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Translation and Homogeneous Surfaces in Homogeneous 3-Spaces

Tarcios Andrey Ferreira

Advisor: João Paulo dos Santos

Departamento de Matemática
Universidade de Brasília

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Resumo

Título: Superfícies de Translação e Homogêneas em 3-espaços Homogêneos.

Neste trabalho consideramos superfícies em espaços homogêneos tridimensionais, com foco em superfícies de translação e superfícies extrinsecamente homogêneas.

Inicialmente, apresentamos uma classificação de superfícies mínimas, solitons de translação e solitons conformes como superfícies de translação no espaço hiperbólico, considerando o caso em que uma curva está contida em uma horosfera e a outra, em um plano hiperbólico totalmente geodésico. Para esse cenário, demonstramos a rigidez das três classes, no sentido de que as superfícies resultantes se reduzem a casos conhecidos na literatura, como planos hiperbólicos e cilindros catenários no caso das superfícies mínimas. Para os solitons, além dos planos e das horosferas, identificamos também os exemplos clássicos da literatura, como os cilindros do tipo "grim reaper".

Para superfícies na esfera, investigamos as superfícies de translação quanto à curvatura gaussiana e à curvatura média constantes. Nossos resultados demonstram que não existem superfícies totalmente geodésicas e totalmente umbílicas que também sejam superfícies de translação. Além disso, os resultados apontam para a rigidez dos toros de Clifford como superfícies de translação mínimas e de curvatura média constante (CMC). Apresentamos ainda uma relação útil e interessante entre superfícies de translação na esfera e o espaço Euclidiano tridimensionais.

Outro tema abordado neste trabalho é o da homogeneidade em variedades, isto é, a existência de isometrias que levem qualquer ponto de uma variedade a outro ponto da mesma. Mais especificamente, analisamos a classificação das hipersuperfícies extrinsecamente homogêneas e suas folheações, as quais aparecem como órbitas de ações isométricas transitivas à esquerda de subgrupos do grupo de isometrias das variedades em questão. Consideramos ainda as classificações entre grupos unimodulares e não unimodulares, no contexto de grupos de Lie tridimensionais. Tal classificação nos permite apresentar exemplos de superfícies totalmente geodésicas e mínimas, além, é claro, de fornecer, como consequência da homogeneidade, exemplos de superfícies de curvatura média constante (CMC).

Palavras-chave: superfícies mínimas, translator, solitons conformes, fluxo da curvatura média, curvatura média constante, superfícies de translação, espaço hiperbólico, esfera tridimensional, superfícies homogêneas, grupos de Lie, espaços homogêneos, ação isométrica, ação de cohomogeneidade um, curvaturas principais constantes.

Abstract

In this work we consider surfaces in 3-dimensional homogeneous spaces, focusing on translation surfaces and extrinsically homogeneous surfaces.

We begin by classifying minimal surfaces, translating solitons, and conformal solitons that arise as translation surfaces in hyperbolic space, focusing on the case where one generating curve lies in a horosphere and the other in a totally geodesic hyperbolic plane. For this setting, we prove rigidity results for all three classes, showing that the resulting surfaces reduce to known cases in the literature, such as hyperbolic planes and catenary-type cylinders in the minimal case. For the solitons, in addition to planes and horospheres, we recover classical examples from the literature, such as Grim Reaper-type cylinders.

For surfaces in the sphere, we investigate translation surfaces with constant Gaussian curvature and constant mean curvature (CMC). Our results show that there are no totally geodesic or totally umbilical surfaces that are also translation surfaces. Furthermore, the results point to the rigidity of Clifford tori as minimal and CMC translation surfaces in the sphere. We also present an interesting and useful relationship between translation surfaces in the sphere and in three-dimensional Euclidean space.

Another topic addressed in this work is that of homogeneity in manifolds, that is, the existence of isometries that map any point of a manifold to any other. More specifically, we analyze the classification of extrinsically homogeneous hypersurfaces and their foliations, which arise as orbits of left-transitive isometric actions by subgroups of the isometry group of the ambient manifold. We also consider the distinction between unimodular and non-unimodular groups in the context of three-dimensional Lie groups. This classification allows us to present examples of totally geodesic and minimal surfaces, and, as a consequence of homogeneity, examples of constant mean curvature (CMC) surfaces.

Keywords: minimal surfaces, translators, conformal solitons, mean curvature flow, constant mean curvature, translation surfaces, hyperbolic space, three-dimensional sphere, homogeneous surfaces, Lie groups, homogeneous spaces, isometric action, cohomogeneity one action, constant principal curvatures.

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Introduction

This thesis investigates 2-dimensional surfaces in 3-dimensional metric Lie groups, focusing on two distinct categories: translation surfaces and extrinsically homogeneous surfaces. Roughly speaking, a translation surface in a 3-dimensional Lie group M is constructed from the group product of two smooth curves in the ambient space. Specifically, given two smooth curves α and β in M, the translation surface Σ generated by α and β is defined by the product $\alpha \cdot \beta$, where $(\cdot)$ denotes the Lie group product operation. Within this context, we examine translation surfaces in the standard 3-sphere $\mathbb{S}^3$ and 3-dimensional hyperbolic space $\mathbb{H}^3$, where our primary focus is on their extrinsic geometry, analyzing structures such as minimal surfaces, surfaces with constant mean curvature (CMC), and solitons to the mean curvature flow.

A surface Σ in an ambient space M is defined extrinsically homogeneous if it is an orbit under an isometric action of the ambient space. Our objective in this area is to classify extrinsically homogeneous surfaces within 3-dimensional homogeneous spaces.

The subsequent sections will elaborate on the main topics and problems addressed in this thesis, along with an overview of the obtained results.

Translation surfaces of $\mathbb{H}^3$ and $\mathbb{S}^3$: Minimal and CMC Surfaces, and Solitons to the Mean Curvature Flow

The theory of geometric flows in Riemannian manifolds was widely studied in the last decades, especially the subject of mean curvature flow in Euclidean spaces, giving rise to a vast literature. Following [14], but in the context of Riemannian manifolds, an immersion of a smooth manifold M^n into a Riemannian ambient space $\widetilde{M}^{n+1}$ evolves under the mean curvature flow (MCF for short) if there exists a smooth one-parameter family of immersions that satisfy an specific evolution equation that rely on the mean curvature vector field $\vec{H}$ of such immersions. Extrinsic geometric flows constitute evolution equations that describe hypersurfaces of a Riemannian manifold evolving in the normal direction with velocity given by the corresponding extrinsic curvature.

Maybe the simplest solution to the mean curvature flow is the minimal surfaces that are the solutions where the flow is constant or static. Another special class of solutions is that of the solitons, also known as the self-similar solutions. Self-similar solutions have played an important role in the development of the theory of the MCF, for example, in the euclidean space, where they serve as comparison solutions to investigate the formation of singularities.

We investigate solitons to the MCF considering a particular class of surfaces called translation surfaces. Following [25], the origin of such surfaces is in the classical text of Darboux [13] where they are presented and later known as Darboux surfaces. Such surfaces are defined as the movement of a curve by a uniparametric family of rigid motions of $\mathbb{R}^3$. Hence a parameterization of a such surface is given by $\Phi(s,t) = A(t) \cdot \alpha(s) + \beta(t)$, where $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ and $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$ are two curves and $A(t)$ is an orthogonal matrix. More precisely, if $A(t)$ is the identity map, a surface $S \subset \mathbb{R}^3$ that can be locally written as the sum of two curves $\Phi(s,t) = \alpha(s) + \beta(t)$, is called a translation surface and the curves α and β are called the generating curves of S. Darboux worked with translation surfaces in [13, pp. 137-142] and such name is given due to the fact that the surface S is obtained by the translation of a curve along the other.

Indeed, in a general Lie group G with product $(\cdot)$ one can define a translation surface $S \subset G$ as a surface in G that can be locally written as the product $\Psi(s,t) = \alpha(s) \cdot \beta(t)$ of two curves $\alpha : I \subset \mathbb{R} \to G$ and $\beta : J \subset \mathbb{R} \to G$. These curves α and β are called the generating curves of S. This work draws inspiration from previous works that have appeared on the study of translation surfaces with constant mean curvature (CMC) as [25, 30, 33, 35, 36, 47, 55]. These prior works primarily focused on Thurston 3-dimensional geometries, which are also Lie groups. A central aim of this thesis is to specifically investigate CMC translation surfaces in 3-dimensional non-Euclidean space forms, namely hyperbolic space $\mathbb{H}^3$ and the 3-sphere $\mathbb{S}^3$. Chapters 2 and 3 are dedicated to these respective cases.

Regarding translation surfaces in the hyperbolic 3-space $\mathbb{H}^3$, Chapter 2 begins by outlining the Lie group structure of $\mathbb{H}^3$ that we'll be using. This structure is defined by considering the upper half-space model and the group of similarities of $\mathbb{R}^2$. Once we establish this algebraic framework, we present two kinds of translation surfaces that will be considered. In both cases, the surfaces are parametrized such that one of the generating curves lies within a horosphere and the other in a totally geodesic plane. More precisely, such kinds are given when the generating curves are local graphs. We then apply this structure to obtain results on arbitrary curves, provided they are contained in these specified subsets. Our results are categorized in three classes of surfaces in $\mathbb{H}^3$: minimal surfaces, hyperbolic translators (see [14]) and conformal solitons to the mean curvature flow (see [40]). The results serve as both classification and rigidity results, since they provide conclusive information about the type of surface obtained when the generating curves are as described above, and these conclusions align with known results in the existing literature. Specifically, we prove that minimal surfaces are either totally geodesic planes or a class of minimal surfaces in $\mathbb{H}^3$, that were described in [34] as singular minimal surfaces in $\mathbb{R}^3$. Such surfaces are weighted minimal surfaces in $\mathbb{R}^3$ and, when viewed as subsets $\mathbb{H}^3$ in the upper half-space model, they are minimal surfaces. It is important to note that our definition of translation surfaces is significantly more general than the one considered in [33], where translation surfaces in $\mathbb{H}^3$ within the half-space model were given as translation surfaces in $\mathbb{R}^3$, again for particular classes of curves. For the hyperbolic translators and conformal solitons to the mean curvature flow, we show that when translation surfaces are generated by curves contained in a totally geodesic plane and in a horosphere, the resulting surfaces are either the totally geodesic planes or the so-called grim-reapers cylinders in each corresponding context, i.e., the translators as given in [14] and the conformal solitons in [40].

In Chapter 3, we consider translation surfaces in the 3-sphere $\mathbb{S}^3$. It is well-known that the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ carries a structure of a Lie group, with a bi-invariant metric, when viewed through its quaternionic structure. Such a structure plays a crucial role in the context of flat surfaces, from the classical Bianchi-Spivak construction for flat surfaces (see [20] and [53]) to a more sophisticated construction given in [31], still been significant today, as we can see in works such as [1, 21, 38]. In this thesis, our findings on translation surfaces in $\mathbb{S}^3$ are presented as rigidity results. To achieve this results, we first stablish the local geometry of generic translation surfaces in $\mathbb{S}^3$ by means of their generating curves, a critical element for understanding such a local geometry, and for subsequent results, is the introduction of a suitable frame field, which has its own interest. From such a frame, geometric objects like the Gaussian curvature and the mean curvature are fully described, it plays a fundamental role in proving our main results. We highlight the non-existence of minimal surfaces when torsions and non-vanishing curvatures of the generating curves are constant, the rigidity of the CMC Clifford tori by means of conditions on the generating curves, and a correspondence between translating surfaces in the Euclidean 3-space $\mathbb{R}^3$ and in the 3-sphere $\mathbb{S}^3$, which also provides some interesting applications.

Homogeneous surfaces

When studying group actions in Riemannian geometry, it is natural to focus on the isometry group, which consists of all transformations of the manifold that preserve distances. The action of a subgroup of the isometry group on a Riemannian manifold is called an isometric action, and the cohomogeneity of such an action is defined as the minimal codimension among its orbits. Each orbit of an isometric action is referred to as an (extrinsically) homogeneous submanifold, and the collection of all orbits defines the orbit foliation of the action. One of the main objectives of this thesis is to study these orbit foliations in order to determine which submanifolds are homogeneous up to isometries.

A singular Riemannian foliation is a decomposition of a Riemannian manifold into immersed submanifolds, called leaves, which may vary in dimension but are locally equidistant (see [2, 46]). Orbit foliations arising from isometric group actions provide the standard examples of singular Riemannian foliations. In particular, the collection of homogeneous submanifolds forms a special case, which we refer to as a homogeneous foliation.

The study of homogeneous submanifolds has developed into a significant area of research over the past century. The classification of homogeneous hypersurfaces in space forms dates back to the early 20th century, with foundational contributions from Cartan, Levi-Civita, Segre, and Somigliana (see [4, 5, 32, 51, 52]). These works focused on hypersurfaces whose nearby parallel hypersurfaces have constant mean curvature, known as isoparametric hypersurfaces. The corresponding foliations of codimension one are called isoparametric foliations. These structures originally appeared in problems related to geometric optics.

In space forms, Cartan demonstrated that a hypersurface has constant principal curvatures if and only if it is isoparametric, and this property also characterizes homogeneous hypersurfaces in Euclidean and hyperbolic spaces. However, the situation in spheres is notably different. The classification of homogeneous hypersurfaces in spheres began with [27], where it was shown that such hypersurfaces have exactly 1, 2, 3, 4, or 6 distinct constant principal curvatures. This result was later extended by Münzner [48], who proved that the same numerical restriction applies to general isoparametric hypersurfaces, though not all of them are homogeneous. Notably, [19] provided counterexamples using representations of Clifford algebras. These examples highlighted the complexity of the classification problem and led to its inclusion in Yau's list of open problems in geometry [54]. Subsequent works [6, 9, 10, 29, 44, 45] completed the classification of isoparametric hypersurfaces in spheres.

In the more general three-dimensional setting, the classification of homogeneous surfaces in Thurston geometries that are not space forms has been addressed in different works. In [17], the authors provide a full classification in the so-called $\mathbb{E}(\kappa,\tau)$ spaces. The classification of isoparametric surfaces in the remaining cases, that is, in homogeneous spaces that are neither space forms nor $\mathbb{E}(\kappa,\tau)$ spaces, remains an open problem. However, the relationship between homogeneous and isoparametric surfaces continues to hold in the context of the remaining homogeneous spaces. The classification of homgeneous surfaces considered in Chapter 4 is based on whether the ambient space is a unimodular or non-unimodular Lie group, a distinction that depends on whether its Lie algebra is unimodular, i.e., whether the left and right Haar measures coincide.

A precise correspondence can be established between the subalgebras of the isometry groups of these spaces and their respective Lie algebras. Utilizing this correspondence, we classify the 2-dimensional subalgebras of such Lie algebras up to conjugacy. This classification allows us to identify the connected subgroups that serve as homogeneous orbits passing through the identity element of the group. With this in hand, we construct a foliation by determining the geodesics that intersect these subgroups orthogonally at the identity. The homogeneity of the ambient spaces ensures that this foliation is well-defined. Finally, these geodesics enable us to study the geometry of the equidistant surfaces through the computation of the shape operator at a given distance, thereby completing the classification of the homogeneous surfaces in homogeneous 3-spaces. This classification provides the cohomogeneity one part of the analysis of polar actions on these spaces given in [16].

Chapter 1

Preliminaries and conventions

In this chapter, we introduce the basic notions and conventions necessary for this thesis. Unless otherwise stated, the notations and conventions presented here will be used throughout the work. Section 1.1 covers fundamental concepts related to the geometry of Riemannian submanifolds. In Section 1.2, we discuss solitons of the Mean Curvature Flow in Riemannian manifolds and provide the definition of translation surfaces in metric Lie groups. Finally, Section 1.3 presents key aspects of the theory of isometric group actions and homogeneous manifolds.

1.1 Geometry of submanifolds

This section is based on [53, Chapter 7], and further details can be found there. In this section, we review some foundational concepts from the theory of submanifolds in Riemannian geometry.

Let M be a Riemannian submanifold of a Riemannian manifold $\widetilde{M}$. The *normal bundle* of M, denoted by νM, consists of vectors orthogonal to the tangent bundle TM. The space of smooth sections of νM is denoted by $\Gamma(\nu M)$. At each point $p \in M$, we have the (non-orthogonal) direct sum decomposition $T_p\widetilde{M} = T_pM \oplus \nu_pM$. Given a vector field X on $\widetilde{M}$ along M, we denote by $X^\perp$ and $X^\top$ its projections onto νM and TM, respectively.

While one may study the intrinsic geometry of both $\widetilde{M}$ and M, we are also interested in the *extrinsic geometry* of M, which describes how M is embedded in $\widetilde{M}$. This is captured by the second fundamental form.

Let $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections on $\widetilde{M}$ and M, respectively, and let $\tilde{R}$ and R be their curvature tensors. The second fundamental form II is defined via the *Gauss formula*:

$$\tilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X,Y), \quad \forall X,Y \in \Gamma(TM), \tag{1.1}$$

Hence, $\mathrm{II}(X,Y) = (\tilde{\nabla}_X Y)^{\perp}$.

Let $\xi \in \Gamma(\nu M)$ be a unit normal vector field. The *shape operator* associated with ξ is the self-adjoint endomorphism defined by

$$\langle S_\xi X, Y\rangle = \langle \mathrm{II}(X,Y), \xi\rangle.$$

The *normal connection* is given by $\nabla^{\perp}_X \xi = (\tilde{\nabla}_X \xi)^{\perp}$, leading to the *Weingarten formula*:

$$\tilde{\nabla}_X \xi = -S_\xi X + \nabla^{\perp}_X \xi.$$

The curvature tensors of $\tilde{M}$ and M are related by the *Gauss equation*:

$$\langle \tilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle - \langle \mathrm{II}(Y,Z), \mathrm{II}(X,W)\rangle + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle.$$

A submanifold M is said to be *totally geodesic* if every geodesic in M is also a geodesic in $\tilde{M}$, which is equivalent to $\mathrm{II} = 0$. A submanifold is *totally umbilical* if there exists $\lambda \in \mathbb{R}$ such that $\mathrm{II} = \lambda\langle\cdot,\cdot\rangle$; in particular, $\lambda = 0$ implies that M is totally geodesic.

Definition 1.1. *The* mean curvature vector *H of a Riemannian submanifold $M \subset \tilde{M}$ is defined as the trace of the second fundamental form. With respect to a local orthonormal frame $\{E_i\}$ of TM, we have*

$$H = \sum_i \langle E_i, E_i\rangle \mathrm{II}(E_i, E_i).$$

The submanifold M is called minimal *if and only if $H = 0$.*

Minimal submanifolds naturally arise as critical points of the volume functional and are central objects of study in differential geometry. Moreover, the mean curvature of M with respect to a unit normal field ξ is the trace of the associated shape operator S_ξ.

Now suppose that M is a *hypersurface* in $\tilde{M}$, that is, a submanifold of codimension one. Then, locally and up to sign, there exists a unique unit normal vector field $\xi \in \Gamma(\nu M)$. Since $\langle \xi, \xi\rangle = 1$, the second fundamental form II is simply a multiple of ξ, and the Gauss and Weingarten formulas simplify to:

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle S_\xi X, Y\rangle \xi, \qquad \tilde{\nabla}_X \xi = -S_\xi X.$$

Consequently, the Gauss equation become

$$\langle \tilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle - \langle S_\xi Y, Z\rangle\langle S_\xi X, W\rangle + \langle S_\xi X, Z\rangle\langle S_\xi Y, W\rangle.$$

Since the mean curvature vector H is proportional to the normal vector ξ, it is customary for hypersurfaces to speak of the *mean curvature* of the hypersurface, defined as the trace of the shape operator S_ξ.

Definition 1.2. *Let ξ be a unit normal vector field defined on an open subset $U \subset M$. A function $\lambda : U \to \mathbb{R}$ is called a* principal curvature *of M (associated with ξ) if there exists a nonzero vector field $X \in \Gamma(TU)$ such that $S_\xi X = \lambda X$. The eigenspace at $p \in U$ is denoted $T_\lambda(p)$ and is referred to as the* principal curvature space *corresponding to $\lambda(p)$. Any nonzero vector $X \in T_\lambda(p)$ is a* principal curvature vector.

Since $\tilde{M}$ is Riemannian, the shape operator S_ξ is self-adjoint and hence diagonalizable at each point. In general, the dimension of the principal curvature space (its *geometric multiplicity*) may vary, but in the Riemannian case, it coincides with the *algebraic multiplicity* (the multiplicity of λ as a root of the characteristic polynomial). Thus, we simply speak of the *multiplicity* of a principal curvature.

A connected hypersurface is said to have *constant principal curvatures* if the eigenvalues of S_ξ are constant across the entire hypersurface. In that case, the dimension of each principal curvature space is constant, and we denote the distribution of eigenspaces by T_λ, with $\Gamma(T_\lambda)$ being the space of vector fields $X \in \Gamma(TM)$ such that $S_\xi X = \lambda X$.

1.2 Solitons to the Mean Curvature Flow on Translation Surfaces in metric Lie groups

This section is based on [14], [25], [36] and [40] and further details can be found there.

The study of *geometric flows* in Riemannian manifolds, particularly the mean curvature flow (MCF) in Euclidean space, has flourished over recent decades. Following [14], let $f : M^n \to \widetilde{M}^{n+1}$ be a smooth immersion of a manifold M^n into a Riemannian ambient space $\widetilde{M}^{n+1}$. We say that f evolves by mean curvature flow if there exists a smooth one–parameter family of immersions $F : M \times I \to \widetilde{M}$, $I = [0,T)$, such that $F_0 = f$ and

$$\frac{\partial F}{\partial t}(p,t) = \vec{H}(p,t), \qquad (p,t) \in M \times I, \tag{1.2}$$

where $\vec{H}(p,t)$ denotes the mean-curvature vector of $F_t = F(\cdot,t)$.

The most elementary stationary solutions of (1.2) are *minimal submanifolds*, for which $\vec{H} \equiv 0$. Beyond these, an especially important class of soltuions are the class of the *solitons* (or *self–similar*) solutions, which model singularity formation by evolving through ambient isometries.

Soliton solutions are generated by the Killing field defined by one–parameter subgroups of isometries of the ambient space. A soliton solution to the mean curvature flow (MCF) is given by $F(p,t) = \Gamma_t(f(p))$ where Γ_t is a one-parameter subgroup of the isometry group of $\widetilde{M}$. According to [14], denote by $\xi \in T\widetilde{M}$ the corresponding Killing vector field, it is well known (see [28]) that a smooth immersion $f : M^n \to \widetilde{M}^{n+1}$ evolves under a soliton solution to the MCF associated with Γ_t if and only if its mean curvature H and unit normal vector N satisfy, up to tangential diffeomorphisms,

$$H = \langle N, \xi \rangle. \tag{1.3}$$

An important class of such solutions is given by the translating solitons in Euclidean space $\mathbb{R}^{n+1}$, that is, solutions that evolve by translation in a fixed direction $v \in \mathbb{R}^{n+1}$. In this case, an immersion $f : M^n \to \mathbb{R}^{n+1}$ that satisfies equation (1.3) with $\xi = v$ is called a translator.

Another noteworthy family, not necessarily associated to Killing fields, consists of *conformal solitons*. However, such solitons are not related to the *translators* approached previously, since their generating fields are not necessarily Killing. Indeed, all killing fields are trivially conformal, but not all conformal fields are killing. Following [40], let $f : M^n \to \widetilde{M}^{n+1}$ be an isometric immersion. If there exists a vector field ξ such that $\vec{H} = \xi^{\perp}$, then f is called a *soliton with respect to* ξ. When ξ is conformal, f is a *conformal soliton*.

In particular, for a tridimensional manifold $\widetilde{M}^3$, solitons where ξ is a conformal field that satisfy

$$H = \langle N, \xi \rangle. \tag{1.4}$$

Recall that a vector field ξ is called *conformal* if its flow consists of conformal transformations, meaning that it preserves angles, though not necessarily distances. Formally, ξ is conformal if the Lie derivative of the metric g with respect to ξ satisfies

$$\mathcal{L}_\xi g = 2\lambda g,$$

where λ is a smooth function known as the potential function. If λ is constant, then ξ is called a homothetic vector field and in particular, if λ is identically zero, then ξ is a Killing vector field."

One of the main objectives of this work is to investigate solutions to the mean curvature flow (MCF) that arise as *translation surfaces*. Following [25], the concept of translation surfaces originates from the classical work of Darboux [13], where they were first introduced and later became known as *Darboux surfaces*. Translation surfaces can be constructed via a one-parameter family of rigid motions in $\mathbb{R}^3$. Given such a family, these surfaces are generated by the motion of a curve under rigid transformations. More precisely, for an orthogonal matrix $A(t)$ and curves $\alpha(s)$ and $\beta(t)$, the surface can be parametrized as $\Phi(s,t) = A(t) \cdot \alpha(s) + \beta(t)$. In the special case where $A(t)$ is the identity for all t, this expression simplifies to $\Phi(s,t) = \alpha(s) + \beta(t)$, which defines a *translation surface*. The curves α and β are referred to as the *generating curves* of the surface S. Darboux studied such surfaces in detail in [13, pp. 137–142], and the name translation surface reflects the fact that S is obtained by translating one curve along another. We now present a classical example of such a surface:

Example 1.1 ([25]). *Let $\alpha(s) = (0, s, p(s))$ and $\beta(t) = (t, 0, q(t))$. Then the corresponding translation surface is given by*

$$X(s,t) = \alpha(s) + \beta(t) = (t, s, p(s) + q(t)).$$

Scherk showed that this surface is minimal only in two cases: either it is a plane, or the functions p and q satisfy

$$p(s) = \frac{\ln|\cos(rs)|}{r}, \quad q(t) = -\frac{\ln|\cos(rt)|}{r},$$

for some constant $r > 0$. Such a surface is known as a Scherk-type surface.

We now extend the concept of translation surfaces to a more general setting. Let G be a Lie group. Recall the following definition:

Definition 1.3. *Let G be a Lie group with differentiable product*

$$\begin{array}{rcccc} p & : & G \times G & \to & G \\ & & (g,h) & \mapsto & g * h. \end{array}$$

Then, for a fixed $g \in G$, the left *and* right translations *by g are defined respectively as*

$$L_g(h) = g * h, \quad R_g(h) = h * g.$$

Definition 1.4. *A surface $S \subset G$ is called a* translation surface *in the Lie group G if, locally around regular points, it can be written as*

$$\Psi(s,t) = \alpha(s) \cdot \beta(t),$$

where $\alpha : I \subset \mathbb{R} \to G$ and $\beta : J \subset \mathbb{R} \to G$ are smooth curves, and "$\cdot$" denotes the Lie group operation. The curves α and β are called the generating curves *of S.*

This work is inspired by various contributions in the literature concerning translation surfaces with constant mean curvature, including [25, 30, 33, 35, 36, 47, 24, 55].

1.3 Isometric Actions and Homogeneous Manifolds

This section is based on [11, Chapter 3] and further details can be found there. In this section, we review fundamental concepts related to isometric group actions on Riemannian manifolds.

We begin with the definition of an isometric action:

Definition 1.5. *Let $\widetilde{M}$ be a Riemannian manifold and G a Lie group. An* isometric action *is a smooth map*

$$\begin{aligned} \varphi : G \times \widetilde{M} &\to \widetilde{M} \\ (g,p) &\mapsto g(p), \end{aligned}$$

where for each $g \in G$, the map $\varphi_g : \widetilde{M} \to \widetilde{M}$ defined by $\varphi_g(p) = gp$ is an isometry of $\widetilde{M}$ and also satisfying

$$(g_1 g_2)(p) = g_1(g_2(p)), \text{ for all } g_1, g_2 \in G, p \in \widetilde{M}.$$

Let $\text{Isom}(\widetilde{M})$ denote the group of isometries of $\widetilde{M}$, which is a Lie group [49]. Then every isometric action induces a Lie group homomorphism:

$$\begin{aligned} \rho : G &\to \text{Isom}(\widetilde{M}) \\ g &\mapsto \varphi_g. \end{aligned}$$

For a point $p \in \widetilde{M}$, the *orbit* of the action through p is $G \cdot p = \{gp \mid g \in G\}$, and the *isotropy group* or *stabilizer* at p is $G_p = \{g \in G \mid gp = p\}$.

Definition 1.6. *Let G act smoothly on $\widetilde{M}$. Then, such action is:*

1. *Trivial if every orbit consists of a single point;*

2. *Effective if the homomorphism ρ is injective, i.e., G is isomorphic to a subgroup of* $\text{Isom}(\widetilde{M})$*;*

3. *Transitive if $G \cdot p = \widetilde{M}$ for some (and hence for every) $p \in \widetilde{M}$, making $\widetilde{M}$ a homogeneous G-space;*

4. *Free if $gp = hp$ implies $g = h$ for all $p \in \widetilde{M}$ and $g, h \in G$;*

5. *Simply transitive if the action is both free and transitive.*

In Chapter 4, we focus on the extrinsic geometry of orbits of isometric actions:

Definition 1.7. *An extrinsically homogeneous submanifold of $\widetilde{M}$ is an orbit of an isometric action on $\widetilde{M}$.*

In general, such orbits are immersed submanifolds. Endowed with the induced metric, each orbit $G \cdot p$ becomes a Riemannian homogeneous space isometric to the quotient G/G_p, on which G acts transitively by isometries.

Definition 1.8. *Let $G \times \widetilde{M} \to \widetilde{M}$ and $G' \times \widetilde{M}' \to \widetilde{M}'$ be two isometric actions. They are said to be Conjugate (or equivalent) if there exists a Lie group isomorphism $\psi : G \to G'$ and an isometry $f : \widetilde{M} \to \widetilde{M}'$ such that*

$$f(gp) = \psi(g)f(p), \quad \forall p \in \widetilde{M},\ g \in G.$$

They are called orbit equivalent if there is an isometry $f : \widetilde{M} \to \widetilde{M}'$ mapping orbits of the G-action to orbits of the G'-action. Every conjugate action is orbit equivalent.

Let $\varphi : G \times \widetilde{M} \to \widetilde{M}$ be an isometric action, and fix a point $p \in \widetilde{M}$. Since the isotropy group G_p fixes p and preserves the orbit $G \cdot p$, the differential of each φ_g with $g \in G_p$ leaves invariant both the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$. This motivates

Definition 1.9. *Under the above conditions, define:*

- *The isotropy representation at p:*

$$\begin{array}{rcl} G_p \times T_p(G \cdot p) & \to & T_p(G \cdot p) \\ (g, X) & \mapsto & (\varphi_g)_{*p} X. \end{array}$$

- *The slice representation at p:*

$$\begin{array}{rcl} G_p \times \nu_p(G \cdot p) & \to & \nu_p(G \cdot p) \\ (g, \xi) & \mapsto & (\varphi_g)_{*p} \xi. \end{array}$$

Let $\widetilde{M}/G$ denote the space of orbits endowed with the quotient topology from the projection $\widetilde{M} \to \widetilde{M}/G$, $p \mapsto G \cdot p$. In general, $\widetilde{M}/G$ is not Hausdorff, which motivates

Definition 1.10. *The action of G on $\widetilde{M}$ is called proper if for any $p, q \in \widetilde{M}$ there exist neighborhoods U_p and U_q such that the set $\{g \in G : (gU_p) \cap U_q \neq \emptyset\}$ is relatively compact in G. Equivalently, the map*

$$\begin{array}{rcl} G \times \widetilde{M} & \to & \widetilde{M} \times \widetilde{M} \\ (g, p) & \mapsto & (p, gp), \end{array}$$

is a proper map (i.e., the preimage of each compact set is compact).

Every compact Lie group action is proper. If $G \leq \mathrm{Isom}(\widetilde{M})$, the G-action is proper if and only if G is closed in $\mathrm{Isom}(\widetilde{M})$. Proper actions guarantee that $\widetilde{M}/G$ is Hausdorff, each orbit

$G \cdot p$ is closed and embedded, and each isotropy group G_p is compact. Moreover, the orbits of an isometric action are closed if and only if the action is orbit equivalent to a proper one.

Definition 1.11. *An orbit $G \cdot p$ is called a principal orbit if for every $q \in \widetilde{M}$, the isotropy group G_p is conjugate to a subgroup of G_q. The codimension of a principal orbit is called the cohomogeneity of the action.*

The union of all principal orbits is open and dense in $\widetilde{M}$. A principal orbit has maximal dimension, and is principal if and only if the slice representation is trivial. We also define:

Definition 1.12.

1. *A non-principal orbit of maximal dimension is called an exceptional orbit.*
2. *An orbit of strictly smaller dimension than a principal orbit is called a singular orbit.*

Definition 1.13. *A Riemannian metric on a Lie group G is said to be left-invariant if left multiplication $L_g : G \to G$ is an isometry for all $g \in G$.*

Such a metric is uniquely determined by a fixed inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$. We denote both the inner product on $\mathfrak{g}$ and the associated metric on G by $\langle \cdot, \cdot \rangle$. Lie groups equipped with a left-invariant metric are called metric Lie groups and serve as fundamental examples of Riemannian homogeneous spaces.

Chapter 2

Solitons to the Mean Curvature Flow as Translation Surfaces in $\mathbb{H}^3$

In this chapter, we consider translation surfaces in the hyperbolic 3-space $\mathbb{H}^3$. The Lie group structure of $\mathbb{H}^3$ is given considering the upper half-space model and the group of similarities of $\mathbb{R}^2$. Once the algebraic setup is established we present two kinds of translation surfaces that will be considered. Both are parametrized as translation surfaces where one of the generating curves is contained in a totally geodesic plane and the other is contained in a horosphere. Our results are divided in three classes of surfaces in $\mathbb{H}^3$: minimal surfaces, hyperbolic translators (see [14]) and conformal solitons to the mean curvature flow (see [40]). The results can be viewed both as classification and rigidity results, as they provide conclusive information regarding the type of surface obtained when the generating curves are as described above. Specifically, we prove that minimal surfaces are either totally geodesic planes or a class of minimal surfaces in $\mathbb{H}^3$, that were described in [34] as singular minimal surfaces in $\mathbb{R}^3$. For the hyperbolic translators and conformal solitons to the mean curvature flow, we show that, the resulting surfaces are the totally geodesic planes and the so-called grim-reapers cylinders in each corresponding context (see [14] and [40]). This chapter is mainly motivated by [14], [33] and [40].

The results presented in this chapter will compose a joint work with João Paulo dos Santos.

2.1 Preliminary concepts

Initially, consider the upper half-space $\mathbb{R}^3_+ = \{(x,y,z) \mid z > 0\}$, where, for each point $p \in \mathbb{R}^3_+$, we define the inner product

$$\langle \cdot, \cdot \rangle_H = \frac{1}{z^2} \langle \cdot, \cdot \rangle, \tag{2.1}$$

with $\langle\cdot,\cdot\rangle$ denoting the standard inner product of $\mathbb{R}^3$. The inner product $\langle\cdot,\cdot\rangle_H$ induces a Riemannian metric on $\mathbb{R}^3_+$ and the resulting Riemannian manifold $(\mathbb{R}^3_+,\langle\cdot,\cdot\rangle_H)$ is called Hyperbolic space, denoted by $\mathbb{H}^3$. This particular representation is referred to as the *upper half-space model* of $\mathbb{H}^3$. Throughout this chapter, any reference to $\mathbb{H}^3$ will be understood to mean this model.

We recall that hyperbolic space $\mathbb{H}^n$, for $n \geq 2$, is a non-commutative metric Lie group. Following [41], in the upper half-space model, $\mathbb{H}^n$ can be identified with the group of similarities of $\mathbb{R}^{n-1}$ via the isomorphism

$$\begin{array}{rccc} \psi_{(x,z)}: & \mathbb{R}^{n-1} & \to & \mathbb{R}^{n-1} \\ & y & \mapsto & zy+x \end{array}.$$

In general, the group operation $*$ for a semidirect product of the form $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ is given by

$$(x_1,z_1)*(x_2,z_2) = (x_1+\varphi_{z_1}(x_2), z_1+z_2), \tag{2.2}$$

where φ is given by the exponential of some matrix $A \in \mathcal{M}_2(\mathbb{R})$, that is,

$$\varphi_z(x) = e^{zA}x = \sum_{k=0}^{\infty} \frac{1}{k!}(zA)^k.$$

We denote the corresponding group by $\mathbb{R}^2 \rtimes_A \mathbb{R}$. If A is de identity matrix $I_2 \in \mathcal{M}_2(\mathbb{R})$, then $e^{zA} = e^z I_2$ and we recover the group of similarities of $\mathbb{R}^2$. Moreover, the map

$$(x,y,z) \in \mathbb{R}^3 \rtimes_{I_2} \mathbb{R} \overset{\Phi}{\mapsto} (x,y,e^z) \in \mathbb{R}^3_+,$$

gives an isomorphism between the group $\mathbb{R}^2 \rtimes_{I_2} \mathbb{R}$ and $\mathbb{H}^3$, endowed with the group structure described above. Thus

$$\begin{aligned}(x_1,y_1,w_1)*(x_2,y_2,w_2) = (x_1+e^{w_1}x_2, y_1+e^{w_1}y_2, w_1+w_2) \overset{\Phi}{\mapsto}& \\ \overset{\Phi}{\mapsto} (x_1+e^{w_1}x_2, y_1+e^{w_1}y_2, e^{w_1+w_2}) = (x_1,y_1,e^{w_1})*(x_2,y_2,e^{w_2}),&\end{aligned}$$

which implies that for any $(x,y,z),(x',y',z') \in \mathbb{H}^3$ we can define the Lie product

$$(x,y,z)*(x',y',z') = (zx'+x, zy'+y, zz').$$

In this set, considering two regular curves α, β in $\mathbb{H}^3$, we define their group product $\alpha*\beta$ as a *translation surface* in $\mathbb{H}^3$.

Returning to the definition of upper halfspace model, we say that two Riemannian metrics g and $\tilde{g}$, defined in a smooth manifold M, are locally conformally equivalent if there exists a function $\varphi : \Omega \subset M \to \mathbb{R}$ such that $\tilde{g} = e^{2\varphi} g$ on Ω. In what follows, let $\varphi : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ be a differential function, where Ω is an open subset of $\mathbb{R}^3$. Let $g = \langle \cdot, \cdot \rangle$ be the standard Euclidean metric on $\mathbb{R}^3$, and consider surfaces $S \subset \mathbb{R}^3$ and $\tilde{S} \subset \mathbb{R}^3_+$. For a point $p = (X_1, X_2, X_3) \in S$, setting $\varphi = -\ln z$ implies that $e^{2\varphi} = \dfrac{1}{z^2}$, which gives

$$\langle \cdot, \cdot \rangle_H = e^{-2\varphi} \langle \cdot, \cdot \rangle = \frac{1}{X_3^2} \langle \cdot, \cdot \rangle.$$

Hence, we obtain equation (2.1).

Let $N(p)$ be the unit normal vector field to the surface $S \subset \mathbb{R}$ at a point $p = (X_1, X_2, X_3) \in S \subset \mathbb{R}^3$. It is clear that this vector field remains normal to the corresponding surface $\tilde{S} \subset \mathbb{R}^3_+$. Moreover, we have

$$\tilde{g}(N(p), N(p)) = e^{2\varphi} \langle N, N \rangle = \frac{1}{X_3^2}.$$

Thus, the unit normal field $\tilde{N}(p)$ to the surface $\tilde{S} \subset \mathbb{H}^3$ at $p = (X_1, X_2, X_3) \in \tilde{S}$ is given by

$$\tilde{N}(p) = \frac{1}{e^{\varphi}} N(p) = X_3 N(p).$$

Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections compatible with the metrics g and $\tilde{g}$, respectively. Then, for two vector fields $X, Y \in T\mathbb{R}^3$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + S(X, Y),$$

where

$$S(X, Y) = d\varphi(X) Y + d\varphi(Y) X - \langle X, Y \rangle \nabla \varphi,$$

and $\nabla \varphi$ denotes the gradient of the function φ with respect to the metric g.

Recalling that, for a regular surface $S \subset \mathbb{R}^3$, the principal directions at $p \in S$ are given by orthonormal vectors $\{e_1, e_2\}$ such that $dN_p(e_i) = -k_i e_i$. Moreover, since $\tilde{N} = e^{-\varphi} N$, it follows that

$$\tilde{\nabla}_{e^{-\varphi} e_i} \tilde{N} = \nabla_{e^{-\varphi} e_i} (e^{-\varphi} N) + S(e^{-\varphi} e_i, e^{-\varphi} N).$$

Now, we compute

$$\begin{aligned}\nabla_{e^{-\varphi}e_i}(e^{-\varphi}N) &= -d(e^{-\varphi}N)(e^{-\varphi}e_i) = -e^{-2\varphi}\left[d\varphi(e_i)N + k_ie_i\right],\\ S(e^{-\varphi}e_i, e^{-\varphi}N) &= d\varphi(e^{-\varphi}e_i)(e^{-\varphi}N) + d\varphi(e^{-\varphi}N)(e^{-\varphi}e_i) - \langle e^{-\varphi}e_i, e^{-\varphi}N\rangle\nabla\varphi\\ &= e^{-2\varphi}\left[d\varphi(e_i)N + d\varphi(N)e_i\right].\end{aligned}$$

Thus

$$\tilde{\nabla}_{e^{-\varphi}e_i}\tilde{N} = -e^{-2\varphi}\left[k_ie_i - d\varphi(N)e_i\right] = e^{-\varphi}\left[k_i - d\varphi(N)\right]\tilde{e}_i.$$

Thus $\{\tilde{e_1}, \tilde{e_2}\} = \{e^{-\varphi}e_1, e^{-\varphi}e_2\}$ are principal directions for $\tilde{S}$, and the corresponding principal curvatures $\tilde{k}_i$ are given by

$$\tilde{k}_i = e^{-\varphi}\left[k_i - d\varphi(N)\right].$$

It follows that the mean curvature $\tilde{H}$ of the surface $\tilde{S} \subset \mathbb{H}^3$ is given by

$$\tilde{H} = \frac{\tilde{k}_1 + \tilde{k}_2}{2} = e^{-\varphi}\left[\frac{k_1 + k_2}{2} - d\varphi(N)\right] = e^{-\varphi}(H - d\varphi(N)),$$

where $d\varphi(w) = \langle\nabla\varphi, w\rangle$, for all $w \in T_pS$, and $\nabla\varphi(p) = (0, 0, -1/X_3)$. Hence

$$\tilde{H} = X_3(H + N_3/X_3). \tag{2.3}$$

Therefore, a surface in $\mathbb{H}^3$ is minimal if and only it satisfies

$$X_3H + N_3 = 0. \tag{2.4}$$

We now remember that a particularly important class of solutions to the mean curvature flow (MCF) is that of solitons. These are special solutions that evolve under the flow by the action of a one-parameter group of isometries of the ambient space. More precisely, a soliton is generated by a Killing vector field associated with such a group. Following the exposition in [14], in the context of hyperbolic space $\mathbb{H}^3$, let $\mathcal{G} := \{\Gamma_t \ : \ t \in \mathbb{R}\}$ be the one-parameter subgroup of the isometry group of $\mathbb{H}^3$, where $\mathcal{G}$ comprises the hyperbolic translations along the x_3-axis defined by $\Gamma_t(p) = e^t p$, $p \in \mathbb{H}^3$. A solution to the (MCF) given by $\frac{\partial}{\partial t}F_t(p) = e^t p$ will be called a *translating soliton*, or simply a *translator*, with respect to the group $\mathcal{G}$. This name reflects the self-similar evolution under the flow generated by the group action. To describe the associated Killing field, we make a slight abuse of notation by identifying $\mathbb{H}^3$ with $T_p\mathbb{H}^3$. With his identification, the Killing field associated to the group $\mathcal{G}$ is given by $\xi(p) = p \in \mathbb{H}^3$.

Let ξ be the Killing field determined by the subgroup $\mathcal{G}$. It can be shown (see, e.g., [28]) that a surface $\tilde{S}$ with unit normal $\tilde{N}$ is the initial condition of a $\mathcal{G}$-soliton generated by ξ if and only if the equality

$$\tilde{H} = \langle \xi, \tilde{N} \rangle, \tag{2.5}$$

holds everywhere on $\tilde{S}$. Thus, it follows from equation (2.5) that a surface $\tilde{S} \subset \mathbb{H}^3$ is a translation soliton if and only if

$$\tilde{H} = \frac{1}{X_3} \langle p, N(p) \rangle. \tag{2.6}$$

By equation (2.3), this condition is equivalent to

$$X_3^2 H = X_1 N_1 + X_2 N_2. \tag{2.7}$$

Another class of solutions that we will approach here is that of the *conformal solitons* to the MCF in $\mathbb{H}^3$. Following [40], in the hyperbolic space $\mathbb{H}^3$, these solitons satisfy equation (1.4) with ξ as a conformal field. In particular, for the conformal field $\xi(p) = -e_3$, for all $p \in \mathbb{H}^3$, a surface $\tilde{S}$ that satisfies $\tilde{H} = -\tilde{g}(e_3, N(p))$. Using the conformal metric structure introduced earlier, this equation can be rewritten as

$$\tilde{H} = -\frac{1}{X_3} \langle e_3, N(p) \rangle = -\frac{N_3}{X_3},$$

that is equivalent to

$$X_3^2 H = -(X_3 + 1) N_3. \tag{2.8}$$

Indeed, instead of e_3, this analysis can be extended to a more general conformal vector field ξ. However, since the class of conformal fields in $\mathbb{H}^3$ is particularly rich, we focus here on the case $\xi = e_3$ case.

Now, let $\alpha : I \subset \mathbb{R} \to \mathbb{H}^3$ and $\beta : J \subset \mathbb{R} \to \mathbb{H}^3$ be two curves in the half-space model of $\mathbb{H}^3$. Suppose that α is contained in a horosphere, more precisely in the horosphere $\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3_+ : z = 1\}$. Then, such a curve can be locally parameterized as

$$\alpha(s) = (s, f(s), 1),$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth function.

Now, consider two cases where β lies in a totally geodesic hyperbolic plane. Up to parabolic translations, such curves can be parameterized as

$$\beta(t) = (at, bt, g(t)), \qquad g(t) > 0$$
$$\beta(t) = (ag(t), bg(t), t), \qquad t > 0$$

where $a^2+b^2=1$ and $g: J \subset \mathbb{R} \to \mathbb{R}$ is a smooth function. Thus, we consider the following parameterizations:

1. For $\beta(t)=(at,bt,g(t))$, with $g(t)>0$, we have

$$\alpha(s)*\beta(t)=(s+at,bt+f(s),g(t)) \quad \text{(First kind)}. \tag{2.9}$$

2. For $\beta(t)=(ag(t),bg(t),t)$, with $t>0$, we have

$$\alpha(s)*\beta(t)=(ag(t)+s,bg(t)+f(s),t) \quad \text{(Second kind)}. \tag{2.10}$$

Under the conditions stated above, we shall prove the following results in the next section.

Theorem 2.1 (Minimal case)**.** *Let $S \subset \mathbb{H}^3$ be a translation surface of the first or the second kind, with $a^2+b^2=1$. Then S is minimal if and only if it is either contained in a totally geodesic plane or is a minimal translation cylinder. Moreover, it can be parameterized as*

$$X(s,t)=(s+at,p(s)+bt,q(t)),$$

where $p(s)=cs+d$, with $c,d\in\mathbb{R}$, $b\neq 0$, and $q(t)$ is the solution of the differential equation

$$q'^2(t)=\frac{m}{q^4(t)}-\frac{(b-ac)^2}{c^2+1}, \quad m>0, \;\; c\neq\frac{b}{a}. \tag{2.11}$$

Furthermore, a solution of such ODE with initial conditions

$$g(0)=y_0>0, \quad m=\frac{y_0^4(b-ac)^2}{c^2+1}, \tag{2.12}$$

is defined on a interval $(-r,r)$, is concave and symmetric with respect to the z axis, and attains a maximum at $t=0$, Additionally, $\lim_{t\to\pm r} g(t)=0$ and $\lim_{t\to\pm r} g'(t)=\pm\infty$. (see figure 2.1).

Remark 2.1. Following [34], these surfaces are referred as α-catenary translation cylinders in $\mathbb{R}^3$. The case where $\alpha=-2$ corresponds to a minimal surface in $\mathbb{H}^3$, when viewed in the upper halfspace model.

Remark 2.2. By choosing $b=1$, and hence $a=0$ in the second kind theorem 2.1, we obtain the parameterization $X(s,t)=(s,g(t)+f(s),t)$, which corresponds to one of the cases discussed in [33], specifically the one he refers to as "type 2". In fact, there is a missing case in his classification that coincides with the second case of our theorem, namely, the one that is a minimal translation cylinder.

As a consequence of the structure provided by Theorem 2.1, we now present a more general result.

Theorem 2.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{H}^3$ be a curve contained in a horosphere $\mathcal{H}_a = \{(x,y,a) \in \mathbb{H}^3 :$ a is constant$\}$, and let $\beta : J \subset \mathbb{R} \to \mathbb{H}^3$ be a planar curve, that is, contained in a totally geodesic hyperbolic plane. Then the translation surface $X : I \times J \to \mathbb{H}^3$, defined by $X(s,t) = \alpha(s) * \beta(t)$ is minimal if and only if it is either a totally geodesic hyperbolic plane or a minimal translation cylinder.*

Proof. By Theorem 2.1, we initially assume that $\alpha : I \subset \mathbb{R} \to \mathbb{H}^3$, contained in the horosphere $\mathcal{H}_1$ and $\beta : J \subset \mathbb{R} \to \mathbb{H}^3$, contained in a totally geodesic hyperbolic plane, are locally parameterized as graphs in the respective subsets of $\mathbb{H}^3$ in which they lie. It turns out that the only solution in the minimal case is when α is a straight line segment and β is given by the solution of equation (2.11). Since α can be extended to a straight line defined on all $\mathbb{R}$ and β is shown to be a closed curve within $\mathbb{H}^3$, the result can be extended to any curves contained in these subsets of $\mathbb{H}^3$. Moreover, choosing a different height horosphere $\mathcal{H}_a$ leads to the same conclusion, and since parabolic translations are isometries of hyperbolic space, the result follows. □

In particular, for the soliton and conformal soliton cases, we will focus on specific cases: in the first case, we choose $a = 0$, and hence $b = 1$; in the second case we chose $a = 1$, and hence $b = 0$. For the translation soliton we have

Theorem 2.3 (Soliton case)**.** *Let $S \subset \mathbb{H}^3$ be a translation surface of the first or second kind with $b = 1$, and hence $a = 0$. Then S is a translation soliton if and only if it is contained in a horosphere, a totally geodesic hyperbolic plane, or it is a connected translator in $\mathbb{H}^3$ which, up to an ambient isometry, is an open subset of a grim reaper surface (see Figure 2.2).*

Remark 2.3. Following [14, Theorems 3.20 and 3.23], the so-called grim reaper surfaces are described as a one-parameter family of non-congruent, complete translators, which are horizontal parabolic cylinders.

Theorem 2.4 (Conformal soliton case)**.** *Let $S \subset \mathbb{H}^3$ be a translation surface of the first or second kind with $b = 1$, and hence $a = 0$. Then S is a conformal translation soliton if and only if it is contained in a totally geodesic hyperbolic plane or a Grim-reaper cylinder. Moreover, for constants $p, q, c \in \mathbb{R}$, it can be parameterized as*

$$X(s,t) = (s,\ t + f(s),\ g(t)),$$

where $f(s) = ps + q$, with $p \neq 0$, and $g(t)$ is either constant or a solution of the equation

$$g'_{\pm}(t) = \pm\sqrt{\frac{(p^2+1)t^4}{ce^{4/t} - p^2t^4}}, \quad c > 0.$$

This solution is well defined for $0 < t < 1/W\left(\sqrt[4]{p^2/c}\right)$, where $W(t)$ denotes the Lambert W function.

Furthermore, given the initial conditions

$$g_{\pm}(t_0) = g_0, \quad g'_{\pm}(t_0) = y_0,$$

we have $c = c(y_0)$, the functions $g_{\pm}(t)$ are bounded, g_+ is convex, g_- is concave, and the following limits hold:

$$\lim_{t\to A} g_{\pm}(t) = L, \quad \lim_{t\to 0} g_{\pm}(t) = L \mp r, \quad \lim_{t\to 0} g'_{\pm}(t) = 0, \quad \lim_{t\to A} g'_{\pm}(t) = \pm\infty,$$

for some constants $L, r < \infty$.

Remark 2.4. The so-called Grim-reaper cylinders described in [40, Theorem C] as conformal solitons with respect to the conformal vector field e_n in the upper half-space model of $\mathbb{H}^{n+1}$, are analogous to those described in Theorem 2.4. The solitons presented there are graphs of the form

$$\Gamma = \{(x, u(x)) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty) : x \in \Omega \subset \mathbb{R}^n\},$$

with $u \in C^\infty(\Omega)$, although not within the context of translation surfaces.

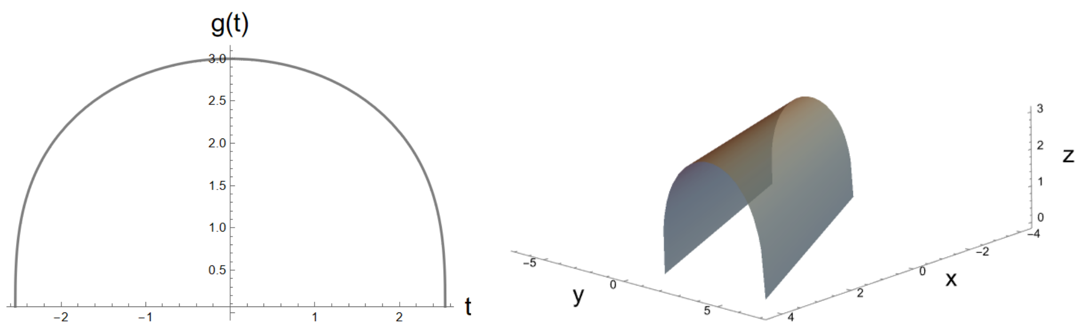

Fig. 2.1 ODE numerical solution and surface of Theorem 2.1 .

Remark 2.5. It seems at first that minimal and conformal translation Grim-reaper cylinders are alike but this is not true, is possible to see by the proof of theorem 2.4 that the conformal ones are never minimal.

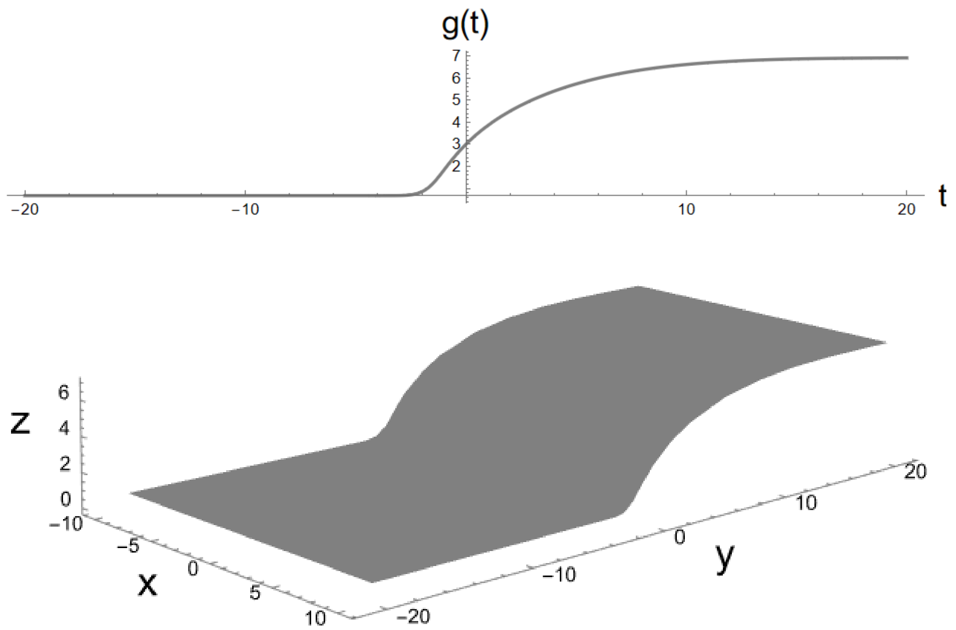

Fig. 2.2 ODE numerical solution and Surface of Theorem 2.3.

2.1.1 Geometry of translation surfaces in $\mathbb{H}^3$

To establish the corresponding equations for each class of surfaces we are considering, we first compute H and N in the Euclidean space and then use equation (2.7) to obtain $\tilde{H}$. Thus for a surface of the form

$$\begin{array}{rccc} X: & I \times J & \to & \mathbb{R}^3 \\ & (s,t) & \mapsto & \alpha(s) * \beta(t) \end{array},$$

we use the classical definition to obtain the mean curvature, that is

$$H = \frac{lG - 2nF + Em}{2(EG - F^2)}.$$

Thus, let X_s and X_t be the derivatives of X with respect to s and t respectively we compute the first and second fundamental forms coefficients in the standard way

$$\begin{array}{rclcrcl} E & = & \langle X_s, X_s \rangle & , & l & = & \langle X_{ss}, N \rangle, \\ G & = & \langle X_t, X_t \rangle & , & m & = & \langle X_{tt}, N \rangle, \\ F & = & \langle X_s, X_t \rangle & , & n & = & \langle X_{st}, N \rangle, \end{array}$$

where $N(s,t)$ is the normal Gauss application of $X(s,t)$.

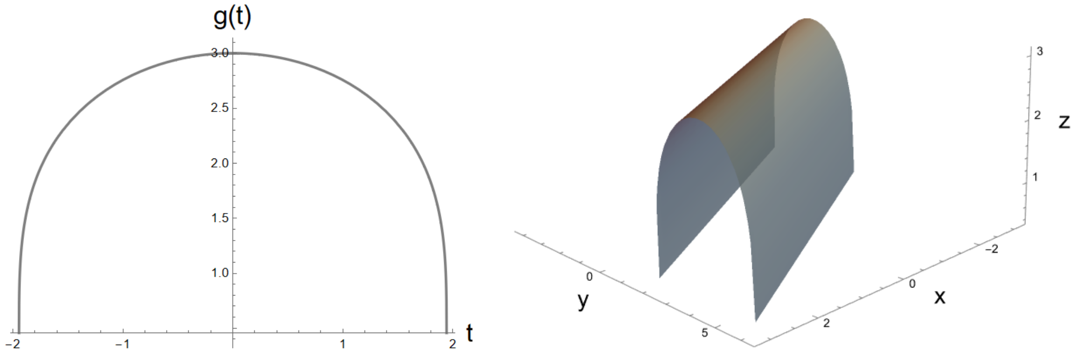

Fig. 2.3 ODE numerical solution and Surface of Theorem 2.4 .

To compute N, we use the classical definition of cross product in $\mathbb{R}^3$, that is, with $x,y \in \mathbb{R}^3$ we have

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Consider now the curves in $\mathbb{H}^3$

$$\alpha(s) = (s, f(s), 1) \;\; , \;\; \beta(t) = (at, bt, g(t)),$$

where $a^2 + b^2 = 1$. For such curves we have

$$X(s,t) = \alpha(s) * \beta(t) = (s + at, bt + f(s), g(t))$$

So we compute

$$\begin{array}{rclrcl} X_s &=& (1, f', 0), & X_{ss} &=& (0, f'', 0), \\ X_t &=& (a, b, g'), & X_{tt} &=& (0, 0, g''), \\ X_{st} &=& (0,0,0). \end{array}$$

Hence we have

$$N = \frac{X_s \times X_t}{|X_s \times X_t|} = \frac{(f'g', -g', b - af')}{\sqrt{(g')^2((f')^2+1) + (b-af')^2}}. \tag{2.13}$$

Thus, the coefficients of the first fundamental form

$$E = 1 + (f')^2. \quad G = 1 + (g')^2, \quad F = f',$$

and the coefficients of the second fundamental form

$$l = \frac{-f''g'}{\sqrt{(g')^2((f')^2+1)+(b-af')^2}}, \quad m = \frac{g''(b-af')}{\sqrt{(g')^2((f')^2+1)+(b-af')^2}}, \quad n = 0.$$

Hence

$$lG = \frac{-f''g'(1+(g')^2)}{\sqrt{(g')^2((f')^2+1)+(b-af')^2}}, \quad Em = \frac{g''(1+(f')^2)(b-af')}{\sqrt{(g')^2((f')^2+1)+(b-af')^2}}, \quad nF = 0.$$

Therefore

$$H = \frac{-f''g'(1+(g')^2)+g''(1+(f')^2)(b-af')}{2[(g')^2((f')^2+1)+(b-af')^2]^{3/2}}. \tag{2.14}$$

In particular, choosing $a = 0$, and thus $b = 1$, yields

$$H = \frac{-f''g'(1+(g')^2)+g''(1+(f')^2)}{2[(g')^2((f')^2+1)+1]^{3/2}}. \tag{2.15}$$

Now consider the curves in $\mathbb{H}^3$

$$\alpha(s) = (s, f(s), 1) \quad , \quad \beta(t) = (ag(t), bg(t), t),$$

with $a^2 + b^2 = 1$. For these curves we have

$$X(s,t) = \alpha(s) * \beta(t) = (ag(t)+s, bg(t)+f(s), t).$$

We compute

$$\begin{aligned} X_s &= (1, f', 0), & X_{ss} &= (0, f'', 0), \\ X_t &= (ag', bg', 1), & X_{tt} &= (ag'', bg'', 0), \\ X_{st} &= (0,0,0). \end{aligned}$$

The unit normal is given by

$$N = \frac{X_s \times X_t}{|X_s \times X_t|} = \frac{(f', -1, g'(b-af'))}{\sqrt{(f')^2+1+(g')^2(b-af')^2}}. \tag{2.16}$$

Thus, the coefficients of the first fundamental form are

$$E = 1+(f')^2, \quad G = 1+(a^2+b^2)(g')^2 = 1+(g')^2, \quad F = g'(a+bf'),$$

and the coefficients of the second fundamental form are

$$l = \frac{-f''}{\sqrt{(f')^2+1+(g')^2(b-af')^2}}, \quad m = \frac{g''(af'-b)}{\sqrt{(f')^2+1+(g')^2(b-af')^2}}, \quad n = 0.$$

Hence

$$lG = \frac{-f''(1+(g')^2)}{\sqrt{(f')^2+1+(g')^2(b-af')^2}}, \quad Em = \frac{g''(1+(f')^2)(af'-b)}{\sqrt{(f')^2+1+(g')^2(b-af')^2}}, \quad nF = 0.$$

Therefore

$$H = \frac{-f''(1+(g')^2)+g''(1+(f')^2)(af'-b)}{2[(f')^2+1+(g')^2(b-af')^2]^{3/2}}. \tag{2.17}$$

In particular, choosing $a = 1$ and thus $b = 0$ provides

$$H = \frac{-f''(1+(g')^2)+g''f'(1+(f')^2)}{2[(f')^2((g')^2+1)+1]^{3/2}}. \tag{2.18}$$

2.1.2 Auxiliary results

Initially, consider the following well known result

Theorem 2.5 (Theorem 10.12 of [15])**.** *Let E be an open set of $\mathbb{R}^n$ that contains x_0, $f \in C^1(E)$ and (α, β) a maximal interval of existence of the following initial value problem (I.V.P. for short)*

$$\dot{x} = f(x), \quad x(0) = x_0. \tag{2.19}$$

If $\beta < \infty$ (similarly $\alpha > -\infty$), then for every compact set $K \subset E$ there exists $t \in (\alpha, \beta)$ such that $x(t) \notin K$.

We now proceed to present some auxiliary results.

Lemma 2.1 ([34, Theorem 3])**.** *Let g be a solution of*

$$g''g(1+c^2) = -2[(g')^2(1+c^2)+(b-ac)^2], \tag{2.20}$$

with the following initial conditions

$$g(0) = y_0 > 0, \quad g'(0) = 0. \tag{2.21}$$

Then g is defined in an interval $(-r, r)$, is concave and symmetrical with relation to the z axis, with a point of maximum in $t = 0$. Also, $\lim_{t \to \pm r} g(t) = 0$ and $\lim_{t \to \pm r} g'(t) = \pm\infty$.

Lemma 2.2 ([14, Lemma 3.19])**.** *Given $\lambda \geq 0$ and $k > 0$, the initial value problem*

$$g'' = -g'(k+(g')^2)\frac{2t}{g^2}, \tag{2.22}$$

with initial conditions

$$g(0) = 1, \quad g'(0) = \lambda.$$

has a unique smooth solution $g : \mathbb{R} \to (0, +\infty)$ *which has the following properties:*

1. g *is constant if* $\lambda = 0$.

2. g *is increasing, convex in* $(-\infty, 0)$*, and concave in* $(0, +\infty)$ *if* $\lambda > 0$.

3. g *is bounded from above by a positive constant.*

4. g *is bounded from below by a positive constant*

The next Lemma presents a similar ODE that is given in the proof of [40, Theorem C]. However, the structure of the proof and the assumptions are different. For the proof here, we argue in a similar way as given in the proof of [34, Theorem 3].

Lemma 2.3. *Let* g *be a solution of*

$$g'' + 2\frac{(g+1)}{g^2}(g')^2 + 2\frac{(g+1)}{g^2(1+a^2)} = 0, \tag{2.23}$$

with the following initial conditions

$$g(0) = y_0 > 0, \quad g'(0) = 0. \tag{2.24}$$

Then g *is defined in a interval* $(-r, r)$*, is concave and symmetrical with relation to* z *axis , with a point of maximum in* $t = 0$*. Also* $\lim\limits_{t \to \pm r} g(t) = 0$ *and* $\lim\limits_{t \to \pm r} g'(t) = \pm\infty$.

Proof. We rewrite equation (2.23) as

$$g''(t) = -2\frac{(g(t)+1)}{g^2(t)}\left[(g'(t))^2 + \frac{1}{(1+a^2)}\right].$$

Now, since g is the solution of (2.23), define $h(t) = g(-t)$. Thus $h'(t) = -g'(-t)$ and $h''(t) = g''(-t)$. Also, we have

$$h''(t) = -2\frac{h(t)+1}{h^2(t)}\left(h'^2(t) + \frac{1}{1+a^2}\right).$$

Also, h satisfies the initial conditions (2.24). Therefore, by the theorem of existence and uniqueness of ODE solutions, we conclude that $g(t) = g(-t)$, so g is symmetric with respect to the z-axis. Hence, g is defined on an interval of the form $(-r, r)$. Since g is a positive function and satisfies $g''(t) < 0$, it follows that g is concave. By symmetry, g attains a unique maximum at $t = 0$. Now, since g is descending in $(0,r)$, it follows from the Theorem 2.5 that

$$\lim_{t\to r} g(t) = 0 \text{ and } \lim_{t\to r} g'(t) = -\infty.$$

Similarly, we have

$$\lim_{t\to -r} g(t) = 0 \text{ and } \lim_{t\to -r} g'(t) = \infty.$$

□

Finally, other results that we use here are the following

Theorem 2.6 ([14, Theorem 3.20]). *In hyperbolic space $\mathbb{H}^3$, there exists a one-parameter family*

$$\mathscr{G} = \{\Sigma_\lambda \mid \lambda \in [0,\infty)\},$$

of noncongruent, complete translators (to be called grim reapers) which are horizontal parabolic cylinders generated by the solutions of (30). Σ_0 is the horosphere $\mathscr{H} \subset \mathbb{H}^3$ at height one, and for $\lambda > 0$, each $\Sigma_\lambda \in \mathscr{G}$ is an entire graph over $\mathbb{R}^2$ which is contained in a slab determined by two horospheres $\mathscr{H}_{\lambda-}$ and $\mathscr{H}_{\lambda+}$. Furthermore, there exist open sets Σ_λ^- and Σ_λ^+ of Σ_λ such that Σ_λ^- is asymptotic to $\mathscr{H}_{\lambda-}$, Σ_λ^- is asymptotic to $\mathscr{H}_{\lambda+}$, and $\Sigma_\lambda = \text{closure}(\Sigma_\lambda^-) \cup \text{closure}(\Sigma_\lambda^+)$.

and

Theorem 2.7 (Theorem 3.23. of [14]). *Any connected translator in $\mathbb{H}^3$ which is a parabolic cylinder is, up to an ambient isometry, an open subset of a grim reaper or of a totally geodesic plane containing the z-axis.*

2.2 Proof of results

Before we begin this section, it is important to highlight that the technique employed here to solve such partial differential equations (PDEs) is the method of separation of variables. This classical technique is widely used to find local solutions for PDEs and involves specific assumptions and procedures. For instance, in this work, it is common to encounter equations of the form $F_1(s)G_1(t) = 0$, where, since we are seeking local solutions, we assume that either $F_1(s) \equiv 0$ for all s in some interval $I \subset \mathbb{R}$, or $G_1(t) \equiv 0$ for all t in some interval $J \subset \mathbb{R}$. Another typical scenario involves equations like $F_1(s)G_1(t) = F_2(s)G_2(t)$, which we aim to rewrite as

$$\frac{F_1(s)}{F_2(s)} = \frac{G_2(t)}{G_1(t)}.$$

To proceed, we first ensure that $F_2(s) \not\equiv 0$ for all s in some interval $I \subset \mathbb{R}$, and $G_1(t) \not\equiv 0$ for all t in some interval $J \subset \mathbb{R}$. Thus, we begin by assuming the contrary and proceed accordingly. The objective is to express each side of the equation in terms of a single variable, leading to

$$\frac{F_1(s)}{F_2(s)} = \frac{G_2(t)}{G_1(t)} = M, \tag{2.25}$$

for some real constant M.

Furthermore, when we refer to integrating equations like (2.25), we mean integrating, for instance, $F_1(s) = MF_2(s)$ with respect to s, not in the sense of the PDE.

We also establish the notation

$$f'(s) = \frac{d}{ds}f(s), \;\; g'(t) = \frac{d}{dt}g(t),$$

which will be used throughout this work. These derivatives will often appear in the simplified form f' and g'. The purpose of this simplification is to make the notation more concise in the computations.

In this work, the assumptions and procedures outlined above are consistently followed, unless stated otherwise.

2.2.1 Proof of Theorem 2.1

Proof. We begin considering parameterizations of the first kind (2.9). Using equations (2.4), (2.13) and (2.14), we have

$$g\frac{-f''g'(1+(g')^2)+g''(1+(f')^2)(b-af')}{2[(g')^2((f')^2+1)+(b-af')^2]^{3/2}} + \frac{(b-af')}{\sqrt{(g')^2((f')^2+1)+(b-af')^2}} = 0.$$

That is

$$-f''gg'(1+(g')^2)+g''g(1+(f')^2)(b-af') = -2(b-af')[(g')^2((f')^2+1)+(b-af')^2]. \tag{2.26}$$

We first consider $f'' \equiv 0$. Thus $f = ct + d$, with $c, d \in \mathbb{R}$. Supposing that $c = b/a$, $a \neq 0$. then equation (2.26) is trivially satisfied and we have $N = (b, -a, 0)$, which implies that $X(I,J)$ is contained in a totally geodesic plane.

Suppose now that $c \neq b/a$ and the equation (2.26) becomes

$$g''g(1+c^2) = -2[(g')^2(1+c^2)+(b-ac)^2]. \tag{2.27}$$

This equation is equivalent to equation (2.20) from Lemma 2.1. Observe that if $g'' \equiv 0$ then $g(t) = mt + n$, thus equation (2.27) becomes

$$m^2(1+c^2) + (b-ac)^2 = 0.$$

Then $m = 0$ and thus $c = b/a$, a contradiction. On the other hand, if $g \not\equiv 0$ then, setting $v(g) = (g')^2$ so that $v' = 2g''$, gives

$$v'g = -4(v+m), \quad m = \frac{(b-ac)^2}{c^2+1}.$$

A first integration of this equation gives

$$v(g) = b_1 g^{-4} - m.$$

Hence

$$(g')^2(t) = \frac{b_1}{g^4} - \frac{(b-ac)^2}{c^2+1}. \tag{2.28}$$

If $g(t) = ct + d$ and $c = 0$, then $g(t) = d$ and equation (2.26) becomes $(b - af')^3 = 0$, which implies that $|X_s \times X_t| = 0$, a contradiction. Suppose now that $c \neq 0$ then equation (2.26) becomes

$$-f''(ct+d)c(1+c^2) = -2(b-af')[c^2((f')^2+1) + (b-af')^2].$$

Differentiating with respect to t gives

$$f''c^2(1+c^2) = 0.$$

Thus, $f(s) = ms + n$. As we saw above, this case leads to a contradiction.

From now on, suppose that $f'' \not\equiv 0$ and $g'' \not\equiv 0$. Differentiating equation (2.26) with respect to t gives

$$-f''[gg'(1+(g')^2)]' + (1+(f')^2)(b-af')[(g''g)' + 4g'g''] = 0. \tag{2.29}$$

Since $f'' \not\equiv 0$, we divide both sides of the equation by f'' and differentiate with respect to s to obtain

$$[(g''g)' + 4g'g'']\left[\frac{(1+(f')^2)(b-af')}{f''}\right]' = 0.$$

If $(g''g)' = -4g'g''$ then $g''g = -2(g')^2 + P$. Set $v(g) = (g')^2$ so that $v' = 2g''$ and we have

$$v' = \frac{2}{g}(-2v+P).$$

Integrating the previous equation gives

$$(g')^2 = \frac{k_1}{g^4} + \frac{P}{2},$$

where $k_1 \neq 0$, otherwise $g'' \equiv 0$ and we have a contradiction. Furthermore, if $(g''g)' = -4g'g''$, it follows from (2.29) that $[gg'(1+(g')^2)]' = 0$. Thus

$$M_1^2 = [gg'(1+(g')^2)]^2 = \frac{(g^4p+2k)\left(g^4(p+2)+2k\right)^2}{8g^{10}}.$$

A polynomial equation in g, which implies that g is constant, a contradiction. Thus we must have

$$(1+(f')^2)(b-af') = M_2 f''.$$

If $M_2 = 0$ then $(b-af') = 0$, a contradiction as by hypothesis $f'' \not\equiv 0$. Hence, $M \neq 0$. In this case, equation (2.26) becomes

$$f''[-gg'(1+(g')^2) + g''gM + 2M(g')^2] = -2(b-af')^3.$$

Since $f'' \not\equiv 0$, we divide both sides by it and differentiate with respect to s to get

$$Nf'' = -2(b-af')^3,$$

with $N = 0$. Then

$$1+(f')^2 + \frac{2M}{N}(b-af')^2 = 0.$$

A polynomial equation in f', which leads to the fact that f' is constant, a contradiction with $f'' \neq 0$.

Now we consider parameterizations of the second kind (2.10). Using equations (2.4), (2.16) and (2.17), we have

$$t\frac{-f''(1+(g')^2)+g''(1+(f')^2)(af'-b)}{2[(f')^2+1+(g')^2(b-af')^2]^{3/2}} + \frac{g'(b-af')}{[(f')^2+1+(g')^2(b-af')^2]^{1/2}} = 0,$$

that simplifies as

$$-tf''(1+(g')^2)+tg''(1+(f')^2)(af'-b) = 2g'(af'-b)[((f')^2+1)+(g')^2(b-af')^2]. \tag{2.30}$$

Initially suppose that $f(s) = c \in \mathbb{R}$, then the equation (2.30) turns into

$$-2g'b[1+(g')^2b^2] = 0.$$

If $b = 0$ then $X(s,t) = (ag(t)+s, c, t)$, and the surface is contained in a totally geodesic plane. If $b \neq 0$ then $g(t) \equiv d \in \mathbb{R}$ and we have $X(s,t) = (ad+s, bd+c, t)$, and the surface is again contained in a totally geodesic plane.

Suppose now that $g(t) = m$. Then equation (2.30) turns into $-tf'' = 0$. Thus $f(s) = cs+d$ and $X(s,t) = (ad+s, bd+cs+d, t)$. Hence, the surface is contained in a totally geodesic plane.

On the other hand, suppose that $f(s) = cs+d$, $c \neq 0$. Hence, equation (2.30) turns into

$$tg''(1+c^2)(ac-b) = 2g'(ac-b)[(c^2+1)+(g')^2(b-ac)^2].$$

Initially, if $c = b/a$ then $N = (b, -a, 0)$. Thus, the surface $X(s,t)$ is contained in the totally geodesic plane. Suppose now that $c \neq b/a$. Since $t > 0$ we have

$$g'' = 2\frac{g'}{t} + 2\frac{g'^3}{t}\frac{(b-ac)^2}{1+c^2}. \tag{2.31}$$

Set $v(t) = g'(t)$, we obtain

$$v' = \frac{2}{t}v + 2\frac{(b-ac)^2}{(1+c^2)t}v^3,$$

which is a Bernoulli equation. Let $w(t) = v^{-2}$. Thus, $w' = -2v^{-3}v'$, that leads to

$$w' = -\frac{4}{t}w - \frac{4(b-ac)^2}{(1+c^2)t}.$$

This equation is linear, which we can solve by computing

$$F(t) = -\int \frac{4}{t}\,dt = \ln\left(t^{-4}\right).$$

Thus

$$w(t) = Pe^F - e^F \int e^{-F}\frac{4(b-ac)^2}{(1+c^2)t}\,dt = Pt^{-4} - \frac{4(b-ac)^2}{(1+c^2)}t^{-4/c}\int t^{4-1}\,dt = Pt^{-4} - \frac{(b-ac)^2}{(1+c^2)}.$$

Hence

$$(g')^2(t) = \frac{t^4(1+c^2)}{P-(b-ac)^2t^4}, \quad P > 0, \tag{2.32}$$

that is defined for $P > (b-ac)^2t^4$, that is $0 < t < \sqrt[4]{P/(b-ac)^2}$.

From now on suppose that $g' \not\equiv 0$ and $f'f'' \not\equiv 0$. Since $t(1+(g')^2) > 0$ by hypothesis, we divide both sides of equation (2.30) by $t(1+(g')^2) > 0$ to obtain

$$-f'' + \frac{g''}{(1+(g')^2)}(1+(f')^2)(af'-b) = 2\frac{g'}{t(1+(g')^2)}(af'-b)[((f')^2+1)+(g')^2(af'-b)^2].$$

Differentiating it with respect to t gives

$$\left(\frac{g''}{(1+(g')^2)}\right)'(1+(f')^2)(af'-b) = 2\left(\frac{g'}{t(1+(g')^2)}\right)'(af'-b)((f')^2+1)+2\left(\frac{g'^3}{t(1+(g')^2)}\right)'(af'-b)^3. \tag{2.33}$$

Since $f'f'' \neq 0$, we divide both sides by $(1+(f')^2)(af'-b)$ and differentiate again with respect to s to get

$$\left(\frac{g''}{(1+(g')^2)}\right)' = 2\left(\frac{g'^3}{t(1+(g')^2)}\right)'\left(\frac{(af'-b)^2}{((f')^2+1)}\right)'. \tag{2.34}$$

A further derivative with respect to s provides

$$2\left(\frac{g'^3}{t(1+(g')^2)}\right)'\left(\frac{(af'-b)^2}{((f')^2+1)}\right)'' = 0.$$

Now if

$$\left(\frac{(b-af')^2}{((f')^2+1)}\right)'' = 0,$$

then

$$\frac{(b-af')^2}{((f')^2+1)} = Ms+N.$$

If $M=0$, then $(b-af')^2 - N((f')^2+1) = 0$, a polynomial equation, which implies that f' is constant and thus $f'' \equiv 0$ a contradiction.

Suppose that $M \neq 0$, then we return to (2.33) to get

$$\left(\frac{g''}{(1+(g')^2)}\right)'(1+(f')^2)(af'-b) =$$

$$= 2\left(\frac{g'}{t(1+(g')^2)}\right)'(af'-b)((f')^2+1)+2\left(\frac{g'^3}{t(1+(g')^2)}\right)'(af'-b)((f')^2+1)(Ms+N),$$

that is

$$\left(\frac{g''}{(1+(g')^2)}\right)' = 2\left(\frac{g'}{t(1+(g')^2)}\right)' + 2\left(\frac{g'^3}{t(1+(g')^2)}\right)'(Ms+N).$$

Differentiating with respect to s gives

$$M\left(\frac{g'^3}{t(1+(g')^2)}\right)' = 0. \tag{2.35}$$

As $M \neq 0$, we conclude by equation (2.34), that

$$\left(\frac{g'^3}{t(1+(g')^2)}\right)' = \left(\frac{g''}{(1+(g')^2)}\right)' = 0. \tag{2.36}$$

Nevertheless, by equation (2.33) we have

$$\left(\frac{g'}{t(1+(g')^2)}\right)' = 0. \tag{2.37}$$

From equations (2.35), (2.36) and (2.37), we have

$$g' = C_1 t(1+(g')^2), \quad g'^3 = C_2 t(1+(g')^2), \quad g'' = C_3(1+(g')^2).$$

As $g' \neq 0$ then $C_1 \neq 0$ and $C_2 \neq 0$. Thus we have

$$t(1+(g')^2) = \frac{g'}{C_1} = \frac{g'^3}{C_2}.$$

Hence, $C_1(g')^2 = C_2$, which implies that $g'' \equiv C_3 = 0$, We return to (2.30) with $g(t) = ct+d$ to get

$$-tf''(1+c^2) = 2c(af'-b)[((f')^2+1)+c^2(b-af')^2].$$

A derivative with respect to t gives $-f'' = 0$, a contradiction.

We now return to equations (2.31) and (2.32). In the following equation

$$g'_{\pm}(t) = \pm\sqrt{\frac{t^4(1+c^2)}{m-(b-ac)^2t^4}}, \quad p > 0.$$

Set $z = g(t)$. Since $g' \neq 0$, there is a differential inverse of g. Hence, let $t = h(z) = g^{-1}(g(t))$. We have

$$[h'(z)]^2 = \frac{1}{[g'(t)]^2} = \frac{1}{[g'(h(z))]^2} = \frac{m-(b-ac)^2(h(z))^4}{(h(z))^4(1+c^2)} = \frac{m}{(1+c^2)}\frac{1}{h^4(z)} - \frac{(b-ac)^2}{(1+c^2)}.$$

We return to equation (2.28) to conclude that these solutions must be the same.

□

2.2.2 Proof of Theorem 2.3

Proof of theorem 2.3. We begin by considering the parameterizations of the first kind (2.9). Using equations (2.7), (2.13) and (2.15), we have

$$g^2\frac{-f''g'(1+g'^2)+g''(1+f'^2)}{2[g'^2(f'^2+1)+1]^{3/2}}=\frac{sf'g'-g'(t+f)}{\sqrt{g'^2(f'^2+1)+1}},$$

or equivalently

$$-f''g^2g'(1+g'^2)+g^2g''(1+f'^2)=2g'[g'^2(f'^2+1)+1][(sf'-f)-t]. \tag{2.38}$$

Suppose initially that $g\equiv c>0$. Then $N=e_3$ and the surface is contained in a horosphere.

Suppose now that $(sf'-f)'\equiv 0$. We have that $f(s)=bs+a$, with $a,b\in\mathbb{R}$. Thus equation (2.38) becomes

$$g^2g''(b^2+1)=-2g'[g'^2(b^2+1)+1][a+t], \tag{2.39}$$

that has at least the constant solution. Also, Equation (2.39) with the change of variables $v=a+t$ and $(b^2+1)^{-1}=k$ gives $[g(a+t)]'=g'(v)$ and $[g(a+t)]''=g''(v)$. Thus

$$g''=-g'[g'^2+k]\frac{2v}{g^2}.$$

The solution of this ODE satisfy the conditions of Lemma 2.2.

Suppose from now on that $g'\not\equiv 0$, $f'\not\equiv 0$ and $(sf'-f)'\not\equiv 0$. We divide both sides by g' to obtain

$$-f''g^2(1+g'^2)+\frac{g^2g''}{g'}(1+f'^2)=2[g'^2(f'^2+1)+1][(sf'-f)-t].$$

Expanding the right-hand side we get

$$-f''g^2(1+g'^2)+\frac{g^2}{g'}g''(1+f'^2)=2g'^2(f'^2+1)(sf'-f)-2tg'^2(f'^2+1)+2(sf'-f)-2t.$$

We now differentiate with respect to t to obtain

$$-f''[g^2(1+g'^2)]'+\left[\frac{g^2g''}{g'}\right]'(1+f'^2)=2(g'^2)'(f'^2+1)(sf'-f)-2[tg'^2]'(f'^2+1)-2.$$

Since $f'^2+1\neq 0$, we divide both sides by it to get

$$-\frac{f''}{f'^2+1}[g^2(1+g'^2)]'+\left[\frac{g^2g''}{g'}\right]'=2(g'^2)'(sf'-f)-2[tg'^2]'-\frac{2}{f'^2+1}.$$

A differentiation with respect to s provides

$$-\left[\frac{f''}{f'^2+1}\right]'[g^2(1+g'^2)]' = 2(g'^2)'(sf'-f)' - \left[\frac{2}{f'^2+1}\right]'.$$

Now, taking the derivative with respect to t, we have

$$-\left[\frac{f''}{f'^2+1}\right]'[g^2(1+g'^2)]'' = 2(g'^2)''(sf'-f)'. \tag{2.40}$$

Consider the following notation

$$\begin{array}{rclcrcl} F_1 & = & -\left[\dfrac{f''}{f'^2+1}\right]', & & G_1 & = & [g^2(1+g'^2)]'', \\ F_2 & = & (sf'-f)', & & G_2 & = & (g'^2)''. \end{array}$$

Thus, equation (2.40) becomes

$$F_1G_1 = 2G_2F_2.$$

Suppose initially that $G_2 \neq 0$ and $F_1 \neq 0$. We have

$$\frac{F_2}{F_1} = \frac{G_1}{2G_2} = -\overline{P}, \quad \overline{P} \in \mathbb{R}.$$

Since each side rely on its on variable, we get

$$-\overline{P}\left[\frac{f''}{f'^2+1}\right]' = -(sf'-f)'.$$

If $\overline{P} = 0$ we have $F_2 = (sf'-f)' \equiv 0$, a contradiction. Suppose that $\overline{P} \neq 0$ and writing $P = 1/\overline{P}$. Then, after a first integration, the previous equation becomes

$$f'' = (f'^2+1)[P(sf'-f)+Q]. \tag{2.41}$$

We return to the equation (2.38) to get

$$\begin{aligned} -(f'^2+1)[P(sf'-f)+Q]g^2(1+g'^2) + \frac{g^2}{g'}g''(1+f'^2) = \\ = 2g'^2(f'^2+1)(sf'-f) - 2tg'^2(f'^2+1) + 2(sf'-f) - 2t. \end{aligned}$$

Dividing both sides by $(1+f'^2)$ and differentiating them with respect to s gives

$$-P(sf'-f)'g^2(1+g'^2) = 2g'^2(sf'-f)' + \left[\frac{2(sf'-f)}{f'^2+1}\right]' - 2t\left[\frac{1}{f'^2+1}\right]'.$$

A further differentiation, now with respect to t, provides

$$-P(sf'-f)'[g^2(1+g'^2)]' = 2(g'^2)'(sf'-f)' - 2\left[\frac{1}{f'^2+1}\right]'.$$

As $(sf'-f)' \not\equiv 0$ by hypothesis, on the previous equation we divide both sides by $(sf'-f)'$ to obtain

$$-[g^2(1+g'^2)]' = 2(g'^2)' - 2\frac{1}{(sf'-f)'}\left[\frac{1}{f'^2+1}\right]'.$$

Differentiating with respect to s gives

$$\left[\frac{1}{f'^2+1}\right]' = M(sf-f)'.$$

If $M=0$, then f' is constant, a contradiction as $F_1 \neq 0$ by hypothesis. Thus $M \neq 0$ and we have

$$\frac{1}{P}\left[\frac{f''}{f'^2+1}\right]' = (sf'-f)' = \frac{1}{M}\left[\frac{1}{f'^2+1}\right]',$$

that is

$$f'' = \frac{P}{M} + N(f'^2+1), \quad \text{and} \quad (sf'-f) = \frac{1+R(f'^2+1)}{M(f'^2+1)}. \tag{2.42}$$

Observe that if $N=0$, then f'' is constant, which implies that $f(s) = \frac{P}{2M}s^2 + bs + c$ and $f' = \frac{P}{M}s + b$. In view of (2.41), we have

$$\left(\left(\frac{P}{M}s+b\right)^2+1\right)\left(P\left(\frac{P}{M}s+b\right)+\left(\frac{P}{2M}s^2+bs+c\right)+Q\right) = \frac{P}{M},$$

that is

$$\frac{P^3}{2M}s^4 + D(s) = 0,$$

where $D(s)$ is a polynomial of degree 3. As this equation must be true for every s we conclude that $P=0$, a contradiction.

Suppose that $N \neq 0$. Then we differentiate the second equation of (2.42) to obtain

$$sf'' = -\frac{2}{M}\frac{f'f''}{(1+f'^2)^2}.$$

Since $f'' \not\equiv 0$, we have

$$Ms = -2\frac{f'}{(1+f'^2)^2}.$$

Differentiating again with respect to s gives

$$M = -2f''\left[\frac{1}{(1+f'^2)^2} - 4\frac{f'^2}{(1+f'^2)^3}\right].$$

By the first equation of (2.42), we have

$$M(1+f'^2)^3 = -2\left[\frac{P}{M} + N(f'^2+1)\right]\left[(1+f'^2) - 4f'^2\right],$$

that is

$$-M^2f'^6 + (6MN - 3M^2)f'^4 + (4MN + 6P - 3M^2)f'^2 - M^2 - 2MN - 2P = 0.$$

Since $M \neq 0$, this equation leads to the fact that f' is constant, a contradiction.

It remains to evaluate the case where $F_1 \equiv 0$ and/or $G_2 \equiv 0$. If $F_1 \not\equiv 0$ and $G_2 \equiv 0$, from equation (2.40), we must have $G_1 \equiv 0$. $G_2 \equiv 0$ implies $g'^2 = K_2t + N_2$, which has solution given by

$$g(t) = \frac{d - 2(K_2t + N_2)^{3/2}}{3K_2}.$$

Moreover, $G_1 \equiv 0$ implies $g^2(1+g'^2) = K_3t + N_3$. Thus we have

$$(d - 2(K_2t + N_2)^{3/2})^2 - 9K_2^2K_3t - 9K_2^2N_3 = 0,$$

that vanishes for every t if $K_2 = 0$, which means that $g'^2 = N_2 > 0$. Unless g is constant, which is not the case, then $g' = \pm\sqrt{N_2}$ and $g^2(1+N_2) = K_3t + N_3$. This implies that $g^2 = K_4t + N_4$, that is

$$2gg' = \pm 2g\sqrt{N_2} = K_4,$$

which means that g is constant, a contradiction.

If $F_1 \equiv 0$ we must have $G_2 \equiv 0$ or $F_2 \equiv 0$. As $F_2 = (sf' - f)' \not\equiv 0$ by hypothesis, we must have $G_2 \equiv 0$. We remember also that $g' \not\equiv 0$. If $(g'^2)'' = 0$, then $g'^2 = Kt + L$ and we have two cases

1. If $K = 0$, then $g(t) = \sqrt{L}t + b$. Hence, equation (2.38) becomes

$$-f''(\sqrt{L}t+b)^2\sqrt{L}(1+L) = 2\sqrt{L}[L(f'^2+1)+1][(sf'-f)-t].$$

If $L = 0$ we have g constant, a contradiction. Suppose now that $L \neq 0$, the solution of the equations is $g(t) = \sqrt{L}t + b$, with $L > 0$. Differentiating again with respect to t yields

$$-f''2(\sqrt{L}t+b)L(1+L) = -2\sqrt{L}[L(f'^2+1)+1].$$

A further differentiation with respect to t gives

$$-f''2\sqrt{L}L(1+L) = 0,$$

which implies that $f'' = 0$, a contradiction.

2. If $K \neq 0$, we remember that as $F_1 \equiv 0$, then $f'' = M(1+f'^2)$. Also, the solution g of $g'^2 = Kt + L$, is given by

$$g(t) = \frac{b - 2(Kt+L)^{3/2}}{3K}$$

If $M = 0$ we have $f'' = 0$, a contradiction as $(sf' - f)' \not\equiv 0$ by hypothesis. Thus $M \neq 0$, we return to the equation (2.38) with $g'^2 = Kt + L$ to obtain

$$-M(f'^2+1)g^2g'(1+g'^2)+g^2g''(1+f'^2) = 2g'[g'^2(f'^2+1)+1][(sf'-f)-t].$$

We divide both sides by $f'^2 + 1$ to get

$$-Mg^2g'(1+g'^2)+g^2g'' = 2g'[(sf'-f)-t]+2g'\frac{(sf'-f)-t}{1+f'^2}.$$

Differentiating with respect to s gives

$$2g'^2(sf'+f)'+2g'\left[\frac{(sf'-f)}{1+f'^2}\right]'-2g't\left[\frac{1}{1+f'^2}\right]' = 0.$$

Since $g' \not\equiv 0$, we have

$$2g'(sf'-f)'+2\left[\frac{(sf'-f)}{1+f'^2}\right]'-2t\left[\frac{1}{1+f'^2}\right]'=0.$$

We now differentiate with respect to t to obtain

$$2g''(sf'-f)'=2\left[\frac{1}{1+f'^2}\right]'.$$

Since $(sf'-f)' \not\equiv 0$ we have

$$g''=-\frac{1}{(sf'-f)'}\left[\frac{1}{1+f'^2}\right]'.$$

This implies that g'' is constant. Remembering that $g'(t)=\sqrt{Kt+L}$, we have

$$g''(t)=\frac{1}{2}\frac{K}{\sqrt{Kt+L}},$$

which implies that $K=0$, a contradiction.

We now consider parameterizations of the second kind (2.10). Using equations (2.7), (2.16) and (2.18), we have

$$t^2\frac{-f''(1+g'^2)+g''f'(1+f'^2)}{2[f'^2(g'^2+1)+1]^{3/2}}=\frac{f'(g+s)-f}{\sqrt{f'^2(g'^2+1)+1}},$$

or equivalently

$$-f''t^2(1+g'^2)+g''t^2f'(1+f'^2)=2[f'^2(g'^2+1)+1][f'(g+s)-f]. \tag{2.43}$$

Suppose initially that $f'=0$. The equation (2.43) becomes $0=-2f$ which implies that $f\equiv 0$. Thus the surface is contained in a hyperbolic plane. Suppose now that $f''=0$ but $f'\neq 0$. We have $f(s)=as+b$, $a\neq 0$, and the equation (2.43) becomes

$$g''t^2a(1+a^2)=2[a^2g'^2+a^2+1][ag-b], \tag{2.44}$$

that has at least the constant solution.

Suppose from now on that $f'f''\not\equiv 0$. We expand the right-hand side to obtain

$$-f''t^2(1+g'^2)+g''t^2f'(1+f'^2)=2f'^3(g'^2+1)(g+s)-2ff'^2(g'^2+1)+2f'(g+s)-2f.$$

Differentiating with respect to t yields

$$-2f''t(1+g'^2)-2f''t^2g'g''+g'''t^2f'(1+f'^2)+2g''tf'(1+f'^2)= \\ =4f'^3g'g''(g+s)+2g'f'^3(g'^2+1)-4ff'^2g'g''+2g'f'.$$

Since $f' \neq 0$, we divide both sides by f'^3 to obtain

$$-2\frac{f''}{f'^3}[t(1+g'^2)+t^2g'g'']+\frac{(1+f'^2)}{f'^2}[g'''t^2+2g''t]= \\ =4g'g''(g+s)+2g'(g'^2+1)-4\frac{f}{f'}g'g''+2g'\frac{1}{f'^2}.$$

Differentiating with respect to s gives

$$-2\left(\frac{f''}{f'^3}\right)'[t(1+g'^2)+t^2g'g'']-2\frac{f''}{f'^3}[g'''t^2+2g''t]=-4\left(\frac{f}{f'}\right)'g'g''-4g'\frac{f''}{f'^3}.$$

We now multiply both sides by $\frac{f'^3}{f''}$ to get

$$-2\frac{f'^3}{f''}\left(\frac{f''}{f'^3}\right)'[t(1+g'^2)+t^2g'g'']-2[g'''t^2+2g''t]=-4\frac{f'^3}{f''}\left(\frac{f}{f'}\right)'g'g''-4g'. \tag{2.45}$$

Finally, we differentiate again with respect to s to obtain

$$\left[-2\frac{f'^3}{f''}\left(\frac{f''}{f'^3}\right)'\right]'[t(1+g'^2)+t^2g'g'']=-4\left[\frac{f'^3}{f''}\left(\frac{f}{f'}\right)'\right]'g'g''. \tag{2.46}$$

We introduce now the following notation

$$F_1(s)=\left[-2\frac{f'^3}{f''}\left(\frac{f''}{f'^3}\right)'\right]',\ F_2(s)=-4\left[\frac{f'^3}{f''}\left(\frac{f}{f'}\right)'\right]',\ G(t)=[t(1+g'^2)+t^2g'g''].$$

Then, we can rewrite the previous equation as

$$F_1(s)G(t)=F_2(s)g'g''. \tag{2.47}$$

Now, we present some particular cases:

1. If $G(t) \equiv 0$, then $t(1+g'^2) = -t^2 g' g''$. If $g'g'' \equiv 0$ we have a contradiction. Put $v = g'$ and since $t > 0$, we have
$$v' \frac{v}{1+v^2} = \frac{1}{t}.$$
A first integration of this equation gives $g'(t) = \pm\sqrt{ct^2-1}$, with $c > 0$. Thus
$$g'' = \pm ct/\sqrt{ct^2-1}, \quad g'''(t) = \pm\left(\frac{c}{\sqrt{ct^2-1}} - \frac{c^2t^2}{(ct^2-1)^{3/2}}\right).$$
Returning to equation (2.45) we have
$$\pm\left[-2\left(\left(\frac{c}{\sqrt{ct^2-1}} - \frac{c^2t^2}{(ct^2-1)^{3/2}}\right)t^2 + \frac{2ct^2}{\sqrt{ct^2-1}}\right)\right] = \pm\left[cMt - 4\sqrt{ct^2-1}\right].$$
Simplifying we have
$$4(ct^2-1)^{1/2}\left(ct^2-1\right)^{3/2} = 2ct^2\left(2ct^2-3\right) + cMt\left(ct^2-1\right)^{3/2},$$
which implies that t is constant, a contradiction.

2. If $F_1(s) \equiv 0$, by the previous case we must have $g'g'' = 0$ and

   (a) If $g' = 0$ the equation (2.43) becomes
$$-f''t^2 = 2[f'^2+1][f'(c+s)-f].$$
   Differentiating with respect to t yields
$$-2f''t = 0.$$
   Since $t > 0$, we must have $f'' = 0$, again a contradiction.

   (b) If $g'' = 0$ but $g' \neq 0$ we have $g(t) = ct + d$, $c \neq 0$, and the equation (2.43) becomes
$$-f''t^2(1+c^2) = 2[f'^2(c^2+1)+1][f'(ct+d+s)-f].$$
   We differentiate with respect to t to get
$$-2f''t(1+c^2) = 2[f'^2(c^2+1)+1][cf'].$$

Differentiating again with respect to t yields

$$-2f''(1+c^2)=0.$$

Thus, we must have $f''=0$, once again a contradiction.

We conclude that

$$F_1(s)=MF_2(s) \quad , \quad G(t)=Mg'g'', \quad M\neq 0.$$

From the first equality we get

$$2M\frac{f'^3}{f''}\left(\frac{f''}{f'^3}\right)'+N=\frac{f'^3}{f''}\left(\frac{f}{f'}\right)'.$$

From the second one we obtain

$$\frac{t}{t^2+2M}=\frac{g'g''}{1+g'^2}.$$

A first integration of both sides of this equation gives

$$\frac{1}{2}\ln\big(t^2+M\big)+d=\frac{1}{2}\ln\big(1+g'^2\big),$$

that we may rewrite as

$$g'^2=e^{2d}(t^2+M)-1.$$

We also have

$$g''=e^{2d}\frac{t}{\sqrt{t^2+M-1}}=e^{2d}\frac{t}{g'}. \tag{2.48}$$

Thus, the equation (2.43) becomes

$$-f''g'e^{2d}t^2(t^2+M)+e^{2d}t^3f'(1+f'^2)=2[e^{2d}f'^2(t^2+M)+1][f'(g+s)-f].$$

We expand the right-hand side to obtain

$$-f''g'e^{2d}t^2(t^2+M)+e^{2d}t^3f'(1+f'^2)=2e^{2d}f'^3(t^2+M)(g+s)-2e^{2d}ff'^2(t^2+M)+f'(g+s)-f.$$

Now, differentiating with respect to t gives

$$-e^{2d}f''[g't^2(t^2+M)]'+3e^{2d}t^2f'(1+f'^2)=2e^{2d}f'^3[(t^2+M)(g+s)]'-4e^{2d}ff'^2t+f'g'.$$

Dividing both sides by t, yields

$$-e^{2d}f''\frac{[g't^2(t^2+M)]'}{t}+3e^{2d}tf'(1+f'^2)=2e^{2d}f'^3\frac{[(t^2+M)(g+s)]'}{t}-4e^{2d}ff'^2+f'\frac{g'}{t}.$$

We differentiate twice with respect to t to obtain

$$-e^{2d}f''\left(\frac{[g't^2(t^2+M)]'}{t}\right)''=2e^{2d}f'^3\left(\frac{[(t^2+M)(g+s)]'}{t}\right)''+f'\left(\frac{g'}{t}\right)'',$$

that we rewrite as

$$f''G_1(t)=f'^3G_2(t)+f'\left(\frac{g'}{t}\right)''.$$

Since $f'\not\equiv 0$, if $G_1(t)\equiv 0$, we divide both sides of the previous equation by f' and differentiate it with respect to s to obtain $G_2(t)\equiv 0$. Thus $(g'/t)''\equiv 0$, that is $g'=Pt^2+Qt$. Squaring both sides gives $g'^2=P^2t^4+2PQt^3+Q^2t^2$. In view of (2.48), we must have $P=Q=0$. Thus $g'\equiv 0$, a contradiction.

Suppose that $G_1(t)\not\equiv 0$ and divide both sides by it. We have

$$f''=f'^3\frac{G_2(t)}{G_1(t)}+f'\frac{1}{G_1(t)}\left(\frac{g'}{t}\right)''.$$

Differentiating with respect to t gives

$$f'^3\left(\frac{G_2(t)}{G_1(t)}\right)'+f'\left(\frac{1}{G_1(t)}\left(\frac{g'}{t}\right)''\right)'=0.$$

Since $f'\not\equiv 0$, we divide both sides by f' and differentiate with respect to s to obtain

$$2f'f''\left(\frac{G_2(t)}{G_1(t)}\right)'=0.$$

If $[G_2(t)/G_1(t)]'\not\equiv 0$, then $f''\equiv 0$, a contradiction. Thus

$$\left(\frac{G_2(t)}{G_1(t)}\right)'\equiv 0 \quad \text{and} \quad \left(\frac{1}{G_1(t)}\left(\frac{g'}{t}\right)''\right)'\equiv 0.$$

Using equation (2.48), we have

$$\left(\frac{1}{G_1(t)}\left(\frac{g'}{t}\right)''\right)'=\frac{((-1+e^{2d}(M+t^2))(2+e^{4d}M(2M+3t^2)-d(4M+3t^2)))}{G_3(t)}\not\equiv 0,$$

where

$$G_3(t) = (t^3(-8+28e^{2d}(M+2t^2)-e^{4d}(32M^2+113Mt^2+75t^4)+ \\ +3e^{6d}(4M^3+19M^2t^2+25Mt^4+10t^6))).$$

A contradiction.

We now return to equation (2.44). Set $z = g(t)$. Since $g' \neq 0$, there is a differential inverse of g. Hence, let $t = h(z) = g^{-1}(g(t))$. Thus we have

$$h'(z) = \frac{1}{g'(t)} = \frac{1}{g'(h(z))}.$$

Differentiating with respect to z gives

$$h''(z) = -\frac{g''(h(z))h'(z)}{[g'(h(z))]^2} = -g''(h(z))h'^3(z).$$

Since $t = h(z) > 0$, equation (2.44) becomes

$$\frac{h''}{h'^3}h^2a(1+a^2) = -2\left(a^2\frac{1}{h'^2}+a^2+1\right)(b-az),$$

that is

$$h^2h''a(1+a^2) = -2h'\left(a^2+(a^2+1)h'^2\right)(b-az).$$

Since $a \neq 0$, and choosing $1/a^2 = c^2$ and $b/a = -d$, we have

$$h^2h''\left(c^2+1\right) = -2h'\left(h'^2\left(1+c^2\right)+1\right)(-d-z).$$

We return to equation (2.39) to conclude that these solutions must be the same.

□

2.2.3 Proof of Theorem 2.4

Proof. We begin by considering the parameterizations of the first kind (2.9). Using equations (2.8), (2.13) and (2.15), we have

$$g^2\frac{-f''g'(1+g'^2)+g''(1+f'^2)}{2[g'^2(f'^2+1)+1]^{3/2}} = \frac{-(g+1)}{\sqrt{g'^2(f'^2+1)+1}},$$

or equivalently,

$$-f''g^2g'(1+g'^2)+g^2g''(1+f'^2) = -2(g+1)[g'^2(f'^2+1)+1]. \tag{2.49}$$

Suppose initially that $f'f'' \equiv 0$. Then $f(s) = as+b$ and equation (2.49) becomes

$$g^2g''(1+a^2)+2(g+1)[g'^2(a^2+1)+1] = 0.$$

Since $g(t) > 0$ for all t, by hypothesis we have

$$g''+2\frac{(g+1)}{g^2}g'^2+2\frac{(g+1)}{g^2(1+a^2)} = 0. \tag{2.50}$$

This equation is the same as (2.23) of Lemma 2.3. Set $v(g) = (g')^2$, that is $v'g' = 2g'g''$ and equation (2.23) becomes

$$v' = -4\frac{g+1}{g^2}v-4\frac{g+1}{g^2(1+a^2)}.$$

The equation above is a first order linear equation that we solve by computing

$$F(g) = \int -4\frac{g+1}{g^2}\,dg = -4\ln(g)+\frac{4}{g}.$$

Hence

$$v = Ce^F+e^F\frac{4}{1+a^2}\int e^{-F}\frac{g+1}{g^2}\,dg = C\frac{e^{4/g}}{g^4}-\frac{1}{1+a^2}.$$

Thus

$$g'^2(t) = \frac{Ce^{4/g(t)}}{g^4(t)}-\frac{1}{1+a^2}. \tag{2.51}$$

Suppose from now on that $f'f'' \not\equiv 0$, we differentiate the equation (2.49) with respect to s to obtain

$$-f'''g^2g'(1+g'^2)+2f'f''g^2g'' = -4(g+1)g'^2f'f''.$$

Since $f'f'' \not\equiv 0$, we divide both sides of the previous equation by $f'f''$ to get

$$-\frac{f'''}{f'f''}g^2g'(1+g'^2)+2g^2g'' = -4(g+1)g'^2. \tag{2.52}$$

Now, differentiating again with respect to s gives

$$\left[\frac{f'''}{f'f''}\right]'g^2g'(1+g'^2) = 0.$$

Thus, we must have that either

$$\left[\frac{f'''}{f'f''}\right]' = 0 \quad \text{or} \quad g^2g'(1+g'^2) = 0.$$

If $g^2g'(1+g'^2) = 0$ we have that $g' \equiv 0$, as $g(t) > 0$ for all t. Thus the equation (2.49) becomes $g(t) \equiv -1$, a contradiction. Thus, we must have

$$f''' = 2Mf'f'', \quad M \in \mathbb{R}. \tag{2.53}$$

1. If $M = 0$ we have $f(s) = as^2 + bs + c$ where $a, b, c \in \mathbb{R}$ with $a \neq 0$. Thus the equation (2.49) becomes

   $$-2ag^2g'(1+g'^2) + g^2g''(1+(2as+b)^2) = -2(g+1)[g'^2((2as+b)^2+1)+1].$$

   A differentiation with respect to s and a simplification leads to $g^2g'' = -2(g+1)g'^2$. Set $v(g) = g'^2$, then $v'g' = 2g'g''$ and as $g(t) > 0$ for all t, we obtain

   $$\frac{1}{v}v' = -4\frac{(g+1)}{g^2},$$

   that has solution

   $$v(g) = g'^2(t) = \frac{C^2}{e^{4g(t)}g^4(t)}, \quad C > 0.$$

   Thus

   $$g'(t) = \pm\frac{C}{e^{2g(t)}g^2(t)}, \quad C > 0.$$

   Returning to (2.49) and since $g^2g'' = -2(g+1)g'^2$, we have

   $$ag^2g'(1+g'^2) = g+1.$$

   Hence

   $$\pm a\frac{C}{e^{2g}}\left(\frac{e^{4g}g^4+C}{e^{4g}g^4}\right) = g+1,$$

   that is

   $$\pm aC(e^{4g}g^4+C) = e^{6g}g^4(g+1).$$

   A polynomial equation that leads into the fact that g is constant, a contradiction.

2. If $M \neq 0$, then a first integration of (2.53) gives $f'' = Mf'^2 + B$. Thus, equation (2.49) becomes

$$-(Mf'^2+B)g^2g'(1+g'^2)+g^2g''(1+f'^2) = -2(g+1)[g'^2(f'^2+1)+1],$$

that is

$$f'^2[-Mg^2g'(1+g'^2)+g^2g''+2(g+1)g'^2] = [Bg^2g'-2(g+1)](1+g'^2)-g^2g''.$$

Since by equation (2.52) we have $-Mg^2g'(1+g'^2) = -g^2g''-2(g+1)g'^2$, then

$$[-Bg^2g'+2(g+1)](1+g'^2) = -g^2g''.$$

Replacing this in equation (2.52) gives

$$(B-M)g^2g'(1+g'^2) = 2(g+1).$$

Clearly if $B = M$ we have $g \equiv -1$, a contraction. Since $g(t) > 0$ for all t, we have

$$(B-M)g'(1+g'^2) = 2(g+1)\frac{1}{g^2}.$$

Replacing this in equation (2.52) provides

$$g'' = -\frac{M}{M-B}\frac{g+1}{g^2} - 2\frac{g+1}{g^2}g'^2.$$

Set $v(g) = g'^2$. Then $v'g' = 2g'g''$ and we have

$$v' = -4\frac{g+1}{g^2}\left(\frac{M}{2(M-B)}+v\right),$$

that has a solution

$$v = g'^2(t) = \frac{Ce^{4/g(t)}-M}{2(M-B)g^4(t)}-, \quad C \neq 0.$$

Thus we have

$$(B-M)^2g^4g'^2(1+g'^2)^2 = (B-M)^2g^4\left(\frac{Ce^{4/g}-M}{2(M-B)g^4}\right)\left(1+\frac{Ce^{4/g}-M}{2(M-B)g^4}\right)^2 = 4(g+1)^2,$$

that is
$$(Ce^{4/g}-M)[(M-B)g^4+Ce^{4/g}-M]^2=4(g+1)^2(M-B)g^8.$$

An equation that leads to the fact that g is constant, a contradiction.

We now approach the parameterizations of the second kind (2.10). Using (2.8) , (2.16) and (2.18), we have

$$t^2\frac{-f''(1+g'^2)+g''f'(1+f'^2)}{2[f'^2(g'^2+1)+1]^{3/2}}=\frac{(t+1)f'g'}{\sqrt{f'^2(g'^2+1)+1}},$$

or equivalently

$$-f''t^2(1+g'^2)+g''t^2f'(1+f'^2)=2f'g'(t+1)[f'^2(g'^2+1)+1]. \tag{2.54}$$

Suppose initially that $f'\equiv 0$, the equation (2.54) is trivially satisfied and the surface is contained in a hyperbolic plane. Supposing now that $f'\not\equiv 0$ and $f''\equiv 0$. We have $f(s)=as+b$, $a\neq 0$ and the equation (2.54) becomes

$$g''t^2(1+a^2)=2g'(t+1)[a^2(g'^2+1)+1].$$

This ODE has at least the constant solution $g'\equiv 0$. Suppose then that $g'\not\equiv 0$ and consider now the substitution $v(t)=g'(t)$, we have

$$\frac{1}{v(a^2(v^2+1)+1)}v'=2\frac{(t+1)}{t^2(1+a^2)}.$$

A separable differential equation that has a solution

$$\frac{2\ln(v)-\ln\big(a^2(v^2+1)+1\big)}{2(a^2+1)}=\frac{2}{1+a^2}\left(\ln(t)-\frac{1}{t}\right)+d,$$

that is
$$\ln\left(\frac{v^2}{a^2(v^2+1)+1}\right)=4\left(\ln(t)-\frac{1}{t}\right)+d,$$

which implies that

$$\frac{v^2}{a^2(v^2+1)+1}=\frac{t^4}{Ce^{4/t}},\ \ C=1/e^d>0,\ d\in\mathbb{R}.$$

Simplifying, we have
$$v^2=\frac{(a^2+1)t^4}{Ce^{4/t}-a^2t^4}.$$

Hence

$$g'^2(t) = \frac{(a^2+1)t^4}{Ce^{4/t} - a^2t^4}.$$

Suppose from now on that $f'f'' \not\equiv 0$. As $t^2(1+g'^2) \neq 0$ we divide both sides by it in the equation (2.54) to get

$$-f'' + \frac{g''}{(1+g'^2)} f'(1+f'^2) = 2f' \frac{g'(t+1)}{t^2(1+g'^2)}[f'^2(g'^2+1)+1].$$

As $f' \not\equiv 0$, we divide both sides by it to obtain

$$-\frac{f''}{f'} + \frac{g''}{(1+g'^2)}(1+f'^2) = 2\frac{g'(t+1)}{t^2(1+g'^2)}[f'^2(g'^2+1)+1].$$

Differentiating with respect to s gives

$$\left[-\frac{f''}{f'}\right]' + 2f'f''\frac{g''}{(1+g'^2)} = 4f'f''\frac{g'(t+1)}{t^2}.$$

Since $f'' \not\equiv 0$, we have

$$\frac{1}{2f'f''}\left[-\frac{f''}{f'}\right]' = -\frac{g''}{(1+g'^2)} + 2\frac{g'(t+1)}{t^2}.$$

As each side depends only on its own variable, we have

$$\frac{1}{2f'f''}\left[-\frac{f''}{f'}\right]' = M = -\frac{g''}{(1+g'^2)} + 2\frac{g'(t+1)}{t^2}, \quad M \in \mathbb{R}. \tag{2.55}$$

If $M = 0$, then $f'' = -Pf'$, $P \neq 0$ and $g''t^2 - 2g'(t+1)(1+g'^2) = 0$. Thus, equation (2.54) becomes

$$Pf't^2(1+g'^2) + g''t^2 f'(1+f'^2) = 2f'g'(t+1)[f'^2g'^2 + (f'^2+1)].$$

That can be reduced to

$$Pf't^2(1+g'^2) = 2f'^3g'^3(t+1).$$

Again, as $f' \not\equiv 0$, we have

$$Pt^2(1+g'^2) = 2f'^2g'^3(t+1),$$

which implies that f' is constant, a contradiction.

If $M \neq 0$, a first integration of equation (2.55) with relation to s gives $-f'' = Mf'^3 + Nf'$. Thus, equation (2.54) becomes

$$(Mf'^3 + Nf')t^2(1+g'^2) + g''t^2 f'^3 + g''t^2 - 2f'^3 g'(t+1)(g'^2+1) - 2f'g'(t+1) = 0.$$

Since $g''t^2 - 2g'(t+1)(1+g'^2) = -Mt^2(1+g'^2)$, we get

$$Nf't^2(1+g'^2) + g''t^2 - 2f'g'(t+1) = 0.$$

Since $f' \not\equiv 0$ is a contradiction, then we divide both sides of the previous equation by f' to obtain

$$Nt^2(1+g'^2) + \frac{1}{f'}g''t^2 - 2g'(t+1) = 0.$$

Differentiating with respect to s gives

$$-\frac{f''}{f'^2}g''t^2 = 0,$$

which implies that $g'' \equiv 0$, that is $g(t) = ct + d$. Then equation (2.54) becomes

$$f'[Nt^2(1+c^2) - 2c(t+1)] = 0.$$

Since $f' \not\equiv 0$, we must have

$$Nt^2(1+c^2) - 2c(t+1) = 0,$$

which implies that t is constant or $c = N = 0$, both contradictions.

We remember the following equation

$$g'^2(t) = \frac{(a^2+1)t^4}{Ce^{4/t} - a^2t^4}.$$

Let $c = m/a^2 > 0$ and consider the following change of variables $z = g(t)$ and $t = h(z) = g^{-1}(g(t))$. Thus we have

$$[h'(z)]^2 = \frac{1}{[g'(t)]^2} = \frac{1}{[g'(h(z))]^2} = \frac{Ce^{4/t} - a^2t^4}{(a^2+1)t^4} = \frac{Ce^{4/h(z)} - a^2h^4(z)}{(a^2+1)h^4(z)},$$

with $(1+m^2) = (1+a^2)/a^2$, and $p = c/(1+a^2)$ we have

$$h'^2(z) = \pm\sqrt{\frac{pe^{4/h(z)}}{h^4(z)} - \frac{1}{1+m^2}}.$$

An ODE equivalent to equation (2.51). Thus, the solutions are equivalent. □

Chapter 3

Translation surfaces in $\mathbb{S}^3$

A translation surface in the three-dimensional sphere $\mathbb{S}^3$ is a surface generated by the quaternionic product of two curves, called generating curves. In this chapter, we present rigidity results for such surfaces. We introduce an associated frame for curves in $\mathbb{S}^3$, and by means of it, we describe the local intrinsic and extrinsic geometry of translation surfaces in $\mathbb{S}^3$. The rigidity results, concerning minimal and constant mean curvature (CMC) surfaces, are given in terms of the curvature and torsion of the generating curves and their proofs rely on the associated frame of such curves. Finally, we present a correspondence between translation surfaces in $\mathbb{S}^3$ and translation surfaces in $\mathbb{R}^3$. We show that these surfaces are locally isometric, and we present a relation between their mean curvatures.

The content of this chapter constitutes a joint work with João Paulo do Santos [18], entitled "Rigidity of Translation Surfaces in the Three-Dimensional Sphere $\mathbb{S}^3$".

3.1 Preliminary Concepts

In this section, we present the quaternionic model for $\mathbb{S}^3$, which equips it with the structure of a Lie group endowed with a bi-invariant metric. We also introduce basic concepts and properties that will be useful throughout this work. For further details, we refer [20, 53] to the interested reader.

We begin by identifying $\mathbb{R}^4$ with the nonzero quaternions $\mathbb{H}^* = \mathbb{H}\backslash\{0\}$ in the standard way: (x_1,x_2,x_3,x_4) is viewed as the quaternion $x_1 + ix_2 + jx_3 + kx_4$. Hence, for $x = (x_1,x_2,x_3,x_4)$

and $y = (y_1, y_2, y_3, y_4)$, we have

$$x \cdot y = \begin{bmatrix} x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \\ x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 \\ x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 \\ x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1 \end{bmatrix}.$$

We also define the conjugate of $x \in \mathbb{R}^4$ as $\overline{x} = (x_1, -x_2, -x_3, -x_4)$.

Now, let $x, y, a \in \mathbb{H}^*$. The following summarizes the properties of this group and they follow from the definition of quaternions and the usual metric of $\mathbb{R}^4$

$$\begin{array}{llll} 1. & \overline{x \cdot y} = \overline{y} \cdot \overline{x} & 3. & \langle x \cdot y, x \cdot y \rangle = \langle x, x \rangle \langle y, y \rangle. \\ 2. & \langle x \cdot a, y \cdot a \rangle = \langle x, y \rangle. & 4. & x^{-1} = \overline{x}/|x|^2. \end{array} \tag{3.1}$$

Therefore, since $\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, it follows from the previously listed properties that, for all $x, y \in \mathbb{S}^3$, we have $\langle x \cdot y, x \cdot y \rangle = 1$, that is, the product is closed in $\mathbb{S}^3$. Since this product is differentiable, it endows $\mathbb{S}^3$ with the structure of a Lie group, whose identity element is $e_1 = (1, 0, 0, 0)$. We also point out that $\mathcal{S} = (\{0\} \times \mathbb{R}^3) \cap \mathbb{S}^3$ can be seen as the space of purely imaginary unit quaternions and this notation will be important as the set $\mathcal{S}$ appears recursively throughout this work. Finally, we will use the notation $x \perp y$, for $x, y \in \mathbb{S}^3$, to indicate that $\langle x, y \rangle = 0$.

By the property 4 in (3.1) we conclude that $x^{-1} = \overline{x}$ whenever $x \in \mathbb{S}^3$. Therefore, if $x \perp y$, we have $\langle x \cdot \overline{y}, e_1 \rangle = 0$ and $\langle \overline{x}, \overline{y} \rangle = 0$. Moreover, if $x_1 = y_1 = 0$, then

$$x \cdot y = (0, x_3y_4 - x_4y_3, x_4y_2 - x_2y_4, x_2y_3 - x_3y_2).$$

Now let $\tilde{x} = (x_2, x_3, x_4)$, $\tilde{y} = (y_2, y_3, y_4) \in \mathbb{R}^3$, and set $x = (0, \tilde{x}), y = (0, \tilde{y}) \in \mathbb{R}^4$. Thus

$$x \cdot y = (0, \tilde{x} \times \tilde{y}),$$

where $\times$ denotes the cross product in $\mathbb{R}^3$.

Remark 3.1. One can also see this group product as a canonical product of a linear map with a vector in $\mathbb{R}^4$, in the following sense

$$x \cdot y = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_1 & -y_2 & -y_3 & -y_4 \\ y_2 & y_1 & y_4 & -y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

It is also well known that any rotation A in 4-dimensional space can be expressed in terms of a pair of unit quaternions x and y. More precisely, given a nonzero vector $v \in \mathbb{R}^4$, a rotation can be written as $A_{x,y}v = x \cdot v \cdot y$. Arbitrary rotations in four dimensions have 6 degrees of freedom; each matrix represents 3 of those 6 degrees. Furthermore, for a translation surface $\alpha(s) \cdot \beta(t)$, we have

$$A_{x,y}(\alpha(s) \cdot \beta(t)) = (x \cdot \alpha(s)) \cdot (\beta(t) \cdot y) = \tilde{\alpha}(s) \cdot \tilde{\beta}(t).$$

In particular, let $z, w \in \mathbb{S}^3$, then

$$A_{x,y}\alpha(s) \cdot A_{z,w}\beta(t) = x \cdot \alpha(s) \cdot y \cdot z \cdot \beta(t) \cdot w,$$

that is not necessarily an isometry applied to $\alpha(s) \cdot \beta(t)$. Moreover, we can write the conjugacy of an element $\alpha \in \mathbb{S}^3$ as

$$A\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \overline{\alpha}.$$

Since $\det A = -1$ and A is an isometry, then A inverts the orientation. Since the sign of the torsion of a curve defines such orientation, then A changes the sign of the torsion. Thus $\tau_\alpha = -\tau_{\overline{\alpha}}$.

3.1.1 Frenet-Serret equations and special frames for curves in $\mathbb{S}^3$

In what follows, let ∇ be the standard Levi-Civita connection in $\mathbb{S}^3$. Let $\alpha : I \subset \mathbb{R} \to \mathbb{S}^3$ be a smooth curve parametrized by the arc length. Following [53, Chapter 7, Part B], we denote the tangent vector of α. by $t_\alpha = \alpha'$. The curvature of α is defined as $\kappa_\alpha(s) := \left|\nabla_{\alpha'(s)}\alpha'(s)\right|$. At the points s where $\kappa_\alpha(s) \neq 0$, we define $n_\alpha(s)$ as $n_\alpha(s) = \kappa_\alpha^{-1}(s)\nabla_{\alpha'(s)}t_\alpha(s)$. Finally, at the points where both t_α and n_α are well defined, we define the binormal vector field b_α to α as the unit vector in $T_\alpha\mathbb{S}^3$ that is orthogonal to both t_α and n_α, and such that the frame $\{t_\alpha, n_\alpha, b_\alpha\}$ is positively oriented with respect to the orientation of $\mathbb{S}^3$. Throughout this paper, we will consider the orientation on $\mathbb{S}^3$ such that the unit normal field is given by $N(p) = p$. In this case, $b_\alpha \in T_\alpha\mathbb{S}^3$ defined so that $\det(\alpha, t_\alpha, n_\alpha, b_\alpha) > 0$.

The well-known Frenet-Serret equations for smooth curves in $\mathbb{S}^3$, parametrized by the arc length are given by

$$\begin{cases} \nabla_{t_\alpha} t_\alpha &= \kappa_\alpha n_\alpha, \\ \nabla_{t_\alpha} n_\alpha &= -\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha, \\ \nabla_{t_\alpha} b_\alpha &= -\tau_\alpha n_\alpha, \end{cases}$$

where κ_α and τ_α are the curvature and torsion of α, respectively. Thus, from the definition of ∇, we derive the following equations

$$\begin{cases} \alpha' = t_\alpha, \\ \alpha'' = \kappa_\alpha n_\alpha - \alpha. \end{cases} \qquad \begin{cases} {t_\alpha}' = \kappa_\alpha n_\alpha - \alpha, \\ n'_\alpha = -\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha, \\ b'_\alpha = -\tau_\alpha n_\alpha. \end{cases} \tag{3.2}$$

Since t_α is a unit vector field, we define the vector field T_α as the product $T_\alpha := \overline{\alpha} \cdot t_\alpha$. If $\kappa_\alpha \neq 0$, the Frenet frame $\{t_\alpha, n_\alpha, b_\alpha\}$ is well defined, and we can extend this construction to define the vector fields $N_\alpha := \overline{\alpha} \cdot n_\alpha$ and $B_\alpha := \overline{\alpha} \cdot b_\alpha$. It follows from (3.1) that $\{T_\alpha, N_\alpha, B_\alpha\}$ provides an orthonormal frame. In the context of translation surfaces, it will be also useful to consider the frame $\{\hat{T}_\alpha, \hat{N}_\alpha, \hat{B}_\alpha\}$ defined by $\hat{T}_\alpha = \alpha \cdot \overline{t_\alpha}$, $\hat{N}_\alpha = \alpha \cdot \overline{n_\alpha}$ and $\hat{B}_\alpha = \alpha \cdot \overline{b_\alpha}$. Let us formalize this construction with the following definition:

Definition 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{S}^3$ be an arc length curve with curvature $\kappa_\alpha \neq 0$ everywhere. A quaternionic frame associated to α is defined as the orthonormal set $\{T_\alpha, N_\alpha, B_\alpha\}$, where $T_\alpha = \overline{\alpha} \cdot t_\alpha$, $N_\alpha = \overline{\alpha} \cdot n_\alpha$ and $B_\alpha = \overline{\alpha} \cdot b_\alpha$. Similarly, we define the frame $\{\hat{T}_\alpha, \hat{N}_\alpha, \hat{B}_\alpha\}$, where $\hat{T}_\alpha = \alpha \cdot \overline{t_\alpha}$, $\hat{N}_\alpha = \alpha \cdot \overline{n_\alpha}$ and $\hat{B}_\alpha = \alpha \cdot \overline{b_\alpha}$. We call these frames the left and right frames, respectively.*

The next proposition provides useful identifications for the frames $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{\hat{T}_\alpha, \hat{N}_\alpha, \hat{B}_\alpha\}$.

Proposition 3.1. *Let $\alpha(s)$, be an arc length curve in $\mathbb{S}^3$ with $\kappa_\alpha \neq 0$. Then we have*

$$\begin{aligned} &T_\alpha = \overline{b_\alpha} \cdot n_\alpha, \quad && N_\alpha = \overline{t_\alpha} \cdot b_\alpha, \quad && B_\alpha = \overline{n_\alpha} \cdot t_\alpha, \\ &\hat{T}_\alpha = -b_\alpha \cdot \overline{n_\alpha}, \quad && \hat{N}_\alpha = -t_\alpha \cdot \overline{b_\alpha}, \quad && \hat{B_\alpha} = -n_\alpha \cdot \overline{t_\alpha}. \end{aligned}$$

Proof. Since $x \perp y$ implies in $x \cdot \overline{y} \in \mathcal{S}$, $\langle \overline{x}, \overline{y} \rangle = 0$ and $x \cdot \overline{y} = -y \cdot \overline{x}$. Thus $\left\{\overline{n_\alpha} \cdot b_\alpha, \overline{b_\alpha} \cdot t_\alpha, \overline{t_\alpha} \cdot n_\alpha\right\} \subset \mathcal{S}$ is an orthonormal frame and we have the following

$$\langle \overline{\alpha} \cdot t_\alpha, \overline{t_\alpha} \cdot \alpha \rangle = \langle \overline{\alpha} \cdot t_\alpha, -\overline{\alpha} \cdot t_\alpha \rangle = -1,$$

Thus, by the same properties, we have

$$\langle \overline{\alpha} \cdot t_\alpha, \overline{t_\alpha} \cdot n_\alpha \rangle = \langle \overline{\alpha} \cdot t_\alpha, \overline{t_\alpha} \cdot b_\alpha \rangle = 0.$$

Then, we proceed to compute the following

$$\begin{array}{lclclclcl} \langle \overline{\alpha} \cdot n_\alpha, \overline{t_\alpha} \cdot b_\alpha \rangle & = & \pm 1 & , & \langle \overline{\alpha} \cdot n_\alpha, \overline{t_\alpha} \cdot \alpha \rangle & = \langle \overline{\alpha} \cdot n_\alpha, \overline{t_\alpha} \cdot t_\alpha \rangle & = \langle \overline{\alpha} \cdot n_\alpha, \overline{t_\alpha} \cdot n_\alpha \rangle & = 0, \\ \langle \overline{\alpha} \cdot b_\alpha, \overline{t_\alpha} \cdot n_\alpha \rangle & = & \pm 1 & , & \langle \overline{\alpha} \cdot b_\alpha, \overline{t_\alpha} \cdot \alpha \rangle & = \langle \overline{\alpha} \cdot b_\alpha, \overline{t_\alpha} \cdot t_\alpha \rangle & = \langle \overline{\alpha} \cdot b_\alpha, \overline{t_\alpha} \cdot b_\alpha \rangle & = 0. \end{array}$$

Hence, repeating this process for other possible products, we obtain the following table

$\langle \cdot, \cdot \rangle$	$\overline{t_\alpha} \cdot \alpha$	$\overline{t_\alpha} \cdot n_\alpha$	$\overline{t_\alpha} \cdot b_\alpha$	$\overline{n_\alpha} \cdot \alpha$	$\overline{n_\alpha} \cdot t_\alpha$	$\overline{n_\alpha} \cdot b_\alpha$	$\overline{b_\alpha} \cdot \alpha$	$\overline{b_\alpha} \cdot t_\alpha$	$\overline{b_\alpha} \cdot n_\alpha$
$\overline{\alpha} \cdot t_\alpha$	-1	0	0	0	0	± 1	0	0	± 1
$\overline{\alpha} \cdot n_\alpha$	0	0	± 1	-1	0	0	0	± 1	0
$\overline{\alpha} \cdot b_\alpha$	0	± 1	0	0	± 1	0	-1	0	0

The process is simplified in determining the signs of the following products

$$\langle \overline{\alpha} \cdot t_\alpha, \overline{b_\alpha} \cdot n_\alpha \rangle = \pm 1, \quad \langle \overline{\alpha} \cdot n_\alpha, \overline{t_\alpha} \cdot b_\alpha \rangle = \pm 1, \quad \langle \overline{\alpha} \cdot b_\alpha, \overline{n_\alpha} \cdot t_\alpha \rangle = \pm 1.$$

We have chosen these ones as they are more relevant to the context of the computations of this work. Hence, in order to define the correct signs of the above products, observe that it must be true for every configuration of the frame $\{t_\alpha, n_\alpha, b_\alpha\}$. Up to rigid motion, we can consider that, at a given point s_0, we have $\alpha(s_0) = e_1$, $t_\alpha(s_0) = e_2$, $n_\alpha(s_0) = e_3$ and $b_\alpha(s_0) = e_4$. In this case, $T_\alpha = e_2$ and $\overline{b}_\alpha \cdot n_\alpha = e_2$. The other cases are similar. Proceeding this way, and repeating the process to the other possible vectors formed by $\alpha, t_\alpha, n_\alpha, b_\alpha$ and their products, we have

$\langle \cdot, \cdot \rangle$	$\overline{b}_\alpha \cdot n_\alpha$	$\overline{t}_\alpha \cdot b_\alpha$	$\overline{n}_\alpha \cdot t_\alpha$
T_α	1	0	0
N_α	0	1	0
B_α	0	0	1

The procedure to the frame $\{\hat{T}_\alpha, \hat{N}_\alpha, \hat{B}_\alpha\}$ is analogous. □

Using the equations (3.2) we derive the following Frenet-Serret type equations for the quaternionic frame of α:

$$\begin{cases} T'_\alpha = & \overline{t_\alpha} \cdot t_\alpha + \overline{\alpha} \cdot (\kappa_\alpha n_\alpha - \alpha) & = & \kappa_\alpha N_\alpha, \\ N'_\alpha = & \overline{t_\alpha} \cdot n_\alpha + \overline{\alpha} \cdot (-\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha) & = & -\kappa_\alpha T_\alpha + (\tau_\alpha - 1) B_\alpha, \\ B'_\alpha = & \overline{t_\alpha} \cdot b_\alpha + \overline{\alpha} \cdot (-\tau_\alpha n_\alpha) & = & -(\tau_\alpha - 1) N_\alpha. \end{cases} \tag{3.3}$$

Moreover, for the frame $\{\hat{T}_\alpha, \hat{N}_\alpha, \hat{B}_\alpha\}$, we have

$$\begin{cases} \hat{T}'_\alpha = & t_\alpha \cdot \overline{t_\alpha} + \alpha \cdot \overline{(\kappa_\alpha n_\alpha - \alpha)} & = & \kappa_\alpha \hat{N}_\alpha, \\ \hat{N}'_\alpha = & t_\alpha \cdot \overline{n_\alpha} + \alpha \cdot \overline{(-\kappa_\alpha t_\alpha + \tau_\alpha b_\alpha)} & = & -\kappa_\alpha \hat{T}_\alpha + (\tau_\alpha + 1)\hat{B}_\alpha, \\ \hat{B}'_\alpha = & t_\alpha \cdot \overline{b_\alpha} + \alpha \cdot \overline{(-\tau_\alpha n_\alpha)} & = & -(\tau_\alpha + 1)\hat{N}_\alpha. \end{cases} \tag{3.4}$$

3.1.2 Geometry of translation surfaces in $\mathbb{S}^3$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{S}^3$, $\alpha(s)$ and $\beta : J \subset \mathbb{R} \to \mathbb{S}^3$, $\beta(t)$ be two arc length curves. Let $X : I \times J \to \mathbb{S}^3$ be the map given by $X(s,t) = \alpha(s) \cdot \beta(t)$. Since $\partial_s X(s,t) = \alpha'(s) \cdot \beta(t)$ and $\partial_t X(s,t) = \alpha(s) \cdot \beta'(t)$ are non-null vectors, the condition for X to be a regular parametrization for a surface in $\mathbb{S}^3$ is given by $\langle \alpha'(s) \cdot \beta(t), \alpha(s) \cdot \beta'(t) \rangle \neq \pm 1$.

Let $X : I \times J \to \mathbb{S}^3$, $X(s,t) = \alpha(s) \cdot \beta(t)$ be a parametrization of a translation surface. From now on, we will always use the left frame for the curve α and the right frame for the curve β. In order to simplify the notation, the structure of the following computations will allow us to denote the right frame $\{\hat{T}_\beta, \hat{N}_\beta, \hat{B}_\beta\}$ of the curve β as $\{T_\beta, N_\beta, B_\beta\}$ without chance of confusion. The parameters s and t will be omitted to make the presentation of the calculations simpler and more pleasant for the reader. Also, from now on, we will always assume that α and β are parametrized by the arc length.

Furthermore, throughout this work, the results are stated for $\alpha(s) \cdot \beta(t)$ but they are also true for $\beta(t) \cdot \alpha(s)$, unless said otherwise. In particular, for the results that use $\tau_\alpha = 1$, we have the same changing the role of α β but with $\tau_\beta = -1$.

Now we present the following

Theorem 3.1. *Let $X : I \times J \to \mathbb{S}^3$, $X(s,t) = \alpha(s) \cdot \beta(t)$, be a parametrization of a translation surface. Then the regularity condition is given by*

$$\langle T_\alpha, T_\beta \rangle \neq \pm 1. \tag{3.5}$$

The unit normal field at $X(s,t)$ in $\mathbb{S}^3$ is

$$N(s,t) = \frac{\alpha'(s) \cdot \beta'(t) - \langle T_\alpha(s), T_\beta(t) \rangle \; \alpha(s) \cdot \beta(t)}{\sqrt{1 - \langle T_\alpha(s), T_\beta(t) \rangle^2}}. \tag{3.6}$$

The mean curvature is given by

$$H = \frac{\kappa_\alpha \langle B_\alpha, T_\beta \rangle - \kappa_\beta \langle T_\alpha, B_\beta \rangle - 2\langle T_\alpha, T_\beta \rangle [\langle T_\alpha, T_\beta \rangle^2 - 1]}{2[1 - \langle T_\alpha, T_\beta \rangle^2]^{3/2}}, \tag{3.7}$$

and the Gaussian curvature is

$$K = \frac{\kappa_\alpha \kappa_\beta \langle B_\alpha, T_\beta \rangle \langle T_\alpha, B_\beta \rangle}{(1 - \langle T_\alpha, T_\beta \rangle^2)^2} \tag{3.8}$$

Proof. Initially, with equations (3.3) and (3.4), we compute the coefficients of the first fundamental form

$$\begin{aligned} E &= \langle X_s, X_s \rangle &&= \langle \alpha' \cdot \beta, \alpha' \cdot \beta \rangle &&= 1, \\ G &= \langle X_t, X_t \rangle &&= \langle \alpha \cdot \beta', \alpha \cdot \beta' \rangle &&= 1, \\ F &= \langle X_s, X_t \rangle &&= \langle \alpha' \cdot \beta, \alpha \cdot \beta' \rangle &&= \langle T_\alpha, T_\beta \rangle. \end{aligned} \tag{3.9}$$

Let now $Y(s,t) = \alpha'(s) \cdot \beta'(t)$, we have

$$\begin{aligned} \langle X_s(s,t), Y(s,t) \rangle &= \langle \alpha'(s) \cdot \beta(t), \alpha'(s) \cdot \beta'(t) \rangle &= \langle \beta(t), \beta'(t) \rangle &= 0, \\ \langle X_t(s,t), Y(s,t) \rangle &= \langle \alpha(s) \cdot \beta'(t), \alpha'(s) \cdot \beta'(t) \rangle &= \langle \alpha(s), \alpha'(s) \rangle &= 0. \end{aligned}$$

Thus, $Y(s,t)$ is orthogonal to X_s e X_t, for every $s \in I, t \in J$. Also, by a similar argument, one can see that $X(s,t)$ is also orthogonal to X_s and X_t, for every $s \in I, t \in J$. Hence, X and Y are contained in a plane that is orthogonal to the plane span$\{X_s, X_t\}$ in $\mathbb{R}^4$.

Let $N(s,t)$ be the unit normal field at $X(s,t)$ in $\mathbb{S}^3 \subset \mathbb{R}^4$ that is at the same time orthogonal to X, X_s and X_t. Thus $N = aX + bY$, with $a^2 + b^2 + 2ab\langle X, Y \rangle = 1$ and we have $\langle N, X \rangle = a + b\langle X, Y \rangle = 0$. Then $a = -b\langle X, Y \rangle$, which implies that

$$b^2(1 + \langle X, Y \rangle^2) - 2b^2 \langle X, Y \rangle^2 = b^2(1 - \langle X, Y \rangle^2) = 1.$$

As we may choose $b = 1/\sqrt{1 - \langle X, Y \rangle^2}$, and since $\langle X, Y \rangle = \langle \alpha \cdot \beta, \alpha' \cdot \beta' \rangle = \langle \beta \cdot \overline{\beta'}, \overline{\alpha} \cdot \alpha' \rangle$. we get

$$N(s,t) = \frac{\alpha'(s) \cdot \beta'(t) - \langle T_\alpha(s), T_\beta(t) \rangle \; \alpha(s) \cdot \beta(t)}{\sqrt{1 - \langle T_\alpha(s), T_\beta(t) \rangle^2}}.$$

We now recall that ∇ and $\tilde{\nabla}$ are the Levi-Civita connections in $\mathbb{S}^3$ and $\mathbb{R}^4$, respectively. Thus, since $p = X(s_0, t_0)$ is orthogonal to the surface T_pX for every s and t (as the surface is contained in $\mathbb{S}^3 \subset \mathbb{R}^4$), and $\tilde{\nabla}$ is known to be equivalent to the usual differentiation, we have

$$\begin{aligned} X_{ss} &= \nabla_{X_s} X_s + \langle X_{ss}, X \rangle X, \\ X_{st} &= \nabla_{X_s} X_t + \langle X_{st}, X \rangle X, \\ X_{tt} &= \nabla_{X_t} X_t + \langle X_{tt}, X \rangle X. \end{aligned}$$

With these equations, we compute the coefficients of the second fundamental form of the surface $X(s,t)$ as

$$e = \langle X_{ss}, N\rangle, \;\; g = \langle X_{tt}, N\rangle, \;\; f = \langle X_{st}, N\rangle.$$

Remembering that $\langle N, X\rangle = 0$, $\langle \alpha\cdot\beta, \alpha''\cdot\beta\rangle = -1$, properties (3.1) and system (3.2), we begin computing the coefficients of the second fundamental form

$$e = \langle N, \nabla_{X_s} X_s\rangle = \frac{\langle \alpha'\cdot\beta' - \langle \alpha'\cdot\beta', \alpha\cdot\beta\rangle\alpha\cdot\beta, \alpha''\cdot\beta\rangle}{\sqrt{1-\langle \alpha'\cdot\beta', \alpha\cdot\beta\rangle^2}} = \frac{\langle \alpha'\cdot\beta', \alpha''\cdot\beta + \alpha\cdot\beta\rangle}{\sqrt{1-\langle \alpha'\cdot\beta', \alpha\cdot\beta\rangle^2}} =$$
$$= \frac{\langle t_\alpha\cdot t_\beta, \kappa_\alpha n_\alpha\cdot\beta - \alpha\cdot\beta + \alpha\cdot\beta\rangle}{\sqrt{1-\langle t_\alpha\cdot t_\beta, \alpha\cdot\beta\rangle^2}} = \frac{\kappa_\alpha\langle \overline{n_\alpha}\cdot t_\alpha, \beta\cdot\overline{t_\beta}\rangle}{\sqrt{1-\langle \overline{\alpha}\cdot t_\alpha, \beta\cdot\overline{t_\beta}\rangle^2}}.$$

If $\kappa_\alpha \equiv 0$, then $\alpha'' = -\alpha$ and $e = 0$. Symmetrically we have

$$g = \langle N, \nabla_{X_t} X_t\rangle = \frac{\langle \alpha'\cdot\beta', \alpha\cdot\beta'' + \alpha\cdot\beta\rangle}{\sqrt{1-\langle \alpha'\cdot\beta', \alpha\cdot\beta\rangle^2}} = \frac{\kappa_\beta\langle \overline{\alpha}\cdot t_\alpha, n_\beta\cdot\overline{t_\beta}\rangle}{\sqrt{1-\langle \overline{\alpha}\cdot t_\alpha, \beta\cdot\overline{t_\beta}\rangle^2}}.$$

Again, if $\kappa_\beta \equiv 0$, then $\beta'' = -\beta$ and $g = 0$. Also we have

$$f = \langle N, \nabla_{X_t} X_s\rangle = \frac{\langle \alpha'\cdot\beta' - \langle \alpha'\cdot\beta', \alpha\cdot\beta\rangle\alpha\cdot\beta, \alpha'\cdot\beta'\rangle}{\sqrt{1-\langle \alpha'\cdot\beta', \alpha\cdot\beta\rangle^2}} = \sqrt{1-\langle \overline{\alpha}\cdot t_\alpha, \beta\cdot\overline{t_\beta}\rangle^2}.$$

In case $\kappa_\alpha \not\equiv 0$ and $\kappa_\beta \not\equiv 0$, it follows from the first system in (3.2), Definition 3.1 and Proposition 3.1 that the coefficients of the second fundamental form can be written as

$$e = \frac{\kappa_\alpha\langle B_\alpha, T_\beta\rangle}{\sqrt{1-\langle T_\alpha, T_\beta\rangle}}, \quad g = -\frac{\kappa_\beta\langle T_\alpha, B_\beta\rangle}{\sqrt{1-\langle T_\alpha, T_\beta\rangle}}, \quad f = \sqrt{1-\langle T_\alpha, T_\beta\rangle^2}. \tag{3.10}$$

Also, using the usual mean curvature formula, we obtain

$$H = \frac{1}{2}\frac{eG - 2fF + Eg}{EG - F^2} = \frac{\kappa_\alpha\langle B_\alpha, T_\beta\rangle - \kappa_\beta\langle T_\alpha, B_\beta\rangle - 2\langle T_\alpha, T_\beta\rangle[\langle T_\alpha, T_\beta\rangle^2 - 1]}{2[1-\langle T_\alpha, T_\beta\rangle^2]^{3/2}}.$$

In order to obtain the Gaussian curvature we must compute the extrinsic curvature by the classical equation $K_{ext} = \dfrac{eg - f^2}{EG - F^2}$, that is

$$K_{ext} = -\frac{\kappa_\alpha\kappa_\beta\langle B_\alpha, T_\beta\rangle\langle T_\alpha, B_\beta\rangle}{1-\langle T_\alpha, T_\beta\rangle^2} - 1. \tag{3.11}$$

then the Gaussian curvature is $K = K_{ext} + 1$.

□

Remark 3.2. From Theorem 3.1 we obtain some important equations that will be useful throughout this work. A translation surface is minimal if and only if

$$\kappa_\alpha \langle B_\alpha, T_\beta \rangle - \kappa_\beta \langle T_\alpha, B_\beta \rangle = 2\langle T_\alpha, T_\beta \rangle [\langle T_\alpha, T_\beta \rangle^2 - 1]. \tag{3.12}$$

Furthermore, a translation surface is flat if and only if

$$\kappa_\alpha \kappa_\beta \langle B_\alpha, T_\beta \rangle \langle T_\alpha, B_\beta \rangle = 0. \tag{3.13}$$

3.2 Translation surfaces with constant Gaussian curvature: revisiting the flat case

It is well-known from the Bianchi-Spivak construction [20, 53] that a flat surface in $\mathbb{S}^3$ is recovered locally as the quaternionic product of two curves in $\mathbb{S}^3$. In other words, every flat surface is locally a translation surface. On the other hand, the only translation surfaces in $\mathbb{S}^3$ with constant Gaussian curvature are the flat ones. This is the content of the following recent result:

Proposition 3.2 ([24]). *Let G be an n-dimensional ($n \geq 3$) Lie group with a bi-invariant metric, and M be a translation surface in G with constant Gaussian curvature, then M must be flat.*

This means that, as we have $\mathbb{S}^3 \subset \mathbb{R}^4$ with the usual metric induced by the four-dimensional Euclidean space, which is a bi-invariant metric, the classification of translation surfaces with constant Gaussian curvature is reduced to the flat case.

Remark 3.3. Let $x, y \in G = \mathbb{S}^3$, and let $A, B \in \mathfrak{g}$, with $A, B \neq 0$. The standard metric on $\mathbb{R}^4$ is bi-invariant if and only if

$$\langle A, B \rangle_e = \langle (dL_x)_e A, (dL_x)_e B \rangle = \langle (dR_x)_e A, (dR_x)_e B \rangle.$$

Since each element $x \in \mathbb{S}^3 \subset \mathbb{H}$ can be interpreted as a rotation of $\mathbb{R}^4$, we may identify left and right translations by x with linear isometries X_l and X_r, respectively. That is, for $y \in \mathbb{S}^3$,

$$L_x(y) = x \cdot y = X_l(y), \quad \text{and} \quad R_x(y) = y \cdot x = X_r(y),$$

so the differentials satisfy $(dL_x)_* = X_l$ and $(dR_x)_* = X_r$, respectively (see Remark 3.1).

Identifying $\mathfrak{g}\setminus\{0\}$ with the subspace $\{0\}\times\mathbb{R}^3\subset\mathbb{R}^4$, we can further identify A and B with purely imaginary quaternions $a,b\in\mathbb{H}^*$. Using the multiplicative properties of quaternions, we obtain

$$\langle A,B\rangle = \langle (dL_x)_e A, (dL_x)_e B\rangle = \langle x\cdot a, x\cdot b\rangle = \langle a,b\rangle,$$

which shows that the metric is left-invariant. The same argument applies to the right multiplication case, thus proving that the metric is bi-invariant.

A first direct consequence of Proposition 3.2 is the non-existence of totally umbilic and totally geodesic translation surfaces in $\mathbb{S}^3$. In particular, the question of whether totally geodesic spheres are minimal translation surfaces is natural due to the fact that their analogues in $\mathbb{R}^3$, i.e., the planes, are a trivial example of such surfaces. In this sense, we present the following

Theorem 3.2. *There is no totally umbilic surfaces or totally geodesic surfaces in $\mathbb{S}^3$ given as a translation surface.*

Proof. As a totally umbilic surface or a totally geodesic surface has constant principal curvatures equal to $\lambda\in\mathbb{R}$, it has constant Gaussian curvature K by the Gauss Equation:

$$\lambda^2+1=K$$

By proposition 3.2 K must be zero, which is a contradiction with the equation above. □

It is also a consequence of the Gauss Equation that flat surfaces in $\mathbb{S}^3$ have negative extrinsic curvature $K_{ext}=\lambda_1\lambda_2$, where λ_i, $i=1,2$ denote the principal curvatures. In this sense, we have the following well-known result

Theorem 3.3 ([20])**.** *Let Σ be a surface and $\psi:\Sigma\to\mathbb{M}^3(c)$ an immersion with negative constant extrinsic curvature K_{ext} in a space form. Then the asymptotic curves of ψ have constant torsion τ, with $\tau^2=-K_{ext}$ at points where the curvature of the curve does not vanish. Moreover, two asymptotic curves through a point have torsions of opposite signs if they have nonvanishing curvature at that point.*

Such a result is particularly important when we recover a flat surface $\Sigma\subset\mathbb{S}^3$ by means of its asymptotic lines. In fact, it is shown that the asymptotic lines are congruent to each other within their family [20, Proposition 3], and the curves that provide the translation structure are exactly a representative of each class [20, Theorem 9]. Our next result provides a kind of converse of these facts

Proposition 3.3. *Let $X:I\times J\to\mathbb{S}^3$, $X(s,t)=\alpha(s)\cdot\beta(t)$, be a translation surface. in $\mathbb{S}^3$. Suppose that $\kappa_\alpha\neq 0$ and $\alpha(s)\cdot\beta(t_0)$ is an asymptotic line for all $t_0\in J\subset\mathbb{R}$. Then $\tau_\alpha=1$ and either $\kappa_\beta=0$ or $\tau_\beta=-1$. If we have also $\kappa_\beta\not\equiv 0$ then $g=0$, $\tau_\alpha\equiv 1$ and $\tau_\beta\equiv -1$.*

Proof. Suppose that $\alpha(s)\cdot\beta(t_0)$ is an asymptotic line of X for all $t_0 \in J \subset \mathbb{R}$. This implies that $e=0$ and, by Theorem 3.1 we have $\kappa_\alpha\langle B_\alpha, T_\alpha\rangle = 0$ and $K_{ext}\equiv -1$. It follows from Theorem 3.3 that α has torsion $\tau = \pm 1$ where the curvature does not vanishes, since α is congruent to $\alpha\cdot\beta(t_0)$, for all $t_0 \in J$.

Suppose now that $\kappa_\alpha \neq 0$. In order to have $e=0$ we need that $\langle B_\alpha, T_\beta\rangle = 0$. Differentiating the previous equation with respect to s gives $-(\tau_\alpha - 1)\langle N_\alpha, T_\beta\rangle = 0$. Suppose that $\tau_\alpha = 1$, we then differentiate again to obtain

$$-(\tau_\alpha - 1)\langle(-\kappa_\alpha T_\alpha + (\tau_\alpha - 1)B_\alpha), T_\beta\rangle = \kappa_\alpha(\tau_\alpha - 1)\langle T_\alpha, T_\beta\rangle = 0.$$

Since $\kappa_\alpha \not\equiv 0$, we must have

$$\langle T_\alpha, T_\beta\rangle = \langle N_\alpha, T_\beta\rangle = \langle B_\alpha, T_\beta\rangle = 0,$$

a contradiction as T_α, N_α, B_α, $T_\beta \in \mathcal{S}$. Thus we must have $\tau_\alpha = 1$.

Now if $\tau_\alpha = 1$ then $B_\alpha = C$ a constant vector in $\mathcal{S}$. If $\langle B_\alpha, T_\beta\rangle = 0$ and as we have also $T_\beta \in \mathcal{S}$, then T_β is contained in the intersection of a two dimensional plane that passes through the origin with $\mathcal{S}$, that means that either B_β or T_β is constant and thus either $\kappa_\beta \equiv 0$ or $\tau_\beta \equiv -1$. As the conjugacy inverts the orientation (Remark 3.1), this implies the change of sign of the torsion (as this sign defines such orientation), which means that $\tau_\alpha = -\tau_{\overline{\alpha}}$.

Suppose now also $\kappa_\beta \not\equiv 0$, then $\tau_\alpha = -1$ and $B_\alpha = C$ so we have

$$ef = \kappa_\alpha\langle C, T_\beta\rangle = 0.$$

Differentiating with respect to t gives $\kappa_\beta\langle C, N_\beta\rangle = 0$. As $\kappa_\beta \neq 0$, we have that $C \perp N_\beta$ and $C \perp T_\beta$. But $C \perp N_\alpha$ and $C \perp T_\alpha$ and also $T_\alpha, T_\beta, N_\alpha, N_\beta \in \mathcal{S}$, which means that they are all contained in a two dimensional plane in $\mathbb{R}^4$ that is, at the same time, orthogonal to e_1 and C. As $B_\beta \in (\{0\}\times\mathbb{R}^3)\cap\mathbb{S}^3$, $B_\beta \perp T_\beta$ and $B_\beta \perp N_\beta$ we must have $B_\beta = \pm C$ and thus

$$\langle T_\alpha, B_\beta\rangle = \pm\langle T_\alpha, C\rangle = 0,$$

that is $g=0$. As shown before this means that $\tau_\beta = -1$.

□

Following [3], a curve $\gamma(s)$ in $\mathbb{S}^3$ is called a *general helix* if there exists a Killing vector field $V(s)$ with constant length along γ such that the angle between V and γ' is a non-zero constant along γ. It is established in [3, Theorem 3], that a curve γ in $\mathbb{S}^3$ is a general helix if and only if either $\tau \equiv 0$ and γ is a curve in some unit 2-sphere $\mathbb{S}^2$ or there exists a constant b

such that $\tau = b\kappa \pm 1$. Therefore, a curve $\gamma \subset \mathbb{S}^3$ with constant curvature and torsion is a general helix since it satisfies the more general statement $\tau = b\kappa \pm 1$. In particular, when κ and τ are constant we call this curve a *proper helix*. Now, we present

Lemma 3.1. *Let α be an arc-length general helix in $\mathbb{S}^3$ with arc length parameter s. Then T_α, N_α and B_α describe circles in $\mathcal{S}$.*

Proof. Let $\alpha(s)$ be an helix in $\mathbb{S}^3$, then there exists a constant $b \in \mathbb{R}$ such that $\tau_\alpha = b\kappa_\alpha \pm 1$. We may suppose without loss of generality that $\tau_\alpha = b\kappa_\alpha + 1$. Initially, if $\kappa_\alpha \equiv 0$ then T_α is constant and N_α and B_α are not defined. If $\tau_\alpha \equiv 1$ then B_α is constant and T_α and N_α describe the great circle that is orthogonal to B_α and e_1. Thus, suppose from now on that $\kappa_\alpha \not\equiv 0$ and $\tau_\alpha \not\equiv 1$ and consider the curve $\hat{\alpha}(s) = T_\alpha(s)$. Since $\hat{\alpha}$ is in $\mathcal{S}$, then is immediate that $\tau_{\hat{\alpha}}$ vanishes. We compute

$$\frac{d}{ds}\hat{\alpha}(s) = T'_\alpha(s) = \kappa_\alpha(s)N_\alpha(s).$$

Hence, for such a curve, consider the arc length parameter

$$\hat{s}(s) = \int_0^s |\hat{\alpha}'(s)|\, ds = \int_0^s |\kappa_\alpha(s)N_\alpha(s)|\, ds = \int_0^s \kappa_\alpha(s)\, ds = f(s).$$

Then $s(\hat{s}) = f^{-1}(\hat{s})$. Hence

$$\frac{d}{d\hat{s}}s(\hat{s}) = \frac{d}{d\hat{s}}f^{-1}(\hat{s}) = \frac{1}{f'(f^{-1}(\hat{s}))} = \frac{1}{\kappa_\alpha(\hat{s})}.$$

Since $\tilde{s}$ is the arc length parameter of $\hat{\alpha}$, we conclude that

$$\frac{d}{d\hat{s}}\hat{\alpha}(\hat{s}) = \frac{1}{\kappa_\alpha(\hat{s})}[\kappa_\alpha(\hat{s})N_\alpha(\hat{s})] = N_\alpha(\hat{s}).$$

Thus, $t_{\hat{\alpha}}(\hat{s}) = N_\alpha(\hat{s})$. Now, we compute

$$t'_{\hat{\alpha}} = \frac{d}{d\hat{s}}N_\alpha(\hat{s}) = \frac{1}{\kappa_\alpha(\hat{s})}[-\kappa_\alpha(\hat{s})T_\alpha(\hat{s}) + (\tau_\alpha(\hat{s})-1)B_\alpha(\hat{s})] = \frac{\tau_\alpha(\hat{s})-1}{\kappa_\alpha(\hat{s})}B_\alpha(\hat{s}) - \hat{\alpha}(\hat{s}).$$

Since $B_\alpha \perp \hat{\alpha}$, $B_\alpha \perp t_{\hat{\alpha}}$ and equation (3.2), we get that $n_{\hat{\alpha}} = B_\alpha$ and $\kappa_{\hat{\alpha}}(\hat{s}) = (\tau_\alpha(\hat{s})-1)/\kappa_\alpha(\hat{s})$. Since $\tau_\alpha(\hat{s}) = b\kappa_\alpha(\hat{s}) + 1$ we have

$$\kappa_{\hat{\alpha}} = \frac{(\tau_\alpha(\hat{s})-1)}{\kappa_\alpha(\hat{s})} = \frac{b\kappa_\alpha(\hat{s})}{\kappa_\alpha(\hat{s})} = b.$$

As b is constant by hypothesis, we conclude that T_α describes a circle in $\mathcal{S}$.

We remember that in $\mathbb{S}^3$, a small circle is always contained in a small sphere, that is, for some $v \in \mathbb{S}^3$ we describe a small sphere as $\mathcal{S}_v = \{w \in \mathbb{S}^3 : \ w \perp v\}$. Thus, for some $u \in \mathcal{S}_v$ and a constant $\theta \in \mathbb{R}$, a small circle $\mathcal{C}_{v,u,\theta} = \{w \in \mathcal{S}_v : \langle w, u\rangle = \cos(\theta)\}$ has pole (or spherical center) given by v and u. Since T_α is a circle in $\mathcal{S}$, then suppose that for some $u \in \mathcal{S}$, it describes the circle $\mathcal{C}_{e_1,u,\theta} = \{w \in \mathcal{S} : \langle w, u\rangle = \cos(\theta)\}$, then $\langle T_\alpha, u\rangle = \cos(\theta)$. Differentiating with respect to s gives $\kappa_\alpha \langle N_\alpha, u\rangle = 0$. As $\kappa_\alpha \not\equiv 0$, it follows that $t_{\tilde{\alpha}} = N_\alpha$ describes the great circle that is orthogonal to u and e_1.

Now, differentiating $\langle N_\alpha, u\rangle$ again with respect to s gives

$$-\kappa_\alpha \langle T_\alpha, u\rangle + (\tau_\alpha - 1)\langle B_\alpha, u\rangle = 0.$$

Thus

$$\langle B_\alpha, u\rangle = \frac{\kappa_\alpha}{\tau_\alpha - 1}\cos(\theta) = \frac{1}{b}\cos(\theta),$$

which implies that $\langle B_\alpha, u\rangle$ is constant. Thus, B_α describes a circle in $\mathcal{S}$ and T_α, N_α and B_α have the same pole.

□

The definition of *general helix* provides a nice geometric description of a translation surface generated by curves α and β where T_α and T_β make a constant angle. Firstly, it follows from Theorem 3.1 that the metric components of a translation surface are given by $E = G = 1$ and $F = \langle T_\alpha, T_\beta\rangle$. Therefore, when $\langle T_\alpha, T_\beta\rangle$ is constant, such a surface is flat. We can go further and characterize the curves α and β in this case:

Furthermore, we also have

Proposition 3.4. *Let $X : I \times J \to \mathbb{S}^3$, $X(s,t) = \alpha(s)\cdot\beta(t)$, be a translation surface. If $\langle T_\alpha, T_\beta\rangle = C$, then this surface is flat. Moreover, one has $\kappa_\beta \equiv 0$ and either $\kappa_\alpha \equiv 0$ or α is a general helix satisfying $\tau_\alpha = \dfrac{C}{\eta}\kappa_\alpha + 1$ with , $C, \eta \in \mathbb{R}$.*

Proof. Supposing that $F = \langle T_\alpha, T_\beta\rangle = C$ and knowing that T_α and T_β are curves contained in $\mathcal{S}$, we may see this elements as curves in $\mathbb{S}^2 \subset \mathbb{R}^3$. Now, fixing t_0, the condition $\langle T_\alpha(s), T_\beta(t_0)\rangle = C$ for every s implies that either

1. T_α is constant, which implies that $\kappa_\alpha \equiv 0$, Symmetrically fixing s_0 we get that T_β is constant or contained in a cone with center T_α.

2. T_α is contained in a cone centered in $T_\beta(t_0)$. In this case the angle must remain constant if we choose $t_1 \neq t_0$, which implies by the geometry of the sphere $\mathbb{S}^2$ that T_β is constant and thus $\kappa_\beta \equiv 0$.

Now, differentiating $\langle N_\alpha, T_\beta \rangle$ with respect to s gives

$$\kappa_\alpha \langle N_\alpha, T_\beta \rangle = 0. \tag{3.14}$$

If $\kappa_\alpha \not\equiv 0$ we have that $\langle N_\alpha, T_\beta \rangle = 0$, and differentiating again it with respect to s gives

$$-\kappa_\alpha \langle T_\alpha, T_\beta \rangle + (\tau_\alpha - 1)\langle B_\alpha, T_\beta \rangle = -\kappa_\alpha C + (\tau_\alpha - 1)\langle B_\alpha, T_\beta \rangle = 0.$$

If $C = 0$, then $(\tau_\alpha - 1)\langle B_\alpha, T_\beta \rangle = 0$. Thus either $\tau_\alpha = 1$ or $\langle B_\alpha, T_\beta \rangle = 0$. If the second one is true then $T_\alpha \| T_\beta$ and $C = \pm 1$, a contradiction. Hence $\tau_\alpha = 1$.

If $C \neq 0$, we have two cases:

1. If $\tau_\alpha = 1$, then $\kappa_\alpha C = 0$, which implies that $C = 0$ or $\kappa_\alpha = 0$, both contradictions.

2. If $\tau_\alpha \neq 1$, then
$$\langle B_\alpha, T_\beta \rangle = C \frac{\kappa_\alpha}{\tau_\alpha - 1}.$$
Differentiating this equation with respect to s, and using equation (3.14), yields
$$-(\tau_\alpha - 1)\langle N_\alpha, T_\beta \rangle = \left[C \frac{\kappa_\alpha}{\tau_\alpha - 1} \right]' = 0,$$
which implies that $C\kappa_\alpha + \eta = \eta \tau_\alpha$, $\eta \in \mathbb{R}$. It is clear that $\eta \neq 0$, otherwise $\kappa_\alpha C = 0$, which implies that $C = 0$ or $\kappa_\alpha = 0$, both contradictions. Hence we have
$$\frac{C}{\eta}\kappa_\alpha + 1 = \tau_\alpha.$$

□

As a consequence of this result, we can extend this result to a general context, i.e., for any two curves in $\mathbb{S}^3$, in which $\langle \overline{\alpha} \cdot t_\alpha, \beta \cdot \overline{t_\beta} \rangle = C$.

Corollary 3.1. *Let $\alpha(s)$ and $\beta(t)$ be curves in $\mathbb{S}^3$ such that $\langle \overline{\alpha} \cdot t_\alpha, \beta \cdot \overline{t_\beta} \rangle = C$, then $\kappa_\alpha \equiv 0$ and either $\kappa_\beta \equiv 0$ or β is a general helix with $\tau_\beta = \frac{C}{\eta}\kappa_\beta - 1$, $C, \eta \in \mathbb{R}$.*

3.3 On CMC translation surfaces

In the sense of the first case of Theorem 3.1, we begin this section with the following example

Example 3.1. *We know that some of the classically known examples are the so-called Clifford tori C_{R_1,R_2}, given by*

$$C_{R_1,R_2} = \left\{ (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \;\middle|\; x_1^2 + x_2^2 = R_1^2,\;\; x_3^2 + x_4^2 = R_2^2 \right\}.$$

Such a surface can be parameterized as

$$X(s,t) = (R_1\cos(s+t), R_1\sin(s+t), R_2\cos(s-t), R_2\sin(s-t)), \quad R_1^2 + R_2^2 = 1.$$

Consider the rotation

$$M_{R_1,R_2} = \begin{bmatrix} R_1 & 0 & R_2 & 0 \\ 0 & R_1 & 0 & R_2 \\ R_2 & 0 & -R_1 & 0 \\ 0 & R_2 & 0 & -R_1 \end{bmatrix}, \quad \mathrm{Det}(M_{R_1,R_2}) = 1.$$

Thus

$$M_{R_1,R_2}(X(s,t)) = \begin{bmatrix} R_1^2\cos(s+t) + R_2^2\cos(s-t) \\ R_1\sin(s+t) + R_2\sin(s-t) \\ R_1R_2[\cos(s+t) - \cos(s-t)] \\ R_1R_2[\sin(s+t) - \sin(s-t)] \end{bmatrix} = \begin{bmatrix} \cos(s)\cos(t) - (R_1^2 - R_2^2)\sin(s)\sin(t) \\ \sin(s)\cos(t) + (R_1^2 - R_2^2)\cos(s)\sin(t) \\ -2R_1R_2\sin(s)\sin(t) \\ 2R_1R_2\cos(s)\sin(t) \end{bmatrix}.$$

Thus $M_{R_1,R_2}(X(s,t)) = \alpha(s)\cdot\beta(t)$, where

$$\begin{aligned} \alpha(s) &= (\cos(s), \sin(s), 0, 0), \\ \beta(t) &= (\cos(t), (R_1^2 - R_2^2)\sin(t), 0, 2R_1R_2\sin(t)). \end{aligned}$$

Since $T_\alpha = e_2$, we compute

$$\langle T_\alpha, T_\beta \rangle = -(R_1^2 - R_2^2)[\cos^2(t) + \sin^2(t)] = -(R_1^2 - R_2^2).$$

Also $\kappa_\alpha = \kappa_\beta = 0$, then by Theorem 3.1 we have

$$H = \frac{-\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}} = \frac{(R_1^2 - R_2^2)}{\sqrt{1 - (R_1^2 - R_2^2)^2}}.$$

as R_1 and R_2 are constant, we have that $\alpha(s)\cdot\beta(t)$ is a CMC Surface. It is clear that when we have $R_1 = R_2 = 1/\sqrt{2}$, we have the minimal and flat Clifford torus (See Figure 3.1).

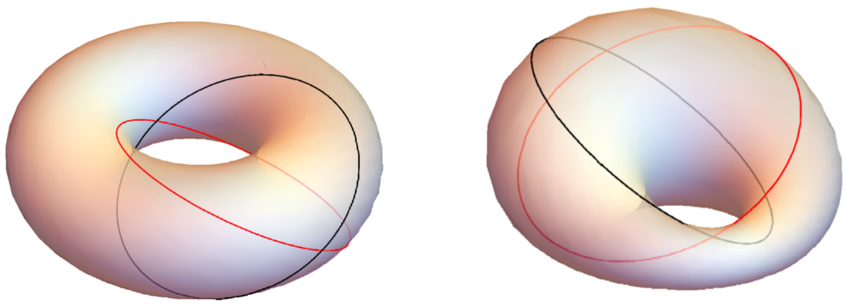

Fig. 3.1 Clifford torus with mean curvatures $H=0$ and $H=1/\sqrt{3}$, respectively. Also with curves α in red and β in black.

We present initially the following result

Theorem 3.4. *Let $X: I\times J\to\mathbb{S}^3$, $X(s,t)=\alpha(s)\cdot\beta(t)$, be a translation surface. If $\kappa_\alpha=\kappa_\beta=0$, then it is a CMC Clifford torus. Moreover, we have $\langle T_\alpha,T_\beta\rangle=C\in(-1,1)$ and the mean curvature is given by*

$$H=\frac{-C}{\sqrt{1-C^2}}.$$

Proof. If $\kappa_\alpha=\kappa_\beta=0$, then T_α and T_β are constant vectors and also, this surface is flat. Thus, $\langle T_\alpha,T_\beta\rangle=C\in(-1,1)$ and equation (3.7) becomes

$$H=\frac{-\langle T_\alpha,T_\beta\rangle}{\sqrt{1-\langle T_\alpha,T_\beta\rangle^2}}=\frac{-C}{\sqrt{1-C^2}}.$$

Moreover, from the proof of [26, Proposition 3.4], we know that such surfaces must be a standard product of circles, $\mathbb{S}^1(r)\times\mathbb{S}^1(\rho)$, and thus CMC Clifford tori. □

We also prove the following

Theorem 3.5. *Let $X: I\times J\to\mathbb{S}^3$, $X(s,t)=\alpha(s)\cdot\beta(t)$, be a CMC translation surface. If $F=\langle T_\alpha,T_\beta\rangle=C\in(-1,1)$ or $\kappa_\alpha=0$ (symmetrically $\kappa_\beta=0$), then this surface is a CMC Clifford Torus.*

Proof. If $F=\langle T_\alpha,T_\beta\rangle=C\in\mathbb{R}$, then this surface is flat. By theorem 3.1, the same occurs when $\kappa_\alpha\equiv 0$ (or $\kappa_\beta\equiv 0$). Again, from the proof of [26, Proposition 3.4], this surface is a CMC Clifford torus. □

3.4 Correspondence between translation surfaces in $\mathbb{S}^3$ and $\mathbb{R}^3$

In this section we present a result that establishes a connection between translation surfaces in $\mathbb{S}^3$ and translation surfaces in $\mathbb{R}^3$. The objective here is to understand the relationship between these surfaces and analyze them through both their intrinsic and extrinsic geometry. We then present some applications, drawing on examples and results from [25, 36]. Accordingly, we state the following

Theorem 3.6. *Let $M \subset \mathbb{S}^3$ be a translation surface generated by curves α and β with curvatures κ_α, κ_β and, when $\kappa_\alpha \not\equiv 0$ and $\kappa_\beta \not\equiv 0$, torsions τ_α and τ_β. Then this surface is locally isometric to a translation surface $\tilde{M} \subset \mathbb{R}^3$, generated by curves $\tilde{\alpha}$ and $\tilde{\beta}$, whose curvatures and torsions satisfy $\tilde{\kappa}_\alpha = \kappa_\alpha$, $\tilde{\kappa}_\beta = \kappa_\beta$, $\tilde{\tau}_\alpha = (\tau_\alpha - 1)$ and $\tilde{\tau}_\beta = (\tau_\beta + 1)$, respectively. The reciprocal identification also holds. Moreover, the mean curvatures $\tilde{H}$ and H, of $\tilde{M}$ and M, respectively, satisfy*

$$\tilde{H} = H + \frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}}.$$

Proof. Suppose initially that $\kappa_\alpha = \kappa_\beta \equiv 0$, then we write $\overline{\alpha} \cdot t_\alpha = T_\alpha$, which is constant. We associate with this a curve $\tilde{\alpha}$ in $\mathbb{R}^3$, which is a straight line in the direction of T_α. By symmetry, we proceed similarly for β and $\tilde{\beta}$. Now, consider a translation surface in $\mathbb{R}^3$ defined by $\Psi(s,t) = \tilde{\alpha}(s) + \tilde{\beta}(t)$. We have that

$$\begin{array}{rcccccccl}
\tilde{E} & = & \langle \Psi_s(s,t), \Psi_s(s,t) \rangle & = & \langle \tilde{\alpha}'(s), \tilde{\alpha}'(s) \rangle & = & \langle T_\alpha, T_\alpha \rangle & = & 1, \\
\tilde{G} & = & \langle \Psi_t(s,t), \Psi_t(s,t) \rangle & = & \langle \tilde{\beta}'(t), \tilde{\beta}'(t) \rangle & = & \langle T_\beta, T_\beta \rangle & = & 1, \\
\tilde{F} & = & \langle \Psi_s(s,t), \Psi_t(s,t) \rangle & = & \langle \tilde{\alpha}'(s), \tilde{\beta}'(t) \rangle & = & \langle T_\alpha, T_\beta \rangle. & &
\end{array}$$

By [25], we know also that

$$\tilde{N}(s,t) = \frac{T_\alpha \times T_\beta}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}}.$$

Thus

$$\begin{array}{rcccccl}
\tilde{e} & = & \langle \Psi_s s(s,t), N(s,t) \rangle & = & \langle \tilde{\alpha}'', N \rangle & = & 0, \\
\tilde{g} & = & \langle \Psi_t t(s,t), N(s,t) \rangle & = & \langle \tilde{\beta}'', N \rangle & = & 0, \\
\tilde{f} & = & \langle \Psi_s t(s,t), N(s,t) \rangle & = & \langle 0, N \rangle & = & 0.
\end{array}$$

By theorem 3.1, we have $\tilde{E} = E = \tilde{G} = G = 1$, $\tilde{F} = F$ and $\tilde{e} = e = \tilde{g} = g = 0$. Consequently

$$\tilde{K} = -\frac{\tilde{e}\tilde{g} - \tilde{f}}{\tilde{E}\tilde{G} - \tilde{F}^2} = 0 = K.$$

where $\tilde{K}$ and K are the Gaussian curvatures in $\mathbb{R}^3$ and $\mathbb{S}^3$ respectively. Moreover, we have

$$\tilde{H} = \frac{\tilde{e}\tilde{G} - 2\tilde{f}\tilde{F} + \tilde{E}\tilde{g}}{2(\tilde{E}\tilde{G} - \tilde{F}^2)} = H + \frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}} = 0.$$

where $\tilde{H}$ and H are the mean curvatures in $\mathbb{R}^3$ and $\mathbb{S}^3$ respectively.

Suppose now that $\kappa_\alpha \equiv 0$ and $\kappa_\beta \not\equiv 0$ (or symmetrically $\kappa \equiv 0$, $\kappa_\alpha \not\equiv 0$). With the vectors T_β, N_β, B_β and equations of Frenet-frame kind

$$\begin{aligned} T'_\beta &= \kappa_\beta N_\beta, \\ N'_\beta &= -\kappa_\beta T_\beta + (\tau_\beta + 1) B_\beta, \\ B'_\alpha &= -(\tau_\beta + 1) N_\beta. \end{aligned}$$

We can associate a unique arc length curve $\tilde{\beta}$ in $\mathbb{R}^3$, up to rigid motion, whose curvature and torsion satisfy $\tilde{\kappa}_\beta = \kappa_\beta$ and $\tilde{\tau}_\beta = (\tau_\beta + 1)$, respectively, and whose Frenet frame is given by $\{T_\beta, N_\beta, B_\beta\}$. As before, we associate to α a curve $\tilde{\alpha}$ in $\mathbb{R}^3$ that is a straight line parallel to T_α. Consider now a translation surface in $\mathbb{R}^3$ defined by $\Psi(s,t) = \tilde{\alpha}(s) + \tilde{\beta}(t)$. We have that

$$\tilde{E} = \langle T_\alpha, T_\alpha \rangle = 1, \;\; \tilde{G} = \langle T_\beta, T_\beta \rangle = 1, \;\; \tilde{F} = \langle T_\alpha, T_\beta \rangle.$$

Again $\tilde{N}(s,t) = T_\alpha \times T_\beta / \sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}$. Thus

$$\tilde{e} = \langle \tilde{\alpha}'', N \rangle = 0, \;\; \tilde{g} = \langle \tilde{\beta}'', N \rangle = -\frac{\kappa_\beta \langle T_\alpha, B_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}}, \;\; \tilde{f} = \langle 0, N \rangle = 0.$$

Once again, by theorem (3.1), we conclude that $\tilde{E} = E = \tilde{G} = G = 1$, $\tilde{F} = F$ and $\tilde{e} = e = 0$, $\tilde{g} = g$. Hence, $\tilde{K} = 0 = K$ and also

$$\tilde{H} = \frac{-\kappa_\beta \langle T_\alpha, B_\alpha \rangle}{2(1 - \langle T_\alpha, T_\beta \rangle^2)^{3/2}} - \frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}} + \frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}} = H + \frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}}.$$

From now on, suppose that $\kappa_\alpha \not\equiv 0$ and $\kappa_\beta \not\equiv 0$. Consider now a translation surface in $\mathbb{R}^3$ defined by $\Psi(s,t) = \tilde{\alpha}(s) + \tilde{\beta}(t)$. We have that

$$\tilde{E} = \langle T_\alpha, T_\alpha \rangle = 1, \;\; \tilde{G} = \langle T_\beta, T_\beta \rangle = 1, \;\; \tilde{F} = \langle T_\alpha, T_\beta \rangle.$$

One more time, by [25], we know that $\tilde{N}(s,t) = (T_\alpha \times T_\beta)/\sqrt{1-\langle T_\alpha, T_\beta\rangle^2}$. Thus

$$\tilde{e} = \langle \tilde{\alpha}'', N\rangle = \frac{\kappa_\alpha \langle B_\alpha, T_\alpha\rangle}{\sqrt{1-\langle T_\alpha, T_\beta\rangle^2}}, \quad \tilde{g} = \langle \tilde{\beta}'', N\rangle = -\frac{\kappa_\beta \langle T_\alpha, B_\beta\rangle}{\sqrt{1-\langle T_\alpha, T_\beta\rangle^2}}, \quad \tilde{f} = \langle 0, N\rangle = 0.$$

Thus $\tilde{E} = E = \tilde{G} = G = 1$, $\tilde{F} = F$ and $\tilde{e} = e$ and $\tilde{g} = g$. Hence

$$\tilde{K} = -\frac{\kappa_\alpha \kappa_\beta \langle B_\alpha, T_\alpha\rangle \langle T_\alpha, B_\beta\rangle}{1-\langle T_\alpha, T_\beta\rangle^2} = K,$$

and

$$\tilde{H} = \frac{\kappa_\alpha \langle B_\alpha, T_\alpha\rangle - \kappa_\beta \langle T_\alpha, B_\alpha\rangle}{2(1-\langle T_\alpha, T_\beta\rangle^2)^{3/2}} = H + \frac{\langle T_\alpha, T_\beta\rangle}{\sqrt{1-\langle T_\alpha, T_\beta\rangle^2}}.$$

Now, consider a curve $\tilde{\alpha}$ in $\mathbb{R}^3$. If $\tilde{\alpha}$ is a straight line, we correspond it with a geodesic circle in $\mathbb{S}^3$. Otherwise, let $\{T_{\tilde{\alpha}}, N_{\tilde{\alpha}}, B_{\tilde{\alpha}}\}$ denote its Frenet frame, $\tilde{\kappa}_\alpha$ and $\tilde{\tau}_\alpha$ its curvature and torsion, respectively. Let α be the unique curve in $\mathbb{S}^3$, up to rigid motion, with curvature $\kappa_\alpha = \tilde{\kappa}_\alpha$ and torsion $\tau_\alpha = \tilde{\tau}_\alpha + 1$. We know that $\{T_\alpha, N_\alpha, B_\alpha\}$ satisfy (3.3), which correspond to the Frenet formulas for $\tilde{\alpha}$ in $\mathbb{R}^3$. Now, using Theorem (3.1) and [25], we conclude the result. □

As applications of this theorem we have the following

Corollary 3.2. *Let $\tilde{M} \subset \mathbb{R}^3$ and $M \subset \mathbb{S}^3$ be translation surfaces in the conditions of Theorem 3.6. If they both are CMC, then they are also flat, M is a CMC flat torus and $\tilde{M}$ is a plane or a cylinder.*

and

Example 3.2. *Let $M \subset \mathbb{S}^3$ be a translation surface such that $\langle T_\alpha, T_\alpha\rangle = C$. Then, by Theorem 3.5, this surface is a flat CMC torus, and thus the associated surface $\tilde{M} \subset \mathbb{R}^3$ is also flat and CMC. Therefore, $\tilde{M}$ must be a plane or a cylinder. Moreover, by Proposition 3.4, the condition $\langle T_\alpha, T_\beta\rangle = C$ implies that either $\kappa_\alpha \equiv \kappa_\beta \equiv 0$, or $\kappa_\beta \equiv 0$ and α is a general helix. In the case $\kappa_\alpha \equiv \kappa_\beta \equiv 0$, $\tilde{M}$ is a plane. If α is a helix, then by the proof of Proposition 3.4, $\langle B_\alpha, T_\beta\rangle$ is constant. Thus, by theorem (3.1), if M is CMC, then κ_α is constant, which implies that τ_α is constant as well. Hence, $\tilde{\alpha}$ is a helix in $\mathbb{R}^3$.*

The next ones are motivated by the works [36, 25], in particular, they approach the case of circular helices in $\mathbb{R}^3$.

Corollary 3.3. *Let $\tilde{M} \subset \mathbb{R}^3$ and $M \subset \mathbb{S}^3$ be translation surfaces in the conditions of Theorem 3.6. If $\kappa_\alpha \neq 0$ and $|\tau_\alpha| \neq 1$ are constant then $\alpha(s) \cdot \alpha(t)$ is not minimal in $\mathbb{S}^3$.*

Proof. Following [25, Theorem 3.2], let α in $\mathbb{R}^3$ be an arc length curve with constant curvature and torsion. Then a translation surface $\tilde{M} \subset \mathbb{R}^3$, locally parameterized as $\tilde{X}(s,t) = \alpha(s) + \alpha(t)$, is minimal if and only if it is a helicoid. By Theorem 3.6, we conclude that $M \subset \mathbb{S}^3$ cannot be minimal. □

Remark 3.4. For corollary 3.3, it is important to keep the regularity condition, that is, $\langle T_\alpha, T_\beta \rangle \neq 1$.

About curves with constant curvature and torsion we have

Corollary 3.4. *If $\tilde{M} \subset \mathbb{R}^3$ is minimal and one of the curves is a circular helix. Then the surface $M \subset \mathbb{S}^3$ is neither CMC nor flat and its mean curvature is given by*

$$H = -\frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}}.$$

Proof. By [25, Theorem 3.2], if $\tilde{M} \subset \mathbb{R}^3$ is a minimal translation surface and one of the generating curves has constant curvature and torsion, then the other curve is a congruent curve with the same curvature and torsion, and $\tilde{M}$ is the helicoid. By Theorem 3.6, the surface $M \subset \mathbb{S}^3$ is neither CMC nor is flat, and its mean curvature is given by

$$H = -\frac{\langle T_\alpha, T_\beta \rangle}{\sqrt{1 - \langle T_\alpha, T_\beta \rangle^2}}.$$

□

3.5 On minimal translation surfaces

The first theorem of this section can be viewed as a generalization of Corollary 3.3.

Theorem 3.7. *There are no minimal translation surfaces $X : I \times J \to \mathbb{S}^3$, $X(s,t) = \alpha(s) \cdot \beta(t)$ with $\kappa_\alpha \equiv C \in \mathbb{R}$, $C \neq 0$ and $\tau_\alpha = 1$.*

Proof. Differentiating equation (3.12) with respect to s gives

$$\kappa_\alpha' \langle B_\alpha, T_\beta \rangle - \kappa_\alpha(\tau_\alpha - 1)\langle N_\alpha, T_\beta \rangle - \kappa_\beta \kappa_\alpha \langle N_\alpha, B_\beta \rangle = 2\kappa_\alpha \langle N_\alpha, T_\beta \rangle [3\langle T_\alpha, T_\beta \rangle^2 - 1]. \quad (3.15)$$

If $\tau_\alpha = 1$ and $\kappa_\alpha \equiv C \neq 0$, $C \in \mathbb{R}$, then equation (3.15) becomes

$$-\kappa_\beta \langle N_\alpha, B_\beta \rangle = 2\langle N_\alpha, T_\beta \rangle [3\langle T_\alpha, T_\beta \rangle^2 - 1].$$

Differentiating again with respect to s gives

$$\kappa_\beta \kappa_\alpha \langle T_\alpha, B_\beta \rangle = -2\kappa_\alpha \langle T_\alpha, T_\beta \rangle [3\langle T_\alpha, T_\beta \rangle^2 - 6\langle N_\alpha, T_\beta \rangle^2 - 1]. \tag{3.16}$$

We rewriting equation (3.12) as

$$\kappa_\beta \langle T_\alpha, B_\beta \rangle = -2\langle T_\alpha, T_\beta \rangle [\langle T_\alpha, T_\beta \rangle^2 - 1] + \kappa_\alpha \langle B_\alpha, T_\beta \rangle.$$

Replacing the previous equation in equation (3.15) gives

$$\kappa_\alpha \langle B_\alpha, T_\beta \rangle = -2\langle T_\alpha, T_\beta \rangle [2\langle T_\alpha, T_\beta \rangle^2 - 6\langle N_\alpha, T_\beta \rangle^2].$$

We differentiate once again with respect to s and simplify to obtain

$$12\kappa_\alpha \langle N_\alpha, T_\beta \rangle^3 + 36\kappa_\alpha \langle T_\alpha, T_\beta \rangle^2 \langle N_\alpha, T_\beta \rangle = 0.$$

Suppose that $\langle N_\alpha, T_\beta \rangle \neq 0$. Then we have $\langle N_\alpha, T_\beta \rangle^2 = -3\langle T_\alpha, T_\beta \rangle^2$, and differentiating with respect to s gives

$$-2\kappa_\alpha \langle T_\alpha, T_\beta \rangle \langle N_\alpha, T_\beta \rangle = -6\kappa_\alpha \langle N_\alpha, T_\beta \rangle \langle T_\alpha, T_\beta \rangle,$$

which implies that $\langle T_\alpha, T_\beta \rangle = \langle N_\alpha, T_\beta \rangle = 0$, a contradiction.

Suppose now that $\langle N_\alpha, T_\beta \rangle = 0$. Then differentiating with respect to s gives $\kappa_\alpha \langle T_\alpha, T_\beta \rangle = 0$. As $\kappa_\alpha \neq 0$ by hypothesis, then $\langle T_\alpha, T_\beta \rangle = 0$. Moreover, as $T_\alpha, T_\beta, N_\alpha, N_\beta, B_\alpha, B_\beta \in \mathcal{S}$, and we have here that $T_\beta \perp T_\alpha$ and $T_\beta \perp N_\alpha$, then $T_\beta = \pm B_\alpha$ that is a constant vector. This implies that $\kappa_\beta = 0$.

Now returning to equation (3.7) we obtain

$$\kappa_\alpha \langle B_\alpha, T_\beta \rangle = 0.$$

Thus $\langle B_\alpha, T_\beta \rangle = 0$, a contradiction.

□

When τ_α is any constant, not necessarily equal to 1, we impose conditions on the curve β to obtain the following result:

Theorem 3.8. *The are no minimal translation surface $X : I \times J \to \mathbb{S}^3$, $X(s,t) = \alpha(s) \cdot \beta(t)$, where $\kappa_\alpha \neq 0$, $\kappa_\beta \neq 0$, τ_α and τ_β are constant.*

Proof. Since the case where $\tau_\alpha = 1$ and $\tau_\beta = -1$ was treated in Theorem 3.7, we assume from now on that $\tau_\alpha \neq 1$ and $\tau_\beta \neq -1$. Let $\alpha(s)$ and $\beta(t)$ be arc-length curves that are also proper helices in $\mathbb{S}^3$. By lemma 3.1, the curves $T_\alpha(s)$, $N_\alpha(s)$, $B_\alpha(s)$, $T_\beta(t)$, $N_\beta(t)$ and $B_\beta(t)$ all trace circles in $\mathcal{S}$. Moreover, assuming without loss of generality that $(1-\tau_\alpha) > 0$ and $-(1+\tau_\beta) > 0$, the curves $\tilde{\alpha}(s) = B_\alpha(s)$ and $\tilde{\beta}(t) = B_\beta(t)$ have curvatures $\tilde{\kappa}_\alpha = \kappa_\alpha/(1-\tau_\alpha)$ and $\tilde{\kappa}_\beta = -\kappa_\beta/(1+\tau_\beta)$, respectively.

Now, since $T_\alpha, N_\alpha, B_\alpha, T_\beta, N_\beta, B_\beta \in \mathcal{S}$, to compute $\langle T_\alpha, T_\beta \rangle$, $\langle B_\alpha, T_\beta \rangle$ and $\langle T_\alpha, B_\beta \rangle$, we reduce the problem to one in $\mathbb{S}^2 \subset \mathbb{R}^3$. This reduction is valid because the values of these inner products depend only on the angles between the vectors involved, not on their specific positions (see Figure 3.2).

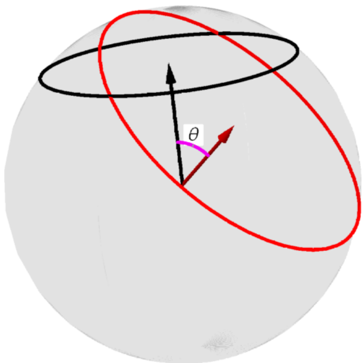

Fig. 3.2 Relative position of the circles in Theorem 3.8.

Thus, let $x(s) = (1-\tau_\alpha)s$ and $y(t) = -(1+\tau_\beta)t$ be arc lengths parameters for $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. We can then choose convenient coordinates and parameterize these curves as follows

$$\begin{aligned}
\tilde{\alpha}(x) = B_\alpha(x) &= \left(Q\cos\left(\frac{1}{Q}x\right), Q\sin\left(\frac{1}{Q}x\right), P \right), \\
\tilde{\beta}(y) = B_\beta(y) &= \left(S\cos\left(\frac{1}{S}y\right), S\cos\theta\sin\left(\frac{1}{S}y\right) - R\sin\theta, S\sin\theta\sin\left(\frac{1}{S}y\right) + R\cos\theta \right),
\end{aligned}$$

where $0 < \theta < \pi$ is a constant angle and $P, Q, R, S > 0$, $P^2 + Q^2 = S^2 + R^2 = 1$.

Now, in order to better describe the curvatures of the generating curves, we differentiate B_α with respect to x to obtain

$$B'_\alpha = N_\alpha = \left(-\sin\left(\frac{1}{Q}x\right), \cos\left(\frac{1}{Q}x\right), 0\right),$$

A second differentiation with respect to x gives

$$N'_\alpha = -\frac{1}{Q}\left(\cos\left(\frac{1}{Q}x\right), \sin\left(\frac{1}{Q}x\right), 0\right).$$

Since $N'_\alpha = \kappa_{\tilde{\alpha}} n_{\tilde{\alpha}} - \tilde{\alpha}$, we have $\langle N'_\alpha, N'_\alpha \rangle = \kappa^2_{\tilde{\alpha}} + 1 = 1/Q^2$. Thus

$$\kappa^2_{\tilde{\alpha}} = \frac{1-Q^2}{Q^2} = \frac{P^2}{Q^2}.$$

Now, for the curve β, we differentiate $B_\beta(y)$ with respect to y to obtain

$$B'_\beta = N_\beta = \left(-\sin\left(\frac{1}{S}y\right), \cos\theta\cos\left(\frac{1}{S}y\right), \sin\theta\cos\left(\frac{1}{S}y\right)\right).$$

A second differentiation with respect to y gives

$$N'_\beta = -\frac{1}{S}\left(\cos\left(\frac{1}{S}y\right), \cos\theta\sin\left(\frac{1}{S}y\right), \sin\theta\sin\left(\frac{1}{S}y\right)\right).$$

Since $N'_\beta = \kappa_{\tilde{\beta}} n_{\tilde{\beta}} - \tilde{\beta}$, we have $\langle N'_\alpha, N'_\alpha \rangle = \kappa^2_{\tilde{\alpha}} + 1 = 1/S^2$. Thus

$$\kappa^2_{\tilde{\beta}} = \frac{1-S^2}{S^2} = \frac{R^2}{Q^2}.$$

It follows that

$$\frac{\kappa_\alpha}{1-\tau_\alpha} = \kappa_{\tilde{\alpha}} = \frac{P}{Q}, \qquad \frac{\kappa_\beta}{1+\tau_\beta} = \kappa_{\tilde{\beta}} = \frac{R}{S}.$$

Now, since $\frac{d}{ds}N_\alpha = -\kappa_\alpha T_\alpha + (\tau_\alpha - 1)B_\alpha$, then

$$T_\alpha = \frac{1}{\kappa_\alpha}\left[-\frac{d}{ds}N_\alpha + (\tau_\alpha - 1)B_\alpha\right] = \frac{1}{\kappa_\alpha}\left[-(1-\tau_\alpha)N'_\alpha + (\tau_\alpha - 1)B_\alpha\right] =$$
$$= \frac{1-\tau_\alpha}{\kappa_\alpha}\left(\frac{1-Q^2}{Q}\cos\left(\frac{1}{Q}x\right), \frac{1-Q^2}{Q}\sin\left(\frac{1}{Q}x\right), -P\right) =$$
$$= \frac{Q}{P}\left(\frac{P^2}{Q}\cos\left(\frac{1}{Q}x\right), \frac{P^2}{Q}\sin\left(\frac{1}{Q}x\right), -P\right) = \left(P\cos\left(\frac{1}{Q}x\right), P\sin\left(\frac{1}{Q}x\right), -Q\right).$$

Analogously, we have

$$T_\beta = \frac{1}{\kappa_\beta}\left[-\frac{d}{dt}N_\beta + (\tau_\beta + 1)B_\beta\right] = \frac{1}{\kappa_\beta}\left[-(\tau_\beta+1)N'_\beta + (\tau_\beta + 1)B_\beta\right] =$$
$$= \frac{S}{R}\left(\frac{R^2}{S}\cos\left(\frac{1}{S}y\right), \frac{R^2}{S}\cos\theta\sin\left(\frac{1}{S}y\right) + R\sin\theta, \frac{R^2}{S}\sin\theta\sin\left(\frac{1}{S}y\right) - R\cos\theta\right)$$
$$= \left(R\cos\left(\frac{1}{S}y\right), R\cos\theta\sin\left(\frac{1}{S}y\right) + S\sin\theta, R\sin\theta\sin\left(\frac{1}{S}y\right) - S\cos\theta\right).$$

Since κ_α, κ_β, τ_α and τ_β are constant, we differentiate (3.15) with respect to t to obtain

$$-\kappa_\alpha\kappa_\beta(\tau_\alpha - 1)\langle N_\alpha, N_\beta\rangle + \kappa_\alpha\kappa_\beta(\tau_\beta + 1)\langle N_\alpha, N_\beta\rangle = 2\kappa_\alpha\kappa_\beta\langle N_\alpha, N_\beta\rangle[3\langle T_\alpha, T_\beta\rangle^2 - 1] +$$
$$12\kappa_\alpha\kappa_\beta\langle N_\alpha, T_\beta\rangle\langle T_\alpha, T_\beta\rangle\langle T_\alpha, N_\beta\rangle.$$

As $\kappa_\alpha \neq 0$ and $\kappa_\beta \neq 0$, we have

$$[(\tau_\beta + 1) - (\tau_\alpha - 1)]\langle N_\alpha, N_\beta\rangle =$$
$$= 2\langle N_\alpha, N_\beta\rangle[3\langle T_\alpha, T_\beta\rangle^2 - 1] + 12\langle N_\alpha, T_\beta\rangle\langle T_\alpha, T_\beta\rangle\langle T_\alpha, N_\beta\rangle. \quad (3.17)$$

Evaluating T_α, N_α, B_α, T_β, N_β and B_β in $(x/Q, y/S) = (0, \pi/2)$ gives

$$\begin{aligned} T_\alpha &= (P, 0, -Q), & T_\beta &= (0, R\cos\theta + S\sin\theta, R\sin\theta - S\cos\theta), \\ N_\alpha &= (0, 1, 0), & N_\beta &= (-1, 0, 0), \\ B_\alpha &= (Q, 0, P), & B_\beta &= (0, S\cos\theta - R\sin\theta, S\sin\theta + R\cos\theta). \end{aligned}$$

Hence,

$$\begin{aligned} \langle T_\alpha, T_\beta\rangle &= Q(S\cos\theta - R\sin\theta), & \langle N_\alpha, N_\beta\rangle &= 0, \\ \langle N_\alpha, T_\beta\rangle &= R\cos\theta + S\sin\theta, & \langle T_\alpha, N_\beta\rangle &= -P. \end{aligned}$$

Thus, equation (3.17) becomes

$$12PQ(S\cos\theta - R\sin\theta)(R\cos\theta + S\sin\theta) = 0. \tag{3.18}$$

If $S\cos\theta = R\sin\theta$ then $\cos\theta = (R/S)\sin\theta$, with $\theta \neq \pi$. Hence

$$R\cos\theta + S\sin\theta = \left(\frac{R^2}{S} + S\right)\sin\theta = \frac{1}{S}\sin\theta \neq 0.$$

Thus, $(S\cos\theta - R\sin\theta) = 0$ and $(R\cos\theta + S\sin\theta) \neq 0$. We have

$$\begin{array}{rclrcl} T_\alpha & = & (P,0,-Q), & T_\beta & = & (0,\frac{1}{S}\sin\theta,0), \\ N_\alpha & = & (0,1,0), & N_\beta & = & (-1,0,0), \\ B_\alpha & = & (Q,0,P), & B_\beta & = & (0,0,\frac{1}{S}\sin\theta). \end{array}$$

It follows that $\langle T_\alpha, T_\beta\rangle = \langle B_\alpha, T_\beta\rangle = 0$ and $\langle T_\alpha, B_\beta\rangle = -(Q/S)\sin\theta$. Evaluating in equation (3.12) gives

$$\kappa_\alpha \frac{Q}{S}\sin\theta = 0,$$

a contradiction.

Suppose now that $R\cos\theta = -S\sin\theta$, then $\cos\theta = -(S/R)\sin\theta$, with $\theta \neq \pi$. Hence

$$S\cos\theta - R\sin\theta = \left(-\frac{S^2}{R} - R\right)\sin\theta = -\frac{1}{R}\sin\theta \neq 0.$$

Thus, $(S\cos\theta - R\sin\theta) \neq 0$ and $(R\cos\theta + S\sin\theta) = 0$. We have

$$\begin{array}{rclrcl} T_\alpha & = & (P,0,-Q), & T_\beta & = & (0,0,\frac{1}{R}\cos\theta), \\ N_\alpha & = & (0,1,0), & N_\beta & = & (-1,0,0), \\ B_\alpha & = & (Q,0,P), & B_\beta & = & (0,-\frac{1}{R}\sin\theta,0). \end{array}$$

It follows that $\langle N_\alpha, T_\beta\rangle = 0$, $\langle N_\alpha, B_\beta\rangle = -\frac{1}{R}\sin\theta$ and $\langle T_\alpha, T_\beta\rangle = -(Q/R)\sin\theta$. Evaluating in equation (3.15) gives

$$\frac{1}{R}\kappa_\alpha\kappa_\beta\sin\theta = 0,$$

a contradiction. Thus, we must have $(S\cos\theta - R\sin\theta) = (R\cos\theta + S\sin\theta) = 0$, a contradiction with equation (3.18).

□

Chapter 4

Homogeneous surfaces in Homogeneous tri-spaces

In this chapter, we present a classification of homogeneous surfaces in homogeneous 3-spaces. Such classification is based on whether the ambient space is a unimodular or non-unimodular Lie group. We use a precise correspondence between the Lie algebras of the isometry groups of these spaces and their respective Lie algebras to classify the 2-dimensional subalgebras of such Lie algebras up to conjugacy. This classification allows us to identify the connected subgroups that serve as homogeneous orbits passing through the identity element of the group. With this in hand, we construct a foliation by determining the geodesics that intersect these subgroups orthogonally at the identity. Finally, these geodesics enable us to study the geometry of the equidistant surfaces through the computation of the shape operator at a given distance, thereby completing the classification of the homogeneous surfaces in homogeneous 3-spaces. For further information on homogeneous manifolds, see [39, 41, 43].

The results of this chapter compose part of a joint work with Miguel Dominguez-Vazquez and Tomas Otero [16], entitled "Polar actions on homogeneous 3-spaces".

4.1 Riemannian homogeneous 3-manifolds and metric Lie groups

A Riemannian metric on a Lie group G is said to be left-invariant if the left multiplication by g, L_g, is an isometry of G for all $g \in G$ (Definition 1.13). Therefore, a left-invariant metric on G is determined by the choice of an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$. We will denote by $\langle \cdot, \cdot \rangle$ both the inner product on $\mathfrak{g}$ and the associated Riemannian metric on G. Lie groups equipped with a left-invariant metric are often called metric Lie groups, and provide perhaps the simplest examples of Riemannian homogeneous spaces.

In this special case, that is of dimension 3, it turns out that metric Lie groups are close to exhausting all examples of Riemannian homogeneous manifolds: any simply connected homogeneous 3-manifold different from $\mathbb{S}^2 \times \mathbb{R}$ is isometric to a 3-dimensional metric Lie group (see Theorem 2.4 of [41]). It should be noted, however, that non-isomorphic Lie groups can give rise to isometric Riemannian homogeneous spaces. We also note that, among simply connected homogeneous 3-manifolds, the subclass of Riemannian symmetric spaces is constituted by the space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ and the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, where the latter correspond to $\mathbb{E}(\kappa, \tau)$-spaces with $\tau = 0$ and $\kappa \neq 0$.

We will summarize below Milnor's description of 3-dimensional Lie algebras [43]. For the sake of convenience, one distinguishes two cases depending on the unimodularity of the group:

Definition 4.1. *Let G be a Lie group. G is called unimodular if its left and right invariant Haar measures coincide Otherwise, it is called non-unimodular.*

Being unimodular is equivalent to asking that the adjoint transformation

$$\begin{array}{rcl} \mathrm{ad}_X : \mathfrak{g} & \to & \mathfrak{g} \\ Y & \mapsto & [X,Y] \end{array},$$

has trace 0 for every $X \in \mathfrak{g}$. Thus, a Lie algebra $\mathfrak{g}$ that satisfies $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{g}$ is called a unimodular Lie algebra.

4.1.1 Unimodular Lie groups of dimension 3

Let G be a connected 3-dimensional Lie group with left invariant metric $\langle \cdot, \cdot \rangle$. Choosing an orientation for the Lie algebra $\mathfrak{g}$ of G, we can define a cross product $\times$ and present the following result

Proposition 4.1 ([43]). *The bracket product operation in this Lie algebra $\mathfrak{g}$ is related to the cross product operation by the formula*

$$L(X \times Y) = [X,Y], \quad \text{for all } X,Y \in \mathfrak{g}.$$

where L is a uniquely defined linear mapping from $\mathfrak{g}$ to itself. The Lie group G is unimodular if and only if this linear transformation L is self adjoint.

Therefore, there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ such that

$$[E_1, E_2] = \lambda_3 E_3, \qquad [E_2, E_3] = \lambda_1 E_1, \qquad [E_3, E_1] = \lambda_2 E_2, \tag{4.1}$$

for some constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. These constants determine the geometry and group structure of the unique simply connected metric Lie group G with Lie algebra $\mathfrak{g}$ and left invariant metric $\langle \cdot, \cdot \rangle$.

Note that changing the sign of all the λ_i corresponds to a change in the orientation in $\mathfrak{g}$, but the metric structure does not change. Also, if one multiplies all the λ_i by a positive number, it corresponds to rescaling the metric in G, but again, the underlying group structure remains the same. From now on, we will assume without loss of generality that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and at most one $\lambda_i < 0$. A list of the corresponding Lie groups is given in Table 4.1.

Lie group	SU_2	$\widetilde{\mathsf{E}}_2$	$\widetilde{\mathsf{SL}}_2(\mathbb{R})$	Sol_3	Nil_3	$\mathbb{R}^3$
Signs of $\lambda_1, \lambda_2, \lambda_3$	$+\,+\,+$	$+\,+\,0$	$+\,+\,-$	$+\,0\,-$	$+\,0\,0$	$0\,0\,0$

Table 4.1 Three-dimensional simply-connected unimodular Lie groups in terms of the structure constants.

We remember the Koszul formula, which derives from the Levi-Civita connection of a Riemannian metric $\langle \cdot, \cdot \rangle$

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}\left[X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\right]. \quad (4.2)$$

Thus, since $\{E_1, E_2, E_3\}$ is an orthonormal basis for a Riemannian metric, and with equation (4.1), we have

$$\begin{aligned}
\langle \nabla_{E_1} E_1, E_1 \rangle &= \frac{1}{2}\big[E_1\langle E_1,E_1\rangle + E_1\langle E_1,E_1\rangle - E_1\langle E_1,E_1\rangle - \\
&\quad -\langle [E_1,E_1],E_1\rangle - \langle [E_1,E_1],E_1\rangle - \langle [E_1,E_1],E_1\rangle\big] = 0, \\
\langle \nabla_{E_1} E_1, E_2 \rangle &= \frac{1}{2}\big[E_1\langle E_1,E_2\rangle + E_1\langle E_1,E_2\rangle - E_2\langle E_1,E_1\rangle - \\
&\quad -\langle [E_1,E_2],E_1\rangle - \langle [E_1,E_2],E_1\rangle - \langle [E_1,E_1],E_2\rangle\big] = -\langle \lambda_3 E_3, E_1\rangle = 0, \\
\langle \nabla_{E_1} E_1, E_3 \rangle &= \frac{1}{2}\big[E_1\langle E_1,E_3\rangle + E_1\langle E_1,E_3\rangle - E_3\langle E_1,E_1\rangle - \\
&\quad -\langle [E_1,E_3],E_1\rangle - \langle [E_1,E_3],E_1\rangle - \langle [E_1,E_1],E_3\rangle\big] = \langle \lambda_2 E_2, E_1\rangle = 0.
\end{aligned}$$

We proceed in the same way to compute the other possible inner products to obtain $\nabla_{E_i} E_j$, for $i, j = 1, 2, 3$. Now, let μ_1, μ_2 and μ_3 be real numbers given by

$$\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3), \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3), \quad \mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3).$$

Hence, we have

$$\begin{array}{llllllll} \nabla_{E_1}E_1 & = & 0 & , & \nabla_{E_1}E_2 = \mu_1 E_3 & , & \nabla_{E_1}E_3 = -\mu_1 E_2, \\ \nabla_{E_2}E_1 & = & -\mu_2 E_3 & , & \nabla_{E_2}E_2 = 0 & , & \nabla_{E_2}E_3 = \mu_2 E_1, \\ \nabla_{E_3}E_1 & = & \mu_3 E_2 & , & \nabla_{E_3}E_2 = -\mu_3 E_1 & , & \nabla_{E_3}E_3 = 0. \end{array} \tag{4.3}$$

Remark 4.1.

1. If $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 \neq \lambda_1$, then $(G, \langle\cdot,\cdot\rangle)$ is isometric to $\mathbb{E}\left(\lambda_1\lambda_3, \dfrac{\lambda_1}{2}\right)$, and $\mathrm{Isom}(G)$ has dimension 4.

2. If $\lambda_1 = \lambda_2 \neq \lambda_3 = 0$ then G is isomorphic to the universal cover $\widetilde{\mathrm{E}}_2$ of the Euclidean group of the plane (which in turn is isometric to the Euclidean space $\mathbb{R}^3$).

3. If $\lambda_1 = \lambda_2 = \lambda_3$, then G is isomorphic to $\mathbb{R}^3$ or SU_2, it has non-negative constant sectional curvature and 6-dimensional isometry group.

4. When the three structure constants are distinct, then $\mathrm{Isom}(G)$ has dimension 3.

A description of the full isometry group of each unimodular group G can be found in [23].

The following groups are the object of study of this work as they all have isometry group of dimension 3 and a full description of this groups and their left invariant metrics can be found in [41].

- The universal cover $\widetilde{\mathrm{E}}_2$ of the group E_2 of orientation-preserving rigid motions of the Euclidean plane is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- The projective special linear group is

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\},$$

where $\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \det A = 1\}$ is the special linear group. Both groups $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{R})$ have the same universal cover, which we denote by $\widetilde{\mathrm{SL}}_2(\mathbb{R})$.

- Sol_3 can be described as the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

4.1.2 Non-unimodular Lie groups of dimension 3

A way to see some 3-dimensional Lie groups is to consider semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where A is a 2×2 real matrix. In this case, a group structure is defined by

$$(p_1,z_1)\cdot(p_2,z_2) = (p_1 + e^{z_1 A}p_2, z_1+z_2), \quad (p_1,z_1),(p_2,z_2)\in\mathbb{R}^2\times\mathbb{R},$$

where the exponential matrix is defined by

$$e^{zA} = \sum_{k=0}^{\infty} \frac{z^k}{k!}A^k.$$

Let $\mathfrak{g}$ be a non-unimodular 3-dimensional Lie algebra of the group G. Then it can be defined as a semi-direct product $\mathfrak{g} = \mathbb{R}^2 \rtimes \mathbb{R}$ and, up to rescaling the metric, there exists an orthonormal basis $\{E_1,E_2,E_3\}$ of $\mathfrak{g}$ such that $\mathbb{R}^2 = \mathrm{span}\{E_1,E_2\}$, and ad_{E_3} is given by the matrix

$$A = \begin{bmatrix} (1+\alpha) & -(1-\alpha)\beta \\ (1+\alpha)\beta & (1-\alpha) \end{bmatrix},$$

in the basis $\{E_1,E_2\}$, for some constants $\alpha,\beta \geq 0$. That is, the Lie bracket of $\mathfrak{g}$ is given by

$$\begin{aligned} [E_1,E_2] &= 0, \\ [E_2,E_3] &= (1-\alpha)\beta E_1 + (\alpha-1)E_2, \\ [E_3,E_1] &= (1+\alpha)E_1 + (1+\alpha)\beta E_2. \end{aligned} \tag{4.4}$$

If $A \neq \mathrm{Id}$, then its determinant is given by $\det A = (1-\alpha^2)(1+\beta^2)$ and provides a complete isomorphism invariant for the Lie algebra $\mathfrak{g}$, that is, if $\det(A) = \det(B)$ and $A \neq \mathrm{Id} \neq B$, then $\mathbb{R}^2 \rtimes_A \mathbb{R} \cong \mathbb{R}^2 \rtimes_B \mathbb{R}$ as Lie algebras (although their metrics may differ).

Using the Koszul formula (4.2), we get

$$\begin{aligned} &\nabla_{E_1}E_1 = (1+\alpha)E_3, && \nabla_{E_1}E_2 = \alpha\beta E_3, && \nabla_{E_1}E_3 = -(1+\alpha)E_1 - \alpha\beta E_2, \\ &\nabla_{E_2}E_1 = \alpha\beta E_3, && \nabla_{E_2}E_2 = (1-\alpha)E_3, && \nabla_{E_2}E_3 = -\alpha\beta E_1 - (1-\alpha)E_2, \\ &\nabla_{E_3}E_1 = \beta E_2, && \nabla_{E_3}E_2 = -\beta E_1, && \nabla_{E_3}E_3 = 0. \end{aligned} \tag{4.5}$$

Remark 4.2. It is important to distinguish some cases:

1. If $\alpha = 0$, then G is isometric to $\mathbb{H}^3(-1)$.

2. If $\alpha = 1$, then G is isometric to $\mathbb{E}(-4,\beta)$, whose isometry group has dimension 4.

3. If $\alpha \notin \{0,1\}$ it is possible to show that $\dim(\mathrm{Isom}(G)) = 3$ by looking at the eigenvalues of the Ricci operator.

A description of the full isometry group of G can be found in [12].

4.2 Cohomogeneity one actions on 3-dimensional groups

The aim of this section is to classify cohomogeneity one actions on 3-dimensional simply connected metric Lie groups. Along with the known result for space forms and $\mathbb{E}(\kappa,\tau)$-spaces (see Remark 4.4 below), this completes the classification problem on all simply connected 3-dimensional homogeneous spaces.

Remark 4.3 (Cohomogeneity one actions on space forms). The classification of cohomogeneity one H-actions on simply connected 3-dimensional space forms M is summarized in Table 4.2. This result is well known and can be obtained directly or as a consequence of classical results on isoparametric hypersurfaces by Segre and Cartan in the 30s (see [11, p. 84] or [7, pp. 96-99]). Actually, isoparametric hypersurfaces in Euclidean and real hyperbolic spaces are homogeneous; this is no longer true for round spheres, but it still holds for $\mathbb{S}^3(\kappa)$.

In the case of $\mathbb{H}^3(\kappa)$ in Table 4.2, we use certain notation coming from the Iwasawa decomposition of the simple Lie group $G = \mathrm{SO}^0_{1,3}$, which is the connected component of the identity of $\mathrm{Isom}(\mathbb{H}^3(\kappa))$. This result states that a semisimple Lie group G is diffeomorphic to a product manifold $K \times A \times N$, see [11, pp. 340 and 343]. In this case, $K \cong \mathrm{SO}_3$ is the isotropy at some basepoint o, $A \cong R$ is a certain 1-dimensional subgroup of G, and N is a 2-dimensional abelian subgroup of G. The K-action on $\mathbb{H}^3(\kappa)$ fixes the basepoint o and the other orbits are geodesic spheres around it, the A-action gives rise to a geodesic through the basepoint o and equidistant curves to it, and the N-action produces a horosphere foliation whose common point at infinity is one of the limit points of the geodesic $A \cdot o$. The combination of the A-action with certain rotational 1-dimensional subgroup of K (precisely, the identity connected component of the centralizer of A in K), or with any 1-dimensional subgroup of N yields the second and third actions in Table 4.2. for $\mathbb{H}^3(\kappa)$, respectively.

In all cases in Table 4.2, all H-orbits are surfaces, except for at most two singular H-orbits. If there are singular orbits, the 2-dimensional orbits are tubes of different radii around each one of the singular orbits. By tube of radius r we mean the subset of the ambient space obtained by traveling a fixed distance r in all normal directions to a submanifold of codimension ≥ 2.

Remark 4.4 (Cohomogeneity one actions on $\mathbb{E}(\kappa,\tau)$-spaces). The classification of cohomogeneity one actions in $\mathbb{E}(\kappa,\tau)$-spaces, up to orbit equivalence, is summarized in Table 4.3 and it can be obtained from the classification of isoparametric surfaces in these spaces [17]. As

M	H	Orbits
$\mathbb{R}^3$	$\mathbb{R}^2$	Parallel affine planes
	$\mathsf{SO}_2 \times \mathbb{R}$	A straight line and coaxial cylinders around it
	SO_3	A fixed point and spheres around it
$\mathbb{S}^3(\kappa)$	SO_3	Two antipodal fixed points and tot. geodesic 2-spheres around them
	$\mathsf{SO}_2 \times \mathsf{SO}_2$	Two maximal circles and tori around them
$\mathbb{H}^3(\kappa)$	$K \cong \mathsf{SO}_3$	A fixed point and geodesic spheres around it
	$\mathsf{SO}_2 \times \mathbb{R}$	A geodesic and tubes around it
	$\mathbb{R} \ltimes \mathbb{R}$	Totally geodesic $\mathbb{H}^2(\kappa)$ and its equidistant surfaces
	$N \cong \mathbb{R}^2$	Horosphere foliation

Table 4.2 Cohomogeneity one actions on 3-dimensional space forms.

stated there, a complete hypersurface in one of these spaces is homogeneous if and only if it is isoparametric.

Recall that an $\mathbb{E}(\kappa,\tau)$-space is the total space of a fiber bundle $\pi\colon \mathbb{E}(\kappa,\tau) \to \mathbb{M}^2(\kappa)$ over a complete, simply-connected surface of constant curvature κ. The parameter τ codifies the curvature of the bundle. Thus $\mathbb{E}(\kappa,\tau)$ is a product $\mathbb{M}^2(\kappa) \times \mathbb{R}$ if and only if $\tau = 0$. Moreover, one requires $\kappa - 4\tau^2 \neq 0$, so that $\dim(\mathrm{Isom}(\mathbb{E}(\kappa,\tau))) = 4$. Actually, one of the Killing fields of $\mathbb{E}(\kappa,\tau)$ is tangent to the fibers of π, which then turns out to be a Killing (Riemannian) submersion.

According to [17], the only examples of homogeneous surfaces of $\mathbb{E}(\kappa,\tau)$, $\kappa - 4\tau^2 \neq 0$, are: vertical cylinders over a complete curve of constant curvature in $\mathbb{M}^2(\kappa)$, a horizontal slice $\mathbb{M}^2(\kappa) \times \{t_0\}$, $t_0 \in \mathbb{R}$, when $\tau = 0$, or a so-called parabolic helicoid when $\kappa < 0$. By vertical cylinder over a subset of the base $\mathbb{M}^2(\kappa)$ we understand the preimage of such a subset under π. Recall that a curve of constant curvature in $\mathbb{M}^2(\kappa)$ is a geodesic circle if $\kappa > 0$; a circle or a straight line if $\kappa = 0$; or a geodesic, an equidistant curve to a geodesic, a geodesic circle or a horocycle if $\kappa < 0$. Finally, we refer to [17] for the explicit parameterization of parabolic helicoids (see also [50, Example 4.4] for an alternative description in the case of $\mathbb{H}^2(\kappa) \times \mathbb{R}$).

Let G be a simply connected 3-dimensional Lie group with a left invariant metric $\langle\cdot,\cdot\rangle$ (Definition 1.13). Since the action of G on itself by left translations is a proper isometric free action, a subgroup H of G acts on G with orbits of dimension $d = \dim(H)$. For an arbitrary metric Lie group, one can recover its isometry group as

$$\mathrm{Isom}(G,\langle\cdot,\cdot\rangle) = G \cdot \mathrm{Isom}(G,\langle\cdot,\cdot\rangle)_e,$$

M	H	Orbits
$\mathbb{S}^2(\kappa)\times\mathbb{R}$	$\mathsf{SO}_2\times\mathbb{R}$	Two vertical lines and vertical cylinders around them
	SO_3	Parallel horizontal slices $\mathbb{S}^2(\kappa)\times\{t_0\}$
$\mathbb{H}^2(\kappa)\times\mathbb{R}$ $\widetilde{\mathsf{SL}_2}(\mathbb{R})$	$\mathsf{SO}_2\times\mathbb{R}$	A vertical line and vertical cylinders around it
	$\mathbb{R}\times\mathbb{R}$	Vertical cylinders over a geodesic and over its parallel curves
	$\mathbb{R}\times\mathbb{R}$	Vertical cylinders over a horocycle foliation
	$\mathsf{SL}_2(\mathbb{R})$	Parallel horizontal slices $\mathbb{H}^2(\kappa)\times\{t_0\}$ (only for $\mathbb{H}^2(\kappa)\times\mathbb{R}$)
	$\mathbb{R}\ltimes\mathbb{R}$	Parallel parabolic helicoids
Nil_3	$\mathsf{SO}_2\times\mathbb{R}$	A vertical geodesic and tubes around it
	$\mathbb{R}\times\mathbb{R}$	Vertical cylinders over a foliation by parallel lines
$\mathbb{S}^3_{\text{Berger}}$	$\mathsf{SO}_2\times\mathsf{SO}_2$	Two vertical circles and vertical tori around them

Table 4.3 Cohomogeneity one actions on $\mathbb{E}(\kappa,\tau)$-spaces of non-constant curvature

where G is identified with its left translations and $\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)_e$ denotes the group of isometries of G fixing the identity element.

Since cohomogeneity one actions on space forms and $\mathbb{E}(\kappa,\tau)$-spaces are well understood, (see Remark 4.4), from now on we will suppose that $\dim(\mathrm{Isom}(G))=3$. Thus, the isometry group $\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)_e$ can be identified with a discrete (and hence finite) subgroup of O_3 via the isotropy representation.

We now present the following Proposition.

Proposition 4.2. *Let G be a simply connected metric Lie group with* $\dim(\mathrm{Isom}(G))=3$*. Then, there is a one-to-one correspondence between isometric actions of connected subgroups on G up to orbit equivalence and subalgebras of* $\mathfrak{g}$ *up to conjugacy and isometric automorphisms.*

Proof. Let $\mathrm{Aut}(G)$ be the group of automorphisms of G. It follows from [23, Corollary 2.8] that $\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)_e=\mathrm{Aut}(G)\cap\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)$ is precisely the group of isometric automorphisms of G, and so, by [23, p. 192], we have

$$\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)=G\rtimes(\mathrm{Aut}(G)\cap\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)).$$

Thus, any effective isometric action of a connected Lie group on G is equivalent to the action by left translations of some connected Lie subgroup H of G. Moreover, two subgroups $H,\tilde{H}$ of G give rise to orbit equivalent actions if and only if $H=g\varphi(\tilde{H})g^{-1}$ for some $g\in G$ and $\varphi\in\mathrm{Aut}(G)\cap\mathrm{Isom}(G,\langle\cdot,\cdot\rangle)$. □

Remark 4.5. Let H be a subgroup of a 3-dimensional simply connected Lie group G. If G is solvable, then H is closed in G (see [8, p.670]). Otherwise, $G=\widetilde{\mathsf{SL}_2}(\mathbb{R})$ or $G=\mathsf{SU}_2$, and every abelian subgroup of G is one-dimensional and closed. It follows from [37, Theorem 15] that any subgroup H of G is closed. Therefore, the action of H on G by left translations is proper, which implies that its orbits are closed embedded submanifolds, see for example [42, pp. 66-67].

In the following, we will classify codimension one subalgebras of $\mathfrak{g}$ up to conjugacy and isometric automorphisms. For this, we will make use of the structure results of Section 4.1 in order to give an explicit description of the corresponding Lie algebras $\mathfrak{g}$. It should be noted that there exist classifications of 2-dimensional subgroups of 3-dimensional Lie groups in the literature (see [41, Theorem 3.6]). However, our explicit description in terms of an orthonormal basis of $\mathfrak{g}$ will allow us to easily compute the geometry of the orbits and determine whether the actions of two subgroups are orbit equivalent.

Recall that if $\mathfrak{g}$ is a unimodular 3-dimensional Lie algebra, there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ such that

$$[E_1, E_2] = \lambda_3 E_3, \qquad [E_2, E_3] = \lambda_1 E_1, \qquad [E_3, E_1] = \lambda_2 E_2,$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$, and the corresponding Lie group structure can be recovered from the signs of the λ_i.

Let $\mathfrak{h}$ be a two dimensional subalgebra of $\mathfrak{g}$, and write $\mathfrak{h} = \operatorname{span}\{A, B\}$ for some $A, B \in \mathfrak{g}$ where $A = a_1 E_1 + a_2 E_2 + a_3 E_3$ and $B = b_1 E_1 + b_2 E_2 + b_3 E_3$. Then, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ if, and only if

$$\langle [A, B], A \times B \rangle = (a_2 b_3 - a_3 b_2)^2 \lambda_1 + (a_3 b_1 - a_1 b_3)^2 \lambda_2 + (a_1 b_2 - a_2 b_1)^2 \lambda_3 = 0. \tag{4.6}$$

Now, if $G = \mathbb{R}^3$ or $G = \mathsf{Nil}_3$, then any left invariant metric on G has an isometry group of dimension ≥ 4. Also, the only connected subgroups of SU_2 are the trivial subgroup or SO_2, which is one-dimensional. Thus, we will exclude these cases from our study, and restrict to the Lie groups $\tilde{\mathrm{E}}_2$, $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ and Sol_3 . Thus, we present the following

Theorem 4.1. *Let G be a 3-dimensional simply-connected unimodular Lie group with 3-dimensional isometry group and Lie algebra $\mathfrak{g}$. Then $\mathfrak{h}$ is a two dimensional subalgebra of $\mathfrak{g}$ if and only if one of the following cases happens*

1. *If $\mathfrak{g} = \mathfrak{e}_2$ then $\mathfrak{h} = \operatorname{span}\{E_1, E_2\}$.*
2. *If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ then $\mathfrak{h} = \operatorname{span}\{\sqrt{\lambda_1} E_1 + \sqrt{-\lambda_3} E_3, E_2\}$.*
3. *If $\mathfrak{g} = \mathfrak{sol}_3$ then $\mathfrak{h} = \operatorname{span}\{E_1, E_3\}$ or $\mathfrak{h} = \operatorname{span}\{\sqrt{\lambda_1} E_1 + \sqrt{-\lambda_3} E_3, E_2\}$.*

Proof. We will prove this Theorem by approaching the cases separately

1. For $\mathfrak{g} = \mathfrak{e}_2$, we have that $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 = 0$, so $\mathfrak{e}_2 = \operatorname{span}\{E_1, E_2\} \rtimes \mathbb{R}E_3$. Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{e}_2$, if and only if,

$$(a_2b_3 - a_3b_2)^2\lambda_1 + (a_3b_1 - a_1b_3)^2\lambda_2 = 0.$$

Since both $\lambda_1 > \lambda_2 > 0$, we have that $\mathfrak{h}$ is a subalgebra if, and only if,

$$\begin{cases} a_2b_3 - a_3b_2 = 0, \\ a_3b_1 - a_1b_3 = 0. \end{cases}$$

Thus, $\mathfrak{h} = \operatorname{span}\{E_1, E_2\}$ is the only codimension one subalgebra of $\mathfrak{e}_2$.

2. The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is a real simple Lie algebra. Since any 2-dimensional Lie algebra is solvable, it follows that any 2-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ is maximal solvable and thus, by definition, a parabolic subalgebra of $\mathfrak{sl}_2(\mathbb{R})$, see [11, p. 340]. But since $\mathfrak{sl}_2(\mathbb{R})$ has real rank equal to one, it has only one proper parabolic subalgebra, up to conjugacy [11, p. 348]. Choosing $A = \sqrt{\lambda_1}\,E_1 + \sqrt{-\lambda_3}\,E_3$ and $B = E_2$, with $\lambda_1 > 0 > \lambda_3$, one immediately verifies (4.6):

$$\langle [A,B], A \wedge B \rangle = \left(-\sqrt{-\lambda_3}\right)^2\lambda_1 + \left(\sqrt{\lambda_1}\right)^2\lambda_3 = -\lambda_1\lambda_3 + \lambda_1\lambda_3 = 0.$$

Hence $\mathfrak{h} = \operatorname{span}\{\sqrt{\lambda_1}\,E_1 + \sqrt{-\lambda_3}E_3, E_2\}$ gives us a representative for this conjugacy class.

3. For $\mathfrak{g} = \mathfrak{sol}_3$, we have $\lambda_1 > \lambda_2 = 0 > \lambda_3$. Hence, $\mathfrak{h}_0 = \operatorname{span}\{E_1, E_3\}$ is a 2-dimensional abelian subalgebra, and $\mathfrak{sol}_3 = \mathfrak{h}_0 \rtimes \mathbb{R}E_2$.

   Suppose that $\mathfrak{h} \subset \mathfrak{sol}_3$ is some other 2-dimensional subalgebra. Since we assume $\mathfrak{h} \neq \mathfrak{h}_0$, we may write $\mathfrak{h} = \operatorname{span}\{A, B\}$ for $A = a_1E_1 + a_3E_3$ and $B = b_1E_1 + E_2 + b_3E_3$. Let $g = \operatorname{Exp}(C)$, where Exp denotes the Lie exponential map and $C = -\frac{b_3}{\lambda_3}E_1 + \frac{b_1}{\lambda_1}E_3$. Then, $\mathfrak{h}$ is a subalgebra of $\mathfrak{sol}_3$ if and only if $\tilde{\mathfrak{h}} = \operatorname{Ad}_g\mathfrak{h} = \operatorname{span}\{\operatorname{Ad}_g(A), \operatorname{Ad}_g(B)\}$ is also a subalgebra, where Ad_g is the adjoint map. Since $\operatorname{span}\{E_1, E_3\}$ is abelian, we have $\operatorname{Ad}_g(A) = A$. We compute $[C,B] = -b_3E_3 - b_1E_1$ and hence

$$\begin{aligned}\operatorname{Ad}_g(B) = e^{\operatorname{ad}_C}B = B + [C,B] + \frac{1}{2!}[C,[C,B]] + \cdots = B + [C,B] = \\ = b_1E_1 + E_2 + b_3E_3 - \frac{b_3}{\lambda_3}[E_1,E_2] + \frac{b_1}{\lambda_1}[E_3,E_2] = E_2.\end{aligned}$$

Thus, $\mathfrak{h}$ is a subalgebra if and only if $\tilde{\mathfrak{h}} = \operatorname{span}\{a_1E_1 + a_3E_3, E_2\}$ is a subalgebra. Taking now $A = a_1E_1 + a_3E_3$ and $B = E_2$, condition (4.6) is equivalent to $(-a_3)^2\lambda_1 + (a_1)^2\lambda_3 = 0$. By rescaling the basis so that $a_1 = \sqrt{\lambda_1}$, we get $a_3 = \pm\sqrt{-\lambda_3}$, and therefore $\tilde{\mathfrak{h}}_1 = \operatorname{span}\{\sqrt{\lambda_1}E_1 \pm \sqrt{-\lambda_3}E_3, E_2\}$. Note that the map $\varphi\colon \mathfrak{sol}_3 \to \mathfrak{sol}_3$ given by

$$E_1 \longmapsto E_1, \quad E_2 \longmapsto -E_2, \quad E_3 \longmapsto -E_3,$$

is an automorphism of $\mathfrak{sol}_3$ preserving the metric and maps $\operatorname{span}\{\sqrt{\lambda_1}E_1 + \sqrt{-\lambda_3}E_3, E_2\}$ to $\operatorname{span}\{\sqrt{\lambda_1}E_1 - \sqrt{-\lambda_3}E_3, E_2\}$.

Thus, any 2-dimensional subalgebra of $\mathfrak{sol}_3$ is, up to conjugacy and isometric automorphism, either the abelian subalgebra $\mathfrak{h}_0 = \operatorname{span}\{E_1, E_2\}$ or the non-abelian $\mathfrak{h}_1 = \operatorname{span}\{\sqrt{\lambda_1}E_1 + \sqrt{-\lambda_3}E_3, E_2\}$.

□

We now proceed to classify two-dimensional subgroups of non-unimodular groups. If $\mathfrak{g} = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a non-unimodular 3-dimensional Lie algebra, there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ that satisfy (4.4) for some $\alpha, \beta \geq 0$. We will assume that $\alpha \neq 0, 1$, as otherwise one would have that $\dim(\operatorname{Isom}(G)) \geq 4$ (see Remark 4.2).

Theorem 4.2. *Let G be a 3-dimensional simply-connected non-unimodular Lie group with 3-dimensional isometry group and Lie algebra $\mathfrak{g}$. Then $\mathfrak{h}$ is a two dimensional subalgebra of $\mathfrak{g}$ if and only if*

1. $\mathfrak{h}_0 = \operatorname{span}\{E_1, E_2\}$.
2. *If $\beta = 0$ then $\mathfrak{h}_1 = \operatorname{span}\{E_1, E_3\}$ or $\mathfrak{h}_2 = \operatorname{span}\{E_2, E_3\}$*
3. *If $\beta \neq 0$ and with $\det A = (1-\alpha^2)(1+\beta^2)$, we have*

$$\mathfrak{h}_\pm = \operatorname{span}\{E_1 + c_\pm E_2, E_3\}, \quad c_\pm = \frac{\alpha \pm \sqrt{1-\det A}}{(1-\alpha)\beta}.$$

If $\det A = 1$ then clearly $\mathfrak{h}_- = \mathfrak{h}_+$.

Proof. It follows directly from the bracket relations that $\mathfrak{h}_0 = \operatorname{span}\{E_1, E_2\}$ is an abelian subalgebra of $\mathfrak{g}$. Suppose now that $\mathfrak{h} \neq \mathfrak{h}_0$ is a two dimensional subalgebra of $\mathfrak{g}$, and write $\mathfrak{h} = \operatorname{span}\{A, B\}$ where $A = a_1E_1 + a_2E_2$ and $B = b_1E_1 + b_2E_2 + E_3$. Let $g = \operatorname{Exp}(sE_1 + tE_2)$, with $s, t \in \mathbb{R}$. Then, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ if, and only if $\tilde{\mathfrak{h}} = \operatorname{Ad}_g \mathfrak{h} = \operatorname{span}\{\operatorname{Ad}_g(A), \operatorname{Ad}_g(B)\}$

is a subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}_0 = \text{span}\{E_1, E_2\}$ is abelian, then $\text{Ad}_g(A) = A$. We now compute

$$\begin{aligned}\text{Ad}_g(B) = \exp(\text{ad}_{sE_1+tE_2})B = B + [sE_1+tE_2, B] + \frac{1}{2!}[sE_1+tE_2,[sE_1+tE_2,B]] + \cdots = \\ = b_1E_1 + b_2E_2 + E_3 - s[E_1,E_3] + t[E_2,E_3] = \\ = (b_1 - s(1+\alpha) + t\beta(1-\alpha))E_1 + (b_2 - s\beta(1+\alpha) + t(1-\alpha))E_2 + E_3.\end{aligned}$$

Note that since $(1+\beta^2)(1-\alpha^2) \neq 0$, there exist unique $s_0, t_0 \in \mathbb{R}$ such that

$$\begin{cases} b_1 - s_0(1+\alpha) + t_0\beta(1-\alpha) = 0, \\ b_2 - s_0\beta(1+\alpha) + t_0(1-\alpha) = 0. \end{cases}$$

Thus, for $g = \text{Exp}(s_0E_1 + t_0E_2)$ we get that $\tilde{\mathfrak{h}} = \text{Ad}_g\,\mathfrak{h} = \text{span}\{a_1E_1 + a_2E_2, E_3\}$. Now, $\tilde{\mathfrak{h}}$ is a subalgebra if, and only if,

$$\langle [a_1E_1 + a_2E_2, E_3], (a_1E_1 + a_2E_2) \times E_3 \rangle = (1+\alpha)\beta {a_1}^2 + (1-\alpha)\beta {a_2}^2 - 2\alpha a_1 a_2 = 0. \quad (4.7)$$

Note that this is the equation of a degenerate conic passing through the origin, and so one might study it according to its discriminant $1-(1-\alpha^2)(1+\beta^2) = 1 - \det A$:

- If $1 - \det A < 0$, equation (4.7) has only the real solution $a_1 = a_2 = 0$, wich contradicts the fact that $\tilde{\mathfrak{h}}$ is 2-dimensional.

- If $1 - \det A > 0$ equation (4.7) corresponds to a pair of non-coincident intersecting lines, and thus we get two distinct subalgebras. For $\beta = 0$, these are $\mathfrak{h}_1 = \text{span}\{E_1, E_3\}$ and $\mathfrak{h}_2 = \text{span}\{E_2, E_3\}$. For $\beta \neq 0$, the solutions are given by

$$a_2 = c_\pm a_1, \quad c_\pm = \frac{\alpha \pm \sqrt{1-\det A}}{(1-\alpha)\beta}.$$

  The corresponding subalgebras of $\mathfrak{g}$ will be denoted by $\mathfrak{h}_\pm = \text{span}\{E_1 + c_\pm E_2, E_3\}$.

- If $1 - \det A = 0$, the only solution to equation (4.7) corresponds to the straight line $\alpha a_1 = (1-\alpha)\beta a_2$. As $\beta = 0$ contradicts $\det A = 1$, then $\beta \neq 0$ and and we get the subalgebra $\mathfrak{h}_\pm$ presented before with $c_+ = c_- = \alpha/(1-\alpha)\beta)$.

We have proved that any 2-dimensional Lie subalgebra of $\mathfrak{g}$ is conjugate to the abelian subalgebra $\mathfrak{h}_0$, to one of the non-abelian subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ (arising when $\beta = 0$ and hence $\det L < 1$), to the non-abelian subalgebra $\mathfrak{h}_+$ (when $\beta \neq 0$ and $\det L = 1$), or to $\mathfrak{h}_+$, $\mathfrak{h}_-$ (if $\beta \neq 0$ and $\det L < 1$).

Now we will show that no two of these subalgebras are conjugate. Fix one of the non-abelian subalgebras $\mathfrak{h} \in \{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_+, \mathfrak{h}_-\}$. First note that $\mathfrak{h} = \operatorname{span}\{X, E_3\}$, for some $X \in \operatorname{span}\{E_1, E_2\}$. Since $\mathfrak{h}$ is a subalgebra and ad_{E_3} leaves $\operatorname{span}\{E_1, E_2\}$ invariant, that is, by eq. (4.4) we have

$$\operatorname{ad}_{E_3}(X) = [E_3, X] = Y \in \operatorname{span}\{E_1.E_2\}, \ \ \forall X \in \operatorname{span}\{E_1.E_2\}.$$

As $\mathfrak{h}$ is subalgebra by hypothesis, it follows that $\operatorname{ad}_{E_3}(X) = \lambda X$, that is, X is an eigenvector of ad_{E_3}. Let $Y = sE_1 + tE_2 + rE_3$, with $s, t, r \in \mathbb{R}$, and let $g = \operatorname{Exp}(Y)$ be an arbitrary element in a neighborhood of the identity element of G. Then

$$[Y, X] = [sE_1 + tE_2, X] + [rE_3, X] = r\lambda X.$$

Hence,

$$\operatorname{Ad}_g(X) = e^{\operatorname{ad}_Y} X = X + [Y, X] + \frac{1}{2!}[Y, [Y, X]] + \cdots = X + [Y, X] + 0 = (1 + r\lambda X),$$

that is, $[Y, X]$ and $\operatorname{Ad}_g(X)$ are proportional to X. Since any neighborhood of the identity of G generates G, an arbitrary element $g \in G$ can be written as $g = g_1 \cdots g_\ell$, with $g_i = \operatorname{Exp}(Y_i)$, $i = 1, \ldots, \ell$, as before. But then $\operatorname{Ad}_g(\mathfrak{h})$ would contain X. However X does not belong to any of the subalgebras $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_+, \mathfrak{h}_-$ different from $\mathfrak{h}$ itself. Hence, $\mathfrak{h}$ cannot be conjugate to any of the other subalgebras that we have determined. Finally, since we are assuming that the isometry group of G is 3-dimensional, according to [12] we have $\operatorname{Isom}(G, \langle\cdot,\cdot\rangle) = G$. Thus, by virtue of Proposition 4.2, the number of orbit equivalence classes of cohomogeneity one actions on G is one in the case $\det L > 1$, two when $\det L = 1$, and three in the case $\det L < 1$.

□

See second column of Table 4.4 that summarizes the results of the previous theorems.

4.3 Geometry of the 2-dimensional homogeneous foliations

If H is a Lie group acting properly by isometries on a Riemannian manifold, its orbits are mutually equidistant submanifolds. In particular, if H acts with cohomogeneity one and no singular orbits, one may recover the geometry of all orbits by determining the geometry of the orbit through a fixed (but arbitrary) basepoint, and then investigating the geometry of its parallel hypersurfaces. In this section, we will make use of this idea to study the geometry of homogeneous hypersurfaces of 3-dimensional metric Lie groups with 3-dimensional isometry group, thus proving in particular the geometric information in the last column of Table 4.4.

Let G be a simply connected 3-dimensional metric Lie group with $\dim(\mathrm{Isom}(G)) = 3$. Recall that, by Proposition 4.2, there is a one-to-one correspondence between cohomogeneity one actions on G up to orbit equivalence and Lie subalgebras of $\mathfrak{g}$ up to conjugacy and isometric automorphisms. In particular, cohomogeneity one actions on G are induced by connected 2-dimensional subgroups of G.

Thus, let H be a codimension one Lie subgroup of G. Then, the orbit through the identity element e is precisely H. One can compute its shape operator at e with a computation at the Lie algebra level. Note that the orbits of H are precisely the right cosets $H \cdot g$, with $g \in G$. Thus, if $v \in T_gG$ is a normal vector to $H \cdot g$ at a point $g \in G$, then $(L_h)_* v$ is a normal vector to $H \cdot g$ at hg. Thus, the left-invariant field $\xi \in \mathfrak{g}$ with $\xi_g = v$ is a unit normal field to $H \cdot g$.

Let γ be a unit-speed normal geodesic to H with $\gamma(0) = e$ and $\gamma'(0) = v$. Since G is a Riemannian homogeneous space, it is geodesically complete, so we can assume that γ is defined in all $\mathbb{R}$. Let $\xi \in \mathfrak{g}$ be the left-invariant field with $\xi_e = v$, and write H^t for the parallel displacement of H in the direction of ξ at distance t, that is, $H^t = \{\exp_h(t\xi_h) : h \in H\}$ is the parallel (or equidistant) surface to H at distance t, where exp is the Riemannian exponential map. Then, $H^t = H \cdot \gamma(t)$, and the left-invariant vector field ξ^t with $\xi^t_{\gamma(t)} = \gamma'(t)$ is a unit normal field to H^t. In order to calculate the shape operator S^t of H^t with respect to ξ^t, we just need to determine the tangent vector γ' to the normal geodesic γ, and then use the formulas (4.3) and (4.5) for the Levi-Civita connection of G in terms of left-invariant fields to calculate $S^t = -\nabla_\cdot \xi^t$ for each $t \in \mathbb{R}$.

Before proceeding with a case-by-case analysis, we will calculate the system of ordinary differential equations defining the geodesic equation on G, depending on whether G is unimodular or non-unimodular. Thus, let $\{E_1, E_2, E_3\}$ be some left-invariant orthonormal frame, and write

$$\gamma'(t) = x(t)E_1 + y(t)E_2 + z(t)E_3,$$

for some real functions $x(t)$, $y(t)$, $z(t)$. Denoting by ∇ the Levi-Civita connection of G, we have

$$\nabla_{\gamma'}\gamma' = x'E_1 + x\nabla_{\gamma'}E_1 + y'E_2 + y\nabla_{\gamma'}E_2 + z'E_3 + z\nabla_{\gamma'}E_3 = 0. \tag{4.8}$$

If G is unimodular, $\{E_1, E_2, E_3\}$ is an orthonormal frame as in (4.1) and using (4.3), we compute

$$\begin{aligned}
\nabla_{\gamma'}E_1 &= x\nabla_{E_1}E_1 + y\nabla_{E_2}E_1 + z\nabla_{E_3}E_1 = -y\mu_2E_3 + z\mu_3E_2,\\
\nabla_{\gamma'}E_2 &= x\nabla_{E_1}E_2 + y\nabla_{E_2}E_2 + z\nabla_{E_3}E_2 = x\mu_1E_3 - z\mu_3E_1,\\
\nabla_{\gamma'}E_3 &= x\nabla_{E_1}E_3 + y\nabla_{E_2}E_3 + z\nabla_{E_3}E_3 = -x\mu_1E_2 + y\mu_2E_1,
\end{aligned}$$

Recalling that $\mu_i = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)-\lambda_i$, $i=1,2,3$, we have that $(x(t),y(t),z(t))$ is a solution of the following system of differential equations

$$\begin{cases} x' + yz(\lambda_3-\lambda_2) = 0, \\ y' + xz(\lambda_1-\lambda_3) = 0, \\ z' + xy(\lambda_2-\lambda_1) = 0. \end{cases} \tag{4.9}$$

If G non-unimodular, $\{E_1,E_2,E_3\}$ satisfies (4.4) and using (4.5), we have

$$\begin{aligned}
\nabla_{\gamma'}E_1 &= x\nabla_{E_1}E_1 + y\nabla_{E_2}E_1 + z\nabla_{E_3}E_1 = x(1+\alpha)E_3 + y\alpha\beta E_3 + z\beta E_2,\\
\nabla_{\gamma'}E_1 &= x\nabla_{E_1}E_2 + y\nabla_{E_2}E_2 + z\nabla_{E_3}E_2 = x\alpha\beta E_3 + y(1-\alpha)E_3 - z\beta E_1,\\
\nabla_{\gamma'}E_1 &= x\nabla_{E_1}E_3 + y\nabla_{E_2}E_3 + z\nabla_{E_3}E_3 = -x((1+\alpha)E_1+\alpha\beta E_2) - y(\alpha\beta E_1 + (1-\alpha)E_2).
\end{aligned}$$

Thus, $(x(t),y(t),z(t))$ is a solution of the following system of differential equations

$$\begin{cases} x' - (1+\alpha)(x+\beta y)z = 0, \\ y' - (1-\alpha)(y-\beta x)z = 0, \\ z' + (1+\alpha)x^2 + 2\alpha\beta xy + (1-\alpha)y^2 = 0. \end{cases} \tag{4.10}$$

This way we present the next theorems as the main result of this chapter. Also, these results are summarized in the last column of Table 4.4.

Initially, we present the results for the unimodular case. Following the Theorem 4.1 we state

Theorem 4.3. *Let G be a 3-dimensional simply-connected unimodular Lie group with 3-dimensional isometry group and Lie algebra $\mathfrak{g}$. Then a homogeneous surface $H\cdot\gamma(t)$ of G , where t is the parameter of a geodesic that intersects H at e, satisfies the following*

1. *If $G=\widetilde{\mathrm{E}}_2$, then the principal curvatures of $H\cdot\gamma(t)$ are $\pm\mu_1$. The H-orbits are minimal, but not totally geodesic.*

2. *If $G=\mathrm{Sol}_3$, then*

   (a) If $\mathfrak{h}_0=\mathrm{span}\{E_1,E_3\}$, the principal curvatures of $H_0\cdot\gamma(t)$ are $\pm\mu_3$. All H_0-orbits are minimal, but none of them is totally geodesic (since $\mu_3\neq 0$).

   (b) If $\mathfrak{h}_1=\mathrm{span}\{\sqrt{\lambda_1}E_1+\sqrt{-\lambda_3}E_3,E_2\}$, the principal curvatures of $H_1\cdot\gamma(t)$ are given by

$$\pm\sqrt{\frac{(\lambda_1+\lambda_3)^2}{4} - \lambda_1\lambda_3\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right)^2}.$$

All H_1-orbits are minimal. The only H_1-orbit that is totally geodesic is $H_1 \cdot e$, when $\lambda_3 = -\lambda_1$.

3. *If $G = \widetilde{\mathrm{Sl}}_2(\mathbb{R})$, then of $H \cdot \gamma(t)$ is a minimal surface and has shape operator of the form*

$$S^t \equiv \frac{1}{x^2+z^2}\begin{pmatrix} xyz(\mu_1-\mu_3) & -(x^2\mu_3+z^2\mu_1) \\ -(x^2\mu_3+z^2\mu_1) & -xyz(\mu_1-\mu_3) \end{pmatrix}.$$

where y is the solution of the ODE

$$y'^2(t) = [(\lambda_3-\lambda_2)y^2(t)-\lambda_3][\lambda_1+(\lambda_2-\lambda_1)y^2(t)],$$

and $x(t)$ and $z(t)$ can be defined respectively as

$$x(t) = \sqrt{\frac{(\lambda_3-\lambda_2)y^2(t)-\lambda_3}{\lambda_1-\lambda_3}}, \quad z(t) = \sqrt{\frac{(\lambda_2-\lambda_1)y^2(t)+\lambda_1}{\lambda_1-\lambda_3}}.$$

The orbits of $H\cdot\gamma(t)$ are minimal submanifolds for all t. The orbit $H\cdot e$ is totally geodesic if and only if $\mu_2 = 0$, or equivalently, $\lambda_2 = \lambda_1+\lambda_3$. The other orbits are not totally geodesic as S^t, $t \neq 0$, does not vanish.

Proof. In this proof, we will treat each case separately

1. Homogeneous surfaces of $\widetilde{\mathrm{E}}_2$.

   Recall that the only cohomogeneity one action on $\widetilde{\mathrm{E}}_2 = \mathbb{R}^2 \rtimes \mathbb{R}$ is the action of the abelian normal subgroup $H = \mathbb{R}^2$. If $\{E_1,E_2,E_3\}$ is an orthonormal basis of $\mathfrak{e}_2$ satisfying the bracket relations described in (4.1), where $\lambda_1 > \lambda_2 > 0$ and $\lambda_3 = 0$, then $\mathfrak{h} = \mathrm{span}\{E_1,E_2\}$. It follows from (4.3) that $\gamma(t) = \mathrm{Exp}(tE_3)$ is a normal geodesic to H through e, where Exp is the Lie exponential map. Since $\gamma'(t) = E_3$ for all $t \in \mathbb{R}$, the tangent space to $H\cdot\gamma(t)$ at $\gamma(t)$ is the subspace $\mathfrak{h}_{\gamma(t)} \subset T_{\gamma(t)}\widetilde{\mathrm{E}}_2$ corresponding to the left-invariant distribution $\mathfrak{h}$. Using (4.3), we have that the shape operator of $H\cdot\gamma(t)$ is given by

$$S^tE_1 = -\nabla_{E_1}E_3 = \mu_1E_2, \quad S^tE_2 = -\nabla_{E_2}E_3 = -\mu_2E_1.$$

   Since $\lambda_3 = 0$, we have that $\mu_1 = -\mu_2 = (\lambda_2-\lambda_1)/2 \neq 0$, and the shape operator of $H\cdot\gamma(t)$ at $\gamma(t)$ in terms of the basis $\{E_1,E_2\}$ is given by

$$S^t \equiv \begin{pmatrix} 0 & \mu_1 \\ \mu_1 & 0 \end{pmatrix}.$$

Thus, the principal curvatures of $H \cdot \gamma(t)$ are $\pm\mu_1 \neq 0$, with respective principal directions $E_1 \pm E_2$. Therefore, all H-orbits are minimal, but not totally geodesic.

2. Homogeneous surfaces of Sol_3.

   Any 2-dimensional subalgebra of $\mathfrak{sol}_3$ is, up to conjugacy and isometric automorphism, either the abelian subalgebra $\mathfrak{h}_0 = \text{span}\{E_1, E_3\}$ or the non-abelian $\mathfrak{h}_1 = \text{span}\{\sqrt{\lambda_1}E_1 + \sqrt{-\lambda_3}E_3, E_2\}$, where $\{E_1, E_2, E_3\}$ is an orthonormal basis of $\mathfrak{sol}_3$ satisfying the bracket relations described in (4.1) with $\lambda_1 > \lambda_2 = 0 > \lambda_3$.

   For $\mathfrak{h}_0 = \text{span}\{E_1, E_3\}$, $\gamma(t) = \text{Exp}(tE_2)$ is a normal geodesic to H_0 trough e, where H_0 is the connected subgroup H_0 of Sol_3 with Lie algebra $\mathfrak{h}_0$. Thus, $\gamma'(t) = E_2$ for all $t \in \mathbb{R}$, and the tangent space to $H_0 \cdot \gamma(t)$ at $\gamma(t)$ is $(\mathfrak{h}_0)_{\gamma(t)}$. We have

$$S^t E_1 = -\nabla_{E_1} E_2 = -\mu_1 E_3, \quad S^t E_3 = -\nabla_{E_3} E_2 = \mu_3 E_1.$$

   As $\lambda_2 = 0$, then $\mu_1 = -\mu_3$, and we have

$$S^t \equiv \begin{pmatrix} 0 & \mu_3 \\ \mu_3 & 0 \end{pmatrix},$$

   in terms of the orthonormal basis $\{E_1, E_3\}$ of $T_{\gamma(t)}(H_0 \cdot \gamma(t))$. The principal curvatures of $H_0 \cdot \gamma(t)$ are $\pm\mu_3$, and the corresponding principal directions are $E_1 \pm E_3$. Hence, all H_0-orbits are minimal, but none of them is totally geodesic (since $\mu_3 \neq 0$).

   If $\mathfrak{h}_1 = \text{span}\{\sqrt{\lambda_1}E_1 + \sqrt{-\lambda_3}E_3, E_2\}$ and $\gamma(t)$ is a geodesic to H_1 through e with $\gamma'(0) = \dfrac{1}{\sqrt{\lambda_1 - \lambda_3}}\left(\sqrt{-\lambda_3}E_1 - \sqrt{\lambda_1}E_3\right) \perp \mathfrak{h}_1$. Solving the differential equation (4.9) with initial conditions

$$x(0) = \sqrt{\frac{-\lambda_3}{\lambda_1 - \lambda_3}}, \quad y(0) = 0 \quad \text{and} \quad z(0) = -\sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_3}},$$

   gives us

$$\gamma'(t) = \frac{\operatorname{sech}\left(\sqrt{-\lambda_1\lambda_3 t}\right)}{\sqrt{\lambda_1 - \lambda_3}}(\sqrt{-\lambda_3}E_1 - \sqrt{\lambda_1}E_3) + \tanh\left(\sqrt{-\lambda_1\lambda_3 t}\right)E_2.$$

   Let $V(t) = \dfrac{1}{\sqrt{\lambda_1 - \lambda_3}}(\sqrt{\lambda_1}E_1 + \sqrt{-\lambda_3}E_3)$ and

$$W(t) = -\frac{\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right)}{\sqrt{\lambda_1-\lambda_3}}(\sqrt{-\lambda_3}E_1 - \sqrt{\lambda_1}E_3) + \operatorname{sech}\left(\sqrt{-\lambda_1\lambda_3}t\right)E_2.$$

Then, $\{V,W\}$ is an orthonormal basis of $T_{\gamma(t)}(H_1\cdot\gamma(t))$. We have

$$S^tV(t) = -\nabla_{V(t)}\gamma'(t) = \sqrt{-\lambda_1\lambda_3}\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right)V(t) + \frac{1}{2}(\lambda_1+\lambda_3)W(t),$$
$$S^tW(t) = -\nabla_{W(t)}\gamma'(t) = \frac{1}{2}(\lambda_1+\lambda_3)V(t) - \sqrt{-\lambda_1\lambda_3}\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right)W(t).$$

Thus, the shape operator of $H_1\cdot\gamma(r)$ with respect to this basis is given by

$$S^t \equiv \begin{pmatrix} \sqrt{-\lambda_1\lambda_3}\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right) & \frac{1}{2}(\lambda_1+\lambda_3) \\ \frac{1}{2}(\lambda_1+\lambda_3) & -\sqrt{-\lambda_1\lambda_3}\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right) \end{pmatrix}.$$

The principal curvatures of $H_1\cdot\gamma(t)$ are

$$\pm\sqrt{\frac{(\lambda_1+\lambda_3)^2}{4} - \lambda_1\lambda_3\tanh\left(\sqrt{-\lambda_1\lambda_3}t\right)^2}.$$

Note that all H_1-orbits are minimal. The only H_1-orbit that can be totally geodesic is $H_1\cdot e$, which happens precisely when $\lambda_3 = -\lambda_1$.

3. Homogeneous surfaces of $\widetilde{\mathsf{SL}_2}(\mathbb{R})$.

The Lie group $\widetilde{\mathsf{SL}_2}(\mathbb{R})$ has a unique codimension one subgroup H up to conjugation. If $\{E_1,E_2,E_3\}$ is an orthonormal basis of $\mathfrak{sl}_2(\mathbb{R})$ satisfying the bracket relations described in (4.1), where $\lambda_1 > \lambda_2 > 0 > \lambda_3$, a representative for the conjugacy class is given by the connected subgroup of $\widetilde{\mathsf{SL}_2}(\mathbb{R})$ with Lie algebra $\mathfrak{h} = \operatorname{span}\{\sqrt{\lambda_1}E_1 + \sqrt{-\lambda_3}E_3, E_2\}$.

Let $\gamma(t)$ denote a unit normal geodesic to the corresponding connected subgroup H passing through $\gamma(0) = e$ with $\gamma'(0) = \frac{1}{\sqrt{\lambda_1-\lambda_3}}\left(\sqrt{-\lambda_3}E_1 - \sqrt{\lambda_1}E_3\right)$, and write $\gamma'(t) = x(t)E_1 + y(t)E_2 + z(t)E_3$. Since $(x(t),y(t),z(t))$ is a solution to (4.9), it satisfies

$$\frac{xx'}{\lambda_3-\lambda_2} = \frac{yy'}{\lambda_1-\lambda_3} = \frac{zz'}{\lambda_2-\lambda_1}.$$

Integrating the previous equation, we get that $y(t)$ is the solution of the following differential equation

$$y'^2(t) = [(\lambda_3-\lambda_2)y^2(t) - \lambda_3][\lambda_1 + (\lambda_2-\lambda_1)y^2(t)],$$

and $x(t)$ and $z(t)$ can be defined respectively as

$$x^2(t) = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_3} y^2(t) + k_1, \;\; z^2(t) = \frac{\lambda_2 - \lambda_1}{\lambda_1 - \lambda_3} y^2(t) + k_2,$$

for some real constants k_1, k_2. It follows from the initial conditions that

$$x(t) = \sqrt{\frac{(\lambda_3 - \lambda_2) y^2(t) - \lambda_3}{\lambda_1 - \lambda_3}}, \quad z(t) = -\sqrt{\frac{(\lambda_2 - \lambda_1) y^2(t) + \lambda_1}{\lambda_1 - \lambda_3}}. \tag{4.11}$$

In particular, $x(t) > 0 > z(t)$ for all t, so

$$\begin{aligned} V(t) &= \frac{1}{\sqrt{x(t)^2 + z(t)^2}} (z(t)E_1 - x(t)E_3), \\ W(t) &= \frac{1}{\sqrt{x(t)^2 + z(t)^2}} \left(x(t)y(t)E_1 - (x(t)^2 + z(t)^2)E_2 + y(t)z(t)E_3 \right). \end{aligned}$$

are an orthonormal basis of $T_{\gamma(t)}(H \cdot \gamma(t))$. Then, the shape operator of $H \cdot \gamma(t)$ is given by

$$\begin{aligned} S^t V(t) = -\nabla_{V(t)} \gamma'(t) &= \frac{(\mu_1 - \mu_3) x(t) y(t) z(t)}{x^2(t) + z^2(t)} V(t) - \frac{\mu_3 x^2(t) + \mu_1 z^2(t)}{x^2(t) + z^2(t)} W(t), \\ S^t W(t) = -\nabla_{W(t)} \gamma'(t) &= -\frac{\mu_3 x^2(t) + \mu_1 z^2(t)}{x^2(t) + z^2(t)} V(t) - \frac{(\mu_1 - \mu_3) x(t) y(t) z(t)}{x^2(t) + z^2(t)} W(t). \end{aligned}$$

Thus, the matrix expression of S^t in terms of the orthonormal basis $\{V(t), W(t)\}$ is given by

$$S^t \equiv \frac{1}{x^2 + z^2} \begin{pmatrix} xyz(\mu_1 - \mu_3) & -(x^2\mu_3 + z^2\mu_1) \\ -(x^2\mu_3 + z^2\mu_1) & -xyz(\mu_1 - \mu_3) \end{pmatrix}.$$

Therefore, the orbits of H are all minimal surfaces of $\widetilde{\mathsf{SL}_2}(\mathbb{R})$. Using (4.11), one can verify that the orbit through e is totally geodesic if and only if $\mu_2 = 0$, or equivalently, $\lambda_2 = \lambda_1 + \lambda_3$. The other orbits are not totally geodesic. Indeed, since $\lambda_1 > \lambda_3$ and $x(t) > 0 > z(t)$ for all t, by (4.9) we get that $y'(t) > 0$ for all $t \in \mathbb{R}$, as $y(0) = 0$, then $y(t) \neq 0$ for all $t \in \mathbb{R} \setminus \{0\}$. Thus S^t does not vanishes for all $t \neq 0$, since $\mu_1 - \mu_3 = \lambda_3 - \lambda_1 \neq 0$.

□

Now we present the geometry of Homogeneous surfaces of non-unimodular groups. Following Theorem 4.2 we state our final result

Theorem 4.4. *Let G be a 3-dimensional simply-connected non-unimodular Lie group with 3-dimensional isometry group and Lie algebra $\mathfrak{g}$. Then a homogeneous surface $H \cdot \gamma(t)$ of G, where t is the parameter of a geodesic that intersects H at e, satisfies the following*

1. *For $\mathfrak{h}_0 = \operatorname{span}\{E_1, E_2\}$, the principal curvatures of $H_0 \cdot \gamma(t)$ are $1 \pm \alpha\sqrt{1+\beta^2}$. In particular, the mean curvature is equal to 1, and none of the orbits of H_0 are minimal.*

2. *If $\beta = 0$ then we have two particular cases*

   (a) *For $\mathfrak{h}_1 = \operatorname{span}\{E_1, E_3\}$, the principal curvatures of $H_1 \cdot \gamma(t)$ are*

   $$(\pm\alpha - 1)\tanh((1-\alpha)t)$$

   and it has constant mean curvature $-\tanh((1-\alpha)t) \in (-1, 1)$.

   (b) *For $\mathfrak{h}_2 = \operatorname{span}\{E_2, E_3\}$, the principal curvatures of $H_2 \cdot \gamma(t)$ are*

   $$(\pm\alpha - 1)\tanh((1+\alpha)r)$$

   and it has constant mean curvature $-\tanh((1+\alpha)t) \in (-1, 1)$.

   In particular, the only minimal orbits are $H_i \cdot e$ with $i = 1, 2$, which are actually totally geodesic.

3. *If $\beta \neq 0$ then for $\mathfrak{h}_\pm = \operatorname{span}\{E_1 + c_\pm E_2, E_3\}$, the principal curvatures are given by*

   $$-\tanh((1-\alpha)(1+\beta c_\pm)t) \pm \sqrt{(1-\det L)\tanh^2((1-\alpha)(1+\beta c_\pm)t) + \beta^2}.$$

   and the mean curvature of $H_\pm \cdot \gamma(t)$ is $-\tanh((1-\alpha)(1+\beta c_\pm)t)$. Since $\alpha \neq 1$ and $1 + \beta c_\pm \neq 0$, the only minimal $H_\pm$-orbit is $H_\pm \cdot e$ but it is not totally geodesic.

Proof. The proof will be given again in separate cases

Initially for $\mathfrak{h}_0$, we have that $\gamma(t) = \operatorname{Exp}(tE_3)$ is a normal geodesic to H_0 through e. Thus, $\gamma'(t) = E_3$ for all $t \in \mathbb{R}$, and the tangent space to $H_0 \cdot \gamma(t)$ at $\gamma(t)$ is $\mathfrak{h}_{\gamma(t)}$. Hence, using (4.5), we get

$$S^t E_1 = -\nabla_{E_1} E_3 = (1+\alpha)E_1 + \alpha\beta E_2, \qquad S^t E_2 = -\nabla_{E_2} E_3 = \alpha\beta E_1 + (1-\alpha)E_2.$$

Therefore, the principal curvatures of $H_0 \cdot \gamma(t)$ are $1 \pm \alpha\sqrt{1+\beta^2}$. In particular, the mean curvature is 1, and none of the orbits of H_0 is minimal

Now for the case with $\beta = 0$, let $\mathfrak{g}$ be a non-unimodular 3-dimensional Lie algebra satisfying (4.4) in terms of some orthonormal basis $\{E_1, E_2, E_3\}$. If $\alpha \neq 0, 1$ and $\beta \neq 0$, any 2-dimensional subalgebra of $\mathfrak{g}$ is conjugate to the abelian $\mathfrak{h}_0 = \operatorname{span}\{E_1, E_2\}$, or the non abelian cases $\mathfrak{h}_1 = \operatorname{span}\{E_1, E_3\}$ and $\mathfrak{h}_2 = \operatorname{span}\{E_2, E_3\}$.

We then have the following cases

($\mathfrak{h}_1$) For $\mathfrak{h}_1 = \operatorname{span}\{E_1, E_3\}$, let γ be the unit normal geodesic to the corresponding connected Lie subgroup H_1 of G with $\gamma'(0) = E_2 \perp \mathfrak{h}_1$. Solving equation (4.10) we have $\gamma'(t) = \operatorname{sech}((1-\alpha)t)E_2 - \tanh((1-\alpha)t)E_3$. Thus, an orthonormal basis of $T_{\gamma(t)}(H_1 \cdot \gamma(t))$ is given by

$$V(t) \equiv E_1 \quad , \quad W(t) = \tanh((1-\alpha)t)E_2 + \operatorname{sech}((1-\alpha)t)E_3.$$

Using (4.5) we obtain

$$\begin{aligned} S^t V(t) = -\nabla_{V(t)}\gamma'(t) &= -(1+\alpha)\tanh((1-\alpha)t)V(t), \\ S^t W(t) = -\nabla_{W(t)}\gamma'(t) &= -(1-\alpha)\tanh((1-\alpha)t)W(t), \end{aligned}$$

so $V(t)$ and $W(t)$ are principal directions of $H_1 \cdot \gamma(t)$. Thus, the H_1-orbit through $\gamma(t)$ has constant mean curvature $-\tanh((1-\alpha)t) \in (-1, 1)$. In particular, the only minimal orbit is the one through e, which is actually totally geodesic.

($\mathfrak{h}_2$) For $\mathfrak{h}_2$, if γ is the normal geodesic to H_2 with $\gamma'(0) = E_1 \perp \mathfrak{h}_2$, we have that $\gamma'(t) = \operatorname{sech}((1+\alpha)t)E_1 - \tanh((1+\alpha)t)E_3$, and so $V = E_2$ and $W = \tanh((1+\alpha)t)E_1 + \operatorname{sech}((1+\alpha)t)E_3$ provide an orthonormal basis for $T_{\gamma(t)}(H_2 \cdot \gamma(t))$. In this case,

$$\begin{aligned} S^t V(t) = -\nabla_{V(t)}\gamma'(t) &= -(1-\alpha)\tanh((1+\alpha)t)V(t), \\ S^t W(t) = -\nabla_{W(t)}\gamma'(t) &= -(1+\alpha)\tanh((1+\alpha)t)W(t). \end{aligned}$$

Therefore, the orbit $H_2 \cdot \gamma(t)$ has constant mean curvature $-\tanh((1+\alpha)t) \in (-1, 1)$, and $H_2 \cdot e$ is totally geodesic.

Finally, we consider a non-unimodular Lie algebra $\mathfrak{g}$ with $\beta \neq 0$, and the Lie subalgebras $\mathfrak{h}_\pm = \operatorname{span}\{E_1 + c_\pm E_2, E_3\}$, where

$$c_\pm = \frac{\alpha \pm \sqrt{1-(1-\alpha^2)(1+\beta^2)}}{(1-\alpha)\beta} = \frac{\alpha \pm \sqrt{1-\det L}}{(1-\alpha)\beta}.$$

We will treat both cases $\mathfrak{h}_+$ and $\mathfrak{h}_-$ simultaneously. We have that $\frac{1}{\sqrt{1+c_\pm^2}}(c_\pm E_1 - E_2)$ is a unit normal vector to $\mathfrak{h}_\pm$. Solving the initial value problem (4.10) with initial conditions $x(0) = \frac{c_\pm}{\sqrt{1+c_\pm^2}}$, $y(0) = \frac{-1}{\sqrt{1+c_\pm^2}}$ and $z(0) = 0$ yields

$$\gamma'(t) = \frac{\operatorname{sech}((1-\alpha)(1+\beta c_\pm)t)}{\sqrt{1+c_\pm^2}}(c_\pm E_1 - E_2) - \tanh((1-\alpha)(1+\beta c_\pm)t)E_3.$$

For each $t \in \mathbb{R}$ we define $V(t) = \frac{1}{\sqrt{1+c_\pm^2}}(E_1 + c_\pm E_2)$ and compute $W(t) = V(t) \wedge \gamma'(t)$, that is

$$W(t) = \frac{-\tanh((1-\alpha)(1+\beta c_\pm)t)}{\sqrt{1+c_\pm^2}}(c_\pm E_1 - E_2) - \operatorname{sech}((1-\alpha)(1+\beta c_\pm)t)E_3.$$

Thus, $\{V(t), W(t)\}$ is an orthonormal basis of $H_\pm \cdot \gamma(t)$. Let $D = (1-\alpha)(1+\beta c_\pm)$ in order to simplify the notation. We compute

$$\overline{\nabla}_{V(t)}\gamma'(t) = \frac{1}{\sqrt{c_\pm^2+1}}\left[\frac{\operatorname{sech}(Dt)}{\sqrt{c_\pm^2+1}}\left(c_\pm(1+\alpha) - \alpha\beta + c_\pm^2\alpha\beta - c_\pm(1-\alpha)\right)E_3 + \right.$$
$$\left. + \tanh(Dt)[(1+\alpha+c_\pm\alpha\beta)E_1 + (\alpha\beta + c_\pm(1-\alpha))E_2]\right].$$

But $\alpha\beta c_\pm^2 - \alpha\beta + 2\alpha c_\pm = \beta(c_\pm^2+1)$ and

$$(\alpha + \alpha\beta c_\pm)E_1 + (\alpha\beta - \alpha c_\pm)E_2 = \beta(c_\pm E_1 - E_2) + A(E_1 + c_\pm E_2),$$

with $A_\pm = \mp\sqrt{\alpha^2(1-\beta^2) - \beta^2}$. Then

$$\overline{\nabla}_{V(t)}\gamma'(t) = \beta\left[\frac{\tanh(Dt)}{\sqrt{c_\pm^2+1}}(c_\pm E_1 - E_2) + \operatorname{sech}(Dt)E_3\right] +$$
$$+ \frac{\tanh(Dt)}{\sqrt{c_\pm^2+1}}(1+A_\pm)(E_1 + c_\pm E_2). = (1+A_\pm)\tanh(Dt)V(t) - \beta W(t).$$

Now we compute

$$\overline{\nabla}_{W(t)}\gamma'(t) = \operatorname{sech}(Dt)\frac{\tanh(Dt)}{c_\pm^2+1}[-c_\pm^2(1+\alpha)+2c_\pm\alpha\beta-(1-\alpha)]E_3 - \frac{\tanh^2(Dt)}{\sqrt{c_\pm^2+1}}[c_\pm E_1 - E_2]+$$
$$+\frac{\tanh^2(Dt)}{\sqrt{c_\pm^2+1}}[(\alpha\beta-c_\pm\alpha)E_1-(\alpha+c_\pm\alpha\beta)E_2]-\beta\operatorname{sech}^2(Dt)\frac{1}{\sqrt{c_\pm^2+1}}[E_1+c_\pm E_2].$$

But

$$(\alpha\beta-c_\pm\alpha)E_1-(\alpha+c_\pm\alpha\beta)E_2 = B(c_\pm E_1-E_2)-\beta(E_1+c_\pm E_2),$$

with $B_\pm = \mp\sqrt{\alpha^2(\beta+1)-\beta^2} = A_\pm$ and also

$$\frac{-c_\pm^2(1+\alpha)+2c_\pm\alpha\beta-(1-\alpha)}{1+c_\pm^2} = B_\pm - 1 = -(1-\alpha)(\beta c_\pm+1).$$

We then get

$$\overline{\nabla}_{W(t)}\gamma'(t) = \tanh(Dt)(1-\alpha)(bc_\pm+1)\left[-\frac{\tanh(Dt)}{\sqrt{c_\pm^2+1}}(c_\pm E_1-E_2)-\operatorname{sech}(Dt)E_3\right]+$$
$$-\frac{\beta}{\sqrt{c_\pm^2+1}}\left[\tanh^2(Dt)+\frac{1}{\cosh^2(Dt)}\right][E_1+c_\pm E_2] = -\beta V(t)+\tanh(Dt)(1-A_\pm)W(t).$$

Hence, we conclude that

$$S_\pm^t = \begin{bmatrix} -\tanh(Dt)(1+A_\pm) & \beta \\ \beta & -(1-A_\pm)\tanh(Dt) \end{bmatrix}.$$

And the principal curvatures are $-\tanh(Dt)\pm\sqrt{A^2\tanh^2(Dt)+\beta^2}$. The mean curvature of $H_\pm\cdot\gamma(t)$ is $-\tanh(Dt)$. Since $\alpha\neq 1$ and $1+\beta c_\pm \neq 0$, we deduce that the only minimal $H_\pm$-orbit is the one through the identity element, but it is not totally geodesic since $\beta\neq 0$. □

G	$\mathfrak{g}$	$\mathfrak{h}$	Orbits
$\widetilde{\mathsf{E}}_2$	$[E_2,E_3]=\lambda_1 E_1$ $[E_3,E_1]=\lambda_2 E_2$ $\lambda_1>\lambda_2>0$	$\operatorname{span}\{E_1,E_2\}$	All orbits are minimal, but no orbit is totally geodesic.
$\widetilde{\mathsf{SL}_2}(\mathbb{R})$	$[E_2,E_3]=\lambda_1 E_1$ $[E_3,E_1]=\lambda_2 E_2$ $[E_1,E_2]=\lambda_3 E_3$ $\lambda_1>\lambda_2>0>\lambda_3$	$\operatorname{span}\{\sqrt{\lambda_1}E_1+\sqrt{-\lambda_3}E_3,E_2\}$	All orbits are minimal. No orbit is totally geodesic, except $H\cdot e$ precisely when $\lambda_2=\lambda_1+\lambda_3$.
Sol_3	$[E_2,E_3]=\lambda_1 E_1$ $[E_1,E_2]=\lambda_3 E_3$ $\lambda_1>0>\lambda_3$	$\operatorname{span}\{E_1,E_3\}$	All orbits are minimal, but not totally geodesic.
		$\operatorname{span}\{\sqrt{\lambda_1}E_1+\sqrt{-\lambda_3}E_3,E_2\}$	All orbits are minimal. No orbit is totally geodesic, except $H\cdot e$ when $\lambda_3=-\lambda_1$.
$\mathbb{R}^2\rtimes\mathbb{R}$	$[E_2,E_3]=(1-\alpha)(\beta E_1-E_2)$ $[E_3,E_1]=(1+\alpha)(E_1+\beta E_2)$ $\alpha,\beta\geq 0$, $\alpha\neq 0,1$ $\det L=(1-\alpha^2)(1+\beta^2)$	$\operatorname{span}\{E_1,E_2\}$	All orbits have constant mean curvature 1.
		$\operatorname{span}\{E_1+\frac{\alpha}{(1-\alpha)\beta}E_2,E_3\}$, when $\det L=1$	Each orbit has constant mean curvature in $(-1,1)$, and the map sending each orbit to its mean curvature is a bijection from the orbit space to $(-1,1)$. The orbit $H\cdot e$ is minimal, and it is totally geodesic if and only if $\beta=0$.
		$\operatorname{span}\{E_1,E_3\}$, when $\beta=0$	
		$\operatorname{span}\{E_2,E_3\}$, when $\beta=0$	
		$\operatorname{span}\{E_1+\frac{\alpha+\sqrt{1-\det L}}{(1-\alpha)\beta}E_2,E_3\}$, when $\beta\neq 0$ and $\det L<1$	
		$\operatorname{span}\{E_1+\frac{\alpha-\sqrt{1-\det L}}{(1-\alpha)\beta}E_2,E_3\}$, when $\beta\neq 0$ and $\det L<1$	

Table 4.4 Cohomogeneity one actions on 3-dimensional metric Lie groups with $\dim(\mathrm{Isom}(G))=3$.

Conclusions and open problems

Regarding Chapters 2 and 3, it is natural to consider some remaining cases. For translation surfaces in $\mathbb{H}^3$, it would be interesting to investigate minimal surfaces and solitons of the mean curvature flow generated by generic curves in $\mathbb{H}^3$, that is, without assuming the curves lie in specific subsets. This problem appears to be particularly difficult to address using the approach adopted here, due to the Lie group structure of $\mathbb{H}^3$. Concerning translation surfaces in $\mathbb{S}^3$, a natural question related to the final theorem is whether there exist minimal surfaces generated by curves that are general helices but not proper helices. One could study both situations: when the generating curves are proper helices and when they are general helices, in the context of constant mean curvature. Moreover, it would be worthwhile to explore how the structure of translation surfaces can be used to construct examples and obtain classification results for solitons of the mean curvature flow in $\mathbb{S}^3$.

It is natural to consider the classification of homogeneous hypersurfaces in four-dimensional Thurston geometries that are not space forms or products of lower-dimensional space forms. In fact, the author of this thesis, in joint work with his advisor J.P. dos Santos and M. Domínguez-Vázquez, has partially carried out such a classification and hopes to complete this work in the future. Another possible direction is the classification of codimension-two actions on these geometries, that is, the study of two-dimensional homogeneous manifolds and foliations.

It is also natural to ask about the classification of homogeneous hypersurfaces in five-dimensional Thurston geometries. Indeed, according to [22], there is a classification of such geometries, but the number of examples is significantly larger than in lower dimensions.

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