cdots & t_n D^2_{u_nu_1}\Phi(tu)u_nu_1 \\ t_1 D^2_{u_1u_2}\Phi(tu)u_1u_2 & t_2 D^2_{u_2u_2}\Phi(tu)u_2u_2 & \cdots & t_n D^2_{u_nu_2}\Phi(tu)u_nu_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 D^2_{u_1u_n}\Phi(tu)u_1u_n & t_2 D^2_{u_2u_n}\Phi(tu)u_2u_n & \cdots & t_n D^2_{u_nu_n}\Phi(tu)u_nu_n \end{bmatrix} = \mathcal{H}(\Phi(tu)),$$

$$\mathcal{H}(\Phi(tu)) \cdot \overline{1_n} = \begin{bmatrix} \sum_{j=1}^{n} t_j D^2_{u_ju_1}\Phi(tu)u_ju_1 \\ \sum_{j=1}^{n} t_j D^2_{u_ju_2}\Phi(tu)u_ju_2 \\ \vdots \\ \sum_{j=1}^{n} t_j D^2_{u_ju_n}\Phi(tu)u_ju_n \end{bmatrix}$$

analogously for $tu \in \mathcal{N}$, $\nabla_t\Phi(tu)(tu) = \mathcal{H}(\Phi(tu)) \cdot 1_n$. □

Notice that $\lambda = r_u(t)$ for $tu \in \mathcal{N}$. Thus, if t is a critical point of $r_u(t)$ and $tu \in \mathcal{N}$, then $\det\mathcal{H}(\Phi(tu)) = 0$ with $\lambda = r_u(t)$. However, the converse assertion is not always satisfied. Proceeding from (A.1.1), we just may conclude that to have equality $\nabla_t r_u(t) = 0_n$ for $tu \in \mathcal{N}$, the condition $1_n \in Ker\mathcal{H}(\Phi(u))$ is required. Our basic assumption is the following:

(R) For $u \in \mathcal{N}$, if $det\mathcal{H}(\Phi(u)) = 0$ then $1_n \in Ker\mathcal{H}(\Phi(u))$

Lemma (A.0.1) implies the following

Corollary A.1.1. *Assume (A_1), (A_2) and (R) are satisfied. Let $\lambda \in \mathbb{R}$. Suppose that $\mathcal{N} \neq \emptyset$ and $r_u(t)$ does not have critical points in $(\dot{\mathbb{R}}^+)$ such that $tu \in \mathcal{N}$. Then $\mathcal{N}$ is a C^1- manifold of codimension n, $W = T_u(\mathcal{N}) \oplus \mathbb{R}^n u$ for every $u \in \mathcal{N}$ and any solution of (A.0.3) satisfies (A.0.1).*

Proof. Let $\lambda \in \mathbb{R}$ and $u \in \mathcal{N}$. To obtain a contradiction, suppose that $det\mathcal{H}(\Phi(u)) = 0$. Then (A.1.1) and (R) imply that the point $t = 1_n$ is a critical point for $r_u(t)$. But $1_n u \in \mathcal{N}$ and we get a contradiction. Thus $det\mathcal{H}(\Phi(u)) \neq 0$ and the proof follows from Lemma (A.0.1). □

Proposition A.1.2. *For any $u \in \dot{W}$ and $t \in (\dot{\mathbb{R}}^+)^n$ there hold:*

(a) $r_u(t) = \lambda$ *if and only* $\partial\Phi(tu)\backslash\partial t = 0$*;*

(b) if $tu \in \mathcal{N}$*, then* $\lambda = r_u(t)$*.*
Futhermore, if $\partial G(tu)\backslash\partial t > 0$ $(\partial G(tu)\backslash\partial t < 0)$ *for* $u \in \dot{W}$ *and* $t \in (\dot{\mathbb{R}}^+)^n$*, then:*

(c) $r_u(t) > \lambda$ *if only if* $\partial\Phi(tu)\backslash\partial t > 0 \quad (\partial\Phi(tu)\backslash\partial t < 0)$*;*

(d) $r_u(t) < \lambda$ *if only if* $\partial\Phi(tu)\backslash\partial t < 0 \quad (\partial\Phi(tu)\backslash\partial t > 0)$*.*

Proof. Observe that

$$r_u(t) = \frac{\partial T(tu)\backslash\partial t}{\partial G(tu)\backslash\partial t}, \quad u \in \dot{W}, t \in (\dot{\mathbb{R}^+})^n.$$

Thus, to obtain the proof it is sufficient to note that $r_u(t) := \lambda$ is nothing else but the root of the equation

$$\frac{\partial\Phi(tu)}{\partial t} \equiv \frac{\partial T(tu)}{\partial t} - \lambda\frac{\partial G(tu)}{\partial t} = 0$$

for (b) note that $\dfrac{\partial\Phi(tu)}{\partial t} = \sum_{i=1}^{n} D_{t_iu_i}\Phi(tu)t_iu_i = 0$ well, for being in the Nehari $\nabla_{tu}\Phi(tu)tu = \overline{0}$ that is to say

$$\nabla_u\Phi(tu)tu := (D_{t_1u_1}\Phi(tu)(t_1u_1), \ldots, D_{t_nu_n}\Phi(tu)(t_nu_n))^T = \overline{0}$$

for (c) and (d) it is only greater or less than zero and find the desired □

Remark A.1.1. *In view of Proposition (A.0.1), all of the above statements still hold after making a change of variable* $t = \theta(\overline{\tau})$*, where* $\theta : (\dot{\mathbb{R}}^+)^n \longrightarrow (\dot{\mathbb{R}}^+)^n$ *is a* C^1-*map such that the* $detJ_{\overline{\tau}}(\theta(\overline{\tau})) \neq 0$ *for all* $\overline{\tau} \in (\dot{\mathbb{R}^+})^n$*. Furthermore,* (R) *is satisfied if and only if the same assumption* (R) *holds after making a change of variable* $t = \theta(\overline{\tau})$

A.2 Extreme values

$$\lambda_i(u) := \inf_{t \in (\mathbb{R}^+)^n} r_u(t), \quad u \in \dot{W},$$
$$\lambda_s(u) := \sup_{t \in \dot{W}} r_u(t), \quad u \in \dot{W},$$

and we restrict our main attention to the extremal values:

$$\lambda_{ii} = \inf_{u \in \dot{W}} \lambda(u), \qquad \lambda_{ss} = \sup_{u \in \dot{W}} \Lambda(u), \tag{A.2.1}$$

$$\lambda_{si} = \sup_{u\in\dot{W}} \lambda(u), \qquad \lambda_{is} = \inf_{u\in\dot{W}} \Lambda(u). \tag{A.2.2}$$

Once again, by analogy with the analysis carried out in the previous chapter, we examine the fiber map and establish the conditions it must satisfy. We will also show how to extract valuable information from it, now taking into account that $t \in \mathbb{R}^n$.

(S) For all $u \in \dot{W}$, $r_u(t)$ does not have critical points in $(\dot{\mathbb{R}}^+)^n$ such that $t\,u \in \mathcal{N}$ except points of global minimum or maximum of $r_u(t)$ on $(\mathbb{R}^+)^n$.

Theorem A.2.1. *Assume $(A_1),(A_2)$ and (A_3) hold. Suppose $r_u(t)$ satisfies $(R),(S)$ and $\lambda_{si} < \lambda_{is}$. Then for each $\lambda \in (\lambda_{si}, \lambda_{is})$ the vector Nehari manifold method is applicable to (A.0.1) so that if $\mathcal{N} \neq \emptyset$, then $\mathcal{N}$ is a C^1-manifold of codimension n and any solution of (A.0.3) satisfies (A.0.1).*

Proof. Let $\lambda \in (\lambda_{si}, \lambda_{is})$ and $u \in \mathcal{N}$. Suppose by contradiction that $\det \mathcal{H}(\Phi(u)) = 0$. Then for (A.1.1) and (R)

$$\nabla_t r_u(1_n) = \frac{\mathcal{H}(\Phi(1_n u)).1_n}{D_u G(1_n u)(1_n u)} = \frac{\mathcal{H}(\Phi(u)).1_n}{D_u G(u)(u)} = 0$$

then $t = 1_n$ is a critical for $r_u(t)$, and by (S) the function $r_u(t)$ attains its global minimum or \ and maximum at $t = 1_n$. Assume, for instance, that this is a global minimum point. Since $\lambda > \lambda_{si}$ and $\lambda = r_u(1_n)$ for $u \in \mathcal{N}$, (A.2.2) implies

$$r_u(1_n) = \min_{t\in(\mathbb{R}^+)^n} r_u(t) = \lambda > \lambda_{si} = \sup_{u\in\dot{W}} \left(\inf_{t\in(\mathbb{R}^+)^n} r_u(t) \right) \geq \inf_{t\in(\dot{\mathbb{R}}^+)^n} r_u(t) = r_u(1_n).$$

Thus we get a contradiction and the proof follows from Theorem (A.0.1). □

In the case of enhancing condition (S) by introducing additional restrictions, one should except to recesive more precise estimations of the extreme values of NM-method. Let us consider the following special case of S.

(S_0) For any $u \in \dot{W}$ one of the following holds:

(a) $r_u(t)$ has no critical point $t \in (\dot{\mathbb{R}}^+)^n$ such that $t\,u \in \mathcal{N}$;

(b) $\nabla_t r_u(t) = 0_n$ for all $t \in (\dot{\mathbb{R}}^+)^n$.

Theorem A.2.2. *Assume $(A_1),(A_2)$ and (A_3) hold. Suppose $r_u(t)$ satisfies $(R),(S_0)$ and $\lambda_{ii} < \lambda_{is}$ $(\lambda_{si} < \lambda_{ss})$. Then for each $\lambda \in (\lambda_{ii}, \lambda_{is})$ $(\lambda \in (\lambda_{si}, \lambda_{ss}))$ the vector Nehari manifold* *method is applicable to (A.0.1) so that if $\mathcal{N} \neq \emptyset$, then $\mathcal{N}$ is a C^1-manifold of (A.0.3) satisfies (A.0.1).*

Proof. We prove the statement for the case $\lambda_{ii} < \lambda_{is}$. The proof in the case $\lambda_{si} < \lambda_{ss}$ is similar, given $u \in \mathcal{N}$. Suppose by contradiction that $\det \mathcal{H}(\Phi(u)) = 0$. Then (A.1.1) and (R)

$$\nabla_t r_u(1_n) = \frac{\mathcal{H}(\Phi(1_n u)).1_n}{D_u G(1_n u)(1_n u)} = \frac{\mathcal{H}(\Phi(u)).1_n}{D_u G(u)(u)} = 0$$

then $t = 1_n$ is a critical point of the function $r_u(t)$. Hence, (S_0) entails that the function $r_u(t)$ identically equals to the constant λ in $(\mathbb{R}^+)^n$ and attains its global minimum and maximum at any point $t \in (\mathbb{R}^+)^n$. However, the assumption $\lambda < \lambda_{is}$ yields that

$$\lambda < \lambda_{is} = \inf_{u \in \dot{W}} \left(\sup_{t \in (\mathbb{R}^+)^n} r_u(t) \right) \leq \sup_{t \in (\mathbb{R}^+)^n} r_u(t) = \max_{t \in (\mathbb{R}^+)^n} r_u(t) = r_u(1_n) \equiv R(u) = \lambda.$$

□

Thus we get a contradiction and proof follows from (A.0.1).

A.3 Extremal values for a one-parameter elliptic system

In what follows, we analyze a system in the case $n = 2$, considering two Banach spaces and a fiber map defined on the positive quadrant of $\mathbb{R}^2$. Our aim is to characterize the extremal values of the fiber maps associated with the previously introduced Rayleigh quotient and to establish the conditions under which the Nehari manifold is empty or nonempty. Thus, consider the following system:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + \alpha f |u|^{\alpha-2} u |v|^{\beta}, & \text{en } \Omega, \\ -\Delta_q v = \lambda |v|^{q-2} v + \beta f |u|^{\alpha} |v|^{\beta-2} v, & \text{en } \Omega, \\ u|_{\partial\Omega} = 0, v|_{\partial\Omega} = 0, \end{cases} \tag{A.3.1}$$

where $\lambda \in \mathbb{R}$, $1 < p < +\infty, 1 < q < +\infty$.

$$\alpha, \beta > 0, \frac{\alpha}{p} + \frac{\beta}{q} > 1, \frac{\alpha}{p^*} + \frac{\beta}{q^*} \leq 1 \tag{A.3.2}$$

We suppose

(f_1) $f \in L^d(\Omega)$, where $d \geq p^* q^*/(p^* q^* - \alpha q^* - \beta p^*)$ if $p < N$ or/and $q < N$, $\alpha/p^* + \beta/q^* < 1$; $d = +\infty$ if $p < N$, $q < N$ and $\alpha/p^* + \beta/q^* = 1; d > 1$ if $p \geq N$, $q \geq N$.

Furthermore, the function f may change the sign in Ω, i.e . problem (A.3.1) has the nonlinearity indefinite in sign. By a solutions of (A.3.1) we shall mean a weak solution $(u,v) \in W := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. Let us study (A.3.1) using the vector NM-method. Consider the corresponding Nehari manifold problem:

$$\begin{cases} \Phi(u,v) \to \text{Crítico}, \\ (u,v) \in \mathcal{N}. \end{cases} \tag{A.3.3}$$

Where

$$\Phi(u,v) = \frac{1}{p}\int (|\nabla u|^p - \lambda |u|^p)dx + \frac{1}{q}\int (|\nabla v|^q - \lambda |v|^q)dx - F(u,v)$$

then

$$\Phi(tu,sv) = \frac{t^p}{p}\int (|\nabla u|^p - \lambda |u|^p)dx + \frac{s^q}{q}\int (|\nabla v|^q - \lambda |v|^q)dx - F(tu,sv)$$

note that

$$\mathcal{N} := \left\{(u,v) \in \dot{W} : \nabla_{(u,v)}\Phi(u,v)(u,v) = \nabla_{(t,s)}\Phi(tu,sv)|_{(t,s)=(1,1)}\right\}$$

$$\mathcal{N} := \left\{ (u,v) \in \dot{W} : \begin{bmatrix} \int (|\nabla u|^p - \lambda |u|^p)dx - \alpha F(u,v) \\ \int (|\nabla v|^q - \lambda |v|^q)dx - \beta F(u,v) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \tag{A.3.4}$$

Here $F(u,v) = \int f|u|^\alpha |v|^\beta$ é $\dot{W} := (W_0^{1,p}(\Omega) \setminus \{0\} \times (W_0^{1,p}(\Omega) \setminus \{0\})$ The corresponding vector fibering Rayleigh quotient is given as follows:

$$r_{(u,v)}(t,s) := R(tu,sv) = \frac{D_{u,v}T(tu,sv)(tu,sv)}{D_{u,v}G(tu,sv)(tu,sv)}, \quad t,s > 0, u \in \dot{W},$$

$$r_{(u,v)}(t,s) = \frac{t^p \int |\nabla u|^p dx + s^q \int |\nabla v|^q dx - t^\alpha s^\beta (\alpha+\beta) F(u,v)}{t^p \int |u|^p dx + s^q \int |v|^q dx}$$

for $t,s \in \dot{\mathbb{R}}^+$ and $(u,v) \in \dot{W}$. Evidently conditions $(A_1)-(A_3)$ are satisfied.
Consider the Hessian matrix

$$\mathcal{H}(\Phi)(u,v) = \begin{bmatrix} D^2_{uu}\Phi(u,v)(h,k) & D^2_{uv}\Phi(u,v)(h,k) \\ D^2_{vu}\Phi(v,u)(h,k) & D^2_{vv}\Phi(u,v)(h,k) \end{bmatrix}$$

$$, D_u\Phi(u,v)(h) = \int (|\nabla u|^{p-2}\nabla u\nabla h - \lambda |u|^{p-2}uh)dx - \alpha \int f|u|^{\alpha-2}uh|v|^{\beta}dx,$$

$$\begin{aligned} D^2_{uu}\Phi(u,v)(h,k) = &\int [\nabla h\nabla u(p-2)|\nabla u|^{p-4}\nabla u\nabla k + |\nabla u|^{p-2}\nabla h\nabla k]dx \\ &- \lambda \int [uh|u|^{p-4}uk + |u|^{p-2}hk]dx \\ &- \int f\alpha[uh|v|^{\beta}(\alpha-2)|u|^{\alpha-4}uk + |v|^{\beta}h|u|^{\alpha-2}k]dx. \end{aligned}$$

Thus,

$$D^2_{uv}\Phi(u,v)(h,k) = -\beta\alpha \int f|u|^{\alpha-2}uh|v|^{\beta-2}vkdx$$

for $(h,k) = (u,u)$

$$D^2_{uu}\Phi(u,v)(u,v) = \int (p-1)[|\nabla u|^p - \lambda |u|^p]dx - \int \alpha(\alpha-1)f|v|^{\beta}|u|^{\alpha}dx$$

for $(h,k) = (u,v)$

$$D^2_{uv}\Phi(u,v)(h,k) = -\beta\alpha \int f|u|^{\alpha}|v|^{\beta}dx = \alpha\beta F(u,v)$$

$$D_v\Phi(u,v)(h) = \int (|\nabla v|^{q-2}\nabla v\nabla h - \lambda |v|^{q-2}vh)dx - \beta \int |u|^{\alpha}|v|^{\beta-2}vhdx$$

$$\begin{aligned} D^2_{vv}\Phi(u,v)(h,k) = &\int [\nabla h\nabla v(q-2)|\nabla v|^{q-4}\nabla v\nabla k + |\nabla v|^{q-2}\nabla h\nabla k]dx \\ &- \lambda \int [vh|v|^{q-4}vk + |v|^{q-2}hk]dx \\ &- \int \beta[vh|u|^{\alpha}(\beta-2)|v|^{\beta-4}vk + h|v|^{\beta-2}k]dx \end{aligned}$$

$$D^2_{vu}\Phi(u,v)(h,k) = -\beta\alpha \int f|u|^{\alpha-2}uh|v|^{\beta-2}vkdx$$

for $(h,k) = (v,v)$

$$D^2_{vv}\Phi(u,v)(u,v) = \int (q-1)[|\nabla v|^q - \lambda |v|^q]dx - \int \beta(\beta-1)f|u|^{\alpha}|v|^{\beta}dx$$

for $(h,k)=(v,u)$

$$D^2_{uv}\Phi(u,v)(h,k)=-\beta\alpha\int f|u|^\alpha|v|^\beta dx=\alpha\beta F(u,v)$$

then

$$\mathcal{H}(\Phi)(u,v)=\begin{bmatrix}(p-1)P(u)-\alpha(\alpha-1)F(u,v) & -\alpha\beta F(u,v)\\ -\alpha\beta F(u,v) & (q-1)Q(v)-\beta(\beta-1)F(u,v)\end{bmatrix}.$$

Here we denote

$$P:=\int|\nabla u|^p dx-\lambda\int|u|^p dx,\quad Q(v)=\int|\nabla v|^q dx-\lambda\int|v|^q dx.$$

Then, for $(u,v)\in\mathcal{N}$ we $\int|\nabla u|^p dx-\lambda\int|u|^p dx=\alpha F(u,v)$ and $\int|\nabla v|^q dx-\lambda\int|v|^q dx=\beta F(u,v)$ then

$$\mathcal{H}(\Phi)(u,v)=F(u,v)\begin{bmatrix}\alpha(p-\alpha) & -\alpha\beta\\ -\alpha\beta & \beta(q-\beta)\end{bmatrix}.$$

Proposition A.3.1. *$r_{u,v}(t,s)$ satisfies (R) and S_0.*

Proof. Observe that $\det\mathcal{H}(\Phi)(u,v)=\alpha\beta(pq-p\beta-q\alpha)F^2(u,v)$ for $(u,v)\in\mathcal{N}$. By (A.3.2), $pq-p\beta-q\alpha\neq0$. Hence $\det\mathcal{H}(\Phi)(u,v)=0$ for $(u,v)\in\mathcal{N}$ if and only if $F(u,v)=0$. However, $\mathcal{H}(\Phi)(u,v)1_2=0_2$ if $F(u,v)=0$. Thus, condition (R) holds.
Observe, for $(u,v)\in\dot{W},t>0,s>0$ we have

$$\frac{\partial}{\partial t}r_{(u,v)}(t,s)=\frac{1}{t\left(t^p\int|u|^p dx+s^q\int|v|^q dx\right)}(pP(tu)-\alpha(\alpha+\beta)F(tu,sv)),$$

$$\frac{\partial}{\partial s}r_{(u,v)}(t,s)=\frac{1}{s\left(t^p\int|u|^p dx+s^q\int|v|^q dx\right)}(qQ(sv)-\beta(\alpha+\beta)F(tu,sv)).$$

Thus, if $(t_0u,s_0v)\in\mathcal{N}_{r_{(u,v)}(t_0s_0)}$ and $\partial r_{(u,v)}(t_0,s_0)/\partial t=0$, $\partial r_{(u,v)}(t_0,s_0)/\partial s=0$ for some $t_0>0,s_0>0$, then $P(t_0u)=0,Q(s_0v)=0$ and $F(t_0u,s_0v)=0$. Hence we have successively $P(u)=0$, $Q(v)=0,F(u,v)=0$ and $\partial r_{(u,v)}(t,s)/\partial t\equiv0$, $\partial r_{(u,v)}(t,s)/\partial s\equiv0$ for all $t>0,s>0$. Thus, condition (S_0) holds. $\square$

Now, let us prove

Lemma A.3.1. *The extreme value λ_{is} of Nehari manifold (A.3.4) is expressed by :*

$$\lambda_{is} = \inf_{(u,v)\in\dot{W}} \left\{ \max\left\{ \frac{\int|\nabla u|^p dx}{\int|u|^p dx}, \frac{\int|\nabla v|^q dx}{\int|v|^q dx} \right\} : F(u,v) \geq 0 \right\}. \tag{A.3.5}$$

Proof. We claim that

$$\lambda_s = \sup_{t,s\in\dot{\mathbb{R}}^+} r_{(u,v)}(t,s) = \begin{cases} \max\left\{ \frac{\int|\nabla u|^p dx}{\int|u|^p dx}, \frac{\int|\nabla v|^q dx}{\int|v|^q dx} \right\}, & \text{if } F(u,v) \geq 0, \\ +\infty, & \text{if } F(u,v) < 0, \end{cases}$$

and

$$\lambda_i(u,v) = \inf_{t,s\in\dot{\mathbb{R}}^+} r_{(u,v)}(t,s) = \begin{cases} -\infty, & \text{if } F(u,v) > 0, \\ \min\left\{ \frac{\int|\nabla u|^p dx}{\int|u|^p dx}, \frac{\int|\nabla v|^q dx}{\int|v|^q dx} \right\}, & \text{if } F(u,v) \leq 0. \end{cases}$$

Let us show, as an example, the first equality. Assume $F(u,v) < 0$. Then setting $t = \sigma^q, s = \sigma^p$ we obtain

$$\frac{\int|\nabla u|^p dx + \int|\nabla v|^q dx - \sigma^{pq(\alpha/p+\beta/q-1)} F(u,v)}{\int|u|^p dx + \int|v|^q dx} \to \infty$$

as $\sigma \to +\infty$, since $\alpha/p + \beta/q > 1$. Consider now the case $F(u,v) \geq 0$. Without loss of generality, we can suppose that

$$\frac{\displaystyle\int|\nabla u|^p dx}{\displaystyle\int|u|^p dx} \geq \frac{\displaystyle\int|\nabla v|^q dx}{\displaystyle\int|v|^q dx}.$$

This implies that

$$\frac{\displaystyle\int|\nabla u|^p dx + \tau\int|\nabla v|^q dx}{\displaystyle\int|u|^p dx + \tau\int|v|^q dx} \leq \frac{\displaystyle\int|\nabla u|^p dx}{\displaystyle\int|u|^p dx}$$

for any $\tau \geq 0$. Since $F(u,v) \geq 0$,

$$r_{(u,v)}(t,s) = \frac{\displaystyle\int|\nabla u|^p dx + s^q t^{-p}\int|\nabla v|^q dx - t^{\alpha-p} s^{\beta}(\alpha+\beta)F(u,v)}{\displaystyle\int|u|^p dx + s^q t^{-p}\int|v|^q dx} \leq \frac{\displaystyle\int|\nabla u|^p dx}{\displaystyle\int|u|^p dx}$$

for any $s \geq 0$ and $t > 0$. Taking into account that this inequality becomes equality if $s = 0$, we get the proof of the assertion and the lemma. $\square$

Observe, that $\lambda_{si} = \sup_{(u,v)\in\dot{W}} \lambda_i(u,v) = +\infty$ if the set $\{x \in \Omega : f(x) \leq 0\}$ contais an open domain uo to asubset of Lebesgue measure zero. Consider $\lambda_{ii} = \inf_{(u,v)\in\dot{W}} \lambda_i(u,v)$. Simple analisis shows that $\mathcal{N} \neq \emptyset$ as $\lambda \in (\lambda_{ii}, \lambda_{is})$.

Lemma A.3.2. *Assume (A.3.2), (f_1) are satisfied. Then $\lambda_{ii} < \lambda_{is}$ and for $\lambda \in (\lambda_{ii}, \lambda_{is})$, the vector NM$-$ method (A.3.3) is applicable to (A.3.1) so that (A.3.4) is a C^1- manifold of codimension 2 and solution of (A.3.3) satisfies (A.3.1).*

Proof. Consider $\lambda_1^l := \min\{\lambda_{1,p}, \lambda_{1,q}\}$, $\lambda_1^u := \max\{\lambda_{1,p}, \lambda_{1,q}\}$. Clearly,

$$\lambda_1^l = \inf_{(u,v)\in\dot{W}} \min\left\{\frac{\displaystyle\int|\nabla u|^p dx}{\displaystyle\int|u|^p dx}, \frac{\displaystyle\int|\nabla v|^q dx}{\displaystyle\int|v|^q dx}\right\}$$

,

$$\lambda_1^u = \inf_{(u,v)\in\dot{W}} \max\left\{\frac{\displaystyle\int|\nabla u|^p dx}{\displaystyle\int|u|^p dx}, \frac{\displaystyle\int|\nabla v|^q dx}{\displaystyle\int|v|^q dx}\right\}$$

. Hence $\lambda_{is} \geq \lambda_1^u \geq \lambda_1^l$. Observe that

$$\lambda_{ii} = \begin{cases} -\infty, & \text{if there exists } (u,v) \in \dot{W} \text{ such that } F(u,v) > 0, \\ \lambda_1^l, & \text{if for all } (u,v) \in \dot{W},\ F(u,v) \leq 0. \end{cases}$$

Now taking into account that $\lambda_{is} = +\infty$ if $F(u,v) \leq 0$, for all $(u,v) \in \dot{W}$, we get $\lambda_{ii} < \lambda_{is}$. By Proposition (A.3.1) condition $(R), (S_0)$ are satisfied. Thus, the proof of the Lemma follows from Theorem (A.2.2). $\square$

Remark A.3.1. *Note that if $f > 0$ almost everywhere in Ω, then $-\infty = \lambda_{si} < \lambda_{ss} = +\infty$. Thus, in this case, we can apply Theorem (A.2.2) with the extreme values $\lambda_{si}, \lambda_{ss}$ that is (A.3.3) is applicable to (A.3.1) for any $\lambda \in \mathbb{R}$.*

A.3.1 Multiplicity of nonnegative solutions for a convex-concave elliptic p-Laplacian system.

In this, subsection, using the Theorem (2.1.1) proved in Chapter 2 we obtain a result on the existence of multiple sign-constant solutions for problems with a general convex-concave type nonlinearity and $p-$Laplacian, where we shall work with a vectorial system.
Consider the following of the equations:

$$\begin{cases} -\Delta_p u_1 = \lambda |u|^{q-2} u_1 + f_1(x,u) & \text{in } \Omega \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots \\ -\Delta_p u_n = \lambda |u|^{q-2} u_n + f_n(x,u) & \text{in } \Omega \\ u_i | \partial\Omega = 0 & i = 1,2,\dots n \end{cases} \tag{A.3.6}$$

where $1 < q < p < +\infty$, $f_i : \Omega \times \mathbb{R}^n \to \mathbb{R}^+, i = 1,\dots,n$ are Carathéodory functions such that $f_i(x,0_n) = 0, f_i(x,.) \in C^1(\mathbb{R}^n,\mathbb{R})$ for almost all $x \in \Omega$, with primitive $F(x,u)$ so that $f_i(x,u) = \partial F(x,u)/\partial u_i$ for almost all $x \in \Omega, u \in \mathbb{R}^n, i = 1,\dots,n$.
We will suppose that $f = (f_1,\dots,f_n)$ satisfies the following conditions:

(F_1) there exist $\gamma_1,\gamma_2 \in (p,p^*)$ with $\gamma_1 \le \gamma_2$, and there exist $g_j \in L^{\beta_j}(\Omega)$, $g_j \ge 0$, $j = 1,2$, such that: for $i = 1,\dots,n$, for all $u \in \mathbb{R}^n \setminus \{0_n\}$, and for all $s \in \mathbb{R} \setminus \{0\}$,

$$0 < s\frac{\partial}{\partial s} f_i(x,su) \le g_1(x)|su|^{\gamma_1-1} + g_2(x)|su|^{\gamma_2-1} \quad \text{a.e. in } \Omega$$

where $\beta_j > p^*/(p^*-\gamma_j)$, if $N > p$ and $\beta_j > 1$, if $N \le p$, $j = 1,2.$;

(F_2) there exist $\theta > p, K_1 > 0$ such that

$$0 < \theta F(x,u) \le \sum_{i=1}^{n} f_i(x,u)u_i, \quad \text{a.e. in } \Omega, |u| \ge K_1;$$

(F_3) for all $u \in \mathbb{R}^n \setminus 0_n$ and for almost all $x \in \Omega$

$$\rho(s) := \sum_{i=1}^{n} \frac{\frac{\partial}{\partial s} s^{1-q} f_i(x,su)u_i}{s^{p-q-1}}$$

is a monotone functions such that $\rho(s) \to +\infty$ as $s \to +\infty$.

By solutions of (A.3.6) we shall mean a weak $u \in W := (W_0^{1,p}(\Omega))^n$.
Construction of the energy functional of the system (A.3.6).
Consider the system for the i-th component

$$\begin{cases} -\Delta_p u_i = \lambda |u|^{q-2} u_i + f_i(x,u), & \text{in } \Omega, \\ u_i = 0, & \text{in } \partial\Omega, \end{cases}$$

where $u = (u_1, \dots, u_n)$ and $|u| = (\sum_j u_j^2)^{1/2}$ then we say that $u_i \in W_0^{1,p}(\Omega)$ is a weak solution of this component if

$$\int_\Omega |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi_i \, dx = \lambda \int_\Omega |u|^{q-2} u_i \, \varphi_i \, dx + \int_\Omega f_i(x,u) \, \varphi_i \, dx, \quad \forall \varphi_i \in W_0^{1,p}(\Omega).$$

The energy functional associated to u_i If there exists a potential $F_i(x,u)$ such that $\partial F_i / \partial u_i = f_i(x,u)$, then the energy functional for this component is

$$\Phi_i(u_i) = \frac{1}{p} \int_\Omega |\nabla u_i|^p \, dx - \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \int_\Omega F_i(x,u) \, dx.$$

Its Fréchet derivative in the direction φ_i is

$$D_{u_i} \Phi(u)(\varphi_i) = \langle \Phi_i'(u_i), \varphi_i \rangle = \int_\Omega |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi_i \, dx - \lambda \int_\Omega |u|^{q-2} u_i \, \varphi \, dx - \int_\Omega f_i(x,u) \, \varphi_i \, dx,$$

for $\varphi_i = u_i$

$$D_{u_i} \Phi(u)(u_i) = \int_\Omega |\nabla u_i|^p \, dx - \lambda \int_\Omega |u|^q \, dx - \int_\Omega f_i(x,u) \, u_i \, dx.$$

Then problem (A.3.6) has a variational form with the Euler- Lagrange functional

$$\Phi(u) = \frac{1}{p} \int |\nabla u|^p dx - \lambda \frac{1}{q} \int |u|^q dx - \int F(x,u) dx. \tag{A.3.7}$$

Here $\nabla u := (\nabla u_1, \dots, \nabla u_n)$ and $|\nabla u|^p = \sum_{i=1}^n |\nabla u_i|^p$. i.e

$$\Phi(u) = \frac{1}{p} \sum_{i=1}^n \int_\Omega |\nabla u_i|^p \, dx \; - \; \frac{\lambda}{q} \int_\Omega |u|^q \, dx \; - \; \int_\Omega F(x,u) \, dx.$$

As $\Phi(u) = T(u) - \lambda G(u)$ then $T(u) = \frac{1}{p} \int |\nabla u|^p dx - \int F(x,u) dx$ and $G(u) = \frac{1}{q} \int |u|^q dx$

of that $D_uT(u)(u) = \int |\nabla u|^p dx - \sum_{i=1}^{n} \int f_i(x,u)u_i dx$ and $D_uG(u)(u) = \int |u|^q dx$ then the $NG-$Rayleigh quotient

$$R(u) = \frac{\int |\nabla u|^p dx - \sum_{i=1}^{n} \int f_i(x,u)u_i dx}{\int |u|^q dx} \tag{A.3.8}$$

and the corresponding fibering map

$$r_u(s) = \frac{s^{p-q} \int |\nabla u|^p dx - s^{1-q} \sum_{i=1}^{n} \int f_i(x,su)u_i dx}{\int |u|^q dx} \tag{A.3.9}$$

for $u \in W \setminus 0_n$, $s > 0$.

Proposition A.3.2. (F_1) *implies, for all* $u \in \mathbb{R}^n \setminus 0_n$,

$$0 < f_i(x,u) \leq g_1'(x)|u|^{\gamma_1 - 1} + g_2'(x)|u|^{\gamma_2 - 1} \quad \text{for a.a. } x \in \Omega, i = 1, \dots n. \tag{A.3.10}$$

Here $g_j' = g_j/(\gamma_j - 1), j = 1,2.$

Proof. By $(F_1), s > 0$ and and for each i

$$0 < s\frac{\partial}{\partial s} f_i(x,su) \leq g_1(x)|su|^{\gamma_1 - 1} + g_2(x)|su|^{\gamma_2 - 1}$$

then

$$0 < \frac{\partial}{\partial s} f_i(x,su) \leq g_1(x)|u|^{\gamma_1 - 1}s^{\gamma_1 - 2} + g_2(x)|u|^{\gamma_2 - 1}s^{\gamma_2 - 2}.$$

Since it holds for all $s > 0$ then we can integrate for $[0,1]$ then

$$0 < \int_0^1 \frac{\partial}{\partial s} f_i(x,su)ds \leq g_1(x)|u|^{\gamma_1 - 1} \int_0^1 s^{\gamma_1 - 2} ds + g_2(x)|u|^{\gamma_2 - 1} ds \int_0^1 s^{\gamma_2 - 2} ds$$

so

$$0 < f_i(x,u) \leq \frac{g_1(x)}{\gamma_1 - 1}|u|^{\gamma_1 - 1} + \frac{g_2(x)}{\gamma_2 - 1}|u|^{\gamma_2 - 1} \quad \text{for a.a. } x \in \Omega, i = 1, \dots n.$$

□

Lemma A.3.3. *For $u \in W, j = 1,2$, then have*

$$\left|\int g_j(x)|u|^{\gamma_j}dx\right| \leq C||u||_W^{\gamma_j}\left(\int |g_j'(x)|^{p^*/(p^*-\gamma_i)}dx\right)^{(p^*-\gamma_j)/p^*} \leq \hat{C}||u||_W^{\gamma_j}||g_j'(x)||_{L^{\beta_j}} \quad \text{(A.3.11)}$$

Where $\hat{C} < \infty$.

Proof. We apply Holder's inequality to $r = \dfrac{p^*}{\gamma_j}$ and $r' = \dfrac{p^*}{p^* - \gamma}$ note thet $\dfrac{1}{r} + \dfrac{1}{r'} = 1$

$$\left|\int g_j(x)|u|^{\gamma_j}dx\right| \leq \left(\int |u|^{\gamma_j.(p^*/\gamma_j)}dx\right)^{\gamma_j/p^*}\left(\int |g_j'(x)|^{p^*/(p^*-\gamma_i)}dx\right)^{(p^*-\gamma_j)/p^*}$$
$$= ||u||_{(L^{p^*})^n}^{\gamma_j}\left(\int |g_j'(x)|^{p^*/(p^*-\gamma_i)}dx\right)^{(p^*-\gamma_j)/p^*}.$$

on the other hand $W = (W^{1,p}(\Omega)^n \hookrightarrow (L^{p^*}(\Omega))^n$ then $||u||_{L^{p^*}} \leq C||u||_W$ so we have

$$\left|\int g_j(x)|u|^{\gamma_j}dx\right| \leq C||u||_W^{\gamma_j}\left(\int |g_j'(x)|^{p^*/(p^*-\gamma_i)}dx\right)^{(p^*-\gamma_j)/p^*}$$

besides $L^{\beta_j}(\Omega) \hookrightarrow L^{p^*/p^*-\gamma_j}(\Omega)$ then

$$\left|\int g_j(x)|u|^{\gamma_j}dx\right| \leq \hat{C}||u||_W^{\gamma_j}||g_j'(x)||_{L^{\beta_j}} < \infty$$

for $j = 1,2$. □

This implies that Φ and r are well defined on W and $W \setminus 0_n$ respectively.

Here and in what follws we denote $(L^d)^n = (L^d(\Omega))^n, 1 < d < \infty$. Consider the extreme value

$$\lambda_{is} = \inf_{u \in W \setminus 0_n} \sup_{s>0} \frac{s^{p-q}\int |\nabla u|^p dx - s^{1-q}\sum_{i=1}^{n}\int f_i(x,su)u_i dx}{\int |u|^q dx}. \quad \text{(A.3.12)}$$

We prove

Theorem A.3.1. *Assume $1 < q < p < +\infty$, $f_i{:}\mathbb{R}^n \to \mathbb{R}^+$, $i = 1,\ldots,n$, are Carathéodory functions such that $f_i(x,0_n) = 0$, $f_i(x,\cdot) \in C^1(\mathbb{R}^n,\mathbb{R})$ for almost all x, and that conditions (F_1)–(F_3) hold. Then $0 < \lambda_{is}$ and for any $\lambda < \lambda_{is}$, problem (A.3.6) admits a weak solution $u_\lambda^1 \neq 0$. Furthermore, when $\lambda \in (0,\lambda_{is})$, problem (A.3.6) has a second weak solution $u_\lambda^2 \neq 0$.*

Moreover:

(a) $\displaystyle \frac{d^2}{ds^2}\Phi(su^1_\lambda)\Big|_{s=1} < 0, \quad \frac{d^2}{ds^2}\Phi(su^2_\lambda)\Big|_{s=1} > 0, \quad \Phi(u^2_\lambda) < 0;$

(b) if $\lambda \in (-\infty, 0]$, *then* u^1_λ *is a ground state of (A.3.6);*

(c) if $\lambda \in (0, \lambda_{is})$, *then* u^2_λ *is a ground state of (A.3.6).*

Proff. We will obtain the proof by applying Theorem (2.1.1) and Corollary (2.1.1). First, we verify conditions (1)–(4) of Theorem (2.1.1).

Hypothesis 1. Note that

$$\frac{d}{ds}r_u(s) = \frac{(p-q)s^{p-q-1}\int |\nabla u|^p\,dx - \int \frac{\partial}{\partial s}\left(s^{1-q}\sum_{i=1}^{n} f_i(x,su)u_i\right)dx}{\int |u|^q dx}. \tag{A.3.13}$$

Affirmation.

$$\frac{\int \frac{\partial}{\partial s}\left(s^{1-q}\sum_{i=1}^{n} f_i(x,su)u_i\right)dx}{s^{p-q-1}} \to 0$$

as $s \to 0$ for any $u \in W$.

Proof. Note that

$$\frac{\int \frac{\partial}{\partial s}\left(s^{1-q}\sum_{i=1}^{n} f_i(x,su)u_i\right)dx}{s^{p-q-1}} = \frac{\int \left[(1-q)s^{-q}\sum_{i=1}^{n} f_i(x,su)u_i + s^{1-q}\sum_{i=1}^{n}\frac{\partial}{\partial s}f_i(x,su)u_i\right]dx}{s^{p-q-1}}.$$

By Proposition (A.3.2) and $u_i \leq |u_i|$ then

$$0 < f_i(x,su)|u_i| \leq [g'_1(x)|su|^{\gamma_1-1} + g'_2(x)|su|^{\gamma_2-1}]|u_i|$$

$$\sum_{i=1}^{n} f_i(x,su)|u_i| \leq [g'_1(x)|su|^{\gamma_1-1} + g'_2(x)|su|^{\gamma_2-1}]\sum_{i=1}^{n}|u_i|$$

as $\displaystyle\sum_{i=1}^{n}|u_i|.1 \leq \left(\sum_{i=1}^{n}|u_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}1\right) = \sqrt{n}|u|$ (for Cauchy Schwarz)

$$\sum_{i=1}^{n} f_i(x,su)|u_i| \leq [s^{\gamma_1-1}g'_1(x)|u|^{\gamma_1} + s^{\gamma_2-1}g'_2(x)|u|^{\gamma_2}]\sqrt{n}.$$

Then

$$\left|\int (1-q)s^{-q}\sum_{i=1}^{n} f_i(x,su)\,u_i\,dx\right| \le \sqrt{n}|1-q|s^{-q}\left(\int [s^{\gamma_1-1}g_1'(x)|u|^{\gamma_1}+s^{\gamma_2-1}g_2'(x)|u|^{\gamma_2}]dx\right)$$

$$\le (1-q)\hat{C}\left[s^{-q+\gamma_1-1}\|u\|_W^{\gamma_1}\left(\int |g_1'(x)|^{\frac{p^*}{p^*-\gamma_1}}dx\right)^{\frac{p^*-\gamma_1}{p^*}}\right.$$

$$\left.+s^{-q+\gamma_2-1}\|u\|_W^{\gamma_2}\left(\int |g_2'(x)|^{\frac{p^*}{p^*-\gamma_2}}dx\right)^{\frac{p^*-\gamma_2}{p^*}}\right]$$

where $||u||_W = \sum_{i=1}^{n}||u_i||_{W_i}$, $\hat{C}$ is the maximum of the constants, besides $\gamma_1,\gamma_2 \in (p,p^*)$ and $q < p < \gamma_1 \le \gamma_2 \le p^*$ then when $s \to 0$

$$\frac{\int\left[(1-q)s^{1-q}\sum_{i=1}^{n} f_i(x,su)u_i\right]dx}{s^{p-q-1}} \to 0$$

on the other hand by (F_1) and $u_i \le |u_i|$ then

$$0 < s\frac{\partial}{\partial s}f_i(x,su)|u_i| \le [g_1(x)|su|^{\gamma_1-1}+g_2(x)|su|^{\gamma_2-1}]|u_i|$$

then for $s > 0$

$$\sum_{i=1}^{n} s\frac{\partial}{\partial s}f_i(x,su)|u_i| \le [g_1'(x)|su|^{\gamma_1-1}+g_2'(x)|su|^{\gamma_2-1}]\sum_{i=1}^{n}|u_i|$$

as $\sum_{i=1}^{n}|u_i|.1 \le \left(\sum_{i=1}^{n}|u_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}1\right) = \sqrt{n}|u|$ (for Cauchy Schwarz)
then

$$\sum_{i=1}^{n} s\frac{\partial}{\partial s}f_i(x,su)|u_i| \le [s^{\gamma_1-1}g_1(x)|u|^{\gamma_1}+s^{\gamma_2-1}g_2(x)|u|^{\gamma_2}]\sqrt{n}$$

besides $u_i \leq |u_i|$ then

$$\int \frac{s^{1-q}}{s} \sum_{i=1}^{n} s \frac{\partial}{\partial s} f_i(x,su)u_i dx \leq s^{-q} \left(\int [s^{\gamma_1 - 1} g_1(x)|u|^{\gamma_1} + s^{\gamma_2 - 1} g_2(x)|u|^{\gamma_2}] dx \right)$$

$$\leq \hat{C} \left[s^{-q+\gamma_1-1} \|u\|_W^{\gamma_1} \left(\int |g_1(x)|^{\frac{p^*}{p^*-\gamma_1}} dx \right)^{\frac{p^*-\gamma_1}{p^*}} + s^{-q+\gamma_2-1} \|u\|_W^{\gamma_2} \left(\int |g_2(x)|^{\frac{p^*}{p^*-\gamma_2}} dx \right)^{\frac{p^*-\gamma_2}{p^*}} \right]$$

where where $||u||_W = \sum_{i=1}^{n} ||u_i||_{W_i}$, $\hat{C}$ is the maximum of the constants, besides $\gamma_1, \gamma_2 \in (p, p^*)$ and
$q < p < \gamma_1 \leq \gamma_2 \leq p^*$ then when $s \to 0$

$$\frac{\int \sum_{i=1}^{n} \frac{\partial}{\partial s} f_i(x,su)u_i dx}{s^{p-q-1}} \to 0.$$

Therefore

$$\frac{\int \frac{\partial}{\partial s} \left(s^{1-q} \sum_{i=1}^{n} f_i(x,su)u_i \right) dx}{s^{p-q-1}} \to 0.$$

□

Affirmation. For (F_3) the equation

$$\int |\nabla u|^p \, dx - \frac{\int \frac{\partial}{\partial s} \left(s^{1-q} \sum_{i=1}^{n} f_i(x,su)u_i \right) dx}{s^{p-q-1}} = 0.$$

Proof. As $\int |\nabla u|^p dx > 0$ besides

$$\lim_{s \to \infty} \rho(s) = \infty > \rho(s) > \lim_{s \to 0} \rho(s) = 0.$$

Since ρ is continuous then by the intermediate value theorem there exists an $s' > 0$ tal que $\int |\nabla u|^p dx = \int \rho(s') dx$ and since ρ is monotonically increasing, that s' is unique.

then $s' = s_{\max}$ is a global maximum point of $r_u(s)$. □

Hypothesis 2. Suppose, by contradiction, that there exists a sequence $(v_m) \subset S$ such that $s_m := s_{\max}(v_m) \to 0$ as $m \to \infty$. In view of (A.3.13), we have

$$(p-q)s_m^{p-q-1}\|v_m\|_W^p = \int \frac{\partial}{\partial s}\left(s_m^{1-q}\sum_{i=1}^{n} f_i(x, s_m v_m)v_{m,i}\,dx\right).$$

Now, using (F_1) and (A.3.10)–(A.3.11), we obtain

$$\int \frac{\partial}{\partial s}\left(s^{1-q}\sum_{i=1}^{n} f_i(x,su)u_i\right)dx \leq \sqrt{n}|1-q|s^{-q}\left(\int [s^{\gamma_1-1}g_1'(x)|u|^{\gamma_1}+s^{\gamma_2-1}g_2'(x)|u|^{\gamma_2}]\,dx\right)$$
$$+s^{-q}\left(\int [s^{\gamma_1-1}g_1(x)|u|^{\gamma_1}+s^{\gamma_2-1}g_2(x)|u|^{\gamma_2}]\,dx\right).$$

Then

$$(p-q)s_m^{p-q-1} - c_1 s_m^{\gamma_2-q-1} - c_2 s_m^{\gamma_1-q-1} \leq 0$$
$$\Longleftrightarrow s_m^{p-q-1}[(p-q) - c_1 s_m^{\gamma_2-p} - c_2 s_m^{\gamma_1-p}] \leq 0$$

where c_1, c_2 do not depend on $s > 0$ and $m \in \mathbb{N}$. However, since $q < p < \gamma_1 \leq \gamma_2$, where $s_m \to 0^+$ $s_m^{\gamma_2-p} \to 0$ and $s_m^{\gamma_1-q} \to 0$.
Therefore, there exists $s_0 > 0$ such that for all $0 < s_m < s_0$,

$$\left[(p-q) - c_1 s_m^{\gamma_2-p} - c_2 s_m^{\gamma_1-p}\right] > \frac{p-q}{2} > 0,$$

and since $s^{p-q-1} > 0$, the whole left-hand side is strictly positive for small s. This contradicts the assumed ≤ 0 for all $s > 0$. Thus, we get (2).
Hypothesis 3. Assume that $(v_m) \subset S$ is weakly separated from 0_n in W.
Since (v_m) is bounded in W and W is a reflexive Banach space, we may assume that $v_m \rightharpoonup v_0$ weakly in W for some $v_0 \in W$. Furthermore, by the Rellich–Kondrachov theorem, $||v_m||_{(L^d)^n} \leq C||u||_W < \infty$ then $||v_m||_{(L^d)^n} < C_1$, for $m = 1, 2, \ldots 1 \leq d \leq p^*$ and $v_m \to v_0$ in $(L^d(\Omega))^n$ for $d < p^*$.
Since $(v_m) \subset S$ is weakly separated from 0_n in W, $v_0 \neq 0$, and consequently, there exists $\delta_1 > 0$ such that $||v_0||_{(L^q)^n}^q = \int |v_0|^q dx > \delta_1^q$, using the triangular inequality $||v_m||_{(L^q)^n} \geq$

$||v_0||_{(L^q)^n} - ||v_m - v_0||_{(L^q)^n} > \delta_1 - \varepsilon$ then $\int |v_m|^q dx \geq \delta_0 = \delta_1^q$ for all $m = 1, 2 \ldots.$ As

$$R(su) = \frac{s^{p-q}\int |\nabla u|^p dx - s^{1-q}\sum_{i=1}^{n}\int f_i(x,su)u_i dx}{\int |u|^q dx}.$$

Hence, by (F_1), (A.3.10) and (A.3.11), we have for any $s \in [\sigma, T], \sigma, T \in (0, \infty)$,

$$\begin{aligned}|R(sv_m)| &\leq \frac{1}{\delta_0}\left[s^{p-q}\left|\int |\nabla v_m|^p dx\right| + s^{1-q}\sum_{i=1}^{n}\int |f_i(x,sv_m)v_{m,i}|dx\right] \\ &\leq \frac{1}{\delta_0}\left[s^{p-q}||v_m||_W^p + s^{1-q}\sum_{i=1}^{n}\int |f_i(x,sv_m)v_{m,i}|dx\right]\end{aligned}$$

but

$$\int s^{1-q}\sum_{i=1}^{n} f_i(x,sv_m)\, v_{m,i}\, dx \leq s^{1-q}\left(\int [g_1'(x)s^{\gamma_1-1}|v_m|^{\gamma_1} + g_2'(x)s^{\gamma_2-1}|v_m|^{\gamma_2-1}]\, dx\right)$$

then

$$|R(sv_m)| \leq c_1 s^{p-q} + c_2 s^{\gamma_1-q} + c_3 s^{\gamma_2-q}$$

when $s \leq T$ then $\{R(sv_m)\}_{m=1}^{\infty}$ is bounded, and

$$\begin{aligned}\left|\frac{d}{ds}R(sv_m)\right| &\leq \delta_0^{-1}\left((p-q)s^{p-q-1} + (q-1)s^{-q}\sum_{i=1}^{n}\int |f_i(x,sv_m)v_{m,i}|dx + s^{1-q}\sum_{i=1}^{n}\int\left|\frac{\partial}{\partial s}f_i(x,sv_m)v_{m,i}dx\right|\right) \\ &\leq c_4 s^{p-q-1} + c_5 s^{\gamma_1-q-1} + c_6 s^{\gamma_2-q-1}\end{aligned}$$

where $c_1, c_2, \ldots, c_6$ do not depend on $s > 0$ and m and $s \in [\sigma, T]$ then $\{\frac{d}{ds}R(sv_m)\}_{m=1}^{\infty}$ is bounded Thus, we get (3).

Hypothesis 4. Observe that (A.3.10) and (A.3.11) imply

$$r_u(s) \geq \frac{s^{p-q}||u||_W^p - C_1' s^{\gamma_2-q}||u||_{(L^{\gamma_2})^n}^{\gamma_2} - C_2' s^{\gamma_1-q}||u||_{(L^{\gamma_1})^n}^{\gamma_1}}{||u||_{(L^q)^n}} \qquad \text{(A.3.24)}$$

for $s > 0$, $u \in W \setminus \{0\}$, where C_1', C_2' do not depend on $s > 0$. Suppose, by contradiction, that there exists $(s_m v_m) \subset \mathcal{N}, \lambda \in \mathbb{R}$ such that $(v_m) \subset S$, $\sigma < s_m < T$, $m = 1, \ldots,$ for some $\sigma, T \in (0, +\infty)$, and $v_m \rightharpoonup 0$ weakly in W. Then, we may assume that $v_m \to 0$ in $(L^q)^n$, $(L^{\gamma_1})^n$

and $(L^{\gamma_2})^n$.

Afirmattion. By (A.3.24), we have

$$s_m^{\gamma_1-q} \geq \frac{\sigma^{p-q}-\lambda||v_m||^q_{(L^q)^n}}{C_1'T^{\gamma_2-\gamma_1}||v_m||^{\gamma_2}_{(L^{\gamma_2})^n}+C_2'||v_m||^{\gamma_1}_{(L^{\gamma_1})^n}} \to \infty \quad \text{as } m\to\infty.$$

Proof. As $v_m \subset S \to ||v_m||_W = 1$ then by (A.3.24)

$$r_{v_m}(s_m) \geq \frac{s_m^{p-q}-C_1's_m^{\gamma_2-q}\|v_m\|^{\gamma_2}_{(L^{\gamma_2})^n}-C_2's_m^{\gamma_1-q}\|v_m\|^{\gamma_1}_{(L^{\gamma_1})^n}}{\|v_m\|_{(L^q)^n}}$$

note that as $\sigma < s_m < T \implies \sigma^{p-q} < s_m^{p-q}$ and $s_m^{\gamma_2-q} = s_m^{\gamma_1-q}s_m^{\gamma_2-\gamma_1} < s_m^{\gamma_1-q}T^{\gamma_2-\gamma_1}$ then

$$r_{v_m}(s_m) \geq \frac{\sigma^{p-q}-C_1'T^{\gamma_2-\gamma_1}s_m^{\gamma_1-q}\|v_m\|^{\gamma_2}_{(L^{\gamma_2})^n}-C_2's_m^{\gamma_1-q}\|v_m\|^{\gamma_1}_{(L^{\gamma_1})^n}}{\|v_m\|_{(L^q)^n}}$$

besides $s_mv_m \in \mathcal{N} \iff r_{v_m}(s_m) = \lambda$ then

$$\lambda \geq \frac{\sigma^{p-q}-C_1'T^{\gamma_2-\gamma_1}s_m^{\gamma_1-q}\|v_m\|^{\gamma_2}_{(L^{\gamma_2})^n}-C_2's_m^{\gamma_1-q}\|v_m\|^{\gamma_1}_{(L^{\gamma_1})^n}}{\|v_m\|_{(L^q)^n}}$$

then

$$C_1'T^{\gamma_2-\gamma_1}s_m^{\gamma_1-q}\|v_m\|^{\gamma_2}_{(L^{\gamma_2})^n}C_2's_m^{\gamma_1-q}\|v_m\|^{\gamma_1}_{(L^{\gamma_1})^n} \geq \sigma^{p-q}-\lambda\|v_m\|_{(L^q)^n}$$

therefore

$$s_m^{\gamma_1-q} \geq \frac{\sigma^{p-q}-\lambda||v_m||^q_{(L^q)^n}}{C_1'T^{\gamma_2-\gamma_1}||v_m||^{\gamma_2}_{(L^{\gamma_2})^n}+C_2'||v_m||^{\gamma_1}_{(L^{\gamma_1})^n}} \to \infty \quad \text{as } m\to\infty$$

□

but this last affirmation contradicts the assumption $s_m < T, m = 1,2,\ldots.$ Thus, (4) also holds.

It is readily seen that (F_1), (A.3.10) and (A.3.11) imply that (A.3.8) satisfies condition (2.1.13) of Corollary (2.1.1). Finally, let us show that conditions (a)–(b) of Theorem (2.1.1) are satisfied.

For $u \in \mathcal{N}$, we have

$$\Phi(u) = \Phi(u) - \frac{1}{\theta} D_u\Phi(u)(u) = \left(\frac{1}{p} - \frac{1}{\theta}\right)\int |\nabla u|^p dx - \lambda\left(\frac{1}{q} - \frac{1}{\theta}\right)\int |u|^q dx - \left[\int F(u)dx - \frac{1}{\theta}\int \sum_{i=1}^{n} f_i(x,u)u_i dx\right]$$

$$\Phi(u) = \left(\frac{\theta - p}{\theta p}\right)\int |\nabla u|^p dx - \lambda\left(\frac{\theta - q}{\theta q}\right)\int |u|^q dx - \frac{1}{\theta}\left[\int \theta F(u)dx - \sum_{i=1}^{n} f_i(x,u)u_i dx\right].$$

Hence, (F_2) and Sobolev inequalities yield

$$\Phi(u) \geq \frac{(\theta - p)}{\theta p}\|u\|_W^p - C\frac{\lambda(\theta - q)}{\theta q}\|u\|_W^q$$

for $\|u\|_W > K_1$. Since $q < p$, this implies $\Phi(u) \to +\infty$ as $\|u\|_W \to +\infty$, and thus condition (a) of Theorem (2.1.1) is satisfied.

Afirmattion. The functional $\Phi(u) = \frac{1}{p}\int |\nabla u|^p dx - \lambda\int |u|^q dx - \int F(u)dx$ is sequentially weakly lower semicontinuous on W.

Proof. Indeed, we will prove that each terminal is sequentially weakly lower semicontinuous on W.

Given $u_m \rightharpoonup u$ then $\int |u_n|^q dx \to \int |u|^q dx$ then $\int |u|^q dx \leq \liminf \int |u_m|^q dx$. Note that $u_m \to u \in (L^{\gamma_1})^n, (L^{\gamma_2})^n$ this is by the immersion $W(\Omega) \hookrightarrow (L^{\gamma_1}(\Omega))^n, (L^{\gamma_2}(\Omega))^n$ besides $\int f_i(x,u_m)u_{m,i}dx \to \int f_i(x,u)u_i dx$, indeed

$$\left|\int [f_i(x,u_m)u_{m,i} - f_i(x,u)u_i]dx\right| \leq \int |f_i(x,u_m)u_{m,i} - f_i(x,u)u_{m,i} + f_i(x,u)u_{m,i} - f_i(x,u)u_i]|dx$$
$$\leq \int |f_i(x,u_m) - f_i(x,u)||u_{m,i}|dx + \int |f_i(x,u)||u_{m,i} - u_i|dx$$

$$\int |f_i(x,u_m) - f(x,u)||u_{m,i}|dx \leq \left(\int |f_i(x,u_m) - f(x,u)|^{\gamma_1'}dx\right)^{1/\gamma_1'}\left(\int |u_{m,i}|^{\gamma_1}dx\right)^{1/\gamma_1}$$

besides $\left(\int |u_{m,i}|^{\gamma_1}dx\right)^{1/\gamma_1} \leq C||u_{m,i}||_{W_0^{1,p}} < \infty$ It is uniformly limited, for being a Caratheodory function $f_i(x,u_m) \to f_i(x,u)$ by the dominated convergence theorem $\int |f_i(x,u_m) - f(x,u)||u_{m,i}|dx \to 0$ and

$$\int |f_i(x,u)||u_{m,i}-u_i|dx \le \left(\int |f_i(x,u)|^{\gamma_1'}\right)^{1/\gamma_1'} \left(\int |u_{m,i}-u_i|^{\gamma_1} dx\right)^{1/\gamma_1}$$

then

$$\int |f_i(x,u)||u_{m,i}-u_i|dx \to 0.$$

$\int F(x,u_m)dx \to \int F(x,u)dx$. Indeed , by (F_2)

$$\left|\int F(x,u_m)-F(x,u)dx\right| \le \frac{1}{\theta}\sum_{i=1}^{n}\left|\int f_i(x,u_m)u_{m,i}-f_i(x,u)u_i dx\right| \to 0.$$

Therefore

$$\int F(x,u)\,dx, \quad \int f_i(x,u)u_i\,dx, \quad \int |u|^q\,dx$$

are weakly continuous on W, and

$$||u||_W \le \lim_{m\to\infty}\inf ||u_m||_W$$

is a weakly lower semi-continuous functional on W. The functional Φ ans R are sequentially weakly lower semicontinuous on W. □

This last result and (A.3.7) and (A.3.8) satisfy condition (b) of Theorem (2.1.1).

Note that (A.3.24), (F_1) and Sobolev's inequalities yield

$$\begin{aligned}\lambda_{is} &\ge \inf_{\|v\|_W=1}\sup_{s>0}\frac{s^{p-q}||v||_W^p - \hat{c}_1 s^{\gamma_2-q}||v||_W^{\gamma_2} - \hat{c}_2 s^{\gamma_1-q}||v||_W^{\gamma_1}}{\hat{c}_3||v||_W^q}\\ &= \max_{s>0}\{s^{p-q}-\hat{c}_1 s^{\gamma_2-q}-\hat{c}_2 s^{\gamma_1-q}\}/\hat{c}_3 > 0\end{aligned}$$

for some constants $\hat{c}_1,\hat{c}_2,\hat{c}_3 > 0$. Note that when s is close to zero, the dominant term is the one with the lowest exponent. Hence $\lambda_{is} > 0$. Thus, all assumptions of Theorem (2.1.1) and Corollary (2.1.1) are satisfied.

Denote $|u| = (|u_1|,\dots,|u_n|)$. The next corollary on the existence of sign-constant solutions follows in the standard way.

Corollary A.3.1. *Suppose the assumptions of Theorem (A.3.1) are satisfied and* $F(x,u) = F(x,|u|)$ *almost everywhere in* Ω*, for any* $u \in \mathbb{R}^n$*. Then, for* $\lambda < \lambda_{is}$*, the system* *of equations (A.3.6) admits a pair of non-trivial weak solutions* $u_\lambda^{1,+} \geq 0_n \geq u_\lambda^{1,-}$ *and for* $\lambda \in (0, \lambda_{is})$*, the system of equations (A.3.6) has a second pair of non-trivial weak solutions* $u_\lambda^{2,+} \geq 0_n \geq u_\lambda^{2,-}$*. Furthermore, assertions* (a)–(c) *of Theorem (A.3.1) are satisfied for* $u_\lambda^{1,\pm}$ *and* $u_\lambda^{2,\pm}$.

In the case $n = 1$ of problem (A.3.6), this result can be strengthened. Let us consider.

$$\begin{cases} -\Delta_p u = \lambda\,|u|^{q-2}u + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{A.3.25}$$

Consider the extreme value

$$\lambda_{is} = \inf_{v \in W \setminus 0} \sup_{s>0} \frac{s^{p-q}\displaystyle\int |\nabla u|^p dx - s^{1-q} \int f(x,su)u dx}{\displaystyle\int |u|^q dx}. \tag{A.3.26}$$

Theorem A.3.2. *Assume* $1 < q < p < +\infty, f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function such that* $f(x,.) \in C^1(\mathbb{R},\mathbb{R}), f(x,0) = 0, \partial f(x,s)/\partial s|_{s=0} = 0$ *for almost all* $x \in \Omega$ *and* $(F_1) - (F_3)$ *(whit* $n = 1$*) hold. Then* $0 < \lambda_{is}$ *and for any* $\lambda < \lambda_{is}$*, problem (A.3.25) admits a pair of non-trivial weak solutions* $u_\lambda^{1,+} \geq 0 \geq u_\lambda^{1,-}$*. Futheromre, when* $\lambda \in (0, \lambda_{is})$ *equations (A.3.25) has a second pair of non-trivial weak solutions*
$u_\lambda^{2,+} \geq 0 \geq u_\lambda^{2,-}$*. Moreover:*

(a) $d^2\Phi(su_\lambda^{1,\pm})/ds^2|_{s=1} < 0$, $d^2\Phi(su_\lambda^{2,\pm})/ds^2|_{s=1} > 0$, $\Phi(u_\lambda^{2,\pm}) < 0$;

(b) if $\lambda \in (-\infty, 0]$*, then one of the solutions* $u_\lambda^{1,+}$ *or* $u_\lambda^{1,-}$ *is a ground state of (A.3.25);*

(c) if $\lambda \in (0, \lambda_{is})$*, then one of the solutions* $u_\lambda^{2,+}$ *or* $u_\lambda^{2,-}$ *is a ground state of (A.3.25).*

Proof. In order to obtain sign-constant solutions $u_\lambda^{1,+} \geq 0 \geq u_\lambda^{1,-}$ and $u_\lambda^{2,+} \geq 0 \geq u_\lambda^{2,-}$, we truncate and reflect $f(x,u)$ as follows:

$$f^{\pm}(x,u) = \begin{cases} f(x,u), & \text{if } \pm u \geq 0, \\ -f(x,-u), & \text{if } \pm u < 0. \end{cases} \tag{A.3.27}$$

Let $F^{\pm}(x,u)$ denote the primitive of $f^{\pm}(x,u)$ and consider

$$\Phi^{\pm}(u) = \frac{1}{p}\int |\nabla u|^p dx - \lambda \frac{1}{q} \int |u|^q dx - \int F^{\pm}(x,u) dx. \tag{A.3.28}$$

Clearly, $\Phi^{\pm}(u) \in C^1(W \setminus 0, \mathbb{R})$. Futhermore, since $f(x,0) = 0, \partial f(x,s)/ds|_{s=0} = 0$, for almost all $x \in \Omega$, $\frac{\partial}{\partial s} \int F^{\pm}(x,su)dx$ is a map of class C^1 on $\dot{\mathbb{R}}^+ \times (W \setminus 0)$. As above in the proof of Theorem (A.3.1), it can be shown that all the other assumptions of Theorem (2.1.1) and Corollary (2.1.1) are also satisfied. Thus there exist weak solutions $u_\lambda^{1,\pm}, u_\lambda^{2,\pm} \in W_0^{1,p}(\Omega)$ of

$$\Delta_p u = \lambda |u|^{q-2}u + f^{\pm}(x,u)$$

for $\lambda < \lambda_{is}$ and $\lambda \in (0, \lambda_{is})$, respectively. Since $\Phi_\lambda^{\pm}(|u|) = \Phi^{\pm}(u)$ we may assume that the minimizers $u_\lambda^{1,+}, u_\lambda^{2,+}$ of $\Phi_{j,+} := \min\{\Phi^+(u) : u \in \mathcal{N}^{\pm}\}, j = 1,2$, respectively, are non-negative, whereas the minimizers $u_\lambda^{1,-}, u_\lambda^{2,-}$ of $\Phi_{j,-} := \min\{\Phi^-(u) : u \in \mathcal{N}^{\pm}\}, j = 1,2$, respectively, are non-positive. Now taking into account (A.3.27) we get that the functions $u_\lambda^{1,\pm}, u_\lambda^{2,\pm}$ in fact are weak solutions of the original problem (A.3.25). Finally, assertions $(a),(c)$ of Theorem (A.3.25) follow from Theorem (2.1.1) and Corollary (2.1.1). □

Appendix B

Classical results

In this appendix, we present several results that have been used throughout this dissertation and that have made it possible to understand and develop each of the problems addressed. Results from functional analysis, partial differential equations, and manifold theory, among other relevant topics, will be included.

B.0.1 Smooth Manifolds

Definition B.0.1. *Let $f : U(u_0) \subset W \to Y$ by a C^1-map on an open neighborhood of u_0, where X and Y are Banach spaces over $\mathbb{K}$. Then:*

(i) f is called a submersion at u_0 if and only if $f'(u_0) : W \to Y$ is surjective and the null space $N(f'(u_0))$ splits W;

(ii) f is called an inmersion at u_0 if and only if $f'(u_0) : W \to Y$ is injective and the range $R(f'(u_0))$ splits Y;

(iii) f is called a subimmersion at u_0 if and only if either- W and Y have finite dimensions (1) and rank $f'(u_0)$ is constant on some open neighborhood of u_0 or condition (1) is not satisfied and $N(f'(u_0))$ splits W, $R(f'(u_0))$ splits Y, as well as

$$f'(u_0)(N_c) = f'(u_0)(W)$$

for all u on some open neighborhood of u_0, where N_c is given in such a way that $W = N(f'(u_0)) \oplus N_c$ is fixed topological sum.

Definition B.0.2. *Suppose X and Y are manifolds and $f : X \to Y$ is of class C^r, with $r \geq 1$. A point $y \in Y$ is called a* regular value *of f if, for each $x \in f^{-1}(\{y\})$, the differential $T_x f : T_x X \to T_{f(x)} Y$ is surjective with split kernel. Let R_f denote the set of regular values of $f : X \to Y$; note that $Y \setminus f(X) \subset R_f \subset Y$.*

If, for each x in a set $S \subset X$, the map $T_x f$ is surjective with split kernel, we say that f is a submersion on S*. Thus, $y \in R_f$ if and only if f is a submersion on $f^{-1}(\{y\})$. If $T_x f$ is not surjective, the point $x \in X$ is called a* critical point*, and $y = f(x) \in Y$ is called a* critical value *of f.*

Theorem B.0.1. *Let $f : X \to Y$ be a C^∞ map and let $y \in R_f$. Then the level set*

$$f^{-1}(y) = \{x \in M \mid f(x) = y\}$$

is a closed submanifold of X, with tangent space at each point $x \in f^{-1}(y)$ given by

$$T_x f^{-1}(y) = \ker T_x f.$$

Proof. See [1] Submersion Theorem, Theorem 3.5.4. □

Remark B.0.1. *If $\varphi : X^x \longrightarrow Y^y$ is an immersion, then $x \leq y$; the difference $y - x$ is called the codimension of the immersion φ.*

B.0.2 Fréchet Differentiability

Definition B.0.3. *Let W and Y be normed spaces over $\mathbb{R}$, let $U \subseteq W$ be an open set, and let $J : U \to Y$ be a map. We say that J is **Fréchet differentiable** at $u_0 \in U$ if there exists a bounded linear operator $A_{u_0} \in \mathscr{L}(W, Y)$ such that:*

$$\lim_{\|h\|_W \to 0} \frac{\|J(u_0 + h) - J(u_0) - A_{u_0}(h)\|_Y}{\|h\|_W} = 0.$$

Definition B.0.4. *Let W and Y be normed spaces over $\mathbb{R}$, let $U \subseteq W$ be an open set, and let $J : U \to Y$ be a map. We say that J is **Gâteaux differentiable** at $u_0 \in U$ if there exists a bounded linear operator $A_{u_0} \in \mathscr{L}(W, Y)$ such that:*

$$\lim_{\lambda \to 0} \frac{J(u_0 + \lambda v) - J(u_0)}{\lambda} = A_{u_0}(v), \quad \text{for all } v \in W.$$

Proposition B.0.1. *Let $J : U \subseteq W \to \mathbb{R}$ be a map that is Gâteaux differentiable on U, and suppose that the Gâteaux derivative $J'_G : U \to \mathscr{L}(W, \mathbb{R})$ is continuous. Then J is Fréchet differentiable on U, and the Fréchet derivative coincides with the Gâteaux derivative, i.e.,*

$$J'_F = J'_G.$$

Proposition B.0.2. *If f possesses the partial derivate with respect to u and v in a neighbourhood $\mathcal{N}$ of (u, v) and the maps $u :\to \partial_u f$ and $v :\to \partial_v f$ are continuous in $\mathcal{N}$, them f is differentiable at (u, v) and*

$$Df(u,v)[h,k] = \partial_u f(u,v)[h] + \partial_v f(u,v)[k].$$

Proof. See [2] proposition 1.2. □

Remark B.0.2. *Let $f : W \times Y \longrightarrow Z$, and consider the map $f_v : u \longrightarrow f(u,v)$, respectively $f_u : v \longrightarrow f(u,v)$. The partial derivate of f with respect to u, rerspectively v, at $(u,v) \in W \times Y$ is defined by $\partial_u f(u,v) = Df_v(u)$, respectively $\partial_v f(u,v) = Df_u(v)$. In particular particular, $\partial_u f(u,v) \in L(W,Z)$ and $\partial_v f(u,v) \in L(Y,Z)$. It is easy to see that if $f : W \times Y \longrightarrow Z$ é differentiable at (u,v), them f is partially differentiable and $\partial_u f(u,v)[h] = Df_v(u)[h] = Df(u,v)[h,0]$, respectively $\partial_v f(u,v)[k] = Df_u(v)[k] = Df(u,v)[0,k]$. Furthermore, the following result holds.*

The Implicit Function Theorem

We want to solve the Operator equation

$$F(u,v) = 0 \tag{B.0.1}$$

in a neighborhood of the point (u_0, v_0), where we assume that

$$F(u_0, v_0) = 0. \tag{B.0.2}$$

In particular, we are interested in a *locally unique* solution. Condition (B.0.3) is decisive. Set $U := \{u \in X : ||u - u_0|| < \rho\}$

Theorem B.0.2. *Let W, Y, and Z de Banach space over $\mathbb{K}$, and let*

$$F : U(u_0, v_0) \subseteq W \times Y \to Z$$

be a C^n-map on an open neighborhood of the point (u_0, v_0) such that (B.0.1) holds and $1 \leq n \leq \infty$. Suppose that the operator

$$F_v(u_0, v_0) : Y \to Z \quad \textit{is bijective.} \tag{B.0.3}$$

Then the following statements hold true:

(i) There exists numbers $r > 0$ and $\rho > 0$ such that, for each given $u \in U$, the original equation (B.0.1) has a unique solution $v \in Y$ with $||v - v_0|| \leq r$. Denote this solution by $v(u)$;

(ii) The function $u \to v(u)$ is C^n on U. In particular,

$$v'(u) = -F_v(u, v(u))^{-1} F_u(u, v(u)) \quad \textit{for all} \quad u \in U. \tag{B.0.4}$$

Proof. See Theorem 4.E. [33]. □

Applications to the Lagrange Multiplier Rule

Let us consider the minimum problem

$$f(u_0) = \min! \tag{B.0.5}$$

along with the side condition

$$G(u_0) = 0. \tag{B.0.6}$$

Our goal is to justify the necessary solvability condition

$$f'(u_0) + \lambda G'(u_0) = 0 \tag{B.0.7}$$

where λ is called a *Lagrange multiplier*

Proposition B.0.3. *Let $f : U(u_0) \subseteq W \to \mathbb{R}$ and $G : U(u_0) \subseteq W \to \mathbb{R}$ be C^1 on an open neighborhood of u, where W is Banach space. Suppose that u is asolution of (B.0.5), (B.0.6), where*

$$G'(u) : W \to \mathbb{R} \quad \textit{is surjective}.$$

Then there exists a functional $\lambda \in \mathbb{R}$ such that (B.0.7) hold true.

Proof. See Proposition 4.1 [33]. □

A smooth map

$$f : X \to Y$$

between Banach spaces is called Freholm (see [26]) if the differential $Df(x) : X \to Y$ is a Fredholm operator for every $x \in X$. Since the Fredholm index is invariant under small perturbations the index of $Df(x)$ is independent of the choice of x. It will by denoted by index(f). For any smooth map $f : X \to Y$, Fredholm or not, a vector $y \in Y$ is called a regular value of f if $Df(x) : X \to Y$ is onto and has a right inverse for every $x \in f^{-1}(y)$. The implicit function theorem asserts that $f^{-1}(y)$ is a smooth manifold for every regular value of f. Moreover, if f is a Fredholm map then the dimension of $f^{-1}(y)$ is finite and agrees with the Fredholm index of f.

Theorem B.0.3. *Let W and Y be Banach spaces, $U \subset W$ be an open set, and ℓ be a positive integer. If $f : U \to Y$ is of class C^ℓ and y is a regular value of f then*

$$\mathcal{M} := f^{-1}(y) \subset W$$

is a C^ℓ Banach manifold and

$$T_x\mathcal{M} = \ker Df(x)$$

for every $x \in M$. Hence, if f is a Fredholm map, M is finite dimensional and

$$\dim \mathcal{M} = \text{index}(f).$$

Proof. See Theorem A.3.3 [26]. □

Theorem B.0.4 (Characterization of codimension–one subspaces in Banach spaces)**.** *Let W be a real Banach space and let $M \subset W$ be a C^1 submanifold. For a point $u \in M$, the following statements are equivalent:*

1.

$$\dim(W/T_u(M)) = 1.$$

2. *There exists a nonzero continuous linear functional*

$$L \in W^* \setminus \{0\}$$

such that

$$T_u(M) = \ker L.$$

In particular, a closed linear subspace $Y \subset W$ has codimension 1 if and only if there exists a nonzero functional $L \in W^*$ such that

$$Y = \ker L.$$

(See [38], Chapter I, Section 2)

Theorem B.0.5 (Rellich-Kondrachov Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, and let $1 \le p \le \infty$. Then the following embeddings are compact:*

1. *$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q < p^*$ if $p \le n$;*
2. *$W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ if $p > n$,*

where

$$p^* = \begin{cases} \dfrac{np}{n-p}, & \text{if } p < n, \\ \infty, & \text{if } p = n. \end{cases}$$

Proof. [15] □

Theorem B.0.6 (Lebesgue's Dominated Convergence Theorem)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy:*

(a) $f_n(x) \to f(x)$ *a.e* *on* Ω;

(b) there is a function $g \in L^1$ such that for all n, $|f_n(x)| \le g(x)$ a.e on Ω.
Then $f \in L^1$ and $||f_n - f||_1 \to 0$.

Proof. See [6] □

Theorem B.0.7. *Let $\Omega \subset \mathbb{R}^N$ be an open domain, $1 < p < \infty$, and let $(u_n) \subset W^{1,p}(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega).$$

Then, the following lower semicontinuity property holds:

$$\int_\Omega |\nabla u|^p \, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx.$$

Proof. [6] Teorema 8.4. □

Theorem B.0.8. *Let $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_m\}_{m\in\mathbb{N}}$ be two sequences such that*

$$a_m \ge 0, \qquad b_m > 0 \quad \text{for all } m \in \mathbb{N}.$$

Then the following inequality holds:

$$\frac{\liminf_{m\to\infty} a_m}{\limsup_{m\to\infty} b_m} \le \liminf_{m\to\infty} \frac{a_m}{b_m}.$$

Proof. For every m we have

$$\frac{a_m}{b_m} \ \ge \ \frac{\inf_{k\ge m} a_k}{\sup_{k\ge m} b_k}.$$

Taking the $\liminf$ on both sides gives

$$\liminf_{m\to\infty} \frac{a_m}{b_m} \ \ge \ \liminf_{m\to\infty} \frac{\inf_{k\ge m} a_k}{\sup_{k\ge m} b_k} = \frac{\liminf_{m\to\infty} a_m}{\limsup_{m\to\infty} b_m},$$

which proves the claim. □

Theorem B.0.9. *Let $\Omega \subset \mathbb{R}^n$ with $\mu(\Omega) < \infty$. Suppose $\{f_n\} \subset L^1(\Omega)$ and $f \in L^1(\Omega)$. Then:*

(a) If $f_n \to f$ almost everywhere in Ω, and if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $E \subset \Omega$ with $\mu(E) < \delta$ one has

$$\int_E |f_n| \, d\mu < \varepsilon, \quad \forall n \in \mathbb{N},$$

then $f_n \to f$ in $L^1(\Omega)$.

(b) If $f_n \to f$ in $L^1(\Omega)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $E \subset \Omega$ with $\mu(E) < \delta$ one has

$$\int_E |f_n| \, d\mu < \varepsilon, \quad \forall n \in \mathbb{N},$$

and, up to subsequences, $f_n \to f$ *almost everywhere in* Ω.

Proof. See [5]. □

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