UNIVERSIDADE DE BRASÍLIA
Instituto de Ciências Exatas
Departamento de Matemática

Symmetric solutions for some elliptic equations

por

Thiago Guimarães Melo

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Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos para obtenção do título de Doutor em Matemática sob orientação do Prof. Dr. Marcelo F. Furtado.

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PROGRAMA DE PÓS GRADUAÇÃO EM MATEMÁTICA

Ata Nº: 06

Aos dezoito dias do mês de junho do ano de dois mil e vinte e cinco, instalou-se a banca examinadora de Tese de Doutorado do aluno Thiago Guimarães Melo, matrícula 20/0113097. A banca examinadora foi composta pelos professores Dr. Ricardo Ruviaro (membro interno - MAT/UnB), Dr. João Pablo Pinheiro da Silva (membro externo à instituição - UFPA), Dr. Everaldo Souto Medeiros (membro externo à instituição - UFPB), Dr. Giovany de Jesus Malcher de Figueiredo (Suplente - membro interno -MAT/UnB) e Dr. Marcelo Fernandes Furtado (orientador/presidente - MAT/UnB). O discente apresentou o trabalho intitulado "*Symmetric solutions for some elliptic equations*".

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Ao meu pai, Anselmo Martins Melo, que sempre esteve comigo, ainda que em outro plano, e me ensinou a trilhar os caminhos do bem.

"O impossível existe até que alguém
duvide dele e prove o contrário"
(Albert Einstein)

Título em português: Soluções simétricas para algumas equações elípticas.

Palavras-chave: Operador de Grushin, equação de Hénon, quebra de simetria, problemas supercríticos, métodos variacionais, equações elípticas, operador biharmônico, sistema elíptico, desigualdade de Trudinger-Moser.

Resumo expandido

Este trabalho aborda várias equações diferenciais não lineares e sistemas envolvendo pesos radiais, o operador de Grushin, o operador biharmônico e um sistema acoplado do tipo FitzHugh–Nagumo. Considera não linearidades com crescimento superlinear ou supercrítico, concentrando-se em como a presença de pesos afeta as propriedades de simetria das soluções. Novos lemas de simetria radial, adaptados aos pesos específicos, são estabelecidos, juntamente com resultados referentes à existência e multiplicidade de soluções fracas e regulares. Em certos contextos, também se demonstra a quebra de simetria para soluções de energia mínima.

A seguir, explicaremos brevemente o que foi provado em cada capítulo da tese. No Capítulo 2, intitulado *“Equação de Hénon para o operador de Grushin”*, estudamos a equação do tipo Hénon

$$\begin{cases} -(\mathcal{G}_\alpha u)(z) = |z|^\ell f(z,u), & B, \\ u = 0, & \partial B, \end{cases} \tag{P_α}$$

onde B é a bola unitária em $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, f se comporta como $|t|^{p-2}t$, $\ell > 0$, $\alpha \geqslant 0$, $2 < p < 2^*_\alpha = \dfrac{2\tilde{N}}{\tilde{N}-2}$, com $\tilde{N} = N_1 + (1+\alpha)N_2$, e

$$\mathcal{G}_\alpha u(x,y) = \Delta_x u + |x|^{2\alpha}\Delta_y u$$

é o operador de Grushin. O expoente crítico 2^*_α garante que a imersão $H^1_{0,\alpha}(B) \hookrightarrow L^p(B)$ é compacta para $2 < p < 2^*_\alpha$, onde $H^1_{0,\alpha}(B)$ generaliza o espaço de Sobolev usual $H^1_0(B)$.

A principal dificuldade em demonstrar um resultado de imersão compacta, a partir do qual obtemos soluções fracas radiais, foi estender uma desigualdade famosa para funções radiais de [60]. Precisamente, provamos que

$$|u(z)| \leqslant C\frac{\|u\|}{|z|^{(\tilde{N}-2)/2}}, \quad \text{para q.t.p. } z \in B,$$

é válida para u radialmente simétrica em $C_0^\infty(B)$, onde $\|\cdot\|$ é a norma em um certo espaço X de funções radiais. Outra dificuldade foi provar que a solução obtida é não-negativa, já que $\mathcal{G}_\alpha$ não é invariante sob aplicações ortogonais. Por fim, é bem conhecido na literatura que, para $\ell > 0$ grande e $\alpha = 0$, a solução de menor energia (ground state) não é radial. Para demonstrar um resultado semelhante quando $\alpha > 0$, utilizamos uma família de dilatações inerente ao operador de Grushin, o que é particularmente interessante.

No Capítulo 3, intitulado *"Equação de Hénon para o operador biharmônico"*, também decompomos $\mathbb{R}^N$ como no Capítulo 2 e estudamos a seguinte classe de problemas do tipo Hénon:

$$\begin{cases} \Delta^2 u = [W(z)]^\ell f(u), & B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \partial B, \end{cases} \tag{$P_{x,y}$}$$

onde $\Delta^2 u = \Delta(\Delta u)$ é o operator biharmônico, $3 \leqslant N_2 \leqslant N_1$, $\ell > 0$, $W(x,y)$ se comporta como o produto $|x||y|$, e f possui crescimento do tipo $|t|^{p-2}t$ para $2^*_{\ell,N_1} := [2N_1/(N_1 - 2)] + [2\ell/(N_1 - 2)]$. O peso W nos leva a considerar o subespaço $H^2_{0,x,y}(B)$ de $H^2_0(B)$, constituído por funções radiais em cada uma das direções x e y.

Um dos pontos principais do capítulo foi a demonstração de uma desigualdade análoga à já mencionada de [60], agora envolvendo o peso em questão, a qual não foi encontrada na literatura. Especificamente, provamos a seguinte interessante desigualdade:

$$|u(x,y)| \leqslant C \frac{\|\Delta u\|_{L^2(B)}}{|x|^{\frac{N_1-2}{2}} |y|^{\frac{N_2-2}{2}}}, \quad \text{para quase todo } (x,y) \in B,$$

onde $u \in H^2_{0,x,y}(B)$, e a constante C foi também explicitada. A técnica utilizada na demonstração dessa desigualdade foi inspirada em [60], mas parece ser inovadora. Além disso, tal desigualdade dá origem a um produto interno em $H^2_{0,x,y}(B)$, aparentemente desconhecido até então, a saber,

$$B_{x,y}[u,v] := \int_B \Delta_x u(z) \Delta_y v(z)\, \mathrm{d}z, \quad \forall\, u, v \in H^2_{0,x,y}(B).$$

Essa desigualdade nos permite provar um resultado de imersão compacta em espaços de Lebesgue com o peso W; da qual segue a existência de solução fraca para ($P_{x,y}$), superando inclusive o expoente crítico usual de $H^2(B)$, a saber, $2^{**} = 2N/(N-4)$. Por fim, mostramos como nosso resultado de imersão compacta complementa um análogo apresentado em [20].

No Capítulo 4, intitulado *"Kirchhoff-Boussinesq equation with Hénon nonlinearity"*, damos continuidade ao estudo de equações do tipo Hénon. No entanto, diferentemente dos capítulos anteriores, não realizamos aqui nenhuma decomposição em

$\mathbb{R}^N$, sendo que um vetor neste espaço é denotado por x. Estudamos o seguinte problema do tipo Kirchhoff-Boussinesq:

$$\begin{cases} \Delta^2 u \pm \operatorname{div}(|x|^\kappa |\nabla u|^{p-2}\nabla u) = |x|^\ell f(u), & \text{in } B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \tag{$\mathcal{P}_\pm$}$$

onde $\kappa, \ell \geqslant 0$, $2 \leqslant p < 2^*_\kappa := 2^* + \dfrac{2\kappa}{N-2}$, e a função f possui crescimento do tipo $|t|^{q-2}t$, com $p < q < 2^{**}_\ell := 2^{**} + \dfrac{2\ell}{N-4}$. Assim como no Capítulo 1, buscamos soluções fracas radiais para o problema ($\mathcal{P}_\pm$), ou seja, soluções pertencentes ao subespaço X de $H^2_0(B)$ formado por funções radiais.

Para tratar da existência de soluções fracas radiais para esse problema, empregamos uma desigualdade de interpolação de Gagliardo-Nirenberg com pesos, a fim de contornar a dificuldade imposta pelo termo $\operatorname{div}(|x|^\kappa |\nabla u|^{p-2}\nabla u)$. A principal dificuldade consistiu em determinar parâmetros adequados para κ, p e q que nos permitissem obter soluções fracas em X.

Demonstramos que, para $N > 4$ e $\ell > 0$, existe $\kappa_* = \kappa_*(\ell, N) > 0$ tal que, para qualquer $\kappa \in [0, \kappa_*)$ e $2 \leqslant p < 2^*_\kappa$, existem constantes $\theta_* \in [1/2, 1)$ e $q_* \in [2, 2^{**}_\ell)$ satisfazendo

$$\|\nabla u\|_{L^p_\kappa(\mathbb{R}^N)} \leqslant C \|\nabla^2 u\|^{\theta_*}_{L^2(\mathbb{R}^N)} \|u\|^{1-\theta_*}_{L^{q_*}_\ell(\mathbb{R}^N)}, \qquad \forall\, u \in C^\infty_0(\mathbb{R}^N),$$

sendo que os valores de κ_* e q_* são explicitamente determinados. O intervalo obtido para q não é arbitrário, sendo este de modo a garantir a validade da imersão compacta de X em espaços de Lebesgue com peso $|x|^\ell$, conforme [20].

Por fim, supondo certa simetria em f, pudemos empregar uma versão simétrica do Teorema do Passo da Montanha em conjunto com a teoria espectral do operador Δ^2 associada ao problema de autovalor com peso $|x|^\ell$, a fim de demonstrar a existência de infinitas soluções para o problema ($\mathcal{P}_\pm$).

No Capítulo 5, intitulado *"FitzHugh-Nagumo system with exponential growth"*, deixamos as equações do tipo Hénon de lado para estudar um problema no $\mathbb{R}^2$, porém ainda buscando soluções radiais. Precisamente, estudamos o sistema do tipo FitzHugh-Nagumo

$$\begin{cases} -\Delta u = \lambda Q(|x|) f(u) - V(|x|) v, & \text{in } \mathbb{R}^2, \\ -\Delta v = V(|x|) u - V(|x|) v, & \text{in } \mathbb{R}^2, \end{cases} \tag{$\mathcal{S}_\lambda$}$$

onde $\lambda > 0$, os potenciais V e Q são contínuos e positivos em $(0, +\infty)$ e se comportam como potências de r. Precisamente, existem $a, b, b_0 > -2$ tais que

$$\liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0, \quad \limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < +\infty \quad \text{e} \quad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < +\infty.$$

A não linearidade f é contínua e tem crescimento exponencial.

Para lidar com o Problema $(\mathcal{S}_\lambda)$, primeiro fixamos $u \in E$; em que E é o espaço apropriedade do tipo $H^2(\mathbb{R}^2)$ que envolve o peso V, e ,na segunda equação, obtemos via Teorema da Representação de Riez, uma solução $B[u]$. Assim, definimos um certo espaço com base em $B[u]$ e provamos uma desigualdade do tipo Trudinger-Moser especialmente para este, a qual é interessante por si só.

Com a estrutura variacional pronta, mostramos que existe $\lambda_0 > 0$ tal que, para todo $\lambda \geqslant \lambda_0$, o sistema $(\mathcal{S}_\lambda)$ admite uma solução fraca radial não nula (u, v). Além disso, com uma hipótese adicional de V perto da origem, mostramos um resultado de regularidade. Finalmente, supondo que f tem certa simetria, mostramos que o problema $(\mathcal{S}_\lambda)$ possui infinitas soluções.

Abstract

This work addresses various nonlinear differential equations and systems involving radial weights, the Grushin operator, the biharmonic operator, and a coupled FitzHugh–Nagumo-type system. It considers nonlinearities with superlinear or supercritical growth, focusing on how the presence of weights affects the symmetry properties of solutions. New radial symmetry lemmas tailored to the specific weights are established, along with results concerning the existence and multiplicity of weak and regular solutions. In certain settings, symmetry breaking is also demonstrated for minimal energy solutions.

Key words: Grushin operator, Hénon equation, symmetry breaking, supercritical problems, variational methods, elliptic equations, biharmonic operator, elliptic system, Trudinger-Moser inequality.

Contents

Chapter 1

Introduction

Many physical phenomena can be described by partial differential equations (PDE's). In fluid mechanics and gas dynamics, the Navier-Stokes equations model fluid flow, including air, water, and blood, as well as weather patterns (*cf.* [19]). In electromagnetism and potential theory, Maxwell's equations govern electromagnetic waves and optics (*cf.* [44]). The heat equation, a parabolic PDE, describes time-dependent diffusion processes like thermal conduction. Meanwhile, the wave equation, a hyperbolic PDE, represents wave propagation in vibrations, sound, and light. Poisson's equation, an elliptic PDE, models steady-state phenomena such as electric potential and heat distribution (*cf.* [40]). PDE's also play a crucial role in elasticity, solid mechanics, quantum mechanics, general relativity, cosmology, biology, medicine, geophysics and earth sciences.

This thesis is specifically focused on certain types of non linear elliptic PDE's and our approach is variational. We briefly comment on this branch of Analysis. Generally, the problem of finding a weak solution u of an elliptic PDE is the same as finding some u, in a Banach space X of functions, which verifies an identity of the form

$$J(u, \varphi) = 0, \quad \forall \varphi \in X. \tag{1.1}$$

The space X is generally related to the nature of the problem. The techniques in Variational Methods mostly involve associate a functional $I \in C^1(X, \mathbb{R})$ to (1.1) such that

$$I'(u)\varphi = J(u, \varphi), \quad \forall u, \varphi \in X$$

and so, critical points of I would be weak solutions of the problem of interest. One of the most important results in this line is the so called Mountain Pass Theorem, duo Antonio Ambrosetti and Paul Rabinowitz (*cf.* [5]), which will be used in this thesis several times.

In this thesis, we investigate three distinct elliptic problems. So, the main part of the work is organized into three chapters, each dedicated to one of these problems, which are briefly described in the following sections. We would like to comment that, as each chapter is independent, we may use the same notation for properties of the non linearity term and some weighted Lebesgue space in different chapters. We also mention that the thesis includes an appendix with general results used throughout the text.

Hénon equation with Grushin operator

The Hénon equation

$$-\Delta u(z) = |z|^{\ell} u(z)^{p-1}, z \in B, \quad u(z) = 0, z \in \partial B, \tag{1.2}$$

was introduced in [43] as a model for investigating spherically symmetric clusters of stars. In the equation, B is the unit ball of $\mathbb{R}^N$, $N \geqslant 3$, $\ell > 0$ and $p > 2$. Its mathematical significance grew after Ni's paper [60], where the existence of positive radial solutions was established for $2 < p < 2^* + 2\ell/(N-2)$. The crucial idea for the existence of solutions beyond the critical Sobolev exponent $2^* := 2N/(N-2)$ lies in the following inequality

$$|u(z)| \leqslant \frac{\|\nabla u\|_{L^2(B)}}{\sqrt{\omega_N(N-2)}|z|^{(N-2)/2}}, \quad z \in B, \tag{1.3}$$

which holds for any radially symmetric $u \in C^1(B)$ vanishing in the boundary of B. Here ω_N denotes the surface area of the unit ball in $\mathbb{R}^N$ for $N \geqslant 3$. This inequality is known in the literature as Ni's Radial Lemma.

Since Ni's work, equation (1.2) has been studied in several different perspectives. Since it is impossible to present a complete list, we only cite [7, 10, 11, 13, 21, 36, 37, 50, 52, 53, 57, 67, 70, 79] and references therein for further exploration. In addition to Ni's paper, the work of Smets, Su and Willem [71] strongly influenced our investigation in Chapter 2. They proved that, for large $\ell > 0$ and $p \in (2, 2^*)$, the ground state solution of (1.2) is non-radial. This phenomena was first observed via numerical computations in [17] and it is called in literature as symmetry breaking. For more recent works on this kind of result for Hénon type equations we refer to [6, 42, 64].

In Chapter 2, we split each vector $z \in \mathbb{R}^N$ as $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and define the *Grushin operator*

$$(\mathcal{G}_\alpha u)(z) := \Delta_x u(z) + |x|^{2\alpha} \Delta_y u(z), \tag{1.4}$$

where $\alpha \geqslant 0$, N_1, $N_2 \in \mathbb{N}$, and Δ_x, Δ_y denote the usual Laplacian in the variables x, y, respectively. When $\alpha = 0$ this operator is the usual Laplacian operator.

The operator (1.4) was first studied by Grushin in [34, 35] when $\alpha \in \mathbb{Z}$ and first addressed in [29, 30] for cases where $\alpha \notin \mathbb{Z}$. Since then, it has been the subject of intensive research, *cf. e.g.*, [4, 24, 51, 56] and references therein. In addition, we remark that this operator belongs to two more generals class of elliptic operators, *cf. e.g.*[39, 47, 54].

We propose the study of the following Hénon type equation

$$\begin{cases} -(\mathcal{G}_\alpha u)(z) = |z|^\ell f(z,u), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{P_α}$$

with $f(z,s)$ behaving like $|s|^{p-2}s$, for suitable values of $p > 2$, which may exhibit supercritical growth with respect to the critical exponent associated to the Grushin operator.

More specifically, we fix $\alpha \geqslant 0$, define

$$\tilde{N} := N_1 + (1+\alpha)N_2, \qquad 2^*_\alpha := \frac{2\tilde{N}}{\tilde{N}-2},$$

and require that the nonlinear term f satisfies the following assumptions:

(f_1) $f \in C(\overline{B} \times \mathbb{R}, \mathbb{R})$;

(f_2) there exists $\ell > \alpha N_2$,

$$2 < p < 2^*_\alpha + \frac{2(\ell - \alpha N_2)}{\tilde{N}-2}$$

and $C > 0$ such that

$$|f(z,s)| \leqslant C\left(1 + |s|^{p-1}\right), \quad \forall\, (z,s) \in \overline{B} \times \mathbb{R}.$$

The natural space to deal with the Grushin operator is

$$H^1_\alpha(B) := \Big\{u \in L^2(B) : |\nabla_x u| \in L^2(B),\, |x|^\alpha |\nabla_y u| \in L^2(B)\Big\},$$

which becomes a Banach space (*cf.* [77, Theorem 4]) when endowed with the norm

$$||u||_{H^1_\alpha(B)} := \left(\int_B \left[|\nabla_\alpha u|^2 + |u|^2\right] dz\right)^{1/2},$$

where

$$\nabla_\alpha u(z) := (\nabla_x u(z), |x|^\alpha \nabla_y u(z)) \in \mathbb{R}^N.$$

We also define the subspace $H^1_{0,\alpha}(B) := \overline{C_0^\infty(B)}^{||\cdot||_{H^1_\alpha(B)}}$ with the norm

$$\|u\| := \left(\int_B |\nabla_\alpha u(z)|^2 \, \mathrm{d}z\right)^{1/2},$$

which is equivalent to $\|\cdot\|_{H^1_\alpha(B)}$ (*cf.* [47, (1.8)]). The space $H^1_{0,\alpha}(B)$ is continuously and compactly embedded in $L^p(B)$, for $p \in [1, 2^*_\alpha]$ and $p \in [1, 2^*_\alpha)$, respectively (*cf.* [47, Theorem 3.3 and (1.8)]). In addition, let

$$C_{0,r}^{\infty}(B) := \{u \in C_0^{\infty}(B) : u \text{ is radial}\}$$

and $X := \overline{C_{0,r}^{\infty}(B)}^{H_{0,\alpha}^1(B)}$.

In line with the preceding studies concerning the Grushin operator, we aim to identify critical points of the Euler-Lagrange functional associated with Problem (P_α). The main novelty here is that we are able to consider nonlinearities which grows beyond the Grushin critical exponent 2_α^*, since we are assuming $\ell > \alpha N_2$. This can be done by a careful estimate of the decay rate of radial functions of our working space together with an embedding result in an weighted L^p-Lebesgue space. More specifically, in Section 2.1 we prove that

$$|u(z)| \leqslant C\frac{\|u\|}{|z|^{(\tilde{N}-2)/2}}, \qquad \text{for a.e. } z \in B,$$

where $C = C(N) > 0$ does not depends on $u \in X$.

After establishing the variational framework, we revisit Problem (P_α) and impose conditions on f to utilize the full apparatus of Critical Point Theory (*cf.* [66]) for obtaining solutions. The range of problems that can be addressed is extensive. As an illustrative example, we focus on the classical superlinear setting, wherein we assume:

(f_3) there holds

$$\lim_{s\to 0}\frac{f(z,s)}{s} = 0, \quad \text{uniformly in } \overline{B};$$

(f_4) there exists $\mu > 2$ and $s_0 > 0$ such that

$$0 < \mu F(z,s) \leqslant sf(z,s), \quad \forall\, z \in \overline{B},\ s \geqslant s_0,$$

where $F(z,s) := \displaystyle\int_0^s f(z,\tau)d\tau$.

As an application of the variational setting developed in Section 2.1, we shall prove in Section 2.2 the following existence result:

Theorem 2.1. *Suppose that $\alpha > 0$, $N_2 \geqslant 2$ and f satisfies (f_1)-(f_4). Then Problem (P_α) has a nonzero and radial weak solution. If in addition $\alpha \geqslant 1$, this solution is nonnegative.*

To the best of our knowledge, there is limited literature on the Grushin operator with Hénon-type nonlinearities. We could mention the paper by Duong and Nguyen [24], where a nonexistence result is established for stable weak solutions of the equation:

$$-\mathcal{G}_\alpha u + [\nabla_\alpha w \cdot \nabla_\alpha u] = |z|_\alpha^{\ell}|u|^{p-2}u, \quad z = (x,y) \in \mathbb{R}^{N_1}\times\mathbb{R}^{N_2},$$

with $p > 2$, $\ell \geqslant 0$, $|z|_\alpha := (|x|^{2(\alpha+1)}+|y|^2)^{1/2(\alpha+1)}$, and the function w satisfying appropriate decay properties at infinity. Similar results can be found in [65, 80, 81]. Our first main result differs from and complements these findings.

In the homogeneous case $f(z,s) = |s|^{p-2}s$, it is evident that the solution can be obtained through constrained minimization. When $p \in (2, 2^*_\alpha)$, it is natural to inquire whether the ground state solutions of the problem are always radial. In our final result of Chapter 2, we leverage the action of a semigroup of dilations associated with the Grushin operator to obtain a symmetry breaking result for large values of $\ell > 0$. More specifically, in Section 2.3, after performing careful estimates of the asymptotic behavior of the ground state levels of our problem in the whole space and in the space of radial functions, we complement [71, Theorem 3.1] by proving the following result:

Theorem 2.2. *Suppose that $\alpha > 0$, $N_2 \geqslant 2$ and $f(z,s) = |s|^{p-2}s$, for any $(z,s) \in \overline{B} \times \mathbb{R}$ and some $p \in (2, 2^*_\alpha)$. Then there exists $\ell^* > 0$ such that the ground state solution of Problem (P_α) is not radial provided $\ell \geqslant \ell^*$.*

We remark that a ground state solution of Problem (P_α) is a solution that minimizes the associated Euler–Lagrange functional over $H^1_{0,\alpha}(B) - \{0\}$.

Hénon equation for the biharmonic operator

In Chapter 3, we consider the Hénon equation

$$-\Delta u(z) = |z|^\ell u^{p-1}(z), z \in B, \quad u(z) = 0, z \in \partial B, \tag{1.5}$$

where B is the unit ball of $\mathbb{R}^N$, $N \geqslant 3$, $\ell > 0$ and $p > 2$. We recall that the crucial aspect used in [60] to obtain existence of positive weak solutions for $2 < p < 2^* + 2\ell/(N-2)$, with $2^* := 2N/(N-2)$ lies in obtaining a constant $C > 0$, such that

$$|u(z)| \leqslant \frac{\|\nabla u\|_{L^2(B)}}{\sqrt{\omega_N(N-2)}|z|^{(N-2)/2}}, \quad z \in B, \tag{1.6}$$

for any radially symmetric $u \in C^1(B)$ vanishing in the boundary of B. With this inequality in hands, it is possible to embed the subspace of $H^1_0(B)$ of radial functions $H^1_{0,r}(B)$ into Lebesgue spaces $L^s(B)$ with the number s beyond the critical Sobolev exponent 2^*.

The first aim of Chapter 3 is establishing a version of inequality (1.6) that involves the $H^2_0(B)$ norm, when the space $\mathbb{R}^N$ has an specific decomposition. In order to be more specific, in next lines we will define the appropriate spaces which will be used in Chapter 3.

We decompose $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with $3 \leqslant N_2 \leqslant N_1$ and write a vector $z \in \mathbb{R}^N$ as $z = (x, y)$ with $x \in \mathbb{R}^{N_1}$ and $y \in \mathbb{R}^{N_2}$. For any $1 \leqslant p < +\infty$ and $\ell \geqslant 0$, we set

$$L^p_\ell(B) := \left\{ u \in L^1_{loc}(B) : \int_B |u(z)|^p [W(z)]^\ell dz < +\infty \right\},$$

where W verifies

(W_1) $W \in L^1_{loc}(B)$ and there exists $c_W > 0$, such that

$$0 < W(z) \leqslant c_W |x||y|, \qquad \text{for a.e. } z \in B.$$

This is a Banach space with the norm

$$\|u\|_{L^p_\ell(B)} := \left(\int_B |u(z)|^p [W(z)]^\ell dz \right)^{1/p}.$$

We define $H^2_0(B)$ as the closure of $C^\infty_0(B)$ under the $H^2(B)$ norm. Using Poincaré Inequality and integration by parts, one can see that the usual norm induced by that of $H^2(B)$ is equivalent to

$$\|u\|_{H^2_0(B)} := \left(\sum_{|\alpha|=2} |D^\alpha u(z)|^2 \, \mathrm{d}z \right)^{1/2} = \left(\int_B |\Delta u(z)|^2 \, \mathrm{d}z \right)^{1/2}.$$

Finally, we denote by $O(k)$ the group of real orthogonal $k \times k$ matrices and define

$$H^2_{0,x,y}(B) := \overline{C^\infty_{0,x,y}(B)}^{H^2_0(B)},$$

where

$$C^\infty_{0,x,y}(B) := \left\{ u \in C^\infty_0(B) : u(x, y) = u(T_1(x), T_2(y)) \ \forall \ T_i \in O(N_i), i = 1, 2 \right\},$$

is the set of compactly supported functions in B with are coordinate-radial.

Our radial type inequality is the following:

Theorem 3.1. *For any $u \in H^2_{0,x,y}(B)$, there holds*

$$|u(x, y)| \leqslant C \frac{\|\Delta u\|_{L^2(B)}}{|x|^{\frac{N_1-2}{2}} |y|^{\frac{N_2-2}{2}}}, \quad \text{for a.e. } (x, y) \in B, \tag{1.7}$$

with

$$C := \sqrt{\frac{\Gamma\left(\frac{N_1}{2}\right)\Gamma\left(\frac{N_2}{2}\right)}{4\pi^{\frac{N}{2}}(N_1 - 2)(N_2 - 2)}},$$

and $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt$ *being the Gamma function.*

Theorem 3.1 was inspired by some ideas contained in [60, p. 802] and [20, Corollary 4.1], which considered versions of the last inequality for the Laplacian and biharmonic operator, respectively, but with a power of $|(x,y)|$ depending on N in the denominator. We also learn from [48, Lemma 2.1], where it is proved that there exists $C_1 = C_1(N)$, such that

$$u(x,y) \leqslant C_1 \frac{\|u\|^{1/2}_{L^2(\mathbb{R}^N)} \|\nabla_x u\|^{1/2}_{L^2(\mathbb{R}^N)}}{|x|^{\frac{N_1-1}{2}}|y|^{\frac{N_2}{2}}},$$

for any $u \in C_0^\infty(\mathbb{R}^N)$ such that $u(x,y) = \varphi(|x|,|y|)$, for φ non-increasing in $|y|$, and $N_1 \geqslant 2, N_2 \geqslant 1$. Notice that no monotonicity conditions are assumed in our work.

As a consequence of Theorem 3.1, we prove an embedding result for the space $H^2_{0,x,y}(B)$. Actually, if we set

$$2^*_{\ell,N_1} := \frac{2N_1}{N_1-2} + \frac{2\ell}{N_1-2},$$

for any $N_1 > 2$ and $\ell > 0$, we have the following:

Theorem 3.2. *Suppose that* $N = N_1 + N_2$*, with* $3 \leqslant N_2 \leqslant N_1$*,* $\ell \geqslant 0$ *and* $1 \leqslant p < 2^*_{\ell,N_1}$*. Then the embedding* $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ *is compact.*

It is important to analyze situations in which the last result allows us to consider exponents beyond the critical Sobolev exponent $2^{**} := 2N/(N-4)$. We have that

$$2^{**} < 2^*_{\ell,N_1} \quad \Longleftrightarrow \quad \ell > \frac{2(N_1-N_2)}{N-4}, \tag{1.8}$$

and therefore we can consider supercritical growth. The most favorable situation occurs when $N_1 = N_2$ because, in this case, the exponent $2^*_{\ell,N_1}$ is supercritical for any $\ell > 0$. Even when the dimensions are not equal, the condition on ℓ does not seem very restrictive, since it can be easily shown that $3 \leqslant N_2 < N_1$ implies

$$\frac{2(N_1-N_2)}{N-4} < 2,$$

and, therefore, supercritical growth is possible for any $\ell \geqslant 2$.

The embeddings obtained in Theorem 3.2 are closely related to, and complement, the results presented in [20, Theorem 1.4 and Corollary 1.5]. For a detailed comparison, we refer to Subsection 3.2.1, where it is shown that our result covers a strictly larger range of the parameters p and ℓ.

Sobolev embeddings like that in Theorem 3.2 can be used to derive the existence of solutions for nonlinear PDE's. As an simple example, we consider the problem

$$\begin{cases} \Delta^2 u = [W(z)]^\ell f(u), & \text{in} B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \tag{$P_{x,y}$}$$

with the supercritical nonlinearity f satisfying:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) there exists $c_f > 0$ and $p \in (2, 2^*_{\ell,N_1})$, such that

$$|f(s)| \leqslant c_f \left(1 + |s|^{p-1}\right), \quad \forall\, s \in \mathbb{R};$$

(f_3) there holds

$$\lim_{s \to 0} \frac{f(s)}{s} = 0;$$

(f_4) there exist $\mu > 2$ and $s_0 > 0$, such that

$$0 < \mu F(s) \leqslant s f(s), \quad \forall\, |s| \geqslant s_0 > 0,$$

where $F(s) := \displaystyle\int_0^s f(t)dt$.

We prove the following:

Theorem 3.3. *Suppose that $\ell \geqslant 0$, $p \in (2, 2^*_{\ell,N_1})$ and f, W satisfy (f_1)-(f_4) and (W_1), respectively. Then Problem ($P_{x,y}$) has a nonzero weak solution in $H^2_{0,x,y}(B)$.*

As far we know, equation (1.5) was not studied before with the weight $|x||y|$ and the biharmonic operator. As we know from Section 1.1, since Ni's work, numerous researchers have approached equation (1.5) from various perspectives. Specifically, we refer to [20, 38, 84, 85] for studies involving the biharmonic operator, which had a significant impact on our investigation.

Kirchhoff-Boussinesq equation with Hénon nonlinearity

In this subsection, we present the mean results developed in Chapter 4.

The class of elliptic Kirchhoff-Boussinesq-type problems can be regarded as the stationary counterpart of the following class of time-dependent problems (see, e.g., [8, 9, 45]).

$$u_{tt} + ku_t + \Delta^2 u - \nabla.\left(|\nabla u|^{p-2}\nabla u\right) + \sigma\Delta\left(u^2\right) = \mathcal{F}(u),$$

where $k \geqslant 0$ denotes the damping parameter and σ is a nonnegative constant, characterizing the behavior of a scalar field within a bounded domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary. This model finds its roots in structural dynamics, notably within the framework of the Mindlin-Timoshenko equations, which account for transverse shear effects in plate dynamics. The reaction term $\mathcal{F}(u)$ captures nonlinear phenomena intrinsic to the system, representing a feedback force acting on the plate. For a more comprehensive understanding and further motivation, interested readers are directed to [18] and additional references therein.

In recent years, the analysis of biharmonic elliptical equations with p-Laplacian has been extensively investigated. In [14], the authors consider existence and multiplicity of solutions for the problem

$$\begin{cases} \Delta^2 u \pm \Delta_p u = f(u) + \beta|u|^{2^{**}-2}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 5$, $\beta \in \{0, 1\}$, f is continuous function and

$$2 < p < 2^* := \frac{2N}{N-2}, \qquad 2^{**} := \frac{2N}{N-4}.$$

In the proof, it was used minimization arguments, the Nehari method and genus theory. Also with the Nehari method, the authors in [15] obtained existence of solution for the problem

$$\begin{cases} \Delta^2 u \pm \Delta_p u = f(u), & \text{in } \Omega, \\ \Delta u = u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $N = 4$, $2 < p < 4$, $\Omega \subset \mathbb{R}^N$ and f continuous with exponential subcritical or critical growth.

The authors in [49] studied the existence of ground state solutions for weighted elliptic Kirchhoff–Boussinesq type problems with supercritical exponential growth of the following equation

$$\begin{cases} \Delta(w_\beta(x)\Delta u) \pm \operatorname{div}(w_\beta(x)|\nabla u|^{p-2}\nabla u) = f(x, u), & \text{in } B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases}$$

where $B \subset \mathbb{R}^4$ is the unit ball, $w_\beta(x) = \log\left(e/|x|\right)^\beta$ or $w_\beta(x) = \log\left(1/|x|\right)^\beta$, $\beta \in (0,1)$, $2 < p < 4$, $f(x,t) \sim \exp\left(\alpha t^{\frac{2}{1-\beta}+h(|x|)}\right)$ is a continuous function and $h : [0,1) \to [0,\infty)$ satisfy some mild conditions.

The fourth-order Hénon type equation

$$\begin{cases} \Delta^2 u = |x|^\ell |t|^{p-2} t, & \text{in } B, \\ u > 0, & \text{in } B, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \tag{1.9}$$

was considered in [84]. Regarding the p-Laplacian operator, the authors in [23] studied local and global properties of the equation,

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = |x|^\ell u^q, \text{ in } \Omega, \tag{1.10}$$

where $1 < p < N$, $q > p-1$, $\ell > 0$ and Ω is an open domain containing the origin. Specifically, local properties refer to local behavior of solutions near a certain point, like removable singularity and the order of isolated singularity and global properties refer to properties of solutions in $\mathbb{R}^N$.

Motivated by the aforementioned works, in Chapter 4, we aim to study the existence and multiplicity of radial solutions for the following class of problems

$$\begin{cases} \Delta^2 u \pm \text{div}(|x|^\kappa |\nabla u|^{p-2}\nabla u) = |x|^\ell f(u), & \text{in } B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \tag{$\mathcal{P}_\pm$}$$

where $\ell, \kappa \geqslant 0$, B is a unit ball in $\mathbb{R}^N$ and $p > 2$. We require that the nonlinear term f satisfies the following assumptions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) there exists $\ell > 0$,

$$2 < q < 2^{**}_\ell := 2^{**} + \frac{2\ell}{N-4}$$

and $C > 0$, such that

$$|f(s)| \leqslant C\left(1 + |s|^{q-1}\right), \quad \forall\, s \in \mathbb{R};$$

(f_3) there holds

$$\lim_{s \to 0} \frac{f(s)}{s} = 0;$$

(f_4) there exists $\mu > p$ and $s_0 > 0$ such that

$$0 < \mu F(s) \leqslant sf(s), \quad \forall\ |s| \geqslant s_0,$$

where $F(s) := \int_0^s f(\tau)d\tau$.

Now we are able to state our first result:

Theorem 4.1. *Suppose that $\ell > 0$, $N > 4$ and that f satisfies (f_1)-(f_4). Then there exists $\kappa_* = \kappa_*(\ell, N) > 0$ such that, for any $\kappa \in [0, \kappa_*)$, Problem ($\mathcal{P}_\pm$) has a nontrivial radial weak solution provided*

$$2 < p < 2^*_\kappa := 2^* + \frac{2\kappa}{N-2}, \qquad p < q.$$

We note that condition (f_2) requires $2 < q < 2^{**}_\ell$. To apply the last theorem, we also need $2 < p < 2^*_\kappa$ and $p < q$. Therefore, it is desirable to ensure that the inequality $2^*_\kappa < 2^{**}_\ell$ holds, so that the admissible range for q is non-empty for any choice of p. We prove in Proposition 4.5 that this condition is indeed satisfied.

In our second application, we prove that under symmetric conditions on f we can obtain multiple solutions. More specifically, the following holds:

Theorem 4.2. *Let $\kappa_* > 0$ be given by Theorem 4.1. Suppose that $\ell > 0$, $N > 4$, $\kappa \in [0, \kappa_*)$, $2 < p < 2^*_\kappa$, $2 < q < 2^{**}_\ell$ and f is an odd function satisfying (f_1), (f_2) and (f_4). Then Problem ($\mathcal{P}_\pm$) admits infinity many radial weak solutions.*

This work was carried out based on an initial idea proposed by Romulo Diaz Carlos, a postdoctoral researcher at UEMA, to whom we express our sincere gratitude.

FitzHugh-Nagumo system with exponential growth

In this subsection, we present the results of Chapter 5, obtained in collaboration with my advisor and Prof. Jônison Lucas dos Santos Carvalho (UFS). These results can also be found in the accepted paper [16].

We analyze the existence, multiplicity, and regularity of solutions to the following planar FitzHugh–Nagumo system:

$$\begin{cases} -\Delta u = \lambda Q(|x|)f(u) - V(|x|)v, & \text{in } \mathbb{R}^2, \\ -\Delta v = V(|x|)u - V(|x|)v, & \text{in } \mathbb{R}^2, \end{cases} \tag{$\mathcal{S}_\lambda$}$$

where $\lambda > 0$, the potentials V, $Q : (0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions meeting certain conditions specified later. This type of system, derived from activator-inhibitor dynamics, is significant in neurobiology for modeling nerve conduction and the transmission of electrical signals in neurons. Relevant background and studies can be found in [27, 41, 59, 75].

More broadly, our problem examines the steady-state of FitzHugh–Nagumo systems, which are described by the following ODE:

$$u_t = u^3 - v, \quad \tau v_t = u + a - bv, \tag{1.11}$$

initially proposed by Richard FitzHugh [27] and further developed by Jinichi Nagumo and collaborators [59]. This system models nerve impulse propagation through a simplified activator-inhibitor framework, capturing essential neurobiological processes. Further details on the physical background are available in [75].

Authors in [22, 46] point out that system (1.11) belongs to a more general class of reaction-diffusion systems, namely

$$\begin{cases} u_t = D_1 \Delta u + g(u) - v, & \text{in } (0, \infty) \times \Omega, \\ v_t = D_2 \Delta v + \varepsilon(u - \gamma v), & \text{in } (0, \infty) \times \Omega, \end{cases}$$

where Ω is a bounded domain and D_1, D_2, ε and γ are positive constants. This type of problem has motivated the study of the system

$$\begin{cases} u_t = D_1 \Delta u + g(u) - kv, & \text{in } (0, \infty) \times \mathbb{R}^N, \\ v_t = D_2 \Delta v + u - \gamma v, & \text{in } (0, \infty) \times \mathbb{R}^N. \end{cases}$$

See, for example, [46, 62] for the one-dimensional case and more recently [26] for the n-dimensional case, which has strongly influenced our investigation.

From a mathematical perspective, researchers have focused on problems involving potentials and weights that may be either unbounded or vanish at infinity. We especially emphasize the paper by Su, Wang, and Willem [73] (*cf.* [1–3]), which suppose, among other conditions, that V and Q satisfy the following:

(V_1) $V : (0, +\infty) \to (0, +\infty)$ is continuous and there exists $a > -2$, such that

$$\liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0;$$

(Q_1) $Q : (0, +\infty) \to (0, +\infty)$ is continuous and there exist b_0, $b > -2$, such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < +\infty, \qquad \limsup_{r\to +\infty} \frac{Q(r)}{r^{b}} < +\infty.$$

In their paper, the authors consider the Schrödinger equation

$$-\Delta u + V(|x|)u = Q(|x|)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

for $N \geqslant 2$, with an additional condition concerning the behavior of V near the origin. After establishing the appropriate functional framework involving radially symmetric functions, they proved some existence and non-existence results for solutions that approach zero at infinity.

Before presenting our main results, let us briefly outline our strategy for addressing the system $(\mathcal{S}_\lambda)$, which will be more detailed in Section 5.1. For a fixed radial function u in an appropriate subspace of $W^{1,2}(\mathbb{R}^2)$, we consider the linear problem

$$-\Delta v + V(|x|)v = V(|x|)u, \quad \text{in } \mathbb{R}^2.$$

After finding a solution $v = B[u]$ to this problem, we return to system $(\mathcal{S}_\lambda)$ and replace v with $B[u]$ in the first equation. This substitution transforms the system into the following problem:

$$-\Delta u + V(|x|)B[u] = \lambda Q(|x|)f(u), \quad \text{in } \mathbb{R}^2,$$

in such a way that the solutions of this scalar equation provides solutions $(u, B[u])$ for System $(\mathcal{S}_\lambda)$.

The aim of Chapter 5 is twofold: we show how to adapt the abstract ideas from [74] to address the System $(\mathcal{S}_\lambda)$, and we also consider the problem in the two-dimensional case. In this setting, we expect to allow nonlinearities with exponential growth. Specifically, we shall assume the following conditions on f:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$, such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if} \quad \alpha > \alpha_0, \\ \infty, & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_2) there holds

$$\lim_{s\to 0} \frac{f(s)}{|s|^{\gamma-1}} = 0,$$

where

$$\gamma := \max\left\{2, \frac{4(b-a)}{(a+2)} + 2\right\};$$

(f_3) there exists $\mu > \gamma$, such that

$$0 < \mu F(s) := \mu \int_0^s f(t)\,\mathrm{d}t \leqslant f(s)s, \qquad \forall\, s \neq 0;$$

(f_4) there exist $C > 0$ and $\nu > \gamma$, such that

$$F(s) \geqslant C|s|^\nu, \qquad \forall\, s \in \mathbb{R}.$$

Before stating the main results of Chapter 5, we present some examples of functions satisfying our hypothesis. First notice that, for any $a > -2$ and $\overline{a} \geqslant a$, the function $V : (0, +\infty) \to (0, +\infty)$ defined by $V(r) = r^{\overline{a}}$ verifies (V_1). Also, for $-2 < b, b_0$ and $s_0 \geqslant b_0$, $s \leqslant b$, the function

$$Q(r) = \begin{cases} r^{s_0}, & \text{if } 0 < r \leqslant 1 \\ r^{s}, & \text{if } r > 1, \end{cases}$$

verifies (Q_1). More simply, in the case $-2 < b_0 \leqslant b$, we can take $b_0 \leqslant \beta \leqslant b$ and see the function $Q(s) = r^\beta$ also verifies the same condition. Finally, a typical example of a function f verifying conditions (f_1)-(f_4) is

$$f(s) = |s|^{p-2} s e^{\alpha_0 s^2}, \qquad s \in \mathbb{R},$$

with $p > \gamma$, $\alpha_0 > 0$ and $\mu = \nu = p$.

The main results of Chapter 5 are:

Theorem 5.1. *Suppose that (V_1), (Q_1) and (f_1)-(f_4) hold. Then there exists $\lambda_0 > 0$ such that the System ($\mathcal{S}_\lambda$) has a radial non-zero weak solution, provided $\lambda \geqslant \lambda_0$. Moreover, if we call (u, v) this solution, the following hold:*

(a) if there exists $a_0 > -2$, such that

$$\limsup_{r\to 0} \frac{V(r)}{r^{a_0}} < \infty,$$

then $u, v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p > 1$ such that $pa_0, pb_0 > -2$. In particular, the functions u, v are locally Hölder continuous;

(b) if V is locally Hölder continuous, then $v \in C^{2,\sigma}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\sigma \in (0, 1)$.

Theorem 5.2. *Suppose that* (V_1), (Q_1) *and* (f_1)-(f_4) *hold. If additionally* f *is odd then, for any given* $m \in \mathbb{N}$*, there exists* $\lambda_m > 0$ *such that the System* $(\mathcal{S}_\lambda)$ *has at least* $2m$ *radial nonzero weak solutions, provided* $\lambda \geqslant \lambda_m$.

For the proof of the first theorem, we apply the classical Mountain Pass Theorem. It is important to establish the variational framework to correctly define the energy functional. In particular, we prove a Trudinger-Moser type inequality (*cf.* Theorem 5.10), which is interesting in itself (*cf.* Remark 5.11). Our abstract results actually complement those of [71] and can be applied to other types of problems with exponential growth. For the second theorem, we exploit the symmetry of the functional to obtain multiple critical points. As the associated functional is even, the strategy is to obtain m distinct nonzero critical points as the parameter λ becomes large.

Chapter 2

Hénon equation for the Grushin operator

In this chapter, we study the following Hénon-type problem

$$\begin{cases} -(\mathcal{G}_\alpha u)(z) = |z|^\ell f(z,u), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{P_α}$$

where $B \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is the unit ball, N_1, $N_2 \geqslant 1$, $\ell > 0$ and, for each $\alpha \geqslant 0$ and $z = (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$,

$$(\mathcal{G}_\alpha u)(z) := \Delta_x u(z) + |x|^{2\alpha} \Delta_y u(z),$$

is the Grushin operator. We also recall that the critical exponent associated to Grushin operator is

$$2^*_\alpha := \frac{2\widetilde{N}}{\widetilde{N}-2}, \quad \widetilde{N} := N_1 + (1+\alpha)N_2.$$

Concerning the nonlinear term f, we suppose the following:

(f_1) $f \in C(\overline{B} \times \mathbb{R}, \mathbb{R})$;

(f_2) there exists $\ell > \alpha N_2$,

$$2 < p < 2^*_\alpha + \frac{2(\ell - \alpha N_2)}{\widetilde{N}-2}$$

and $C > 0$, such that

$$|f(z,s)| \leqslant C\left(1 + |s|^{p-1}\right), \quad \forall\, (z,s) \in \overline{B} \times \mathbb{R};$$

(f_3) there holds

$$\lim_{s \to 0} \frac{f(z,s)}{s} = 0, \quad \text{uniformly in } \overline{B};$$

(f_4) there exists $\mu > 2$ and $s_0 > 0$, such that

$$0 < \mu F(z,s) \leqslant s f(z,s), \quad \forall\, z \in \overline{B},\ s \geqslant s_0,$$

where $F(z,s) := \displaystyle\int_0^s f(z,\tau)\,d\tau$.

In Section 2.1, we follow some ideas presented in the famous paper of Wei-Ming Ni [60], where the authors proved that

$$|u(z)| \leqslant \frac{\|\nabla u\|_{L^2(B)}}{\sqrt{\omega_N(N-2)}|z|^{(N-2)/2}},$$

for any radially symmetric $u \in C^1(B)$ vanishing on the boundary of B. Our version of this inequality leads to some compact immersion results (*cf.* Proposition 2.4 and Theorem 2.3) when considering an appropriated Banach space $(X, \|\cdot\|)$ to deal with the Grushin operator.

As an application of the variational setting developed in the first section, we define $u \in X$ as being a weak solution of (P_α) if

$$\int_B [\nabla_\alpha u \cdot \nabla_\alpha \varphi] \, \mathrm{d}z = \int_B |z|^\ell f(z,u)\varphi \, \mathrm{d}z, \qquad \forall\, \varphi \in X,$$

where

$$\nabla_\alpha u(z) := (\nabla_x u(z), |x|^\alpha \nabla_y u(z)) \in \mathbb{R}^N.$$

With this definition in hands, we prove in Section 2.2 the following:

Theorem 2.1. *Suppose that $\alpha > 0$, $N_2 \geqslant 2$ and f satisfies (f_1)-(f_4). Then Problem (P_α) has a nonzero and radial weak solution. If in addition $\alpha \geqslant 1$, this solution is nonnegative.*

We finish this chapter in Section 2.3 by providing a detailed analysis of the asymptotic behavior of the ground state levels of our problem, both in the whole space and within the space of radial functions. This allows us to complement [71, Theorem 3.1] proving the following result:

Theorem 2.2. *Suppose that $\alpha > 0$, $N_2 \geqslant 2$ and $f(z,s) = |s|^{p-2}s$, for any $(z,s) \in \overline{B} \times \mathbb{R}$ and some $p \in (2, 2^*_\alpha)$. Then there exists $\ell^* > 0$ such that the ground state solution of Problem (P_α) is not radial provided $\ell \geqslant \ell^*$.*

We remark that a ground state solution of Problem (P_α) is a solution that minimizes the associated Euler–Lagrange functional over $H^1_{0,\alpha}(B) - \{0\}$. In other words, any minimizer of

$$\inf \left\{ \int_B |\nabla_\alpha u(z)|^2 : \int_B |u|^p |z|^\ell \, \mathrm{d}z = 1,\ u \in H^1_{0,\alpha}(B) \right\}.$$

2.1 A compact embedding for radial functions

Recall that the points $z \in \mathbb{R}^N$ are writing as $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. The ball centered at $z \in \mathbb{R}^N$ with radius $t > 0$ is denoted by $B_t(z)$, as well as $B = B_1(0)$. We denote by $\|u\|_{L^t(B)}$ the L^t-norm of a measurable function $u \in L^t(B)$.

As stated in the introduction, we consider

$$H^1_\alpha(B) := \Big\{u \in L^2(B) : |\nabla_x u| \in L^2(B),\ |x|^\alpha |\nabla_y u| \in L^2(B)\Big\},$$

which is a Banach space when endowed with the norm

$$||u||_{H^1_\alpha(B)} := \left(\int_B \left[|\nabla_\alpha u|^2 + |u|^2\right] dz\right)^{1/2}.$$

We also define the subspace $H^1_{0,\alpha}(B) := \overline{C_0^\infty(B)}^{||\cdot||_{H^1_\alpha(B)}}$ with the norm

$$\|u\| := \left(\int_B |\nabla_\alpha u(z)|^2 \, \mathrm{d}z\right)^{1/2},$$

which is equivalent to $\|\cdot\|_{H^1_\alpha(B)}$ (*cf.* [47, (1.8)]). In addition, let

$$C^\infty_{0,r}(B) := \{u \in C_0^\infty(B) : u \text{ is radial}\}$$

and $X := \overline{C^\infty_{0,r}(B)}^{H^1_{0,\alpha}(B)}$. The space X is continuously embedded in $L^p(B)$, for $p \in [1, 2^*_\alpha]$ (*cf.* [47, Theorem 3.3]).

Finally, for any $1 \leqslant p + \infty$ and $m \geqslant 0$, we set

$$L^p_m(B) := \left\{u \in L^1_{loc}(B) \ : \ \int_B |u(z)|^p |z|^{mp} \, \mathrm{d}z < +\infty\right\},$$

which is a Banach space with the norm

$$\|u\|_{L^p_m} := \left(\int_B |u(z)|^p |z|^{mp} \, \mathrm{d}z\right)^{1/p}.$$

The main result of this section reads as:

Theorem 2.3. *Suppose that $m \geqslant 0$, $N_2 \geqslant 2$ and define*

$$\widetilde{m} := \begin{cases} \dfrac{2N}{\widetilde{N} - 2 - 2m}, & \text{if } m < (\widetilde{N} - 2)/2, \\ +\infty, & \text{if } m \geqslant (\widetilde{N} - 2)/2. \end{cases}$$

Then, for any $p \in [1, \widetilde{m})$, the embedding $X \hookrightarrow L^p_m(B)$ is compact.

The key ingredient for the proof is the following radial lemma which is a version, for our setting, of a result presented in [60, p. 802]:

Proposition 2.4. *Suppose that $\alpha \geqslant 0$ and $N_2 \geqslant 2$. Then, for any $u \in X$, there holds*

$$|u(z)| \leqslant C \frac{\|u\|}{|z|^{(\tilde{N}-2)/2}}, \qquad \text{for a.e. } z \in B, \tag{2.1}$$

with $C = C(N) > 0$, given by

$$C := \left[(\tilde{N} - 2) \int_{\mathbb{S}^{N-1}} |\xi|^{2\alpha} |\eta|^2 \, d\sigma_{N(\xi,\eta)} \right]^{-1},$$

and σ_N is the surface measure on $\mathbb{S}^{N-1}$.

Proof. Let $u \in C^\infty_{0,r}(B)$ and consider $\varphi \in C^\infty_0(-1,1)$ such that $u(z) = \varphi(|z|)$. For any given $z_0 \in B - \{0\}$ and $\widetilde{z} \in \partial B$ we have that

$$-u(z_0) = u(\widetilde{z}) - u(z_0) = \int_{|z_0|}^1 \varphi'(t) \, \mathrm{d}t.$$

From $N_2 \geqslant 2$, we obtain $\widetilde{N} = N_1 + (1+\alpha)N_2 > 2$, and therefore Hölder's inequality yields

$$\begin{aligned} |u(z_0)| &\leqslant \int_{|z_0|}^1 |\varphi'(t)| \, \mathrm{d}t = \int_{|z_0|}^1 |\varphi'(t)| t^{(\tilde{N}-1)/2} t^{-(\tilde{N}-1)/2} \, \mathrm{d}t \\ &\leqslant \left(\int_{|z_0|}^1 |\varphi'(t)|^2 t^{\tilde{N}-1} \, \mathrm{d}t \right)^{1/2} \left(\int_{|z_0|}^1 t^{-(\tilde{N}-1)} \, \mathrm{d}t \right)^{1/2}. \end{aligned}$$

After computing the last integral above, we get

$$|u(z_0)|^2 \leqslant \frac{1}{(\widetilde{N} - 2)|z_0|^{\tilde{N}-2}} \int_{|z_0|}^1 |\varphi'(t)|^2 t^{\tilde{N}-1} \, \mathrm{d}t, \qquad \forall \ z_0 \in B.$$

We now notice that $N_2 \geqslant 2$ implies

$$\int_{|z_0|}^1 |\varphi'(t)|^2 t^{\tilde{N}-1} \, \mathrm{d}t = \int_{|z_0|}^1 t^{\alpha N_2} |\varphi'(t)|^2 t^{N-1} \, \mathrm{d}t \leqslant \int_{|z_0|}^1 t^{2\alpha} |\varphi'(t)|^2 t^{N-1} \, \mathrm{d}t.$$

We claim that

$$\int_{|z_0|}^1 t^{2\alpha} |\varphi'(t)|^2 t^{N-1} \, \mathrm{d}t = C_1 \int_{\{|z_0| \leqslant |z| \leqslant 1\}} |x|^{2\alpha} |\nabla_y u(z)|^2 \, \mathrm{d}z, \tag{2.2}$$

for some $C_1 = C_1(N) > 0$, independent of u. If this is true, it follows from the inequalities above that

$$|u(z_0)|^2 \leqslant \frac{C_1}{(\tilde{N}-2)} \frac{\int_B |x|^{2\alpha}|\nabla_y u(z)|^2 \, \mathrm{d}z}{|z_0|^{\tilde{N}-2}} \leqslant C_2^2 \frac{\|\nabla_\alpha u\|^2_{L^2(B)}}{|z_0|^{(\tilde{N}-2)}}, \qquad \text{for a.e. } z_0 \in B.$$

with $C_2 := [C_1/(\tilde{N}-2)]^{1/2} > 0$. So, the proposition holds for smooth functions and the result follows from the density of $C^\infty_{0,r}(B)$ in X.

It remains to prove (2.2). To do this, we denote by $w = (\xi, \eta) \in \partial B$ a general vector in the unit sphere of $\mathbb{R}^N$, with $\xi \in \mathbb{R}^{N_1}$ and $\eta \in \mathbb{R}^{N_2}$. Then, for any $i = 1, 2, \ldots, N_2$,

$$\begin{aligned}\int_{\{|z_0|\leqslant|z|\leqslant 1\}} |x|^{2\alpha} u_{y_i}^2(z) \, \mathrm{d}z &= \int_{|z_0|}^1 \int_{\partial B_t(0)} |x|^{2\alpha} u_{y_i}^2(z) \, \mathrm{d}\sigma_z \, \mathrm{d}t \\ &= \int_{|z_0|}^1 \int_{\mathbb{S}^{N-1}} |t\xi|^{2\alpha} u_{y_i}^2(tw) t^{N-1} \, \mathrm{d}\sigma_w \, \mathrm{d}t.\end{aligned}$$

Thus, since $u_{y_i}(x, y) = \varphi'(|z|)\dfrac{y_i}{|z|}$ for $z = (x, y) \in \mathbb{R}^N$, it holds

$$\begin{aligned}\int_{\{|z_0|\leqslant|z|\leqslant 1\}} |x|^{2\alpha} u_{y_i}^2(z) \, \mathrm{d}z &= \int_{|z_0|}^1 \int_{\mathbb{S}^{N-1}} t^{2\alpha}|\xi|^{2\alpha} \left[\varphi'(|tw|)\frac{t\eta_i}{|tw|}\right]^2 t^{N-1} \, \mathrm{d}\sigma_w \, \mathrm{d}t \\ &= \int_{\mathbb{S}^{N-1}} |\xi|^{2\alpha}\eta_i^2 \, \mathrm{d}\sigma_w \int_{|z_0|}^1 t^{2\alpha}|\varphi'(t)|^2 t^{N-1} \, \mathrm{d}t.\end{aligned}$$

After summing on i we get

$$\int_{\{|z_0|\leqslant|z|\leqslant 1\}} |x|^{2\alpha}|\nabla_y u(z)|^2 \, \mathrm{d}z = C_3 \int_{|z_0|}^1 t^{2\alpha}|\varphi'(t)|^2 t^{N-1} \, \mathrm{d}t,$$

where

$$C_3 := \int_{\mathbb{S}^{N-1}} |\xi|^{2\alpha}|\eta|^2 \, \mathrm{d}\sigma_w = \int_{\mathbb{S}^{N-1}} |\xi|^{2\alpha}(1 - |\xi|^2) \, \mathrm{d}\sigma_w,$$

depends only on N.

We need only to check that the above integral is positive. In order to prove it, take $0 < r < s < 1$ and consider the spherical annular region

$$\mathbb{A} = \{(\xi, \eta) \in \mathbb{S}^{N-1} : \sqrt{1-s^2} < |\xi| < \sqrt{1-r^2}\}.$$

Since this nonempty set is open in $\mathbb{S}^{N-1}$, we have that $\sigma_N(\mathbb{A}) > 0$. For $(\xi, \eta) \in \mathbb{A}$, $|\eta|^2 = 1 - |\xi|^2 > r^2$, $|\xi|^2 > 1 - s^2 > 0$ and $|\xi|^{2\alpha} > (1-s^2)^\alpha$. Thus

$$\int_{\mathbb{S}^{N-1}} |\xi|^{2\alpha}|\eta|^2 \, \mathrm{d}\sigma_{(\xi,\eta)} \geqslant \int_{\mathbb{A}} |\xi|^{2\alpha}|\eta|^2 \, \mathrm{d}\sigma_{(\xi,\eta)} > (1-s^2)^\alpha r^2 \sigma_N(\mathbb{A}) > 0$$

and the proof is concluded. □

Before presenting the proof of Theorem 2.3 we recall that, for any $\gamma \in \mathbb{R}$, there holds

$$\int_B |z|^\gamma \, \mathrm{d}z = \begin{cases} \dfrac{\omega_N}{(\gamma + N)}, & \text{if } \gamma > -N, \\ +\infty, & \text{if } \gamma \leqslant -N, \end{cases} \tag{2.3}$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$.

Proof of Theorem 2.3. We first prove the continuity of the embedding for $p \in [1, \widetilde{m})$. To do this, pick $u \in X$ and apply (2.1) to get

$$\|u\|_{L^p_m}^p = \int_B |u|^p |z|^{mp} \, \mathrm{d}z \leqslant C^p \|u\|^p \int_B |z|^\gamma \, \mathrm{d}z,$$

where

$$\gamma := \frac{p}{2}\left(2m - \widetilde{N} + 2\right).$$

If $m < (\widetilde{N} - 2)/2$, we may use the definition of $\widetilde{m}$ and $p \in [1, \widetilde{m})$ to show that $\gamma + N > 0$. Hence, we obtain from (2.3) a constant $C_1 = C_1(m, p, \alpha, N_1, N_2) > 0$, such that

$$\|u\|_{L^p_m}^p \leqslant C_1 \|u\|^p, \quad \forall\, u \in X. \tag{2.4}$$

When $m \geqslant (\widetilde{N} - 2)/2$, we have that $\gamma \geqslant 0$ and therefore the embedding holds for any $1 \leqslant p < +\infty$.

For the compactness, we pick $\beta \in (0, 1)$ to be chosen later and apply Hölder's inequality with exponents $1/\beta$ and $1/(1-\beta)$, to get

$$\begin{aligned} \int_B |u|^p |z|^{mp} \, \mathrm{d}z &= \int_B |u|^\beta |u|^{p-\beta} |z|^{mp} \, \mathrm{d}z \\ &\leqslant \|u\|_{L^1(B)}^\beta \left(\int_B |u|^{(p-\beta)/(1-\beta)} |z|^{mp/(1-\beta)} \, \mathrm{d}z \right)^{1-\beta}. \end{aligned}$$

So, if we set

$$p_\beta := \frac{p - \beta}{1 - \beta}, \quad m_\beta := \frac{mp}{p - \beta},$$

we have that

$$\|u\|_{L^p_m}^p \leqslant \|u\|_{L^1(B)}^\beta \|u\|_{L^{p_\beta}_{m_\beta}}^{p-\beta}. \tag{2.5}$$

Since $m_\beta \to m$, as $\beta \to 0^+$, it is clear that, if $m < (\tilde{N} - 2)/2$, then $m_\beta < (\tilde{N} - 2)/2$, for any $\beta > 0$ small. So, we can use $p < \tilde{m}$ to get

$$\lim_{\beta \to 0^+} (p_\beta - \widetilde{m_\beta}) = \lim_{\beta \to 0} \left(\frac{p - \beta}{1 - \beta} - \frac{2N}{\tilde{N} - 2 - 2m_\beta} \right) = p - \frac{2N}{\tilde{N} - 2 - 2m} < 0,$$

and therefore we can pick $\beta_0 > 0$ small in such a way that $p_{\beta_0} \in [1, \widetilde{m_{\beta_0}})$. The same holds if $m \geqslant (\tilde{N} - 2)/2$, because in this case the inequality $m_\beta > m$ implies $\widetilde{m_\beta} = +\infty$.

Let $(u_n) \subset X$ be a sequence such that $u_n \rightharpoonup 0$ weakly in X. Since $p_{\beta_0} \in [1, \widetilde{m_{\beta_0}})$, we can use (2.5) and the embedding proved in the first part to get

$$\|u_n\|^p_{L^p_m} \leqslant C_1 \|u_n\|^{\beta_0}_{L^1(B)} \|u_n\|^{p-\beta_0} \leqslant C_2 \|u_n\|^{\beta_0}_{L^1(B)},$$

where we also have used the boundedness of (u_n) in X. Since the embedding $X \hookrightarrow L^1(B)$ is compact (*cf.* [47, Theorem 3.3]), up to a subsequence $u_n \to 0$ strongly in $L^1(B)$. This and the above expression imply that $u_n \to 0$ strongly in X. This finishes the proof of the theorem. □

Given $u \in X$, we define $u^-(x) := \max\{-u(x), 0\}$ and $u^+ := u + u^-$, the positive and negative part of u, respectively. We finish this subsection proving that $u^\pm \in X$, whenever $u \in X$. Since we did not find a clear state of this fact in the literature, we sketch the proof here for the sake of completeness.

Lemma 2.5. *Suppose that $\alpha \geqslant 1$ and $u \in X = \overline{C^\infty_{0,r}(B)}^{H^1_{0,\alpha}(B)}$. Then $u^\pm \in X$.*

Proof. Let $(u_n) \subset C^\infty_{0,r}(B)$ be such that $u_n \to u$ in $H^1_\alpha(B)$. Since $\alpha \geqslant 1$, the function $f(x, y) = |x|^\alpha$ is locally Lipschitz. This allows us to use [39, Corollary 2.2.], in order to obtain $|u_n|, |u| \in H^1_\alpha(B)$ with

$$\nabla_\alpha |u_n| = \begin{cases} \nabla_\alpha u_n, & \text{if } u_n > 0, \\ 0, & \text{if } u_n = 0, \\ -\nabla_\alpha u_n, & \text{if } u_n < 0, \end{cases} \quad \text{and} \quad \nabla_\alpha (u_n)^+ = \begin{cases} \nabla_\alpha u_n, & \text{if } u_n > 0, \\ 0, & \text{if } u_n \leqslant 0, \end{cases} \tag{2.6}$$

and the same kind of equality for u.

We intend to prove $|u_n| \to |u|$ in $H^1_\alpha(B)$. Of course $|u_n| \to |u|$ in $L^2(B)$. For checking that $\nabla_\alpha |u_n| \to \nabla_\alpha |u|$ in $L^2(B)$, it is enough to prove, up to an arbitrary subsequence, point convergence occurs a.e. in B, because there exists $g \in L^1(B)$ such that $|\nabla_\alpha u_n| \leqslant g$, a.e. in B, and $\nabla_\alpha |u_n| = \pm \nabla_\alpha u_n$. We will consider an arbitrary subsequence of $\nabla_\alpha |u_n|$ and use, without loss of generality, the same notation for it. If $u(z) > 0$ or

$u(z) < 0$, we can use (2.6) in order to obtain $\nabla_\alpha |u_n|(z) \to \nabla_\alpha |u|(z)$. However, if $u(z) = 0$, we remark that $u = 2u^+ - |u|$ a.e. from which we obtain

$$|\nabla_\alpha |u_n|(z) - \nabla_\alpha |u|(z)| = | \pm \nabla_\alpha u_n(z)| \to |\nabla_\alpha u(z)| = |2\nabla_\alpha u^+(z) - \nabla_\alpha |u|(z)| = 0.$$

Thus $|u_n| \to |u|$ in $H^1_\alpha(B)$.

From the convergence just proved, we obtain $u_n^- = (|u_n| - u_n)/2 \to (|u| - u)/2 = u^-$ in $H^1_\alpha(B)$. By [76, p. 6513 and 6514], $u_n^- \in H^1_{0,\alpha}(B)$ and its extension by zero $v_n := u_n^-$ in B, $v_n := 0$ outside B, belongs to $H^1_\alpha(\mathbb{R}^N)$. Of course v_n is radial. Taking $k > 0$ and considering $\eta_{1/k}$ a standard modifier, we can use [31, Proposition 1.4] to see that, for each $n \in \mathbb{N}$, $v_n * \eta_{1/k} \to v_n$ in $H^1_{\alpha,\mathrm{loc}}(\mathbb{R}^N)$ as $k \to +\infty$. In particular, $v_n * \eta_{1/k} \to u_n^-$ in $H^1_\alpha(B)$ as $k \to +\infty$. We now take $k_n \in \mathbb{N}$ such that $\overline{\mathrm{supp}(v_n) + \mathrm{supp}(\eta_{1/k})} \subset B$, for any $k \geqslant k_n$. If we define $w_{n,m} := v_n * \eta_{1/(m+k_n-1)}$, for $m \geqslant 1$, as $v_n \to u^-$ as $n \to +\infty$ and $w_{n,m} \to v_n$ as $m \to +\infty$; both convergences in $H^1_\alpha(B)$, we can use a diagonal argument to obtain a subsequence of $\{w_{n,m} : n, m \in \mathbb{N}\}$ which converges to u^- in $H^1_\alpha(B)$. Once $w_{n,m} \in C^\infty_{0,r}(B)$, we have proved that $u^- \in X$. Finally, as $u^+ = u + u^-$, then $u^+ \in X$. □

2.2 Existence of solution

In this section, we use Theorem 2.3 to prove our existence result.

Proof of Theorem 2.1. Since we are interested in nonnegative solutions, we can suppose without loss of generality that $f(z, s) = 0$ for $z \in B$ and $s < 0$. Formally, the energy functional associated to our problem is

$$I(u) := \frac{1}{2}\|u\|^2 - \int_B F(z, u)|z|^\ell \,\mathrm{d}z, \qquad u \in X.$$

To prove that I is well defined we notice that, for any given $\varepsilon > 0$, we may use (f_1)-(f_3) to obtain a positive constant C_1 such that

$$|F(z, s)| \leqslant \frac{\varepsilon}{2}|s|^2 + C_1|s|^p, \quad \forall\, (z, s) \in \overline{B} \times \mathbb{R}.$$

Hence,

$$\int_B F(x, u)|z|^\ell \,\mathrm{d}z \leqslant \frac{\varepsilon}{2}\|u\|^2_{L^2(B)} + C_1\|u\|^p_{L^p_{\ell/p}}. \tag{2.7}$$

The last integral above is finite whenever $1 \leqslant p < (\widetilde{\ell/p})$. If $\ell/p \geqslant (\tilde{N}-2)/2$, this always happen, since $\widetilde{\ell/p} = +\infty$. In the case $\ell/p < (\tilde{N}-2)/2$, the condition on p reads as

$$p < \widetilde{\left(\frac{\ell}{p}\right)} = \frac{2N}{\tilde{N} - 2 - 2\frac{\ell}{p}} \quad \Longleftrightarrow \quad p < 2^*_\alpha + \frac{2(\ell - \alpha N_2)}{\tilde{N} - 2}.$$

So, we conclude that the function is well defined. Moreover, standard arguments shows that $I \in C^1(X, \mathbb{R})$ with

$$I'(u)v = \int_B (\nabla_\alpha u \cdot \nabla_\alpha v)\, \mathrm{d}z - \int_B f(z,u)v|z|^\ell\, \mathrm{d}z, \quad \forall\, u, v \in X,$$

and therefore critical points of I are precisely the weak solutions of Problem (P_α).

Using (2.4) and (2.7) with $\varepsilon = 1/(2C_1)$, we get

$$I(u) \geqslant \frac{1}{4}\|u\|^2 - C_2\|u\|^p = \|u\|^2\left(\frac{1}{4} - C_2\|u\|^{p-2}\right)$$

and therefore, since $p > 2$, we can find ρ, $\eta > 0$, such that

$$I(u) \geqslant \eta, \quad \forall\, u \in X,\ \|u\| = \rho.$$

Moreover, using (f_1) and (f_4), we obtain a constant $C_2 > 0$, such that

$$F(z,s) \geqslant C_2|s|^\mu - C_2, \quad \forall\, (z,s) \in \overline{B} \times \mathbb{R}.$$

So, if we choose a nonnegative function $u_0 \in C^\infty_{0,r}(B) - \{0\}$, we have that

$$I(su_0) \leqslant \frac{s^2}{2}\|u_0\|^2 - C_2 s^\mu \int_B |u_0|^\mu |z|^\ell\, \mathrm{d}z + C_3.$$

Once we can assume without loss of generality that $2 < \mu < 2^*_\alpha$, it follows that $I(su_0) \to -\infty$, as $s \to +\infty$. Thus, there exists $e \in X$ such that $I(e) \leqslant 0$ and $\|e\| > \rho$.

The above considerations shows that it is well defined

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \geqslant \eta > 0,$$

with $\Gamma := \{\gamma \in C([0,1], X : \gamma(0) = 0, \gamma(1) = e\}$. Hence, according to the Mountain Pass Theorem (*cf.* [5]), there exists $(u_n) \subset X$ such that

$$\lim_{n\to+\infty} I(u_n) = c, \qquad \lim_{n\to+\infty} I'(u_n) = 0. \tag{2.8}$$

We claim that, along a subsequence, $u_n \to u$ strongly in X. Actually, we first use (f_1) and (f_4) to get

$$\begin{aligned} I(u_n) - \frac{1}{\mu} I'(u_n)u_n &\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 \\ &\quad + \frac{1}{\mu} \int_{\{0 \leqslant |u_n| \leqslant s_0\}} \left[f(z,u_n)u_n - \mu F(z,u_n)\right] |z|^\ell \, \mathrm{d}z \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 - C_4. \end{aligned}$$

Since $\mu > 2$, we conclude that (u_n) is bounded in X. Hence, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X. Thus,

$$o_n(1) = I'(u_n)(u_n - u) = \|u_n\|^2 - \langle u_n, u\rangle - A_n = \|u_n\|^2 - \|u\|^2 - A_n + o_n(1),$$

where $o_n(1)$ denotes a quantity approaching zero as $n \to +\infty$ and

$$A_n := \int_B f(z,u_n)(u_n - u)|z|^\ell \, \mathrm{d}z.$$

To get the strong convergence claimed, we need only to check that $A_n \to 0$. Since the embedding $H^1_{0,\alpha}(B) \hookrightarrow L^1(B)$ is compact, we may assume $u_n \to u$ strongly in $L^1(B)$. Hence, it follows from (f_1) and Hölder's inequality with exponents p and $p' := p/(p-1)$ that

$$\begin{aligned} |A_n| &\leqslant C_5 \int_B \left(1 + |u_n|^{p-1}\right) |u_n - u||z|^\ell \, \mathrm{d}z \\ &= o_n(1) + C_5 \int_B |u_n|^{p-1}|z|^{\ell/p'}|u_n - u||z|^{\ell/p} \, \mathrm{d}z \\ &\leqslant o_n(1) + C_5 \left(\int_B |u_n|^p |z|^\ell \, \mathrm{d}z\right)^{1/p'} \left(\int_B |u_n - u|^p |z|^\ell \, \mathrm{d}z\right)^{1/p} \\ &\leqslant C_5 \|u_n\|^{p/p'}_{L^p_{\ell/p}} \|u_n - u\|_{L^p_{\ell/p}}. \end{aligned}$$

Recalling that $p < (\widetilde{\ell/p})$, we may use the boundedness of (u_n) in X and the compactness of the embedding $X \hookrightarrow L^p_{\ell/p}$, to conclude that the right-hand side above goes to zero, as $n \to +\infty$. This concludes the proof that $u_n \to u$ strongly in X.

From (2.8), the regularity of I and the strong convergence just proved, we obtain $I(u) = c > 0$ and $I'(u) = 0$. This shows that $u \in X$ is a nonzero solution of Problem (P_α). Moreover, assuming $\alpha \geqslant 1$, we can use Lemma 2.5 in order to obtain

$$0 = I'(u)u^- = \int_B \left[\nabla_\alpha(u^+ - u^-) \cdot \nabla_\alpha u^-\right] \mathrm{d}z - \int_{\{u \geqslant 0\}} f(z,u)u^-|z|^\ell \, \mathrm{d}z = -\|u^-\|^2,$$

which implies $u \geqslant 0$ a.e. in B and concludes the proof of the theorem. □

2.3 Symmetry breaking

In this section, our goal is to prove Theorem 2.2. Along all this section, we assume that $2 < p < 2^*_\alpha$, $f(z,t) = |t|^{p-2}t$ and consider the ratio

$$R(u) := \frac{\|u\|^2}{\left(\int_B |u|^p |z|^\ell \, \mathrm{d}z\right)^{2/p}}, \qquad u \in H^1_{0,\alpha}(B) - \{0\}.$$

We also introduce the minimizers

$$S_{\ell,p} := \inf_{\substack{u\in H^1_{0,\alpha}(B)\\ u\neq 0}} R(u) = \inf\left\{\|u\|^2 : \int_B |u|^p|z|^\ell \, \mathrm{d}z = 1,\, u \in H^1_{0,\alpha}(B)\right\}$$

and

$$S^R_{\ell,p} := \inf_{\substack{u\in X\\ u\neq 0}} R(u) = \inf\left\{\|u\|^2 : \int_B |u|^p|z|^\ell \, \mathrm{d}z = 1,\, u \in X\right\}.$$

The principal ingredients for the proof of our last main theorem of the chapter are the following estimates, which are versions of [71, Theorems 4.1 and 4.2]:

Proposition 2.6. *Suppose that $p \in (2, 2^*_\alpha)$. Then*

1. *there exists $\mathcal{C}_{rad} = \mathcal{C}_{rad}(N,p,\alpha) > 0$, such that*

$$S^R_{\ell,p} \geqslant \mathcal{C}_{rad}\ell^{(p+2)/p}, \quad \forall\, \ell > N_2\alpha;$$

2. *for any given $\ell_0 > 2$, there exists $\mathcal{C} = \mathcal{C}(p,\ell_0,\alpha) > 0$, such that*

$$S_{\ell,p} \leqslant \mathcal{C}\ell^{2-\tilde{N}+(2\tilde{N}/p)}, \quad \forall\, \ell \geqslant \ell_0.$$

Before proving this proposition, let us show how we can use it to get our symmetry breaking result:

Proof of Theorem 2.2. For any $p \in (2, 2^*_\alpha)$ fixed, it is sufficient to obtain $\ell^* > 0$ such that $S_{\ell,p} < S^R_{\ell,p}$, for any $\ell \geqslant \ell^*$. Suppose, by contradiction, that this is not the case. Then, since $S_{\ell,p} \leqslant S^R_{\ell,p}$ is always true, there exists a sequence $(\ell_n) \subset (0,+\infty)$ such that $S_{\ell_n,p} = S^R_{\ell_n,p}$ and $\ell_n \to +\infty$. Using Proposition 2.6 we get

$$\mathcal{C}_{rad}\ell_n^{(p+2)/p} \leqslant S^R_{\ell_n,p} = S_{\ell_n,p} \leqslant \mathcal{C}\ell_n^{2-\tilde{N}+(2\tilde{N}/p)},$$

for any $n \in \mathbb{N}$ large. This implies

$$0 < \frac{\mathcal{C}_{rad}}{\mathcal{C}} \leqslant \ell_n^{2-\tilde{N}+(2\tilde{N}/p)-1-(2/p)} = \ell_n^{(p-2)(1-\tilde{N})/p},$$

which is impossible since $\ell_n \to +\infty$ and $(p-2)(1-\tilde{N})/p < 0$. □

We devote the next two subsections to the proof of Proposition 2.6.

2.3.1 Estimating $S_{\ell,p}^R$

We start this subsection with a technical lemma which is a consequence of the chain rule:

Lemma 2.7. *Suppose that $N \geqslant 2$, $\varphi \in C_0^\infty(-1,1)$ and $\beta \in (0,1]$. Then the functions*

$$v(z) := \varphi(|z|^\beta), \qquad w(z) := \varphi(|z|^{1/\beta})$$

belong to $H_{0,r}^1(B)$, the space of radial functions in $H_0^1(B)$.

Proof. Of course v, w are radial, continuous and null on ∂B. Since φ is a C^1-function with bounded derivatives, we need only to check that the maps $g_1(z) := |z|^\beta$ and $g_2(z) := |z|^{1/\beta}$ are in $H^1(B)$ (*cf.* [25, p. 308]). Clearly g_1, $g_2 \in L^2(B)$. Moreover, for any $z \neq 0$, we have that

$$|\nabla g_1(z)|^2 = \beta^2 |z|^{2(\beta-1)}, \qquad |\nabla g_2(z)|^2 = \frac{1}{\beta^2}|z|^{2(1-\beta)/\beta}$$

and the result follows from (2.3), because $N/2 > 1 > 1-\beta$ and $2(1-\beta)/\beta > 0 > -N$. □

Proposition 2.8. *Suppose that $\beta \in (0,1]$, $u \in C_{0,r}^\infty(B)$ and $\varphi \in C_0^\infty(-1,1)$ is such that $u(z) = \varphi(|z|)$. If*

$$v(z) := \varphi(|z|^\beta), \qquad z \in B,$$

then

$$\int_B |\nabla_\alpha u|^2 \,\mathrm{d}z \geqslant \frac{C}{\beta\omega_N} \int_B |\nabla v|^2 |z|^{(\beta-1)(N-2)} \,\mathrm{d}z, \tag{2.9}$$

for some constant $C = C(N) > 0$. Moreover, if $\beta = N/(\ell+N)$, then

$$\int_B |u|^p |z|^\ell \,\mathrm{d}z = \beta \int_B |v|^p \,\mathrm{d}z. \tag{2.10}$$

Proof. One can easily check that

$$|\nabla_\alpha u(z)|^2 = [\varphi'(|z|)]^2 \left[\frac{|x|^2 + |x|^{2\alpha}|y|^2}{|z|^2}\right].$$

If we call $g_\alpha(z)$ the expression into brackets above and denote $w = (\xi, \eta) \in \mathbb{S}^{N-1}$, we get

$$\begin{aligned}\int_B |\nabla_\alpha u(z)|^2 \,\mathrm{d}z &= \int_0^1 \int_{\mathbb{S}^{N-1}} [\varphi'(|rw|)]^2 g_\alpha(rw) r^{N-1} \,\mathrm{d}\sigma_w \,\mathrm{d}r \\ &= \int_0^1 \int_{\mathbb{S}^{N-1}} [\varphi'(r)]^2 \left(|\xi|^2 + |r\xi|^{2\alpha}|\eta|^2\right) r^{N-1} \,\mathrm{d}\sigma_w \,\mathrm{d}r \\ &\geqslant \left(\int_{\mathbb{S}^{N-1}} |\xi|^2 \,\mathrm{d}\sigma_w\right) \int_0^1 [\varphi'(r)]^2 r^{N-1} \,\mathrm{d}r. \\ &= C \int_0^1 [\varphi'(r)]^2 r^{N-1} \,\mathrm{d}r,\end{aligned}$$

with $C := \int_{\mathbb{S}^{N-1}} |\xi|^2 \,\mathrm{d}\sigma_w$. To check that $C > 0$, it is sufficient to pick $0 < \delta < 1$ and notice that the set $\mathbb{A}_\delta := \{(\xi, \eta) \in \mathbb{S}^{N-1} : |\xi| \geqslant \delta\}$ has positive $(N-1)$-dimensional measure.

Using the change of variable $r = \rho^\beta$ in the last integral above and defining $\widetilde{v}(r) := \varphi(r^\beta)$, we obtain

$$\begin{aligned}\int_B |\nabla_\alpha u(z)|^2 \,\mathrm{d}z &\geqslant \frac{C}{\beta} \int_0^1 [\varphi'(\rho^\beta)\beta\rho^{(\beta-1)}]^2 \frac{\rho^{\beta(N-1)}\rho^{\beta-1}}{\rho^{2(\beta-1)}} d\rho \\ &= \frac{C}{\beta} \int_0^1 [\widetilde{v}'(\rho)]^2 \frac{\rho^{\beta(N-1)}\rho^{\beta-1}\rho^{1-N}}{\rho^{2(\beta-1)}} \rho^{N-1} d\rho \\ &= \frac{C}{\beta} \int_0^1 [\widetilde{v}'(\rho)]^2 \rho^{[\beta(N-1)+\beta-N-2(\beta-1)]} \rho^{N-1} d\rho \\ &= \frac{C}{\beta\omega_N} \int_B [\widetilde{v}'(|z|)]^2 |z|^{(\beta-1)(N-2)} \,\mathrm{d}z.\end{aligned}$$

Since $v(z) = \tilde{v}(|z|)$, it follows that $|\nabla v(z)|^2 = [\widetilde{v}'(|z|)]^2$ and the last expression yields (2.9).

Using the change of variables $r = \rho^\beta$ again, we get

$$\begin{aligned}\int_B |u(z)|^p |z|^\ell \,\mathrm{d}z &= \omega_N \int_0^1 |\varphi(r)|^p r^\ell r^{N-1} \,\mathrm{d}r \\ &= \beta\omega_N \int_0^1 |\varphi(\rho^\beta)|^p \rho^{[\beta(N-1)+\beta\ell+\beta-1+(1-N)]} \rho^{N-1} d\rho.\end{aligned}$$

Thus, if $\beta = N/(\ell + N)$, we have that

$$\beta(N-1) + \beta\ell + \beta - 1 + (1-N) = \beta(N+\ell) - N = 0,$$

and therefore

$$\int_B |u(z)|^p |z|^\ell \,\mathrm{d}z = \beta\omega_N \int_0^1 |\widetilde{v}(\rho)|^p \rho^{N-1} d\rho = \beta \int_B |v(z)|^p \,\mathrm{d}z,$$

which gives (2.10). This finishes the proof. □

We are ready to present the estimation of $S^R_{\ell,p}$.

Proof of item (1) in Proposition 2.6. We first notice that, since the function $g(s) := 2s/(s-2)$ is decreasing in $(2,+\infty)$ and $2 < N \leqslant \tilde{N}$, we have that $2 < p < 2^*_\alpha \leqslant 2^*$.

Let $u \in C^\infty_{0,r}(B) - \{0\}$ and $\varphi \in C^\infty_0(-1,1)$ such that $u(z) = \varphi(|z|)$. We also define $v(z) := \varphi(|z|^\beta)$, with $\beta = N/(\ell+N)$. By Lemma 2.7, we know that $v \in H^1_{0,r}(B)$. Hence, it follows from Proposition 2.8 that

$$R(u) \geqslant \frac{C}{\beta\omega_N}\frac{\displaystyle\int_B |\nabla v|^2|z|^{(\beta-1)(N-2)}\,\mathrm{d}z}{\beta^{2/p}\left(\displaystyle\int_B |v|^p\,\mathrm{d}z\right)^{2/p}} = C_1\beta^{-(p+2)/p}\frac{\displaystyle\int_B |\nabla v|^2|z|^{(\beta-1)(N-2)}\,\mathrm{d}z}{\left(\displaystyle\int_B |v|^p\,\mathrm{d}z\right)^{2/p}},$$

where $C_1 = C\omega_N^{-1}$. Since $|z|^{(\beta-1)(N-2)} \geqslant 1$ for $|z| < 1$,

$$R(u) \geqslant C_1\beta^{-(p+2)/p}\frac{\displaystyle\int_B |\nabla v|^2\,\mathrm{d}z}{\left(\displaystyle\int_B |v|^p\,\mathrm{d}z\right)^{2/p}} \geqslant C_1 D_p\beta^{-(p+2)/p},$$

with

$$D_p := \inf_{\phi\in H^1_{0,r}(B)}\left\{\int_B |\nabla\phi|^2 : \int_B |\phi|^p = 1\,\mathrm{d}z\right\} > 0,$$

because $2 < p < 2^*_\alpha \leqslant 2^*$ and $v \in H^1_{0,r}(B) - \{0\}$ (*cf.* Lemma 2.7). We now recall that $\beta = N/(\ell+N)$ to obtain, for any $u \in C^\infty_{0,r}(B) - \{0\}$,

$$R(u) \geqslant C_1 D_p\left(\frac{\ell+N}{N}\right)^{(p+2)/p} = \frac{C_1 D_p}{N^{(p+2)/p}}(\ell+N)^{(p+2)/p} \geqslant \mathcal{C}_{rad}\ell^{(p+2)/p},$$

with

$$\mathcal{C}_{rad} := C_1 D_p N^{-(p+2)/p}.$$

Using the continuous embedding $X \hookrightarrow L^p(B)$ and a density argument, we can prove that the above inequality also holds in X. Thus $S^R_{\ell,p} \geqslant \mathcal{C}_{rad}\ell^{(p+2)/p}$, for any $\ell > N_2\alpha$, and the proof is complete. □

2.3.2 Estimating $S_{\ell,p}$

Given $\ell > 1$, we introduce the map

$$\delta_\ell(z) := \left(\ell x, \ell^{1+\alpha}y\right), \quad \forall\, z = (x,y) \in \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}.$$

It is a linear transformation with linear inverse given by

$$\delta_\ell^{-1}(z) := \left(\frac{x}{\ell}, \frac{y}{\ell^{1+\alpha}}\right).$$

We now consider

$$z_\ell := \left(0_x, \ell^\alpha - \ell^{\alpha-1}, 0_{y'}\right) \in \mathbb{R}^N,$$

where 0_x and $0_{y'}$ are the null vector of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2-1}$, respectively. Define the maps $T, T^{-1} : \mathbb{R}^N \to \mathbb{R}^N$ by

$$T(z) := \delta_\ell^{-1}(z + \ell z_\ell), \qquad T^{-1}(z) := \delta_\ell(z) - \ell z_\ell. \tag{2.11}$$

A simple computation shows that T^{-1} is the inverse of T.

Since $\ell > 1$ and δ_ℓ^{-1} is linear, we have for $z \in B$

$$\begin{aligned} |T(z)| &\leqslant |\delta_\ell^{-1}(z)| + |\delta_\ell^{-1}(\ell z_\ell)| = \ell^{-1}\left(|x|^2 + \frac{|y|^2}{\ell^{2\alpha}}\right)^{1/2} + (1 - \ell^{-1}) \\ &\leqslant \ell^{-1}\left(|x|^2 + |y|^2\right)^{1/2} + (1 - \ell^{-1}) < \ell^{-1} + 1 - \ell^{-1} = 1. \end{aligned}$$

Hence,

$$B \subset A_\ell := \left\{z \in \mathbb{R}^N : |T(z)| < 1\right\}.$$

Another easy consequence of the definition of T and A_ℓ is that $T(B) \subset T(A_\ell) = B$. Moreover, if $z \in B - T(B)$, then $T^{-1}(z) \in A_\ell - B$, which implies $|T^{-1}(z)| \geqslant 1$ (*cf.* Figure 2.1).

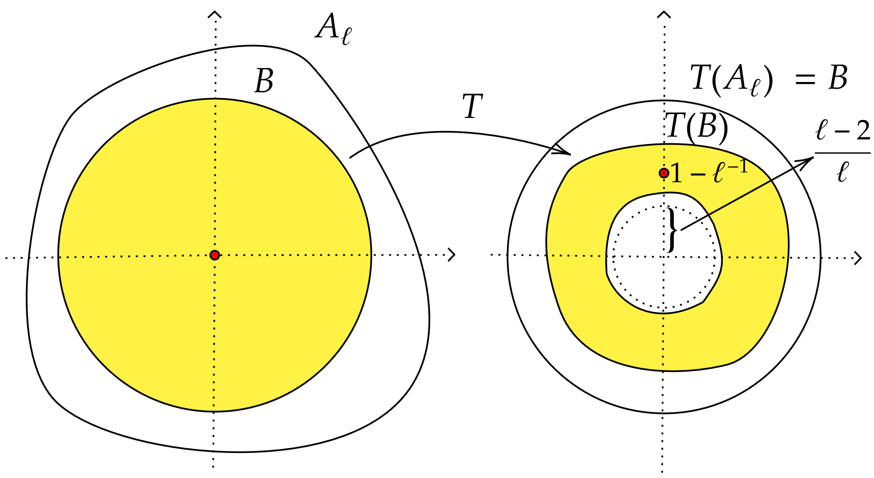

Figure 2.1: Application T

We finally notice that, if $w = (\xi, \eta) \in B$ and $z = T(w)$, then

$$|z| \geqslant \left|\delta_\ell^{-1}(\ell z_\ell)\right| - \left|\delta_\ell^{-1}(\ell z_\ell) - T(w)\right| = 1 - \frac{1}{\ell} - \frac{1}{\ell}\sqrt{\xi^2 + \frac{\eta^2}{\ell^{2\alpha}}} > 1 - \frac{1}{\ell} - \frac{1}{\ell}\sqrt{\xi^2 + \eta^2},$$

and therefore (cf. Figure 2.1)

$$|z| > \frac{\ell - 2}{\ell}, \quad \forall\, z \in T(B). \tag{2.12}$$

The more useful property of the map T is:

Proposition 2.9. *Suppose that $u \in C_0^\infty(B)$, $\ell > 2$, T is as in (2.11) and define $v := (u \circ T^{-1})$. Then $v \in C_0^\infty(B)$ and*

$$\|v\|^2 = \ell^{2-\tilde{N}}\|u\|^2, \qquad \int_B |v|^p |z|^\ell \,\mathrm{d}z > \left(\frac{\ell - 2}{\ell}\right)^\ell \ell^{-\tilde{N}} \|u\|^p_{L^p(B)}.$$

Proof. We start proving that $v \in C_0^\infty(B)$. Indeed, let $K \subset B$ be the support of u and notice that, if $z \notin T(K)$, then $u(T^{-1}(z)) = 0$. Thus, $\mathrm{supp}(v) \subset \overline{T(K)} = T(K) \subset T(B) \subset B$. We now compute

$$v_{x_i}(z) = \ell u_{x_i}(T^{-1}z), \qquad v_{y_j}(z) = \ell^{1+\alpha} u_{y_j}(T^{-1}(z)),$$

for any $i = 1, \ldots, N_1$ and $j = 1, \ldots, N_2$. Thus,

$$\begin{aligned}
|\nabla_\alpha v(z)|^2 &= |\nabla_x v(z)|^2 + |x|^{2\alpha}|\nabla_y v(z)|^2 \\
&= \ell^2 |\nabla_x u(T^{-1}(z))|^2 + \ell^{2(1+\alpha)}|x|^{2\alpha}|\nabla_y u(T^{-1}(z))|^2 \\
&= \ell^2 \left[|\nabla_x u(T^{-1}(z))|^2 + |\ell x|^{2\alpha}|\nabla_y u(T^{-1}(z))|^2\right] \\
&= \ell^2 |\nabla_\alpha u(T^{-1}(z))|^2
\end{aligned}$$

and therefore, since the Jacobian of the transformation T is $\ell^{-(N_1 + N_2(1+\alpha))} = \ell^{-\tilde{N}}$, we can use $B = T(A_\ell)$ to obtain

$$\begin{aligned}
\int_B |\nabla_\alpha v(z)|^2 \,\mathrm{d}z &= \ell^2 \int_{T(A_\ell)} |\nabla_\alpha u(T^{-1}(z))|^2 \,\mathrm{d}z \\
&= \ell^{2-\tilde{N}} \int_{A_\ell} |\nabla_\alpha u(z)|^2 \,\mathrm{d}z \\
&= \ell^{2-\tilde{N}} \int_B |\nabla_\alpha u(z)|^2 \,\mathrm{d}z + \ell^{2-\tilde{N}} \int_{A_\ell - B} |\nabla_\alpha u(z)|^2 \,\mathrm{d}z \\
&= \ell^{2-\tilde{N}} \int_B |\nabla_\alpha u(z)|^2 \,\mathrm{d}z,
\end{aligned}$$

since the derivatives u_{x_i}, u_{y_j} have support inside B. This proves the first statement of the proposition.

To check the second statement of the proposition, we recall that $|T^{-1}(z)| \geqslant 1$, for any $z \in B - T(B)$ and so $u(T^{-1}(z)) = 0$ for any $z \in B - T(B)$. Thus,

$$\begin{aligned}\int_B |v(z)|^p |z|^\ell \, \mathrm{d}z &= \int_{T(B)} |u(T^{-1}(z))|^p |z|^\ell \, \mathrm{d}z + \int_{B-T(B)} |u(T^{-1}(z))|^p |z|^\ell \, \mathrm{d}z \\ &= \int_{T(B)} |u(T^{-1}(z))|^p |z|^\ell \, \mathrm{d}z,\end{aligned}$$

and it follows from (2.12) that

$$\begin{aligned}\int_B |v(z)|^p |z|^\ell \, \mathrm{d}z &> \left(\frac{\ell-2}{\ell}\right)^\ell \int_{T(B)} |u(T^{-1}(z))|^p \, \mathrm{d}z \\ &= \left(\frac{\ell-2}{\ell}\right)^\ell \ell^{-\tilde{N}} \int_B |u(z)|^p \, \mathrm{d}z,\end{aligned}$$

which is exactly the second statement of the proposition. □

We are ready to prove the last result of this chapter.

Proof of item (2) in Proposition 2.6. Let $u_0 \in C_0^\infty(B) - \{0\}$ be fixed and $v_0 := (u_0 \circ T^{-1})$ as in Proposition 2.9. Since it is clear that $v_0 \in H^1_{0,\alpha}(B)$, it follows from Proposition 2.9 that, for any $\ell > 2$,

$$S_{\ell,p} \leqslant R(v_0) < \frac{\ell^{2-\tilde{N}} \displaystyle\int_B |\nabla_\alpha u_0|^2 \, \mathrm{d}z}{\left[\left(\frac{\ell-2}{\ell}\right)^\ell \ell^{-\tilde{N}} \displaystyle\int_B |u_0|^p \, \mathrm{d}z\right]^{2/p}} = \frac{\ell^{2-\tilde{N}+(2\tilde{N}/p)}}{\left(\frac{\ell-2}{\ell}\right)^{2\ell/p}} \frac{\|u_0\|^2}{\|u_0\|^2_{L^p(B)}}. \tag{2.13}$$

We now define

$$g(s) := \left(\frac{s-2}{s}\right)^s, \qquad \forall\, s \geqslant \ell_0 > 2.$$

By a straightforward computation one gets

$$g'(s) = g(s)\left[\ln\left(\frac{s-2}{s}\right) + \frac{2}{s-2}\right].$$

If we call $h(s)$ the expression into the brackets above, we have that

$$h'(s) = -\frac{4}{s(s-2)^2} < 0, \quad s > \ell_0.$$

and therefore h is decreasing. Moreover, $\lim_{s\to+\infty} h(s) = \ln(1) = 0$. Hence, we conclude that h is positive, which implies that g is increasing in $[\ell_0, +\infty)$. Thus, for $\ell \geqslant \ell_0$, we have that $[g(\ell_0)]^{2/p} \leqslant [g(\ell)]^{2/p}$. In other words,

$$0 < \left(\frac{\ell_0 - 2}{\ell_0}\right)^{2\ell_0/p} < \left(\frac{\ell - 2}{\ell}\right)^{2\ell/p}, \qquad \forall\, \ell \geqslant \ell_0.$$

Coming back to (2.13), we get

$$S_{\ell,p} \leqslant \mathcal{C}\ell^{2-\tilde{N}+(2\tilde{N}/p)}, \quad \forall\, \ell \geqslant \ell_0,$$

where

$$\mathcal{C} := \left(\frac{\ell_0 - 2}{\ell_0}\right)^{-2\ell_0/p} \frac{\|u_0\|^2}{\|u_0\|^2_{L^p(B)}},$$

and the proof is complete. □

Hénon equation for the biharmonic operator

The aim of this chapter is to consider the problem

$$\begin{cases} \Delta^2 u = [W(z)]^\ell f(u), & \text{in } B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \tag{$P_{x,y}$}$$

where $B \subset \mathbb{R}^N$ is the unit ball, and the functions W and f satisfy certain conditions that will be specified later. To proceed, we will introduce a Sobolev space consisting of symmetric functions and establish a pointwise inequality that holds for functions belonging to this space.

First, as it is well know, in order to prove existence of a weak solution to the problem

$$-\Delta u(z) = |z|^\ell u^{p-1}(z), z \in B, \quad u(z) = 0, z \in \partial B,$$

for $2 < p < 2^* + 2\ell/(N-2)$, Ni in [60] proved the inequality

$$|u(z)| \leqslant \frac{\|\nabla u\|_{L^2(B)}}{\sqrt{\omega_N(N-2)}|z|^{(N-2)/2}}, \quad z \in B, \tag{3.1}$$

for any radially symmetric $u \in C^1(B)$ vanishing in the boundary of B. Our idea here is to follow a similar approach introducing a functional space with symmetry and proving radial type lemmas.

In order to define such a space, we recall that $H_0^2(B)$ is the closure of $C_0^\infty(B)$ under the $H^2(B)$ norm and, using Poincaré Inequality and integration by parts, one can see that the usual norm induced by $H^2(B)$ is equivalent to

$$\|u\|_{H_0^2(B)} := \left(\sum_{|\alpha|=2} |D^\alpha u(z)|^2 \, \mathrm{d}z \right)^{1/2} = \left(\int_B |\Delta u(z)|^2 \, \mathrm{d}z \right)^{1/2}.$$

Now, we decompose $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, denote by $O(k)$ the group of real orthogonal $k \times k$ matrices, consider

$$C_{0,x,y}^\infty(B) := \left\{ u \in C_0^\infty(B) : u(x,y) = u(T_1(x), T_2(y)), \ \forall \ T_i \in O(N_i), i = 1, 2 \right\},$$

the set of compactly supported functions in B with are coordinate-radial and define

$$H^2_{0,x,y}(B) = \overline{C^\infty_{0,x,y}(B)}^{H^2_0(B)}.$$

Our first main result is a version, for our setting, of the inequality (3.1):

Theorem 3.1. *For any $u \in H^2_{0,x,y}(B)$ and $N_1, N_2 \geqslant 3$, there holds*

$$|u(x,y)| \leqslant C \frac{\|\Delta u\|_{L^2(B)}}{|x|^{\frac{N_1-2}{2}} |y|^{\frac{N_2-2}{2}}}, \quad \text{for a.e. } (x,y) \in B, \tag{3.2}$$

with

$$C = \sqrt{\frac{\Gamma\left(\frac{N_1}{2}\right)\Gamma\left(\frac{N_2}{2}\right)}{4\pi^{\frac{N}{2}}(N_1-2)(N_2-2)}},$$

and $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t$ being the Gamma function.

In order to state our second mean result, we set, for any $1 \leqslant p < +\infty$ and $\ell \geqslant 0$

$$L^p_\ell(B) := \left\{ u \in L^1_{loc}(B) : \int_B |u(z)|^p [W(z)]^\ell \,\mathrm{d}z < +\infty \right\}, \tag{3.3}$$

with the weight W satisfying:

(W_1) $W \in L^1_{loc}(B)$ and there exists $c_W > 0$, such that

$$0 < W(z) \leqslant c_W |x||y|, \qquad \text{for a.e. } z \in B.$$

This is a Banach space with the norm

$$\|u\|_{L^p_\ell(B)} := \left(\int_B |u(z)|^p [W(z)]^\ell \,\mathrm{d}z \right)^{1/p}.$$

As a consequence of Theorem 3.1, we can prove an embedding result for the space $H^2_{0,x,y}(B)$. Actually, if we set

$$2^*_{\ell,N_1} := \frac{2N_1}{N_1-2} + \frac{2\ell}{N_1-2},$$

for any $N_1 > 2$ and $\ell \geqslant 0$, we have the following:

Theorem 3.2. *Suppose that $N = N_1 + N_2$, with $3 \leqslant N_2 \leqslant N_1$, $\ell \geqslant 0$ and $1 \leqslant p < 2^*_{\ell,N_1}$. Then the embedding $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ is compact.*

It is important to analyze situations in which the last result allows us to consider exponents beyond the critical Sobolev exponent $2^{**} := 2N/(N-4)$. We have that

$$2^{**} < 2^*_{\ell,N_1} \quad \Longleftrightarrow \quad \ell > \frac{2(N_1 - N_2)}{N-4},$$

and therefore we can consider supercritical growth. The most favorable situation occurs when $N_1 = N_2$ because, in this case, the exponent $2^*_{\ell,N_1}$ is supercritical for any $\ell > 0$. Even when the dimensions are not equal, the condition on ℓ does not seem very restrictive, since it can be easily shown that $3 \leqslant N_2 < N_1$ implies

$$\frac{2(N_1 - N_2)}{N-4} < 2,$$

and, therefore, supercritical growth is possible for any $\ell \geqslant 2$.Also, we remark that when $N_1 \leqslant N_2$, analogous results can be obtained by simply replacing $2^*_{\ell,N_1}$ with $2^*_{\ell,N_2}$.

As an application of the Theorem 3.2, we consider the problem $(P_{x,y})$ with the supercritical nonlinearity f satisfying:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) there exists $c_f > 0$ and $p \in (2, 2^*_{\ell,N_1})$ such that

$$|f(s)| \leqslant c_f \left(1 + |s|^{p-1}\right), \quad \forall\, s \in \mathbb{R};$$

(f_3) there holds

$$\lim_{s \to 0} \frac{f(s)}{s} = 0;$$

(f_4) there exist $\mu > 2$ and $s_0 > 0$, such that

$$0 < \mu F(s) \leqslant s f(s), \quad \forall\, |s| \geqslant s_0,$$

where $F(s) := \displaystyle\int_0^s f(t)\,\mathrm{d}t$.

We say that a function $u \in H_0^2(B)$ is a weak solution of Problem $(P_{x,y})$ if

$$\int_B \Delta u \Delta \varphi \,\mathrm{d}z = \int_B [W(z)]^\ell f(u)\varphi \,\mathrm{d}z, \quad \forall\, \varphi \in H^1_{0,x,y}(B).$$

Finally, we shall prove the following:

Theorem 3.3. *Suppose that $\ell \geqslant 0$, $p \in (2, 2^*_{\ell,N_1})$ and f, W satisfy (f_1)-(f_4) and (W_1), respectively. Then Problem $(P_{x,y})$ has a nonzero weak solution in $H^2_{0,x,y}(B)$.*

This chapter is organized as follows. In Section 3.1, we present some notations and preliminary results, including the proof of an integral identity in the space $H^2_{0,x,y}(B)$. In Section 3.2, we provide the proofs of Theorems 3.1 and 3.2. Finally, Section 3.3 is devoted to the study of problem $(P_{x,y})$.

3.1 Some technical results

From now on, we shall denote

$$D := \left\{(s,t) \in \mathbb{R}^2 \,:\, s^2 + t^2 < 1\right\}$$

the unitary ball in $\mathbb{R}^2$ and

$$D_+ := \{(s,t) \in D \,:\, s,\, t \geqslant 0\}.$$

Notice that, if $u \in H^1_0(B)$ is such that $u = u \circ (T_1, T_2)$, for any $T_i \in O(i)$, $i = 1, 2$, then it is well defined the function

$$v(s,t) := u(se, tf), \quad (s,t) \in D,$$

with $|e| = |f| = 1$. From the symmetry properties of u it is clear that v is radial in each of this components and $u(x,y) = v(|x|, |y|)$, for any $(x,y) \in B$.

For the reader's convenience, we state and prove below an integral identity that was used in [7] .

Lemma 3.4. *Suppose that $N = N_1 + N_2 \geqslant 2$ with N_1, $N_2 \in \mathbb{N}$, and $u \in L^1(B)$ be such that $u(x,y) = v(|x|, |y|)$, for some function v defined on D_+. Then*

$$\int_B u(z)\,\mathrm{d}z = C \int_{D_+} v(s,t) s^{N_1 - 1} t^{N_2 - 1}\,\mathrm{d}(s,t)$$

where

$$C = \frac{4\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N_1}{2}\right)\Gamma\left(\frac{N_2}{2}\right)} > 0,$$

and $\Gamma(s) := \displaystyle\int_0^\infty t^{s-1} e^{-t}\,\mathrm{d}t$ *is the Gamma function.*

Proof. We have that

$$\int_B u(z)\,\mathrm{d}z = \int\limits_{\mathbb{R}^N} v(|x|,|y|)\chi_B(z)\,\mathrm{d}z = \int\limits_{\mathbb{R}^{N_1}} \left[\int\limits_{\mathbb{R}^{N_2}} v(|x|,|y|)\chi_B(z)\,\mathrm{d}y\right] \mathrm{d}x$$

and there we can use polar coordinates (*cf.* [28, Theorem 2.9]) to get

$$
\begin{aligned}
\int_B u(z)\,\mathrm{d}z &= \int\limits_{\mathbb{R}^{N_1}} \left[\int_0^\infty \int\limits_{\mathbb{S}^{N_2-1}} v(|x|,|y't|)\chi_B(x,y't)t^{N_2-1} d\sigma^2_{y'}\,\mathrm{d}t \right] \mathrm{d}x \\
&= \int_0^\infty \int_{\mathbb{S}^{N_2-1}} \left[\int\limits_{\mathbb{R}^{N_1}} v(|x|,t)\chi_B(x,y't)t^{N_2-1}\,\mathrm{d}x \right] d\sigma^2_{y'}\,\mathrm{d}t \\
&= \int_0^\infty \int\limits_{\mathbb{S}^{N_2-1}} \left[\int_0^\infty \int\limits_{\mathbb{S}^{N_1-1}} v(|x's|,t)\chi_B(x's,y't)s^{N_1-1}t^{N_2-1} d\sigma^1_{x'}\,\mathrm{d}s \right] d\sigma^2_{y'}\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \left[\int\limits_{\mathbb{S}^{N_2-1}} \int\limits_{\mathbb{S}^{N_1-1}} v(s,t)\chi_B(x's,y't)s^{N_1-1}t^{N_2-1} d\sigma^1_{x'} d\sigma^2_{y'} \right] \mathrm{d}s\,\mathrm{d}t,
\end{aligned}
$$

where $\mathbb{S}^{N_i-1}$ is the surface of the unit ball $B^i \subset \mathbb{R}^{N_i}$ and $d\sigma^i$ is the surface element.

Since for any $(x',y') \in \mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$ and $s,t \geqslant 0$ there holds

$$
\chi_B(x's,y't) = \chi_{D_+}(s,t),
$$

we obtain

$$
\begin{aligned}
\int_B u(z)\,\mathrm{d}z &= \int_0^\infty \int_0^\infty \left[\int\limits_{\mathbb{S}^{N_2-1}} \int\limits_{\mathbb{S}^{N_1-1}} v(s,t)\chi_B(x's,y't)s^{N_1-1}t^{N_2-1} d\sigma^1_{x'} d\sigma^2_{y'} \right] \mathrm{d}s\,\mathrm{d}t \\
&= \omega_{N_1}\omega_{N_2} \int_0^\infty \int_0^\infty v(s,t)\chi_{D_+} s^{N_1-1}t^{N_2-1}\,\mathrm{d}s\,\mathrm{d}t \\
&= \omega_{N_1}\omega_{N_2} \int_{(0,\infty)\times(0,\infty)} v(s,t)\chi_{D_+} s^{N_1-1}t^{N_2-1} d(s,t) \\
&= \omega_{N_1}\omega_{N_2} \int_{D_+} v(s,t)s^{N_1-1}t^{N_2-1} d(s,t),
\end{aligned}
$$

with $\omega_{N_i} = \sigma^i(\mathbb{S}^{N_i-1})$, $i = 1,2$. As proved in [28, Proposition 2.54], we have that

$$
\omega_{N_i} = \frac{2\pi^{\frac{N_i}{2}}}{\Gamma\left(\frac{N_i}{2}\right)}, \quad i = 1,2,
$$

and the proof is concluded. □

We present and prove in what follows two auxiliary results which will be essential in the proof of Theorem 3.1.

Lemma 3.5. *Suppose that* $(s_0, t_0) \in D$*. Then there exists* $\tilde{t}$ *such that* $|t_0| < |\tilde{t}| \leqslant 1$ *and* $(s_0, \tilde{t}) \in \partial D$*. Moreover, for each* $\beta \in (|t_0|, |\tilde{t}|)$*, there exists* s_β *such that* $|s_0| < |s_\beta| < 1$ *and* $(s_\beta, \beta) \in \partial D$*.*

Proof. Let $(s_0, t_0) \in D$, define the vertical line $r_1(t) := (|s_0|, t)$ and notice that $|r_1(\tilde{t})| = 1$ for $\tilde{t} = \pm\sqrt{1 - s_0^2}$, that is, $(s_0, \tilde{t}) \in \partial D$. Notice that $|t_0| < |\tilde{t}| < 1$. Analogously, fixing $|t_0| < \beta < |\tilde{t}|$ and defining the horizontal line $r_2(\eta) := (\eta, \beta)$, we can notice that $|r_2(s_\beta)| = 1$ for $s_\beta = \pm\sqrt{1 - \beta^2}$ and $|s_0| < |s_\beta| < 1$, because $s_0^2 + \beta^2 < s_0^2 + \tilde{t}^2 = 1$ (*cf.* Figure 3.1). ☐

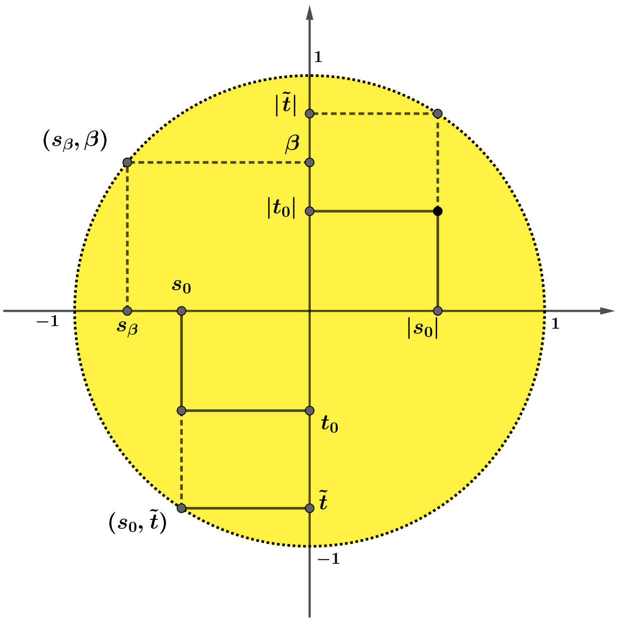

Figure 3.1: Construction of $(s_0, \tilde{t})$ and (s_β, β)

Lemma 3.6. *Suppose that* $N = N_1 + N_2$ *with* $N_1, N_2 \geqslant 3$ *and* $\varphi \in C_0^2(D)$ *verifies*

$$\varphi(s_1, t_1) = \varphi(s_2, t_2), \quad \textit{if } (|s_1|, |s_2|) = (|t_1|, |t_2|).$$

Then

$$|\varphi(s_0, t_0)| \leqslant C \frac{\left(\int_B |\partial_{st} \varphi(|x|, |y|)|^2 \, \mathrm{d}z \right)^{1/2}}{|s_0|^{\frac{N_1 - 2}{2}} |t_0|^{\frac{N_2 - 2}{2}}}, \quad \forall \ (s_0, t_0) \in D,$$

with

$$C = \sqrt{\frac{\Gamma\left(\frac{N_1}{2}\right)\Gamma\left(\frac{N_2}{2}\right)}{4\pi^{\frac{N}{2}}(N_1 - 2)(N_2 - 2)}}.$$

Proof. Given $(s_0, t_0) \in D$, we can use Lemma 3.5 to obtain $\tilde{t}$ such that $|t_0| < |\tilde{t}| < 1$ and $(s_0, \tilde{t}) \in \partial D$. From the equation

$$-\varphi(s_0, t_0) = \varphi(s_0, |\tilde{t}|) - \varphi(s_0, |t_0|) = \int_{|t_0|}^{|\tilde{t}|} \partial_t \varphi(s_0, \beta) \, \mathrm{d}\beta,$$

we get

$$|\varphi(s_0, t_0)| \leqslant \int_{|t_0|}^{|\tilde{t}|} |\partial_t \varphi(s_0, \beta)| \, \mathrm{d}\beta. \tag{3.4}$$

Now, for each $\beta \in (|t_0|, |\tilde{t}|)$, we can use Lemma 3.5 again to choose s_β such that $|s_0| < |s_\beta| < 1$ and $(s_\beta, \beta) \in \partial D$. Notice that, as $\varphi(s_1, t) = \varphi(s_2, t)$, if $|s_1| = |s_2|$ then $\partial_t \varphi(s_1, t) = \partial_t \varphi(s_2, t)$. So, as the support of φ and $\partial_t \varphi$ are in D

$$\partial_t \varphi(s_0, \beta) = -[\partial_t \varphi(|s_\beta|, \beta) - \partial_t \varphi(|s_0|, \beta)] = -\int_{|s_0|}^{|s_\beta|} \partial_{st} \varphi(\alpha, \beta) \, \mathrm{d}\alpha,$$

which implies

$$|\partial_t \varphi(s_0, \beta)| \leqslant \int_{|s_0|}^{|s_\beta|} |\partial_{st} \varphi(\alpha, \beta)| \, \mathrm{d}\alpha \leqslant \int_{|s_0|}^{1} |\partial_{st} \varphi(\alpha, \beta)| \, \mathrm{d}\alpha, \tag{3.5}$$

for any $\beta \in (|t_0|, |\tilde{t}|)$. Finally, combining (3.4) with (3.5), we obtain

$$\begin{aligned} |\varphi(s_0, t_0)| &\leqslant \int_{|t_0|}^{|\tilde{t}|} \int_{|s_0|}^{1} |\partial_{st} \varphi(\alpha, \beta)| \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ &\leqslant \int_{|t_0|}^{1} \int_{|s_0|}^{1} |\partial_{st} \varphi(\alpha, \beta)| \, \mathrm{d}\alpha \, \mathrm{d}\beta. \end{aligned}$$

If we define $I_{t_0,s_0} := (|t_0|, 1) \times (|s_0|, 1)$, we can use Hölder's inequality to get

$$\begin{aligned} \int_{|t_0|}^{1} \int_{|s_0|}^{1} |\partial_{st} \varphi(\alpha, \beta)| \, \mathrm{d}\alpha \, \mathrm{d}\beta &= \int_{I_{t_0,s_0}} |\partial_{st} \varphi(\alpha, \beta)| \alpha^{\frac{N_1-1}{2}} \beta^{\frac{N_2-1}{2}} \alpha^{\frac{1-N_1}{2}} \beta^{\frac{1-N_2}{2}} \, \mathrm{d}(\alpha, \beta) \\ &\leqslant \left(\int_{I_{t_0,s_0}} |\partial_{st} \varphi(\alpha, \beta)|^2 \alpha^{N_1-1} \beta^{N_2-1} \, \mathrm{d}(\alpha, \beta) \right)^{1/2} \cdot \\ &\quad \left(\int_{I_{t_0,s_0}} \alpha^{1-N_1} \beta^{1-N_2} \, \mathrm{d}(\alpha, \beta) \right)^{1/2} . \end{aligned}$$

As $|s_0|, |t_0| < 1$

$$\int_{I_{t_0,s_0}} \alpha^{1-N_1}\beta^{1-N_2}\,\mathrm{d}(\alpha,\beta) = \frac{1}{(N_1-2)(N_2-2)}\left(\frac{1}{|s_0|^{N_1-2}}-1\right)\left(\frac{1}{|t_0|^{N_2-2}}-1\right) \leqslant \frac{1}{(N_1-2)(N_2-2)|s_0|^{N_1-2}|t_0|^{N_2-2}}.$$

Therefore, for

$$C := \frac{1}{\sqrt{(N_1-2)(N_2-2)}},$$

we have that

$$\begin{aligned} |\varphi(s_0,t_0)| &\leqslant C\frac{\left(\int_{I_{t_0,s_0}} |\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\right)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}} \\ &\leqslant C\frac{\left(\int_{(0,1)\times(0,1)} |\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\right)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}} \\ &= C\frac{\left(\int_{D_+} |\partial_{st}\varphi(\alpha,\beta)|^2\alpha^{N_1-1}\beta^{N_2-1}\,\mathrm{d}(\alpha,\beta)\right)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}}, \end{aligned}$$

because, as the support of φ is on D, then the same happens to $\partial_{st}\varphi$. Now, we apply Lemma 3.4 to obtain

$$|\varphi(s_0,t_0)| \leqslant C_1\frac{\left(\int_B |\partial_{st}\varphi(|x|,|y|)|^2\,\mathrm{d}z\right)^{1/2}}{|s_0|^{\frac{N_1-2}{2}}|t_0|^{\frac{N_2-2}{2}}},$$

where

$$C_1 = \sqrt{\frac{\Gamma\left(\frac{N_1}{2}\right)\Gamma\left(\frac{N_2}{2}\right)}{4\pi^{\frac{N}{2}}\sqrt{(N_1-2)(N_2-2)}}},$$

completing the proof. □

3.2 Proofs of Theorems 3.1 and 3.2

We start this section by proving our version of the radial lemma.

Proof of Theorem 3.1. It suffices to prove the result for a function $u \in C^\infty_{0,x,y}(B)$. To that end, we consider a function φ such that $u(x,y) = \varphi(|x|,|y|)$ and $\varphi(s_1,t_1) = \varphi(s_2,t_2)$ whenever $(|s_1|,|t_1|) = (|s_2|,|t_2|)$.

For each $i = 1, ..., N_1$ and $j = 1, ..., N_2$

$$u_{x_i y_j}(x, y) = \partial_{st}\varphi(|x|, |y|)\frac{x_i}{|x|}\frac{y_j}{|y|}$$

and therefore

$$\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}[u_{x_i y_j}(x, y)]^2 = [\partial_{st}\varphi(|x|, |y|)]^2. \tag{3.6}$$

Since

$$\int_B \Delta_x u \Delta_y u \, \mathrm{d}z = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B u_{x_i x_i} u_{y_j y_j} \, \mathrm{d}z = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B [u_{x_i y_j}(x, y)]^2 \, \mathrm{d}z$$

we can use identity (3.6) to get

$$\int_B \Delta_x u \Delta_y u \, \mathrm{d}z = \int_B [\partial_{st}\varphi(|x|, |y|)]^2 \, \mathrm{d}z.$$

and therefore we can use Lemma 3.6 to get

$$|u(x, y)| \leqslant C\frac{\left(\int_B \Delta_x u \Delta_y u \, \mathrm{d}z\right)^{1/2}}{|x|^{\frac{N_1-2}{2}}|y|^{\frac{N_2-2}{2}}}, \quad \text{for a.e. } (x, y) \in B. \tag{3.7}$$

Using this inequality together with

$$|\Delta u|^2 = |\Delta_x u + \Delta_y u|^2 = |\Delta_x u|^2 + 2\Delta_x u \Delta_y u + |\Delta_y u|^2,$$

we conclude that (3.2) holds. □

Remark 3.7. *As a consequence of the previous result, we can see that the bilinear form*

$$B_{x,y}[u, v] := \int_B \Delta_x u(z)\Delta_y v(z) \, \mathrm{d}z, \quad \forall\, u, v \in H^2_{0,x,y}(B)$$

is an inner product. Indeed, for any $u, v \in C^\infty_{0,x,y}(B)$, we have that

$$B_{x,y}[u, v] = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B u_{x_i x_i} v_{y_j y_j} \, \mathrm{d}z = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B u v_{x_i x_i y_j y_j} \, \mathrm{d}z = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\int_B u_{y_j y_j} v_{x_i x_i} \, \mathrm{d}z,$$

which implies $B_{x,y}[u,v] = B_{x,y}[v,u]$. *For* $u, v \in H^2_{0,x,y}(B)$ *let* $u_n, v_n \in C^\infty_{0,xy}(B)$ *such that* $u_n \to u$ *and* $v_n \to v$ *in* $H^2(B)$. *Then,*

$$\int_B \Delta_x u_n \Delta_y v_n \, \mathrm{d}z = \int_B \Delta_x v_n \Delta_y u_n \, \mathrm{d}z.$$

As $(u_n)_{x_ix_i} \to u_{x_ix_i}$, $(u_n)_{y_jy_j} \to u_{y_jy_j}$, $(v_n)_{x_ix_i} \to v_{x_ix_i}$, $(v_n)_{y_jy_j} \to v_{y_jy_j}$ *in* $L^2(B)$, *there exists* $f_1, f_2, f_3, f_4 \in L^2(B)$ *such that, up to a subsequence,*

$$|(u_n)_{x_ix_i}| \leqslant f_1, \quad |(u_n)_{y_jy_j}| \leqslant f_2, \quad |(v_n)_{x_ix_i}| \leqslant f_3, \qquad |(v_n)_{y_jy_j}| \leqslant f_4,$$

almost everywhere in B. *So,* $|u_{x_ix_i} v_{y_jy_j}| \leqslant f_1 f_4 \in L^1(B)$ *and* $|v_{x_ix_i} u_{y_jy_j}| \leqslant f_3 f_2 \in L^1(B)$. *Using the Lebesgue Theorem we conclude that* $B_{x,y}[u,v] = B_{x,y}[v,u]$. *Moreover*

$$B_{x,y}[u,u] = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_B [u_{x_iy_j}]^2 dz \geqslant 0$$

and it follows from inequality (3.7) *that, if* $u \neq 0$, *then* $B_{x,y}[u,u] \neq 0$.

Now we have proved our radial-type result, we can obtain embedding properties for the space $H^2_{0,x,y}(B)$.

Proof of Theorem 3.2. We first prove that the embedding $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ is continuous, for any $1 \leqslant p < 2^*_{\ell,N_1}$. To that end, we pick $u \in H^2_{0,x,y}(B)$ and use (W_1) and (3.2) to obtain

$$\begin{aligned}
\|u\|^p_{L^p_\ell(B)} &= \int_B |u(z)|^p [W(z)]^\ell \, \mathrm{d}z \\
&\leqslant c^\ell_W \int_B |u(z)|^p (|x||y|)^\ell \, \mathrm{d}z \\
&\leqslant c^\ell_W \|u\|^p_{H^2_0(B)} \int_B |x|^{-\frac{(N_1-2)p}{2}} |y|^{-\frac{(N_2-2)p}{2}} (|x||y|)^\ell \, \mathrm{d}z \\
&\leqslant c^\ell_W \|u\|^p_{H^2_0(B)} \int_{B^1} \int_{B^2} |x|^{\left[\ell - \frac{(N_1-2)p}{2}\right]} |y|^{\left[\ell - \frac{(N_2-2)p}{2}\right]} \, \mathrm{d}x \, \mathrm{d}y \\
&= c^\ell_W \|u\|^p_{H^2_0(B)} \left(\int_{B^1} |x|^{\left[\ell - \frac{(N_1-2)p}{2}\right]} \, \mathrm{d}x \right) \left(\int_{B^2} |y|^{\left[\ell - \frac{(N_2-2)p}{2}\right]} \, \mathrm{d}y \right),
\end{aligned}$$

where we have used the inclusion $B \subset B^1 \times B^2$ (recall that B^i is the unit ball of $\mathbb{R}^{N_i}$). Thus

$$\|u\|^p_{L^p_\ell(B)} \leqslant C_1 \|u\|^p_{H^2_0(B)} \left(\int_0^1 r^{\left[\ell - \frac{(N_1-2)p}{2} + N_1 - 1\right]} dr \right) \left(\int_0^1 r^{\left[\ell - \frac{(N_2-2)p}{2} + N_2 - 1\right]} dr \right),$$

with $C_1 := c_W^\ell \omega_{N_1}\omega_{N_2}$. Since $p \in [1, 2^*_{\ell,N_1})$ and $N_2 \leqslant N_1$, the integrals above are finite, and thus the continuity of the embedding is established.

To prove the compactness, we first point out that, since $H^2_{0,x,y}(B)$ is a subspace of $W_0^{2,2}(B)$, the Rellich-Kondrachov Theorem assures that it is compactly embedded in $L^1(B)$. Consider $\beta \in (0,1)$ to be chosen. For $u \in H^2_{0,x,y}(B)$, we can use Hölder's inequality with exponents $s = \beta^{-1}$ and $s' = (1-\beta)^{-1}$ to obtain

$$\begin{aligned}\|u\|^p_{L^p_\ell(B)} &= \int_B |u(z)|^\beta |u(z)|^{p-\beta}[W(z)]^\ell \, \mathrm{d}z \\ &\leqslant \|u\|^\beta_{L^1(B)} \left(\int_B |u(z)|^{\frac{p-\beta}{1-\beta}} [W(z)]^{\frac{\ell}{1-\beta}} \, \mathrm{d}z\right)^{1-\beta}\end{aligned}$$

or, equivalently,

$$\|u\|^p_{L^p_\ell(B)} \leqslant \|u\|^\beta_{L^1(B)} \|u\|^{p-\beta}_{L^{q_\beta}_{\ell_\beta}(B)} \tag{3.8}$$

with

$$\ell_\beta := \frac{\ell}{1-\beta}, \qquad q_\beta := \frac{p-\beta}{1-\beta}.$$

Since $p > 1$, it is clear that $q_\beta \geqslant 1$ for any $\beta \in (0,1)$. Moreover,

$$\lim_{\beta\to 0^+}\left(q_\beta - 2^*_{\ell_\beta,N_1}\right) = \lim_{\beta\to 0^+}\left(\frac{p-\beta}{1-\beta} - \frac{2N_1}{N_1-2} - \frac{2\ell}{(1-\beta)(N_1-2)}\right) = \left(p - 2^*_{\ell,N_1}\right) < 0,$$

and therefore $q_\beta \in [1, 2^*_{\ell_\beta,N_1})$, for some $\beta \in (0,1)$ sufficiently close to 0. Considering this choice for β, we can apply the embedding proved in the first part to find $C_2 > 0$ such that

$$\|u\|_{L^{q_\beta}_{\ell_\beta}(B)} \leqslant C_2 \|u\|_{H^2_0(B)}.$$

Combining this estimate with (3.8), we obtain

$$\|u\|^p_{L^p_\ell(B)} \leqslant C_3 \|u\|^{\beta_0}_{L^1(B)} \|u\|^{p-\beta_0}_{H^2_0(B)}, \quad \forall\, u \in H^2_{0,x,y}(B).$$

If $(u_n) \subset H^2_{0,x,y}(B)$ is a bounded sequence, we can extract a subsequence (still denoted by (u_n)) such that $u_n \rightharpoonup u$ weakly in $H^2_{0,x,y}(B)$. By the compact embedding of $H^2_{0,x,y}(B)$ into $L^1(B)$, we also have $u_n \to u$ in $L^1(B)$. Therefore,

$$\|u_n - u\|^p_{L^p_\ell(B)} \leqslant C_3 \|u_n - u\|^\beta_{L^1(B)} \|u_n - u\|^{p-\beta}_{H^2_0(B)} \leqslant C_4 \|u_n - u\|^\beta_{L^1(B)} \to 0,$$

which shows that $u_n \to u$ in $L^p_\ell(B)$, completing the proof. □

3.2.1 Further comments

We begin this section by presenting some results that are complemented by Theorem 3.2. The following result can be found in [20, Theorem 1.4 and Corollary 1.5].

Theorem 3.8. *Suppose that $2 \leqslant N_2 \leqslant N_1$, $2 < q < +\infty$ and define the numbers*

$$p_0 := \begin{cases} \dfrac{2(N_1+1)}{N_1-3}, & \text{if } N_1 > 3, \\ q, & \text{if } N_1 \leqslant 3 \end{cases}$$

and

$$q_0 := N - (N_2+1)\frac{2}{p_0} = \begin{cases} N - \dfrac{(N_2+1)(N_1-3)}{N_1+1}, & \text{if } N_1 > 3, \\ N - \dfrac{2(N_2+1)}{q}, & \text{if } N_1 \leqslant 3. \end{cases}$$

Denoting by B the unit ball of $\mathbb{R}^N$, define also the space

$$H^2_{x,y}(B) := \{u \in H^2(B) : u(x,y) = u(|x|,|y|), \; \forall\, (x,y) \in B\}.$$

Then the embedding $H^2_{x,y}(B) \hookrightarrow L^p(B,|z|^\gamma)$ is continuous for $1 \leqslant p \leqslant p_0$ and compact if $1 \leqslant p < p_0$, provided

$$\gamma > \frac{p_0 q_0}{2} = \begin{cases} \dfrac{N_1^2 + 4N_2 + 3}{N_1 - 3}, & \text{if } N_1 > 3, \\ \dfrac{Nq - 2(N_2+1)}{2}, & \text{if } N_1 \leqslant 3. \end{cases}$$

As a consequence, we have the following result related to the space $L^p_\ell(B)$ (*cf.* (3.3)):

Corollary 3.9. *With the same notation of Theorem 3.8, suppose that $2 \leqslant N_2 \leqslant N_1$, $2 < q < +\infty$ and*

$$\ell > \begin{cases} \dfrac{N_1^2 + 4N_2 + 3}{2(N_1 - 3)}, & \text{if } N_1 > 3, \\ \dfrac{Nq - 2(N_2+1)}{4}, & \text{if } N_1 \leqslant 3. \end{cases}$$

Then the embedding $H^2_{x,y}(B) \hookrightarrow L^p_\ell$ is compact for

$$\begin{cases} 1 \leqslant p < \dfrac{2(N_1+1)}{N_1-3}, & \text{if } N_1 > 3, \\ 1 \leqslant p < q, & \text{if } N_1 \leqslant 3. \end{cases}$$

Proof. Fist, for a given $\ell > 0$, as $|x| \leqslant |z|$ and $|y| \leqslant |z|$, then $(|x||y|)^{\ell} \leqslant |z|^{2\ell}$. This implies, by (W_1), for $1 \leqslant p < +\infty$

$$L^p(B, |z|^{2\ell}) \hookrightarrow L^p(B, (|x||y|)^{\ell}) \hookrightarrow L^p_{\ell}, \quad \forall\ \ell > 0.$$

Now the result follows from Theorem 3.8 with $\gamma = 2\ell$. □

Now we can apply Corollary 3.9 with $3 \leqslant N_2 \leqslant N_1$ to obtain results analogous to those in Theorem 3.2. More precisely, the immersion $H^2_{0,x,y}(B) \hookrightarrow L^p_{\ell}(B)$ is compact for

$$\begin{cases} 1 \leqslant p < \dfrac{4(\ell+2)}{6}, & \text{if } N_1 = N_2 = 3 \text{ and } \ell > 1, \\ 1 \leqslant p < \dfrac{2(N_1+1)}{N_1-3}, & \text{if } 3 \leqslant N_2 < N_1 \text{ and } \ell > \ell_* := \dfrac{N_1^2 + 4N_2 + 3}{2(N_1-3)}. \end{cases}$$

To verify that our results are sharper, we first consider the case $N_1 = N_2 = 3$. In this situation, Theorem 3.2 ensures the compact embedding for any $\ell > 0$ and $1 \leqslant p < 2^*_{\ell,N_1} = 6 + 2\ell$. Since

$$\frac{4(\ell+2)}{6} < 6 + 2\ell,$$

for all $\ell > 0$, we conclude that our result encompasses a strictly wider range. A similar improvement occurs when $3 \leqslant N_2 < N_1$, because

$$\frac{2(N_1+1)}{N_1-3} < 2^*_{\ell,N_1} \iff \ell > \frac{2(N_1-1)}{N_1-3},$$

and a straightforward computation shows that the inequality on the right-hand side is always satisfied whenever $\ell > \ell_*$. In other words, $\ell_* > 2(N_1-1)/(N_1-3)$ is equivalent to $N_1(N_1-4) + 4N_2 + 7 > 0$, which is always true.

3.3 Application

In this section, we prove Theorem 3.3. We first notice that, given $\varepsilon > 0$, we may use (f_1)-(f_3) to obtain

$$|F(s)| \leqslant \varepsilon |s|^2 + C_1 |s|^p, \quad \forall s \in \mathbb{R}.$$

Thus, for any $u \in H^2_{0,x,y}(B)$, we can use Theorem 3.2 to guarantee that

$$\int_B F(u)[W(z)]^{\ell}\,\mathrm{d}z \leqslant \varepsilon \|u\|^2_{L^2_{\ell}(B)} + C_1 \|u\|^p_{L^p_{\ell}(B)} < +\infty. \tag{3.9}$$

So, the functional

$$I(u) := \frac{1}{2}\|u\|^2_{H_0^2(B)} - \int_B F(u(z))[W(z)]^\ell \,\mathrm{d}z, \qquad u \in H^2_{0,x,y}(B),$$

is well defined. Moreover, standard computations shows that $I \in C^1(H^2_{0,x,y}(B), \mathbb{R})$ and its critical points are the weak solution of problem ($P_{x,y}$).

By using (3.9), (W_1), and Theorem 3.2 again, we obtain

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\|u\|^2_{H_0^2(B)} - \varepsilon\|u\|^2_{L^2_\ell(B)} - C_1\|u\|^p_{L^p_\ell(B)} \\ &\geqslant \frac{1}{2}\|u\|^2_{H_0^2(B)}\left(1 - C_2\varepsilon - 2C_3\|u\|^{p-2}_{H_0^2(B)}\right). \end{aligned}$$

Since $p > 2$, we can choose $\varepsilon > 0$ sufficiently small to obtain constants $\rho, \beta > 0$ such that

$$I(u) \geqslant \beta, \qquad \forall\, u \in H^2_{0,x,y}(B),\ \|u\|_{H_0^2(B)} = \rho.$$

Moreover, using (f_1), (f_2), and (f_4), we obtain $C_4 > 0$ such that

$$F(s) \geqslant C_4|s|^\mu - C_4, \quad \forall\, s \in \mathbb{R}.$$

Then, choosing a positive function $u_0 \in H^2_{0,x,y}(B)$, we find that

$$I(su_0) \leqslant \frac{s^2}{2}\|u_0\|^2_{H_0^2(B)} - C_4 s^\mu \int_B |u_0|^\mu [W(z)^\ell] \,\mathrm{d}z - C_5.$$

Since $\mu > 2$, it follows that $\lim\limits_{s\to+\infty} I(su_0) = -\infty$. Therefore, there exists $e \in H^2_{0,x,y}(B)$ such that $I(e) \leqslant 0$ and $\|e\|_{H_0^2(B)} > \rho$.

According to the above considerations, we can define

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \geqslant \beta > 0,$$

with $\Gamma := \{\gamma \in C([0,1], H^2_{0,x,y}(B)) : \gamma(0) = 0,\ \gamma(1) = e\}$, and invoke the Mountain Pass Theorem (*cf.* [5]) to obtain a sequence $(u_n) \subset H^2_{0,x,y}(B)$ such that

$$\lim_{n\to+\infty} I(u_n) = c > 0, \qquad \lim_{n\to+\infty} I'(u_n) = 0. \tag{3.10}$$

From the above convergences and (f_4), we get

$$
\begin{aligned}
C_6 &\geqslant I(u_n) - \frac{1}{\mu} I'(u_n)(u_n) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2_{H_0^2(B)} + \int_B \left[\frac{1}{\mu} f(u_n)u_n - F(u_n)\right] [W(z)]^\ell \, \mathrm{d}z \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2_{H_0^2(B)} + \int_{\{|u_n| \leqslant s_0\}} \left[\frac{1}{\mu} f(u_n)u_n - F(u_n)\right] [W(z)]^\ell \, \mathrm{d}z.
\end{aligned}
$$

Since f is continuous, we can find $C_7 > 0$ such that

$$
\left|\frac{1}{\mu} f(s)s - F(s)\right| \leqslant C_7, \quad \forall\ |s| \leqslant s_0,
$$

which implies

$$
\left|\int_{\{|u_n| \leqslant s_0\}} \left[\frac{1}{\mu} f(u_n)u_n - F(u_n)\right] [W(z)]^\ell \, \mathrm{d}z\right| \leqslant C_7 \int_B [W(z)]^\ell \, \mathrm{d}z =: C_8.
$$

Thus

$$
C_6 \geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2_{H_0^2(B)} - C_8,
$$

and it follows from $\mu > 2$ and (f_1) that the sequence (u_n) is bounded.

Let $u \in H^2_{0,x,y}(B)$ be the weak limit of a subsequence of (u_n). We aim to prove that, along a subsequence, $u_n \to u$ strongly in $H^2_{0,x,y}(B)$. If this is true, then by the regularity of I and (3.10), we conclude that $u \neq 0$ is a weak solution of problem $(P_{x,y})$.

To prove the strong convergence, we compute

$$
I'(u_n)(u_n - u) = \|u_n\|^2_{H_0^2(B)} - \langle u_n, u\rangle - A_n, \tag{3.11}
$$

where

$$
A_n := \int_B f(u_n)(u_n - u)[W(z)]^\ell \, \mathrm{d}z.
$$

Using the boundedness of B, (W_1), and (f_2), we obtain

$$
|A_n| \leqslant c_f \int_B (1 + |u_n|^{p-1})|u_n - u|[W(z)]^\ell \, \mathrm{d}z \leqslant c_f(D_n + E_n),
$$

where

$$
D_n := \int_B |u_n - u|[W(z)]^\ell \, \mathrm{d}z, \qquad E_n := \int_B |u_n|^{p-1}|u_n - u|[W(z)]^\ell \, \mathrm{d}z.
$$

Clearly, $D_n \to 0$ as $n \to +\infty$, because W is bounded in B and $u_n \to u$ in $L^1(B)$. Furthermore, using Hölder's inequality,

$$\begin{aligned} E_n &= \int_B \left(|u_n|^{p-1}[W(z)]^{\ell/p'}\right)\left(|u_n - u|[W(z)]^{\ell/p}\right) \mathrm{d}z \\ &\leqslant \left(\int_B |u_n|^{p'(p-1)}[W(z)]^{\ell} \,\mathrm{d}z\right)^{1/p'} \left(\int_B |u_n - u|^p [W(z)]^{\ell} \,\mathrm{d}z\right)^{1/p} \\ &= \left(\int_B |u_n|^{p}[W(z)]^{\ell} \,\mathrm{d}z\right)^{1/p'} \left(\int_B |u_n - u|^p [W(z)]^{\ell} \,\mathrm{d}z\right)^{1/p} \end{aligned}$$

and therefore

$$E_n \leqslant \|u_n\|_{L^p_\ell(B)}^{\frac{p}{p'}} \|u_n - u\|_{L^p_\ell(B)} \leqslant C_7 \|u_n\|_{H^2_0(B)}^{\frac{p}{p'}} \|u_n - u\|_{L^p_\ell(B)}.$$

Since the embedding $H^2_{0,x,y}(B) \hookrightarrow L^p_\ell(B)$ is compact, we conclude that $E_n \to 0$.

Altogether, these estimates show that $A_n \to 0$ as $n \to +\infty$. Therefore, using the second convergence in (3.10), (3.11), and the weak convergence, we obtain

$$\lim_{n\to+\infty} \|u_n\|^2_{H^2_0(B)} = \|u\|^2_{H^2_0(B)}.$$

This implies that $u_n \to u$ strongly in $H^2_{0,x,y}(B)$, completing the proof.

Chapter 4

Kirchhoff-Boussinesq equation with Hénon nonlinearity

In this chapter, we will study the existence and multiplicity of radial solutions for the following class of problems

$$\begin{cases} \Delta^2 u \pm \operatorname{div}(|x|^\kappa |\nabla u|^{p-2}\nabla u) = |x|^\ell f(u), & \text{in } B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \qquad (\mathcal{P}_\pm)$$

where $\ell > 0$, $\kappa \geqslant 0$, $B \subset \mathbb{R}^N$ is the unit ball and p, $q > 2$. The nonlinear term f satisfies the following assumptions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) there exists $\ell > 0$,

$$2 < q < 2_\ell^{**} := 2^{**} + \frac{2\ell}{N-4}$$

and $C > 0$ such that

$$|f(s)| \leqslant C\left(1 + |s|^{q-1}\right), \quad \forall\, s \in \mathbb{R};$$

(f_3) there holds

$$\lim_{s \to 0} \frac{f(s)}{s} = 0;$$

(f_4) there exists $\mu > p$ and $s_0 > 0$ such that

$$0 < \mu F(s) \leqslant s f(s), \quad \forall\, |s| \geqslant s_0,$$

where $F(s) := \displaystyle\int_0^s f(\tau)d\tau$.

We remark that the growth of f in (f_2) may be supercritical, since $2_\ell^{**} > 2^{**}$ for any $\ell > 0$.

The first main result of this chapter is:

Theorem 4.1. *Suppose that $\ell > 0$, $N > 4$ and that f satisfies (f_1)-(f_4). Then there exists $\kappa_* = \kappa_*(\ell, N) > 0$ such that, for any $\kappa \in [0, \kappa_*)$, Problem $(\mathcal{P}_\pm)$ has a nonzero radial weak solution provided*

$$2 < p < 2^*_\kappa := 2^* + \frac{2\kappa}{N-2}, \qquad p < q.$$

We observe that $p < q$ is always true in view of (f_2) and (f_4). As we are assuming $2 < q < 2^{**}_\ell$, in order to apply the above theorem, we also need $2 < p < 2^*_\kappa$ and $p < q$. Therefore, it is desirable to ensure that the inequality $2^*_\kappa < 2^{**}_\ell$ holds, so that the admissible range for q is non-empty for any choice of p. We prove in Proposition 4.5 that this condition is indeed satisfied.

In our second application, we prove that under symmetric conditions on f we can obtain multiple solutions. More specifically, the following holds:

Theorem 4.2. *Let $\kappa_* > 0$ be given by Theorem 4.1. Suppose that $\ell > 0$, $N > 4$, $\kappa \in [0, \kappa_*)$, $2 < p < 2^*_\kappa$, $2 < q < 2^{**}_\ell$ and f is an odd function satisfying (f_1), (f_2) and (f_4). Then Problem $(\mathcal{P}_\pm)$ admits infinity many radial weak solutions.*

This chapter is structured as follows: Section 4.1 presents the variational framework, where several technical results are provided and proved. In Section 4.2, Theorem 4.1 is proved, while Section 4.3 is dedicated to the proof of Theorem 4.2.

4.1 The variational framework

For $1 \leqslant q < +\infty$, we consider the following weighted Lebesgue space

$$L^q_\ell := \left\{ u : B \to \mathbb{R} : u \text{ is measurable and } \int_B |u(x)|^q |x|^\ell \, \mathrm{d}x < \infty \right\},$$

endowed with the norm

$$\|u\|_{L^q_\ell} := \left(\int_B |u(x)|^q |x|^\ell \, \mathrm{d}x \right)^{1/q}.$$

Analogously, we just write L^q for the usual Lebesgue space $L^q(B)$.

We state in what follows a Gagliardo-Nirenberg interpolation inequality with weights due to [68] (*cf.* [33, 61]):

Theorem 4.3. *Let* $p, q, r \in [1, +\infty)$, $\theta \in [1/2, 1]$, $(1 - N/r) \notin \mathbb{N} \cup \{0\}$ *and* α, β, $\gamma \in \mathbb{R}$ *be such that* $\alpha > -N/r$, $\beta > -N/q$, $\gamma > -N/p$ *and*

$$\begin{cases} \dfrac{1}{p} - \dfrac{1-\gamma}{N} = \theta\left(\dfrac{1}{r} - \dfrac{2-\alpha}{N}\right) + (1-\theta)\left(\dfrac{1}{q} + \dfrac{\beta}{N}\right) \\ 0 \leqslant \theta\alpha + (1-\theta)\beta - \gamma \leqslant 2\theta - 1. \end{cases}$$

Then there exists $C = C(N, p, q, \theta, \gamma, \alpha, \beta) > 0$ *such that*

$$\left\| |x|^\gamma \nabla u \right\|_{L^p(\mathbb{R}^N)} \leqslant C \left\| |x|^\alpha \nabla^2 u \right\|^\theta_{L^r(\mathbb{R}^N)} \left\| |x|^\beta u \right\|^{1-\theta}_{L^q(\mathbb{R}^N)}, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N).$$

As a consequence, we prove:

Theorem 4.4. *Suppose that* $N > 4$ *and* $\ell > 0$. *Then there exists* $\kappa_* = \kappa_*(\ell, N) > 0$ *such that, for any* $\kappa \in [0, \kappa_*)$ *and* $2 \leqslant p < 2^*_\kappa$ *there holds*

$$\|\nabla u\|_{L^p_\kappa(\mathbb{R}^N)} \leqslant C \|\nabla^2 u\|^{\theta_*}_{L^2(\mathbb{R}^N)} \|u\|^{1-\theta_*}_{L^{q_*}_\ell(\mathbb{R}^N)}, \qquad \forall\, u \in C_0^\infty(\mathbb{R}^N),$$

with

$$\theta_* := \frac{2\ell p + 2p^2 + 2\kappa(\ell - \kappa)}{4p^2 + \ell p^2 + p\kappa(4+\ell)}, \qquad q_* := \frac{p(2+\ell) - 2\ell + 2\kappa}{4 - p + \kappa}, \tag{4.1}$$

where $C > 0$ *is a constant independent of* u. *In addition,* $\theta_* \in [1/2, 1)$ *and* $q_* \in [2, 2^{**}_\ell)$.

Proof. Let $\ell > 0, \kappa \geqslant 0$ and $2 \leqslant p < 2^*_\kappa$ be fixed. The idea is to apply Theorem 4.3 with $\alpha = 0$, $\gamma = \kappa/p$, $\beta = \ell/q_*$, $r = 2$ and θ_*, q_* as in the statement. To do this, we first show that, with this choice of the parameters, the couple (θ_*, q_*) verifies

$$\begin{cases} \dfrac{1}{p} - \dfrac{1-\gamma}{N} = \theta_*\left(\dfrac{1}{r} - \dfrac{2-\alpha}{N}\right) + (1-\theta_*)\left(\dfrac{1}{q_*} + \dfrac{\beta}{N}\right) \\ \theta_*\alpha + (1-\theta_*)\beta - \gamma = 2\theta_* - 1. \end{cases}$$

Actually, replacing the values of α, γ, β and r, we can rewrite the above system as

$$\begin{cases} \dfrac{N - p + \kappa}{pN} = \theta_* \dfrac{(N-4)}{2N} + (1-\theta_*)\dfrac{(N+\ell)}{q_* N} \\ (1-\theta_*)\dfrac{\ell}{q_*} - \dfrac{\kappa}{p} = 2\theta_* - 1. \end{cases} \tag{4.2}$$

From the second equation, we obtain

$$\theta_* = \frac{\ell p + q_*(p-\kappa)}{p(2q_* + \ell)}. \tag{4.3}$$

After substitute this expression in the first equation of (4.2) and make a lot of calculations, we get

$$N - p + \kappa = \frac{[\ell p + q_*(p-\kappa)](N-4) + 2(p+\kappa)(N+\ell)}{2(2q_* + \ell)},$$

from which we obtain

$$q_*(4 - p + \kappa) = \ell p + 2p + 2\kappa - 2\ell. \tag{4.4}$$

Since $N > 4$, one has $2^* < 4$, and therefore

$$p \leqslant 2^* + \frac{2\kappa}{N-2} < 4 + \frac{2\kappa}{N-2} < 4 + \kappa,$$

that is, $(4 - p + \kappa) > 0$. So, we can go back to (4.4) to obtain the expression of q_* in (4.1). Replacing this value in (4.3) and performing some calculations, we conclude that θ_* needs to be as in (4.1).

We prove in what follows that all the requirements for applying Theorem 4.3 hold true. The number $p \in [2, 2^*_\kappa)$ is fixed.

Claim 1: $\theta_* \geqslant 1/2$, is $\kappa \geqslant 0$ is sufficiently small.

In fact, using the definition of θ_*, we can see that the claimed inequality is equivalent to

$$h_1(p) := -\ell p^2 + [4\ell - \kappa(4+\ell)]p + 4s(\ell - \kappa) \geqslant 0, \quad \forall\, p \in [2, 2^*_\kappa].$$

The derivative of h_1 vanishes in $p_1 := [4\ell - \kappa(4+\ell)]/(2\ell)$, and therefore we can use $\ell > 0$ to conclude that p_1 is decreasing in $(p_1, +\infty)$. Since an easy calculation shows that $p_1 < 2$, it is sufficient to check that $h_1(2^*_\kappa) \geqslant 0$.

We have that

$$\begin{aligned}
h_1\left(2^*_\kappa\right) &= -\ell\left(2^* + \frac{2\kappa}{N-2}\right)^2 + [4\ell - \kappa(4+\ell)]\left(2^* + \frac{2\kappa}{N-2}\right) + 4\kappa(\ell-\kappa)\\
&= -\ell\left(\frac{2N+2\kappa}{N-2}\right)^2 + [4\ell - \kappa(4+\ell)]\left(\frac{2N+2\kappa}{N-2}\right) + 4\kappa(\ell-\kappa)\\
&= \frac{-\ell(2N+2\kappa)^2 + [4\ell - \kappa(4+\ell)](2N+2\kappa)(N-2) + 4\kappa(\ell-\kappa)(N-2)^2}{(N-2)^2}\\
&= \frac{-\ell(4N^2 + 8N\kappa + 4\kappa^2) + [8\ell N + 8\ell\kappa - 2N(4+\ell)\kappa - 2(4+\ell)\kappa^2](N-2)}{(N-2)^2}\\
&+ \frac{[4\kappa\ell - 4\kappa^2](N-2)^2}{(N-2)^2}
\end{aligned}$$

Thus

$$\begin{aligned} h_1\left(2^*_\kappa\right) = & \frac{(-4\ell - 2(4+\ell)(N-2) - 4(N-2)^2)\kappa^2}{(N-2)^2} \\ & + \frac{[-8\ell N + (8\ell - 2N(4+\ell))(N-2) + 4\ell(N-2)^2]\kappa}{(N-2)^2} \\ & + \frac{-4\ell N^2 + 8\ell N(N-2)}{(N-2)^2} \end{aligned}$$

from which we conclude that

$$h_1\left(2^*_\kappa\right) = -\frac{a(\ell,N)}{(N-2)^2}\kappa^2 + \frac{b(\ell,N)}{(N-2)^2}\kappa + \frac{c(\ell,N)}{(N-2)^2}$$

with

$$\begin{cases} a(\ell,N) := 4N(N-2) + 2\ell N > 0, \\ b(\ell,N) := 16N - 8N^2 - 12\ell N + 2\ell N^2, \\ c(\ell,N) := 4\ell N(N-4) > 0. \end{cases} \tag{4.5}$$

Since $a(\ell,N) > 0$ and $c(\ell,N) > 0$, the roots of the quadratic polynomial $h_1(2^*_\kappa)$ are given by

$$\kappa_\mp := \frac{b(\ell,N) \mp \sqrt{[b(\ell,N)]^2 + 4a(\ell,N)c(\ell,N)}}{2a(\ell,N)}.$$

Notice that $\kappa_- < 0 < \kappa_+$. So, if we define

$$\kappa_* := \frac{b(\ell,N) + \sqrt{[b(\ell,N)]^2 + 4a(\ell,N)c(\ell,N)}}{2a(\ell,N)},$$

we have that $h_1(2^*_\kappa) \geqslant 0$, whenever $\kappa \in [0,\kappa_*)$, and Claim 1 is proved.

Claim 2: $\theta_* < 1$, if $\kappa \in [0,\kappa_*)$.

Arguing as in the former claim, we notice that $\theta_* < 1$ is equivalent to

$$h_2(p) := (\ell+2)p^2 + [\kappa(4+\ell) - 2\ell]p - 2\kappa(\ell-\kappa) > 0, \quad \forall\, p \in [2, 2^*_\kappa).$$

The derivative of h_2 vanishes in $p_2 = [2\ell - \kappa(4+\ell)]/[2(\ell+2)]$ and therefore we can use $\ell > 0$ to conclude that p_2 is increasing in $(p_2, +\infty)$. Since an easy calculation shows that $p_2 \leqslant 2$, it is sufficient to check that $h_2(2) > 0$. But this clearly holds true, because

$$h_2(2) = 2(\kappa^2 + 4\kappa + 4) = 2(\kappa+2)^2$$

and therefore Claim 2 is verified.

We finally note that, for any $\kappa \in [0, \kappa_*)$, there holds

$$(1-\theta_*)\frac{\ell}{q_*} - \frac{\kappa}{p} = 2\theta_* - 1 \geqslant 0$$

and therefore the first statement of the theorem is a direct consequence of Theorem 4.3.

It remains to check that $q_* \in [2, 2_\ell^{**})$. To do this, we first notice $q_* < 2_\ell^{**}$ is equivalent to

$$[p(2+\ell) - 2\ell + 2\kappa](N-4) < (2N+2\ell)(4-p+\kappa),$$

that is,

$$4pN + p\ell N - 2\ell N - 8p - 8\kappa < 8N + 2\ell p + 2\ell\kappa.$$

But this is the same as

$$p(N-2)(4+\ell) < 2N(4+\ell) + 2\kappa(4+\ell).$$

Dividing both sides of last inequality by $(4+\ell)$, we can see the above inequality is equivalent to $p < 2_\kappa^*$, which holds true. Finally, $q_* \geqslant 2$ is equivalent to

$$2(4-p+\kappa) \leqslant p(2+\ell) - 2\ell + 2\kappa,$$

in other words,

$$2(\ell+4) \leqslant p(\ell+4),$$

which is also true. The theorem is proved. □

We now verify that, for any choice of $p \in [2, 2_\kappa^*)$ in Theorem 4.1, the range of possible values for q is non-empty.

Proposition 4.5. *Under the hypotheses of Theorem 4.1, we have that* $2_\kappa^* < 2_\ell^{**}$.

Proof. We first recall that

$$2_\kappa^* = \frac{2(N+\kappa)}{N-2}, \qquad 2_\ell^{**} = \frac{2(N+\ell)}{N-4}$$

and so $2_\kappa^* < 2_\ell^{**}$ is equivalent to

$$\kappa(N-4) < \ell(N-2) + 2N.$$

The above expression is trivially true if $\kappa < \ell$. Recalling that

$$0 < \kappa < \kappa_* = \frac{b(\ell,N) + \sqrt{[b(\ell,N)]^2 + 4a(\ell,N)c(\ell,N)}}{2a(\ell,N)},$$

where the above quantities where defined in (4.5), it is sufficient to verify that the right-hand side of the above expression is less than ℓ.

We first observe that $\kappa_* < \ell$ is equivalent to

$$\sqrt{[b(\ell,N)]^2 + 4a(\ell,N)c(\ell,N)} < 6N^2\ell + 4\ell^2 N - 4N\ell + 8N^2 - 16N. \tag{4.6}$$

Moreover,

$$\begin{aligned}[b(\ell,N)]^2 = {} & 4\ell^2N^4 + 64N^4 + 256N^2 + 144\ell^2N^2 - 32\ell N^4 \\ & + 256\ell N^3 - 48\ell^2N^3 - 256N^3 - 384\ell N^2,\end{aligned}$$

and

$$4a(\ell,N)c(\ell,N) = 64\ell N^4 + 32\ell^2N^3 - 384\ell N^3 - 128\ell^2N^2 + 512\ell N^2.$$

Thus,

$$\begin{aligned}[b(\ell,N)]^2 + 4a(\ell,N)c(\ell,N) = {} & 4\ell^2N^4 + 32\ell N^4 + 64N^4 - 16\ell^2N^3 - 256N^3 \\ & - 128\ell N^3 + 256N^2 + 128\ell N^2 + 16\ell^2N^2.\end{aligned}$$

We may now square both sides of (4.6) and perform straightforward (though lengthy) algebraic manipulations to obtain that $\kappa_* < \ell$ is equivalent to

$$0 \leqslant 64N^3\ell(N-2) + 32N^2\ell^2(N^2+N-4) + 16N^2\ell^3(3N-2) + 16N^2\ell^4,$$

which is clearly true, since $N > 4$. □

We finish this section by presenting an embedding proved in [20], which will be crucial in next section:

Theorem 4.6. *Let $H_r^2(B)$ be the set of all functions of $H^2(B)$ which are radial. Then each function $u \in H_r^2(B)$ is a.e. equal to a function $\overline{u} \in C^1(\overline{B} - \{0\})$. Moreover, for any $i, j = 1, ..., N$, the derivative $\overline{u}_{x_ix_j}(x)$ exists a.e. for $|x| \in (0,1)$. Also, there exists a positive constant C such that*

$$|\overline{u}(x)| \leqslant C\frac{\|\overline{u}\|_{H^2(B)}}{|x|^{(N-4)/2}}, \quad \forall\, x \in \overline{B} - \{0\}.$$

Finally, for any $\ell > 0$, the embedding $H_r^2(B) \hookrightarrow L_\ell^q$ is continuous if $q \in [1, 2_\ell^{**}]$ and compact if $q \in [1, 2_\ell^{**})$.

4.2 Existence of a solution

We begin this section by defining the workspace to deal with Problem ($\mathcal{P}_\pm$). Let $C^\infty_{0,r}(B)$ be the space of all radially symmetric functions in $C^\infty_0(B)$ and denote

$$X := \overline{C^\infty_{0,r}(B)}^{\|\cdot\|}, \qquad \|u\| := \left(\int_B |\Delta u|^2 \, \mathrm{d}x\right)^{1/2}.$$

It is a Hilbert space with inner product given by

$$\langle u, v\rangle := \int_B \Delta u \Delta v \, \mathrm{d}x, \qquad \forall\, u,\, v \in X.$$

In order to define the energy functional associated to our problem, we pick $\varepsilon > 0$ and use (f_1)-(f_3) to obtain

$$|F(s)| \leqslant \frac{\varepsilon}{2}|s|^2 + C_1|s|^q, \quad \forall\, s \in \mathbb{R}.$$

Hence, since $2 < q < 2^{**}_\ell$, we can use Theorem 4.6 to obtain

$$\int_B |x|^\ell F(u)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}\|u\|^2_{L^2} + C_1\|u\|^q_{L^q_\ell} \leqslant C_2\frac{\varepsilon}{2}\|u\|^2 + C_3\|u\|^q < \infty. \tag{4.7}$$

Also, since $2 < p < 2^*_\kappa$, it follows from Theorems 4.4 and 4.6 that

$$\int_B |x|^\kappa |\nabla u|^p \,\mathrm{d}x \leqslant C_4\|u\|^{\theta_*}\|u\|^{1-\theta_*}_{L^{q*}_\ell} \leqslant C_5\|u\|^{\theta_*}\|u\|^{1-\theta_*} = C_5\|u\| < \infty. \tag{4.8}$$

So, it is well defined $I_\pm : X \to \mathbb{R}$ by

$$I_\pm(u) := \frac{1}{2}\|u\|^2 \pm \frac{1}{p}\int_B |x|^\kappa|\nabla u|^p\,\mathrm{d}x - \int_B |x|^\ell F(u)\,\mathrm{d}x, \qquad u \in X.$$

Moreover, standard arguments shows that $I_\pm \in C^1(X, \mathbb{R})$ with

$$I'_\pm(u)\varphi = \int_B \Delta u \Delta\varphi \, \mathrm{d}x \pm \int_B |x|^\kappa|\nabla u|^{p-2}[\nabla u\cdot\nabla\varphi]\,\mathrm{d}x - \int_B |x|^\ell f(u)\varphi\,\mathrm{d}x,$$

for all u, $\varphi \in X$. Thus, the critical points of $I_\pm$ are precisely the weak solution to our problem. We shall obtain these critical as an application of the Mountain Pass Theorem (*cf.* [5]).

We recall that $I_\pm$ satisfies the $(PS)_d$ condition at level $d \in \mathbb{R}$ if any sequence $(u_n) \subset X$ such that

$$\lim_{n\to+\infty} I_\pm(u_n) = d, \qquad \lim_{n\to+\infty} I'_\pm(u_n) = 0 \tag{4.9}$$

has a convergent subsequence.

Proposition 4.7. *Suppose that $\ell > 0$, $N > 4$ and that f satisfies (f_1),(f_2) and (f_4). Then the functional $I_\pm$ satisfies the $(PS)_d$ condition at any level $d \in \mathbb{R}$.*

Proof. Let $(u_n) \subset X$ be as in (4.9). Since $\mu > p$ in (f_4), we have that

$$0 < pF(s) \leqslant sf(s), \quad \forall\ |s| \geqslant s_0,$$

from which we obtain

$$\begin{aligned} I_\pm(u_n) - \frac{1}{p} I'_\pm(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \int_B |x|^\ell \left[\frac{f(u_n)u_n}{p} - F(u_n)\right] \mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \int_{\{|u_n| \leqslant s_0\}} |x|^\ell \left[\frac{f(u_n)u_n}{p} - F(u_n)\right] \mathrm{d}x. \end{aligned}$$

From (f_1) we obtain a constant $M > 0$ verifying

$$\left|\frac{f(s)s}{p} - F(s)\right| \leqslant M, \quad \forall\ |s| \leqslant s_0.$$

Thus, there exists $C_1 > 0$ such that

$$I_\pm(u_n) - \frac{1}{p} I'_\pm(u_n)u_n \geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - C_1.$$

From the convergences in (4.9) we obtain

$$I_\pm(u_n) - \frac{1}{p} I'_\pm(u_n)u_n = d + o_n(1) + o_n(1)\|u_n\|,$$

where $o_n(1)$ denotes a sequence approaching to zero as $n \to +\infty$. Hence,

$$d + o_n(1) + o_n(1)\|u_n\| \geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - C_1.$$

Last inequality combined with $p > 2$ shows that (u_n) is bounded. Thus, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X.

Using (4.9) again, we obtain

$$\begin{aligned} o_n(1) = I'_\pm(u_n)(u_n - u) = \langle u_n, u_n - u\rangle \pm \int_B |x|^\kappa |\nabla u_n|^{p-2}[\nabla u_n \cdot \nabla(u_n - u)]\,\mathrm{d}x \\ - \int_B |x|^\ell f(u_n)(u_n - u)\,\mathrm{d}x \end{aligned}$$

and therefore

$$o_n(1) = \|u_n\|^2 - \langle u_n, u\rangle + a_n + b_n, \tag{4.10}$$

where

$$a_n := \pm \int_B |x|^\kappa |\nabla u_n|^{p-2}[\nabla u_n \cdot \nabla(u_n - u)]\,\mathrm{d}x$$

and

$$b_n := -\int_B |x|^\ell f(u_n)(u_n - u)\,\mathrm{d}x.$$

Hölder Inequality, Theorems 4.4 and 4.6 and the boundedness of (u_n) yields

$$\begin{aligned}
|a_n| &\leqslant \int_B |x|^{\frac{\kappa}{p'}}|\nabla u_n|^{p-1}|x|^{\frac{\kappa}{p}}|\nabla(u_n - u)|\,\mathrm{d}x \\
&\leqslant \|\nabla u_n\|_{L^p_\kappa}^{p/p'}\|\nabla(u_n - u)\|_{L^p_\kappa} \\
&\leqslant C_1\|u_n\|^{\theta_* p/p'}\|u_n - u\|_{L^{q*}_\ell}^{1-\theta_*} \\
&\leqslant C_2\|u_n - u\|_{L^{q*}_\ell}^{1-\theta_*} = o_n(1),
\end{aligned}$$

where we have used $q_* \in [2, 2^{**}_\ell)$ and the weak convergence in the last equality.

Analogously, using (f_2), we obtain

$$\begin{aligned}
|b_n| &\leqslant C\left[\|u_n - u\|_{L^1} + \int_B |x|^{\frac{\ell}{q'}}|u_n|^{q-1}|x|^{\frac{\ell}{q}}|u_n - u|\right] \\
&\leqslant C\left[\|u_n - u\|_{L^1} + \|u_n\|_{L^q_\ell}^{\frac{q}{q'}}\|u_n - u\|_{L^q_s}\right] = o_n(1).
\end{aligned}$$

Replacing the above estimates in (4.10) and recalling the weak convergence of (u_n), we get

$$o_n(1) = \|u_n\|^2 - \langle u_n, u\rangle + o_n(1) = \|u_n\|^2 - \|u\|^2 + o_n(1),$$

and therefore $u_n \to u$ strongly in X. □

We are ready to prove the existence result.

Proof of Theorem 4.1 . For all $u \in X$, we can use (4.7) with $\varepsilon = 1/(2C_2)$ and (4.8), to get

$$I_-(u) \geqslant \frac{1}{4}\|u\|^2 - \frac{C_6}{p}\|u\|^p - C_3\|u\|^q = \|u\|^2\left(\frac{1}{4} - \frac{C_6}{p}\|u\|^{p-2} - C_3\|u\|^{q-2}\right)$$

and

$$I_+(u) \geqslant \frac{1}{4}\|u\|^2 - C_3\|u\|^q = \|u\|^2\left(\frac{1}{4} - C_3\|u\|^{q-2}\right).$$

Since $p, q > 2$, we can find β, $\rho > 0$ such that

$$I_\pm(u) \geqslant \beta, \qquad \forall\, u \in X \cap \partial B_\rho(0).$$

Moreover, using (f_1) and (f_4), we obtain C_7, $C_8 > 0$ such that

$$F(s) \geqslant C_7|s|^\mu - C_8, \qquad \forall\, s \in \mathbb{R}. \tag{4.11}$$

Since $2^*_\kappa < 2^{**}_\ell$ (*cf.* Proposition 4.5) and $p \in [2, 2^*_\kappa)$, we can assume in (f_4), without loss of generality, that $p < \mu < 2^{**}_\ell$. So, for any $u \in X - \{0\}$ fixed, we can use Theorem 4.6 in order to get

$$I_-(tu) \leqslant \frac{t^2}{2}\|u\|^2 - C_9 t^\mu \|u\|^\mu + C_{10}$$

and, by Theorems 4.4 and 4.6,

$$I_+(tu) \leqslant \frac{t^2}{2}\|u\|^2 + C_{11}t^p\|u\|^p - C_9 t^\mu \|u\|^\mu + C_{10}.$$

Recalling that $2 < p < \mu$, we conclude that $I_\pm(tu) \to -\infty$, as $t \to +\infty$. Thus, there exists $e_\pm \in X$ such that $I(e_\pm) \leqslant 0$ and $\|e_\pm\| > \rho$.

The above considerations show that it is well defined

$$c_\pm := \inf_{\gamma \in \Gamma_e} \max_{t \in [0,1]} I_\pm(\gamma(t)) \geqslant \beta > 0,$$

where $\Gamma_{e_\pm} := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e_\pm\}$. It follows from the Mountain Pass Theorem (*cf.* [5]) that there exists $(u_n) \subset X$ such that

$$\lim_{n \to +\infty} I_\pm(u_n) = c_\pm, \qquad \lim_{n \to +\infty} I'_\pm(u_n) = 0.$$

It follows from Proposition 4.7 that, along a subsequence, $u_n \to u$ strongly in X. Using the regularity of $I_\pm$ we conclude that $I \pm (u) = c_\pm \geqslant \beta > 0$ and $I'_\pm(u) = 0$, that is, $u \in X$ is a nonzero solution to Problem $(\mathcal{P}_\pm)$. □

4.3 Multiplicity of solutions

In this section, we prove Theorem 4.2. The idea is take advantage of the symmetry of the even functional. In order to construct the appropriated linking structure, we need some background on the spectral theory of the biharmonic operator involving the weight $|x|^\ell$.

For $\lambda \in \mathbb{R}^+$, we consider the fourth-order eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda |x|^\ell u, & \text{in } B, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial B, \end{cases} \tag{4.12}$$

If it admits a nonzero weak solution $u \in X$, then λ is called an eigenvalue and u a $\lambda|x|^\ell$-eigenfunction. The set of all eigenvalues is called the spectrum of $(\Delta^2, |x|^\ell)$ in X and it is denoted by $\sigma(\Delta^2, |x|^\ell)$.

Using the compact embedding $X \hookrightarrow L^2_\ell$, we can prove that the smallest eigenvalue $\lambda_1(B)$ of problem (4.12) is exactly

$$\lambda_1(B) := \inf_{u \in X \setminus \{0\}} \left\{ \frac{\displaystyle\int_B |\Delta u|^2 \, \mathrm{d}x}{\displaystyle\int_B |x|^\ell |u|^2 \, \mathrm{d}x} \right\}.$$

Moreover, from the spectral theory of self-adjoint compact operators, we obtain a complete sequence of eigenvalues

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$$

such that $\lambda_k \to +\infty$, as $k \to +\infty$.

For any $i \in \{1, 2, \cdots, k\}$, we denote by φ_i a λ_i-eigenfunction and define the subspaces

$$V_k := \operatorname{span}\{\varphi_1, ..., \varphi_k\}, \qquad W_k := V_k^\perp.$$

We have the orthogonal decomposition $X = V_k \oplus W_k$ and, for any $k \in \mathbb{N}$, the following holds

$$\|u\|^2_{L^2_\ell} \leqslant \frac{1}{\lambda_{k+1}} \|u\|^2, \quad \forall\, u \in W_k. \tag{4.13}$$

The following technical result will be essential in the proof of Theorem 4.2.

Lemma 4.8. *Suppose that $2 < r < 2^{**}_\ell$ and $k \in \mathbb{N}$. Then there exists $\alpha \in (0,1)$ and $C > 0$, independent of k, such that*

$$\|u\|^r_{L^r_\ell} \leqslant \frac{C}{\lambda_{k+1}^{(1-\alpha)}} \|u\|^r, \quad \forall\, u \in W_k.$$

Proof. Since $2 < r < 2_\ell^{**}$, there exist $\alpha \in (0,1)$, such that $r = 2(1-\alpha) + 2_\ell^{**}\alpha$. Thus, we can use Hölder inequality for $t = 1/\alpha$ and $t' = 1/(1-\alpha)$ and Theorem 4.6 to get

$$\begin{aligned}\int_B |x|^\ell |u|^r \, \mathrm{d}x &= \int_B |x|^{\frac{\ell}{t'}} |u|^{(1-\alpha)2} |x|^{\frac{\ell}{t}} |u|^{\alpha 2_\ell^{**}} \, \mathrm{d}x \\ &\leqslant \left(\int_B |x|^\ell |u|^2 \, \mathrm{d}x \right)^{1/t'} \left(\int_B |x|^\ell |u|^{2_\ell^{**}} \, \mathrm{d}x \right)^{1/t} \\ &= \|u\|_{L_\ell^2}^{2(1-\alpha)} \|u\|_{L_\ell^{2_\ell^{**}}}^{2_\ell^{**}\alpha} \\ &\leqslant C \|u\|_{L_\ell^2}^{2(1-\alpha)} \|u\|^{2_\ell^{**}\alpha}.\end{aligned}$$

But the second variational inequality in (4.13) implies that

$$\|u\|_{L_\ell^2}^{2(1-\alpha)} \leqslant \frac{1}{\lambda_{k+1}^{(1-\alpha)}} \|u\|^{2(1-\alpha)}, \qquad \forall\, u \in W_k,$$

and therefore

$$\|u\|_{L_\ell^r}^r \leqslant \frac{C}{\lambda_{k+1}^{(1-\alpha)}} \|u\|^{2(1-\alpha)} \|u\|^{2_\ell^{**}\alpha} = \frac{C}{\lambda_{k+1}^{(1-\alpha)}} \|u\|^r.$$

This lemma is proved. □

To establish the existence of infinitely many solutions for Problem ($\mathcal{P}_\pm$), we will use the following version of the Mountain Pass Theorem (*cf.* [55, Theorem 9.12]).

Theorem 4.9 (Symmetric Mountain Pass Theorem)**.** *Suppose that $\mathcal{X}$ is a real Banach space and $\mathcal{I} \in C^1(\mathcal{X}, \mathbb{R})$ is an even functional satisfying $\mathcal{I}(0) = 0$. Suppose also that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$, where $\mathcal{V}$ is finite dimensional subspace*

($\widehat{\mathcal{I}}_1$) *there are constants $\rho, \tau > 0$ such that $\mathcal{I}(u) \geqslant \tau$, for all $u \in \mathcal{W} \cap \partial B_\rho(0)$;*

($\widehat{\mathcal{I}}_3$) *for each finite dimensional subspace $\widehat{\mathcal{X}} \subset \mathcal{X}$, there exists $R = R(\widehat{\mathcal{X}})$ such that*

$$\sup_{u \in \widehat{\mathcal{X}} \backslash B_{R(\widehat{\mathcal{X}})}(0)} \mathcal{I}(u) \leqslant 0$$

and the $(PS)_c$ condition for any $c \in \mathbb{R}$. Then $\mathcal{I}$ has an unbounded sequence of critical values.

We conclude the chapter with the proof of our second main result.

Proof of Theorem 4.2. We are intending to apply Theorem 4.9 for the functional $I_\pm$. Since we are assuming f is odd, the map $u \mapsto \displaystyle\int_B |x|^\ell F(u) \, \mathrm{d}x$ is even in X, the same occurring with $I_\pm$. Of course $I_\pm(0) = 0$ and, by Proposition 4.7, $I_\pm$ satisfies $(PS)_c$ condition for any level $c \in \mathbb{R}$.

It remains to verify the geometric conditions. We first deal with $(\hat{\mathcal{I}}_3)$. Let $\hat{X} \subset X$ be a finite-dimensional subspace. Without loss of generality we can assume $p < \mu < 2_\ell^{**}$, because $2_\kappa^* < 2_\ell^{**}$ (*cf.* Proposition 4.5). Thus the norms $\|\cdot\|$ and $\|\cdot\|_{L_\ell^\mu}$ are equivalent in $\hat{X}$ and we can use (4.11) in order to obtain,

$$I_-(u) \leqslant \frac{1}{2}\|u\|^2 - C_1\|u\|^\mu + C_2, \qquad \forall\, u \in \hat{X}.$$

Analogously, by Theorems 4.4 and 4.6

$$I_+(u) \leqslant \frac{1}{2}\|u\|^2 + C_3\|u\|^p - C_4\|u\|^\mu + C_5, \qquad \forall\, u \in \hat{X}.$$

Since $2 < p < \mu$, we conclude that $I_\pm(u) \to -\infty$, as $\|u\| \to +\infty$, $u \in \hat{X}$. Thus the condition $(\hat{\mathcal{I}}_3)$ also holds.

In order to verify $(\hat{\mathcal{I}}_1)$, we use (f_2) to obtain $C_6 > 0$ such that

$$|F(s)| \leqslant C_6 + C_7|s|^q, \qquad \forall\, s \in \mathbb{R}.$$

Thus, for any $u \in W_k$,

$$I_\pm(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{p}\||\nabla u|\|_{L_\kappa^p}^p - C_8\|u\|_{L_\ell^q}^q - C_9. \tag{4.14}$$

Now we apply Lemma 4.8 with q and q_* in order to obtain $\alpha, \beta \in (0,1)$ such that

$$\|u\|_{L_\ell^q}^q \leqslant \frac{C_{10}}{\lambda_{k+1}^{(1-\alpha)}}\|u\|^q, \quad \|u\|_{L_\ell^{q_*}}^{q_*} \leqslant \frac{C_{11}}{\lambda_{k+1}^{(1-\beta)}}\|u\|^{q_*} \tag{4.15}$$

and use Theorem 4.4 to get

$$\begin{aligned}\||\nabla u|\|_{L_\kappa^p} &\leqslant C_{12}\|u\|^{\theta_*}\|u\|_{L_\ell^{q_*}}^{1-\theta_*}\\ &\leqslant C_{12}\|u\|^{\theta_*}\left(\frac{C_{11}^{1/q_*}}{\lambda_{k+1}^{(1-\beta)/q_*}}\|u\|\right)^{1-\theta_*} = \frac{C_{13}}{\lambda_{k+1}^{(1-\beta)(1-\theta_*)/q_*}}\|u\|.\end{aligned}$$

Hence,

$$\||\nabla u|\|_{L_\kappa^p}^p \leqslant \frac{C_{14}}{\lambda_{k+1}^{(1-\beta)(1-\theta_*)p/q_*}}\|u\|^p.$$

This inequality combined with (4.14) and (4.15) yields

$$
\begin{aligned}
I_{\pm}(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{C_{15}}{\lambda_{k+1}^{(1-\beta)(1-\theta_*)p/q_*}}\|u\|^p - \frac{C_{16}}{\lambda_{k+1}^{(1-\alpha)}}\|u\|^q - C_9 \\
&= \frac{1}{2}\|u\|^2\left(1 - \frac{C_{17}}{\lambda_{k+1}^{(1-\beta)(1-\theta_*)p/q_*}}\|u\|^{p-2} - \frac{C_{18}}{\lambda_{k+1}^{(1-\alpha)}}\|u\|^{q-2}\right) - C_9,
\end{aligned}
$$

for any $u \in W_k$.

We set

$$
\rho_k := \frac{1}{2}\min\left\{\left(\frac{\lambda_{k+1}^{(1-\beta)(1-\theta_*)p/q_*}}{4C_{17}}\right)^{1/(p-2)}, \left(\frac{\lambda_{k+1}^{(1-\alpha)}}{4C_{18}}\right)^{1/(q-2)}\right\}
$$

in such a way that

$$
\frac{C_{17}}{\lambda_{k+1}^{(1-\beta)(1-\theta_*)p/q_*}}\rho_k^{p-2} < \frac{1}{4}, \qquad \frac{C_{18}}{\lambda_{k+1}^{(1-\alpha)}}\rho_k^{q-2} < \frac{1}{4}.
$$

Thus

$$
1 - \frac{C_{17}}{\lambda_{k+1}^{(1-\beta)(1-\theta_*)p/q_*}}\rho_k^{p-2} - \frac{C_{18}}{\lambda_{k+1}^{(1-\alpha)}}\rho_k^{q-2} > \frac{1}{2}
$$

and consequently

$$
I_{\pm}(u) \geqslant \frac{1}{4}\rho_k^2 - C_9, \qquad \forall\, u \in W_k \cap \partial B_{\rho_k}(0).
$$

Recalling that α, $\beta \in (0,1)$ and $\theta_* < 1$ (see Theorem 4.4), we may use $\lambda_k \to +\infty$ to conclude that $\rho_k \to +\infty$, as $k \to +\infty$. Thus we can find $\eta_{k_0} > 0$ such that $I_{\pm}(u) \geqslant \eta_{k_0}$ for any $u \in W_{k_0} \cap \partial B_{\rho_{k_0}}(0)$. Hence, the condition $(\widehat{\mathcal{I}}_1)$ holds for the decomposition $X = V_{k_0} \oplus W_{k_0}$ and Theorem 4.2 is a direct consequence of Theorem 4.9. □

Chapter 5

FitzHugh-Nagumo system with exponential growth

In this chapter, we will study the following planar FitzHugh–Nagumo system:

$$\begin{cases} -\Delta u = \lambda Q(|x|)f(u) - V(|x|)v, & \text{in } \mathbb{R}^2, \\ -\Delta v = V(|x|)u - V(|x|)v, & \text{in } \mathbb{R}^2, \end{cases} \tag{$\mathcal{S}_\lambda$}$$

where $\lambda > 0$ and the potentials V, $Q : (0, +\infty) \to \mathbb{R}$ are such that

(V) $V \in C((0, +\infty), (0, +\infty))$ and there exists $a > -2$, such that :

$$\liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0;$$

(Q) $Q \in C((0, +\infty), (0, +\infty))$ is continuous and there exist b_0, $b > -2$, such that

$$\limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < +\infty, \qquad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < +\infty.$$

Concerning the nonlinearity $f : \mathbb{R} \to \mathbb{R}$, we assume the following:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$, such that

$$\lim_{|s| \to +\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if} \quad \alpha > \alpha_0, \\ +\infty, & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_2) there holds

$$\lim_{s \to 0} \frac{f(s)}{|s|^{\gamma - 1}} = 0,$$

where

$$\gamma := \max\left\{2, \frac{4(b-a)}{(a+2)} + 2\right\};$$

(f_3) there exists $\mu > \gamma$, such that

$$0 < \mu F(s) := \mu \int_0^s f(t)\,\mathrm{d}t \leqslant f(s)s, \qquad \forall\, s \neq 0;$$

(f_4) there exist $C > 0$ and $\nu > \gamma$ such that

$$F(s) \geqslant C|s|^{\nu}, \qquad \forall\, s \in \mathbb{R}.$$

The main results of this chapter are:

Theorem 5.1. *Suppose that (V_1), (Q_1) and (f_1)-(f_4) hold. Then there exists $\lambda_0 > 0$ such that the System $(\mathcal{S}_\lambda)$ has a radial nonzero weak solution, provided $\lambda \geqslant \lambda_0$. Moreover, if we call (u, v) this solution, the following hold:*

(a) if there exists $a_0 > -2$ such that

$$\limsup_{r \to 0} \frac{V(r)}{r^{a_0}} < +\infty,$$

then $u, v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p > 1$ such that pa_0, $pb_0 > -2$. In particular, u, v are locally Hölder continuous;

(b) if V is locally Hölder continuous, then $v \in C^{2,\sigma}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\sigma \in (0, 1)$.

Theorem 5.2. *Suppose that (V_1), (Q_1) and (f_1)-(f_4) hold. If additionally f is odd then, for any given $m \in \mathbb{N}$, there exists $\lambda_m > 0$ such that the System $(\mathcal{S}_\lambda)$ has at least $2m$ radial nonzero weak solutions, provided $\lambda \geqslant \lambda_m$.*

We present now some examples of functions satisfying our hypothesis. First notice that, for any $a > -2$ and $\overline{a} \geqslant a$, the function $V : (0, +\infty) \to (0, +\infty)$ defined by $V(r) = r^{\overline{a}}$ verifies (V_1). Also, for $-2 < b, b_0$ and $s_0 \geqslant b_0$, $s \leqslant b$, the function

$$Q(r) = \begin{cases} r^{s_0}, & \text{if } 0 < r \leqslant 1, \\ r^{s}, & \text{if } r > 1, \end{cases}$$

verifies (Q_1). More simply, in the case $-2 < b_0 \leqslant b$, we can take $b_0 \leqslant \beta \leqslant b$ and see the function $Q(s) = r^{\beta}$ also verifies the same condition. Finally, a typical example of a function f verifying conditions (f_1)-(f_4) is

$$f(s) = |s|^{p-2} s e^{\alpha_0 s^2}, \qquad s \in \mathbb{R},$$

with $p > \gamma$, $\alpha_0 > 0$ and $\mu = \nu = p$.

The chapter is organized in the following way: in Section 5.1, we established the variational framework to correctly define the energy functional. In particular, we prove a Trudinger-Moser type inequality (*cf.* Theorem 5.10 and Remark 5.11), which is interesting in itself. In Section 5.2, we define the Euler Lagrange functional associated to the system

($\mathcal{S}_\lambda$) and prove a version, for out setting, of the Principle of Symmetric Criticality. We also prove a local compactness result for the energy functional and show it has the Mountain Pass geometry. Finally, we reserve Section 5.3 for the proof of Theorems 5.1 and 5.2.

5.1 Variational setting

Along all this chapter, we assume that (V_1) and (Q_1) hold and consider the set

$$E := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} V(|x|)u^2 \,\mathrm{d}x < +\infty \right\}.$$

We are going to show that it is a Hilbert space (*cf.* Proposition 5.3) when endowed with the scalar product

$$\langle u, w \rangle_E := \int_{\mathbb{R}^2} \Big(\nabla u \cdot \nabla w + V(|x|)uw \Big) \,\mathrm{d}x, \qquad \forall\, u,\, w \in E,$$

whose corresponding norm is $\|u\|_E := \langle u, u \rangle_E^{1/2}$. We also denote by E_{rad} the subspace of E consisted of the radial functions, that is,

$$E_{rad} := \{ u \in E : u \circ g = u, \quad \forall\, g \in O(2) \},$$

where $O(2)$ stands for group of real orthogonal 2×2 matrices.

For completeness, we reproduce in what follows some arguments from [3, Proposition 2.1].

Proposition 5.3. *Suppose that $V \in C((0, +\infty), (0, +\infty))$. Then the space E is a Hilbert space.*

Proof. If $(u_n) \subset E$ is a Cauchy sequence, then for each $i = 1, ..., n$, $(u_n)_{x_i}$ and $\sqrt{V(|\cdot|)}u_n$ are also Cauchy sequences in $L^2(\mathbb{R}^2)$. Hence, there exists $u^i, v \in L^2(\mathbb{R}^2)$ such that

$$(u_n)_{x_i} \to u^i, \quad \sqrt{V(|\cdot|)}u_n \to v, \quad \text{in } L^2(\mathbb{R}^2).$$

We define $w = v/(\sqrt{V(|\cdot|)}$ and we are going to show that $w_{x_i} = u^i$, $w \in E$ and $\|u_n - w\|_E \to 0$.

For any $R > 0$, choose $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\varphi \equiv 1$ on B_R and $\text{supp}(\varphi) \subset B_{R+1}$. To prove u_n is a Cauchy sequence in $L^2(B_R)$ observe that

$$\int_{B_R} |u_n - u_m|^2 \,\mathrm{d}x = \int_{B_R} |\varphi(u_n - u_m)|^2 \,\mathrm{d}x \leqslant \int_{B_{R+1}} |\varphi(u_n - u_m)|^2 \,\mathrm{d}x.$$

As $\nabla(\varphi(u_n - u_m)) = (u_n - u_m)\nabla\varphi + \varphi\nabla(u_n - u_m)$ and φ is constant in B_R, Poincaré's inequality yields constants constants $C_1, C_2 > 0$ such that

$$\int_{B_R} |u_n - u_m|^2 \,\mathrm{d}x \leqslant C_1 \int_{B_{R+1}\backslash B_R} |(u_n - u_m)\nabla\varphi|^2 \,\mathrm{d}x + C_2 \int_{B_R} |\varphi\nabla(u_n - u_m)|^2 \,\mathrm{d}x.$$

Rewriting the right-hand side, we obtain

$$\int_{B_R} |u_n - u_m|^2 \,\mathrm{d}x \leqslant C_3 \int_{\overline{B_{R+1}}\backslash B_R} V(|x|)\frac{|u_n - u_m|^2}{V(|x|)} \,\mathrm{d}x + C_4 \int_{B_R} |\nabla(u_n - u_m)|^2 \,\mathrm{d}x.$$

Let M_R the minimum of V in $\overline{B_R}\backslash B_R$. Then,

$$\int_{B_R} |u_n - u_m|^2 \,\mathrm{d}x \leqslant \frac{C_3}{M_R} \int_{\overline{B_{R+1}}\backslash B_R} V(|x|)|(u_n - u_m)|^2 \,\mathrm{d}x + C_4 \int_{B_R} |\nabla(u_n - u_m)|^2 \,\mathrm{d}x,$$

which implies

$$\int_{B_R} |u_n - u_m|^2 \,\mathrm{d}x \leqslant C_5 \|u_n - u_m\|_E^2. \tag{5.1}$$

Since u_n is a Cauchy sequence in E, this inequality implies u_n is a Cauchy sequence in $L^2(B_R)$. Thus, for each $R > 0$, there exists $u_R \in L^2(B_R)$ such that $u_n \to u_R$ in $L^2(B_R)$ and (up to a subsequence) $u_n \to u_R$ a.e. in B_R. Simultaneously, since $\sqrt{V(|\cdot|)}u_n \to v$ a.e. in $\mathbb{R}^2$, it follows that $u_n \to v$ a.e. in $\mathbb{R}^2$. Therefore, $u_R = w$, implying $w \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Finally, for an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^2)$, let $R > 0$ such that $\mathrm{supp}(\varphi) \subset B_R$ and notice that

$$\begin{aligned}\int_{\mathbb{R}^2} w\varphi_{x_i} \,\mathrm{d}x = \int_{B_R} w\varphi_{x_i} \,\mathrm{d}x = \int_{B_R} u_R\varphi_{x_i} \,\mathrm{d}x \\ = \lim_{n\to+\infty} \int_{B_R} u_n\varphi_{x_i} \,\mathrm{d}x = -\lim_{n\to+\infty} \int_{\mathbb{R}^2} (u_n)_{x_i}\varphi \,\mathrm{d}x.\end{aligned}$$

As $(u_n)_{x_i} \to u^i$ in $L^2(\mathbb{R}^2)$, the weak derivative w_{x_i} exists and equals to u^i. Consequently, $\|u_n - w\|_E \to 0$, which completes the proof. □

Corollary 5.4. *The space E is continuously embedded in $H^1_{loc}(\mathbb{R}^2)$. In particular, for any $R > 0$, E is continuously and compactly embedded in $L^q(B_R)$, for all $q \geqslant 1$.*

Proof. Let $u \in E$ and $R > 0$. We can follow the same step to obtain (5.1) to prove

$$\int_{B_R} u^2 \,\mathrm{d}x \leqslant C_1 \|u\|_E^2,$$

what is enough to conclude $\|u\|_{H^1(B)} \leqslant C_2\|u\|_E$. □

For any $u \in E$ fixed, we define the linear functional $T_u : E_{rad} \to \mathbb{R}$ given by

$$T_u(\varphi) := \int_{\mathbb{R}^2} V(|x|)u\varphi \, \mathrm{d}x.$$

Since $|T_u(\varphi)| \leqslant \|u\|_E \|\varphi\|_E$, we may invoke Riesz's Theorem to obtain $B[u] \in E_{rad}$ such that $T_u(\varphi) = \langle B[u], \varphi \rangle_E$, for all $\varphi \in E_{\text{rad}}$. Of course, T_u can be defined on E. Therefore, $B[u]$ is a critical point of the C^1 functional $J_u : E \to \mathbb{R}$ defined by

$$J_u(w) := \frac{1}{2}\|w\|_E^2 - \int_{\mathbb{R}^2} V(|x|)\, uw \, \mathrm{d}x = \frac{1}{2}\|w\|_E^2 - T_u(w), \qquad \forall\, w \in E,$$

restricted to E_{rad}. Indeed, since T_u is linear, we have that $(T_u)'(w) = T_u$ for every $w \in E$.

Given an orthogonal map $g \in O(2)$ and $w \in E$, we can define $(gw)(x) := w(g^{-1}x)$. If we assume in addition that $u \in E_{\text{rad}}$, since $V(|\cdot|)$ is radial there holds $J_u(gw) = J_u(w)$ and $\|gw\|_E = \|w\|_E$, for all $w \in E$. So, by the Principle of Symmetric Criticality, (*cf.* Theorem 5.18) we conclude that $B[u]$ is a radial weak solution of the linear problem

$$-\Delta v + V(|x|)v = V(|x|)u, \quad \text{in } \mathbb{R}^2. \tag{5.2}$$

So, if you come come back to system $(\mathcal{S}_\lambda)$ and make the change of variable $v := B[u]$ in the first equation, we are led to find a radial function u solving the problem

$$-\Delta u + V(|x|)B[u] = \lambda Q(|x|)f(u), \quad \text{in } \mathbb{R}^2. \tag{5.3}$$

Actually, if $u \in E_{rad}$ is a solution of the above equation, the couple $(u, B[u])$ of radial functions solves the system $(\mathcal{S}_\lambda)$.

In order to address Problem (5.3), we consider the bilinear form

$$\langle u, w \rangle_X := \int_{\mathbb{R}^2} \Big(\nabla u \cdot \nabla w + V(|x|)uB[w] \Big) \, \mathrm{d}x.$$

Using $B[u]$ as a test function in equation (5.2), we obtain

$$\|u\|_X^2 = \langle u, u \rangle_X = \|B[u]\|_E^2 + \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x. \tag{5.4}$$

Hence, it is straightforward to prove that $\langle \cdot, \cdot \rangle_X$ defines a scalar product in E (*cf.* Proposition 5.5). From now on, we denote by X the vector space formed by the set E endowed with the norm induced by this inner product, that is

$$\|u\|_X := \left[\int_{\mathbb{R}^2} \Big(|\nabla u|^2 + V(|x|)uB[u] \Big) \, \mathrm{d}x \right]^{1/2}.$$

As before, we set

$$X_{rad} := \{u \in X,\ u \circ g = u, \quad \forall\, g \in O(2)\}$$

the subspace of X consisting of radial functions.

Proposition 5.5. *If $V \in C((0,+\infty),(0,+\infty))$, then X is a Hilbert space.*

Proof. First we prove $\langle \cdot, \cdot \rangle_X$ is symmetric and positive definite. For any u, w, $\varphi \in X$, we know

$$\int_{\mathbb{R}^2} \nabla B[u] \cdot \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)B[u]\varphi \,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|)u\varphi \,\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^2} \nabla B[w] \cdot \nabla \psi \,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)B[w]\psi \,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|)w\psi \,\mathrm{d}x.$$

Taking $\varphi = B[w]$ in the first equality above, and $\varphi = B[u]$ in the second one, we get

$$\int_{\mathbb{R}^2} V(|x|)uB[w] \,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|)wB[u] \,\mathrm{d}x$$

and so $\langle u, w\rangle_X = \langle w, u\rangle_X$. By (5.4), of course $\langle u, u\rangle_X \geqslant 0$ and it is zero if and only if $u = 0$.

Let $u \in X$ and $v := B[u]$. By picking u as a test function in (5.2) and using Young's inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^2} V(|x|)u^2 \,\mathrm{d}x &= \int_{\mathbb{R}^2} (\nabla v \cdot \nabla u)\ \mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x \\
&= \frac{1}{2}\left[\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x\right] + \frac{3}{2}\int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x \\
&= -\frac{1}{2}\int_{\mathbb{R}^2} V(|x|)v^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} Vu^2 \,\mathrm{d}x &\leqslant \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x \\
&= \frac{3}{2}\left[\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)uv \,\mathrm{d}x\right]
\end{aligned}$$

and therefore

$$\|u\|_E^2 \leqslant \frac{3}{2}\|u\|_X^2, \qquad \forall\, u \in X. \tag{5.5}$$

Let (u_n) be a Cauchy sequence in X. From the inequality (5.5) we conclude that (u_n) is also a Cauchy sequence in the norm $\|\cdot\|_E$. Since E is a Hilbert space (*cf.* Proposition 5.3), there exists $u \in E$ such that $\|u_n - u\|_E = o_n(1)$, where $o_n(1) \to 0$ stands for a quantity approaching zero as $n \to +\infty$. Denoting $v_n := B[u_n]$, we can use the linearity of B and (5.4) to get

$$\|v_n - v_m\|_E^2 = \|B[u_n - u_m]\|_E^2 = \|u_n - u_m\|_X^2 - \int_{\mathbb{R}^2} |\nabla(u_n - u_m)|^2 \,\mathrm{d}x \leqslant \|u_n - u_m\|_X^2,$$

which shows that (v_n) is also a Cauchy sequence in the norm $\|\cdot\|_E$. Again, since E is a Hilbert space, there exists $v \in E$ such that $\|v_n - v\|_E = o_n(1)$. We claim that $v = B[u]$. If this is true, we can use (5.4) again and the linearity of B to obtain

$$\|u_n - u\|_X^2 = \|v_n - v\|_E^2 + \int_{\mathbb{R}^2} |\nabla(u_n - u)|^2 \,\mathrm{d}x \leqslant \|v_n - v\|_E^2 + \|u_n - u\|_E^2 = o_n(1),$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. This shows that $u_n \to u$ in X.

To prove that $v = B[u]$ we notice that, for any $\varphi \in E$, one has

$$\langle v_n, \varphi \rangle_E = \int_{\mathbb{R}^2} V(|x|) u_n \varphi \,\mathrm{d}x. \tag{5.6}$$

Since $v_n \to v$ in E, it follows that $\langle v_n, \varphi\rangle_E = \langle v, \varphi \rangle_E + o_n(1)$. Moreover, using Hölder's inequality, we obtain

$$\left| \int_{\mathbb{R}^2} V(|x|)(u_n - u)\varphi \,\mathrm{d}x \right| \leqslant \|u_n - u\|_E \|\varphi\|_E = o_n(1).$$

These convergences combined with (5.6) imply that

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|) v\varphi \,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|) u \varphi \,\mathrm{d}x, \quad \forall\, \varphi \in E,$$

and therefore $v = B[u]$. □

Remark 5.6. *It follows from* (5.5) *that the embedding $X \hookrightarrow E$ is continuous and therefore, by Corollary 5.4, we also conclude that E continuously immersed in $H^1_{loc}(\mathbb{R}^2)$. We also have the continuous embedding $X \hookrightarrow L^q(B_R)$, for any $R > 0$ and $q \geqslant 1$.*

The following result will be useful in the future.

Lemma 5.7. *Suppose that $u \in E$ and $g \in O(2)$. Then $B[u \circ g^{-1}] \circ g = B[u]$.*

Proof. As $u \circ g^{-1} \in E$,

$$\int_{\mathbb{R}^2} \nabla B[u \circ g^{-1}] \nabla \varphi + V(|x|) B[u \circ g^{-1}] \varphi \,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|)(u \circ g^{-1})\varphi \,\mathrm{d}x, \quad \forall\ \varphi \in E_{\mathrm{rad}}.$$

For $\varphi \in E_{\text{rad}}$, we can use $\varphi \circ g^{-1}$ as a test function in last identity in order to obtain

$$\int_{\mathbb{R}^2} V(|x|)(u \circ g^{-1})(\varphi \circ g^{-1})\,\mathrm{d}x = \int_{\mathbb{R}^2} \nabla B[u \circ g^{-1}]\nabla(\varphi \circ g^{-1})\,\mathrm{d}x + \int_{\mathbb{R}^2} V(|x|)B[u \circ g^{-1}](\varphi \circ g^{-1})\,\mathrm{d}x.$$

As $\mathbb{R}^2 = g\mathbb{R}^2$, after a change of variable we get

$$\int_{\mathbb{R}^2} \nabla(B[u \circ g^{-1}] \circ g)\nabla\varphi + V(|x|)(B[u \circ g^{-1}] \circ g)\varphi\,\mathrm{d}x = \int_{\mathbb{R}^2} V(|x|)u\varphi\,\mathrm{d}x.$$

By uniqueness, $B[u \circ g^{-1}] \circ g = B[u]$. □

Next we define, for each $1 \leqslant p < +\infty$, the space

$$L^p_Q(\mathbb{R}^2) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^2} |u|^p Q(|x|)\,\mathrm{d}x < +\infty \right\},$$

which is Banach space when endowed with the norm

$$\|u\|_{L^p_Q} := \left(\int_{\mathbb{R}^2} |u|^p Q(|x|)\,\mathrm{d}x \right)^{1/p}.$$

We notice that a version of the classical Radial Lemma of Strauss [72] holds in X_{rad}. In fact, it is proved in [73, Lemma 1] that there exist constants $C_r > 0$ and $R_0 > 0$ such that, for any $u \in X_{rad}$, the following holds:

$$|u(x)| \leqslant C_r |x|^{-(2+a)/4}\|u\|_E, \qquad \text{for a.e. } |x| \geqslant R_0. \tag{5.7}$$

It is worth noticing that, in [73], the authors prove (5.7) with the additional condition

$$\limsup_{r \to 0} \frac{V(r)}{r^{a_0}} > 0.$$

However, as we can see from the next result, this hypothesis is not necessary.

Lemma 5.8. *Suppose that $a > -2$ and $R > 0$. Then there exists a constant $C > 0$ such that, for any $u \in X_{rad}$, the following holds:*

$$|u(x)| \leqslant C \frac{\left(\int_{|x|>R} |u|^2 |x|^a\,\mathrm{d}x \right)^{1/4} \left(\int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x \right)^{1/4}}{|x|^{(2+a)/4}}, \qquad \text{for a.e. } |x| \geqslant R. \tag{5.8}$$

In particular, if (V_1) holds, there exists constants $C_r > 0$ and $R_0 > 0$ such that (5.7) is true for any $u \in X_{rad}$.

Proof. In fact, arguing as in [69, Lemma 2.1], it is enough proving inequality (5.8) for a radial function $u \in C_0^\infty(\mathbb{R}^2)$. In this case, write $u(x) = \varphi(|x|)$ for some φ. For a fixed $x_0 \in \mathbb{R}^2 \backslash \{0\}$ with $|x_0| > R$, denote by $\overline{R} = |x_0|$ and notice that

$$\frac{d}{dr}(\varphi^2 r^{(\frac{a}{2}+1)}) = 2\varphi(r)\varphi'(r)r^{(\frac{a}{2}+1)} + \varphi^2(r)\frac{d}{dr}(r^{(\frac{a}{2}+1)}).$$

As $a > -2$, the derivative of $r^{(\frac{a}{2}+1)}$ is positive. Thus

$$2\varphi(r)\varphi'(r)r^{(\frac{a}{2}+1)} \leqslant \frac{d}{dr}(\varphi^2 r^{(\frac{a}{2}+1)}).$$

Integrating over $(\overline{R}, +\infty)$ yields

$$\int_{\overline{R}}^{+\infty} 2\varphi(s)\varphi'(s)s^{(\frac{a}{2}+1)}ds \leqslant \int_{\overline{R}}^{+\infty} \frac{d}{dr}(\varphi^2 s^{(\frac{a}{2}+1)})ds = -\varphi^2(\overline{R})\overline{R}^{(\frac{a}{2}+1)},$$

which implies

$$\begin{aligned}
\varphi^2(\overline{R})\overline{R}^{\frac{2+a}{2}} &\leqslant 2\int_{\overline{R}}^{+\infty} |\varphi(s)\varphi'(s)|s^{(\frac{a}{2}+1)}ds \\
&= 2\int_{\overline{R}}^{+\infty} [|\varphi(s)s^{\frac{a}{2}}s^{\frac{1}{2}}][|\varphi'(s)|s^{\frac{1}{2}}]ds \\
&\leqslant 2\left(\int_R^{+\infty} |\varphi(s)|^2 s^a s ds\right)^{1/2}\left(\int_R^{+\infty} |\varphi'(s)|^2 s\right)^{1/2},
\end{aligned}$$

because $R < \overline{R}$. So, denoting by w_2 the measure of the sphere $\mathbb{S}^1$ under its surface measure σ_2, that is, $w_2 = \sigma_2(\mathbb{S}^1)$,

$$u^2(x)|x_0|^{\frac{2+a}{2}} \leqslant \frac{2}{w_2}\left(\int_{|x|>R} |u(x)|^2|x|^a \,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^2} |\nabla u(x)|^2 \,\mathrm{d}x\right)^{1/2}, \quad \forall\, |x_0| > R,$$

which implies (5.8).

By (V_1), denoting $\beta = \liminf_{r\to+\infty} V(r)/r^a$, for a given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $r^a(\beta - \varepsilon) \leqslant V(r)$, for any $r > \delta_\varepsilon$. Thus for a.e. $x \in \mathbb{R}^2$ and $|x| > \delta_\varepsilon$

$$\begin{aligned}
|u(x)| &\leqslant A_2|x|^{-(2+a)/4}\left(\int_{|x|>\delta_\varepsilon} |u(x)|^2|x|^a dx\right)^{1/4}\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{1/4} \\
&\leqslant \frac{A_2}{\beta-\varepsilon}|x|^{-(2+a)/4}\left(\int_{|x|>\delta_\varepsilon} |u(x)|^2 V(|x|)dx\right)^{1/4}\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{1/4} \\
&\leqslant \frac{A_2}{\beta-\varepsilon}|x|^{-(2+a)/4}\left(\int_{\mathbb{R}^2} |u(x)|^2 V(|x|)dx\right)^{1/4}\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{1/4},
\end{aligned}$$

which implies (5.7) for $R_0 = \delta_\varepsilon$ and $C_r = A_2/(\beta - \varepsilon)$. □

By taking advantage of (5.7), we can obtain a range of compactness for the embedding of X_{rad} into the above weighted Lebesgue spaces. More specifically, the following holds:

Lemma 5.9. *Let γ be defined in (f_2). Then the embedding $X_{rad} \hookrightarrow L^p_Q(\mathbb{R}^2)$ is continuous for any $\gamma \leqslant p < +\infty$ and compact if $p > \gamma$.*

Proof. Let $R_0 > 0$ be like in (5.7) and $R \geqslant R_0$. In view of (Q_1) and (V_1), there exists $C_1 > 0$, $C_2 > 0$ such that

$$\begin{cases} Q(x) \leqslant C_1|x|^{b_0}, & \text{if } |x| \leqslant R, \\ Q(|x|) \leqslant C_1|x|^b,\ V(|x|) \geqslant C_2|x|^a, & \text{if } |x| \geqslant R. \end{cases} \tag{5.9}$$

For any $u \in X_{\mathrm{rad}}$ and $\gamma \leqslant p < +\infty$, we have that

$$\|u\|^p_{L^p_Q} = \int_{B_R} |u|^p Q(|x|)\,\mathrm{d}x + \int_{\mathbb{R}^2\setminus B_R} |u|^p Q(|x|)\,\mathrm{d}x.$$

In order to prove the continuous immersion we estimate each these integrals. Let $q_1 > 1$ be such that $b_0 q_1 > -2$. From (5.9), Hölder's inequality, Remark 5.6 and (5.5) to obtain

$$\int_{B_R} |u|^p Q(|x|)\,\mathrm{d}x \leqslant C_1 \left(\int_{B_R} |x|^{b_0 q_1}\,\mathrm{d}x\right)^{1/q_1} \|u\|^p_{L^{pq_2}(B_R)} \leqslant C_2\|u\|^p_E \leqslant C_3\|u\|^p_X. \tag{5.10}$$

On the other hand, using (5.7) and (5.9) again, we get

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R} |u|^p Q(|x|)\,\mathrm{d}x &= \int_{\mathbb{R}^2\setminus B_R} |x|^{-a}|u|^{p-2}|x|^a|u|^2 Q(|x|)\,\mathrm{d}x \\ &\leqslant \frac{C_r^{(p-2)}C_1}{C_2}\|u\|^{p-2}_E \int_{\mathbb{R}^2\setminus B_R} |x|^\lambda V(|x|)u^2\,\mathrm{d}x,\end{aligned}$$

with

$$\lambda := (b-a) - \frac{(2+a)(p-2)}{4}.$$

As $p \geqslant \gamma$, one deduces $\lambda \leqslant 0$. Hence, last estimate combined with (5.5) yields

$$\int_{\mathbb{R}^2\setminus B_R} |u|^p Q(|x|)\,\mathrm{d}x \leqslant C_4 R^\lambda \|u\|^p_X,$$

which establishes the continuity of the embedding.

We now prove that the embedding is compact. Let $(u_n) \subset X_{\text{rad}}$ be a bounded sequence such that $u_n \rightharpoonup u$ weakly in X. By Corollary 5.4, we conclude that $u_n \to u$ in $L^q(B_R)$, for any $q \geqslant 1$. Replacing u by $(u_n - u)$ in (5.10) yields

$$\int_{B_R} |u_n - u|^q Q(|x|)\,\mathrm{d}x = o_n(1). \tag{5.11}$$

As $\lambda \leqslant 0$, for $\varepsilon > 0$, we can take R large enough such that $R^\lambda < \varepsilon$. After replacing u by $(u_n - u)$ in (5.11), we obtain

$$\int_{\mathbb{R}^2 \backslash B_R} |u_n - u|^p Q(|x|)\,\mathrm{d}x \leqslant C_5 \varepsilon,$$

because (u_n) is bounded. This proves that $u_n \to u$ in X. □

We study now the embedding of the space X_{rad} into weighted Orlicz spaces. So, we pick $\alpha > 0$ and define the Young function

$$\Phi_\alpha(s) := e^{\alpha s^2} - \sum_{j=0}^{j_0 - 1} \frac{\alpha^j}{j!} s^{2j}, \qquad \forall\, s \in \mathbb{R},$$

where $j_0 := \inf\{j \in \mathbb{N} : j \geqslant \gamma/2\}$ and $\gamma > 0$ was defined in (f_2). We have that

$$\Phi_\alpha(s) = \Phi_{\alpha r^2}\left(\frac{s}{r}\right), \quad (\Phi_\alpha(s))^t \leqslant \Phi_{t\alpha}(s), \qquad \forall\, s \in \mathbb{R}, r \in \mathbb{R} - \{0\},\, t \geqslant 1, \tag{5.12}$$

and

$$\Phi_\alpha(s) \leqslant \Phi_\beta(s), \qquad \forall\, 0 < \alpha \leqslant \beta,\, \forall s \in \mathbb{R}. \tag{5.13}$$

Indeed, the second inequality in (5.12) was proved in [83, Lemma 2.1] and the other inequalities above follow directly from the definition of Φ_α.

The following Trudinger-Moser type inequality complements the abstract results stated in [73]:

Theorem 5.10. *Suppose that $\alpha > 0$ and $u \in X_{rad}$. Then $Q(|\cdot|)\Phi_\alpha(u) \in L^1(\mathbb{R}^2)$. Moreover,*

$$\sup_{\{u \in X_{rad}\,:\, \|u\|_X \leqslant 1\}} \int_{\mathbb{R}^2} \Phi_\alpha(u) Q(|x|)\,\mathrm{d}x < +\infty, \tag{5.14}$$

whenever $0 < \alpha < 4\pi(b_0/2 + 1)$.

Proof. Considering $R_0 > 0$ as in (5.7), we fix a number $R > R_0$ and divide the proof into three steps:

First step: for any $\alpha > 0$ and $u \in X_{rad}$, we have that $Q(|\cdot|)\Phi_\alpha(u) \in L^1(B_R)$.

Following [71] (*cf.* also [12]), we consider the function

$$v(|x|) := \beta^{-1/2} u(|x|^{\beta}), \qquad x \in \mathbb{R}^2,$$

with $\beta := 2/(b_0 + 2) > 0$. We claim that $v \in H^1(B_{R^{1/\beta}})$. In fact, a straightforward computation shows that

$$\int_{B_{R^{1/\beta}}} |\nabla v(x)|^2 \, \mathrm{d}x = 2\pi \int_0^{R^{1/\beta}} |v'(s)|^2 s \, \mathrm{d}s = 2\pi\beta \int_0^{R^{1/\beta}} |u'(s^{\beta})|^2 s^{2\beta-1} \, \mathrm{d}s$$

and therefore the change of variables $t = s^{\beta}$ yields

$$\int_{B_{R^{1/\beta}}} |\nabla v(x)|^2 \, \mathrm{d}x = 2\pi \int_0^R |u'(t)|^2 t \, \mathrm{d}t = \int_{B_R} |\nabla u(x)|^2 \, \mathrm{d}x < +\infty. \tag{5.15}$$

On the other hand,

$$\int_{B_{R^{1/\beta}}} v^2(x) \, \mathrm{d}x = 2\pi\beta^{-1} \int_0^{R^{1/\beta}} u^2(s^{\beta}) s \, \mathrm{d}s = 2\pi\beta^{-2} \int_0^R t^{2(1-\beta)/\beta} u^2(t) t \, \mathrm{d}t,$$

where we have used the change of variables $t = s^{\beta}$ again. It follows from $2(1-\beta)/\beta = b_0$ that

$$\int_{B_{R^{1/\beta}}} v^2(x) \, \mathrm{d}x = \beta^{-2} \int_{B_R} |x|^{b_0} u^2(x) \, \mathrm{d}x. \tag{5.16}$$

We now recall that

$$\int_{B_R} |x|^t \, \mathrm{d}x = 2\pi \int_0^R s^{t+1} \, \mathrm{d}s < +\infty,$$

whenever $t > -2$. Since the parameter b_0 in (Q_1) verifies $b_0 > -2$, we can pick $t_1 > 1$ close to 1 in such a way that $|x|^{t_1 b_0} \in L^1(B_R)$. Thus, we may use (5.16), Hölder's inequality and Remark 5.6 to obtain

$$\int_{B_{R^{1/\beta}}} v^2(x) \, \mathrm{d}x \leqslant \beta^{-2} \left(\int_{B_R} |x|^{t_1 b_0} \, \mathrm{d}x \right)^{1/t_1} \left(\int_{B_R} |u(x)|^{2t_2} \, \mathrm{d}x \right)^{1/t_2} < +\infty,$$

where $1/t_1 + 1/t_2 = 1$ and, of course, $2t_2 \geqslant 1$. This and (5.15) prove that $v \in H^1(B_{R^{1/\beta}})$, as claimed.

Arguing as in the proof of (5.16),

$$\int_{B_{R^{1/\beta}}} e^{\alpha\beta v^2(x)} \, \mathrm{d}x = \int_{B_{R^{1/\beta}}} e^{\alpha u^2(|x|^{\beta})} \, \mathrm{d}x = 2\pi\beta^{-1} \int_0^R e^{\alpha u^2(t)} t^{b_0} t \, \mathrm{d}t = \beta^{-1} \int_{B_R} |x|^{b_0} e^{\alpha u^2(x)} \, \mathrm{d}x,$$

and therefore it follows from (5.9) that

$$\int_{B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x \leqslant C\int_{B_R} |x|^{b_0} e^{\alpha u^2(x)}\,\mathrm{d}x = C\beta \int_{B_{R^{1/\beta}}} e^{\alpha\beta v^2(x)}\,\mathrm{d}x. \tag{5.17}$$

We now define $\widetilde{v} \in H_0^1(B_{R^{1/\beta}})$ as

$$\widetilde{v}(|x|) := \begin{cases} v(|x|) - v(R^{1/\beta}), & \text{if } |x| \leqslant R^{1/\beta}, \\ 0, & \text{if } |x| \geqslant R^{1/\beta}. \end{cases}$$

For any $\varepsilon > 0$, since $v(|x|) = \tilde{v}(|x|) + v(R^{1/\beta})$ for $|x| \leqslant R^{1/\beta}$, we can use Young's inequality in order to obtain

$$\begin{aligned} v^2(|x|) &= \widetilde{v}^2(|x|) + 2\widetilde{v}(|x|)v(R^{1/\beta}) + v^2(R^{1/\beta}) \\ &\leqslant \widetilde{v}^2(|x|) + 2\left[\varepsilon^{1/2}|\widetilde{v}(|x|)|\right]\left[\varepsilon^{-1/2}|v(R^{1/\beta}|)\right] + v^2(R^{1/\beta}) \\ &\leqslant (1+\varepsilon)\widetilde{v}^2(|x|) + C(\varepsilon)v^2(R^{1/\beta}), \end{aligned}$$

with $C(\varepsilon) := (\varepsilon+1)/\varepsilon$. This inequality, (5.17) and the classical Trudinger-Moser inequality for bounded domain (*cf.* Theorem 5.19) imply that

$$\int_{B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x \leqslant C_2 e^{\alpha\beta C(\varepsilon)v^2(R^{1/\beta})} \int_{B_{R^{1/\beta}}} e^{\alpha\beta(1+\varepsilon)\widetilde{v}^2(|x|)}\,\mathrm{d}x < +\infty, \tag{5.18}$$

where $C_2 := C\beta$. The first step is proved.

Second step: if $0 < \alpha < 4\pi(b_0/2+1)$, then

$$\sup_{\{u \in X_{rad} : \|u\|_X \leqslant 1\}} \int_{B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x < +\infty.$$

Let $0 < \alpha < 4\pi(b_0/2+1)$ and $u \in X_{rad}$, with $\|u\|_X \leqslant 1$. In this case, it is possible to take $\varepsilon > 0$ such that $\alpha(1+\varepsilon) < 4\pi(b_0/2+1)$. Recalling that $\beta = 2/(b_0+2)$, we get

$$\alpha\beta(1+\varepsilon) < 4\pi. \tag{5.19}$$

Moreover, by the definition of v, (5.7), the assumption $R > R_0$ and (5.5), one deduces

$$\alpha\beta C(\varepsilon)v^2(R^{1/\beta}) = \alpha C(\varepsilon)u^2(R) \leqslant \frac{3}{2}\alpha C(\varepsilon)C_r^2 R^{-(2+a)/2}\|u\|_X^2 \leqslant C_3 R^{-(2+a)/2}.$$

with $C_3 = 3/2\alpha C(\varepsilon)C_r^2$. This and (5.18) imply that

$$\int_{B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x \leqslant C_2 e^{C_3 R^{-(2+a)/2}} \int_{B_{R^{1/\beta}}} e^{\alpha\beta(1+\varepsilon)\tilde{v}^2(x)}\,\mathrm{d}x. \tag{5.20}$$

From the definition of $\tilde{v}$ and (5.15), we get

$$\int_{B_{R^{1/\beta}}} |\nabla\tilde{v}|^2\,\mathrm{d}x = \int_{B_{R^{1/\beta}}} |\nabla v|^2\,\mathrm{d}x = \int_{B_R} |\nabla u|^2\,\mathrm{d}x \leqslant 1.$$

Since $\tilde{v} \in H_0^1(B_R^{1/\beta})$ and $\|\nabla\tilde{v}\|_{L^2(B_{R^{1/\beta}})} \leqslant 1$, we may use (5.19)-(5.20) and the classical Trudinger-Moser inequality (*cf.* Theorem 5.19) to obtain

$$\sup_{\{u\in X_{rad}:\|u\|_X\leqslant 1\}} \int_{B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x < +\infty.$$

The second step is finalized.

Third step: for any $\alpha > 0$, we have that

$$\sup_{\{u\in X_{rad}:\|u\|_X\leqslant 1\}} \int_{\mathbb{R}^2\setminus B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x < +\infty.$$

Given $u \in X_{rad}$, we first prove that $Q(|\cdot|)\Phi_\alpha(u) \in L^1(\mathbb{R}^2\setminus B_R)$. To do this, we recall that

$$\Phi_\alpha(s) = \sum_{j=j_0}^{+\infty} \frac{\alpha^j s^{2j}}{j!}.$$

Thus we can use the Monotone Convergence Theorem to get

$$\int_{\mathbb{R}^2\setminus B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x = \sum_{j=j_0}^{+\infty} \frac{\alpha^j}{j!} \int_{\mathbb{R}^2\setminus B_R} |u|^{2j-\gamma}|u|^\gamma Q(|x|)\,\mathrm{d}x.$$

For $j \geqslant j_0$, notice $2j-\gamma \geqslant 0$, because $j_0 \geqslant \gamma/2$. In this case, since $R \geqslant R_0$, it follows from (5.7) combined with (5.5) that, for $|x| > R$,

$$|u(x)|^{2j-\gamma} \leqslant C_3^{2j-\gamma}|x|^{-(2j-\gamma)(2+a)/4}\|u\|_X^{2j-\gamma},$$

where $C_3 = \sqrt{3/2}C_r$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x &\leqslant \left(\frac{C_3}{R^{(2+a)/4}}\right)^{-\gamma} \left[\sum_{j=j_0}^{+\infty} \frac{\left(\alpha C_3^2 R^{-(2+a)/2}\|u\|_X^2\right)^j}{j!}\right] \left(\frac{\|u\|_{L_Q^\gamma}}{\|u\|_X}\right)^\gamma \\ &\leqslant \left(\frac{C_3}{R^{(2+a)/4}}\right)^{-\gamma} e^{\alpha C_3^2 R^{-(2+a)/2}\|u\|_X^2} \left(\frac{\|u\|_{L_Q^\gamma}}{\|u\|_X}\right)^\gamma.\end{aligned}$$

This, together with Lemma 5.9 and (5.5) provides C_4, $C_5 > 0$ such that

$$\int_{\mathbb{R}^2\setminus B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x \leqslant C_5 e^{\alpha C_3^2 R^{-(2+a)/2}\|u\|_X^2} \leqslant C_3 e^{\alpha C_4\|u\|_X^2} < +\infty.$$

Moreover,

$$\sup_{\{u\in X_{rad}:\|u\|_X\leqslant 1\}} \int_{\mathbb{R}^2\setminus B_R} \Phi_\alpha(u)Q(|x|)\,\mathrm{d}x \leqslant C_4 e^{\alpha C_4},$$

and the proof is finished. □

Remark 5.11. *Let $\beta_* := 4\pi(b_0/2+1)$. As proved in [1, Proposition 2.5], we have*

$$C_0 := \sup_{\{u\in E_{rad}\,:\,\|u\|_E\leqslant 1\}} \int_{\mathbb{R}^N} \Phi_\beta(u)Q(|x|)\,\mathrm{d}x < +\infty,$$

for any $0 < \beta < \beta_*$. This inequality combined with (5.5) and (5.12) provides the conclusion of Theorem 5.10 for $0 < \alpha < 2\beta_*/3$. To see this, take $u \in X_{rad}$ with $\|u\|_X \leqslant 1$ and use (5.5) to obtain $\|\sqrt{2/3}u\|_E \leqslant 1$. So, for $0 < \beta < \beta_*$

$$\int_{\mathbb{R}^N} \Phi_\beta\left(\sqrt{\frac{2}{3}}u\right) Q(|x|)\,\mathrm{d}x \leqslant C_0, \quad \forall\, u \in X_{rad},\ \|u\|_X \leqslant 1.$$

Using the first identity of (5.12) with $r = \sqrt{2/3}$, we obtain

$$\int_{\mathbb{R}^N} \Phi_{(2\beta)/3}(u)Q(|x|)\,\mathrm{d}x \leqslant C_0, \quad \forall\, u \in X_{rad},\ \|u\|_X \leqslant 1,$$

for any $0 < \beta < \beta_*$. If we make the change $\alpha = 2\beta/3$ in this last inequality, we can see that (5.14) works for $0 < \alpha < 2\beta_*/3$. The main point is that we provided here a different proof to encompass the entire range $(0, 4\pi(b_0/2+1))$.

5.2 Mountain Pass structure

Using the abstract results of the former section, we are able to define the Euler-Lagrange functional associated to equation (5.3). The first step in our analysis is proving that $Q(|\cdot|)F(u) \in L^1(\mathbb{R}^2)$, for any $u \in X_{rad}$. We shall use the following basic lemma:

Lemma 5.12. *Suppose that (f_1) and (f_2) hold. Then, for any given $\varepsilon > 0$, $\alpha > \alpha_0$, and $q \geqslant 1$, there exists $C_f > 0$ such that, for any $s \in \mathbb{R}$,*

$$\begin{cases} |f(s)| \leqslant \varepsilon|s|^{\gamma-1} + C_f|s|^{q-1}\Phi_\alpha(s), \\ |F(s)| \leqslant \varepsilon|s|^{\gamma} + C_f|s|^{q}\Phi_\alpha(s). \end{cases} \tag{5.21}$$

Proof. By (f_1),

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{|s|^{q-1}\Phi_\alpha(s)}=0,\quad \forall\ q\geqslant 1,\alpha>\alpha_0, \tag{5.22}$$

because

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{|s|^{q-1}\Phi_\alpha(s)}=\lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}\frac{1}{|s|^{q-1}}\frac{e^{\alpha s^2}}{(e^{\alpha s^2}-\sum_{j=0}^{j_0-1}\frac{\alpha^j}{j!}s^{2j})}=0,$$

where we used that

$$\lim_{|s|\to+\infty}\frac{e^{\alpha s^2}}{(e^{\alpha s^2}-\sum_{j=0}^{j_0-1}\frac{\alpha^j}{j!}s^{2j})}=1.$$

For a given $\varepsilon>0$, we can use the limit (5.22) combined with (f_2) to obtain a constant $C_\varepsilon>0$ such that

$$|f(s)|\leqslant\varepsilon|s|^{\gamma-1}+C_\varepsilon|s|^{q-1}\Phi_\alpha(s),\quad \forall\ q\geqslant 1,\alpha>\alpha_0,s\neq 0. \tag{5.23}$$

As Φ_α is even and increasing on $(0,+\infty)$ and $q\geqslant 1$, there holds

$$\int_0^s|t|^{q-1}\Phi_\alpha(t)dt\leqslant\frac{1}{q}|s|^q\Phi_\alpha(s)\leqslant|s|^q\Phi_\alpha(s),\quad \forall\ s\in\mathbb{R}.$$

Last inequality combined with (5.23) yields

$$|F(s)|\leqslant\varepsilon|s|^\gamma+C_\varepsilon|s|^q\Phi_\alpha(s),\quad \forall\ q\geqslant 1,\alpha>\alpha_0$$

and the lemma is proved. □

Given $u\in X_{rad}$, it follows from (5.21) with $q\geqslant\gamma$ that

$$\int_{\mathbb{R}^2}F(u)Q(|x|)\,\mathrm{d}x\leqslant\varepsilon\|u\|^\gamma_{L^\gamma_Q(\mathbb{R}^2)}+C_f\int_{\mathbb{R}^2}|u|^q\Phi_\alpha(u)Q(|x|)\,\mathrm{d}x.$$

For $t_1>1$ such that $qt_1\geqslant\gamma$, we can use Hölder's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^2}|u|^q\Phi_\alpha(u)Q(|x|)\,\mathrm{d}x&=\int_{\mathbb{R}^2}[Q(|x|)]^{1/t_1}|u|^q[Q(|x|)]^{1/t_2}\Phi_\alpha(u)\,\mathrm{d}x\\&\leqslant\|u\|^q_{L^{qt_1}_Q}\left(\int_{\mathbb{R}^2}\Phi^{t_2}_\alpha(u)Q(|x|)\,\mathrm{d}x\right)^{1/t_2},\end{aligned}$$

where $1/t_1+1/t_2=1$. Since $qt_1\geqslant\gamma$, we can use last property combined with Lemma 5.9, property (5.12) with t_2 and Theorem 5.10 to obtain

$$\int_{\mathbb{R}^2}|u|^q\Phi_\alpha(u)Q(|x|)\,\mathrm{d}x\leqslant\|u\|^q_X\left(\int_{\mathbb{R}^2}\Phi_{t_2\alpha}(u)Q(|x|)\,\mathrm{d}x\right)^{1/t_2}<+\infty.$$

Therefore, for any $q \geqslant \gamma$

$$\int_{\mathbb{R}^2} F(u)Q(|x|)\,\mathrm{d}x \leqslant \varepsilon\|u\|_{L_Q^\gamma}^\gamma + C_f\|u\|_X^q\left(\int_{\mathbb{R}^2}\Phi_{t_2\alpha}(u)Q(|x|)\,\mathrm{d}x\right)^{1/t_2} < +\infty, \tag{5.24}$$

where $1/t_1 + 1/t_2 = 1$.

According to the above considerations, it is well defined the functional

$$I_\lambda(u) := \frac{1}{2}\|u\|_X^2 - \lambda\int_{\mathbb{R}^2} F(u)Q(|x|)\,\mathrm{d}x, \qquad u \in X_{rad}.$$

Moreover, by standard arguments one may conclude that $I_\lambda \in C^1(X_{rad}, \mathbb{R})$ with Gateaux derivative

$$I_\lambda'(u)\varphi = \langle u, \varphi\rangle_X - \lambda\int_{\mathbb{R}^2} f(u)\varphi Q(|x|)\,\mathrm{d}x, \qquad \forall\, u, \varphi \in X_{rad}.$$

Since the functional I_λ is not defined in the whole space X, we cannot directly apply Principle of Symmetric Criticality (*cf.* Theorem 5.18) to conclude that critical points of I_λ weakly solves the first equation in $(\mathcal{S}_\lambda)$. However, an indirect argument proves the following:

Proposition 5.13. *Suppose that* (f_1)-(f_2) *hold and* $u \in X_{rad}$ *is a critical point of* I_λ*. Then* u *is a weak solution of* (5.3)*.*

Proof. Let $u \in X_{rad}$ be such that $I_\lambda'(u) = 0$ and consider the linear functional

$$T_u(w) := \langle u, w\rangle_X - \lambda\int_{\mathbb{R}^2} f(u(x))w(x)Q(|x|)\,\mathrm{d}x, \qquad \forall\, w \in X.$$

Our goal is to show that $T_u(w) = 0$, for all $w \in X$.

We claim that T_u is continuous. If this is true, we may apply Riesz Representation Theorem to obtain a unique $\widetilde{u} \in X$ such that

$$T_u(w) = \langle \widetilde{u}, w\rangle_X, \qquad \forall\, w \in X. \tag{5.25}$$

It is clear that, for any orthogonal transformation $g \in O(2)$, there holds $gu = u$. Since $g^{-1}\mathbb{R}^2 = \mathbb{R}^2$, we can argue as in the beginning of Section 5.1 and use Proposition 5.7 to conclude that $T_u(g\widetilde{u}) = T_u(\widetilde{u})$ and $\|g\widetilde{u}\|_X = \|\widetilde{u}\|_X$. This implies,

$$\|g\widetilde{u} - \widetilde{u}\|_X^2 = \|g\widetilde{u}\|_X^2 - 2\langle g\widetilde{u}, \widetilde{u}\rangle_X + \|\widetilde{u}\|^2 = 2\|\widetilde{u}\|_X^2 - 2T_u(g\widetilde{u}) = 2\|\widetilde{u}\|_X^2 - 2T_u(\widetilde{u}) = 0$$

and therefore $g\widetilde{u} = \widetilde{u}$. Since $g \in O(2)$ is arbitrary, we conclude that $\widetilde{u} \in X_{rad}$. Hence, $0 = I_\lambda'(u)\widetilde{u} = T_u(\widetilde{u}) = \|\widetilde{u}\|_X^2$ and it follows from (5.25) that

$$I'_\lambda(u)w = T_u(w) = \langle 0, w\rangle_X = 0, \qquad \forall\, w \in X.$$

In order to prove the continuity of T_u, we first pick $\varepsilon = 1$ and $q = \gamma + 1$ in (5.21) to get

$$\left|\int_{\mathbb{R}^2} f(u)wQ(|x|)\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}^2} |u|^{\gamma-1}|w|Q(|x|)\,\mathrm{d}x + C_f \int_{\mathbb{R}^2} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x. \tag{5.26}$$

In what follows, we are going to estimate both terms in the last inequality in the ball B_R and in its complement.

Picking $t_1 > 1$ such that $t_1 b_0 > -2$, it follows that $|x|^{t_1 b_0} \in L^1(B_R)$. So, we can use expression (5.9), Hölder's inequality and Remark 5.6, to get

$$\int_{B_R} |u|^{\gamma-1}|w|Q(|x|)\,\mathrm{d}x \leqslant C_1 \left(\int_{B_R} |x|^{t_1 b_0}\,\mathrm{d}x\right)^{1/t_1} \|u\|^{\gamma-1}_{L^{t_2(\gamma-1)}(B_R)} \|w\|_{L^{t_3}(B_R)},$$

with $1/t_1 + 1/t_2 + 1/t_3 = 1$. Thus,

$$\int_{B_R} |u|^{\gamma-1}|w|Q(|x|)\,\mathrm{d}x \leqslant C_3\|w\|_X, \tag{5.27}$$

with C_3 depending on u and b_0. Moreover, using (5.9) and (5.7) we obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R} |u|^{\gamma-1}|w|Q(|x|)\,\mathrm{d}x &= \int_{\mathbb{R}^2\setminus B_R} |u|^{\gamma-2}|u||w|Q(|x|)\,\mathrm{d}x\\ &\leqslant C_1 C_r^{\gamma-2}\|u\|_E^{\gamma-2} \int_{\mathbb{R}^2\setminus B_R} |x|^b |x|^{-a-(\gamma-2)(2+a)/4}|x|^a|u||w|\,\mathrm{d}x\\ &\leqslant C_1 C_r^{\gamma-2}\|u\|_E^{\gamma-2} \int_{\mathbb{R}^2\setminus B_R} |x|^{\lambda_1}|x|^a|u||w|\,\mathrm{d}x,\end{aligned}$$

where

$$\lambda_1 := (b-a) - (\gamma-2)\left(\frac{a+2}{4}\right).$$

From the definition of γ (*cf.* (f_2)), we deduce that $\lambda_1 \leqslant 0$, and there $|x|^{\lambda_1} \leqslant R^{\lambda_1}$ for $|x| \geqslant R$. Thus, we can use the last estimate, Hölder's inequality, (5.9) and (5.5) to obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R} |u|^{\gamma-1}|w|Q(|x|)\,\mathrm{d}x \leqslant& C_4 \left(\int_{\mathbb{R}^2\setminus B_R} |x|^a u^2\,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^2\setminus B_R} |x|^a w^2\,\mathrm{d}x\right)^{1/2}\\ \leqslant& C_5 \left(\int_{\mathbb{R}^2\setminus B_R} V(|x|)u^2\,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^2\setminus B_R} V(|x|)w^2\,\mathrm{d}x\right)^{1/2}\\ \leqslant& C_6\|w\|_X,\end{aligned}$$

where C_6 depends on u and γ. This inequality, combined with (5.26) and (5.27), imply that

$$\left|\int_{\mathbb{R}^2} f(u)wQ(|x|)\,\mathrm{d}x\right| \leqslant (C_3 + C_6)\,\|w\|_X + C_f \int_{\mathbb{R}^2} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x. \tag{5.28}$$

We now proceed with the estimation of the last integral above. First, for any t_1, t_2, $t_3 > 0$ satisfying $1/t_1 + 1/t_2 + 1/t_3 = 1$, we apply Hölder's inequality, the second statement in (5.12), Lemma 5.9, and Theorem 5.10 to obtain

$$\begin{aligned}\int_{B_R} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x \leqslant & \|u\|^\gamma_{L_Q^{t_1\gamma}(B_R)} \left(\int_{B_R} \Phi_{t_2\alpha}(u)Q(|x|)\,\mathrm{d}x\right)^{1/t_2} \|w\|_{L_Q^{t_3}(B_R)} \\ \leqslant & C_7\|w\|_{L_Q^{t_3}(B_R)}.\end{aligned}$$

where we have used $t_1\gamma > \gamma$, for any $t_1 > 1$. By choosing $t_4 > 1$ such that $|x|^{t_4 b_0} \in L^1(B_R)$, we can combine Hölder's inequality and (5.9) to obtain

$$\int_{B_R} |w|^{t_3} Q(|x|)\,\mathrm{d}x \leqslant C_1 \left(\int_{B_R} |x|^{t_4 b_0}\,\mathrm{d}x\right)^{1/t_4} \|w\|^{t_3}_{L^{t_5 t_3}(B_R)},$$

with $1/t_4 + 1/t_5 = 1$. These last two estimates and Remark 5.6 again imply that

$$\int_{B_R} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x \leqslant C_8\|w\|_X. \tag{5.29}$$

From Hölder's inequality, (5.12) and Theorem 5.10, we get

$$\int_{\mathbb{R}^2\setminus B_R} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x \leqslant C_9 \left(\int_{\mathbb{R}^2\setminus B_R} |u|^{2\gamma} w^2 Q(|x|)\,\mathrm{d}x\right)^{1/2},$$

where

$$C_9 := \left(\int_{\mathbb{R}^2\setminus B_R} \Phi_{2\alpha}(u)Q(|x|)\,\mathrm{d}x\right)^{1/2}.$$

Once again, using (5.7) and (5.9), we can conclude that

$$\int_{\mathbb{R}^2\setminus B_R} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x \leqslant C_{10} \left(\int_{\mathbb{R}^2\setminus B_R} |x|^{\lambda_2}|x|^a w^2\,\mathrm{d}x\right)^{1/2},$$

where

$$\lambda_2 := (b-a) - \gamma\left(\frac{a+2}{2}\right).$$

The definition of γ (*cf.* (f_2)) and $a > -2$, yields $\lambda_2 \leqslant 0$. To see this, notice that $\gamma = 2$ if, and only if, $b \leqslant a$, and $\gamma = 4(b-a)/(a+2)+2$ if, and only if, $b > a$. So, we may argue as before to conclude that

$$\int_{\mathbb{R}^2 \setminus B_R} |u|^\gamma \Phi_\alpha(u)|w|Q(|x|)\,\mathrm{d}x \leqslant C_{11}\|w\|_X.$$

This, (5.28), (5.29) and the fact that $\lambda > 0$ imply that T_u is continuous on X. □

We say that I_λ satisfies the $(PS)_c$ condition at level $c \in \mathbb{R}$ if any sequence $(u_n) \subset X_{rad}$ such that

$$\lim_{n\to+\infty} I_\lambda(u_n) = c, \qquad \lim_{n\to+\infty} I'_\lambda(u_n) = 0 \tag{5.30}$$

has a convergent subsequence. We have the following local compactness result holding:

Lemma 5.14. *Suppose that* (f_1)-(f_3) *hold. Then* I_λ *satisfies* $(PS)_c$ *condition at any level*

$$0 < c < \frac{(\mu-2)}{2\mu}\frac{4\pi(b_0/2+1)}{\alpha_0}.$$

Proof. Let $(u_n) \subset X_{rad}$ be as in (5.30). From condition (f_3) we get

$$c + o_n(1)(1+\|u_n\|_X) = I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)u_n \geqslant \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_X^2 \tag{5.31}$$

and therefore we may use $\mu > 2$ to conclude that (u_n) is bounded in X_{rad}. Thus, up to a subsequence, $u_n \rightharpoonup u$ weakly in X_{rad}.

We claim that

$$\int_{\mathbb{R}^2} f(u_n)(u_n-u)Q(|x|)\,\mathrm{d}x = o_n(1). \tag{5.32}$$

If this is true, it follows that

$$o_n(1) = I'_\lambda(u_n)(u_n-u) = \|u_n\|_X^2 - \|u\|_X^2 + o_n(1)$$

and therefore $\|u_n\|_X \to \|u\|_X$. This, together with the weak convergence, implies that $u_n \to u$ strongly in X.

For proving (5.32), we first use the first estimate of (5.21) with $q = 1$ to get

$$\left|\int_{\mathbb{R}^2} f(u_n)(u_n-u)Q(|x|)\,\mathrm{d}x\right| \leqslant \varepsilon A_n + C_f D_n, \tag{5.33}$$

where

$$A_n := \int_{\mathbb{R}^2} |u_n|^{\gamma-1}|u_n - u|Q(|x|)\,\mathrm{d}x, \qquad D_n := \int_{\mathbb{R}^2} \Phi_\alpha(u_n)|u_n - u|Q(|x|)\,\mathrm{d}x.$$

It follows from (5.31) that

$$\limsup_{n\to+\infty} \|u_n\|_X^2 \leqslant \frac{2\mu}{(\mu-2)}c < \frac{4\pi(b_0/2+1)}{\alpha_0}.$$

Hence, there exits $n_0 \in \mathbb{N}$ large and $t_1 \in (1, \gamma/(\gamma-1))$, $\alpha > \alpha_0$ sufficiently close to 1 and α_0, respectively, such that

$$t_1\alpha\|u_n\|_X^2 < 4\pi\left(\frac{b_0}{2}+1\right), \qquad \forall\, n \geqslant n_0.$$

To see this, it is enough notice that we can chose some $0 < \delta < 1/(\gamma-1)$ such that

$$\frac{2\mu}{(\mu-2)}c < \frac{4\pi(b_0/2+1)}{(1+\delta)(\alpha_0+\delta)} < \frac{4\pi(b_0/2+1)}{\alpha_0},$$

and take $t_1 = 1+\delta$ and $\alpha = \alpha_0 + \delta$.

Since we may also assume $u_n \neq 0$, for $n \geqslant n_0$, it follows from Hölder's inequality, (5.12) and Theorem 5.10 that

$$\begin{aligned} D_n &\leqslant \left(\int_{\mathbb{R}^2} \Phi_{t_1\alpha}(u_n)\, Q(|x|)\,\mathrm{d}x\right)^{1/t_1} \|u_n - u\|_{L_Q^{t_2}} \\ &= \left(\int_{\mathbb{R}^2} \Phi_{t_1\alpha\|u_n\|_X^2}\left(\frac{u_n}{\|u_n\|_X}\right) Q(|x|)\,\mathrm{d}x\right)^{1/t_1} \|u_n - u\|_{L_Q^{t_2}} \leqslant C_1\|u_n - u\|_{L_Q^{t_2}}, \end{aligned}$$

where $1/t_1 + 1/t_2 = 1$, with $t_2 > \gamma$. This expression and the compactness of the embedding $X_{rad} \hookrightarrow L_Q^{t_2}(\mathbb{R}^2)$ (*cf.* Lemma 5.9) proves that $D_n = o_n(1)$.

From Hölder's inequality and Lemma 5.9, it follows that

$$A_n \leqslant \|u_n\|_{L_Q^\gamma}^{\gamma-1}\|u_n - u\|_{L_Q^\gamma} \leqslant C_2\|u_n\|_X^{\gamma-1}\|u_n - u\|_X.$$

Thus, there exists $C_3 > 0$ such that $|A_n| \leqslant C_3$, for any $n \in \mathbb{N}$. Hence, we can use $D_n \to 0$ and (5.33) to conclude that

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^2} f(u_n)(u_n - u)Q(|x|)\,\mathrm{d}x\right| \leqslant \varepsilon C_3.$$

Since $\varepsilon > 0$ is arbitrary, it follows that (5.32) holds. □

We now verify that I_λ satisfies the geometry of the Mountain Pass Theorem (*cf.* [5]).

Lemma 5.15. *Suppose that* (f_1)*-*(f_3) *hold. Then,*

(i) there exist $\tau, \rho > 0$ *such that* $I_\lambda(u) \geqslant \tau$*, whenever* $\|u\|_X = \rho$*;*

(ii) there exists $e \in X_{rad}$ *such that* $\|e\|_X > \rho$ *and* $I_\lambda(e) < 0$*.*

Proof. Let $\varepsilon > 0$, $q > \gamma$ and t_1, $t_2 > 1$ be such that $1/t_1 + 1/t_2 = 1$. Using (5.24) and Lemma 5.9, we obtain

$$\int_{\mathbb{R}^2} F(u)Q(|x|)\,\mathrm{d}x \leqslant \varepsilon C_1 \|u\|_X^\gamma + C_1\|u\|_X^q \left(\int_{\mathbb{R}^2} \Phi_{t_2\alpha}(u)Q(|x|)\,\mathrm{d}x\right)^{1/t_2}.$$

If $\rho_1 > 0$ is small in such a way that $t_2\alpha\rho_1^2 < 4\pi(b_0/2+1)$, we can use (5.12), (5.13) and Theorem 5.10 to get

$$\begin{aligned}\int_{\mathbb{R}^2} \Phi_{t_2\alpha}(u)Q(|x|)\,\mathrm{d}x &= \int_{\mathbb{R}^2} \Phi_{t_2\alpha\|u\|_X^2}\left(\frac{u}{\|u\|_X}\right)Q(|x|)\,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^2} \Phi_{t_2\alpha\rho_1^2}\left(\frac{u}{\|u\|_X}\right)Q(|x|)\,\mathrm{d}x \\ &\leqslant C_2,\end{aligned}$$

for all $0 < \|u\|_X \leqslant \rho_1$. If $\|u\|_X \leqslant \rho_1$ and $\varepsilon = 1/(4\lambda C_1)$, we obtain

$$I_\lambda(u) \geqslant \|u\|_X^2\left(\frac{1}{2} - \frac{1}{4}\|u\|_X^{\gamma-2} - C_3\|u\|_X^{q-2}\right).$$

Since $q > \gamma \geqslant 2$, and the expression in parentheses approaches $1/2$ as $\|u\|_X \to 0$, there exists a constant $\nu_0 > 0$ limiting it from below, for any $\|u\|_X = \rho$ sufficiently small, and therefore item (i) holds.

Now, let $K \subset \mathbb{R}^2$ be the support of $\varphi \in C_{0,rad}^\infty(\mathbb{R}^2)$. By (f_2) and (f_3), there exist $C_4, C_5 > 0$ such that $F(s) \geqslant C_4|s|^\mu - C_5$, for any $s \in \mathbb{R}$. Also notice $F(t\varphi(x)) = 0$ for $x \notin K$. Consequently, for $t > 0$,

$$I_\lambda(t\varphi) \leqslant \frac{t^2}{2}\|\varphi\|_X^2 - C_4 t^\mu \int_K |\varphi|^\mu Q(|x|)\,\mathrm{d}x + C_5\int_K Q(|x|)\,\mathrm{d}x.$$

Since $\mu > \gamma \geqslant 2$, item (ii) holds for $e := t_0\varphi$, with $t_0 > 0$ large enough. ∎

5.3 Proofs of Theorems 5.1 and 5.2

We start this section by presenting the proof of our existence (and regularity) result.

Proof of Theorem 5.1. In view of Lemma 5.15, we can define the minimax level

$$c_{MP}^{\lambda} := \inf_{g\in\mathcal{G}} \max_{t\in[0,1]} I_{\lambda}(g(t)) \geqslant \tau > 0,$$

where $\mathcal{G} := \{g \in C([0,1], X_{rad}) : g(0) = 0,\ I_{\lambda}(g(1)) < 0\}$. By using the Mountain Pass Theorem (*cf.* [5]), we obtain a sequence $(u_n) \subset X_{rad}$ such that

$$\lim_{n\to+\infty} I_{\lambda}(u_n) = c_{MP}^{\lambda}, \qquad \lim_{n\to+\infty} I_{\lambda}'(u_n) = 0.$$

We claim that, for $\lambda > 0$ large,

$$c_{MP}^{\lambda} < \frac{(\mu-2)}{2\mu}\frac{4\pi(b_0/2+1)}{\alpha_0}.$$

If this is true, it follows from Lemma 5.14 that, along a subsequence, $u_n \to u$ strongly in X. From the regularity of I_{λ} we obtain $I_{\lambda}'(u) = 0$ and $I_{\lambda}(u) \geqslant \tau > 0$, and therefore it follows from Proposition 5.13 that $u \neq 0$ is a weak solution of Problem (5.3).

For proving the existence of solution, it remains to prove the upper bound on c_{MP}^{λ}. In order to do that, we consider $\nu > \gamma$ given by (f_4). A standard minimization argument together with the compactness of the embedding $X_{rad} \hookrightarrow L_Q^{\nu}(\mathbb{R}^2)$ provides $w_0 \in X_{rad}$ such that

$$\|w_0\|_X^2 = S_{\nu} := \inf\left\{\|u\|_X^2 : u \in X_{rad},\ \int_{\mathbb{R}^2} |u|^{\nu} Q(|x|)\,\mathrm{d}x = 1\right\}.$$

It follows from (f_4) that

$$I_{\lambda}(w_0) \leqslant \frac{1}{2}\|w_0\|_X^2 - \lambda C \int_{\mathbb{R}^2} |w_0|^{\nu} Q(|x|)\,\mathrm{d}x = \frac{1}{2}S_{\nu} - \lambda C < 0,$$

whenever $\lambda > S_{\nu}/2C$. This shows that the curve $g_0(t) := tw_0$ belongs to $\mathcal{G}$. Therefore

$$c_{MP}^{\lambda} \leqslant \max_{t\in[0,1]} I_{\lambda}(g_0(t)) \leqslant \max_{t\geqslant 0}\left\{\frac{t^2}{2}S_{\nu} - \lambda\int_{\mathbb{R}^2} F(tw_0)Q(|x|)\,\mathrm{d}x\right\}.$$

To show that the maximum of $I_{\lambda}(tw_0)$ over $[0,+\infty)$ is well defined, one can proceed as in item (ii) of Lemma 5.15, using (f_4) to establish that $I_{\lambda}(tw_0) < 0$ for large t, while the fact that $I_\lambda(tw_0) > 0$ for small t can be shown as in item (i) of the same lemma. By (f_4), we have that $F(tw_0) \geqslant Ct^\nu |w_0|^\nu$, for any $t \geqslant 0$. Thus,

$$c_{MP}^\lambda \leqslant \max_{t\geqslant 0}\left\{\frac{t^2}{2}S_\nu - \lambda C t^\nu\right\} = h(\lambda) := \frac{\nu}{(\lambda C)^{2/(\nu-2)}}\left(\frac{S_\nu}{\nu}\right)^{\nu/(\nu-2)}\left(\frac{\nu-2}{2\nu}\right),$$

where we have used that the maximum of the function $t \mapsto t^2/2S_\nu - \lambda C t^\nu$ is attained in $t_\lambda = S_\nu/(\lambda C\nu)^{1/(\nu-2)}$. Since $\nu > \gamma \geqslant 2$, we have that $h(\lambda) \to 0$, as $\lambda \to +\infty$, and the claim is proved.

In order to obtain the regularity result, we call $(u, v) \in X_{rad} \times X_{rad}$ the solution given by the former argument. For a fixed $R > 0$, define the function $\widetilde{v}(|x|) := v(|x|) - v(R)$, for $x \in B_R$. From Remark 5.6, we can infer that $\widetilde{v} \in H_0^1(B_R)$ weakly solves

$$-\Delta\widetilde{v} = h, \text{ in } B_R, \qquad \widetilde{v} = 0, \text{ on } \partial B_R, \tag{5.34}$$

where $h(x) := V(|x|)u(|x|) - V(|x|)v(|x|)$. We shall prove that $h \in L^p(B_R)$ for a fixed $p > 1$ such that $pa_0, pb_0 > -2$. Indeed, using that $\limsup_{r\to 0} V(r)/r^{a_0} < +\infty$, we obtain $C_1 > 0$ such that

$$\int_{B_R} |h(x)|^p \,\mathrm{d}x \leqslant C_1 \int_{B_R} |x|^{pa_0}|u|^p \,\mathrm{d}x + C_1 \int_{B_R} |x|^{pa_0}|v|^p \,\mathrm{d}x.$$

Since $pa_0 > -2$, we can pick $t_1 > 1$ such that $|x|^{t_1 pa_0} \in L^1(B_R)$. This, together with Hölder's inequality and Remark 5.6, yield

$$\int_{B_R} |h(x)|^p \,\mathrm{d}x \leqslant C_1 \left(\int_{B_R} |x|^{t_1 pa_0} \,\mathrm{d}x\right)^{1/t_1} \left(\|u\|^{t_2}_{L^{t_2 p}(B_R)} + \|v\|^{t_2}_{L^{t_2 p}(B_R)}\right) < +\infty,$$

where $1/t_1 + 1/t_2 = 1$, proving the claim. Therefore, by classical elliptic regularity theory we conclude that $v = \widetilde{v} + v(R) \in W^{2,p}(B_R)$.

Now, considering $\widetilde{u}(|x|) := u(|x|) - u(R)$, then $\widetilde{u} \in H_0^1(B_R)$ is a solution of problem

$$-\Delta\widetilde{u} = g, \text{ in } B_R, \qquad \widetilde{u} = 0, \text{ on } \partial B_R,$$

where $g(x) := \lambda Q(|x|)f(u(|x|)) - V(|x|)v(|x|)$. Arguing as above, we can prove that $V(|\cdot|)v \in L^p(B_R)$. Moreover, from (5.21) with $q = 1$ and (5.12), we obtain

$$\int_{B_R} |f(u)Q(|x|)|^p \,\mathrm{d}x \leqslant C_2 \int_{B_R} |u|^{p(\gamma-1)}|Q(|x|)|^p \,\mathrm{d}x + C_2 \int_{B_R} \Phi_{p\alpha}(u)|Q(|x|)|^p \,\mathrm{d}x. \tag{5.35}$$

Using (5.9), Hölder's inequality and Remark 5.6, we get

$$\begin{aligned}\int_{B_R} |u|^{p(\gamma-1)}|Q(|x|)|^p \, dx &\leqslant C_3 \int_{B_R} |x|^{pb_0}|u|^{p(\gamma-1)} \, dx \\ &\leqslant C_3 \left(\int_{B_R} |x|^{t_3pb_0} \, dx\right)^{1/t_3} \|u\|^{p(\gamma-1)}_{L^{t_4p(\gamma-1)}(B_R)} < +\infty,\end{aligned} \tag{5.36}$$

where $1/t_3 + 1/t_4 = 1$ and $t_3pb_0 > -2$. On other hand, Young's inequality yields

$$u(|x|)^2 \leqslant 2\widetilde{u}(|x|)^2 + 2u(R)^2.$$

So, we can use (5.9), the inequality $\Phi_\alpha(s) \leqslant e^{\alpha s^2}$, Hölder's inequality and the classical Trudinger-Moser inequality (*cf.* Theorem 5.19) to obtain

$$\begin{aligned}\int_{B_R} \Phi_{p\alpha}(u)|Q(|x|)|^p \, dx &\leqslant C_4 e^{2p\alpha u(R)^2} \int_{B_R} |x|^{pb_0} e^{2p\alpha\widetilde{u}^2} \, dx \\ &\leqslant C_5 \left(\int_{B_R} e^{2t_4p\alpha\widetilde{u}^2} \, dx\right)^{1/t_4} < +\infty.\end{aligned}$$

The above estimate, (5.35) and (5.36), show that $Q(|\cdot|)f(u) \in L^p(B_R)$. Hence, we conclude as before that $u \in W^{2,p}(B_R)$. Since the embedding $W^{2,p}(B_R) \hookrightarrow C^\sigma(\overline{B_R})$ is continuous, for some $\sigma \in (0,1)$, then u, v are locally Hölder continuous.

Suppose now that V is locally Hölder continuous. By the former proof, the functions u, v are locally Hölder continuous, and hence $h(x) := V(|x|)u(|x|) - V(|x|)v(|x|)$ belongs to $C^\sigma(\overline{B_R})$, for some $\sigma \in (0,1)$. Since $\widetilde{v}$ solves (5.34), by classical elliptic regularity theory $v = \widetilde{v} + v(R) \in C^{2,\sigma}(\overline{B_R})$. □

For the proof of our multiplicity result we shall the following abstract result (see [5, 32]).

Theorem 5.16 (Symmetric Mountain Pass Theorem)**.** *Suppose that $\mathcal{X}$ is a real Banach space and $\mathcal{I} \in C^1(\mathcal{X},\mathbb{R})$ is an even functional satisfying $\mathcal{I}(0) = 0$,*

($\mathcal{I}_1$) *there are constants $\rho, \tau > 0$ such that $\mathcal{I}(u) \geqslant \tau$, for all $u \in \partial B_\rho(0)$;*

($\mathcal{I}_3$) *there are $\kappa > 0$ and a subspace $\mathcal{V} \subset \mathcal{X}$ such that $\dim \mathcal{V} = m \in \mathbb{N}$ and*

$$\max_{u\in\mathcal{V}} \mathcal{I}(u) \leqslant \kappa$$

and the $(PS)_c$ condition for any $0 < c < \kappa$. Then it possesses at least m pairs of nonzero critical points.

We are ready to finish this chapter:

Proof of Theorem 5.2. We are intending to apply the above theorem for the functional I_λ. It is clear that $I_\lambda(0) = 0$ and I_λ is even, since we are supposing f odd. Moreover, condition $(\mathcal{I}_1)$ is a consequence of the first statement in Lemma 5.15.

Given $m \in \mathbb{N}$, consider

$$V_m := \text{span}\{\varphi_1, ..., \varphi_m\},$$

where $\{\varphi_i\}_{i=1}^m \subset C_0^\infty(\mathbb{R}^2)$ have disjoint supports, and notice that $\{\phi_1, ..., \phi_m\}$ is an orthogonal set in X_{rad}. Since all norms are equivalent in V_m, we obtain a positive constant $C_1 = C_1(m) > 0$ such that

$$\|u\|_X^\nu \leqslant C_1 \|u\|_{L_Q^\nu}^\nu, \qquad \forall\, u \in V_m.$$

Hence, it follows from (f_4) that

$$I_\lambda(u) \leqslant \frac{1}{2}\|u\|_X^2 - \lambda C \|u\|_{L_Q^\nu}^\nu \leqslant \frac{1}{2}\|u\|_X^2 - \lambda \frac{C_2}{\nu}\|u\|_X^\nu, \qquad \forall\, u \in V_m,$$

where $C_2 = C_1 C$.

We now consider the function

$$g(t) := \frac{t^2}{2} - \lambda \frac{C_2}{\nu} t^\nu, \qquad t \geqslant 0.$$

Since $\nu > 2$, it attains its maximum value at the point $t_* = (\lambda C_2)^{-1/(\nu-2)}$, which implies

$$I_\lambda(u) \leqslant A_{m,\lambda} := g(t_*) = \left(\frac{1}{2} - \frac{1}{\nu}\right)\left(\frac{1}{\lambda C_2}\right)^{2/(\nu-2)}, \qquad \forall\, u \in V_m.$$

Since $A_{m,\lambda} \to 0$, as $\lambda \to +\infty$, we can find $\lambda_m > 0$ such that

$$0 < A_{m,\lambda} < \frac{(\mu - 2)}{2\mu} \frac{4\pi(b_0/2 + 1)}{\alpha_0},$$

for any $\lambda > \lambda_m$. It follows from Lemma 5.14 and Theorem 5.16 that I_λ has at lest m pairs of nonzero critical points. $\square$

5.4 Appendix

In order to present the **Principle of the Symmetric Criticality**, we need the following:

Definition 5.17. *The action of a topological group $(G,*)$ with identity* $\mathbf{e}$ *on a normed space $(X,\|\cdot\|)$ is a continuous map*

$$A : G \times X \to X,$$

with $A(g,u)$ denoted by gu, satisfying for any $g,h \in G$ and $u \in X$

$$\mathbf{e}u = u, \quad (g*h)u = g(hu), \quad u \mapsto gu \textit{ is linear.}$$

The action is isometric if

$$\|gu\| = \|u\|, \quad \forall g \in G \textit{ and } u \in X$$

The space of invariant points is defined by

$$Fix(G) := \{u \in X : gu = u, \quad \forall g \in G\}.$$

A function $I : X \to \mathbb{R}$ is invariant if $I(gu) = I(u)$ for all $g \in G$.

Sometimes it is of interest to find a critical point of a functional restricted to a subspace of a Banach space that satisfies certain symmetry properties. The following result establishes the conditions under which such a critical point is also a critical point of the functional on the entire space. It is very important to deal with the radial solutions of system $(\mathcal{S}_\lambda)$, in Section 5.1.

Theorem 5.18 (Principle of Symmetric Criticality, [63, 82])**.** *Suppose that the action of a topological group G on the Hilbert space X is isometric, $I \in C^1(X,\mathbb{R})$ is invariant and u is a critical point of I restricted to $Fix(G)$, that is $I'(u)\varphi = 0$, for all $\varphi \in Fix(G)$. Then u is a critical point of I.*

The following result was used in the proof of Theorems 5.10 and 5.1.

Theorem 5.19 (Classical Trudinger-Moser inequality [58, 78])**.** *Suppose that $\alpha > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in W_0^{1,N}(\Omega)$, with $N \geqslant 2$. Then $e^{\alpha|u|^{\frac{N}{N-1}}} \in L^1(\Omega)$. Moreover, there exists a constant $C = C(N) > 0$ such that*

$$\sup_{\int_\Omega |\nabla u|^N \, dx \leqslant 1} \int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}}\, dx \begin{cases} \leqslant C & \textit{if } \alpha \leqslant \alpha_N; \\ = \infty & \textit{if } \alpha > \alpha_N, \end{cases}$$

where $\alpha_N = N\omega_N^{\frac{1}{N-1}}$, and ω_N is the area of the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$.

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