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Fractional-laplacian diffusion in logistic parabolic problem with harvesting term

Problema logístico parabólico com laplaciano fracionário e termo de colheita

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Resumo

Este trabalho apresenta um estudo sobre problemas parabólicos logísticos com coeficientes de colheita regidos pelo operador laplaciano fracionário. São introduzidos os espaços de Sobolev-Bochner bem como os espaços de Sobolev fracionários com o objetivo de estabelecer os espaços funcionais mais adequado para o estudo dessas equações. Analisamos o problema de existência e unicidade de soluções para uma classe desses problemas em domínios limitados de $\mathbb{R}^N$, sob condição de fronteira de Dirichlet. Para esse fim, empregamos um método de comparação e o método de iteração monótona de modo a construir um par ordenado de sub e supersolução a partir dos problemas elípticos associados.

Abstract

This work presents a study of a logistic parabolic problem with harvesting term, governed by a fractional Laplacian operator. The Sobolev-Bochner spaces as well as fractional Sobolev spaces are introduced with the aim of establishing the most suitable functional framework for the analysis of such problems. We investigate the existence and uniqueness of solutions for a class of these problems in bounded domains of $\mathbb{R}^N$ under Dirichlet boundary conditions. To this end, we employ a comparison method and the monotone iteration technique in order to construct an ordered pair of sub- and supersolutions, making use of the associated elliptic problems.

Contents

Introduction

In this dissertation, we present a parabolic mathematical model governed by partial differential equations arising from population dynamics, based on a logistic perturbation of fractional diffusion with a harvesting term.

Population dynamics is the field that studies variations in the size of certain populations over time. It is currently a central pillar of mathematical ecology theory. This area began to be developed in the mid-eighteenth and nineteenth centuries, with several prominent figures, among them *Thomas Robert Malthus, Pierre François Verhulst, Alfred James Lotka, Vito Volterra*, among others. The works of these authors, individually and collectively, represented a major advance in the mathematical modeling of population dynamics and logistic models. In 1798, Malthus published his work *An Essay on the Principle of Population*, which may be regarded as the starting point for the systematic study of population dynamics. In this work, *Malthus* introduced the ***Malthusian Model*** to describe population growth. However, this model did not accurately reflect what was observed experimentally. In 1838, *Verhulst* introduced the ***Logistic Model***, the first realistic mathematical model for population dynamics, since it made it possible to incorporate ecological mechanisms such as interspecific competition, resource density etc. It is also worth noting that the ***Lotka–Volterra*** model, developed between 1925–1926, were of great importance, as this system of equations allowed for the description of how prey and predator populations oscillate over time.

With respect to diffusion processes, the nineteenth century was the most prominent one, in which we may highlight the great importance of the work of the French physicist–mathematician *Jean Baptiste Joseph Fourier*, in particular his studies on the behavior of heat in solids and his heat diffusion equations. Such works consolidated the Laplacian operator as the fundamental operator in the modeling of diffusion phenomena. However, just as the ***Malthusian Model*** did not accurately describe real ecological models, the Laplacian operator was not suitable for modeling natural processes of anomalous nature. Much time passed, and only in the twentieth century, with the advent of ***Lévy Processes***, was it realized that many natural processes are governed by a nonlocal dynamics and exhibit irregular behaviors—features that could not be captured by simpler mathematical models. These processes opened the way for the introduction of *fractional operators* in the modeling of diffusion phenomena. Reflections on these new perspectives led to viewing the *Fractional Laplacian Operator* as the most natural generalization of the *Classical Laplacian Operator*, due to its nonlocal behavior. This allowed mathematical models of diffusion dynamics to incorporate long-range interactions and nonlocal effects observed in many anomalous diffusion processes. In this way, fractional diffusion became the mathematical analogue for processes that depend on global long-range interactions, interactions that are present in many advanced ecological processes.

The transition between the concepts of classical dispersion and nonlocal dispersion provided a fertile setting for the refinement of logistic models. It is well known that in real systems, the behavior of populations is not solely one of growth; there are also effects that directly induce a decrease in population size. These effects are known as *harvesting* and are directly related to external agents that reduce the number of individuals in a given population. Thus, incorporating such a term into the modeling is essential for studying population dynamics when these populations are directly affected by external influences.

In this context, we consider $\Omega \subset \mathbb{R}^N$ a bounded domain with boundary $\partial\Omega$ of class $C^{1,1}(\Omega)$ if $N \geq 2$, or an open bounded interval if $N = 1$. Fix $T > 0$ and define $Q_T := \Omega \times (0,T)$. Given $s \in (0,1)$, the following parabolic problem

$$\begin{cases} u_t + (-\Delta)^s u = f(\lambda,x,t,u), & \text{in } Q_T, \\ \qquad u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \qquad u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{0.0.1}$$

is the most general model used to describe population dynamics based on a general nonlinear reaction in fractional diffusion with a harvesting term. The term u_t denotes the derivative $\dfrac{\partial u}{\partial t}$ with respect to time, $u = u(x,t)$ is the population density of a given species in the region Ω, $k \geq 0$ is a constant, and $(-\Delta)^s$ is the *fractional Laplacian*, in the variable x, of the function $u(x,t)$, given by

$$(-\Delta)^s u(x,t) := C_{N,s}\; P.V. \int_{\mathbb{R}^N} \frac{u(x,t)-u(y,t)}{|x-y|^{N+2s}}\, dy,$$

where $C_{N,s} > 0$ is a normalization constant, and *P.V.* denotes the Cauchy principal value of the possibly singular integral.

Following [7], we study a particular case of (0.0.2) given by the following parabolic problem

$$\begin{cases} u_t + (-\Delta)^s u = \lambda\,[a(x)u - bu^2 - h(x)], & \text{in } Q_T, \\ \qquad u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \qquad u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{0.0.2}$$

where $\lambda > 0$ is the dispersal rate (or the inverse of density), a is the resource term, b is the interaction coefficient, and h is the harvesting rate, assumed to have constant yield. Moreover, we assume that the following hypotheses hold:

$$\begin{cases} 0 \le a \in L^\infty(\Omega), \text{ with } a > l \text{ on } S_a := \overline{\{x \in \Omega \mid a(x) > 0\}}, \\ b > 0 \text{ a constant}, \\ 0 \le h \in L^\infty(\Omega). \end{cases} \tag{H}$$

where S_a is the compact support of a.

In order to obtain results concerning existence and uniqueness of solutions to (0.0.2), we introduce some auxiliary elliptic problems. Following [6], given a as in (H), it can be shown that for the weighted eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi = \lambda\, a(x)\, \varphi, & \text{in } \Omega, \\ \qquad \varphi = 0, & \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases} \tag{0.0.3}$$

there exists a principal eigenpair $(\lambda_{1,a},\, \varphi_{1,a})$ with $\lambda_{1,a} > 0$ and $0 < \varphi_{1,a} \in H_0^s(\Omega)$ in Ω, satisfying $\|\varphi_{1,a}\|_\infty = 1$. Moreover, there exist positive constants C_1, C_2 such that

$$C_1\, \delta^s \le \varphi_{1,a} \le C_2\, \delta^s,$$

a.e. in Ω, where $\delta(x) = d(x, \partial\Omega)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$. See [6], Proposition 3.1.

The second auxiliary elliptic problem we introduce is defined as

$$\begin{cases} (-\Delta)^s u = 1, & \text{in } \Omega, \\ \qquad u = 0, & \text{in } (\mathbb{R}^N \setminus \Omega). \end{cases} \tag{0.0.4}$$

It is known that (0.0.4) has a unique positive solution $e \in H_0^s(\Omega)$, and there exist positive constants C_3, C_4 such that

$$C_3\,\delta^s \le e \le C_4\,\delta^s,$$

a.e. in Ω. See [17], Theorem 1.2, and [16], Lemma 7.3.

We aim to establish the following main result:

Theorem 0.0.1. *Let a, b and h satisfy hypothesis (H) above. For every $\lambda > \lambda_{1,a}$, there exists a positive function $h^* \in L^\infty(\Omega)$ and $\underline{u}_0 \in L^\infty(\Omega)$ such that for all h satisfying*

$$\operatorname*{ess\,sup}_{x\in\Omega} \frac{h}{h^*} < 1$$

and for all $u_0 \in L^\infty(\Omega)$ satisfying

$$\underline{u}_0 \le u_0 \le Ke$$

for some constant $K > 0$, the problem (0.0.2) has a positive solution $u \in L^\infty(\Omega)$. Moreover, it is unique in the class of essentially bounded solutions of (0.0.2).

The proof of this result will be presented in Chapter 5, Section 5.2. Our interest is to obtain existence and uniqueness results for *finite-energy solutions* to the parabolic problem with harvesting term (0.0.2). The definition of a *finite-energy solution* appears in Chapter 4, Definition 4.1.2.

To obtain such results of existence and uniqueness for problem (0.0.2), we employ the method of sub- and supersolutions together with the method of monotone iterations. In this regard, we investigate the existence and uniqueness of a solution to the following parabolic problem:

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,u), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{P_1}$$

where $k \ge 0$ is a real parameter, and $g : Q_T \to \mathbb{R}$ is a Carathéodory function, that is

(i) $u \mapsto g(x,t,u)$ is continuous for a.e. $(x,t) \in Q_T$, and

(ii) $(x,t) \mapsto g(x,t,u)$ is measurable for each $u \in \mathbb{R}$.

Moreover, we assume the existence of a function $\kappa \in L^2(\Omega)$ and a continuous increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ such that:

(H_1) $|g(x,t,u)| \le \kappa(x,t)\,(1 + \alpha(|u|))$ for all $u \in \mathbb{R}$ and a.e. $(x,t) \in Q_T$;

(H_2) g is monotone increasing in the third variable; that is, for every $v,w \in \mathbb{R}$,

$$v \leq w \implies g(x,t,v) \leq g(x,t,w).$$

The analysis of this problem will be carried out in Chapter 4, where we prove, following [7], the following theorem:

Theorem 0.0.2. *Let $u_0 \in L^\infty(Q_T)$ and suppose that hypotheses (H_1) and (H_2) hold. Additionally, suppose the existence of $\underline{u}$, $\overline{u} \in L^\infty(Q_T)$, ordered sub- and supersolutions, respectively, a.e. in Q_T of the problem*

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,u), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{0.0.5}$$

Then, there exists a solution u of (0.0.5) such that $\underline{u} \leq u \leq \overline{u}$ a.e. in Q_T. Moreover, if g is locally Lipschitz in the third variable, then (0.0.5) possesses a unique solution in $L^\infty(Q_T)$.

We prove Theorem 0.0.1 by means of Theorem 0.0.2. To this end, we construct a pair of sub- and supersolutions for the logistic parabolic problem (0.0.2) by using the elliptic problem associated with (0.0.2):

$$\begin{cases} (-\Delta)^s u(x) = \lambda \left[a(x)u - bu^2 - h(x)\right], & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{0.0.6}$$

The construction of a pair of sub- and supersolutions for problem (0.0.6) will be obtained by analyzing the associated problem

$$\begin{cases} (-\Delta)^s v(x) = \beta(x) + \varphi_{1,a}^2(x)\, \chi_{S_a^c}, & \text{in } \Omega, \\ \varphi = 0, & \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases} \tag{0.0.7}$$

where β is any function satisfying the following condition:

$(\mathscr{A}_\beta)$ $\beta \in L^\infty(\Omega)$, $\beta \geq 0$, and there exists $c > 0$, such that

$$\inf_{x \in \Omega} \{\beta(x) + \varphi_{1,a}(x)\} \geq c > 0. \tag{0.0.8}$$

We denote by $v := v(\alpha,\beta) \in H_0^s(\Omega)$ the unique positive solution of (0.0.7). Similarly to problems (0.0.3) and (0.0.4), and following [17], Theorem 1.2, and [16], Lemma 7.3, there exist positive constants C_5, C_6 such that

$$C_5\, \delta^s(x) \leq v(x) \leq C_6\, \delta^s(x), \tag{0.0.9}$$

for almost every $x \in \Omega$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. The presentation of the associated elliptic problems discussed above will be given in Chapter 5, Section 5.1.

Based on the results obtained for problem (0.0.6), we prove—following [7]—the following theorem, which is of crucial importance for establishing Theorem 0.0.1:

Theorem 0.0.3. *Consider problem* (0.0.7). *For every $\lambda \geq \lambda_{1,a}$ and every β satisfying condition ($\mathscr{A}_\beta$), there exists a positive function*

$$h^* = h(\lambda, a, b, \beta, \Omega) \in L^\infty(\Omega)$$

such that, for all $h \in L^\infty(\Omega)$ satisfying $0 \leq h \leq h^*$ almost everywhere in Ω, problem (0.0.7) *admits a positive solution.*

Moreover, if h satisfies

$$\mu := \operatorname*{ess\,sup}_{x\in\Omega} \frac{h}{h^*} < 1,$$

then there exists $m^* > 0$ such that for all m satisfying

$$\mu m^* \leq m \leq m^*,$$

the function

$$\underline{u}_m = m\lambda \left(\varphi_{1,a} - \varepsilon\, v_\beta\right)$$

is a subsolution of (0.0.7)*, and it satisfies $0 < \underline{u}_m \leq m\varphi_{1,a}$.*

Furthermore, there exists $k^* > 0$ such that $\overline{u}_k = ke$ is a supersolution of (0.0.7) *for every $k \geq k^*$.*

The proof of this theorem will be presented in Chapter 5, Section 5.2.

With respect to the remaining chapters of this dissertation, we have the following organization: In Chapter 1 we present the Bochner spaces, generalized derivatives in the context of Bochner spaces, and Evolution Triples; in Chapter 2 we briefly introduce the fractional Sobolev spaces, the fractional Laplace operator, and the Sobolev–Bochner spaces; in Chapter 3 we present a Comparison Method that is essential in the study of existence and uniqueness of solutions for the parabolic problem (P_1).

Chapter 1

The Bochner spaces

In this section we introduce the Bochner spaces $L^p(0,T;X)$. Such spaces are generalizations of the usual Lebesgue spaces on classical measure theory. Instead of function $u:\Omega\subset\mathbb{R}^N\to\mathbb{R}$, the Bochner spaces considers a more general class of function $u:\Omega\subset\mathbb{R}^N\to X$, where X is a given Banach space. This spaces play a fundamental role in constructing the functional framework that is most appropriate for the problem under consideration.

1.1 Preliminary Results

In this section we present some preliminary results concerning integration of real valued functions or functions taking values on Banach spaces.

Definition 1.1.1. *Let X be a Banach space, and $f:\Omega\subset\mathbb{R}^N\to X$ be a function. We say that f is measurable if:*

1. *Ω is measurable, i.e, $0<\mu(\Omega)<\infty$, where μ is the Lebesgue Measure;*
2. *there exists a sequence $(f_n)_{n=1}^{\infty}$ of step functions $f_n:\Omega\to X$ such that*

$$\lim_{n\to\infty} f_n(x)=f(x)$$

a.e. $x\in\Omega$.

Theorem 1.1.1. *(Theorem of Pettis) . Let X be a separable real Banach space and $f:\Omega\subset\mathbb{R}^N\to\mathbb{R}$ a function. The following are equivalent:*

1. *f is measurable;*

2. *the real functions* $\varphi : \Omega \to \mathbb{R}$, $\varphi(x) = \langle g\,,f(x)\rangle_{X',X}$ *are measurable on* Ω*, for any linear functional* $g \in X'$.

Remark 1.1.1. *The importance of Pettis' Theorem lies in the fact that the measurability of functions taking values in Banach spaces reduces to the measurability of real-valued functions.*

Theorem 1.1.2. *(Theorem of Luzin and the caracterization of measurable funtions). Let* $\Omega \subset \mathbb{R}^N$ *be measurable,* $\mu(\Omega) < \infty$*, and* $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ *a function. The following are equivalent:*

1. f *is measurable;*

2. f *is continuous up to small sets, i.e, given* $\delta > 0$ *there exists an open set* $\Omega_\delta \subset \Omega$ *such that* f *is continuous in the closed set* $\Omega - \Omega_\delta$ *and* $\mu(\Omega_\delta) < \delta$.

Definition 1.1.2. *(Measurable functions via substitution). Let* U *and* X *be real and separable Banach spaces. Let* $u : \Omega \subset \mathbb{R}^N \to U$ *and* $F : \Omega \to X$ *be functions. Set*

$$F(x) = f(x, u(x))$$

and suppose that

1. Ω *is measurable, and*

2. $f : \Omega \times U \to X$ *is a Carathéodory function, that is*

$$x \mapsto f(x,u) \text{ is measurable on } \Omega \text{ for all } u \in U,$$

$$u \mapsto f(x,u) \text{ is continuous on } U \text{ a.e } x \in \Omega.$$

If the function $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ *is measurable, then the function* $f : \Omega \times U \to X$ *is measurable.*

Theorem 1.1.3. *Let* X *be a real Banach space, and* $f, g : \Omega \subset \mathbb{R}^N \to X$*. Supose that*

1. $f : \Omega \subset \mathbb{R}^N \to X$ *is measurable;*

2. $\|f(x)\|_X \le g(x)$*, a.e* $x \in \Omega$*, and*

3. $\int_\Omega g\,dx$ *exists.*

Then the integrals $\left|\int_\Omega f\,dx\right|$ *and* $\int_\Omega \|f\|_X\,dx$ *exist, and we have*

$$\left|\int_\Omega f\,dx\right| \leq \int_\Omega \|f\|_X\,dx. \tag{1.1.1}$$

Theorem 1.1.4. *(Norm Criterium). If* $f : \Omega \subset R^N$ *is a measurable function, then* $\int_\Omega f\,dx$ *exist if, and only if,* $\int_\Omega \|f\|_X\,dx$ *exists.*

Theorem 1.1.5. *(Lebesgue Dominated Convergence Theorem). Let* $f, f_n, g : \Omega \subset \mathbb{R}^N \to X$ *where* $n \in \mathbb{N}$ *and* X *is a Banach space. Suppose that:*

1. $\|f_n(x)\|_X \leq g(x)$ *a.e* $x \in \Omega$ *for all* $n \in \mathbb{N}$ *and* $\int_\Omega g\,dx$ *exist;*
2. $\lim_{n\to\infty} f_n(x)$ *exist a.e and* f_n *is measurable for every* $n \in \mathbb{N}$.

Then $\int_\Omega f_n\,dx$ *exists for every* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \int_\Omega f_n\,dx = \int_\Omega \lim_{n\to\infty} f_n\,dx. \tag{1.1.2}$$

1.2 The Bochner Spaces

In this section, we introduce the space of Lebesgue functions taking values in some Banach space X. More precisely, we will define the spaces $L^p(0,T;\Omega)$, for $1 \leq p \leq \infty$, and introduce properties and results that will be extremely important later. Following [20], we consider the following definition:

Definition 1.2.1. *Let* $(X, \|\cdot\|_X)$ *be a Banach space over a field* $\mathbb{K}$, *where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$, *and* $0 < T < \infty$.

1. *For* $m = 0,1,2\ldots$, *we define the space* $C^m([0,T];X)$ *as the set of all continuous functions* $u : [0,T] \to X$ *that have continuous derivatives up to order* m *on the interval* $[0,T]$, *with a norm given by*

$$\begin{aligned} \|\cdot\|_{C^m(0,T;X)} : C^m([0,T];X) &\to \mathbb{R} \\ u &\mapsto \|u\|_{C^m(0,T;X)} \end{aligned}$$

   where

$$\|u\|_{C^m(0,T;X)} := \sum_{i=0}^{m} \max_{0 \le t \le T} \|u^{(i)}(t)\|. \tag{1.2.1}$$

Here, only the right-hand and the left-hand derivatives need to exists at the boundary points $t = 0$ and $t = T$, respectively.

2. *For $1 \le p < \infty$, we define the space $L^p(0,T;X)$ as the set of all measurable functions $u : (0,T) \to X$ for which*

$$\int_0^T \|u(t)\|_X \, dt < \infty,$$

endowed with the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X \, dt \right)^{1/p}. \tag{1.2.2}$$

3. *For $p = \infty$, we define the space $L^\infty(0,T;X)$ as the set of all measurable functions $u : (0,T) \to X$ such that there exists a number $c > 0$ with $\|u(t)\|_X \le c$, for almost all $t \in (0,T)$, endowed with the norm*

$$\|u\|_{L^\infty(0,T;X)} = \inf\{c > 0 \mid \|u(t)\|_X \le c, \ a.e\ t \in (0,T)\}. \tag{1.2.3}$$

Remark 1.2.1. *It is worthwhile to notice that:*

- *In the case where $m = 0$, we write $C^0([0,T];X)$ as $C([0,T];X)$;*
- *In the above definition for the spaces $L^p(0,T;X)$, $1 \le p < \infty$, it is understood that we identify functions that are equal almost everywhere on $(0,T)$.*
- *The expression "almost everywhere" (or "a.e.") means that the stated property holds for all $t \in (0,T)$ except possibly on a set of Lebesgue measure zero.*

Proposition 1.2.1. *(Properties of Lebesgue Spaces). Let X,Y be Banach spaces over the same field $\mathbb{K}$, $m = 0,1,2,\cdots$ and $1 \le p < \infty$. Then:*

(i) $(C^m([0,T];X), \|\,.\,\|_{C^m([0,T];X)})$ *is a Banach space over $\mathbb{K}$;*

(ii) $(L^p(0,T;X), \|\,.\,\|_{L^p(0,T;X)})$ *is a Banach space over $\mathbb{K}$. Moreover, the set of all step functions $u : [0,T] \to X$ is dense in $L^p(0,T;X)$;*

(iii) $C([0,T];X)$ *is dense in* $L^p(0,T;X)$*. In addition, the embedding*

$$C([0,T];X) \hookrightarrow L^p(0,T;X) \tag{1.2.4}$$

is continuous;

(iv) *the set of all polynomials* $w : [0,T] \to X$*,* $w(t) = a_0 + a_it + \cdots a_nt^n$*, with* $a_i \in X$*, for all* i *and* $n = 0, 1, \cdots$ *is dense in* $L^p(0,T;X)$ *and* $C([0,T];X)$*;*

(v) *if* X *is a Hilbert space with inner product* $(\cdot,\cdot)_X$*, then* $L^2(0,T;X)$ *is a Hilbert space with inner product*

$$(u,v)_H = \int_0^T (u(t),v(t))_X \, dt; \tag{1.2.5}$$

(vi) *if* X *is separable space and* $1 \leq p < \infty$*, then* $L^p(0,T;X)$ *is separable;*

(vii) *if* X *is a uniformly convex space and* $1 < p < \infty$*, then* $L^p(0,T;X)$ *is uniformly convex;*

(viii) *if the embedding* $X \hookrightarrow Y$ *is continuous, then, for every* $1 \leq q \leq r \leq \infty$*, the embedding*

$$L^r(0,T;X) \hookrightarrow L^q(0,T;Y) \tag{1.2.6}$$

is continuous. In particular, for every $1 \leq p \leq \infty$*, the embedding*

$$L^p(0,T;X) \hookrightarrow L^1(0,T;X) \tag{1.2.7}$$

is continuous.

Proof. For a proof of this result, see [20]. □

I view of Hölder's Inequality for the L^p presented in equation (A.0.3), we now state an analogous result for the spaces $L^p(0,T;X)$.

Proposition 1.2.2. *(Hölder's Inequality for the* $L^p(0,T;X)$ *spaces). Let* $(X, \|\,.\,\|_X)$ *be a Banach space. Then*

$$\int_0^T \left|\langle v(t),u(t)\rangle_{X',X}\right| dt \leq \left(\int_0^T \|v(t)\|_{X'}^p \, dt\right)^{1/p} \left(\int_0^T \|u(t)\|_X^q \, dt\right)^{1/q}, \tag{1.2.8}$$

holds for every $u \in L^p(0,T;X)$, $v \in L^q(0,T;X')$, *with* $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *The integrals in* (1.2.8) *are well defined since* $u \in L^p(0,T;X)$ *and* $v \in L^q(0,T;X')$.

Proof. Let $T > 0$ and define $I := (0,T)$. Since $u \in L^p(0,T:X)$ and $v \in L^q(0,T;X)$, both functions are measurable. It follows that there exists step functions $u_n;I \to X$ and $v_n : I \to X'$ such that

$$u_n(t) \to u(t) \text{ and } v_n(t) \to v(t)$$

almost everywhere $t \in I$ as $n \to \infty$. Hence,

$$\langle v_n(t), u_n(t)\rangle_{X',X} \to \langle v(t), u(t)\rangle_{X',X}$$

almost everywhere $t \in I$ as $n \to \infty$. It follows that the function $t \to \langle v(t), u(t)\rangle_{X',X}$ is measurable on I. Moreover, since

$$\left|\langle v(t), u(t)\rangle_{X',X}\right| \le \|v(t)\|_{X'} \|u(t)\|_X. \tag{1.2.9}$$

it follows by the classical Hölder inequality, (A.0.3), for $N = 2$, that

$$\begin{aligned}\int_0^T \left|\langle v(t), u(t)\rangle_{X',X}\right| dt &\le \int_0^T \|v(t)\|_{X'} \|u(t)\|_X \, dt \\ &\le \left(\int_0^T \|v(t)\|_{X'}^q \, dt\right)^{1/q} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}\end{aligned}$$

□

Remark 1.2.2. *Note that inequality* (1.2.9) *follows from the definition of the norm in the dual space* X'. *Indeed, for a.e.* $t \in (0,T)$ *such that* $u(t) \neq 0$, *we have*

$$\left|\langle v(t), u(t)\rangle_{X',X}\right| = \|u(t)\|_X \left|\left\langle v(t), \frac{u(t)}{\|u(t)\|_X}\right\rangle_{X',X}\right| \le \|u(t)\|_X \|v(t)\|_{X'},$$

since

$$\left\|\frac{u(t)}{\|u(t)\|_X}\right\|_X = 1 \quad \text{and} \quad \|v(t)\|_{X'} = \sup_{\|x\|_X = 1} \left|\langle v(t), x\rangle_{X',X}\right|.$$

One of the main goals of this section is to identify, as it is done in the classical case, the space $(L^p(0,T;X))'$, the topological dual of $L^p(0,T;X)$, with the space $L^q(0,T;X')$, where $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. This is done in the following proposition.

Proposition 1.2.3. *Let V be a separable and reflexive Banach space and suppose, additionally, that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Under these assumptions, the following holds:*

(i) for each function $v \in L^q(0,T;V')$ there corresponds a unique linear functional $\bar{v} \in (L^p(0,T;V))'$ such that

$$\langle \bar{v}, u \rangle_{X',X} = \int_0^T \langle v(t), u(t) \rangle_{V',V} \, dt \tag{1.2.10}$$

for all $u \in L^p(0,T;V)$;

(ii) conversely, for each linear functional $\bar{v} \in (L^p(0,T;V))'$ there corresponds a unique $v \in L^q(0,T;V')$ such that

$$\langle \bar{v}, u \rangle_{X',X} = \int_0^T \langle u(t), v(t) \rangle_{V',V} \, dt \tag{1.2.11}$$

for all $u \in X$. Moreover,

$$\|\bar{v}\|_{X'} = \|v\|_{L^q(0,T;V')} \tag{1.2.12}$$

(iii) the space $L^p(0,T;V)$ is reflexive and separable.

Proof. For a proof of this result, see [20]. □

Remark 1.2.3. *According to Proposition 1.2.3, the spaces $(L^p(0,T;V))'$ and $L^q(0,T;V')$ are isometrically isomorphic, i.e, there exists a bijective isometric mapping*

$$\begin{aligned} \mathscr{J} : L^q(0,T;V') &\to (L^p(0,T;V))' \\ v &\mapsto \mathscr{J}(v) : L^p(0,T;V) \to \mathbb{R} \\ &\qquad\qquad u \mapsto \langle \mathscr{J}(v), u \rangle_{X',X} \end{aligned}$$

where

$$\langle \mathscr{J}(v), u \rangle_{X',X} = \int_0^T \langle v(t), u(t) \rangle_{V',V} \, dt$$

for all $u \in X$. Since isometrically isomorphic spaces can be identified, we have $(L^p(0,T;V))' \cong L^q(0,T;V')$. Thus, identifying $\mathscr{J}(v)$ with v, the equations (1.2.11) and (1.2.12) can be written, respectively, as

$$\langle v,u\rangle_{X',X} = \int_0^T \langle u(t),v(t)\rangle_{V',V}\,dt \tag{1.2.13}$$

and

$$\|v\|_{X'} = \|v\|_{L^q(0,T;V')} = \left(\int_0^T \|v(t)\|_{V'}^q\,dt\right)^{1/q} \tag{1.2.14}$$

for all $u \in L^p(0,T;V)$ *and for all* $v \in (L^p(0,T;V))'$.

Let us now consider a open and bounded subset $\Omega \subset \mathbb{R}^N$ and let $T > 0$. Define $Q_T :=$ $\Omega \times (0,T)$ and let V be a Banach space. We want to do an identification of a function $u \in L^q(Q_T)$ with a function $u \in L^q(0,T;V')$. This identification will be of critical importance when, further in this study, we address problems related to temporal evolution. In the sequence, we will introduce the concept of an evolution triple, also referred to as a Gel'fand triple.

Such identification is provided by the following lemma:

Lemma 1.2.1. *Let* $\Omega \subseteq \mathbb{R}^N$ *be open and bounded,* $N \geq 1$*, and* $0 < T < \infty$*. Define* $Q_t :=$ $\Omega \times (0,T)$*. Set*

$$Y = L^p(\Omega)\,, V = W_0^{m,p}(\Omega)$$

and suppose that a function

$$f : Q_T \to \mathbb{R}\,, (x,t) \mapsto f(x,t)$$

is such that $f \in L^q(Q_T)$*. Then the following hold:*

(a) for all $t \in [0,T]$ *there exists a linear functional* $b(t) \in Y' = L^q(\Omega)$ *such that*

$$\langle b(t),v\rangle_{Y',Y} = \int_\Omega f(x,t)\,v(x)\,dx, \quad \forall v \in Y \tag{1.2.15}$$

for a.e. $t \in [0,T]$*. Moreover, the mapping* $b : (0,T) \to Y'\,, t \mapsto b(t)$ *is an element of* $L^q(0,T;Y') = L^q(0,T;L^q(\Omega))$*. More precisely, for almost all* $t \in [0,T]$ *the function* $f(\cdot,t) : \Omega \to \mathbb{R}\,, x \mapsto f(x,t)$ *is such that* $f(\cdot,t) \in L^q(\Omega)$ *and we can identify it with the functional* $b(t) \in Y'$ *in such a way that* (1.2.15) *holds,*

$$\|b(t)\|_{L^q(\Omega)}^q = \int_\Omega |f(x,t)|^q\,dx \tag{1.2.16}$$

and

$$\|f\|^q_{L^q(0,T;Y')} = \int_0^T \left(\int_\Omega |f(x,t)|^q\,dx\right) dt = \int_{Q_T} |f(x,t)|^q\,dx\,dt; \tag{1.2.17}$$

(b) there exists a functional $b_1(t) \in V'$ *such that*

$$\langle b_1(t), v\rangle_{V',V} = \int_\Omega f(x,t)\,v(x)\,dx, \quad \forall v \in V, \tag{1.2.18}$$

for almost every $t \in [0,T]$ *and* $b_1 : (0,T) \to V'$, $t \mapsto b_1(t)$, *is such that* $b_1 \in L^q(0,T;V')$. *Thus, the linear continuous functional* $b_1(t) : V \to \mathbb{R}$ *is the restriction of the linear operator* $b(t) : Y \to \mathbb{R}$ *to the space* V, *by the embedding* $W_0^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$, $m = 1, 2, \cdots$. *Moreover,*

$$\|b\|_{L^q(0,T;V')} \leq \left(\int_{Q_T} |f(x,t)|^q\,dx\,dt\right)^{1/q}. \tag{1.2.19}$$

Proof. (a) Since $f \in L^q(Q_T)$ it follows that f is integrable. By Fubini's Theorem A.0.2 we have

$$\int_{Q_T} |f(x,t)|^q\,dx\,dt = \int_0^T \left(\int_\Omega |f(x,t)|^q\,dx\right) dt \tag{1.2.20}$$

where the integral on the right exists for every $t \in [0,T] \setminus N$, where N is a set of measure zero. For $t \in N$, we set $f(x,t) \equiv 0$, for every $x \in \Omega$. By the Duality Theorem A.0.5, every function in $L^q(\Omega)$ can be identified with a continuous linear functional in $Y' = (L^p(\Omega))'$. Hence, the functional $b(t) \in Y'$ corresponding to f is precised given by equation (1.2.15). Moreover, equation (1.2.16) holds and equation (1.2.17) follows immediately from equations (1.2.17) and (1.2.20). Let us now show that the function $b : [0,T] \to Y'$ is measurable. Since Y' is a reflexive Banach space, it follows from the Theorem of Pettis, Theorem 1.1.1, that is sufficient to prove the mensurability, for each $v \in Y$ and $t \in [0,T]$, of the real functions $t \mapsto \langle b(t), v\rangle_{Y',Y}$. Let $v \in Y = L^p(\Omega)$ be fixed. Since $v \in L^p(\Omega)$ and $f \in L^q(Q_T)$, from Holder's inequality,(1.2.8), it follows that

$$\int_{Q_T} |f(x,t)v(x)|\,dx\,dt \leq \|f\|_{L^q(Q_T)}\,\|v\|_{L^p(\Omega)} < \infty.$$

Hence $fv \in L^1(Q_T)$ and by the Theorem of Fubini, Theorem A.0.2,

$$\int_{Q_T} f(x,t)v(x)\,dx\,dt = \int_0^T \left(\int_\Omega f(x,t)v(x)\,dx\right) dt$$

and the measurability of the integral $\int_\Omega f(x,t)v(x)\,dx$, as a function of t, follows. The measurability of the function $t \mapsto \langle b(t), v\rangle_{Y',Y}$, $t \in [0,T]$, for all $v \in Y$, is a consequence of equation (1.2.15).

(b) by Hölder's inequality, Proposition 1.2.2,

$$\left|\int_\Omega f(x,t)v(x)\,dx\right| \le \left(\int_\Omega |f(x,t)|^q\,dx\right)^{1/q}\left(\int_\Omega |v(x)|^p\,dx\right)^{1/p}$$
$$\le \left(\int_\Omega |f(x,t)|^q\,dx\right)^{1/q} \|v\|_{L^p(\Omega)} \quad \forall v \in V, \tag{1.2.21}$$

thus ensuring the existence of a linear functional $b_1(t) \in V'$ such that

$$\langle b_1(t), v\rangle_{V',V} = \int_\Omega f(x,t)v(x)\,dx, \quad \forall v \in V.$$

Indeed, for each $t \in [0,T]$, we consider the function

$$\begin{aligned} b_1(t) : V &\to \mathbb{R} \\ v &\mapsto \langle b_1(t), v\rangle_{V',V} \end{aligned}$$

where $\langle b_1(t), v\rangle_{V',V}$ is given by equation (1.2.18). Let us show that, indeed, $b_1(t)$ is a continuous linear functional.

- $b_1(t)$ **is linear in** v**.** Indeed, let v_1, $v_2 \in V$ and $\alpha \in \mathbb{R}$. Then,

$$\begin{aligned} \langle b_1(t), v_1 + \alpha v_2\rangle_{V',V} &= \int_\Omega f(x,t)(v_1 + \alpha v_2)(x)\,dx \\ &= \int_\Omega f(x,t)v_1(x)\,dx + \alpha \int_\Omega f(x,t)v_2(x)\,dx \\ &= \langle b_1(t), v_1\rangle_{V',V} + \alpha\langle b_1(t), v_2\rangle_{V',V} \end{aligned}$$

  for all v_1, $v_2 \in V$ and for all $\alpha \in \mathbb{R}$. Hence, $b_1(t)$ is linear in v.

- $b_1(t)$ **is continuous.** In fact, since it is linear, it is sufficient to show that it is bounded. But this follows immediately from equation (1.2.21)

Moreover, if we take $v \in V$ with $v \neq 0$, then $\|v\|_{L^p(\Omega)} \neq 0$ and from equation (1.2.21) follows

$$\frac{\left|\langle b_1(t), v\rangle_{V',V}\right|}{\|v\|_{L^p(\Omega)}} = \frac{\left|\int_\Omega f(x,t)v(x)\,dx\right|}{\|v\|_{L^p(\Omega)}} \leq \left(\int_\Omega |f(x,t)|^q\,dx\right)^{1/q}.$$

Hence,

$$\|b_1(t)\|^q_{L^q(\Omega)} \leq \int_\Omega |f(x,t)|^q\,dx.$$

Thus, by equation (1.2.14)

$$\|b_1\|^q_{L^q(0,T;V')} = \int_0^T \|b_1(t)\|^q_{L^q(\Omega)}\,dt \leq \int_0^T \left(\int_\Omega |f(x,t)|^q\,dx\right)dt < \infty,$$

and $b_1 \in L^q(0,T;V')$.

□

Remark 1.2.4. *The mapping*

$$b : [0,T] \to (L^p(\Omega))'$$

is defined for every $t \in [0,T]$. Indeed, for almost every $t \in (0,T)$ the section $f(\cdot,t)$ belongs to $L^q(\Omega)$ and, via the identification

$$(L^p(\Omega))' \simeq L^q(\Omega),$$

it naturally defines a continuous linear functional. For the exceptional values of t, belonging to a set of measure zero, the functional $b(t)$ may be defined arbitrarily, for instance by setting $b(t) = 0$.

On the other hand, the integral representation

$$\langle b(t), v\rangle_{Y',Y} = \int_\Omega f(x,t)\,v(x)\,dx, \qquad v \in L^p(\Omega),$$

is valid only for almost every $t \in (0,T)$, since the membership of $f(\cdot,t)$ in $L^q(\Omega)$ is guaranteed by Fubini's Theorem only outside a null set with respect to t. Thus, there is no contradiction between the existence of the functional $b(t)$ for every t and the almost everywhere validity of the integral representation, as these statements refer to distinct levels of the construction.

1.3 Generalized derivatives

Just as in the case of real valued functions, we can also talk about *Generalized Derivatives* in the Bochner spaces $L^p(0,T;X)$. Such derivatives will play a central role when we study time-evolution problems using the *Sobolev-Bochner spaces* $W^{1,p,q}(0,T;V;H)$, to be introduced later on, where V an H are the spaces involved in the evolution triple $V \subseteq H \subseteq V'$.

We began by recalling that, given an open and bounded subset $\Omega \subset \mathbb{R}^N$, $C_0^\infty(\Omega)$ denotes the space of all functions $u : \Omega \to \mathbb{R}$ that are infinitely continuously differentiable and have compact support in Ω. For our purpose, we will assume $N = 1$ an take $\Omega = (0,T)$, for $T > 0$. In what follows, unless explicitly stated otherwise, X and Y will always denote Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.3.1. *(Generalized Derivative). Let $u \in L^1(0,T;X)$ and $w \in L^1(0,T;Y)$. We say that the function w is the n-th generalized derivative of the function u on the interval $(0,T)$, $T > 0$, if*

$$\int_0^T \varphi^{(n)}(t)u(t)\,dt = (-1)^n \int_0^T \varphi(t)w(t)\,dt, \tag{1.3.1}$$

for all $\varphi \in C_0^\infty(0,T)$. In such a case, we write $w = u^{(n)}$.

It is worthwhile to note that, for every continuous function having continuous derivatives up to order $m \geq 1$, the concepts of generalized derivative and (classical) derivative coincide, as can be seen in the following porposition.

Proposition 1.3.1. *Suppose that $X = Y$ and let $u \in C^m([0,T],X)$, with, $m \geq 1$. The m-th continuous derivative*

$$u^{(m)} : [0,T] \to X,$$

is also the m-th generalized derivative of u on the interval $(0,T)$.

Proof. The proof is almost straightforward. Consider $u \in C^m([0,T];X)$ and let $\varphi \in C_0^\infty(0,T)$. Note that

$$(\varphi u)' = \varphi' u + \varphi u'. \tag{1.3.2}$$

Integrating equation (1.3.2) over t,

$$(\varphi u)\,|_0^T = \int_0^T (\varphi u)'\,dt = \int_0^T \varphi' u\,dt + \int_0^T \varphi u'\,dt. \tag{1.3.3}$$

Since $\varphi \in C_0^\infty(0,T)$, the leftmost term on (1.3.3) is identically zero. Hence,

$$\int_0^T \varphi' u\,dt = -\int_0^T \varphi u'\,dt. \tag{1.3.4}$$

Note that (1.3.4) is none other than the classical integration by parts formula. Since φ was arbitrary, (1.3.4) holds for all $\varphi \in C_0^\infty(0,T)$, and the first (classical) derivative of u coincides with its first generalized derivative. Now, we consider integral

$$\int_0^T \varphi'' u\,dt.$$

From (1.3.4), and the fact that $\varphi' \in C_0^\infty(0,T)$, we obtain

$$\int_0^T \varphi u''\,dt = -\int_0^T \varphi' u'.\,dt. \tag{1.3.5}$$

Applying (1.3.4) to the right hand side of (1.3.5):

$$\int_0^T \varphi' u'\,dt = (\varphi u')\,|_0^T - \int_0^T \varphi u''\,dt. \tag{1.3.6}$$

Since $\varphi \in C_0^\infty(0,T)$ the first term on the rigth hand side of (1.3.6) is zero. Hence,

$$\int_0^T \varphi' u'\,dt = -\int_0^T \varphi u''\,dt. \tag{1.3.7}$$

Combining (1.3.7) and (1.3.5) we get that

$$\int_0^T \varphi'' u\,dt = (-1)^2 \int_0^T \varphi u''\,dt. \tag{1.3.8}$$

Consequently, the second order (classical) derivative of u is also the second order generalized derivative of u. The result now follows by applying formula (1.3.4) n times successively.

□

Remark 1.3.1. *In the proof of Proposition 1.3.1, we use the fact that if $\varphi \in C_0^\infty(0,T)$, then $\varphi^{(m)} \in C_0^\infty(0,T)$. In fact, let $m \geq 1$, $m \in \mathbb{N}$, and $K := \overline{\{\, t \in (0,T) \mid \varphi(t) \neq 0\}}$ be the support of φ. Since $K \subseteq (0,T)$, K is clearly compact. Let $t \in (0,T) \setminus K$. Since K is closed, there exists $\varepsilon > 0$ such that the ε-neighborhood $\mathcal{U}_\varepsilon(t) =: (t-\varepsilon, t+\varepsilon)$ of t is such that $\mathcal{U}_\varepsilon(t) \cap K = \emptyset$. In particular, for every $0 < h < \varepsilon$ he have that $t+h \in \mathcal{U}_\varepsilon(t)$, and*

$$\varphi(x+t) = 0 \text{ and } \varphi(t) = 0. \tag{1.3.9}$$

It follows from (1.3.9) that the limit

$$\frac{d\varphi(t)}{dt} = \lim_{h\to 0} \frac{\varphi(t+h)-\varphi(t)}{h} = 0. \tag{1.3.10}$$

Hence,

$$supp\left(\frac{d\varphi}{dt}\right) \subseteq K,$$

from witch follows that $\frac{d\varphi}{dt} \in C_0^\infty(0,T)$. *Assume that we have shown that*

$$\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(m)} \in C_0^\infty(0,T).$$

Let $t \in (0,T)\setminus K$. *As above, there exists* $\varepsilon > 0$ *and a* ε*-neighborhood* $\mathcal{U}_\varepsilon(t)$ *such that*

$$\mathcal{U}_\varepsilon(t) \cap K = \emptyset.$$

For every $0 < h < \varepsilon$ *it follows that* $t+h \in \mathcal{U}_\varepsilon(t)$ *and*

$$\varphi^{(m)}(t+h) = 0\,,\ \varphi^{(m)}(t) = 0. \tag{1.3.11}$$

From (1.3.11) it follows that the limit

$$\varphi^{(m+1)(t)} = \frac{d\varphi^{(m)(t)}}{dt} = \lim_{h\to 0} \frac{\varphi^{(m)}(t+h)-\varphi^{(m)}(t)}{h} = 0, \tag{1.3.12}$$

hence,

$$supp\left(\frac{d\varphi^{(m)}}{dt}\right) \subseteq K,$$

and $\varphi^{(m+1)} \in C_0^\infty(0,T)$. *It follows form the Principle of Finite Induction that*

$$\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(m)} \in C_0^\infty(0,T), \tag{1.3.13}$$

for every $m \in \mathbb{N}$.

Notice that, not all functions $u \in L^p(0,T;X)$ posses generalized derivatives that can be represented by a function $w \in L^p(0,T,Y)$ as (1.3.1) requires. An simple example let us consider the so called *Heaviside Function*

Example 1.3.1. *By a Heaviside Function we mean a function*

$$
\begin{aligned}
H : \mathbb{R} &\rightarrow \mathbb{R} \\
t &\mapsto H(t)
\end{aligned}
\tag{1.3.14}
$$

where,

$$
H(t) = \begin{cases} 0, & se\ t < 0, \\ 1, & se\ t \geq 0. \end{cases}
\tag{1.3.15}
$$

Let $T > 0$ and $0 < t_0 < T$ and consider de the Heaviside Function $\tilde{H} := H\,|_{(0,T)}$

$$
\begin{aligned}
\tilde{H} : (0,T) &\rightarrow \mathbb{R} \\
t &\mapsto H(t - t_0)
\end{aligned}
\tag{1.3.16}
$$

where,

$$
H(t - t_0) = \begin{cases} 0, & se\ t < t_0, \\ 1, & se\ t \geq t_0. \end{cases}
\tag{1.3.17}
$$

Clearly $\tilde{H} \in L^1(0,T;\mathbb{R})$; in fact,

$$
\|\tilde{H}\|_{L^1(0,T;\mathbb{R})} = \int_0^T |H(t-t_0)|\,dt = \int_{t_0}^T H(t-t_0)\,dt = T - t_0 < \infty.
\tag{1.3.18}
$$

In the sense of distributions, the Heaviside function possess generalized derivative named "Dirac's delta", denoted by δ_0, and named after the famous British physicist Paul A.M Dirac, who made important contributions to the theory of Quantum Mechanics end Quantum Field Theory. Let us prove that, in fact,

$$
\frac{d\tilde{H}(t)}{dt} = \delta_{t_0}(t)
\tag{1.3.19}
$$

is the generalized derivative of the function $\tilde{H}$. Given any test function $\varphi \in C_0^\infty(0,T)$,

$$
\begin{aligned}
\int_0^T \tilde{H}(t)\varphi'(t)\,dt &= \int_0^{t_0} H(t-t_0)\varphi'(t)\,dt + \int_{t_0}^T H(t-t_0)\varphi'(t)\,dt \\
&= \int_{t_0}^T \varphi'(t)\,dt = \varphi(T) - \varphi(t_0) = -\varphi(t_0) \\
&= -\int_0^T \varphi(t)\delta_{t_0}(t)\,dt.
\end{aligned}
\tag{1.3.20}
$$

From (1.3.20) it follows that (1.3.19) holds, and the Heaviside function posses generalized derivative as we have claimed. Let us now show that

$$\delta_0 \notin L^1(0,T;\mathbb{R}).$$

Suppose, by contradiction, that there exists a function $f \in L^1(0,T;\mathbb{R})$ such that

$$\int_0^T f(t)\varphi(t)\,dt = \varphi(t_0), \tag{1.3.21}$$

for all $\varphi \in C_0^\infty(0,T)$. We choose a sequence of functions $\varphi_n \in C_0^\infty(0,T)$ with the following properties:

(i) for every $n \in \mathbb{N}$, $supp(\varphi_n) = \left(t_0 - \frac{1}{n}, t_0 + \frac{1}{n}\right) \subset (0,T)$;

(ii) $\int_0^T \varphi_n(t)\,dt \geq 1$, $\varphi_n(t) \geq 0$, for all $n \in \mathbb{N}$.

By the hypothesis (1.3.21),

$$\int_0^T f(t)\varphi_n(t)\,dt = \varphi_n(t_0).$$

Since the support of φ_n has length $2/n$ and $\int \varphi_n \geq 1$, we have

$$\varphi_n \geq \frac{n}{2}, \tag{1.3.22}$$

hence

$$\varphi_n(t_0) \to \infty \text{ as } n \to \infty. \tag{1.3.23}$$

On the other hand, because $f \in L^1(0,T;\mathbb{R})$, we have

$$\left|\int_0^T f(t)\varphi_n(t)\,dt\right| \leq \int_0^T |f(t)|\varphi_n(t)\,dt \leq \|f\|_{L^1(0,T;\mathbb{R})}\|\varphi_n\|_\infty. \tag{1.3.24}$$

Thus the sequence

$$\int_0^T f(t)\varphi_n(t)\,dt \tag{1.3.25}$$

is bounded. But simultaneously

$$\int_0^T f(t)\varphi_n(t)\,dt = \varphi_n(t_0) \to \infty,$$

a contradiction.

Following the development of the theory of generalized derivatives on Bocnher spaces $L^1(0,T;X)$, as we have seen, every function $u \in L^1(0,T;X)$ possesses generalized derivatives w but, in general, it is not true that $w \in L^1(0,T;Y)$. The question of existence of such derivatives are now well established due the previous results. Now we will establish the uniqueness of such derivatives. For this we will suppose not only the existence of, for now, a generalized derivative w of $u \in L^1(0,T;X)$ but also that $w \in L^1(0,T;Y)$. To obtain such result about uniqueness we will need the following variational lemma:

Lemma 1.3.1. *If $u \in L^1(0,T;X)$ and*

$$\int_0^T \varphi(t)u(t)\,dt = 0 \tag{1.3.26}$$

for all $\varphi \in C_0^\infty(0,T)$, then $u(t) = 0$ a.e on $(0,T)$.

Proof. For a proof of this result, see [20]. □

Proposition 1.3.2. *(Uniqueness of Generalized Derivatives). Suppose that $u \in L^1(0,T;X)$ and that $v, w \in L^1(0,T;Y)$ are such that $v = u^{(n)}$ and $w = u^{(n)}$ in the sense of generalized derivatives. Then $v(t) = w(t)$ a.e on $(0,T)$. In other words, $v = w$ in $L^1(0,T;Y)$.*

Proof. By the definition of generalized derivative, (1.3.1) we have that

$$\int_0^T \varphi(t)v(t)\,dt = (-1)^n \int_0^T \varphi^{(n)}(t)u(t)\,dt, \tag{1.3.27}$$

and

$$\int_0^T \varphi(t)w(t)\,dt = (-1)^n \int_0^T \varphi^{(n)}(t)u(t)\,dt \tag{1.3.28}$$

for all $n \in \mathbb{N}$ and for all $\varphi \in C_0^\infty(0,T)$. Subtracting (1.3.28) from (1.3.27) we obtain

$$\int_0^T \varphi(t)(v-w)(t)\,dt = 0 \tag{1.3.29}$$

for all $\varphi \in C_0^\infty(0,T)$. By Lemma 1.3.1 it follows that $v(t) = w(t)$ a.e on $(0,T)$, that is, $v = w$ in $L^1(0,T;X)$.

□

To end this section, we present two more results. One concern the generalized derivative and its relation with weak convergence, while the other stablishes the existence of generalized derivative.

Proposition 1.3.3. *Let X, Y be Banach spaces over a over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with the continuous embedding $X \hookrightarrow Y$. If*

$$\begin{aligned} &u_k^{(n)} = v_k, \text{ on } (0,T) \text{ for all } k \text{ and fixed } n \geq 1 \\ &u_k \rightharpoonup u, \text{ in } L^p(0,T;X) \text{ as } k \to \infty, \\ &v_k \rightharpoonup v, \text{ in } L^q(0,T;Y) \text{ as } k \to \infty, 1 \leq p, q < \infty, \end{aligned} \tag{1.3.30}$$

then

$$u^{(n)} = v \text{ on } (0,T). \tag{1.3.31}$$

Proof. For a proof of this result, see [20]. □

Proposition 1.3.4. *Let*

$$V \subseteq H \subseteq V'$$

be an evolution triple and $1 \leq p, q < \infty$, $0 < T < \infty$. The following holds:

(i) given $u \in L^p(0,T;V)$, the generalized derivative $u^{(n)} \in L^q(0,T;V')$ is unique in the following sense: the mapping

$$t \mapsto u^{(n)}(t) \tag{1.3.32}$$

can be modified only on subsets of $(0,T)$ having measure zero;

(ii) given $u \in L^p(0,T;V)$, there exists the n-th generalized derivative $u^{(n)} \in L^q(0,T;V')$ if, and only if, there exists a function $w \in\in L^q(0,T;V')$ such that

$$\int_0^T (u(t),v)_H \, \phi^{(n)}(t)\, dt = (-1)^n \int_0^T \langle w(t),v\rangle_{V',V} \, \phi(t)\, dt, \tag{1.3.33}$$

for all $v \in V$ and for all $\phi \in C_0^\infty(0,T)$. Moreover, if $\dfrac{d^n}{dt^n}$ stands for the n-th generilezed derivative of real functions on $(0,T)$, then $u^{(n)} = w$ and

$$\frac{d^n}{dt^n}(u(t),v)_H = \left\langle u^{(n)}(t),v\right\rangle_{V',V}, \tag{1.3.34}$$

holds for all $v \in V$ an d almost all $t \in (0,T)$.

Proof. For a proof of this result, see [20]. □

Remark 1.3.2. *Equation (1.3.34) will be of extreme importance when working with the weak, or general, formulation of evolution equations, as will be seen later on this work, as well as the following example.*

Example 1.3.2. *Let $V \subseteq H \subseteq V'$ be an evolution triple, and let $u : [0,T] \to V$ be a continuous function such that its derivative*

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}, \tag{1.3.35}$$

exists for all $t \in [0,T]$ as a limiting value in the space H, with the function $u' : [0,T] \to H$ been continuous. Then the generalized derivative of u exists on $(0,T)$ and is equal to u'. Moreover,

$$u \in L^p(0,T;V)\,,\, u' \in L^q(0,T;V'), \tag{1.3.36}$$

for all $1 \leq p\,, q \leq \infty$. Indeed, since $u : [0,T] \to V$ is a continuous map and the embedding $V \hookrightarrow H$ is continuous and dense, the mapping $u : [0,T] \to H$ is also continuous. Hence, $u \in C([0,T];H)$. The result now follows from Proposition 1.3.1 with $X = Y = H$.

1.4 Gelf'fand Triple

One of the most important and fundamental concepts in the development of the subsequent theory is what is know as the evolution triple, or Gel'fend triple — a concept we are about to introduce.

Definition 1.4.1. *(Evolution triple, or Gel'fand triple). We say that the chain of inclusions*

$$V \subseteq H \subseteq V'$$

is an evolution triple, or a Gel'fand triple, if :

(i) V is a reflexive separable real Banach space;

(ii) H is a separable real Hilbert space;

(iii) V is dense in H and the embedding $V \hookrightarrow H$ is contunuous, i.e, there exists a constant $c > 0$ such that

$$\|v\|_H \leq c\|v\|_V,$$

for all $v \in V$.

Example 1.4.1. *Let $N \geq 1$ and consider a bounded subset $\Omega \subset \mathbb{R}^N$. Let $V = W_o^{m,p}(\Omega)$, with $m \geq 1$ and $2 \leq p < \infty$, and $H = L^2(\Omega)$. It is well known that V is a real, separable, reflexive Banach space; H is a real, separable Hilbert space. Moreover, the embbeding $V \hookrightarrow H$ is continuous and V is dense in H. Hence, the chain of inclusions*

$$W_0^{m,p}(\Omega) \subset L^2(\Omega) \subset W_0^{-m,p}(\Omega)$$

defines a evolution triple. For completeness we explain the last inclusion. Let $V := W_0^{m,p}(\Omega)$. By definition, the space $W_0^{-m,p}(\Omega)$ is the topological dual of V, that is,

$$W_0^{-m,p}(\Omega) = V'.$$

Let $h \in L^2(\Omega)$. To each such h we associate the linear functional

$$\ell_h : V \to \mathbb{R}, \qquad \ell_h(v) := \int_\Omega h(x)\, v(x)\, dx.$$

This functional is well defined, since the embedding $V \hookrightarrow L^2(\Omega)$ is continuous. Indeed, by Hölder's inequality and the continuity of this embedding, there exists a constant $C > 0$ such that

$$|\ell_h(v)| \leq \|h\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C \|h\|_{L^2(\Omega)} \|v\|_{W^{m,p}(\Omega)}, \qquad \forall v \in V.$$

It follows that ℓ_h is a continuous linear functional on V, that is, $\ell_h \in V'$. Consequently, every element of $L^2(\Omega)$ can be identified with an element of $W_0^{-m,p}(\Omega)$, which yields the continuous embedding

$$L^2(\Omega) \hookrightarrow W_0^{-m,p}(\Omega).$$

Proposition 1.4.1. *Let*

$$V \subseteq H \subseteq V'$$

be an evolution triple. Then,

(i) To each $h \in H$ there corresponds a unique continuos linear functional $\overline{h} \in V'$ such that

$$\langle \overline{h}, v \rangle_{V',V} = (h, v)_H, \tag{1.4.1}$$

for all $v \in V$, where $(\cdot,\cdot)_H$ stands for the inner product of the Hilbert space H.

(ii) The mapping

$$\varphi : H \to V', \quad \varphi(h) = \overline{h} \tag{1.4.2}$$

is continuous, injective and linear.

Proof. Let us consider $h \in H$ and define a function

$$\begin{aligned} \overline{h} : V &\to \mathbb{R} \\ v &\mapsto \langle \overline{h}, v \rangle_{V',V} \end{aligned}$$

where $\langle \overline{h}, v \rangle_{V',V} = (h, v)_H$, for all $v \in V$. Let us show that $\overline{h}$ is a continuous linear functional. We begin by showing that $\overline{h}$ is a linear functional. Let $u, v \in V$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \overline{h}(u + \alpha v) &= \langle \overline{h}, u + \alpha v \rangle_{V',V} \\ &= \langle \overline{h}, u \rangle_{V',V} + \langle \overline{h}, \alpha v \rangle_{V',V} \\ &= \langle \overline{h}, u \rangle_{V',V} + \alpha \langle \overline{h}, v \rangle_{V',V} \\ &= \overline{h}(u) + \alpha \overline{h}(v) \end{aligned}$$

thus showing that $\overline{h}$ is a linear function. Since $\overline{h}$ is linear, to show that it is continuous is sufficient to show that it is bounded. Remembering that $V \subseteq H \subseteq V'$ is an evolution triple, we have that the embedding $V \hookrightarrow H$ is continuous, i.e, there exists a positive constant c such that $\|v\|_H \leq c \|v\|_V$, for all $v \in V$. Taking the absolute value of (1.4.1) and using the *Cauchy-Bunyakovsky-Schwarz inequality*

$$\left| \langle \overline{h}, v \rangle_{V',V} \right| = |(h, v)_H| \leq \|h\|_H \|v\|_H \leq c \|h\|_H \|v\|_V. \tag{1.4.3}$$

If we assume that $v \in V \setminus \{0\}$ then $\|v\|_V \neq 0$, and

$$\frac{\left| \langle \overline{h}, v \rangle_{V',V} \right|}{\|v\|_V} \leq c \|h\|_H \quad \Longrightarrow \quad \|\overline{h}\|_{V'} \leq c \|h\|_H. \tag{1.4.4}$$

Proving that $\overline{h}$ is a continuous linear function, and hence $\overline{h} \in V'$ and proving (i).

Let us now prove (ii). Notice that from (1.4.1) and (1.4.2), we have that given $h\,,f\in H$ and $\alpha\,,\beta\in\mathbb{R}$

$$\begin{aligned}\langle\varphi(\alpha h+\beta f),v\rangle_{V',V}&=\langle\overline{\alpha f+\beta h},v\rangle_{V',V}=(\alpha f+\beta h,v)_H\\&=\alpha(f,v)_H+\beta(h,v)_H=\alpha\langle\overline{f},v\rangle_{V',V}+\beta\langle\overline{h},v\rangle_{V',V}\\&=\alpha\langle\varphi(f),v\rangle_{V',V}+\beta\langle\varphi(h),v\rangle_{V',V}=\langle\alpha\varphi(f)+\beta\varphi(h),v\rangle_{V',V}\end{aligned}$$

for all $v\in V$. Thus,

$$\varphi(\alpha f+\beta h)=\alpha\varphi(f)+\beta\varphi(h)$$

for all $f\,,h\in H$ and $\alpha\,,\beta\in\mathbb{R}$. Hence, the mapping (1.4.2) is linear. The continuity of φ follows directly from (1.4.4) taking into account that, by (1.4.2), $\varphi(h)=\overline{h}$, for all $h\in H$. Lastly, it remains to show that φ is injective. Suppose that $\overline{h}=\varphi(h)=0$. Then, since φ is linear, it follows that

$$(h,v)_H=\langle\overline{h},v\rangle_{V',V}=\langle\varphi(h),v\rangle_{V',V}=0.$$

Since V is dense in H, for any $v\in H$, there exists a sequence $(v_n)_{n=1}^{\infty}\subset V$ such that $v_n\to v$ strongly in V, Moreover, for every $n\in\mathbb{N}$,

$$\langle\overline{h},v_n\rangle_{V',V}=(h,v_n)_H=0.$$

From the continuity of both the mapping φ and the inner product $(\cdot,\cdot)_H$ it follows that

$$\begin{aligned}\langle\overline{h},v\rangle_{V',V}&=\langle\overline{h},\lim_{n\to\infty}v_n\rangle_{V',V}=\lim_{n\to\infty}\langle\overline{h},v_n\rangle_{V',V}\\&=\lim_{n\to\infty}\int_\Omega h(x)v_n(x)\,dx=\lim_{n\to\infty}0\\&=\int_\Omega h(x)\lim_{n\to\infty}v_n(x)\,dx=\int_\Omega h(x)v(x)\,dx.\end{aligned}$$

for all $v\in\overline{V}=H$. As a consequence of Hahn-Banach Theorem, second geometric form, it follows that $h=0$. Therefore, φ is injective.

□

Remark 1.4.1. *By Proposition 1.4.1 it follows that we can identify $\varphi(h)=\overline{h}$ with h, in the sense that $H\subseteq V'$. Hence, the following relations are valid:*

$$\langle\overline{h},v\rangle_{V',V}=(h,v)_H \tag{1.4.5}$$

for all $v \in V$ *and* $h \in H$*; and*

$$\|h\|_{V'} \leq c\|h\|_H, \tag{1.4.6}$$

where $c > 0$ *is the constant obtained in the continuous embedding* $V \hookrightarrow H$*. Hereafter, we shall understand the evolution triple* $V \subseteq H \subseteq V'$ *as described above. Therefore:*

(i) since V *is dense in* H *and the embedding* $V \hookrightarrow H$ *is continuous, we have that* $H' \hookrightarrow V'$ *is a continuous embedding;*

(ii) by the Riez-Fréchet Theorem, we can identify the real Hilbert space H *with its dual* H'*, and hence* $H \hookrightarrow V'$ *is a continuous embedding;*

(iii) since V *is a reflexive Banach space, it follows that* H' *is dense in* V'*, and hence* H *is dense in* V'*.*

To ensure the completeness of this work, let us prove the claims above.

(i) Since

$$V \subseteq H \subseteq V'$$

is a evolution triple, V *is dense in* H *and the embedding* $V \hookrightarrow H$ *is continuous, then there exists a positive constant* c *such that*

$$\|u\|_H \leq c\|u\|_V \tag{1.4.7}$$

for all $u \in V$*. Let* $\varphi \in Y'$ *be given. Then* φ *is a continuous linear functional; in particular,* φ *is bounded. Moreover, he have that*

$$\|\varphi\|_{H'} = \sup_{\substack{h \in H \\ x \neq 0}} \frac{\left|\langle \varphi, h\rangle_{H',H}\right|}{\|h\|_H} < \infty,$$

hence

$$\left|\langle \varphi, h\rangle_{H',H}\right| \leq \|h\|_H \|\varphi\|_{H'},$$

for all $h \in H$*. Due to (1.4.7), it follows that*

$$\left|\langle\varphi,h\rangle_{H',H}\right|\leq c\|\varphi\|_{H'}\|x\|_{V}. \tag{1.4.8}$$

Let $\overline{\varphi}\in H'$ be the restriction of the functional $\varphi\in H'$ to $V\subseteq H$. We claim that $\overline{\varphi}\in V'$. In fact, from (1.4.8) it follows that

$$\left|\langle\overline{\varphi},h\rangle_{V',V}\right|\leq c\|\varphi\|_{H'}\|h\|_{V} \tag{1.4.9}$$

for all $\varphi\in H'$, once $\overline{\varphi}|_V$ we have $\|\overline{\varphi}\|_{H'}=\|\overline{\varphi}\|_{V'}$, and $\langle\overline{\varphi},x\rangle_{H',H}=0$, for every $h\in H\setminus V$. Moreover,

$$\langle\varphi,h\rangle_{H',H}=\langle\overline{\varphi},h\rangle_{V',V}, \tag{1.4.10}$$

for all $h\in V$, and, from (1.4.9)

$$\|\overline{\varphi}\|_{V'}\leq c\|\varphi\|_{H'} \tag{1.4.11}$$

for all $\varphi\in H'$.

It remains to prove that if $\overline{\varphi}=0$ then $\varphi=0$, i.e., the restriction mapping

$$\begin{aligned} F:H' &\to V' \\ \varphi &\mapsto F(\varphi)=\overline{\varphi} \end{aligned} \tag{1.4.12}$$

is injective. Notice that, since V is dense in H, given $h\in H$ there exists a sequence $(h_n)_{n=1}^{\infty}\subset V$ such that $h_n\to h$ and $\langle\overline{\varphi},h_n\rangle_{V',V}=0$. From the continuity of $\varphi\in H'$ we have that $\langle\varphi,h_n\rangle_{V',V}=0$, for all $n\in\mathbb{N}$, and

$$0=\lim_{n\to\infty}\langle\varphi,h_n\rangle_{H',H}=\left\langle\varphi,\lim_{n\to\infty}h_n\right\rangle_{H',H}=\langle\varphi,h\rangle_{H',H},$$

for all $h\in H$. Hence, $\varphi=0$, which implies that the mapping (1.4.12) is injective, and from (1.4.11) follows its continuity. As a consequence, every $\varphi\in H'$ can be identified with some $\overline{\varphi}\in V'$, and in this manner we have that $H'\subseteq V'$ and the embedding $H'\hookrightarrow V'$

is continuous. Moreover, by identifying $\overline{\varphi} = F(\varphi)$ *with* φ*, it follows from* (1.4.10) *and* (1.4.11) *that*

$$\langle \varphi, h \rangle_{H',H} = \langle \varphi, h \rangle_{V',V}\,, \tag{1.4.13}$$

for all $x \in V$ *and for all* $\varphi \in H'$*, and*

$$\|\varphi\|_{V'} \leq c\|\varphi\|_{H'}, \tag{1.4.14}$$

for all $\varphi \in H'$.

(ii) Recall that the Riez-Fréchet Theorem states that for every $\varphi \in H'$ *there exists a unique* $f \in H$ *such that*

$$\langle \varphi, f \rangle_{H',H} = (f, v)_H\,,$$

for all $v \in H$*. In particular, the mapping*

$$\begin{aligned} T : H &\to H' \\ u &\mapsto T_u : H \to \mathbb{R} \\ &\qquad v \mapsto \langle T_u, v \rangle_{H',H}\,, \end{aligned} \tag{1.4.15}$$

where $\langle T_u, v \rangle_{H',H} = (u, v)_H$*, for all* $v \in H$ *is an isometric isomorphism. Thus, identifying* H *with its dual* H'*, it follows from* (i) *that*

$$H \cong H' \hookrightarrow V',$$

and the embedding is continuous;

(iii) Let us suppose that H' *is not dense in* V'*. Thus,* $\overline{H'}^{\|\cdot\|_{V'}}$ *is a closed proper linear subspace of* V'*. By a consequence of the Hahn-Banach Theorem, first geometric form, there exist a continuous linear functional* $f \in V''$ *such that* $\langle f, \varphi \rangle_{V'',V'} = 0$*, for all* $\varphi \in H'$*, with* $f \not\equiv 0$*. On the other hand, since* $V \subseteq H \subseteq V'$ *is an evolution triple,* V *is a reflexive space, and there exists* $\tilde{x} \in V$ *such that*

$$\langle f, \varphi \rangle_{V'',V'} = \langle \varphi, \tilde{x} \rangle_{V',V}\,,$$

for all $\varphi \in V'$. *It follows from* (1.4.13) *and the definition of* $f \in V''$ *that*

$$\langle f, \varphi \rangle_{V'',V'} = \langle \varphi, \tilde{x} \rangle_{V',V} = 0,$$

for all $\varphi \in V'$. *Since* $V \subseteq H$, *we have that* $\tilde{x} \in H$ *and* $\langle \varphi, \tilde{x} \rangle_{H',H} = 0$, *for all* $\varphi \in H'$, *i.e,* $\tilde{x} = 0$, *and hence* $f = 0$, *a contradiction.* □

We end this chapter with the following definition and the subsequent proposition will play a central role in the treatment of evolution equations.

Definition 1.4.2. *Consider a Gel'fand triple* $V \hookrightarrow H \hookrightarrow V'$, $1 < p < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *We define the Sobolev-Bochner space* $W^{1,p,q}(0,T;V,V')$ *as the set*

$$W^{1,p,q}(0,T;V,V') := \left\{ u \in L^p(0,T;V) \middle| \frac{du}{dt} \in L^q(0,T;V') \right\} \tag{1.4.16}$$

endowed with the norm

$$\|u\|_{W^{1,p,q}(0,T;V,V')} = \|u\|_{L^p(0,T;V)} + \left\| \frac{du}{dt} \right\|_{L^q(0,T;V')}. \tag{1.4.17}$$

Remark 1.4.2. *In Definition* 1.4.2 *the derivative* $\frac{du}{dt}$ *is to be undertood as a weak derivative.*

Proposition 1.4.2. *Let* $V \hookrightarrow H \hookrightarrow V'$ *be a Gel'fand triple,* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $0 < T < \infty$.

(i) $W^{1,p,q}(0,T;V,V'))$ *is a Banach space with norm given by* (1.4.17)*;*

(ii) the embedding $W^{1,p,q}(0,T;V,V') \hookrightarrow C([0,T];H)$ *is continuous;*

(iii) the set of all polynomials $w : [0,T] \to V$, $w(t) = \sum_i a_i t^i$, *with* $a_i \in V$ *is dense on* $L^p(0,T;V)$, $L^q(0,T;H)$ *and* $W^{1,p,q}(0,T;V,V')$*;*

(iv) given $0 \leq s \leq t \leq T$ *the following integration by parts*

$$(u(t), v(t))_H - (u(s), v(s))_H = \int_s^t \left(\left\langle \frac{du}{dt}, v(\tau) \right\rangle_{V',V} + \left\langle \frac{dv}{dt}, u(\tau) \right\rangle_{V',V} \right) dt, \tag{1.4.18}$$

holds for every $u, v \in W^{1,p,q}(0,T;V,V')$.

Proof. For a proof of this result, see [18] or [20] □

Chapter 2

Fractional Sobolev Spaces

In this chapter we introduce the concept of Fractional Sobolev Spaces, the fractional Laplacian operator and the associated eigenvalue associated with such operator.

2.1 A Brief introduction to Fractional Sobolev Spaces

We now present one of the most fundamental and important function spaces for this work. Together with the Sobolev-Bochner and Bochner spaces, it will form the foundation for the developments that follow.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, $1 \leq p < \infty$. We already seen that, for $m \in \mathbb{N}$, the classical Sobolev space is defined by the set

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \forall \alpha : 1 \leq |\alpha| \leq m\}, \tag{2.1.1}$$

endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}, \forall u \in W^{m,p}(\Omega). \tag{2.1.2}$$

We are now led to the following question: given $s \geq 0$, $s \notin \mathbb{N}$, it is possible to define a space $W^{s,p}(\Omega)$ in the same fashion as $W^{m,p}(\Omega)$? The answer is affirmative, and the construction of such a space - known as the fractional Sobolev Space - will now be presented. However, as we shall see later, these spaces will not be sufficiently suitable for the development of the subsequent theory. Therefore, this presentation will not be carried out in depth.

Definition 2.1.1. *Let $s \in (0,1)$, $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open set. We define the fractional Sobolev Space $W^{s,p}(\Omega)$ as the set*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \ : \ \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\}, \tag{2.1.3}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)} \right)^{1/p}, \tag{2.1.4}$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right), \tag{2.1.5}$$

is Gagliardo seminorm of u.

Suppose now that $s > 1$, $s \notin \mathbb{N}$. Notice that s can be written as a sum of integer part $m \in \mathbb{N}$ and a fractional part $\sigma \in (0,1)$, in the following form: given $s > 1$, $s \notin \mathbb{N}$, set $m := \lfloor s \rfloor$, i.e., m is the greatest positive integer such that $m \leq s$. Next we set $\sigma = s - m$. Hence, $s = m + \sigma$, with $m \in \mathbb{N}$, $m \geq 1$ and $\sigma \in (0,1)$. Using this decomposition, we can define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows:

Definition 2.1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, $s > 1$, $s \notin \mathbb{N}$ and $1 \leq p < \infty$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ as the set*

$$W^{s,p}(\Omega) := \{ u \in W^{m,p}(\Omega) \mid D^\alpha u \in W^{\sigma,p}(\Omega), \forall \alpha \ : \ 1 \leq |\alpha| \leq m \}, \tag{2.1.6}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{1/p}, \forall u \in W^{s,p}(\Omega), \tag{2.1.7}$$

where $s = m + \sigma$, with $m \geq 1$, $m \in \mathbb{N}$ and $\sigma \in (0,1)$.

As in the case of classical Sobolev space, we have the following theorem

Theorem 2.1.1. *Let $s > 0$, $s \notin \mathbb{N}$, $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ open. Them:*

(a) $\left(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)} \right)$ *is a Banach space;*

(b) $\overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}} = W^{s,p}(\mathbb{R}^N)$, *i.e, every function in* $W^{s,p}(\mathbb{R}^N)$ *can be approximated by a sequence of smooth, compactly supported functions in* $\mathbb{R}^N$*.*

Proof. A proof of this result can be found in[14] □

In contrast with item (b) of Theorem 2.1.1, and similarly to the case of classical Sobolev spaces, given an open set $\Omega \subset \mathbb{R}^N$, in general $C_c^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Such fact motivates the following definition

Definition 2.1.3. *Let* $s > 0$ *,* $s \notin \mathbb{N}$ *and* $1 \leq p < \infty$*. Given an open set* $\Omega \subset \mathbb{R}^N$*, we define the space* $W_0^{s,p}(\Omega)$ *as the clousure of* $C_c^\infty(\Omega)$ *in* $W^{s,p}(\Omega)$*, i.e,*

$$W_0^{s,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}. \tag{2.1.8}$$

It is also worth noting that, although we require $s > 0$ in the definition of fractional Sobolev spaces, we can also define spaces of the form $W^{s,p}(\Omega)$ with $s < 0$. To that end, let us consider the following

Definition 2.1.4. *Given* $s < 0$ *,* $s \notin \mathbb{Z}$*,* $1 \leq p < \infty$ *and an open set* $\Omega \subset \mathbb{R}^N$*, we have*

$$W^{s,p}(\Omega) = \left(W_0^{-s,p}(\Omega)\right)', \tag{2.1.9}$$

i.e, $W^{s,p}(\Omega)$*, for* $s < 0$*, is the topological dual of* $W^{-s,p}(\Omega)$*, where* $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.1.1. *Note that in Definition 2.1.4 the space* $W_0^{-s,p}(\Omega)$ *is defined as the dual of* $W_0^{\sigma,p}(\Omega)$*, where* $\sigma = -s > 0$*. In particular, we do not introduce* $W_0^{-s,p}(\Omega)$ *as a new space, but merely as a notational convention for the dual of a Sobolev space of positive order.*

Proposition 2.1.1. *Let* $\Omega \subset \mathbb{R}^N$ *be open, and let* $1 \leq p < \infty$*. The following holds:*

(i) if $0 < s' \leq s < 1$*, then* $W^{s,p}(\Omega) \hookrightarrow W^{s',p}(\Omega)$*, that is, there exists a positive constant* $c_1 = c_1(N, s, p) \geq 1$ *such that*

$$\|u\|_{W^{s,p}(\Omega)} \leq c_1 \|u\|_{W^{s',p}(\Omega)} \tag{2.1.10}$$

for all $u \in W^{s',p}(\Omega)$*.*

(ii) if $0 < s < 1$ and Ω is a Lipschitz domain with bounded boundary, then $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$. Hence, there exists a positive constant $c_2 = c_2(N, s, p) \geq 1$ such that

$$\|u\|_{W^{s,p}(\Omega)} \leq c_2 \|u\|_{W^{1,p}(\Omega)}, \tag{2.1.11}$$

for all $u \in W^{1,p}(\Omega)$;

(iii) if $s' \geq s \geq 1$ and Ω is a Lipschitz domain , then $W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$.

Proof. For a proof, see [14] □

As state at the beginning of this section, the treatment of the spaces $W^{s,p}(\Omega)$ presented here will not be extensive or in-depth, since these spaces are not sufficiently suitable for the problem we will study. For the interested reader, we recommend the book [14] which provides a more thorough treatment of these spaces.

Let us now proceed with the presentation of the definitions, properties, and results concerning the fractional Sobolev space that will be more suitable for our work. Before that, however, the following auxiliary definition is necessary:

Definition 2.1.5. *Let $s \in (0,1)$. We define the Sobolev space $H^s(\mathbb{R}^N)$, $N \geq 1$, as the set*

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \mid [u]_{H^s(\mathbb{R}^N)} < \infty \right\}, \tag{2.1.12}$$

with the norm $\|u\|_{H^s(\mathbb{R}^N)} := \left(\|u\|^2_{L^2(\Omega)} + [u]^2_{H^2(\Omega)} \right)^{1/2}$, where

$$[u]_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx\, dy \right)^{1/2} \tag{2.1.13}$$

is the Gagliardo seminorm of u.

For the development of the subsequent theory, we are interested in a subspace of $H^s(\mathbb{R}^N)$. Namely, we are interested in the space of all functions in $H^s(\mathbb{R}^N)$ whose support is contained in the open subset $\Omega \subset \mathbb{R}^N$.

Definition 2.1.6. *Given an open set $\Omega \subset \mathbb{R}^N$ and $s \in (0,1)$, we define*

$$H^s_0(\Omega) := \{ u \in H^s(\mathbb{R}^N) \mid u \equiv 0, \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}, \tag{2.1.14}$$

and we denote $(H^s_0(\Omega))' := H^{-s}(\Omega)$.

Proposition 2.1.2. *Let $H_0^s(\Omega)$ be as in the Definition 2.1.6. Then,*

(a)

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy, \tag{2.1.15}$$

where

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\left(\frac{N}{2}\right)}\Gamma(1-s)}, \tag{2.1.16}$$

is a inner product in $H_0^s(\Omega)$.

(b) the expression

$$\|u\|_{H_0^s(\Omega)} := \left(\frac{C_{N,s}}{2}\right)^{1/2}[u]_{H^s(\mathbb{R}^N)} \tag{2.1.17}$$

defines a norm in $H_0^s(\Omega)$ which is induced by the inner product (3.2.1).

(c) $(H_0^s(\Omega)\,,\mathcal{E}(\cdot,\cdot))$ is a Hilbert space. In particular, $\left(H_0^s(\Omega)\,,\|\cdot\|_{H_0^s(\Omega)}\right)$ is a Banach space.

Proof. For a proof see [14] □

We are interested in constructing a *Evolution triplet* of the form

$$H_0^s(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega). \tag{2.1.18}$$

Remembering the definition of a *Evolution Triplet*, we must show that

(i) $\left(H_o^s(\Omega)\,,\|\cdot\|_{H_0^s(\Omega)}\right)$ is a real, reflexive and separable Banach space;

(ii) $\left(L^2(\Omega)\,,(\cdot\,,\cdot)_{L^2(\Omega)}\right)$ is a separable Hilbert space, and

(iii) the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and dense.

Notice that, the condition (ii) is immediate, since for every $p \in [1\,,\infty)$, $L^p(\Omega)$ is separable, and $L^2(\Omega)$ is a Hilbert space endowed with an inner product given by

$$(u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\,dx. \tag{2.1.19}$$

The fact that $\left(H_0^s(\Omega)\,,\|\cdot\|_{H_0^s(\Omega)}\right)$ is a reflexive Banach space follows directly of the fact that $(H_0^s(\Omega)\,,\mathcal{E}(\cdot,\cdot))$ is a Hilbert space. It remains to show that the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and dense in $L^2(\Omega)$, and that $H_0^s(\Omega)$ is separable. Let us prove that, in fact, the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, For this purpose, we consider the following lemmas:

Lemma 2.1.1. *Let $s \in (0,1)$, $N > 2s$ an $\Omega \subset \mathbb{R}^N$ be an open and limited subset with $\partial\Omega \in C^0$. Let $\{u_n\}_{n=1}^{\infty} \subset H_0^s(\Omega)$ be a bounded sequence. There exists $u_\infty \in L^\nu(\mathbb{R}^N)$ such that, up to subsequences,*

$$u_j \to u_\infty, \text{ in } L^2(\Omega)$$

as $j \to \infty$, for any $\nu \in [1, 2_s^*)$.

Lemma 2.1.2. *If $s \in (0,1)$ and $N > 2s$, then*

(a) if Ω is such that $\partial\Omega \in C^0$, then

$$H_0^s(\Omega) \hookrightarrow\hookrightarrow L^\nu(\Omega), \tag{2.1.20}$$

that is, the embedding of $H_0^s(\Omega)$ into $L^\nu(\Omega)$ is compact for every $\nu \in [1, 2_s^*)$;

(b) the embedding

$$H_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$$

is continuous.

The results presented in Lemma 2.1.1 and Lemma 2.1.2 show us that the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is, indeed, continuous since $2_s^* > 2$. To see that $H_0^s(\Omega)$ is dense in $L^2(\Omega)$ we proceed as follows: first, notice that $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$. On the other hand, $C_c^\infty(\Omega)$ is a subspace of $H_0^s(\Omega)$. Hence, we have the following chain os inclusions

$$L^2(\Omega) = \overline{C_c^\infty(\Omega)}^{L^2(\Omega)} \subseteq \overline{H_0^s(\Omega)}^{L^2(\Omega)} \subseteq \overline{L^2(\Omega)}^{L^2(\Omega)} = L^2(\Omega). \tag{2.1.21}$$

The result concerning the separability of $H_0^s(\Omega)$ is show in section Section 2.3 since we need the definition of the *Fractional Laplacian Operator*.

2.2 The Fractional Laplacian Operator

Last but not least, we present the definition of the Fractional Laplacian operator, denoted by $(-\Delta)^s$. We began by recalling the definition and some properties of the so-called *Schwarts* *space*. Let $\alpha \in \mathbb{N}^N$ be a multi-index, that is,

$$\alpha := (\alpha_1, \cdots, \alpha_N), \, \alpha_i \in \mathbb{N}, \forall i \in \mathbb{N}. \tag{2.2.1}$$

Given $x \in \mathbb{R}^N$, $x = (x_1, \cdots, x_N)$, we define x^α as

$$x^\alpha := x_1^{\alpha_1} . \cdots . x_N^{\alpha_N}. \tag{2.2.2}$$

Given a function $f : \mathbb{R}^N \to \mathbb{R}$, we say that f decays rapidly if, for all multi-index $\alpha \in \mathbb{N}^N$, we have that

$$x^\alpha f(x) \to 0, \ as \ \|x\|_{\mathbb{R}^N} \to \infty. \tag{2.2.3}$$

Definition 2.2.1. *Given $N \geq 1$, define*

$$\mathscr{S}(\mathbb{R}^N) := \left\{ g \in C_c^\infty(\Omega) \mid \forall \alpha, \beta \in \mathbb{N}^N, x^\alpha \frac{\partial^\beta}{\partial x^\beta} \to 0, \ as \ \|x\|_{\mathbb{R}^N} \to \infty \right\}. \tag{2.2.4}$$

with a norm

$$\|\phi\|_{\mathscr{S}(\mathbb{R}^N)} := \sup_{|\alpha|} \sup_{|\beta| \leq N} \left| x^\alpha \frac{\partial^\beta \phi(x)}{\partial x^\beta} \right|, \forall \phi \in \mathscr{S}(\mathbb{R}^N). \tag{2.2.5}$$

we call this pair $\left(\mathscr{S}(\mathbb{R}^N), \|\cdot\|_{\mathscr{S}(\mathbb{R}^N)}\right)$ *a Schwartz space.*

Notice that if a function u belongs to $\mathscr{S}(\mathbb{R}^N)$, then the function itself, as well as all its partial derivatives of any order, decay rapidly at infinity. A classical example of such a function is the Gaussian

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & (0,\infty) \\ & x & \longmapsto & e^{-x^2} \end{array}. \tag{2.2.6}$$

Indeed, for every $k > 0$,

$$xf(x) = xe^{-x^2} = o(|x|^{-k}) \quad \text{as } |x| \to \infty. \tag{2.2.7}$$

With the aid of Definition 2.2.1 we can introduce a topology on $\mathscr{S}(\mathbb{R}^N)$ as follows:

Definition 2.2.2. *Given a sequence $(\phi_k)_{k=1}^\infty \subset \mathscr{S}(\mathbb{R}^N)$, we say that (ϕ_k) converges to $\phi \in \mathscr{S}(\mathbb{R}^N)$, if and only if, for all multi-indexes $\alpha, \beta \in \mathbb{N}^N$,*

$$x^\alpha \frac{\partial^\beta \phi_k(x)}{\partial x^\beta} \to x^\alpha \frac{\partial^\beta \phi(x)}{\partial x^\beta}, \tag{2.2.8}$$

uniformly for all $x \in \mathbb{R}^N$*, that is, if and only if*

$$\|\phi_k - \phi\|_{\mathscr{S}(\mathbb{R}^N)} \to 0, \text{ as } k \to \infty. \tag{2.2.9}$$

Proposition 2.2.1. *Let* $\alpha, \beta \in \mathbb{N}^N$*, and* $1 \leq p < \infty$*. The following holds true:*

(a) if $\phi \in \mathscr{S}(\mathbb{R}^N) \implies x^\alpha \dfrac{\partial^\beta \phi(x)}{\partial x^\beta} \in \mathscr{S}(\mathbb{R}^N), \forall \alpha, \beta \in \mathbb{N}^N$*;*

(b) $\mathscr{S}(\mathbb{R}^N) \subset L^1(\Omega)$ *and there exists a constant* $C_N > 0$ *such that* $\|\phi\|_{L^1(\mathbb{R}^N)} \leq C_N \|\phi\|_{N+1}$*;*

(c) $\overline{\mathscr{S}(\mathbb{R}^N)} = L^p(\mathbb{R}^N), \forall p \in [1, \infty)$*;*

(d) $\overline{C_c^\infty(\mathbb{R}^N)} = \mathscr{S}(\mathbb{R}^N)$.

Once the Schwartz spaces have been established, the fractional Laplacian operator can be rigorously defined:

Definition 2.2.3. *Let* $s \in (0,1)$ *and* $N \geq 1$. *We define the Fractional Laplacian Operator* $(-\Delta)^s$ *as follows*

$$\begin{array}{rccc} (-\Delta)^s : & \mathscr{S}(\mathbb{R}^N) & \longrightarrow & L^2(\mathbb{R}^N) \\ & u & \longmapsto & (-\Delta)^s u \end{array} \tag{2.2.10}$$

where

$$(-\Delta)^s u := \lim_{\varepsilon \to 0^+} C_{N,s} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \, dy, \, x \in \mathbb{R}^N, \tag{2.2.11}$$

and $C_{N,s}$ *is a positive normalization constant given by*

$$C_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\eta_1)}{|\eta|^{N+2s}} \, d\eta \right)^{-1}, \, \eta = (\eta_1, \eta'), \, \eta' \in \mathbb{R}^{N-1}. \tag{2.2.12}$$

Remark 2.2.1. *It is worthwhile mention that the definition of the Principal Value is the following:*

$$P.V \int_{\mathbb{R}^N} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \, dy := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{(u(x) - u(y))}{|x-y|^{N+2s}} \, dy, \, x \in \mathbb{R}^N.$$

In this dissertation the previous identification is used without mention.

2.3 Spectral Theory of the Fractional Laplacian Operator

For the result concerning separability, we consider the following eigenvalue problem for the fractional Laplacian Operator

$$\begin{cases} (-\Delta)^s u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3.1}$$

where $s \in (0,1)$, $N > 2s$ and $\Omega \subset \mathbb{R}^N$ is a open and bounded subset. It is possible to show that there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of eigenvalues of problem (2.3.1), with corresponding eigenfunctions $(\xi_k)_{k\in\mathbb{N}}$, which forms an orthonormal basis of $L^2(\Omega)$ and of $H_0^s(\Omega)$.

Concerning the principal eigenvalue, namely the smallest eigenvalue of problem (2.3.1), and the corresponding eigenfunction, we have the following result:

Proposition 2.3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded subset, $s \in (0,1)$, and $N > 2s$. The following holds:*

(i) the problem (2.3.1) admits a positive eigenvalue λ_1 which is characterized by

$$\lambda_1 := \min_{\substack{u\in H_0^s(\Omega) \\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \tag{2.3.2}$$

$$:= \min_{u\in H_0^s(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}; \tag{2.3.3}$$

(ii) there exist a strictly positive eigenfunction $\xi_1 \in H_0^s(\Omega)$ corresponding to the eigenvalue λ_1, which attains the minimum in (2.3.2), i.e., $\|\xi_1\|_{L^2(\Omega)} = 1$, and

$$\lambda_1 = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\xi_1(x)-\xi_1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy; \tag{2.3.4}$$

(iii) if $u \in H_0^s(\Omega)$ is a solution of

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy = \lambda_1 \int_\Omega u(x)\varphi(x)\,dx, \tag{2.3.5}$$

for every $\varphi \in H_0^s(\Omega)$, then $u = \theta\xi_1$, with $\theta \in \mathbb{R}$. In other words, λ_1 is simple.

Proof. See [14] for a proof. □

In fact, we have a more general result concerning not only the principals eigenvalue and eigenfunction, but all the eigenvalues and eigenfunctions for the fractional Laplacian Operator:

Proposition 2.3.2. *Under the hypothesis of Proposition 2.3.1 the following assertions holds:*

(i) for the problem (2.3.1), the set of eigenvalues is given by $(\lambda_k)_{k\in\mathbb{N}}$ with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \tag{2.3.6}$$

$$\lim_{k\to\infty} \lambda_k = +\infty \tag{2.3.7}$$

and, for every $k \in \mathbb{N}$, they are characterized by

$$\lambda_k := \min_{\substack{u\in\mathbb{P}_{k+1} \\ \|u\|_{L^2(\Omega)}=1}} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \tag{2.3.8}$$

$$:= \min_{u\in\mathbb{P}_{k+1}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}, \tag{2.3.9}$$

where $\mathbb{P}_{k+1}$ is a subspace of $H_0^s(\Omega)$ given by

$$\mathbb{P}_{k+1} := \{u \in H_0^s(\Omega) \mid (u,\xi_i)_{H_0^s(\Omega)} = 0\,,\, \forall i = 1,\cdots,k\} \tag{2.3.10}$$

(ii) given any $k \in \mathbb{N}$ there exists an eigenfunction $\xi_{k+1} \in \mathbb{P}_{k+1}$ corresponding to λ_{k+1}, and which attains the minimum in (2.3.8), i.e, $\|\xi_{k+1}\|_{L^2(\Omega)} = 1$ and

$$\lambda_{k+1} = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\xi_{k+1}(x)-\xi_{k+1}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy. \tag{2.3.11}$$

(iii) the set of eigenfunctions $\{\xi_k\}_{k\in\mathbb{N}} \subset H_0^s(\Omega)$ corresponding to the sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ is a orthonormal basis both for $L^2(\Omega)$ and for $H_0^s(\Omega)$.

(iv) the multiplicity of each eigenvalue λ_k is finite. In other words, if λ_k is such that

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+r} < \lambda_{k+r+1}, \tag{2.3.12}$$

for some $r \in \mathbb{N}$*, then the set of all eigenfunctions corresponding to* λ_k *is given by*

$$\text{span}\{\xi_k, \cdots, \xi_{k+r}\}. \tag{2.3.13}$$

Proof. See [14] for a proof of this result. □

From Proposition 2.3.2 (iii), we have that

$$\overline{span\{\xi_i \mid \in \mathbb{N}\}}^{\|\cdot\|_{H_0^s(\Omega)}} = H_0^s(\Omega), \tag{2.3.14}$$

hence, in fact $H_0^s(\Omega)$ is a separable space. Since $H_0^s(\Omega)$ is a separable, reflexive Banach space, its dual $(H_0^s(\Omega))' = H^{-s}(\Omega)$ is also a reflexive, separable Banach space, Thus, from what was presented above, the chain of inclusions

$$H_0^s(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega) \tag{2.3.15}$$

is a *Gelf'and Triplet*.

Having defined all the necessary tools for the study of our problem, we now establish our working framework. We will consider the Bochner space $L^2(0,T;H_0^s(\Omega))$, which, by Proposition 1.2.1 is a separable Hilbert space with inner product given by

$$(u,v)_{L^2(0,T;H_O^s(\Omega))} = \int_\Omega \mathcal{E}\left(u(\cdot,t)),v(\cdot,t))\right) dt. \tag{2.3.16}$$

By Proposition 1.2.3 and Remark 1.2.3 its dual space is identified with $L^2(0,T;H^{-s}(\Omega))$ and the duality paring is given by

$$\langle g^*,u\rangle_{L^2(0,T;H^{-s}),L^2(0,T;H_0^s)} = \int_0^T \langle g^*(t),u(t)\rangle_{H^{-s},H_0^s}\, dt, \tag{2.3.17}$$

for all $u \in L^2(0,T:H_0^s(\Omega))$. Assuming $\Omega \subset \mathbb{R}^N$ open and

$$Y = L^2(\Omega)\,, V = H_0^s(\Omega)$$

an adaptation of Lemma 1.2.1 tell us that for every $\tilde{g} \in L^2(Q_T)$, $Q_T = \Omega \times (0,T)$, there exists a linear functional $g^* \in L^2(0,T;H^{-s}(\Omega))$ such that

$$\langle g^*(t),u\rangle_{H^{-s},H_0^s} = \int_\Omega \tilde{g}(x,t)u(x,t)\, dx, \tag{2.3.18}$$

a.e $t \in [0,T]$, and the embedding $L^2(0,T;H_0^s(\Omega)) \hookrightarrow L^2(Q_T)$ is continuous. As a *Gel'fand triple* we use

$$H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$$

and we consider the Sobolev-Bochner space

$$W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega)) := \left\{ u \in L^2(0,T;H_0^s(\Omega)) \middle| \frac{du}{dt} \in L^2(0,T;H^{-S}(\Omega)) \right\} \tag{2.3.19}$$

endowed with the norm

$$\|u\|_{W^{1,2,2}(0,T:H_0^s,H^{-s})} = \|u\|_{L^2(0,T;H_0^s)} + \left\| \frac{du}{dt} \right\|_{L^2(0,T:H^{-s}(\Omega))}. \tag{2.3.20}$$

Chapter 3

Existence and Uniqueness of Solution for a Linear Parabolic Problem

3.1 A Comparison principle for a Linear Parabolic Problem

Let us define the following auxiliary linear parabolic problem

$$\begin{cases} u_t + (-\Delta)^s u + ku = f(x,t), \text{in } Q_T := \Omega \times (0,T) \\ \qquad\qquad u(x,t) = 0, \qquad \text{on } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \qquad\qquad u(x,t) = u_0, \qquad \text{in } \Omega, \end{cases} \tag{L.P.B}$$

where $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$ is open and bounded, $N \geq 1$ and $k \geq 0$ is a real parameter. Our focus lies on establishing results regarding the existence and uniqueness of solutions to this problem. To this end, we shall make use of a ***Comparison Principle***, as the underlying strategy is based on the ***Method of Sub- and Supersolutions***. Prior to this, however, it is necessary to introduce certain definitions and preliminary results.

Definition 3.1.1. *Let us consider $u_0 \in L^2(\Omega)$ and $k \geq 0$. We say that a function*

$$u \in W^{1,2,2}(0,T; H_0^s(\Omega), H^{-s}(\Omega))$$

*is a **subsolution** of finite energy of the linear parabolic problem (L.P.B) if and only if*

1. $f \in L^2(0,T; H^{-s}(\Omega))$,

2. *the inequality*

$$\int_0^T \langle u_t, v\rangle_{H^{-s},H_0^s}\,dt + \int_0^T \mathcal{E}(u,v)\,dt + k\int_0^T (u,v)_{L^2}\,dt \le \int_0^T \langle f(x,t), v\rangle_{H^{-s},H_0^s}\,dt$$

holds for all nonnegative function $v \in L^2(0,T;H_0^s(\Omega))$*, and*

3. $u(x,0) \le u_0$ *a.e in* Ω*, where* $u(\cdot,0) = \lim_{t\to 0^+} u(x,t)$ *in* $L^2(\Omega)$*.*

Analogously, one can define a **supersolution** of finite energy for the linear parabolic problem (*L.P.B*) by replacing the conditions (2) and (3) in Definition 3.1.1 with the following conditions

2'. the inequality

$$\int_0^T \langle u_t, v\rangle_{H^{-s},H_0^s}\,dt + \int_0^T \mathcal{E}(u,v)\,dt + k\int_0^T (u,v)_{L^2}\,dt \ge \int_0^T \langle f(x,t), v\rangle_{H^{-s},H_0^s}\,dt$$

holds for all nonnegative function $v \in L^2(0,T;H_0^s(\Omega))$, and

3' $u(x,0) \ge u_0$ a.e in Ω, where $u(\cdot,0) = \lim_{t\to 0^+} u(x,t)$ in $L^2(\Omega)$.

Definition 3.1.2. *Let* $u_0 \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ *and* $k \ge 0$*. We say that a function* $u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ *is a solution of finite energy (or simply a solution) of the parabolic problem* (*L.P.B*) *if and only if*

(i) *u is a subsolution of the lineear parabolic problem* (*L.P.B*)*;*

(ii) *u is a supersolution of the linear parabolic problem* (*L.P.B*)*, and*

(iii) $\lim_{n\to\infty} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = 0.$

Proposition 3.1.1. *Let* $w \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$*. Then*

$$-\int_0^T \left\langle w_t, w^-\right\rangle_{H^{-s},H_0^s} dt = \frac{1}{2}\int_0^T \frac{\partial}{\partial t}\|w^-\|^2_{L^2(\Omega)}\,dt = \frac{1}{2}\left(\|w^-(T)\|^2_{L^2(\Omega)} - \|w^-(0)\|^2_{L^2(\Omega)}\right). \tag{3.1.1}$$

Proof. Refer to [19], Lemma 3.3, adapting it to the case $V = H_0^s(\Omega)$. □

Remark 3.1.1. *The importance of Proposition 3.1.1 relays on the fact that, in general, if* $\underline{u}$ *and* $\overline{u}$ *are sub- and supersolutions, respectively, of* (P_1)*, it is not true that* $w^- := -\min\{\overline{u} - \underline{u}, 0\}$

is an element of $W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$*. Hence, it will not be possible to apply the Integration by Parts*

$$(u(t_2),v(t_2))_{L^2(\Omega)}-(u(t_1),v(t_1))_{L^2(\Omega)}=\int_{t_1}^{t_2}\left(\left\langle\frac{du}{dt},v\right\rangle_{H^{-s},H_0^s}+\left\langle\frac{du}{dt},v\right\rangle_{H^{-s},H_0^s}\right)dt. \tag{3.1.2}$$

Proposition 3.1.2 (*Comparison Principle*)**.** *Suppose that* $f_1, f_2\in L^2(0,T;H^{-s}(\Omega))$*, where* $\int_0^T\langle f_2-f_1,\phi\rangle_{H^{-s},H_0^s}\,dt\geq 0$*, for all nonnegative* $\phi\in L^2(0,T;H_0^s(\Omega))$*, and let* $u_{0_1}, u_{0_2}\in L^2(\Omega)$ *be such that* $u_{0_1}\leq u_{0_2}$ *a.e. in* Ω*. If* $\underline{u}, \overline{u}\in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ *satisfies*

$$\int_0^T\langle\underline{u}_t,\phi\rangle_{H^{-s},H_0^s}\,dt+\int_0^T\mathcal{E}(\underline{u},\phi)\,dt+k\int_0^T(\underline{u},\phi)_{L^2(\Omega)}\,dt\leq\int_0^T\langle f_1,\phi\rangle_{H^{-s},H_0^s}\,dt \tag{3.1.3}$$

with $\lim_{t\to 0^+}\|\underline{u}(t)-u_{0_1}\|_{L^2(\Omega)}=0$*, for all nonnegative test function* $\phi\in L^2(0,T;H_0^s(\Omega))$*, and*

$$\int_0^T\langle\overline{u}_t,\phi\rangle_{H^{-s},H_0^s}\,dt+\int_0^T\mathcal{E}(\overline{u},\phi)\,dt+k\int_0^T(\overline{u},\phi)_{L^2(\Omega)}\,dt\geq\int_0^T\langle f_2,\phi\rangle_{H^{-s},H_0^s}\,dt \tag{3.1.4}$$

with $\lim_{t\to 0^+}\|\overline{u}(t)-u_{0_2}\|_{L^2(\Omega)}=0$*, for all nonnegative test function* $\phi\in L^2(0,T;H_0^s(\Omega))$*, then* $\underline{u}\leq\overline{u}$ *a.e in* Q_T*.*

Proof. As a test function, we consider $w^-:=-\min\{\overline{u}-\underline{u},0\}$. As mentioned in Remark 3.1.1, in general $w^-\notin W^{1,2,2}(0,T;H_0^s,H^{-s}(\Omega))$, and therefore (3.1.2) cannot be applied. We shall overcome this difficulty by employing the integration by parts formula provided in Proposition 3.1.1. Let us take $\phi=w^-$ as a test function and let us define

$$w:=\overline{u}-\underline{u},\ \psi:=f_2-f_1,\ w_0:=u_{0_2}-u_{0_1}. \tag{3.1.5}$$

Subtracting (3.1.3) from (3.1.4) and using (3.1.5), we obtain

$$\int_0^T\left\langle w_t,w^-\right\rangle_{H^{-s},H_0^s}dt+\int_0^T\mathcal{E}\left(w,w^-\right)dt+k\int_0^T\left(w,w^-\right)_{L^2(\Omega)}dt\geq\int_0^T\left\langle\psi,w^-\right\rangle_{H^{-s},H_0^s}dt. \tag{3.1.6}$$

Using (3.1.1) on the first integral on the left, and noticing that

$$w^-(0)=\overline{u}(\cdot,0)-\underline{u}(\cdot,0)=u_0-u_0=0$$

we obtain

$$\int_0^T \left\langle w_t, w^- \right\rangle_{H^{-s}, H_0^s} dt = -\frac{1}{2}\left(\|w^-(T)\|^2_{L^2(\Omega)} - \|w^-(0)\|^2_{L^2(\Omega)}\right) = -\frac{1}{2}\left(\|w^-(T)\|^2_{L^2(\Omega)}\right) \le 0.$$

Let θ , γ be a given functions, and define $\theta^- := -\min\{\theta\,,0\}$ and $\gamma^- := -\min\{\gamma, 0\}$. Then

$$(\theta-\gamma)(\theta^- - \gamma^-) \le -(\theta^- - \gamma^-)^2 \tag{3.1.7}$$

$$\theta \cdot \theta^- = -(\theta^-)^2. \tag{3.1.8}$$

Applying (3.1.7) to the integral involving the energy term in (3.1.6) follows that

$$\begin{aligned} \mathcal{E}\left(w, w^-\right) &= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(w(x)-w(y))(w^-(x)-w^-(y))}{|x-y|^{N+2s}}\, dx dy \\ &\le -\frac{C_{N,s}}{2} \int_{\mathbb{R} \times \mathbb{R}^N} \frac{(w^-(x)-w^-(y))^2}{|x-y|^{N+2s}}\, dx dy \\ &= -\mathcal{E}\left(w^-, w^-\right) = -\|w^-\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.1.9}$$

and hence

$$\int_0^T \mathcal{E}\left(w, w^-\right) dt \le -\int_0^T \|w^-\|^2_{H_0^2(\Omega)}\, dt \le 0. \tag{3.1.10}$$

As for the integral involving the inner product in $L^2(\Omega)$, observe that, from (3.1.8)

$$k\int_0^T \int_\Omega w w^-\, dt = -k\int_0^T \int_\Omega (w^-)^2\, dt = -k\int_0^T \|w^-\|^2_{L^2(\Omega)}\, dt \le 0, \tag{3.1.11}$$

from which follows that

$$k\int_0^T \|w^-\|^2_{L^2(\Omega)}\, dt \ge 0. \tag{3.1.12}$$

Combining (3.1.6), (3.1), (3.1.10), (3.1.12) and the hypothesis that

$$\int_0^T \langle \psi, \phi \rangle_{H^{-s}, H_0^s}\, dt \ge 0$$

for all nonnegative $\phi \in L^2(0,T;H_0^s(\Omega))$, we obtain

$$\begin{aligned}
0 \le k\int_0^T \|w^-\|_{L^2(\Omega)}^2\, dt &= -k\int_0^T \left(w,w^-\right)_{L^2(\Omega)}\, dt \\
&\le \int_0^T \left\langle w_t,w^-\right\rangle_{H^{-s},H_0^s} dt + \int_0^T \mathcal{E}\left(w,w^-\right) dt - \int_0^T \left\langle \psi,w^-\right\rangle_{H^{-s},H_0^s} dt \\
&\le -\frac{1}{2}\|w^-(T)\|_{L^2(\Omega)}^2 - \int_0^T \|w^-\|_{L^2(\Omega)}^2\, dt - \int_0^T \left\langle \psi,w^-\right\rangle_{H^{-s},H_0^s} dt \\
&\le 0. \qquad (3.1.13)
\end{aligned}$$

It follows that $\int_0^T \|w^-\|_{L^2(\Omega)}^2\, dt = 0$ and hence $w^- = 0$ a.e Q_T, i.e, $\overline{u}(x,t) - \underline{u}(x,t) \ge 0$ for a.e $(x,t) \in Q_T$ □

3.2 The Associated Cauchy Problem

In order to establish a result concerning the existence and uniqueness of solutions to the linear parabolic problem (*L.P.B*), we shall rely fundamentally on its generalized formulation. Let $v \in H_0^s(\Omega)$ be a test function. Multiplying the differential equation in (*L.P.B*) by v and integrating over Ω, we obtain

$$\int_\Omega \frac{\partial u}{\partial t} v\, dx + \int_\Omega ((-\Delta)^s u)\, v\, dx + \int_\Omega u\, v\, dx = \int_\Omega f(x,t)\, v\, dx. \qquad (3.2.1)$$

Observe that, by definition of the distributional derivative, if follows that

$$\frac{d}{dt}(u,v)_{L^2(\Omega)} = \int_\Omega \frac{\partial u}{\partial t} v\, dx = \left\langle \frac{du}{dt}, v\right\rangle_{H^{-s},H_0^s}. \qquad (3.2.2)$$

Moreover, the integral involving the fractional Laplacian operator can be rewritten in terms of the Energy inner product as

$$\int_\Omega ((-\Delta)^s u) v\, dx = \mathcal{E}\left(u,v\right). \qquad (3.2.3)$$

Since $f \in L^2(0,T;H^{-s}(\Omega))$, by Remark 1.2.3, there exists a function $\tilde{f} \in L^2(Q_T)$ which, by abuse of notation, will be denoted by f, such that

$$\langle f(t),v\rangle_{H^{-s},H_0^s} = \int_\Omega f(x,t)\, v(x)\, dx. \qquad (3.2.4)$$

From (3.2.2), (3.2.3) and (3.2.4), equality (3.2.1) can be rewritten as

$$\left\langle \frac{du}{dt}, v \right\rangle_{H^{-s},H_0^s} + \langle A(u(t)), v\rangle_{H^{-s},H_0^s} = \langle f(t), v\rangle_{H^{-s},H_0^s}, \forall v \in H_0^s(\Omega), \tag{3.2.5}$$

where A is an operator defined by

$$\begin{aligned} A : H_0^s(\Omega) &\to H^{-s}(\Omega) \\ u(t) &\mapsto A(u(t)) : H_0^s(\Omega) \to \mathbb{R} \\ &\qquad\qquad v \mapsto \langle A(u(t)), v\rangle_{H^{-s},H_0^s(\Omega)}, \end{aligned} \tag{3.2.6}$$

with

$$\langle A(u(t)), v\rangle_{H^{-s},H_0^s} := \mathcal{E}(u(t), v) + k\int_\Omega (u(t), v)_{L^2(\Omega)}\, dx. \tag{3.2.7}$$

Hence, from (3.2.5) we obtain the generalized problem, also know as *Cauchy Problem*, associated to the auxiliary linear parabolic problem (*L.P.B*), given by

$$\begin{cases} \dfrac{du}{dt} + A(u(t)) = f(t), \text{ a.e } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{3.2.8}$$

where the operator A is as defined in (3.2.6) and (3.2.7).

Let us now study the Cauchy problem (3.2.8).

Definition 3.2.1. *A function $u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ is called a strong solution of (3.2.8) if the first equation in (3.2.8) holds in $H^{-s}(\Omega)$ and the initial condition $u(0) = u_0$ holds in $L^2(\Omega)$.*

The next result concerns the existence of a strong solution for the Cauchy Problem (3.2.8) and through it we will prove the existence of a finite energy solution for the problem (*L.P.B*).

Theorem 3.2.1. *Let $A : V \to V'$ be a pseudomonotone and coercive operator, $f \in L^q(0,T;V')$ and $u \in H$. Suppose the existence of a continuous increasing function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ such that*

$$\|A(v)\|_{V'} \le \mathcal{G}(\|v\|_H)(1 + \|v\|_V), \forall v \in V. \tag{3.2.9}$$

Then, the Cauchy Problem (3.2.8) has a strong solution $u \in W^{1,2,2}(0,T;V,V')$.

Proof. See [18], Chapter 8, Theorem 8.9, page 209. □

An important fact related to the subject is that every strong solution of (3.2.8) is also a solution of finite energy of the parabolic problem (*L.P.B*).

Lemma 3.2.1. *Let $f \in L^2(0,T;H^{-s}(\Omega))$. If $u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ is a strong solution of the Cauchy problem (3.2.8), then u is a solution of finite energy of (L.P.B).*

Proof. Suppose $u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ is a strong solution for (3.2.8). Let $v \in L^2(0,T;H^{-s}(\Omega))$ be an arbitrary test function. From (3.2.5) we obtain

$$\left\langle \frac{du}{dt}, v \right\rangle_{H^{-s},H_0^s} + \langle A(u(t)), v\rangle_{H^{-s},H_0^s} = \langle f(t), v\rangle_{H^{-s},H_0^s}, \ a.e\ t \in [0,T], \tag{3.2.10}$$

Integrating over t and using the relation ((3.2.2)), it follows that

$$\int_0^T \left\langle \frac{du}{dt}, v \right\rangle_{H^{-s},H_0^s} dt + \int_0^T \langle A(u(t)), v\rangle_{H^{-s},H_0^s}\, dt = \int_0^T \langle f(t), v\rangle_{H^{-s},H_0^s}\, dt, \tag{3.2.11}$$

which is equivalent to

$$\int_0^T \left\langle \frac{du}{dt}, v \right\rangle_{H^{-s},H_0^s} dt + \int_0^T \mathcal{E}(u,v)\, dt + k\int_0^T (u,v)_{L^2(\Omega)}\, dt = \int_0^T \langle f(t), v\rangle_{H^{-s},H_0^s}\, dt, \tag{3.2.12}$$

for all $v \in L^2(0,T;H_0^s(\Omega))$. In particular, (3.2.12) holds for every nonnegative test function $\phi \in L^2(0,T;H_0^s(\Omega))$. Moreover, since the initial condition in (3.2.8) is fulfilled in $L^2(\Omega)$ and the embedding

$$W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega)) \hookrightarrow C(0,T;L^2(\Omega))$$

is continuous, it follows that

$$\lim_{t\to 0^+} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = \|u(0) - u_0\|_{L^2(\Omega)} = 0, \tag{3.2.13}$$

and u is a finite energy solution of (*L.P.B*). □

Notice that Lemma 3.2.1 only establishes the existence of a finite energy solution of (*L.P.B*), but does not provide any information about the uniqueness of such a solution. To obtain such a result, we need some definitions and auxiliary results.

Definition 3.2.2. *Let V be a separable and reflexive Banach space, and let V' denote its dual. Given an operator*

$$A : V \to V',$$

we say that:

(a) the operator A is pseudomonotone if A is bounded, and whenever $u_k \rightharpoonup u$ in V with

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle_{V',V} \leq 0, \tag{3.2.14}$$

then, for all $v \in V$, it holds that

$$\langle A(u), u-v\rangle_{V',V} \leq \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle_{V',V}\,; \tag{3.2.15}$$

(b) the operator A is monotone if for all $u, v \in V$,

$$\langle A(u) - A(v), u-v\rangle_{V',V} \geq 0; \tag{3.2.16}$$

(c) the operator A is radially continuous if the mapping

$$\forall u, v \in V: \quad t \mapsto \langle A(u+vt), v\rangle_{V',V} \tag{3.2.17}$$

is continuous.

Following [18], let us consider a Banach space V endowed with a seminorm $|\,.\,|_V$, and a *Evolution Triplet*

$$V \hookrightarrow H \hookrightarrow V'$$

in such way that there exists a positive constant $C_p \in \mathbb{R}^+$ such that

$$\|v\|_V \leq C_p(|v|_V + \|v\|_H)\,,\ \forall v \in V. \tag{3.2.18}$$

Taking into account this seminorm, we have the following definition:

Definition 3.2.3. *An operator*

$$A : V \to V'$$

is semi-coercive if there exists constants $c_0 > 0$ and $c_1, c_2 \geq 0$ such that

$$\langle A(u), u\rangle_{V',V} \geq c_0 |u|_V^p - c_1 \|u\|_H^q - c_2, \tag{3.2.19}$$

for all $u \in V$, for some $1 \leq q < p < \infty$.

Considering the definitions above, the next two results will be essential for establishing the existence and uniqueness of a strong solution problem (3.2.8).

Lemma 3.2.2. *Every operator*

$$A : V \to V'$$

which is monotone and radially continuous satisfies the following condition: if $u_k \rightharpoonup u$ *weakly in* V*, and* $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle_{V',V} \le 0$*, then* $\langle A(u), u - v\rangle_{V',V} \le \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle_{V',V}$*. In particular, every monotone bounded radially continuous operator is pseudomonotone.*

Proof. Consider a sequence $(u_k)_{k=1}^{\infty} \in V$, $u \in V$ such that $u_k \rightharpoonup u$ weakly in V. Suppose additionally that

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle_{V',V} \le 0.$$

Since A is monotone, we have

$$\langle A(u_k), u_k - u\rangle_{V',V} \ge \langle A(u), u_k - u\rangle_{V',V} \to 0 \implies \liminf_{k\to\infty} \langle A(u_k), u_k - u\rangle_{V',V} \ge 0. \tag{3.2.20}$$

Since

$$0 \le \liminf_{k\to\infty} \langle A(u), u_k - u\rangle_{V',V} \le \limsup_{k\to\infty} \langle A(u), u_k - u\rangle_{V',V} \le 0,$$

it follows that

$$\lim_{k\to\infty} \langle A(u), u_k - u\rangle_{V',V} = 0. \tag{3.2.21}$$

Fix $\varepsilon > 0$ and take

$$u_\varepsilon := (1-\varepsilon)u + \varepsilon v\,,\; u, v \in V. \tag{3.2.22}$$

It follows that

$$\begin{aligned} 0 &\le \langle A(u_k) - A(u_\varepsilon), u_k - u_\varepsilon\rangle_{V',V} = \langle A(u_k) - A(u_\varepsilon), u_k - (1-\varepsilon)u - \varepsilon v\rangle_{V',V} \\ &= \langle A(u_k) - A(u_\varepsilon), u_k - u + \varepsilon u - \varepsilon v\rangle_{V',V} = \langle A(u_k) - A(u_\varepsilon), \varepsilon(u - v) + u_k - u\rangle_{V',V} \\ &= \langle A(u_k), \varepsilon(u - v) + u_k - u\rangle_{V',V} + \langle A(u_\varepsilon), \varepsilon(u - v) + u_k - u\rangle_{V',V} \\ &= \varepsilon\,\langle A(u_k), u - v\rangle_{V',V} + \langle A(u_k), u_k - u\rangle_{V',V} - \varepsilon\,\langle A(u_\varepsilon), u - v\rangle_{V',V} - \langle A(u_\varepsilon), u_k - u\rangle_{V',V}\,. \end{aligned} \tag{3.2.23}$$

Hence,

$$\varepsilon\,\langle A(u_k), u - v\rangle_{V',V} + \langle A(u_k), u_k - u\rangle_{V',V} - \varepsilon\,\langle A(u_\varepsilon), u - v\rangle_{V',V} - \langle A(u_\varepsilon), u_k - u\rangle_{V',V} \ge 0. \tag{3.2.24}$$

From which follows that

$$\varepsilon \langle A(u_k), u-v\rangle_{V',V} \geq \varepsilon \langle A(u_\varepsilon), u-v\rangle_{V',V} + \langle A(u_\varepsilon), u_k-u\rangle_{V',V} - \langle A(u_\varepsilon), u_k-u\rangle_{V',V}\,. \tag{3.2.25}$$

Taking the limit as $k \to \infty$ on (3.2.25) and using (3.2.21) we obtain

$$\begin{aligned} \lim_{\kappa\to\infty} \varepsilon \langle A(u_k), u-v\rangle_{V',V} &\geq \varepsilon \langle A(u_\varepsilon), u-v\rangle_{V',V} - \lim_{k\to\infty} \langle A(u_k), u_k-v\rangle_{V',V} \\ &+ \lim_{k\to\infty} \langle A(u_\varepsilon), u_k-u\rangle_{V',V} = \varepsilon \langle A(u_\varepsilon), u-v\rangle_{V',V}\,, \end{aligned} \tag{3.2.26}$$

where we use (3.2.21) and the fact that $u_k \rightharpoonup u$ weakly in V. It follows that

$$\liminf_{k\to\infty} \varepsilon \langle A(u_k), u-v\rangle_{V',V} \geq \varepsilon \langle A(u_\varepsilon), u-v\rangle_{V',V}$$

and with $\varepsilon > 0$, we have

$$\begin{aligned} \liminf_{k\to\infty} \langle A(u_k), u-v\rangle_{V',V} &\geq \langle A(u_\varepsilon), u-v\rangle_{V',V} \\ &= \langle A((1-\varepsilon)u+\varepsilon v), u-v\rangle_{V',V} \\ &= \langle A(u+\varepsilon(v-u)), u-v\rangle_{.} \end{aligned} \tag{3.2.27}$$

Using the fact that A is radially continuous and taking $\varepsilon \to 0$ in (3.2.27)

$$\begin{aligned} \liminf_{k\to\infty} \langle A(u_k), u-v\rangle_{V',V} &\geq \lim_{\varepsilon\to 0} \langle A(u+\varepsilon(u-v)), u-v\rangle_{V',V} \\ &= \left\langle A(u+\lim_{\varepsilon\to 0}\varepsilon(u-v)), u-v\right\rangle_{V',V} \\ &= \langle A(u), u-v\rangle_{V',V}\,. \end{aligned} \tag{3.2.28}$$

Hence,

$$\begin{aligned} \liminf_{k\to\infty} \langle A(u_k), u_k-v\rangle_{V',V} &= \liminf_{k\to\infty} \langle A(u_k), u_k-u+u-v\rangle_{V',V} \\ &= \liminf_{k\to\infty} (\langle A(u_k), u_k-u\rangle_{V',V} + \langle A(u_k), u-v\rangle_{V',V}) \\ &= \lim_{k\to\infty} \langle A(u_k), u_k-u\rangle_{V',V} + \lim_{k\to\infty} \langle A(u_k), u-v\rangle_{V',V} \\ &\geq \langle A(u), u-v\rangle_{V',V}\,, \forall v \in V. \end{aligned} \tag{3.2.29}$$

Consequently, if $A : V \to V'$ is bounded then A is pseudomonotone. □

3.3 Solution to the linear parabolic problem via the Cauchy problem

We now state and prove, following [7], the main theorem of this Chapter concerning the existence and uniqueness of a solution of the linear parabolic problem (*L.P.B*).

Theorem 3.3.1. *If $f \in L^2(0,T : H_0^s(\Omega), H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$ then the problem*

$$\begin{cases} u_t + (-\Delta)^s u + ku = f(x,t), \text{ in } Q_T := \Omega \times (0,T), \\ \qquad u(x,t) = 0, \qquad \text{on } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \qquad u(x,t) = u_0, \qquad \text{in } \Omega, \end{cases} \tag{L.P.B}$$

has a unique solution of finite energy $u \in W^{1,2,2}(0,T; H_0^s(\Omega), H^{-s}(\Omega))$.

Proof. In virtue of Lemma 3.2.1 is sufficient to show that the Cauchy Problem

$$\begin{cases} \dfrac{du}{dt} + A(u(t)) = f(t), \text{ a.e } t \in [0,T], \\ \qquad u(0) = u_0, \end{cases}$$

where the operator A is as defined in (3.2.6) and (3.2.7), has a strong solution.

Recall that the operator A is defined by

$$A : H_0^s(\Omega) \to H^{-s}(\Omega), \; \langle A(u(t)), v \rangle_{H_0^s, H^{-s}} = \mathcal{E}(u(t), v) + k \int_\Omega u(t)\, v\, dx. \tag{3.3.1}$$

The proof will be carried out in steps.

Step 1: Let us show that A is monotone. Clearly A is a linear operator. In fact, given $u,\, v \in H_0^s(\Omega)$ and $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned}
\langle A(u+\alpha v), w\rangle_{H_0^s,H^{-s}} &= \mathcal{E}\,(u+\alpha v, w) + k\int_\Omega (u+\alpha v) w\,dx \\
&= \mathcal{E}\,(u,w) + k\int_\Omega uw\,dx + \alpha\mathcal{E}\,(v,w) + \alpha k\int_\Omega vw\,dx \\
&= \mathcal{E}\,(u,w) + k\int_\Omega uw\,dx + \alpha\left(\mathcal{E}\,(v,w) + k\int_\Omega vw\,dx\right) \\
&= \langle A(u), w\rangle_{H_0^s,H^{-s}} + \alpha\,\langle A(v), w\rangle_{H_0^s,H^{-s}} \\
&= \langle A(u), w\rangle_{H_0^s,H^{-s}} + \langle \alpha A(v), w\rangle_{H_0^s,H^{-s}} \\
&= \langle A(u) + \alpha A(v), w\rangle_{H_0^s,H^{-s}}\,, \forall w \in H_0^s(\Omega).
\end{aligned}$$

Hence,

$$A(u+\alpha v) = A(u) + \alpha A(v),\ \forall u, v \in H_0^s(\Omega).$$

Using now the linearity of A we obtain

$$\begin{aligned}
\langle A(u) - A(v), u-v\rangle_{H_0^s,H^{-s}} &= \langle A(u-v), u-v\rangle_{H_0^s,H^{-s}} \\
&= \mathcal{E}\,(u-v, u-v) + k\int_\Omega (u-v)^2\,dx \geq 0,\ \forall u, v \in H_0^s(\Omega).
\end{aligned}$$

Hence, $A : H_0^s(\Omega) \to H^{-s}(\Omega)$ is monotone.

Step 2: The operator $A : H_0^s(\Omega) \to H^{-s}(\Omega)$ is a bounded operator. Indeed, let $u\,,\, v \in H_0^s(\Omega)$ with $v \neq 0$. Then, we have

$$\begin{aligned}
|\langle A(u), v\rangle_{H^{-s},H_0^s}| &= |\mathcal{E}\,(u,v) + k\int_\Omega uv\,dx \\
&\leq |\mathcal{E}\,(u,v)\,| + k|(u,v)_{L^2(\Omega)}| \\
&\leq \|u\|_{H_0^s(\Omega)}\|v\|_{H_0^s(\Omega)} + k\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.
\end{aligned} \tag{3.3.2}$$

Since the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, there exists a constant $c > 0$ such that

$$\|v\|_{L^2(\Omega)} \leq c\|v\|_{H_0^s(\Omega)},$$

for every $v \in H_0^s(\Omega)$. Therefore, we obtain

$$|\langle A(u), v\rangle_{H^{-s},H_0^s}| \leq \|u\|_{H_0^s(\Omega)}\|v\|_{H_0^s(\Omega)} + kc^2\|u\|_{H_0^s(\Omega)}\|v\|_{H_0^s(\Omega)}. \tag{3.3.3}$$

Consequently, for $v \neq 0$, it follows from (3.2.28) that

$$\|A(u)\|_{H^{-s}(\Omega)} = \sup_{\|v\|_{H_0^s(\Omega)}\neq 0} \frac{|\langle A(u), v\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}|}{\|v\|_{H_0^s(\Omega)}} \leq (1+kc^2)\|u\|_{H_0^s(\Omega)}, \tag{3.3.4}$$

for all $u \in H_0^s(\Omega)$, Thus, A is bounded, as claimed.

Step 3: Next, we show that the operator A is radially continuous. Let us consider $(t_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $t \in \mathbb{R}$ such that $t_n \to t$, as $n \to \infty$. Notice that:

$$\begin{aligned}\langle A(u+vt_n), v\rangle_{H^{-s},H_0^s} &= \mathcal{E}(u+vt_n, v) + k(u+vt_n, v)_{L^2(\Omega)}\\ &= \mathcal{E}(u,v) + k(u,v)_{L^2(\Omega)} + t_n\mathcal{E}(v,v) + kt_n(v,v)_{L^2(\Omega)}.\end{aligned} \tag{3.3.5}$$

Passing to the limit on (3.3.5) as $n \to \infty$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \langle A(u+vt_n), v\rangle_{H^{-s},H_0^s} &= \mathcal{E}(u,v) + k(u,v)_{L^2(\Omega)} + t\mathcal{E}(v,v) + kt(v,v)_{L^2(\Omega)}\\ &= \mathcal{E}(u,v) + \mathcal{E}(tv,v) + k(u,v)_{L^2(\Omega)} + k(tv,v)_{L^2(\Omega)}\\ &= \mathcal{E}(u+tv,v) + k(u+tv,v)_{L^2(\Omega)}\\ &= \langle A(u+tv), v\rangle_{H^{-s},H_0^s},\end{aligned} \tag{3.3.6}$$

for all u, $v \in H_0^s(\Omega)$, from which follows that the mapping

$$t \mapsto \langle A(u+tv), v\rangle_{H^{-s},H_0^s}$$

is continuous and, therefore the operator A is radially continuous.

From **Step 1-Step 3** and Lemma 3.2.2 it follows that the operator A is pseudomonotone.

Step 4 : Clearly the operator A is semicoercive. Indeed, given $v \in H_0^s(\Omega)$ and remembering that the constant $k > 0$, we have that

$$\langle A(v), v\rangle_{H^{-s},H_0^s} = \mathcal{E}(v,v) + k\int_\Omega v^2\,dx \geq \|v\|^2_{H_0^s(\Omega)} + k\|v\|^2_{L^2(\Omega)} \geq \|v\|^2_{H_0^s(\Omega)},$$

for all $v \in H_0^s(\Omega)$. Therefore, the operator A is semicoercive with $c_0 = 1$ and $c_1 = c_2 = 0$.

Step 5: Let us now show that there exists a continuous increasing function $\mathscr{G} : \mathbb{R} \to \mathbb{R}$ such that

$$\|A(v)\|_{H^{-s}(\Omega)} \leq \mathscr{G}(\|v\|_{L^2(\Omega)})(1 + \|v\|_{H_0^s(\Omega)}), \forall v \in H_0^s(\Omega).$$

From (3.3.2) it follows that

$$\begin{aligned}
|\langle A(u), v\rangle_{H^{-s},H_0^s}| &\leq \|u\|_{H_0^s(\Omega)}\|v\|_{H_0^s(\Omega)} + k\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq \|u\|_{H_0^s(\Omega)}\|v\|_{H_0^s(\Omega)} + k\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|u\|_{H_0^s}\|v\|_{H_0^s}\|u\|_{L^2} + k\|u\|_{L^2} \\
&= \|u\|_{H_0^s}\|v\|_{H_0^s} + k\|v\|_{L^2} + \|u\|_{L^2}\left(k\|v\|_{L^2} + \|u\|_{H_0^s}\|v\|_{H_0^s}\right) \\
&= (1 + \|u\|_{L^2})\left(\|u\|_{H_0^s}\|v\|_{H_0^s} + k\|v\|_{L^2}\right) \\
&\leq (1 + \|u\|_{L^2})\left(\|u\|_{H_0^s}\|v\|_{H_0^s} + kc\|v\|_{H_0^s(\Omega)}\right) \\
&= (1 + \|u\|_{L^2})\left(\|u\|_{H_0^s} + kc\right)\|v\|_{H_0^s}.
\end{aligned}$$

Therefore, if we suppose that $\|v\|_{H_0^s} \neq 0$ we obtain

$$\|A(u)\|_{H^{-s}} = \sup_{\|v\|_{H_0^s} \neq 0} \frac{|\langle A(u), v\rangle_{H^{-s},H_0^s}|}{\|v\|_{H_0^s}} \leq (1 + \|u\|_{L^2})\left(kc + \|u\|_{H_0^s}\right). \tag{3.3.7}$$

If $k > 0$, then we choose $c > 0$ in such a manner that $kc > 1$, and therefore

$$\begin{aligned}
(1 + \|u\|_{L^2})\left(kc + \|u\|_{H_0^s}\right) &= k^2c^2\left(\frac{1 + \|u\|_{L^2}}{kc}\right)\left(\frac{\|u\|_{H_0^s}}{kc} + 1\right) &(3.3.8)\\
&< k^2c^2(1 + \|u\|_{L^2})\left(kc + \|u\|_{H_0^s}\right) &(3.3.9)\\
&= \mathscr{G}(\|u\|_{L^2})\left(1 + \|u\|_{H_0^s}\right), &(3.3.10)
\end{aligned}$$

with $\mathscr{G}(x) = k^2c^2(1+x)$. Notice that, since $k^2c^2 > 0$ the function $\mathscr{G}$ is indeed increasing. If $k = 0$, then

$$\begin{aligned}
(1 + \|u\|_{L^2})\|u\|_{H_0^s} &\leq (1 + \|u\|_{L^2})\left(1 + \|u\|_{H_0^s}\right) &(3.3.11)\\
&= \mathscr{G}(\|u\|_{L^2})\left(1 + \|u\|_{H_0^s}\right), &(3.3.12)
\end{aligned}$$

with $\mathscr{G}(x) = (1 + x)$ and clearly $\mathscr{G}$ is increasing.

Therefore, the operator A is semicoercive, pseudomonotone and there exists a increasing function $\mathscr{G};\mathbb{R}\to\mathbb{R}$ satisfiyng the condition

$$\|A(v)\|_{H^{-s}(\Omega)}\leq \mathscr{G}(\|v\|_{L^2(\Omega)})(1+\|v\|_{H_0^s(\Omega)}),\ \forall v\in H_0^s(\Omega).$$

By Theorem 3.2.1, the Cauchy problem (3.2.8) admits a strong solution

$$\tilde{u}\in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega)),$$

and by Lemma 3.2.1, it is also a finite energy solution to problem (*L.P.B*). Hence, we have established the existence of a solutions. In particular, $\tilde{u}$ is a finite energy subsolution and supersolution of (*L.P.B*). Now, suppose that there exists another finite energy solution

$$\hat{u}\in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$$

to problem (*L.P.B*). Then $\hat{u}$ is also a finite energy subsolution and supersolution for this problem. Therefore, by the Comparison Principle, Proposition 3.1.2, we obtain that $\hat{u}\leq\tilde{u}$, $\tilde{u}\leq\hat{u}$ almost everywhere in Q_T; consequently, $\tilde{u}=\hat{u}$ almost everywhere in Q_T, and uniqueness follows. $\square$

Chapter 4

The Parabolic Logistic Equation with Harvesting

4.1 The Method of Monotone Iterations

The aim of this chapter is to apply the method of monotone iterations to study the existence and uniqueness of solution of the parabolic problem

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,u), & \text{in } Q_T := \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,t) = u_0, & \text{in } \Omega, \end{cases} \tag{P_1}$$

where $k \geq 0$, and $g : Q_T \to \mathbb{R}$ is a Carathéodory function, that is:

(i) $u \mapsto g(x,t,u)$ is continuous a.e $(x,t) \in Q_T$; and

(ii) $(x,t) \mapsto g(x,t,u)$ is measurable for each $u \in \mathbb{R}$.

Moreover, we will suppose the existence of a function $\kappa \in L^2(\Omega)$ and a continuous increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ such that:

(H_1) $|g(x,t,u)| \leq \kappa(x,t)(1 + \alpha(|u|))$, for all $u \in \mathbb{R}$ and a.e $(x,t) \in Q_T$,

(H_2) g is monotone increasing on in the third variable, that is, for every $v, w \in \mathbb{R}$

$$v \leq w \implies g(x,t,v) \leq g(x,t,w).$$

Definition 4.1.1. *Let us consider $u_0 \in L^2(\Omega)$ and $k \geq 0$. We say that a function*

$$u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$$

*is a **subsolution** of finite energy of the parabolic problem (P_1) if and only if*

1. $g(\cdot,\cdot,u) \in L^2(0,T;H^{-s}(\Omega))$,
2. *the inequality*

$$\int_0^T \langle u_t, v\rangle_{H^{-s},H_0^s}\, dt + \int_0^T \mathcal{E}(u,v)\, dt + k\int_0^T (u,v)_{L^2}\, dt \leq \int_0^T \langle g(\cdot,t,u(\cdot,t)),v\rangle_{H^{-s},H_0^s}\, dt$$

   holds for all nonnegative function $v \in L^2(0,T;H_0^s(\Omega))$, and

3. $u(x,0) \leq u_0$ *a.e in Ω, where $u(\cdot,0) = \lim_{t\to 0^+} u(x,t)$ in $L^2(\Omega)$.*

Analogously, one can define a **supersolution** of finite energy for the parabolic problem (P_1) by replacing the conditions (2) and (3) in Definition 4.1.1 with the following conditions

2'. the inequality

$$\int_0^T \langle u_t, v\rangle_{H^{-s},H_0^s}\, dt + \int_0^T \mathcal{E}(u,v)\, dt + k\int_0^T (u,v)_{L^2}\, dt \geq \int_0^T \langle g(\cdot,t,u(\cdot,t)),v\rangle_{H^{-s},H_0^s}\, dt$$

holds for all nonnegative function $v \in L^2(0,T;H_0^s(\Omega))$, and

3' $u(x,0) \geq u_0$ a.e in Ω, where $u(\cdot,0) = \lim_{t\to 0^+} u(x,t)$ in $L^2(\Omega)$.

Moreover, any sub- or supersolution of finite energy will be denoted, respectively, by $\underline{u}$ and $\overline{u}$.

Remark 4.1.1. *In the Definition 4.1.1, Item 3, the limit in $L^2(\Omega)$ exists since the embedding*

$$W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega)) \hookrightarrow C(0,T;L^2(\Omega))$$

is continuous. The same remark applies in the definition of supersolution.

With the aid of Definition 4.1.1 we can now define the meaning of a function $u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ being a solution of (P_1).

Definition 4.1.2. *Let $u_0 \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ and $k \geq 0$. We say that a function $u \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$ is a solution of finite energy (or simply a solution) of the parabolic problem (P_1) if and only if* *(i)* *u is a subsolution of the parabolic problem (P_1);*

(ii) *u is a supersolution of the parabolic problem (P_1); and*

(iii) $\lim\limits_{n\to\infty} \|u(\cdot,t) - u_0\|_{L^2(\Omega)} = 0.$

Proposition 4.1.1. *Let X and Y be real Banach spaces and $X \hookrightarrow Y$ be a continuous embedding. If,*

$$u_k^{(n)} = v_k \text{ on } (0,T) \text{ for all } k \text{ and fixed } n \geq 1,$$

$$u_k \rightharpoonup u \text{ in } L^p(0,T;X),$$

$$v_k \rightharpoonup v \text{ in } L^q(0,T;Y),$$

for $1 \leq p, q < \infty$, then $u^{(n)} = v$ on $(0,T)$.

Proof. See [20], page 419 for a proof of this result. □

With the above in mind, we can now state and prove, following [7], the main result of this section: the existence and uniqueness of a solution to the parabolic problem (P_1). The proof of this result is divided in two parts. The first one is presented bellow and is related to the constructions of the sequence of solutions thought the application of the method of monotone interactions. The second part, to be presented in Section 4.2, show that the limit of the constructed sequence of solutions is in fact a solution of the parabolic problem (P_1).

Theorem 4.1.1. *Let $u_0 \in L^\infty(Q_T)$ and suppose that the hypothesis (H_1) and (H_2) holds. Additionally, suppose the existence of $\underline{u}, \overline{u} \in L^\infty(Q_T)$, ordered sub and supersolutions, respectively, a.e in Q_T of the problem*

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,u), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{4.1.1}$$

Then, there exist a solution u for (4.1.1) in such a way that $\underline{u} \leq u \leq \overline{u}$ a.e Q_T. Moreover, if g is locally Lipchitz in the third variable, then (4.1.1) posses a unique solution in $L^\infty(Q_T)$.

Proof. **Part 1:** *Constructing the sequence of solutions.*

The strategy to prove the existence of solution consists in using the method of monotone iterations. First we define two measurable functions

$$\overline{u}, \underline{g} : Q_T \to \mathbb{R}$$

where

$$\begin{aligned} \overline{\underline{u}}(x,t) &= \operatorname{ess\,sup}\{|\underline{u}(x,t)|\,,\,|\overline{u}(x,t)|\}, \\ \overline{\underline{g}}(x,t) &= \operatorname{ess\,sup}\{\left|\underline{g}(x,t)\right|\,,\,|\overline{g}(x,t)|\}, \end{aligned} \tag{4.1.2}$$

a.e. in Q_T. Since $\underline{u}$, $\overline{u} \in L^\infty(Q_T)$, it follows immediately that $\overline{\underline{u}} \in L^\infty(Q_T)$. Moreover, by (H_1), we also have that

$$\overline{\underline{g}}(x,t) \leq \beta(x,t)(1+\alpha(\overline{\underline{u}})) \leq C\,.\,\beta(x,t), \tag{4.1.3}$$

a.e. in Q_T. Hence, $\overline{\underline{g}} \in L^2(Q_T)$, since $\beta \in L^2(Q_T)$.

Let us now proceed to the application of the method of monotone iterations. To this end, let us consider the following iterated problem:

$$\begin{cases} u_t^{(n)} + (-\Delta)^s u^{(n)} + ku^{(n)} = g(x,t,u^{(n-1)}), & \text{in } Q_T, \\ u^{(n)}(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u^{(n)}(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{4.1.4}$$

Note that (4.1.4) is a linear parabolic problem. Let us take $u^0 := \underline{u}$, a subsolution of (4.1.1), which exists by assumption, and consider the problem:

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,\underline{u}(x,t)), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{4.1.5}$$

Let us note that $|g(x,t,\underline{u}(x,t))| \leq \overline{\underline{g}}(x,t)$ a.e $(x,t) \in Q_T$, and since $\overline{\underline{g}} \in L^2(Q_T)$, we obtain that $g(x,t,\underline{u}(x,t)) \in L^2(Q_T)$. Consequently, we can identify $g(x,t,\underline{u}(x,t))$ with a function $f \in L^2(0,T;H^{-s}(\Omega))$, given by

$$\langle f(t), \varphi \rangle_{H^{-s}(\Omega),H_0^s(\Omega)} = \int_\Omega g(x,t,\underline{u}(x,t))\varphi(t)\,dx,$$

for all $\varphi \in H_0^s(\Omega)$ and a.e $t \in (0,T)$. Consequently, the problem (4.1.5) can be reduced to the linear problem

$$\begin{cases} u_t + (-\Delta)^s u + ku = f(x,t), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{4.1.6}$$

Taking into account Theorem 3.3.1, problem (4.1.6) admits a unique solution

$$u^1 \in W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega));$$

in particular, the solution $u^1 \in L^2(0,T;H_0^s(\Omega))$. Moreover, by the Comparison Principle, it holds that $\underline{u} \leq u^1 \leq \overline{u}$ for a.e $(x,t) \in Q_T$. Hence, $u^1 \in L^\infty(Q_T)$, since $\overline{\underline{u}} \in L^\infty(Q_T)$ and $\left|u^1(x,t)\right| \leq \overline{\underline{u}}(x,t)$ for a.e $(x,t) \in Q_T$.

Let us consider now the problem

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,u^1(x,t)), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{4.1.7}$$

We have that $\left|g(x,t,u^1(x,t))\right| \leq \underline{g}(x,t)$ for a.e. $(x,t) \in Q_T$. Therefore $g(x,t;u^1(x,t)) \in L^2(Q_T)$ and we can identify it with a function $f_1 \in L^2(0,T;H^{-s}(\Omega))$ defined by

$$\langle f_1(t), \varphi(t) \rangle_{H^{-s}(\Omega),H_0^s(\Omega)} = \int_\Omega g(x,t,u^2(x,t))\varphi(x)\,dx,$$

for all $\varphi \in H_0^s(\Omega)$, for a.e. $t \in [0,T]$.

Hence, problem (4.1.7) can be reduced to the linear problem

$$\begin{cases} u_t + (-\Delta)^s u + ku = f^1(x,t), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{4.1.8}$$

Again by Theorem 3.3.1, there exists a unique solution $u^2 \in L^2(0,T;H_0^s(\Omega))$, and by the Comparison Principle it holds that $\underline{u} \leq u^1 \leq u^2 \leq \overline{u}$ a.e Q_T and $u^2 \in L^\infty(Q_T)$. We proceed by induction. Suppose that $u^1\,, u^2\,, \cdots, u^{n-1}\,, n \geq 2$, satisfying $u^1 \leq u^2 \leq \cdots \leq u^{n-1}\,, u^k \in L^\infty(Q_T)\,, k = 1\,, 2 \cdots, n-1$ are all defined. Consider the following iterated problem:

$$\begin{cases} u_t + (-\Delta)^s u + ku = g(x,t,u^{n-1}(x,t)), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{4.1.9}$$

By the same previous arguments, there exists a unique solution of problem (4.1.9) $u^n \in L^2(0,T;H_0^s(\Omega))$, where $\underline{u} \le u^1 \le u^2 \le \cdots \le n^{n-1} \le u^n \le \bar{u}$, and $u^n \in L^\infty(Q_T)$. Thus we have constructed a monotone increasing sequence $(u^n)_{n=1}^\infty \subset L^\infty(Q_T)$ that is bounded. By the *Bolzano - Weistrass*, there exists a function $u \in L^\infty(Q_T)$ such that

$$u(x,t) = \lim_{n\to\infty} u^n(x,t) \, for \, a.e \, (x,t) \in Q_T. \tag{4.1.10}$$

Moreover, note that u is uniquely determined by $(u^n)_{n=1}^\infty$ and $\underline{u} \le u \le \bar{u}$. Hence $|u| \le \overline{\underline{u}} \in L^\infty(Q_T)$ and $u \in L^\infty(Q_T)$. We obtain with a simple application of the triangle inequality that

$$|u^n - u| \le (|u^n| + |u|)^2 \le (2\overline{\underline{u}})^2. \tag{4.1.11}$$

and by Theorem 1.1.5 we have that

$$u^n \to u, \ strongly \ on \ L^2(Q_T) \ as \ n \to \infty. \tag{4.1.12}$$

□

4.2 The solution Via the Monotone Iterations

In this section we present the remaining part of the proof of Theorem 4.1.1. In Section 4.1 we proved, following [7], that the sequence $(u^n)_{n=1}^\infty \subset L^\infty(Q_T)$ converges strongly in $L^2(Q_T)$ to a function $u \in L^\infty(Q_T)$. It remains to show that this limit function is in fact a solution for the parabolic problem (4.1.1). We state the remain part of Theorem 4.1.1 as a Corollary. For the sake of consistency, we follow the same enumeration of steps as in the proof of Theorem 4.1.1.

Corollary 4.2.1. *Under the hipotesys of theorem 4.1.1, the limit function $u \in L^\infty(Q_T)$ satisfying (4.1.12) is a solution for the parabolic problem (4.1.1). Moreover, if g is locally Lipchitz in the third variable, then (4.1.1) posseses a unique solution in $L^\infty(Q_T)$.*

Proof. **Part 2:** *The limit function as a solution.*

Let us now show that $u \in L^\infty(Q_T)$ and it is a solution for the problem (4.1.1). For this purpose, we have to stablish the following four convergences:

$$\lim_{n\to\infty}\int_0^T \langle u_t^n(t),\varphi(t)\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}\,dt = \int_0^T \langle u_t(t),\varphi(t)\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}\,dt \tag{4.2.1}$$

$$\lim_{n\to\infty}\int_0^T \mathcal{E}\left(u_t^n(t),\varphi(t)\right)dt = \int_0^T \mathcal{E}\left(u_t(t),\varphi(t)\right)dt, \tag{4.2.2}$$

$$\lim_{n\to\infty} k\int_0^T\int_\Omega u^n(x,t)\varphi(x,t)\,dxdt = k\int_0^T\int_\Omega u(x,t)\varphi(x,t)\,dxdt, \tag{4.2.3}$$

and

$$\lim_{n\to\infty}\int_0^T\int_\Omega g(x,t,u^{n-1}(x,t))\varphi(x,t)\,dxdt = \int_0^T\int_\Omega g(x,t,u(x,t))\varphi(x,t)\,dxdt, \tag{4.2.4}$$

for all $\varphi \in L^2(0,T;H_0^s(\Omega))$.

We begin the verification of these convergences by noting that the convergence in (4.2.3) is a immediate consequence of the convergence (4.1.12). In fact, note that

$$\lim_{n\to\infty}\|u^n-u\|_{L^2(Q_T)}=0 \iff \lim_{n\to\infty}\int_0^T\int_\Omega |u^n(x,t)-n(x,t)|^2\,dxdt = 0.$$

Thus, giving any $\varphi \in L^2(0,T;H_0^s(\Omega))$ we have,

$$u^n(x,t)\varphi(x,t) \to u(x,t)\varphi(x,t) \text{ almost everywhere in } Q_T, \tag{4.2.5}$$

and

$$|u^n(x,t)-u(x,t)|^2\,|\varphi(x,t)| \le (2\overline{u})^2\,|\varphi(x,t)|, \tag{4.2.6}$$

a.e all $(x,t)\in Q_T$, with $(2\overline{u})^2\,|\varphi(x,t)|$ integrable over Q_T. Moreover, by Theorem 1.1.5, the convergence in (4.2.3) follows.

Let us now show the others convergences. We begin by showing the convergence (4.2.4). Note that, by hypothesis, g is monotone increasing in the third variable. From the monotonicity

$$\underline{u} \le u^{n-1} \le u^n \le \overline{u},$$

$n \geq 2$, a.e Q_T. So, we have that

$$g(x,t,\underline{u}(x,t)) \leq g(x,t,u^{n-1}(x,t)) \leq g(x,t,\overline{u}(x,t)),$$

and hence

$$\begin{aligned}\left|g(x,t,u^{n-1}(x,t))\right| \leq \max\{|g(x,t,\underline{u}(x,t))|,|g(x,t,\overline{u}(x,t))|\} = \\ = \underline{\overline{g}}(x,t) \in L^2(Q_T)\end{aligned} \tag{4.2.7}$$

From Proposition 1.2.2 we have that,

$$\int_0^T \int_\Omega \left|\underline{\overline{g}}(x,t)\varphi(x,t)\right| dx dt \leq \|\underline{\overline{g}}\|_{L^2(Q_T)} \|\varphi\|_{L^2(Q_T)} < \infty, \tag{4.2.8}$$

hence $\underline{\overline{g}} \in L^1(Q_T)$ with

$$\left|g(x,t,u^{n-1}(x,t))\varphi(x,t)\right| \leq \underline{\overline{g}}(x,t)\,|\varphi(x,t)|,$$

for all $n \in \mathbb{N}$ and a.e $(x,t) \in Q_T$. The convergence (4.2.4) follows from Theorem 1.1.5.

Let us now verify the convergence (4.2.2). First we show that the sequence $(u^n)_{n=1}^\infty$ is bounded on $L^2(0,T,H_0^s(\Omega))$. We already now that the function $u^n \in L^\infty(Q_T)$ is the solution of the iterated problem (4.1.9). In particular, by the definition of solution for such problem, we know that u^n satisfies the relation

$$\begin{aligned}&\int_0^T \int_\Omega g(x,t,u^{n-1})\varphi\, dx dt = \\ &= \int_0^T \langle u_t^n, \varphi\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}\, dt + \int_0^T \mathcal{E}(u^n,\varphi)\, dt + k\int_0^T (u^n,\varphi)_{L^2(Q_T)}\, dt,\end{aligned} \tag{4.2.9}$$

for all $\varphi \in L^2(0,T;H_0^s(\Omega))$. Taking u^n as a test function, (4.2.9) becomes

$$\begin{aligned}&\int_0^T \int_\Omega g(x,t,u^{n-1})u^n\, dx dt \\ &= \int_0^T \langle u_t^n, u^n\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}\, dt + \int_0^T \mathcal{E}(u^n,u^n)\, dt + k\int_0^T (u^n,u^n)_{L^2(Q_T)}\, dt.\end{aligned} \tag{4.2.10}$$

It follows from (4.2.10) that

$$\begin{aligned}\|u^n\|^2_{L^2(0,T,H^s_0(\Omega))} &\leq \|u^n\|^2_{L^2(0,T;H^s_0(\Omega))}+k\|u^n\|^2_{L^2(Q_T)}\\ &= \int_0^T \mathcal{E}(u^n,u^n)\,dt + k\int_0^T\int_\Omega (u^n(x,t))^2\,dx\,dt \\ &= \int_0^T\int_\Omega g(x,t,u^{n-1}(x,t))u^n(x,t)\,dx\,dt - \int_0^T \langle u^n_t,u^n\rangle_{H^{-s}(\Omega),H^s_0(\Omega)}\,dt.\end{aligned} \tag{4.2.11}$$

Since $u^n \in W^{1,2,2}(0,T;H^s_0(\Omega),H^{-s}(\Omega))$, it follows from the integration by parts in Proposition 1.4.2, item (iv), that

$$(u^n(T),u^n(T))_{L^2(\Omega)} - (u^n(0),u^n(0))_{L^2(\Omega)} = 2\int_0^T \langle u^n_t,u^n\rangle_{H^{-s}(\Omega),H^s_0(\Omega)}\,dt. \tag{4.2.12}$$

Hence,

$$\begin{aligned}-\int_0^T \langle u^n_t,u^n\rangle_{H^{-s}(\Omega),H^s_0(\Omega)}\,dt &= \frac{1}{2}\left(\int_\Omega (u^n(x,0))^2\,dx - \int_\Omega (u^n(x,T))^2\,dx\right)\\ &\leq \int_\Omega (u^n(x,0))^2\,dx = \int_\Omega (u_0(x))^2\,dx.\end{aligned} \tag{4.2.13}$$

From (4.2.7) and the fact that $|u^n(x,t)| \leq \overline{u}(x,t) \in L^\infty(Q_T)$, for all $n\in\mathbb{N}$, we have that

$$\begin{aligned}\|u^n\|^2_{L^2(0,T;H^s_0(\Omega))} &\leq \int_0^T\int_\Omega g(x,t,u^{n-1})u^{n-1}\,dx\,dt - \int_0^T \langle u^n_t,u^n\rangle_{H^{-s}(\Omega),H^s_0(\Omega)}\,dt\\ &\leq \int_0^T\int_\Omega \underline{\overline{g}}(x,t)\,\underline{\overline{u}}(x,t)\,dx\,dt + \int_\Omega (u_0(x))^2\,dx\\ &\leq k^*(\underline{\overline{g}},\underline{u},\overline{u},u_0) = k^* \leq \infty.\end{aligned} \tag{4.2.14}$$

In particular, the sequence $(u^n)_{n=1}^\infty$ is bounded in $L^2(0,T;H^s_0(\Omega))$. On the other hand, since $L^2(0,T;H^s_0(\Omega))$ is reflexive, there exists a $w\in L^2(0,T;H^s_0(\Omega))$ such that $u^n \rightharpoonup w$ weakly on $L^2(0,T;H^s_0(\Omega))$, up to subsequences. Since the embedding $L^2(0,T;H^s_0(\Omega)) \hookrightarrow L^2(Q_T)$ is continuous, we have that

$$u^n \rightharpoonup w, \textit{ weakly on } L^2(Q_T). \tag{4.2.15}$$

From (4.1.12) and (4.2.15), we obtain $u=w$. From the uniqueness of the pointwise limit (4.1.10) it follows that

$$u^n \rightharpoonup u, \textit{ weakly on } L^2(0,T;H^s_0(\Omega)) \tag{4.2.16}$$

for the whole sequence $(u^n)_{n=1}^\infty$.

Given any $\varphi \in L^2(0,T;H_0^s(\Omega))$, we define the following functional:

$$A_\varphi : L^2(0,T;H_0^s(\Omega)) \to \mathbb{R}, \quad u \mapsto A_\varphi(u) = \int_0^T \mathcal{E}\,(u,\varphi)\,. \tag{4.2.17}$$

Let us show that $A_\varphi \in L^2(0,T;H^{-s}(\Omega))$, that is, it is a linear and continuous functional. Given any $u,v \in L^2(0,T;H_0^s(\Omega))$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} A_\varphi(u+\alpha v) &= \int_0^T \mathcal{E}\,(u+\alpha v,\varphi)\,dt \\ &= \int_0^T \mathcal{E}\,(u,\varphi) + \alpha\mathcal{E}\,(v,\varphi)\,dt \\ &= \int_0^T \mathcal{E}\,(u,\varphi)\,dt + \int_0^T \mathcal{E}\,(v,\varphi)\,dt \\ &= A_\varphi(u) + \alpha A_\varphi(v), \end{aligned} \tag{4.2.18}$$

for all $\alpha \in \mathbb{R}$ and for all $u,v \in L^2(0,T;H_0^s(\Omega))$. Thus, A_φ is linear. Since A_φ is a linear mapping, to prove the its continuity it is sufficient to prove that is is bounded in the following sense: there exists a constant $C \geq 0$ such that

$$|A_\varphi(u)| \leq C\|u\|_{L^2(0,T;H_0^s(\Omega))}, \tag{4.2.19}$$

for all $u \in L^2(0,T;H_0^s(\Omega))$. Given $u \in L^2(0,T;H_0^s(\Omega))$, we have that

$$\begin{aligned} \left|A_\varphi(u)\right| &= \left|\int_0^T \mathcal{E}\,(u,\varphi)\,dt\right| \leq \int_0^T |\mathcal{E}\,(u,\varphi)|\,dt \\ &\leq \int_0^T \|u\|_{L^2(0,T;H_0^s(\Omega))}\|\varphi\|_{L^2(0,T;H_0^s(\Omega))}\,dt \\ &= \left(T\|\varphi\|_{L^2(0,T;H_0^s(\Omega))}\right)\|u\|_{L^2(0,T;H_0^s(\Omega))} \\ &= k(T,\varphi)\|u\|_{L^2(0,T;H_0^s(\Omega))}, \end{aligned} \tag{4.2.20}$$

for all $u \in L^2(0,T;H_0^s(\Omega))$, where $k(T,\varphi) \geq 0$ is a real constant. It follows from (4.2.19) that A_φ is bounded, and together with (4.2.18), we have that $A_\varphi \in L^2(0,T;H^{-s}(\Omega))$, for all $\varphi \in L^2(0,T;H_0^s(\Omega))$. From (4.2.16), the definition of weak convergence and the fact that $A_\varphi \in L^2(0,T;H^{-s}(\Omega))$, we have that

$$A_\varphi(u^n) \to A_\varphi(u),$$

hence

$$\int_0^T \mathcal{E}(u^n,\varphi)\,dt \to \int_0^T \mathcal{E}(u,\varphi)\,dt, \tag{4.2.21}$$

for all $\varphi \in L^2(0,T;H_0^s(\Omega))$.

To end the proof, it remains to show the convergence (4.2.1). The strategy is similar to that of the proof of the convergence (4.2.2). Let us first show that the sequence $(u_t^n)_{n=1}^{\infty}$ is bounded on $L^2(0,T;H^{-s}(\Omega))$. We note that

$$\begin{aligned} \|u_t^n\|^2_{L^2(0,T;H^{-s}(\Omega))} &= \int_0^T \|u_t^n\|^2_{H^{-s}(\Omega)}\,dt \\ &= \int_0^T \left(\sup_{\xi\in H^{-s}(\Omega)\setminus\{0\}} \frac{\left|\langle u_t^n(t),\xi\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}\right|}{\|\xi\|H_0^s(\Omega)} \right) dt, \end{aligned} \tag{4.2.22}$$

and that u^n is a strong solution of the Cauchy problem (3.2.8), since the results of existence and uniqueness of solution is obtained by means of the existence and uniqueness of solution for the generalized problem associated with the problem (4.1.9). In particular, we have

$$\begin{aligned} &|\langle u_t^n(t),\xi\rangle| = \\ &= \left|\int_\Omega g(x,t,u^{n-1}(x,t))\xi(x)\,dx - \mathcal{E}(u^n(t),\xi) - k\int_\Omega u_t^n(x,t)\xi(x)\,dx\right| \\ &\le |\mathcal{E}(u_t^n(t),\xi)| + k\left|\int_\Omega u_t^n(x,t)\xi(x)\,dx\right| + \left|\int_\Omega g(x,t,u^{n-1}(x,t))\xi(x)\,dx\right| \\ &\le \|u^n(t)\|_{H_0^s}\|\xi\|_{H^{-s}} + k\|u^n(t)\|_{L^2}\|\xi\|_{L^2} + \|g(x,t,u^{n-1}(x,t))\|_{L^2}\|\xi\|_{L^2}. \end{aligned} \tag{4.2.23}$$

Since the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, there exists a constant $c \ge 0$ such that

$$\|u^n(t)\|_{L^2(\Omega)} \le c\|u^n(t)\|_{H_0^s(\Omega)},\ \|\xi\|_{L^2(\Omega)} \le c\|\xi\|_{H_0^s(\Omega)}. \tag{4.2.24}$$

From (4.2.24) and (4.2.23) we have

$$\begin{aligned} &|\langle u_t^n(t),\xi\rangle| \le \\ &\le \|u^n(t)\|_{H_0^s}\|\xi\|_{H_0^s} + kc^2\|u^n(t)\|_{H_0^s}\|\xi\|_{H^{-s}} + c\|g(\cdot,\cdot,u^{n-1})\|_{L^2}\|\xi\|_{H_0^s} \\ &\le (1+kc^2)\|u^n(t)\|_{H_0^s}\|\xi\|_{H_0^s} + c\|\underline{g}(\cdot,\cdot,u^{n-1})\|_{L^2}\|\xi\|_{L^2} \\ &\le (1+kc^2)\|u^n\|_{H_0^s}\|\xi\|_{H_0^s} + c\|\overline{q}(\cdot,t)\|_{L^2}\|\xi\|_{H_0^s} \\ &\le \left[(1+kc^2)\|u^n\|_{H_0^s} + c\|\overline{g}(\cdot,t)\|_{L^2}\right]\|\xi\|_{H_0^s}. \end{aligned} \tag{4.2.25}$$

Notice that

$$(a+b)^2 = 2a^2 + 2b^2 - (a-b)^2, \tag{4.2.26}$$

which implies

$$(a+b)^2 \leq 2a^2 + 2b^2. \tag{4.2.27}$$

From (4.2.25) and the estimative (4.2.27) it follows that

$$\begin{aligned}\left(\sup_{\xi \in H_0^s \setminus \{0\}} \frac{|\langle u_t^n(t), \xi \rangle|}{\|\xi\|_{H_0^s}}\right)^2 &\leq \left[(1+kc^2)\|u^n(t)\|_{H_0^s} + c\|\underline{g}(\cdot,t)\|_{L^2}\right]^2 \\ &\leq 2(1+kc^2)^2\|u^n(t)\|_{H_0^s} + 2c^2\|\underline{g}(\cdot,t)\|_{L^2}^2.\end{aligned} \tag{4.2.28}$$

Substituting (4.2.28) into (4.2.22), it follows immediately that

$$\|u_t^n(t)\|_{L^2(0,T;H^{-s})} \leq 2(1+kc^2)^2 k^* + 2c^2\|\underline{g}\|_{L(Q_T)}^2 < \infty, \tag{4.2.29}$$

where we use (4.2.14) and the fact that

$$\|\underline{g}\|_{L^2(0,T;Q_T)}^2 = \int_0^T \int_\Omega \left|\underline{g}(\cdot,t)\right|^2 dx dt = \int_0^T \|\underline{g}(\cdot,t)\|_{L^2(\Omega)}^2 dt. \tag{4.2.30}$$

Therefore, the sequence $(u_t^n)_{n=1}^\infty$ is bounded in $L^2(0,T;H^{-s}(\Omega))$, which is a reflexive space. Thus, there exists a $v \in L^2(0,T;H^{-s}(\Omega))$ such that $u_t^n \rightharpoonup v$ weakly in $L^2(0,T;H^{-s}(\Omega))$, up to subsequences. Since $H_0^s(\Omega) \hookrightarrow H^{-s}(\Omega)$ is a continuous embedding, due to the fact that $H_0^s(\Omega) \subseteq L^2(\Omega) \subseteq H^{-s}(\Omega)$ is a Gelfand Triplet, from (4.2.16) and Proposition 4.1.1, it follows that $u_t = v$. Hence,

$$u_t^n \rightharpoonup u_t, \ \textit{weakly in } L^2(0,T;H^{-s}(\Omega)). \tag{4.2.31}$$

Given $\varphi \in L^2(0,T;H_0^s(\Omega))$, let us consider the functional

$$H_\varphi : L^2(0,T;H^{-s}(\Omega)) \longrightarrow \mathbb{R}, \quad u \mapsto H_\varphi(u) = \int_0^T \langle u, \varphi \rangle_{H^{-s},H_0^s} dt. \tag{4.2.32}$$

Let us show that $H_\varphi \in \left(L^2(0,T;H^{-s}(\Omega))\right)^*$, i.e, that H_φ is a continuous linear functional on $L^2(0,T;H^{-s}(\Omega))$. We begin by showing thar H_φ is a linear functional. Given any $u, v \in$

$L^2(0,T;H^{-s}(\Omega))$ and $\alpha \in \mathbb{R}$ we have that

$$\begin{aligned} H_\varphi(u+\alpha v) &= \int_0^T \langle u+\alpha v, \varphi\rangle_{H^{-s},H_0^s}\,dt = \int_0^T \int_\Omega (u+\alpha v)(x,t)\varphi(x,t)\,dx\,dt \\ &= \int_0^T \int_\Omega u(x,t)\varphi(x,t)\,dx\,dt + \alpha \int_0^T \int_\Omega v(x,t)\varphi(x,t)\,dx\,dt \\ &= \int_0^T \langle u, \varphi\rangle_{H^{-s},H_0^s} + \alpha \int_0^T \langle v, \varphi\rangle_{H^{-s},H_0^s} = H_\varphi(u) + \alpha H_\varphi(v). \end{aligned} \tag{4.2.33}$$

Hence, H_φ is a linear functional. We shall now prove that H_φ is continuous. Since it is a linear functional, it is sufficient to prove that it is bounded in the following sense: *There exists a constant c>0 such that, for all* $u \in L^2(0,T;H^{-s})$ *it holds that:*

$$\left|H_\varphi(u)\right| \le c\|u\|_{L^2(0,T,H^{-s}(\Omega))}.$$

From (4.2.32) we have

$$\begin{aligned} \left|H_\varphi(u)\right| &= \left|\int_0^T \langle u, \varphi\rangle_{H^{-s},H_0^s}\,dt\right| \le \int_0^T \left|\langle u, \varphi\rangle_{H^{-s},H_0^s}\right| dt \\ &\le \left(\int_0^T \|u\|_{H^{-s}}^2\,dt\right)^{1/2} \left(\int_0^T \|\varphi\|_{H_0^s}^2\,dt\right)^{1/2} \\ &= \|\varphi\|_{L(0,T;H_0^s)}\|u\|_{L^2(0,T;H^{-s})}, \end{aligned} \tag{4.2.34}$$

for all $u \in L^2(0,T;H^{-s}(\Omega))$, and H_φ is continuous. From (4.2.33) and (4.2.34), it follows that H_φ is a continuous linear functional on $L^2(0,T;H^{-s}(\Omega))$, for all $\varphi \in L^2(0,T;H_0^s(\Omega))$. Since,

$$u_t^n, u_t \in L^2(0,T;H^{-s}(\Omega)),\ \textit{and}\ u_t^n \rightharpoonup u_t,\ \textit{weakly in}\ L^2(0,T;H^{-s}(\Omega)),$$

from the fact that $H_\varphi \in L^2(0,T;H^{-s}(\Omega))$ and the definition of weak convergence, it follows that $H_\varphi(n_t^n) \to H_\varphi(u_t)$ strongly on $\mathbb{R}$, as $n \to \infty$, which is equivalent to (4.2.1). The above proves that, in fact, u is a solution of problem (4.1.1), and satisfies the monotonicity relation $\underline{u} \le u \le \overline{u}$ a.e Q_T.

It remains to prove the uniqueness of the solution in the class of solutions of problem (4.1.1) that are essentially bounded. Let us suppose the existence of two solutions $u_1, u_2 \in L^2(Q_T)$ of problem (4.1.1). By definition, both of then satisfies the relation

$$\begin{aligned} &\int_0^T \int_\Omega g(x,t,u_i)\varphi(x,t)\,dx\,dt = \\ &= \int_0^T \langle u_{t,i}, \varphi\rangle \; dt + \int_0^T \mathcal{E}\,(u_i,\varphi)\; dt + k\int_0^T (u_i,\varphi)_{L^2(\Omega)}\,dt, \end{aligned} \tag{4.2.35}$$

and $u_i(x,0) = u_0(x)\,, i = 1,2$. Define $v := u_2 - u_1$ and let us take $\varphi := v\chi_{(0,\tau)} \in L^2(0,T;H_0^s(\Omega))$, $\tau \in (0,T)$ as a test function. From (4.2.35) we then get

$$\begin{aligned} &\int_0^\tau \int_\Omega [g(x,t,u_2) - g(x,t,u_1)]\,v(x,t)\,dx\,dt \\ &= \int_0^\tau \langle v_t, v\rangle \; dt + \int_0^\tau \mathcal{E}\,(v_t,v)\; dt + k\int_0^\tau (v_t,v)_{L^2(\Omega)}\,dt, \end{aligned} \tag{4.2.36}$$

Define

$$\mathscr{I} := \left[-\max\{\|\underline{u}\|_{L^\infty(Q_T)}, \|\overline{u}\|_{L^\infty(Q_T)}\}, \max\{\|\underline{u}\|_{L^\infty(Q_T)}, \|\overline{u}\|_{L^\infty(Q_T)}\}\right], \tag{4.2.37}$$

which is a subset of $\mathbb{R}$. Since g is, by hypothesis, *Locally Lipchitz* in the third variable, for any $\sigma_1, \sigma_2 \in \mathscr{I}$ holds the following inequality

$$|g(x,t,\sigma_2) - g(x,t,\sigma_1)| \le L\,|\sigma_2 - \sigma_1|\,, \tag{4.2.38}$$

from which we get

$$[g(x,t,\sigma_2) - g(x,t,\sigma_1](\sigma_2 - \sigma_1)) \le L(\sigma_2 - \sigma_1)^2, \tag{4.2.39}$$

for some *Lipchitz constant* $L > 0$. Taking into account (4.2.36) and (4.2.39), the following estimative holds

$$\begin{aligned} &\int_0^\tau \int_\Omega [g(x,t,u_2) - g(x,t,u_1)]\,v(x,t)\,dx\,dt \\ &= \int_0^\tau \langle v_t, v\rangle \; dt + \int_0^\tau \mathcal{E}\,(v_t,v)\; dt + k\int_0^\tau (v_t,v)_{L^2(\Omega)}\,dt \\ &\le L\int_0^\tau \int_\Omega v^2\,dx\,dt. \end{aligned} \tag{4.2.40}$$

Applying the following integration by parts to (4.2.40)

$$(u(t),v(t))_{L^2(\Omega)}-(u(s),v(s))_{L^2(\Omega)}=\int_s^t\left(\left\langle\frac{du}{dt},v(\tau)\right\rangle+\left\langle u(\tau),\frac{dv}{dt}\right\rangle\right)d\tau, \tag{4.2.41}$$

where $\langle\cdot,\cdot\rangle:=\langle\cdot,\cdot\rangle_{H_0^s(\Omega),H^{-s}(\Omega)}$, we readily see that

$$\begin{aligned}2\int_0^\tau\langle v_t,v\rangle_{H^{-s},H_0^s}\,ds&=(v(\tau),v(\tau))_{L^2(\Omega)}-(v(0),v(0))_{L^2(\Omega)}\\&=\|v(\tau)\|^2_{L^2(\Omega)}-\|v(0)\|^2_{L^2(\Omega)}\\&=\|v(\tau)\|^2_{L^2(\Omega)},\end{aligned} \tag{4.2.42}$$

since $v(0)=u_2(x,0)-u_1(x,0)=u_0(x)-u_0(x)=0$. From (4.2.40) and (4.2.42), we obtain the following inequality

$$\frac{1}{2}\|v(\tau)\|^2_{L^2(\Omega)}+\int_0^\tau\mathcal{E}(v,v)\,ds+k\int_0^\tau\int_\Omega v^2\,dx\,ds\le L\int_0^\tau\|v\|^2_{L^2(\Omega)}\,ds, \tag{4.2.43}$$

and since all the terms are nonnegative, we obtain the following estimative

$$\|v(\tau)\|^2_{L^2(\Omega)}\le 2L\int_0^\tau\|v\|^2_{L^2(\Omega)}\,ds,\ \|v(0)\|^2_{L^2(\Omega)}=0. \tag{4.2.44}$$

Applying Lemma A.0.2 to (4.2.44) with $\alpha=0$, $u=0$ and $\beta=\|v\|^2_{L^2(\Omega)}$, it follows that

$$\|v(\tau)\|^2_{L^2(\Omega)}\le 0,$$

for all $\tau\in(0,T)$. Hence, $v(t)=0$ in $(0,T)$, and $u_1=u_2$.

□

Chapter 5

Sub- and Supersolution for the Parabolic Problem with Harvesting Term

In view of Theorem 4.1.1 witch ensures the existence and uniqueness of the solution of problem (4.1.1), we realize that the construction of a pair of sub and supersolution for the problem

$$\begin{cases} u_t + (-\Delta)^s u + ku = \lambda\,[a(x)u - bu^2 - h(x)], & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{5.0.1}$$

is of utmost importance. Only then we can apply Theorem 4.1.1. Thus, this chapter focuses on the construction of such a pair.

Let us consider problem (5.0.1), where $\lambda > 0$ is the reciprocal of the density or dispersal coefficient, a is the resource term, b is the interaction term and h is the harvesting rate. Furthermore, we will suppose that the following conditions over a, b and h holds:

$$\begin{cases} 0 \leq a \in L^\infty(\Omega), \text{ with } a > l \text{ on } S_a := \overline{\{x \in \Omega \mid a(x) > 0\}}; \\ b > 0 \text{ constant}; \\ 0 \leq h \in L^\infty(\Omega). \end{cases} \tag{5.0.2}$$

Furthermore, some associated elliptic problems will be necessary to establish the existence and uniqueness of solutions for the problem (5.0.1).

5.1 Associated Elliptic Problems

The first associated elliptic problem we consider is the weighted eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi = \lambda a(x)\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1.1}$$

where the source term a is as in (5.0.2) . For this problem we have the following (see [6]).

Proposition 5.1.1. *Let $s \in (0,1)$ be fixed and $\Omega \subset \mathbb{R}^N$ be open and bounded subset. Then*

(i) there exists a principal eigenvalue $\lambda_{1,a} > 0$ for the problem (5.1.1), which is characterized by

$$\lambda_{1,a} = \inf_{\varphi \in H_0^s \setminus \{0\}} = \frac{\mathcal{E}(\varphi,\varphi)}{\displaystyle\int_\Omega a(x)\|\varphi\|_{H_0^s}^2\,dx}; \tag{5.1.2}$$

(ii) there exists a nonnegative eigenfunction $\varphi_{1,a} \in H_0^s(\Omega)$ which corresponds to the eigenvalue (5.1.2) in such a way that

$$\lambda_{1,a} = \frac{\mathcal{E}(\varphi_{1,a},\varphi_{1,a})}{\displaystyle\int_\Omega a(x)\|\varphi_{1,a}\|_{H_0^s}^2\,dx}. \tag{5.1.3}$$

Moreover,

$$\mathcal{E}(\varphi_{1,a},\varphi) = \lambda_{1,a}\int_\Omega q(x)\varphi_{1,a}(x)\varphi(x)\,dx, \tag{5.1.4}$$

for all $\varphi \in H_0^s(\Omega)$;

(iii) $\lambda_{1,a}$ is simple, that is, if $\psi \in H_0^s(\Omega)$ is a solution of

$$\mathcal{E}(\psi,\varphi) = \int_\Omega q(x)\psi(x)\varphi(x)\,dx, \tag{5.1.5}$$

for all $\varphi \in H_0^s(\Omega)$, then $\psi = k\varphi_{1,a}$, for some $k \in \mathbb{R}$;

(iv) if $\partial\Omega$ is of class $C^{1,1}$, for $N \geq 2$, or is a bounded interval, for $N = 1$, them there exists positive constants C_1 and C_2, that depends on the source term a, such that

$$0 < C_1\delta^s(x) \leq \varphi_{1,a}(x) \leq C_2\delta^s(x), \tag{5.1.6}$$

almost everywhere Ω, where $\delta(x) = d(x,\partial\Omega)$, and d is the distance function;

(v) If $\partial\Omega$ is of class $C^{1,1}$, for $N \geq 2$, or a bounded interval, for $N = 1$, then

$$\lambda_{1,a} = \inf_{\substack{\varphi \in H_0^s(\Omega) \\ \varphi \geq \Delta^s \text{ a.e } \Omega}} \frac{\mathcal{E}(\varphi,\varphi)}{\int_\Omega a(x)\|\varphi\|^2_{H_0^s}\,dx}. \tag{5.1.7}$$

The second elliptic associated problem we consider is given by

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.1.8}$$

For problem (5.1.8) we have that there exists a unique solution $e \in H_0^s(\Omega)$ and positive constants $C_3\,, C_4 > 0$, such that

$$C_3\delta^s(x) \leq e(x) \leq C_4\delta^s(x), \tag{5.1.9}$$

almost everywhere in Ω, where $\delta(x) = d(x,\partial\Omega)$, and d is the distance function (see [17] and [16]).

The third associated elliptic problem we consider is the following:

$$\begin{cases} (-\Delta)^s v(x) = \beta(x) + \varphi_{1,a}^2(x)\chi_{S_a^C} & \text{in } \Omega, \\ \Omega = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1.10}$$

where β is any function that satisfy the following condition

$(\mathscr{A}_\beta)$ $\beta \in L^\infty(\Omega)$, $\beta \geq 0$, and there exist $c > 0$ such that

$$\inf_{x\in\Omega}\{\beta(x) + \varphi_{1,a}(x)\} \geq c > 0. \tag{5.1.11}$$

By $v := v(\alpha,\beta) \in H_0^s(\Omega)$, we denote the unique positive solution of problem (5.1.10). As in the problems (5.1.1) and (5.1.8), there exists positive constants C_5, C_6, such that

$$C_5\delta^s(x) \leq v(x) \leq C_6\delta^s(x), \tag{5.1.12}$$

almost everywhere in Ω (see [17] and [16]).

We intend to obtain a pair of sub- and supersolutions of the problem (5.0.1). For this purpose, we utilize the pair of sub- and supersolutions of the elliptic problem corresponding to

(5.0.1), given by

$$\begin{cases} (-\Delta)^s u(x) = \lambda [a(x)u - bu^2 - h(x), & \text{in } \Omega, \\ u = 0, & \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases} \tag{5.1.13}$$

For this problem we have the following definition:

Definition 5.1.1. *Consider the elliptic problem* (5.1.13)*.*

(i) A function $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ *is said to be a subsolution for the problem* (5.1.13) *if, and only if,*

$$\mathcal{E}(u, \varphi) \leq \lambda \int_\Omega [a(x)u - bu^2 - h(x)]\varphi \, dx, \tag{5.1.14}$$

for all nonnegative $\varphi \in H_0^s(\Omega)$*;*

(ii) Similarly, a function $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ *is said to be a supersoluiton for the problem* (5.1.13) *if, and only if,*

$$\mathcal{E}(u, \varphi) \geq \lambda \int_\Omega [a(x)u - bu^2 - h(x)]\varphi \, dx, \tag{5.1.15}$$

for all nonnegative $\varphi \in H_0^s(\Omega)$*;*

(iii) If $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ *is both a sub- and a supersolution for the problem* (5.1.13)*, then we say that* u *is a solution of the problem.*

It is clear that, if we want to consider sub- and supersolutions of problem (5.1.13) also as sub- and supersolutions of the parabolic problem (5.0.1), we must ensure that problem (5.1.13) has at least one such pair of ordered sub- and supersolutions. The existence of such pair is ensured by results presented in [6], as follows: Consider the elliptic problem

$$\begin{cases} (-\Delta)^s u(x) = g(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } (\mathbb{R}^N \setminus \Omega), \end{cases} \tag{5.1.16}$$

and suppose that the following hypothesis holds:

(H_a) given $r > 0$, there exists a function $a_r \in L^\infty(\Omega)$ such that $|g(x,t)| \leq a_r(x)$, for all $|t| \leq r$ a.e $x \in \Omega$;

(H_b) for all $r > 0$, there exists a continuous nondecreasing function b_r such that $b_r(0) = 0$ and $|g(x,t_1) - g(x,t_2)| \leq b_r(|t_1 - t_2|)$, for all $|t_1|, |t_2| \leq r$ a.e $x \in \Omega$.

Then, the following result holds true:

Theorem 5.1.1. *Suppose that conditions* (H_a) *and* (H_b) *holds, and let* $\underline{u}, \overline{u} \in H^s(\mathbb{R}^N) \cap L^\infty(\Omega)$ *be , respectively, a subsolution and a supersolution of* (5.1.16) *satisfying* $\underline{u} \leq \overline{u}$ *a.e. in* Ω*. Then, there exists a weak solution* u *of problem* (5.1.16) *such that* $\underline{u} \leq u \leq \overline{u}$ *a.e. in* Ω*.*

Proof. The proof of this result is presented in [6]. □

5.2 Existence and Uniqueness of the Solution to the Parabolic Problem with Harvesting Term

Now, we will state and prove, following [7], one of the most crucial theorems of this dissertation. It will be of extreme importance for obtaining solutions for the problem (5.0.1).

Theorem 5.2.1. *Consider the problem* (5.1.13)*. For every* $\lambda \geq \lambda_{1,a}$ *and* β *satisfying the condition* $\mathscr{A}_\beta$*,* (5.1)*, there exists a positive function*

$$h^* = h(\lambda, a, b, \beta, \Omega) \in L^\infty(\Omega) \tag{5.2.1}$$

such that, for all $h \in L^\infty(\Omega)$ *satisfying* $0 \leq h \leq h^*$*, for almost every* $x \in \Omega$*, the problem* (5.1.13) *has a positive solution.*

Moreover, if h *satisfies*

$$\mu := \operatorname*{ess\,sup}_{x \in \Omega} \frac{h}{h^*} < 1, \tag{5.2.2}$$

then there exists $m^* > 0$ *such that for all* m *with*

$$\mu m^* \leq m \leq m^*, \tag{5.2.3}$$

the function

$$\underline{u}_m = m\lambda \left(\varphi_{1,a} - \varepsilon v_\beta\right) \tag{5.2.4}$$

is a subsolution of (5.1.13) *satisfying* $0 < \underline{u}_m \leq m\varphi_{1,a}$*, Furthermore, there exists* $K^* > 0$ *such that* $\overline{u}_K = Ke$ *is a supersolution of* (5.1.13) *for every* $K \geq K^*$*.*

Proof. We begin the proof by showing that a multiple of the solution of (5.1.1) cannot be a subsolution of (5.1.13). Since we are interest on positive solutions, let $m > 0$ be a constant and let $\underline{u}_m = m\varphi_{1,a}$. By Definition 5.1.1, $\underline{u}_m$ is a subsolution of (5.1.13) if, and only if,

$$\mathcal{E}(\underline{u}_m, \varphi) \leq \lambda \int_\Omega \left(a(x)\underline{u}_m - b\underline{u}_m^2 - h(x)\right) \varphi \, dx,$$

for all $0 \leq \varphi \in H_0^s(\Omega)$. In particular, since

$$\mathcal{E}(u,\varphi) = \int_\Omega (-\Delta)^s u\varphi\, dx, \tag{5.2.5}$$

we have that $\underline{u}_m$ is a subsolution of (5.1.13) if, and only if,

$$(-\Delta)^s \underline{u}_m \leq \lambda \left(a\underline{u}_m - b\underline{u}_m^2 - h\right), \tag{5.2.6}$$

a.e in Ω. We now consider two cases:

Case 1: $x \in S_a$. In this case we have that $a(x) > l > 0$. From (5.2.6), (5.1.1) and taking $\underline{u}_m = m\varphi_{1,a}$ it follows that:

$$(-\Delta)^s \underline{u}_m = ma\lambda_{1,a}\varphi_{1,a} \leq \lambda ma\varphi_{1,a} - bm^2\varphi_{1,a}^2 - \lambda h, \tag{5.2.7}$$

which is equivalent, after a rearrangement, to

$$\begin{aligned} 0 < h &\leq ma\left(1 - \frac{\lambda_{1,a}}{\lambda}\right)\varphi_{1,a} - b\frac{m^2}{\lambda}\varphi_{1,a}^2 \\ &\leq ma(1-\alpha^2)\varphi_{1,a} - b\frac{m^2}{\lambda}\varphi_{1,a}^2, \end{aligned} \tag{5.2.8}$$

where $\alpha^2 := \dfrac{\lambda_{1,a}}{\lambda} \in (0,1)$. It follows from (5.2.8) that $\underline{u}_m$ is a subsolution if, and only if,

$$ma(1-\alpha^2) - b\frac{m^2}{\lambda}\varphi_{1,a} > 0 \implies \frac{a(1-\alpha^2)\lambda}{bm} > \varphi_{1,a}, \tag{5.2.9}$$

a.e in Ω.

Case 2: $x \notin S_a$. In this case $a(x) = 0$ and from (5.2.9) we have that

$$0 < \varphi_{1,a} < 0, \tag{5.2.10}$$

a.e in Ω, which is impossible since $\varphi_{1,a} > 0$. Hence, from (5.2.9) and (5.2.10) it follows that no multiple of the eigenfunction $\varphi_{1,a}$ can be a subsolution of problem (5.1.13). In particular, any solution of problem (5.1.13) must be such that, outside the support of a, it rapidly converges to zero, so that the quadratic term u^2 does not dominate in that region. Consequently, if $\underline{u}_m$ is a subsolution of (5.1.13), it must include a correction that is proportional to v, the only positive solution of the elliptic problem (5.1.10).

Let us fix $\lambda > \lambda_{1,a}$ and β satisfying $(\mathscr{A}_\beta)$, (5.1). From the existence of $C_1, C_2, C_5, C_6 > 0$, with

$$C_1\delta^s(x) \leq \varphi_{1,a}(x) \leq C_2\delta^s(x)$$
$$C_5\delta^s(x) \leq v(x) \leq C_6\delta^s(x), \tag{5.2.11}$$

almost everywhere in Ω, it follows that $C_1\delta^s \leq \varphi_{1,a},\ v \leq C_6\delta^s$ a.e Ω and

$$0 < \frac{C_1}{C_6} \leq \frac{\varphi_{1,a}}{v},\ \ a.e\ \Omega. \tag{5.2.12}$$

Define $\alpha := \sqrt{\frac{\lambda_{1,a}}{\lambda}} \in (0,1)$. Since $\alpha \in (0,1)$, we have that

$$\frac{\varphi_{1,a}}{v} \geq \frac{C_1}{C_6} \implies \frac{\varphi_{1,a}}{v} > \frac{\varphi_{1,a}}{v}(1-\alpha) \geq \frac{C_1}{C_6}(1-\alpha) > \varepsilon. \tag{5.2.13}$$

Thus, there exists $\varepsilon > 0$ such that $\varepsilon \in \left(0, \frac{C_1}{C_6}(1-\alpha)\right)$ and

$$\frac{\varphi_{1,a}}{v} - \alpha\frac{\varphi_{1,a}}{v} > \varepsilon \implies \varphi_{1,a} - \alpha\varphi_{1,a} > \varepsilon v$$
$$\implies \varphi_{1,a} - \varepsilon v \geq \alpha\varphi_{1,a} > 0\ a.e\ \Omega. \tag{5.2.14}$$

Let us define $\underline{u}_m := m\left(\varphi_{1,a} - \varepsilon v\right)$ and, as above, consider two cases:

Case 1: $x \in S_a^c$**.**

In this case, (5.2.6) reduces to

$$(-\Delta)^s\underline{u}_m \leq -\lambda b\underline{u}_m^2 - \lambda h, \tag{5.2.15}$$

and

$$(-\Delta)^s\underline{u}_m = m(-\Delta)^s\left(\varphi_{1,a} - \varepsilon v\right) = -m\varepsilon\left(\beta + \varphi_{1,a}^2\right). \tag{5.2.16}$$

From (5.2.15) and (5.2.16) it follows that

$$0 \leq \lambda h \leq m\varepsilon\left(\beta + \varphi_{1,a}^2\right) - \lambda bm^2\left(\varphi_{1,a} - \varepsilon v\right)^2. \tag{5.2.17}$$

If

$$m\varepsilon\left(\beta + \varphi_{1,a}^2\right) - \lambda bm^2\left(\varphi_{1,a} - \varepsilon v\right)^2 > 0, \tag{5.2.18}$$

then we can define a function $h \geq 0$ in order to satisfy (5.2.17). In order to (5.2.18) hold, It is sufficient to choose m in such a way that

$$\frac{\varepsilon\left(\beta+\varphi_{1,a}^{2}\right)}{\lambda b\left(\varphi_{1,a}-\varepsilon v\right)^{2}} \geq \frac{\varepsilon\left(\beta+\varphi_{1,a}^{2}\right)}{\lambda b\|\varphi_{1,a}-\varepsilon v\|_{L^{\infty}(\Omega)}^{2}} > m > 0. \tag{5.2.19}$$

Case 2: $x \in S_a$**.**

In this case, the function $a(x) > l > 0$ and from (5.2.6) we have that $\underline{u}_m$ is a subsolution if, and only if,

$$\begin{aligned} ma\lambda_{1,a}\varphi_{1,a} - \varepsilon m\beta &\leq \lambda a\underline{u}_m - \lambda b\underline{u}_m^2 - \lambda h \\ &\leq \lambda ma\left(\varphi_{1,a}-\varepsilon v\right) - \lambda bm^2\left(\varphi_{1,a}-\varepsilon v\right)^2 - \lambda h. \end{aligned} \tag{5.2.20}$$

From (5.2.20), after a rearrangement, we get that

$$\lambda h \leq m\varepsilon\beta - ma\lambda_{1,a}\varphi_{1,a} + \lambda ma\left(\varphi_{1,a}-\varepsilon v\right) - \lambda bm^2\left(\varphi_{1,a}-\varepsilon v\right)^2. \tag{5.2.21}$$

In particular, using the estimative (5.2.14), we get that, $\underline{u}_m$ is a sub solution of (5.1.13) if, and only if,

$$\begin{aligned} \lambda h &\leq m\varepsilon\beta - ma\lambda_{1,a}\varphi_{1,a} + \lambda am\alpha\varphi_{1,a} - \lambda bm^2\alpha\varphi_{1,a}\left(\varphi_{1,a}-\varepsilon v\right) \\ &\leq m\left[\varepsilon\beta - a\lambda_{1,a}\varphi_{1,a} + \lambda\alpha\varphi_{1,a}\left(a - bm\left(\varphi_{1,a}-\varepsilon v\right)\right)\right]. \end{aligned} \tag{5.2.22}$$

In order to satisfy (5.2.21), we must have

$$m\left[\varepsilon\beta - a\lambda_{1,a}\varphi_{1,a} + \lambda\alpha\varphi_{1,a}\left(a - bm\left(\varphi_{1,a}-\varepsilon v\right)\right)\right] > 0, \tag{5.2.23}$$

and since $a > l$ on S_a, it is sufficient to have

$$\varepsilon\beta + \left(\alpha\lambda - \lambda_{1,a}\right)l\varphi_{1,a} > bm\alpha\lambda\varphi_{1,a}\left(\varphi_{1,a}-\varepsilon v\right), \tag{5.2.24}$$

from which follows that

$$\frac{\varepsilon\beta + \left(\alpha\lambda - \lambda_{1,a}\right)l\varphi_{1,a}}{\lambda\alpha b\left(\varphi_{1,a}-\varepsilon v\right)} \geq \frac{\varepsilon\beta + \left(\alpha\lambda - \lambda_{1,a}\right)l\varphi_{1,a}}{\lambda\alpha b\|\varphi_{1,a}\|_{L^{\infty}(\Omega)}\|\varphi_{1,a}-\varepsilon v\|_{L^{\infty}(\Omega)}} > m > 0. \tag{5.2.25}$$

From (5.2.19) and (5.2.25) we define

$$m^* := \min\left\{ \frac{\inf_{S_a}\{\varepsilon\beta + (\lambda\alpha - \lambda_{1,a})l\varphi_{1,a}\}}{\lambda ba \left\|\varphi_{1,a}\right\|_{L^\infty(\Omega)} \left\|\varphi_{1,a} - \varepsilon v\right\|_{L^\infty(\Omega)}}, \frac{\varepsilon \inf_{S_a^C}\{\beta + \varphi_{1,a}^2\}}{\lambda b \|\varphi_{1,a} - \varepsilon v\|_{L^\infty(\Omega)}^2}, \right\} > 0, \tag{5.2.26}$$

and with the aid of (5.2.17) and (5.2.21) we define the function

$$h(x) = \begin{cases} h_1^*(x), & x \in S_a^c \\ h_2^*(x), & x \in S_a \end{cases} \tag{5.2.27}$$

where,

$$\begin{aligned} h_1^*(x) &:= \frac{m^*}{\lambda}[\varepsilon(\beta + \varphi_{1,a}) - \lambda b m^*(\varphi_{1,a} - \varepsilon v)^2], \\ h_2^*(x) &:= \frac{m^*}{\lambda}[\varepsilon\beta - \lambda_{1,a} a \varphi_{1,a} + \lambda\alpha\varphi_{1,a}(a - bm^*(\varphi_{1,a} - \varepsilon v))]. \end{aligned} \tag{5.2.28}$$

We claim that $h > 0$ a.e Ω. Indeed, note that $h_1^* > 0$ in S_a^c since, from (5.2.19)

$$0 < m^* \leq \frac{\varepsilon(\beta + \varphi_{1,a}^2)}{\lambda b(\varphi_{1.a} - \varepsilon v)^2} \implies 0 < \varepsilon(\beta + \varphi_{1,a}^2) - \lambda b m^*(\varphi_{1,a} - \varepsilon v)^2. \tag{5.2.29}$$

On the other hand, since

$$0 < m^* \leq \frac{\inf_{S_a}\{\varepsilon\beta + (\lambda\alpha - \lambda_{1,a})l\varphi_{1,a}\}}{\lambda ba \left\|\varphi_{1,a}\right\|_{L^\infty(\Omega)} \left\|\varphi_{1,a} - \varepsilon v\right\|_{L^\infty(\Omega)}}, \tag{5.2.30}$$

and $a > l$ a.e Ω, $b > 0$, we have that

$$\begin{aligned} m^* &\leq \frac{\{\varepsilon\beta + (\lambda\alpha - \lambda_{1,a})l\varphi_{1,a}\}}{\lambda ba \left\|\varphi_{1,a}\right\|_{L^\infty(\Omega)} \left\|\varphi_{1,a} - \varepsilon v\right\|_{L^\infty(\Omega)}} \\ &\implies 0 < \varepsilon\beta + (\lambda\alpha - \lambda_{1,a})l\varphi_{1,a} - \lambda b\alpha m^* \varphi_{1,a}(\varphi_{1,a} - \varepsilon v) \end{aligned} \tag{5.2.31}$$

from which follows that $h_2^* > 0$ in S_a. Let take as a candidate for a subsolution of (5.1.13) the function

$$\underline{u}_{m^*} = m^*\left(\varphi_{1,a} - \varepsilon v\right) \in H_0^s(\Omega), \tag{5.2.32}$$

and verify that, for every $h \le h^*$, it is, in fact, a subsolution of (5.1.13). Using the relation (5.2.5), we see that for every $0 \le \psi \in H_0^s(\Omega)$, (5.2.32) satisfies

$$\begin{aligned}\mathcal{E}\left(\underline{u}_{m^*}, \psi\right) &= \int_\Omega (-\Delta)\underline{u}_{m^*}\psi\, dx \\ &= \int_\Omega [m^* a\lambda_{1,a}\varphi_{1,a} - m^*\varepsilon\left(\beta + \varphi_{1,a}^2\chi_{S_a^c}\right)]\psi\, dx.\end{aligned} \tag{5.2.33}$$

From (5.2.6) and (5.2.33), the function $\underline{u}_{m^*}$ is a subsolutions of (4.1.11) if, and only if,

$$\begin{aligned}m^* a\lambda_{1,a}\varphi_{1,a} - &m^*\varepsilon\left(\beta + \varphi_{1,a}^2\chi_{S_a^c}\right) \\ &\le \lambda[m^*\left(\varphi_{1,a} - \varepsilon v\right)\left(a - bm^*\left(\varphi_{1,a} - \varepsilon v\right)\right) - h],\end{aligned} \tag{5.2.34}$$

a.e in Ω. Again, we consider two cases.

Case 1) $x \in S_a$.

In this case, $a > l$ and (5.2.33) reduces to

$$m^* a\lambda_{1,a}\varphi_{1,a} - m^*\varepsilon\beta \le \lambda[m^*\left(\varphi^{1,a} - \varepsilon v\right)\left(a - bm^*\left(\varphi_{1,a} - \varepsilon v\right)\right) - h], \tag{5.2.35}$$

from which follows that,

$$0 \le \lambda h \le m^*\varepsilon\beta - m^* a\lambda_{1,a}\varphi_{1,a} + \lambda[m^*\left(\varphi^{1,a} - \varepsilon v\right)\left(a - bm^*\left(\varphi_{1,a} - \varepsilon v\right)\right)]. \tag{5.2.36}$$

After dividing (5.2.36) by $\lambda > 0$, and using the relation (5.2.13) we obtain

$$0 \le h \le h_2^* = \frac{m^*}{\lambda}[\varepsilon\beta - \lambda_{1,a} a\varphi_{1,a} + \lambda\alpha\left(a - bm^*\left(\varphi_{1,a} - \varepsilon v\right)\right)], \tag{5.2.37}$$

and (5.2.34) is valid in S_a, i.e, $\underline{u}_{m^*}$ is a subsolution in S_a.

Case 2) $x \in S_a^c$.

In this case, $a = 0$ and (5.2.33) reduces to

$$\begin{aligned}-m^*\varepsilon\left(\beta + \varphi_{1,a}^2\right) &\le \lambda[m^*\left(\varphi^{1,a} - \varepsilon v\right)\left(-bm^*\left(\varphi_{1,a} - \varepsilon v\right)\right) - h] \\ &\le \lambda[-b\left(m^*\right)^2\left(\varphi_{1,a} - \varepsilon v\right)^2 - h].\end{aligned} \tag{5.2.38}$$

Rearranging terms and using and using the definition of h_1^* in (5.2.28), we obtain

$$0 \le h \le h_1^* = \frac{m^*}{\lambda}[\varepsilon\left(\beta\varphi_{1,a}\right) - \lambda bm^*\left(\varphi_{1,a} - \varepsilon v\right)], \tag{5.2.39}$$

and (5.2.34) is valid in S_a^c, i.e, $\underline{u}_{m^*}$ is a subsolution in S_a^c.
It follows from (5.2.37) and (5.2.39) that if $h \leq h^*$ and m^* satisfies (5.2.26), then $\underline{u}_{m^*}$ is a subsolution of (5.1.13), for every $\lambda > \lambda_{1,a}$.

We now turn to find a supersolution of problem (5.1.13). For this purpose, we will use the only positive solution $e \in H_0^s(\Omega)$ of the problem (5.1.8). We take as a candidate of such supersolution the function $\overline{u}_K = Ke$, where $K > 0$ is to be determined. Let us suppose that $\overline{u}_K$ is a supersolution of (4.1.11). Them, by (5.1.15) and (5.2.5), we must have

$$\begin{aligned}\mathcal{E}(\overline{u}_K, \psi) &= \int_\Omega (-\Delta)^s \overline{u}_K \psi \, dx = \int_\Omega K \psi \, dx \\ &\geq \int_\Omega \lambda [aKe - bK^2 e^2] \psi \, dx \\ &\geq \int_\Omega \lambda [aKe - bK^2 e^2 - h] \psi,\end{aligned} \tag{5.2.40}$$

for all $0 \leq \psi \in H_0^s(\Omega)$. Notice that (5.2.40) holds if, and only if,

$$K - \lambda [aKe - bK^2 e^2 - h] \geq 0, \tag{5.2.41}$$

a.e in Ω, if, and only if,

$$(\lambda K a)^2 - 4\lambda b K^2 (K - \lambda H) \leq 0, \tag{5.2.42}$$

from which follows that

$$K \geq \frac{\lambda a^2}{4b} - \lambda h. \tag{5.2.43}$$

Since $h \geq 0$, the term $-\lambda h \leq 0$. Hence, if we choose

$$K \geq K^* := \frac{\|a\|^2_{L^\infty(\Omega)}}{4b} \lambda \tag{5.2.44}$$

the function $\overline{u}_K$ is a positive supersolution of (5.1.13), whenever $K \geq K^*$, and from (5.1.9), we have that $C_3 K \delta^s \leq \overline{u}_K$. Moreover, from (5.2.11), (5.2.14) and (5.2.26), it follows that

$$m^* (C_1 - \varepsilon C_6) \delta^s \leq m^* (\varphi_{1,a} - \varepsilon v) = \overline{u}_K \leq m^* (C_2 - \varepsilon C_5) \delta^s, \tag{5.2.45}$$

a.e in Ω. Hence, for sufficiently large K we have $\underline{u}_{m^*} \leq \overline{u}_K$. On the other hand, the function

$$g : \Omega \times \mathbb{R} \to \mathbb{R}, \, g(x,u) = \lambda \left[a(x)u - bu^2 - h(x) \right], \tag{5.2.46}$$

clearly satisfies the conditions (H_a), (H_b), and due to Theorem 5.1.1, there exists a positive solution u for the problem (5.1.13) such that $\underline{u}_{m^*} \leq u \leq \overline{u}_K$ a.e in Ω, proving the existence of a solution.

Let us now fix a function h satisfying (5.2.2) and define $\underline{u}_m := m\lambda\left(\varphi_{1,a} - \varepsilon v\right)$, where m satisfies (5.2.3) with m^* given by (5.2.26). We want to show that $\underline{u}_m$, satisfying such conditions, is a subsolution for (5.1.13). Notice that we already prove that $\underline{u}_{m^*}$ is a subsolution. It remains to prove that $\underline{u}_m$ is a subsolution for all $m \in [m^*\mu\, , m^*)$. First, let us consider the pointwise inequality (5.2.34) with m^* replaced by m, that is

$$ma\lambda_{1,a}\varphi_{1,a} - m\varepsilon\left(\beta + \varphi_{1,a}^2\chi_{S_a^c}\right) \leq \lambda\left[m\left(\varphi_{1,a} - \varepsilon v\right)\left(a - bm\left(\varphi_{1,a} - \varepsilon v\right)\right) - h\right], \tag{5.2.47}$$

a.e in Ω. Next, considering (5.2.2), we fix h such that $h \leq \mu h^*$. Again, we consider two cases:

Case 1. $x \in S_a^c$.

In this case $a = 0$ and using the definition of h_1^* in (5.2.28), together with $h \leq \mu h^*$, we obtain

$$\begin{aligned} 0 < h \leq \mu h_1^* &= \frac{\mu m^*}{\lambda}\left[\varepsilon\left(\beta + \varphi_{1,a}\right) - \lambda b m^*\left(\varphi_{1,a} - \varepsilon v\right)^2\right] \\ &\leq \frac{m}{\lambda}\left[\varepsilon\left(\beta + \varphi_{1,a}\right) - \lambda b m\left(\varphi_{1,a} - \varepsilon v\right)^2\right], \end{aligned} \tag{5.2.48}$$

which is equivalent to $\underline{u}_m$ be a subsoluition of (5.1.13) in S_a^c.

Case 2. $x \in S_a$.

In this case $a > 0$ and using the definition of h_2^* in (5.2.28) with $h \leq \mu h^*$, we have

$$\begin{aligned} 0 < h \leq \mu h_2^*(x) &= \frac{\mu m^*}{\lambda}[\varepsilon\beta - \lambda_{1,a} a\varphi_{1,a} + \lambda\alpha\varphi_{1,a}(a - bm^*(\varphi_{1,a} - \varepsilon v))] \\ &\leq \frac{m}{\lambda}[\varepsilon\beta - \lambda_{1,a} a\varphi_{1,a} + \lambda\alpha\varphi_{1,a}(a - bm(\varphi_{1,a} - \varepsilon v))], \end{aligned} \tag{5.2.49}$$

which is equivalent to $\underline{u}_m$ be a subsolution of (5.1.13) in S_a. Hence $\underline{u}_m$ is a subsolution of (5.1.13), completing the proof. □

We can now combine all the previously presented results to prove the following theorem concerning the existence and uniqueness of positive solutions to the problem (5.0.1).

Theorem 5.2.2. *Let a, b and h satisfies (5.0.2). For every $\lambda > \lambda_{1,a}$, there exists a positive function $h^* \in L^\infty(\Omega)$ and $\underline{u}_0 \in L^\infty(\Omega)$ such that for all h satisfying*

$$\operatorname*{ess\,sup}_{x\in\Omega} \frac{h}{h^*} < 1$$

and for all $u_0 \in L^\infty(\Omega)$ *satisfying*

$$\underline{u}_0 \leq u_0 \leq Ke$$

for some $K > 0$*, the problem* (5.0.1) *has a positive solution* $u \in L^\infty(\Omega)$*. Moreover, it is unique in the class of essential bounded solutions of* (5.0.1).

Proof. Let us fix $\lambda > \lambda_{1,a}$, and let the positive constants μ, m^* and K^*, such as the positive functions h^*, $\underline{u}_{\mu m^*}$, and $\overline{u}_K$, $K \geq K^*$, be as in the Theorem 5.2.1. Moreover, let us define

$$\underline{u}_0 := \mu m^*(\varphi_{1,a} - \varepsilon v), \tag{5.2.50}$$

and fix $K \geq K^*$ in such a way that

$$\overline{u}_K = Ke \geq u_0. \tag{5.2.51}$$

Note that, by Theorem 5.2.1, $\underline{u}_{\mu m^*}$ and $\overline{u}_K$ determines an ordered pair of sub- and supersolution of the elliptic problem (5.1.13). Hence, we define $\underline{u}, \overline{u} \in L^2(0,T;H_0^s(\Omega))$ by

$$\underline{u}(t) := \underline{u}_{\mu m^*}, \overline{u}(t) := \overline{u}_K, \tag{5.2.52}$$

for any $t \in [0,T]$. Notice that, since $\overline{u}_{\mu m^*}$ and $\overline{u}_K$, as sub- and supersolutions of (5.1.13), are constants over the interval $(0,T)$, it follows from Example 1.3.2 that the pointwise derivatives

$$\underline{u}'(t) = \lim_{\Delta t \to 0} \frac{\underline{u}(t+\Delta t) - \underline{u}(t)}{\Delta t}, \quad \overline{u}'(t) = \lim_{\Delta t \to 0} \frac{\overline{u}(t+\Delta t) - \overline{u}(t)}{\Delta t} \tag{5.2.53}$$

exists and coincides with the generalized derivatives $\underline{u}_t, \overline{u}_t \in L^2(0,T;H_0^s(\Omega))$. Hence,

$$\int_0^T \langle \underline{u}_t, \psi \rangle_{H^{-s},H_0^s} \, dt = \int_0^T \langle \overline{u}_t, \psi \rangle_{H^{-s},H_0^s} \, dt = 0, \tag{5.2.54}$$

for all $0 \leq \psi \in L^2(0,T;H_0^s(\Omega))$. Notice that, $\psi(.,t) \in H_0^s(\Omega)$ for a.e. $t \in (0,T)$, hence

(i)

$$\begin{aligned} &\mathcal{E}(\underline{u}, \psi(\cdot,t)) \leq \lambda \left\langle a(\cdot)\underline{u} - b\underline{u}^2 - h(\cdot), \psi(\cdot,t) \right\rangle_{H^{-s},H_0^s}, \\ &\int_0^T \langle \underline{u}_t, \psi \rangle_{H^{-s},H_0^s} \, dt + \int_0^T \mathcal{E}(\underline{u}, \psi) \, dt = 0 + \int_0^T \mathcal{E}(\underline{u}, \psi) \, dt \\ &\leq \int_0^T \left\langle \lambda a(\cdot)\underline{u} - b\underline{u}^2 - h(\cdot), \psi \right\rangle_{H^{-s},H_0^s} dt \end{aligned} \tag{5.2.55}$$

(ii)

$$\mathcal{E}\left(\underline{u}, \psi(\cdot,t)\right) \geq \lambda \left\langle a(\cdot)\underline{u} - b\underline{u}^2 - h(.), \psi(\cdot,t)\right\rangle_{H^{-s},H_0^s},$$
$$\int_0^T \langle \underline{u}_t, \psi\rangle_{H^{-s},H_0^s}\, dt + \int_0^T \mathcal{E}\left(\underline{u}, \psi\right) dt = 0 + \int_0^T \mathcal{E}\left(\underline{u}, \psi\right) dt \tag{5.2.56}$$
$$\geq \int_0^T \left\langle \lambda a(\cdot)\underline{u} - b\underline{u}^2 - h(\cdot), \psi\right\rangle_{H^{-s},H_0^s} dt.$$

In particular, (5.2.55) and (5.2.56) shows that $\underline{u}$ and $\overline{u}$ are, respectively, sub- and supersolutions of problem (5.1.13).

Let $D := \max\{\|\underline{u}\|_\infty, \|\overline{u}\|_\infty\}$ and define

$$\hat{g}(x,t,\sigma) = \begin{cases} \lambda\left(-a(x)D - bD^2 - h(x)\right), & \text{for } \sigma < -D, \\ \lambda(a(x)\sigma - b\sigma^2 - h(x)), & \text{for } -D \leq \sigma \leq D, \\ \lambda\left(a(x)D - bD^2 - h(x)\right), & \text{for } \sigma > D. \end{cases} \tag{5.2.57}$$

Now, the function $\hat{g}$ is Lipschitz with respect to the third variable with Lipschitz constant

$$k := \|a\|_\infty + 2bD, \tag{5.2.58}$$

and the function

$$g(x,t,\sigma) = \hat{g}(x,t,\sigma) + k\sigma \tag{5.2.59}$$

is clearly monotonically increasing in the third variable and satisfies the hypothesis of Theorem 4.1.1. Since $\underline{u}_0 \leq u_0 \leq \overline{u}_K = Ke$, by Theorem 4.1.1, the parabolic problem (4.1.1), with k given by (5.2.58) and g given by (5.2.59), was a unique positive solution u satisfying

$$-D \leq 0 \leq \underline{u}(t) \leq u(t) \leq \overline{u}(t) \leq D, \tag{5.2.60}$$

hance,

$$\lambda\left[a(x)u - bu^2 - h(x)\right] = \hat{g}(x,,t,u(x,t)), \tag{5.2.61}$$

a.e in Q_T. Consequently, u is a positive solution of (5.0.1), since $\underline{u}$ is positive in Q_T. This proves the existence.

Let us now prove the uniqueness in the class of all essentially bounded solutions of (5.0.1). Suppose that $u_1, u_2 \in L^\infty(\Omega)$ are both solutions of (5.0.1). Then, both satisfy the equality

$$\begin{aligned}\int_0^T \langle u_{i,t}, \varphi\rangle\, dt + \int_0^T \mathcal{E}(u_i,\varphi)\, dt = \int_0^T \left\langle \lambda a(\cdot)\underline{u} - b\underline{u}^2 - h(\cdot), \psi\right\rangle dt \\ \lambda \int_0^T \int_\Omega \left[a(x)u_i - bu_i^2 - h(x)\right] \varphi\, dx dt,\end{aligned} \tag{5.2.62}$$

together with $u_i(.,0) = u_0$, for $i = 1,2$. Denote, as in the proof of Theorem 4.1.1, $v = u_2 - u_1$ and define $\varphi := v\chi_{(0,\tau)} \in L^2(0,T; H_0^s(\Omega))$, $\tau \in (0,T)$. Then, from (5.2.62), we have

$$\int_0^\tau \langle v_t, v\rangle\, dt + \int_0^\tau \mathcal{E}(v,v)\, dt = \lambda \int_0^\tau \int_\Omega [a(x) - b(u_2 - u_1)]\,(u_2 - u_1)^2\, dx dt. \tag{5.2.63}$$

Since

$$\lambda\,[a(x) - b(u_2 - u_1)] \leq \lambda \|a\|_\infty + b\left(\|u_1\|_{L^\infty(Q_T)} + \|u_2\|_{L^\infty(Q_T)}\right) \tag{5.2.64}$$

from (5.2.63) and (5.2.64) we have that

$$\int_0^\tau \langle v_t, v\rangle_{H^{-s}, H_0^s}\, dt + \int_0^\tau \mathcal{E}(v,v)\, dt \leq K \int_0^\tau \int_\Omega v^2\, dx dt, \tag{5.2.65}$$

where K is given by (5.2.64). Following the steps of the proof of Theorem 4.1.1,

$$\|v(\tau)\|^2_{L^2(Q_T)} \leq 2K \int_0^T \|v\|^2_{L^2(\Omega)}\, dt\,,\ \|v(0)\|^2_{L^2(\Omega)} = 0. \tag{5.2.66}$$

Using the Grönwall inequality, Lemma A.0.2, we obtain

$$\|v(\tau)\|_{L^2(\Omega)} \leq 0, \tag{5.2.67}$$

for all $\tau \in (0,T)$, with implies that $v(t) \equiv 0$ on $(0,T)$, from which follows the uniqueness. □

Conclusion

In this dissertation, we investigated a class of nonlinear parabolic problems posed on a bounded domain $\Omega \subset \mathbb{R}^N$, whose boundary $\partial\Omega$ is of class $C^{1,1}(\Omega)$ when $N \geq 2$, or an open bounded interval when $N = 1$. For a fixed final time $T > 0$, the analysis was carried out on the cylinder $Q_T := \Omega \times (0,T)$. More precisely, for a fractional order $s \in (0,1)$, we studied a parabolic problem driven by the fractional Laplacian of the form

$$\begin{cases} u_t + (-\Delta)^s u = f(\lambda, x, t, u), & \text{in } Q_T, \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

which serves as a general mathematical model for population dynamics with nonlocal fractional diffusion, nonlinear reaction effects, and harvesting.

The theoretical framework of this work was grounded on classical results in nonlinear functional analysis and evolution equations, mainly drawn from [20] and [18], together with more recent developments such as [7] and [14]. In terms of applications, we followed the approach proposed in [7] to analyze a population dynamics model involving the fractional Laplacian. Within this setting, the dissertation established existence and uniqueness results for solutions describing the evolution of a population subject to nonlocal dispersal, resource-dependent growth, intraspecific interactions, and constant harvesting on a bounded domain $\Omega \subset \mathbb{R}^N$.

The analysis was carried out within the framework of fractional Sobolev spaces $H_0^s(\Omega)$ and the associated Bochner spaces, in particular $L^2(0,T;H_0^s(\Omega))$ and its dual space $L^2(0,T;H^{-s}(\Omega))$, as well as the Sobolev–Bochner space $W^{1,2,2}(0,T;H_0^s(\Omega),H^{-s}(\Omega))$. These functional tools proved to be fundamental for the formulation of the problem and for obtaining the estimates required in the analysis.

A crucial step in the development of this work consisted in the study of a more general parabolic problem involving a Carathéodory-type nonlinearity. Under suitable growth and monotonicity assumptions, we proved the existence of solutions by means of the monotone iteration method. The reflexivity of the functional spaces involved was essential for passing to the limit in monotone sequences of approximate solutions.

Based on these general results, it was possible to prove the main theorem of this dissertation, which guarantees the existence and uniqueness of a positive essentially bounded solution to the original parabolic problem, provided that the parameters satisfy appropriate conditions. The construction of ordered sub- and supersolutions played a central role in this argument and was carried out using the associated elliptic problem.

Although the results obtained ensure the well-posedness of the problem with respect to existence and uniqueness of solutions, questions related to the asymptotic behavior and stability of solutions remain open. To the best of our knowledge, at the time of writing this dissertation there are no results in the literature addressing the stability of solutions to the parabolic problem under study. Thus, the investigation of such properties constitutes a natural direction for future research and a possible continuation of the topics addressed herein.

Appendix A

We present a collection of results on measure and integration of real-valued functions and Bochner Integrable functions. These results, as well as their proofs, can be found in full in the books [2], [3], [9]

First, we will recall some important results about measure and integration of real-valued functions and, then present similar results for Bochner integrable functions. We will assume that the reader is familiar with the concepts of continuity and measurability of real-valued functions.

Theorem A.0.1. *(Monotone Convergence [21]). Let $f_n : \Omega \subset \mathbb{R}^N$, $n \in \mathbb{N}$, be a sequence of functions. If f_n is integrable for every $n \in \mathbb{N}$ and the sequence $(f_n)_{n=1}^{\infty}$ is monotone with $\sup \left| \int_\Omega f_n \, dx \right| < \infty$, then*

$$\lim_{n\to\infty} \int_\Omega f_n \, dx = \int_\Omega \lim_{n\to\infty} f_n \, dx. \tag{A.0.1}$$

Lemma A.0.1. *(Fatou's Lemma [21]). If $f_n : \Omega \subset R^N \to R$ are non-negative, integrable and $\lim_{n\to\infty} \int_\Omega f_n \, dx < \infty$, for every $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \int_\Omega f_n \, dx = \int_\Omega \lim_{n\to\infty} f_n \, dx. \tag{A.0.2}$$

Theorem A.0.2. *(Theorem of Fubini [21]). Let $f : M \subset \mathbb{R}^{N+L} \to \mathbb{R}$ be an integrable function, with $f(x,y) = 0$ outside M. Then the following holds:*

1. $\int_{\mathbb{R}^N} f(x,y) \, dx$ *exists for a.e $x \in \mathbb{R}^N$;*
2. $\int_{\mathbb{R}^L} f(x,y) \, dy$ *exists for a.e $y \in \mathbb{R}^L$; and*
3. *The integrals*

$$\int_M f(x,y) \, dx dy, \int_{\mathbb{R}^L} \left(\int_{\mathbb{R}^N} f(x,y) \, dx \right) dy, \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^L} f(x,y) \, dx \right) dy$$

exists and are equal.

Theorem A.0.3. *(Theorem of Tonelli [21]). Let $f : \Omega \subset \mathbb{R}^{N+L} \to R$ be a measurable function. Then the following are equivalent:*

1. *f is integrable;*

2. $\int_{R^N}\left(\int_{\mathbb{R}^L}|f|\,dy\right)dx$ *exists, or* $\int_{\mathbb{R}^L}\left(\int_{\mathbb{R}^N}f(x,y)\,dx\right)dy$ *exists.*

Remark A.0.1. *If condition (ii) in Tonelli's Theorem holds, then Fubini's Theorem applies.*

Theorem A.0.4. *(The Hölder's Inequality [21]). Let $\Omega\subset\mathbb{R}^N$ be a limited non-empty subset of $\mathbb{R}^N$, $1<p_1,\dots,p_N<\infty$, with $\sum_{i=1}^{N}p_i^{-1}=1$, and $u_i\in L^{p_i}(\Omega)$. Then,*

$$\left|\int_\Omega\prod_{i=1}^{N}u_i\,dx\right|\leq\prod_{i=1}^{N}\left(\int_\Omega|u_i|^{p_i}\,dx\right)^{\frac{1}{p_i}}=\prod_{i=1}^{N}\|u_i\|_{L^{p_i}(\Omega)}. \tag{A.0.3}$$

We recall that, given two normed linear spaces, X and Y, not necessarily over the same field, a linear operator is a function

$$\begin{array}{rcl}T:D(T)\subset X & \to & Y\\ x & \mapsto & T(x)\end{array}$$

where $D(T)$ is a linear subspace of X called the domain of definition of the operator T. If T is a linear operator as above, then T is said to be continuous on its domain of definition if and only if it is bounded; that is, there exists a constant $C\geq 0$ such that

$$\|T(x)\|_Y\leq C\|x\|_X,$$

for every $x\in D(X)$. In such case, we also say that the operator T is bounded on $D(X)$. The set of all bounded linear operators $T:X\to Y$ is denoted by $\mathcal{B}(X,Y)$. If X is a normed linear space over some field $\mathbb{K}$, then a linear functional on X is a linear operator

$$T:X\to\mathbb{K}.$$

The set of all continuous linear functionals on X is denoted by

$$X'=\mathcal{B}(X,\mathbb{K})$$

and is called the topological dual of X. A linear operator $T:X\to Y$ is an isometry if, and only if,

$$\|T(x)\|_Y=\|x\|_X$$

for every $x\in X$.

Theorem A.0.5. *(The dual space of $L^p(\Omega)$ [21]) Let $\Omega\subset\mathbb{R}^N$ be an nonempty limited open subset of R^N, and $p,q\in\mathbb{R}$ be such that $1<p,q<\infty$, with $p^{-1}+q^{-1}=1$. Then, we can identify $(L^p(\Omega))'$ with $L^q(\Omega)$, that is:*

1. *If $u\in L^q(\Omega)$ and $\mathcal{U}(v)=\int_\Omega uv\,dx$, for every $v\in L^P(\Omega)$, then $\mathcal{U}:L^p(\Omega)\to\mathbb{R}$ is a continuous linear functional on $L^p(\Omega)$, i.e., $\mathcal{U}\in(L^p(\Omega))'$. Moreover, we have $\|\mathcal{U}\|_{(L^p(\Omega))'}=\|u\|_{L^p(\Omega)}$.*

2. *Conversely, each $\mathcal{U} \in (L^p(\Omega))'$ is obtained in the same manner, where $u \in L^p(\Omega)$ is uniquely determined by $\mathcal{U}$.*

Hence, there exists an isometric isomorphism $u \mapsto \mathcal{U}$ from $L^q(\Omega)$ onto $(L^p(\Omega))'$. Identifying $\mathcal{U}$ with u, we will write

$$\langle u, v \rangle_{L^q(\Omega), L^p(\Omega)} = \int_\Omega uv\,dx, \forall u \in L^q(\Omega), \forall v \in L^p(\Omega) \tag{A.0.4}$$

Lemma A.0.2. *(Gronwall's lemma [1]). Let $J \subset \mathbb{R}$, $t_0 \in J$ and α, β, $u \in C(J; \mathbb{R}_+)$. Suppose that*

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^t \beta(s) u(s) \right|, \tag{A.0.5}$$

for all $t \in J$. Then,

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^t \alpha(s)\beta(s) e^{\left| \int_s^t \beta(\sigma)\,d\sigma \right|} ds \right| \tag{A.0.6}$$

for all $t \in J$.

Proof. For a proof, see [1], Chapter 2, Lemma 6.1, page 89. □

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