University of Brasília
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Almost periodic solutions for isolated time scales and applications

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Dissertation submitted to the Department of Mathematics of the University of Brasília, as part of the requirements for obtaining a Master's degree in Mathematics.

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by

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"On becoming famililar with difference equations and their close relation to differential equations, I was in hopes that the theory of difference equations could be brought completely abreast with that for ordinary differential equations."
(Hugh L. Turrittin, 1973)

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Ronaldo Pereira Murakami Filho

Abstract

In this work, we introduce a general concept of almost periodicity for functions defined on isolated time scales. Our concept is consistent with the existing concepts of almost periodicity on quantum calculus and on $\mathbb{Z}$. Also, we prove important properties such as the equivalence between different definitions for almost periodic functions, as well as results ensuring the existence of almost periodic solutions for dynamic equations on time scales under certain properties. We present several examples to illustrate our definition and main results. All the results can be found in [4, 5, 7, 9, 10].

Keywords: almost periodicity; isolated time scales.

Resumo

Neste trabalho, introduzimos um conceito geral de quase-periodicidade para funções definidas em escalas temporais isoladas. Nosso conceito é compatível com os já existentes conceitos de quase-periodicidade no cálculo quântico e em $\mathbb{Z}$. Ademais, provamos propriedades importantes, como a equivalência entre diferentes tipos de definições de funções quase-periódicas, juntamente com resultados garantindo a existência de soluções quase-periódicas para equações dinâmicas em escalas temporais sob certas condições. Todos os resultados podem ser encontrados em [4, 5, 7, 9, 10].

Palavras-chave: quase-periodicidade; escalas temporais isoladas.

Notation

$\mathbb{T}$	Time scale
σ	Forward jump operator
μ	Graininess function
$\mathbb{T}^\kappa$	$\mathbb{T}^\kappa$ scale
$x^\Delta(t)$	Delta derivative of x on t
$G(\mathbb{T},\mathbb{R})$	Set of all regulated functions
$C_{\text{rd}}(\mathbb{T},\mathbb{R})$	Set of all rd-continuous functions
$\int_a^b x(t)\Delta t$	Delta integral of x from a to b
$\xi_h(z)$	Cylinder transformation
$\mathcal{R}$	Set of all regressive and rd-continuous functions
$\oplus$	Circle plus addition
$\ominus$	Circle minus subtraction
$\mathcal{R}^+$	Set of all positively regressive and rd-continuous functions
$e_p(t,s)$	Exponential function on time scale
$\nu(t)$	Iterated shift operator
$\mathcal{P}$	Set of all periodic functions
AP_B	Set of all Bochner almost periodic and regressive functions
AP	Set of all almost periodic functions

Contents

Introduction

The class of almost periodic functions was introduced by Harald Bohr [11, 12, 13] in 1925 as a natural extension of periodicity to describe more general kinds of phenomena that may not be captured by the concept of periodicity, by calling a continuous function f almost periodic if for any $\varepsilon > 0$, there exists a length $l(\varepsilon) > 0$ with the property that any interval of length $l(\varepsilon)$ on the real line contains at least one point ω such that $|f(t+\omega) - f(t)| < \varepsilon$ for all $t \in \mathbb{R}$.

Later, in 1927, Salomon Bochner [2] gave another definition of almost periodic functions, stating that a continuous function is almost periodic if it can be approximated by a trigonometric polynomial (see [14, Page 9]). This definition is equivalent to saying that for any sequence $\{f(t+h_n)\}_{n\in\mathbb{N}}$, where $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, one can extract a uniformly convergent subsequence on $\mathbb{R}$. Also, it is equivalent to Bohr's definition (see [14, Page 14]).

A good example of almost periodicity lies in celestial mechanics, as the description of motion of planets. For instance, let us consider the method of epicycles to describe the motion of the moon provided by Ptolemy: Assume E is the earth and M is the moon. For Ptolemy, a good way to describe this motion was by using two circles for his epicycle model, where the motion of M can be described by the function $p(t) = \pi_1 e^{i\lambda_1 t} + \pi_2 e^{i\lambda_2 t}$ for some constants $\lambda_1, \lambda_2 \in \mathbb{R}$.

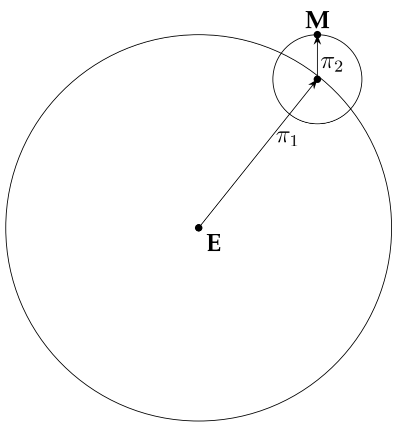

Figure 1: Ptolemy epicycle model

Later, Copernicus showed that by adding a second epicycle, one could get a better approximation to the observed data. This suggests that considering the function $p(t) = \sum_{j=1}^{n} \pi_j e^{i\lambda_j t}$ we can get a better approximation for this motion. In this scenario, if $\lambda_1, \ldots, \lambda_n$ are not all rational multiples of just one real number, then the function p is not periodic. However, p is always almost periodic (in Bohr's and Bochner's sense, see [18]).

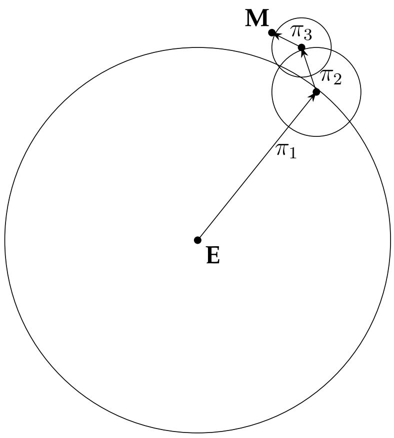

Figure 2: Copernicus epicycle model

The theory of almost periodic functions gave an important input to the development of harmonic analysis on groups, being crucial in this treatment (see [17]). On the other hand, general versions of averaging principles when the function is not periodic were provided, only addressing almost periodicity condition to the function (see [20]).

In recent years, this theory has been developed in connection with problems of differential equations, stability theory, dynamical system, amongs others (see [4, 17, 27]).

On the other hand, the theory of dynamic equations on time scales was first introduced by Stefan Hilger in 1988 in his PhD thesis (see [21]) in order to unify continuous and discrete analysis, as well as all the cases "in between". This theory, which has garnered significant attention for its power of generalization and applicability since then (see [1, 6, 7, 8, 15, 16, 24]), will be explored in the first chapter in order to study the concepts of periodicity and almost periodicity on isolated time scales. Despite this, some basic questions within the theory remain open. For instance, a general definition of almost periodic functions on any time scales is not clear yet. Recent progress in related frameworks includes the adaptation of almost periodicity to quantum calculus by Li [24] and Bohner and Mesquita [4] to deal with q-difference equations, since such equations hold substantial potential for applications in quantum physics, including thermostatics of q-bosons, black hole dynamics and other topics (see [23, 28]), and it is not included among the additivity time scales.

Moreover, the almost periodicity concept on time scales was first introduced in the literature requiring $\mathbb{T}$ to be an almost periodic time scale, that is, $\Pi := \{\tau \in \mathbb{R} \,:\, t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}$ (see [26]). With this definition, for an almost periodic time scale, one can define a function $f\colon \mathbb{T} \times H \to \mathbb{R}$ (where H is open in $\mathbb{R}^n$ or $\mathbb{C}^n$) as almost periodic in t uniformly for $x \in H$ if the ε-translation set of f

$$E\{\varepsilon, f, S\} := \{\tau \in \Pi \,:\, |f(t+\tau, x) - f(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{T} \times S\}$$

is relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset $S \subset H$. This concept of almost periodicity depends on the definition of almost periodic time scale, which is extremely restritive and does not cover one of the most interesting examples of time scales, the quantum time scale. Thus, the main challenge here for the generalization is due to the additive property that appears in the definition of almost periodic functions. Since there are many interesting time scales which do not have such property, it is necessary to find a definition which can fit in all these cases, and can include also the ones that do not have the additivity property.

Alternatively, a specific definition of almost periodic functions for the quantum case was established in 2018 by Bohner and Mesquita [4], based on the conservation of area property for periodic functions (see [3]), to study q-difference equations within this framework. Also, in 2019, Li [24] gave a definition of almost periodicity for the quantum time scale. Moreover, another advance on almost periodicity on time scales includes the new definition of almost periodicity on any time scale provided by Li and Huang, which are equivalent to Bohr's and Bochner's definition on almost periodic time scales (see [25]).

Based on the study of almost periodicity on quantum calculus from [4] and with the analogue techniques used to describe periodicity by means of area in [7] (where both will be explored in Chapter 2 of this work), our primary goal is to generalize the concept of almost periodicity from the quantum case to any isolated time scale (i.e., all points of the time scale are right-scattered and left-scattered, except when the time scale has a minimum or maximum (or both). In this case, the minimum point must be right-scattered and the maximum point must be left-scattered) in the last chapter of this work, in order to extend the study for more tools in the setting of time scale theory. The two classical definitions of almost periodicity (the Bochner's and Bohr's ones) presented here for isolated time scales are consistent with the known ones for the discrete and quantum calculus settings. Establishing those definitions, we prove that any

Bohr almost periodic function is also Bochner almost periodic and the reciprocal also remains true under more general hypothesis, allowing us to call those functions just as almost periodic. Moreover, we prove many properties for this class of almost periodic functions. Also, we show that the set of Bochner almost periodic functions with operation $\oplus$ is a subgroup of $(\mathcal{R}, \oplus)$, which shows that our definition makes sense in the time scale context. In addition, we explicit the relation between the exponential function and the almost periodicity concept, and we state some others equivalences for this class of functions, using almost periodic functions defined on $\mathbb{Z}$ and $\mathbb{R}$. Furthermore, we establish the hypothesis for the first order linear dynamic equation

$$X^{\Delta}(t) = A(t)X(t) + f(t)$$

to have an Bochner almost periodic solution, where $A, B \colon \mathbb{T} \to \mathbb{R}^{n\times n}$. All the results concerning almost periodicity on isolated time scales presented in the last chapter are completely new in the literature and they are contained in [5].

Chapter 1

Preliminaries on time scale theory

In this chapter, we recall fundamental concepts within the framework of time scale theory, which will be necessary through the text. The first section will deal with the calculus theory in this setting and we present the foundations for derivation and integration, the mean value theorem, chain rule theorems and a substitution rule in this context. Moreover, the class of regressive functions and the definition of the exponential function on time scales will be introduced, along with the Variation of Constants Formula which will be useful later in the next chapter. The main references for this chapter are both [9] and [10].

1.1 Calculus on time scales

This first section provides a detailed introduction to differential and integral calculus in the context of time scales.

1.1.1 Basic definitions and the induction principle

First, some fundamental definitions will be presented, followed by the classification of points that will be used in the whole text. After, we provide some examples for illustrate and we prove the induction principle.

Definition 1.1.1 (See [10, Page 1]). *A **time scale** is any nonempty closed subset of real numbers (with the usual topology). The notation $\mathbb{T} \subset \mathbb{R}$ will be used in the whole text.*

To avoid confusion dealing with intervals, we will adopt the notation $(a,b)_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ and $[a,b]_{\mathbb{T}}$ to represent $(a,b) \cap \mathbb{T}$, $[a,b) \cap \mathbb{T}$, $(a,b] \cap \mathbb{T}$ and $[a,b] \cap \mathbb{T}$, respectively.

Some simple examples of time scales are $\mathbb{R}, \mathbb{Z}$ and $h\mathbb{Z} := \{hk \ : \ k \in \mathbb{Z}\}$, for $h > 0$. Furthermore, an interesting example of time scale is $\overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$ for $q > 1$, where $q^{\mathbb{Z}} := \{q^k \ : \ k \in \mathbb{Z}\}$. Also, the time scale $q^{\mathbb{N}_0}$ for $q > 1$ will be widely used in this text and it is known as the *quantum time scale*, which plays a crucial role for *quantum calculus* theory and its applications for quantum physics (see [19]). Also, Cantor set is an example of time scale which is useful to find counterexamples.

Next, we define two important operators used to introduce certain basic concepts in the theory.

Definition 1.1.2 (See [10, Definition 1.1]). *Let $\mathbb{T}$ be a time scale. The* ***forward jump operator*** *is a function $\sigma\colon \mathbb{T} \to \mathbb{T}$ given by*

$$\sigma(t) := \inf\{s \in \mathbb{T} \ : \ s > t\}$$

and, analogously, we also define the ***backward jump operator*** *$\rho\colon \mathbb{T} \to \mathbb{T}$ as*

$$\rho(t) := \sup\{s \in \mathbb{T} \ : \ s < t\}.$$

Since $\mathbb{T}$ is closed, it is clear that these operators are well-defined. Moreover, for the cases when the sets $\{s \in \mathbb{T} \ : \ s > t\}$ and $\{s \in \mathbb{T} \ : \ s < t\}$ are empty, we consider

$$\inf\{s \in \mathbb{T} \ : \ s > t\} = \inf \emptyset = \sup \mathbb{T},$$

$$\sup\{s \in \mathbb{T} \ : \ s < t\} = \sup \emptyset = \inf \mathbb{T}.$$

Once defined the operators, we will denote the following classification for the points $t \in \mathbb{T}$ as follow:

Table 1.1: Classification of points

Definition	Property
t *right-scattered*	$t < \sigma(t)$
t *right-dense*	$t = \sigma(t)$ and $t < \sup \mathbb{T}$
t *left-scattered*	$t > \rho(t)$
t *left-dense*	$t = \rho(t)$ and $t > \inf \mathbb{T}$
t *isolated*	$\rho(t) < t < \sigma(t)$
t *dense*	$\rho(t) = t = \sigma(t)$

Definition 1.1.3 (See [10, Page 2]). *Let $\mathbb{T}$ be a time scale. The* ***graininess function*** *$\mu : \mathbb{T} \to [0, +\infty)$ is defined as $\mu(t) := \sigma(t) - t$.*

In what follows, we give some examples for the forward jump and backward jump operators and also for the graininess function for some time scales.

Table 1.2: Some basic examples

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$h\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$
$\rho(t)$	t	$t-1$	$t-h$	$\frac{t}{q}$
$\sigma(t)$	t	$t+1$	$t+h$	qt
$\mu(t)$	0	1	h	$t(q-1)$

We state and prove the induction principle, which will be crucial for the proof of the mean value theorem presented later.

Theorem 1.1.4 (See [10, Theorem 1.7]). *(Induction Principle). Let $t_0 \in \mathbb{T}$ and assume that*

$$\{S(t) \ : \ t \in [t_0, \infty)_{\mathbb{T}}\}$$

is a family of statements satisfying:

(i) The statement $S(t_0)$ is true;

(ii) If $t \in [t_0, \infty)_{\mathbb{T}}$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is true;

(iii) If $t \in [t_0, \infty)_{\mathbb{T}}$ is right-dense and $S(t)$ is true, then there exists a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t, \infty)_{\mathbb{T}}$;

(iv) If $t \in (t_0, \infty)_{\mathbb{T}}$ is left-dense and $S(s)$ is true for all $s \in [t_0, t)_{\mathbb{T}}$, then $S(t)$ is true.

Then, $S(t)$ is true for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Proof. Let

$$S^* = \{t \in [t_0, \infty)_{\mathbb{T}} \ : \ S(t) \text{ is not true}\}$$

and assume that $S^* \neq \emptyset$. Since S^* is nonempty and limited above, there exists $t^* := \inf S^*$ and since $\mathbb{T}$ is closed, we have $t^* \in \mathbb{T}$. Thus, $S(t^*)$ is true. Indeed, if $t^* = t_0$, then, from (i), $S(t^*)$ is true. If $t^* \neq t_0$ and $\rho(t^*) = t^*$, then $S(t^*)$ is true from (iv). Also, if $\rho(t^*) < t^*$, then $S(\rho(t^*))$ is true and then, from (ii), $S(\sigma(\rho(t^*))) = S(t^*)$ is true. Hence, in any case, $t^* \notin S^*$. Therefore, t^* cannot be right-scattered (because if t^* is right-scattered, from (ii), $S(\sigma(t^*))$ is true and $\sigma(t^*) \in S^*$, which cannot happen) and $t^* \neq \max \mathbb{T}$ either (it follows from the fact that $S^* \neq \emptyset$, t^* is its infimum and $t^* \notin S^*$). Hence t^* is right-dense and it is a contradiction by (iii). Thus, $S^* = \emptyset$ and we have the desired. $\square$

1.1.2 Differentiation and integration

We now introduce the derivative and the integral in this context, the so-called delta derivative and delta integral, respectively, and will be established some of its properties.

Definition 1.1.5 (See [10, Page 2]). *Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function. Then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.*

Definition 1.1.6 (See [10, Page 2]). *We define the $\mathbb{T}^\kappa$ **set** by*

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \textit{if} \;\; \sup \mathbb{T} < +\infty, \\ \mathbb{T}, & \textit{if} \;\; \sup \mathbb{T} = +\infty. \end{cases}$$

In other words, the above definition removes the maximum point of the time scale $\mathbb{T}$ if this maximum exists and is left-scattered. This is needed to have a well-defined derivative in this context, which will be justified later after its definition.

Definition 1.1.7 (See [10, Definition 1.10]). *Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number with the property that given $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t-\delta, t+\delta)_\mathbb{T}$ for some $\delta > 0$) such that*

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon \left|\sigma(t) - s\right|$$

*for all $s \in U$. We call $f^\Delta(t)$ as **delta derivative** of f at t. Moreover, we say that f is **delta differentiable** (or just **differentiable**) on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Also, the function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the **derivative** of f on $\mathbb{T}^\kappa$.*

Remark 1.1.8. *The above definition is well-defined. Indeed, let $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. If $\alpha, \beta \in \mathbb{R}$ both satisfy the definition of delta derivative of f at t, then for every $\varepsilon > 0$, there exists a neighborhood $U = (t-\delta, t+\delta)_\mathbb{T}$, for some $\delta > 0$, such that*

$$\begin{aligned} |(\alpha - \beta)[\sigma(t) - s]| &= |f(\sigma(t)) - f(s) - f(\sigma(t)) + f(s) + (\alpha - \beta)[\sigma(t) - s]| \\ &\leq |f(\sigma(t)) - f(s) - \alpha[\sigma(t) - s]| + |f(\sigma(t)) - f(s) - \beta[\sigma(t) - s]| \\ &\leq 2\varepsilon|\sigma(t) - s| \end{aligned}$$

for all $s \in U$. Since $\varepsilon > 0$ is arbitrary, we have $\alpha = \beta$ or $\sigma(t) = s$ for all $s \in U$. This second option cannot happen for $t \in \mathbb{T}^\kappa$ and hence $\alpha = \beta$.

The next remark shows that the delta derivative is not well defined on $\mathbb{T} \setminus \mathbb{T}^\kappa$.

Remark 1.1.9. *Suppose $\mathbb{T} \setminus \mathbb{T}^\kappa \neq \emptyset$ and let $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$. Thus, t is left-scattered and the maximum of $\mathbb{T}$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $U = (t-\delta, t+\delta)_{\mathbb{T}} = \{t\}$ and for any $\alpha \in \mathbb{R}$, we have*

$$|[f(\sigma(t)) - f(s)] - \alpha[\sigma(t) - s]| = |[f(t) - f(s)] - \alpha[t - s]| = 0 \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

The next theorem plays an important role for computation of the delta derivative explicitly and shows its relation with the well-known derivatives in the continuous and discrete cases.

Theorem 1.1.10 (See [10, Theorem 1.16]). *Let $f \colon \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then, the following statements hold:*

(i) If f is Δ-differentiable at t, then f is continuous at t;

(ii) If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}; \tag{1.1}$$

(iii) If t is right-dense, then f is Δ-differentiable at t if, and only if, the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}; \tag{1.2}$$

(iv) If f is Δ-differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{1.3}$$

Proof. (i) Let $\varepsilon \in (0,1)$, without loss of generality, and define

$$\varepsilon^* := \frac{\varepsilon}{[2\mu(t) + 1 + |f^\Delta(t)|]} \in (0,1).$$

Since f is Δ-differentiable at t, there exists a neighborhood $U = (t - \delta, t +$

$\delta)_{\mathbb{T}},$ for some $\delta > 0$ with $\delta < \varepsilon^*$ (without loss of generality), of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon^* \left|\sigma(t) - s\right|,$$

holds for all $s \in U$. Note that

$$\begin{aligned}
|f(t) - f(s)| &= |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) + f^{\Delta}(t)[\sigma(t) - s] - f^{\Delta}(t)[\sigma(t) - s]| \\
&= |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) + f^{\Delta}(t)[\mu(t) + t - s] \\
&\quad - f^{\Delta}(t)[\sigma(t) - s]| \\
&\leq |f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| + |f(\sigma(t)) - f(t) - f^{\Delta}(t)\mu(t)| \\
&\quad + |f^{\Delta}(t)[t - s]| \\
&\leq \varepsilon^*|\sigma(t) - s| + \varepsilon^*|\mu(t)| + |f^{\Delta}(t)| \cdot |t - s|.
\end{aligned}$$

But since

$$|\sigma(t) - s| = |\sigma(t) - t + t - s| \leq |\sigma(t) - t| + |t - s| = |\mu(t)| + |t - s|,$$

we conclude

$$\begin{aligned}
|f(t) - f(s)| &\leq \varepsilon^*(|\mu(t)| + |t - s|) + \varepsilon^*|\mu(t)| + |f^{\Delta}(t)| \cdot |t - s| \\
&< \varepsilon^*(|\mu(t)| + |t - s|) + \varepsilon^*|\mu(t)| + |f^{\Delta}(t)|\varepsilon^* \\
&< \varepsilon^*[2\mu(t) + 1 + |f^{\Delta}(t)|] = \varepsilon.
\end{aligned}$$

This shows that f is continuous at t.

(ii) Since f is continuous at t, $\lim_{s\to t} f(s) = f(t)$. On the other hand, since t is right-scattered, $\sigma(t) > t$ and $\mu(t) \neq 0$. Hence, we have

$$\lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

By the definition of limit, given $\varepsilon > 0$, there exists a neighborhood $U = (t-\delta, t+\delta)_{\mathbb{T}}$ of t such that if $s \in U$, then

$$\left|\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - \frac{f(\sigma(t)) - f(t)}{\mu(t)}\right| \leq \varepsilon.$$

Thus, multiplying this inequality by $|\sigma(t) - s|$, we conclude that

$$\left| f(\sigma(t)) - f(s) - \frac{f(\sigma(t)) - f(t)}{\mu(t)}[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|,$$

i.e., $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If f is Δ-differentiable at t and t is right-dense, then given $\varepsilon > 0$ there exists a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ of t such that if $s \in U$, then

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|,$$

i.e.,

$$|f(t) - f(s) - f^{\Delta}(t)[t - s]| \leq \varepsilon |t - s|.$$

Multiplying this expression by $\left|\frac{1}{t-s}\right|$ (for $s \neq t$), we have

$$\left| \frac{f(t) - f(s)}{t - s} - f^{\Delta}(t) \right| \leq \varepsilon,$$

for all $s \in U$. Therefore, $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$.

Reciprocally, if the limit $\lim_{s \to t} \frac{f(\sigma(t))-f(s)}{t-s}$ exists, then given $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ of t such that if $s \in U$ with $s \neq t$, then

$$\left| \frac{f(t) - f(s)}{t - s} - f^{\Delta}(t) \right| \leq \varepsilon,$$

for some value $f^{\Delta}(t)$. Thus, note that

$$\begin{aligned} |f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| &= |f(t) - f(s) - f^{\Delta}(t)[t - s]| \frac{|t - s|}{|t - s|} \\ &= \left| \frac{f(t) - f(s)}{t - s} - f^{\Delta}(t) \right| |t - s| \\ &\leq \varepsilon |t - s| \end{aligned}$$

for all $s \in U$ with $s \neq t$. Therefore, f is Δ-differentiable at t and

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s}.$$

(iv) If t is right-dense the equality is trivially satisfied.

Moreover, if t is right-scattered and since f is Δ-differentiable at t, by (i) f is continuous at t and by (ii), we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

i.e.,

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

□

The usual calculus rules for Δ-derivatives also holds with similarity in this context as it can be viewed in the next theorem.

Theorem 1.1.11 (See [10, Theorem 1.20]). *Let $f, g \colon \mathbb{T} \to \mathbb{R}$ be Δ-differentiable functions at $t \in \mathbb{T}^{\kappa}$. Then, the following statements hold:*

(i) $f + g$ is Δ-differentiable at t with

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t);$$

(ii) For any constant $\alpha \in \mathbb{R}$, αf is Δ-differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t);$$

(iii) The product fg is Δ-differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)); \tag{1.4}$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is Δ-differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}; \tag{1.5}$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \tag{1.6}$$

Proof. (i) Since f and g are Δ-differentiable at $t \in \mathbb{T}^\kappa$, given $\varepsilon > 0$, there exist neighborhoods U_1 and U_2 of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \frac{\varepsilon}{2}|\sigma(t) - s| \text{ for all } s \in U_1$$

and

$$|g(\sigma(t)) - g(s) - g^\Delta(t)[\sigma(t) - s]| \leq \frac{\varepsilon}{2}|\sigma(t) - s| \text{ for all } s \in U_2.$$

Thus, for $U = U_1 \cap U_2$ and $s \in U$, we have

$$\begin{aligned}
&|(f+g)(\sigma(t)) - (f+g)(s) - [f^\Delta(t) + g^\Delta(t)][\sigma(t) - s]| \\
&\quad \leq |f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| + |g(\sigma(t)) - g(s) - g^\Delta(t)[\sigma(t) - s]| \\
&\quad \leq \frac{\varepsilon}{2}|\sigma(t) - s| + \frac{\varepsilon}{2}|\sigma(t) - s| \\
&\quad = \varepsilon|\sigma(t) - s|.
\end{aligned}$$

Therefore, $f + g$ is Δ-differentiable at t and $(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.

(ii) If $\alpha = 0$, the equality is trivially satisfied, since $(\alpha f)^\Delta(t) = \alpha f^\Delta(t) = 0$. Now, if $\alpha \neq 0$, since f is Δ-differentiable at $t \in \mathbb{T}^\kappa$, given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \frac{\varepsilon}{|\alpha|}|\sigma(t) - s| \text{ for all } s \in U.$$

Thus,

$$\begin{aligned}
|(\alpha f)(\sigma(t)) - (\alpha f)(s) - [\alpha f^\Delta(t)][\sigma(t) - s]| &= |\alpha| \cdot |f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \\
&\leq |\alpha| \cdot \frac{\varepsilon}{|\alpha|}|\sigma(t) - s| \\
&= \varepsilon|\sigma(t) - s|.
\end{aligned}$$

Therefore, αf is Δ-differentiable at t and $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.

(iii) Let $\varepsilon \in (0, 1)$ and define

$$\varepsilon^* = \frac{\varepsilon}{|g(\sigma(t))| + |f(t)| + 1 + |g^\Delta(t)|} \in (0, 1).$$

Since f and g are Δ-differentiable at $t \in \mathbb{T}^\kappa$, there exist neighborhoods U_1 and U_2

of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon^* \, |\sigma(t) - s| \ \text{ for all } \ s \in U_1$$

and

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon^* \, |\sigma(t) - s| \ \text{ for all } \ s \in U_2.$$

Moreover, by Theorem 1.1.10 (i), f and g are continuous at t and there exists a neighborhood U_3 of t such that

$$|f(t) - f(s)| \leq \varepsilon^* \ \text{ for all } \ s \in U_3.$$

Thus, for $U = U_1 \cap U_2 \cap U_3$ and $s \in U$, then

$$\begin{aligned}
&|(fg)(\sigma(t)) - (fg)(s) - [f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t)][\sigma(t) - s]| \\
&\quad = |f(\sigma(t))g(\sigma(t)) - f(s)g(s) - f^{\Delta}(t)[\sigma(t) - s]g(\sigma(t)) - f(t)[\sigma(t) - s]g^{\Delta}(t)| \\
&\quad = |f(\sigma(t))g(\sigma(t)) - f(s)g(s) - f^{\Delta}(t)[\sigma(t) - s]g(\sigma(t)) - f(t)[\sigma(t) - s]g^{\Delta}(t) \\
&\qquad + f(s)g(\sigma(t)) - f(s)g(\sigma(t)) + g(\sigma(t))f(t) - g(\sigma(t))f(t) + g(s)f(t) \\
&\qquad - g(s)f(t) + g^{\Delta}(t)[\sigma(t) - s][f(s) - f(t)] - g^{\Delta}(t)[\sigma(t) - s][f(s) - f(t)]| \\
&\quad = |[f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]]g(\sigma(t)) + [g(\sigma(t)) - g(s) \\
&\qquad - g^{\Delta}(t)[\sigma(t) - s]]f(t) + [g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t)) - s]][f(s) - f(t)] \\
&\qquad + [\sigma(t) - s]g^{\Delta}(t)[f(s) - f(t)]| \\
&\quad \leq |[f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]]g(\sigma(t))| + |[g(\sigma(t)) - g(s) \\
&\qquad - g^{\Delta}(t)[\sigma(t) - s]]f(t)| + |[g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t)) - s]][f(s) - f(t)]| \\
&\qquad + |[\sigma(t) - s]g^{\Delta}(t)[f(s) - f(t)]| \\
&\quad \leq \varepsilon^*|\sigma(t) - s| \cdot |g(\sigma(t))| + \varepsilon^*|\sigma(t)) - s| \cdot |f(t)| + \varepsilon^*|\sigma(t) - s| \cdot \varepsilon^* \\
&\qquad + |[\sigma(t) - s]g^{\Delta}(t)|\varepsilon^* \\
&\quad = \varepsilon^*|\sigma(t) - s|[|g(\sigma(t))| + |f(t)| + \varepsilon^* + |g^{\Delta}(t)|] \\
&\quad < \varepsilon^*|\sigma(t) - s|[|g(\sigma(t))| + |f(t)| + 1 + |g^{\Delta}(t)|] \\
&\quad \leq \varepsilon|\sigma(t) - s|.
\end{aligned}$$

Therefore, fg is Δ-differentiable at t and $(fg)^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$. The proof of the second equality follows analogous, thus we omit it here.

(iv) Firstly, suppose that t is right-dense. Since f is Δ-differentiable at t, we have

$$f^{\Delta}(t) \overset{(1.2)}{=} \lim_{t\to s} \frac{f(t)-f(s)}{t-s}.$$

Thus, we conclude that the limit

$$\begin{aligned}\lim_{s\to t} \frac{\left(\frac{1}{f}\right)(t) - \left(\frac{1}{f}\right)(s)}{t-s} &= \lim_{s\to t} \frac{f(s)-f(t)}{f(t)f(s)(t-s)} \\ &= -\lim_{s\to t} \frac{1}{f(t)f(s)} \frac{f(t)-f(s)}{t-s} \\ &= -\frac{f^{\Delta}(t)}{f(t)f(t)} = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}\end{aligned}$$

exists and, therefore,

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

Now, if t is right-scattered, we have

$$f^{\Delta}(t) \overset{(1.1)}{=} \frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

Since f is continuous at t, the function $1/f$ is continuous at t and we get

$$\begin{aligned}\left(\frac{1}{f}\right)^{\Delta}(t) &\overset{(1.1)}{=} \frac{\left(\frac{1}{f}\right)(\sigma(t)) - \left(\frac{1}{f}\right)(t)}{\mu(t)} \\ &= \frac{f(t)-f(\sigma(t))}{f(t)f(\sigma(t))\mu(t)} = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.\end{aligned}$$

(v) Since f and g are Δ-differentiable at t and $g(t)g(\sigma(t)) \neq 0$, by the previous items (iii) and (iv), we have

$$\begin{aligned}\left(\frac{f}{g}\right)^{\Delta}(t) &= \left(f \cdot \frac{1}{g}\right)^{\Delta}(t) = f(t)\left(\frac{1}{g}\right)^{\Delta}(t) + f^{\Delta}(t)\left(\frac{1}{g(\sigma(t))}\right) \\ &= f(t)\left[-\frac{g^{\Delta}(t)}{g(t)g(\sigma(t))}\right] + \frac{f^{\Delta}(t)}{g(\sigma(t))} \\ &= -\frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))},\end{aligned}$$

getting the desired result. □

Now, we present the fundamental definitions of functions defined on time scales, in order to introduce the concept of delta integral.

Definition 1.1.12 (See [10, Definition 1.57]). *A function $f\colon \mathbb{T} \to \mathbb{R}$ is called* ***regulated*** *provided its right-sided limit exist at all right-dense points in $\mathbb{T}$ and its left-sided limit exist at left-dense points in $\mathbb{T}$. The set of all regulated functions $f\colon \mathbb{T} \to \mathbb{R}$ will be denoted by $G(\mathbb{T}, \mathbb{R})$ or simply G.*

Definition 1.1.13 (See [10, Definition 1.58]). *A function $f\colon \mathbb{T} \to \mathbb{R}$ is called* ***rd-continuous*** *provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exist at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f\colon \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$ or simply C_{rd}.*

The next theorem states some straightforward, but useful implications.

Theorem 1.1.14 (See [10, Theorem 1.60]). *Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function. Then:*

(i) If f is continuous, then f is rd-continuous;

(ii) If f is rd-continuous, then f is regulated;

(iii) The forward jump operator σ is rd-continuous;

(iv) If f is regulated or rd-continuous, then f^σ has the same property.

Proof. (i) and (ii) follows directly by its definition.

(iii) Let $t \in \mathbb{T}$ be a right-dense point. Given $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ such that $\sigma(s) = s$ for all $s \in U = (t - \delta, t + \delta)$. Then, for $s \in U$, we have

$$\sigma(s) - \sigma(t) = s - t < \delta < \varepsilon.$$

Also, since $t - \delta < s < t + \delta$, we get $-\delta < s - t$. Thus,

$$-\varepsilon < -\delta < s - t = \sigma(s) - \sigma(t).$$

Hence, for $s \in U$, we conclude that $|\sigma(s) - \sigma(t)| < \varepsilon$, i.e., σ is continuous for all its right-dense points. Furthermore, it is easy to see that the left-sided limit always exists for its left-dense points.

Therefore, we conclude that σ is rd-continuous.

(iv) Suppose f is regulated. By the previous item, since σ is also regulated, the lateral limits $\lim_{s\to t^+} f(s)$ and $\lim_{s\to t^+} \sigma(s)$ both exist for all right-dense points t. Also, the lateral limits $\lim_{s\to t^-} f(s)$ and $\lim_{s\to t^-} \sigma(s)$ both exist for all left-dense points t. Thus, the limit $\lim_{s\to t^+} f(\sigma(t))$ exists for all right-dense points t and the limit $\lim_{s\to t^-} f(\sigma(t))$ exists for all left-dense points t, i.e., f^σ is regulated.

If f is rd-continuous, it is easy to prove that f^σ is also rd-continuous. □

Now, we are ready to define the concept of pre-differentiable function.

Definition 1.1.15 (See [10, Definition 1.62]). *A continuous function $f\colon \mathbb{T} \to \mathbb{R}$ is called* ***pre-differentiable*** *with region of differentiation D, provided D satisfy: (i) $D \subset \mathbb{T}^\kappa$; (ii) $\mathbb{T}^\kappa \setminus D$ is countable; (iii) $\mathbb{T}^\kappa \setminus D$ contains no right-scattered elements of $\mathbb{T}$; (iv) f is Δ-differentiable at each $t \in D$.*

The next theorem is an analogue version of the Mean Value Theorem for time scales, which its corollary will be extremely useful.

Theorem 1.1.16 (See [10, Theorem 1.67]). *(Mean Value Theorem). Let $f, g\colon \mathbb{T} \to \mathbb{R}$ be pre-differentiable functions with region of differentiation D satisfying*

$$|f^\Delta(t)| \le g^\Delta(t) \ \textit{for all} \ \ t \in D.$$

Then, for all $r, s \in \mathbb{T}$ such that $r \le s$, we have

$$|f(s) - f(r)| \le g(s) - g(r).$$

Proof. Let $r, s \in \mathbb{T}$ with $r \le s$ and denote $[r, s)_\mathbb{T} \setminus D = \{t_n; n \in \mathbb{N}\}$ ($\mathbb{T}^\kappa \setminus D$ is countable, implying $[r, s)_\mathbb{T} \setminus D$ is also countable). Given $\varepsilon > 0$, we just need to show that

$$S(t):\ |f(t) - f(r)| \le g(t) - g(r) + \varepsilon \left(t - r + \sum_{t_n < t} \frac{1}{2^n} \right)$$

holds for all $t \in [r, s]_\mathbb{T}$. Indeed, note that:

(i) $S(r)$ holds, because

$$|f(r) - f(r)| = 0 \le g(r) - g(r) + \varepsilon \left(r - r + \sum_{t_n < r} \frac{1}{2^n} \right) = \varepsilon \left(\sum_{t_n < r} \frac{1}{2^n} \right).$$

(ii) If t is right-scattered and $S(t)$ holds, then $t \in D$ (because $\mathbb{T}^\kappa \setminus D$ contains no right-scattered elements of $\mathbb{T}$) and we have

$$\begin{aligned}
|f(\sigma(t)) - f(r)| &\overset{(1.3)}{=} |f(t) + \mu(t)f^\Delta(t) - f(r)| \\
&\leq |\mu(t)f^\Delta(t)| + |f(t) - f(r)| \\
&\leq \mu(t)g^\Delta(t) + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} \frac{1}{2^n}\right) \\
&= g(\sigma(t)) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t} \frac{1}{2^n}\right) \\
&< g(\sigma(t)) - g(r) + \varepsilon\left(\sigma(t) - r + \sum_{t_n<\sigma(t)} \frac{1}{2^n}\right).
\end{aligned}$$

Thus, $S(\sigma(t))$ holds.

(iii) If t is right-dense, with $t \neq s$, and $S(t)$ holds, considering $t \in D$, we have that f and g are Δ-differentiable at t, i.e., there exists a neighborhood U of t such that

$$|f(t) - f(\tau) - f^\Delta(t)[t - \tau]| \leq \frac{\varepsilon}{2}|t - \tau| \text{ for all } \tau \in U,$$

and

$$|g(t) - g(\tau) - g^\Delta(t)[t - \tau]| \leq \frac{\varepsilon}{2}|t - \tau| \text{ for all } \tau \in U.$$

Thus, by reverse triangle inequality, we have

$$|f(t) - f(\tau)| \leq \left[|f^\Delta(t)| + \frac{\varepsilon}{2}\right]|t - \tau| \text{ for all } \tau \in U,$$

and also we get

$$g^\Delta(t)[\tau - t] \leq g(\tau) - g(t) + \frac{\varepsilon}{2}|t - \tau| \text{ for all } \tau \in U.$$

Therefore, for all $\tau \in U \cap (t, \infty)_\mathbb{T}$, we have

$$\begin{aligned}
|f(\tau) - f(r)| &\leq |f(\tau) - f(t)| + |f(t) - f(r)| \\
&\leq \left[|f^\Delta(t)| + \frac{\varepsilon}{2}\right]|t - \tau| + |f(t) - f(r)|
\end{aligned}$$

$$\begin{aligned}
&\leq \left[g^{\Delta}(t) + \frac{\varepsilon}{2}\right]|t-\tau| + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t}\frac{1}{2^n}\right)\\
&\leq g(\tau) - g(t) + \frac{\varepsilon}{2}|t-\tau| + \frac{\varepsilon}{2}(\tau - t) + g(t) - g(r) + \varepsilon(t-r) + \varepsilon\sum_{t_n<t}\frac{1}{2^n}\\
&= g(\tau) - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<t}\frac{1}{2^n}\right)\\
&\leq g(\tau) - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<\tau}\frac{1}{2^n}\right).
\end{aligned}$$

Moreover, if $t \notin D$ then, by the definition of pre-differentiable function, $t = t_m$ for some $m \in \mathbb{N}$. Also, by hypothesis, f and g are continuous, i.e, there exists a neighborhood U of t such that

$$|f(\tau) - f(t)| \leq \frac{\varepsilon}{2}\cdot\frac{1}{2^m} \text{ for all } \tau \in U,$$

and also

$$|g(\tau) - g(t)| \leq \frac{\varepsilon}{2}\cdot\frac{1}{2^m} \text{ for all } \tau \in U,$$

i.e.,

$$g(t) \leq g(\tau) + \frac{\varepsilon}{2}\cdot\frac{1}{2^m} \text{ for all } \tau \in U.$$

Therefore, for all $\tau \in U \cap (t,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
|f(\tau) - f(r)| &\leq |f(\tau) - f(t)| + |f(t) - f(r)|\\
&\leq \frac{\varepsilon}{2}\cdot\frac{1}{2^m} + g(t) - g(r) + \varepsilon\left(t - r + \sum_{t_n<t}\frac{1}{2^n}\right)\\
&\leq \frac{\varepsilon}{2}\cdot\frac{1}{2^m} + g(\tau) + \frac{\varepsilon}{2}\cdot\frac{1}{2^m} - g(r) + \varepsilon\left(t - r + \sum_{t_n<t}\frac{1}{2^n}\right)\\
&\leq \frac{\varepsilon}{2}\cdot\frac{1}{2^m} + g(\tau) + \frac{\varepsilon}{2}\cdot\frac{1}{2^m} - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<t}\frac{1}{2^n}\right)\\
&\leq g(\tau) - g(r) + \varepsilon\left(\tau - r + \sum_{t_n<\tau}\frac{1}{2^n}\right).
\end{aligned}$$

Thus, either for $t \in D$ or $t \notin D$, we have concluded that $S(\tau)$ holds for all $\tau \in U \cap (t,\infty)_{\mathbb{T}}$.

(iv) Finally, if t is left-dense and $S(\tau)$ holds for all $\tau < t$, then

$$|f(\tau) - f(r)| \leq g(\tau) - g(r) + \varepsilon\Bigg(\tau - r + \sum_{t_n < t} \frac{1}{2^n}\Bigg).$$

Thus, since f and g are continuous, taking the limit when $\tau \to t^-$ we have

$$\begin{aligned} |f(t) - f(r)| &\leq \lim_{\tau \to t^-} \left[g(\tau) - g(r) + \varepsilon\Bigg(\tau - r + \sum_{t_n < t} \frac{1}{2^n}\Bigg)\right] \\ &= g(t) - g(r) + \varepsilon \left(t - r + \sum_{t_n < t} \frac{1}{2^n}\right), \end{aligned}$$

i.e., $S(t)$ holds.

Therefore, by Theorem 1.1.4, we have the desired. □

Corollary 1.1.17 (See [10, Corollary 1.68]). *Let $f, g \colon \mathbb{T} \to \mathbb{R}$ be pre-differentiable functions with region of differentiation D. Then:*

(i) If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(t)| \right\} |s - r|;$$

(ii) If $f^{\Delta}(t) = 0$ for all $t \in D$, then f is a constant function;

(iii) If $f^{\Delta}(t) = g^{\Delta}(t)$ for all $t \in D$, then

$$g(t) = f(t) + C \ \textit{for all} \ \ t \in \mathbb{T},$$

where C is a constant.

Proof. (i) Let $r, s \in \mathbb{T}$ with $r \leq s$ and define $g \colon \mathbb{T} \to \mathbb{R}$ by

$$g(t) := \left\{ \sup_{\tau \in [r,s]_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(\tau)| \right\} (t - r) \ \text{ for } \ t \in \mathbb{T}.$$

Thus,

$$g^{\Delta}(t) = \sup_{\tau \in [r,s]_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(\tau)| \geq |f^{\Delta}(t)| \ \text{ for all } \ t \in [r, s]_{\mathbb{T}}^{\kappa} \cap D$$

and, by Theorem 1.1.16, we have

$$|f(t) - f(r)| \leq g(t) - g(r) = g(t) \text{ for all } t \in [r, s]$$

and we conclude that

$$|f(s) - f(r)| \leq g(s) = \left\{ \sup_{\tau \in [r,s]_{\mathbb{T}}^{\kappa} \cap D} |f^{\Delta}(\tau)| \right\} (s - r).$$

(ii) Since $f^{\Delta}(t) = 0$ for all $t \in D$ then, by item (i), we have

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U_{\mathbb{T}}^{\kappa} \cap D} |0| \right\} |r - s| = 0,$$

i.e., $f(s) = f(r)$. Thus, changing the endpoints s and r along D it is easy to conclude that f is constant.

(iii) Taking $h(t) = f(t) - g(t)$, since

$$h^{\Delta}(t) = f^{\Delta}(t) - g^{\Delta}(t) = 0,$$

we have by item (ii) that $h(t) \equiv C$, where C is a constant, i.e.,

$$f(t) = g(t) + C.$$ □

With sufficient foundations established, we now introduce the integration on time scales. Firstly, for regulated functions, the following holds:

Theorem 1.1.18 (See [10, Theorem 1.70])**.** *(Existence of Pre-Antiderivatives). Let f be a regulated function. Then, there exists a pre-differentiable function F with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \text{ for all } t \in D.$$

Proof. The proof can be found in [10, Theorem 1.70]. □

The next definition, together with Theorem 1.1.18, provides the concepts of pre-antiderivative, indefinite integral and Cauchy integral.

Definition 1.1.19 (See [10, Definition 1.71])**.** *Let $f \colon \mathbb{T} \to \mathbb{R}$ be a regulated function. Any function F as in Theorem 1.1.18 is called* ***pre-antiderivative*** *of f . We define the* ***indefinite***

integral *of* f *by*

$$\int f(t)\Delta t = F(t) + C,$$

where C *is an arbitrary constant and* F *is a pre-antiderivative of* f.

Definition 1.1.20. *We define the* ***Cauchy integral*** *by*

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for all } r, s \in \mathbb{T}.$$

Definition 1.1.21. *A function* $F\colon \mathbb{T} \to \mathbb{R}$ *is called* ***antiderivative*** *of* $f\colon \mathbb{T} \to \mathbb{R}$ *provided*

$$F^\Delta(t) = f(t) \text{ for all } t \in \mathbb{T}^\kappa.$$

The next theorem provides the existence of antiderivatives for the class of all rd-continuous functions.

Theorem 1.1.22 (See [10, Theorem 1.74])**.** *(Existence of Antiderivatives). Every rd-continuous function has an antiderivative. In particular, if* $t_0 \in \mathbb{T}$, *then* F *defined by*

$$F(t) := \int_{t_0}^t f(\tau)\Delta\tau \text{ for } t \in \mathbb{T}$$

is an antiderivative of f.

Proof. Suppose f is an rd-continuous function. Then, by Theorem 1.1.14 (ii), we have that f is regulated. Also, by Theorem 1.1.18, since f is regulated, there exists a pre-differentiable function F with region of differentiation D such that $F^\Delta(t) = f(t)$ for all $t \in D$. We want to show that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Thus, let $t \in \mathbb{T}^\kappa \setminus D$. It follows that t is right-dense (because $\mathbb{T}^\kappa \setminus D$ contains no right-scattered elements). So, since f is rd-continuous and t is right-dense, f is continuous at t, i.e., given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$|f(s) - f(t)| \leq \varepsilon \text{ for all } s \in U.$$

Now, define

$$h(\tau) := F(\tau) - f(t)(\tau - t_0) \text{ for all } \tau \in \mathbb{T}.$$

Thus, h is pre-differentiable with D and we have

$$h^\Delta(\tau) = F^\Delta(\tau) - f(t) = f(\tau) - f(t) \text{ for all } \tau \in D.$$

Then, $|h^\Delta(s)| = |f(s) - f(t)| \leq \varepsilon$ for all $s \in D \cap U$ and $\sup_{s\in D\cap U} |h^\Delta(s)| \leq \varepsilon$. Thus, since $h(t) = F(t) - f(t)(t - t_0)$, for $r \in U$, we have

$$\begin{aligned}|F(t) - F(r) - f(t)(t - r)| &= |h(t) + f(t)(t - t_0) - [h(r) + f(t)(r - t_0)] - f(t)(t - r)| \\ &= |h(t) + f(t)t - f(t)t_0 - h(r) - f(t)r + f(t)t_0 - f(t)t + f(t)r| \\ &= |h(t) - h(r)|\end{aligned}$$

and, by Corollary 1.1.17 (i), we conclude that

$$|h(t) - h(r)| \leq \left[\sup_{s\in D\cap U} |h^\Delta(s)|\right] |t - r| \leq \varepsilon|t - r|,$$

i.e.,

$$|F(t) - F(r) - f(t)(t - r)| \leq \varepsilon|t - r| \text{ for all } r \in U.$$

Therefore, F is Δ-differentiable at t with $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$. □

Some simple and useful properties for the integral on time scales are given in the next two theorems.

Theorem 1.1.23 (See [10, Theorem 1.75])**.** *If $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t). \tag{1.7}$$

Proof. Since f is rd-continuous, by Theorem 1.1.22, there exists an antiderivative F of f. Thus, we have

$$\begin{aligned}\int_t^{\sigma(t)} f(\tau)\Delta\tau &= F(\sigma(t)) - F(t) \\ &\overset{(1.3)}{=} F(t) + \mu(t)F^\Delta(t) - F(t) \\ &= \mu(t)f(t).\end{aligned}$$

□

Theorem 1.1.24 (See [10, Theorem 1.76])**.** *If $f^\Delta \geq 0$, then f is nondecreasing.*

Proof. Suppose $f^\Delta(t) \geq 0$ for all $t \in \mathbb{T}^\kappa$ and let $r, s \in \mathbb{T}$ such that $r \geq s$. Thus,

$$f(r) = f(s) + f(r) - f(s) = f(s) + \int_s^r f^\Delta(\tau)\Delta\tau \geq f(s),$$

because $f^\Delta(t) \geq 0$ and $r \geq s$ imply $\int_s^r f^\Delta(\tau)\Delta\tau \geq 0$. □

The usual properties for integration also holds for time scales, as in the next result. Since the proof is direct, we omit its proof here.

Theorem 1.1.25 (See [10, Theorem 1.77]). *Let $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then:*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f(t))\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$;

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$;

(vii) $\int_a^a f(t)\Delta t = 0$;

(viii) If $|f(t)| \leq g(t)$ *on* $[a, b)$, *then* $\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t$;

(ix) If $f(t) \geq 0$ *for all* $t \in [a, b)_{\mathbb{T}}$, *then* $\int_a^b f(t)\Delta t \geq 0$.

The following theorem shows how this new recently defined integral depends on the time scale and it gives a precise formula for sets that only has dense points or isolated points, together with the formula for $h\mathbb{Z}$.

Theorem 1.1.26 (See [10, Theorem 1.79]). *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then:*

(i) If $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

where the integral on the right hand side is the usual Riemann integral.

(ii) If $[a, b]_{\mathbb{T}}$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)_{\mathbb{T}}} \mu(t)f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t\in[b,a)_{\mathbb{T}}} \mu(t)f(t) & \text{if } a > b. \end{cases} \tag{1.8}$$

(iii) If $\mathbb{T} = h\mathbb{Z} = \{hk \ : \ k \in Z\}$*, where* $h > 0$*, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \textit{if } a < b, \\ 0 & \textit{if } a = b, \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & \textit{if } a > b. \end{cases}$$

Proof. (i) Since $f \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$, by Theorem 1.1.22, f has an antiderivative F. Thus

$$F^{\Delta}(t) = f(t) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

On the other hand, for $\mathbb{T} = \mathbb{R}$ we have $F^{\Delta}(t) = F'(t)$ and it implies that $F'(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Therefore, by Cauchy integral

$$\int_a^b f(t)\Delta t = F(b) - F(a)$$

and by the Fundamental Theorem of Calculus

$$\int_a^b f(t)dt = F(b) - F(a),$$

concluding that

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

(ii) Considering $a < b$ and denoting $[a, b]_{\mathbb{T}} = \{a = t_0, t_1, t_2, \cdots, t_n = b\}$ (with $t_0 < t_1 < t_2 < \cdots < t_n$) we have, by Theorem 1.1.25 (iv), the following

$$\begin{aligned} \int_a^b f(t)\Delta t &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(t)\Delta t = \sum_{k=0}^{n-1} \int_{t_k}^{\sigma(t_k)} f(t)\Delta t \\ &\overset{(1.7)}{=} \sum_{k=0}^{n-1} \mu(t_k) f(t_k) = \sum_{t \in [a,b)_{\mathbb{T}}} \mu(t) f(t). \end{aligned}$$

Moreover, if $a > b$ the proof is analogous and if $a = b$ the proof follows directly from Theorem 1.1.25 (vii).

(iii) If $a < b$, directly from item (ii), we have

$$\sum_{t\in[a,b)} \mu(t)f(t) = \sum_{t\in[a,b)} f(t)h = \sum_{k=0}^{n-1} f(t_k)h = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h,$$

which implies the desired. The cases $a > b$ and $a = b$ are analogous. □

Some examples for derivative of functions defined on time scales $\mathbb{R}$, $\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$, $q > 1$, are given next. It can be viewed as a direct application of Theorem 1.1.26.

Table 1.3: Examples for $\mathbb{R}$, $\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$
$f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$	$\frac{f(qt)-f(t)}{t(q-1)}$
$\int_a^b f(t)\Delta t$ for $a < b$	$\int_a^b f(t)dt$	$\sum_{t=a}^{b-1} f(t)$	$(q-1)\sum_{t\in[a,b)_{\mathbb{T}}} tf(t)$
f rd-continuous	f continuous	any function f	any function f

1.1.3 Chain rules and substitution rule

The chain and substitution rules are essential calculus tools and now its versions on time scale theory will be examined. We first demonstrate via counterexample that it is not possible to replace the usual derivative by the delta derivative in the usual chain rule. After that, we state a different version of chain rule for this theory. Afterwards, the derivative of inverse will be discussed in order to prove the substitution rule.

Example 1.1.27 (See [10, Example 1.85]). *Let $f, g\colon \mathbb{Z} \to \mathbb{Z}$ be functions given by $f(t) = t^2$ and $g(t) = 2t$. Note that*

$$(f \circ g)^{\Delta}(t) = (4t^2)^{\Delta} = 4(\sigma(t) + t) = 4(2t + 1) = 8t + 4.$$

On the other hand,

$$f^{\Delta}(g(t))g^{\Delta}(t) = (2(2t) + 1)2 = (4t + 1)2 = 8t + 2,$$

i.e., $(f \circ g)^{\Delta}(t) \neq f^{\Delta}(g(t))g^{\Delta}(t)$ for all $t \in \mathbb{Z}$ and the usual chain rule does not hold in this context.

Remark 1.1.28. *Let $\mathbb{T}$ be a time scale. For the remainder of this section, consider $\nu\colon \mathbb{T} \to \mathbb{R}$ as an strictly increasing function such that $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is also a time scale, $\tilde{\sigma}$ as the forward jump operator on $\tilde{\mathbb{T}}$ and $^{\tilde{\Delta}}$ as the delta derivative on $\tilde{\mathbb{T}}$.*
With those information, $\nu(\sigma(t)) = \tilde{\sigma}(\nu(t))$ holds for all $t \in \mathbb{T}$. Indeed, the case when $t = \max \mathbb{T}$ is trivial. Now, let $t \in \mathbb{T}$ and suppose that t is right-scattered, i.e., $\sigma(t) > t$. Then, since ν is strictly increasing, $\nu(\sigma(t)) > \nu(t)$. If we assume that there exists $\tilde{s} \in \tilde{\mathbb{T}}$ such that $\nu(\sigma(t)) > \tilde{s} > \nu(t)$, then there will be a unique (since ν is injective) $s \in \mathbb{T}$ such that $\nu(s) = \tilde{s}$. Thus, $\sigma(t) > s > t$, which is a contradiction, implying that there is no elements between $\nu(t)$ and $\nu(\sigma(t))$, and hence, $\tilde{\sigma}(\nu(t)) = \nu(\sigma(t))$. Now, if t is right-dense, i.e., $\sigma(t) = t$, then $\nu(\sigma(t)) = \nu(t)$ and since $t = \inf\{s \in \mathbb{T} \ : \ s > t\}$ for all $\varepsilon > 0$, there exists $s \in (t, t+\varepsilon) \cap \mathbb{T}$ and $\nu(t) < \nu(s)$ holds for that s. Hence, for all $\varepsilon > 0$, there is an $\nu(s) \in (\nu(t), \nu(t)+\varepsilon) \cap \tilde{\mathbb{T}}$ for some $s \in \mathbb{T}$ with $s > t$, i.e., $\nu(t)$ is right-dense and $\tilde{\sigma}(\nu(t)) = \nu(t) = \nu(\sigma(t))$.

The next theorem is the last chain rule that we will present and it will be the most useful one for this text.

Theorem 1.1.29 (See [10, Theorem 1.93])**.** *(Chain Rule). Let $w\colon \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta}(t) = (w^{\tilde{\Delta}} \circ \nu)(t)\nu^{\Delta}(t). \tag{1.9}$$

Proof. Let $\varepsilon \in (0,1)$ and define

$$\varepsilon^* = \frac{\varepsilon}{1 + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|} \in (0,1).$$

Since $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, there exist neighborhoods U_1 of t and U_2 of $\nu(t)$ such that

$$|\nu(\sigma(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)| \leq \varepsilon^* |\sigma(t) - s| \text{ for all } s \in U_1$$

and

$$|w(\tilde{\sigma}(\nu(t))) - w(s) - (\tilde{\sigma}(\nu(t)) - s)w^{\tilde{\Delta}}(\nu(t))| \leq \varepsilon^* |\tilde{\sigma}(\nu(t)) - s| \text{ for all } s \in U_2.$$

Thus, for $s \in U = U_1 \cap \nu^{-1}(U_2)$, we have that $\nu(s) \in U_2$ and

$$\begin{aligned} &|w(\nu(\sigma(t))) - w(\nu(s)) - (\sigma(t) - s)[w^{\tilde{\Delta}}(\nu(t))\nu^{\Delta}(t)]| \\ &\quad = |w(\nu(\sigma(t))) - w(\nu(s)) - (\sigma(t) - s)[w^{\tilde{\Delta}}(\nu(t))\nu^{\Delta}(t)] \end{aligned}$$

$$
\begin{aligned}
&\quad + (\tilde{\sigma}(\nu(t)) - \nu(s))w^{\tilde{\Delta}}(\nu(t)) - (\tilde{\sigma}(\nu(t)) - \nu(s))w^{\tilde{\Delta}}(\nu(t))| \\
&= |w(\nu(\sigma(t))) - w(\nu(s)) - (\tilde{\sigma}(\nu(t)) - \nu(s))w^{\tilde{\Delta}}(\nu(t)) \\
&\quad + [\tilde{\sigma}(\nu(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)]w^{\tilde{\Delta}}(\nu(t))| \\
&\leq |w(\tilde{\sigma}(\nu(t))) - w(\nu(s)) - (\tilde{\sigma}(\nu(t)) - \nu(s))w^{\tilde{\Delta}}(\nu(t))| \\
&\quad + |\nu(\sigma(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)||w^{\tilde{\Delta}}(\nu(t))| \\
&\leq \varepsilon^*|\tilde{\sigma}(\nu(t)) - \nu(s)| + \varepsilon^*|\sigma(t) - s||w^{\tilde{\Delta}}(\nu(t))| \\
&\leq \varepsilon^*[|\nu(\sigma(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)| + |\sigma(t) - s||\nu^{\Delta}(t)| \\
&\quad + |\sigma(t) - s||w^{\tilde{\Delta}}(\nu(t))|] \\
&= \varepsilon^*|\sigma(t) - s|[\varepsilon^* + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|] \\
&\leq \varepsilon^*|\sigma(t) - s|[1 + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|] \\
&= \frac{\varepsilon}{1 + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|}|\sigma(t) - s|[1 + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|] \\
&= \varepsilon|\sigma(t) - s|,
\end{aligned}
$$

getting the desired result. □

The derivative of the inverse formula is given next.

Theorem 1.1.30 (See [10, Theorem 1.97]). *(Derivative of the Inverse). If $\nu^{\Delta}(t)$ and $(\nu^{-1})^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$ and if $\nu^{\Delta}(t) \neq 0$, then*

$$
\frac{1}{\nu^{\Delta}(t)} = ((\nu^{-1})^{\tilde{\Delta}} \circ \nu)(t). \tag{1.10}
$$

Proof. We have

$$
(\nu^{-1} \circ \nu)^{\Delta}(t) \overset{(1.9)}{=} ((\nu^{-1})^{\tilde{\Delta}} \circ \nu)(t)\nu^{\Delta}(t),
$$

i.e.,

$$
\frac{1}{\nu^{\Delta}(t)} = ((\nu^{-1})^{\tilde{\Delta}} \circ \nu)(t).
$$

□

To complete the calculus section, we state and prove the substitution rule for the time scale theory.

Theorem 1.1.31 (See [10, Theorem 1.98]). *(Substitution). Let $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $a, b \in \mathbb{T}$. If ν is Δ-differentiable with rd-continuous derivative, then*

$$
\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta}s. \tag{1.11}
$$

Proof. Since $f, \nu^\Delta \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$, the product $f\nu^\Delta$ is also rd-continuous and, by Theorem 1.1.22, there exists F such that $F^\Delta = f\nu^\Delta$. Thus, we have

$$\begin{aligned}
\int_a^b f(t)\nu^\Delta(t)\Delta t &= \int_a^b F^\Delta(t)\Delta t \\
&= F(b) - F(a) \\
&= (F \circ \nu^{-1})(\nu(b)) - (F \circ \nu^{-1})(\nu(a)) \\
&= \int_{\nu(a)}^{\nu(b)} (F \circ \nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
&\overset{(1.9)}{=} \int_{\nu(a)}^{\nu(b)} (F^\Delta \circ \nu^{-1})(s)(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
&= \int_{\nu(a)}^{\nu(b)} ((fv^\Delta) \circ \nu^{-1})(s)(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
&= \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)v^\Delta(\nu^{-1}(s))(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta}s \\
&\overset{(1.10)}{=} \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)v^\Delta(\nu^{-1}(s))\frac{1}{\nu^\Delta(\nu^{-1}(s))}\tilde{\Delta}s \\
&= \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta}s.
\end{aligned}$$

□

1.2 First order linear equations

This section establishes the necessary theory to define the exponential function on time scales and examine its properties. We also analyze some basic dynamic equations on time scales, namely the first order homogeneous and inhomogeneous dynamic equations.

1.2.1 Hilger's complex plane

First, we introduce the Hilger's complex plane in order to define the cylinder transformation, which will be useful to define the exponential function.

Definition 1.2.1 (See [10, Definition 2.2]). *For $h > 0$, we define the **Hilger complex numbers** as*

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

Theorem 1.2.2 (See [10, Theorem 2.7]). *Let $z, w \in \mathbb{C}_h$, we define the **circle plus addition** $\oplus$ on $\mathbb{C}_h$ by*

$$z \oplus w := z + w + zwh.$$

Then, $(\mathbb{C}_h, \oplus)$ is an Abelian group.

Proof. Let $z, w \in \mathbb{C}_h$ and note that

$$1 + h(z \oplus w) = 1 + h(z + w + zwh) = (1 + hz)(1 + hw) \neq 0,$$

i.e., $\mathbb{C}_h$ is closed under the circle plus addition $\oplus$. Moreover, it is easy to see that 0 satisfies $z \oplus 0 = z = 0 \oplus z$. Also, the associative and commutative properties follows trivially. Finally, given $z \in \mathbb{C}_h$, to find an element $w \in \mathbb{C}_h$ such that $z \oplus w = 0 = w \oplus z$ we just need to solve $z + w + zwh = 0$ to obtain $w = -\frac{z}{1+zh}$ (it is easy to verify that $w \in \mathbb{C}_h$), which will be denoted as $\ominus z$. Therefore, $(\mathbb{C}_h, \oplus)$ is an Abelian group. □

Definition 1.2.3 (See [10, Definition 2.13]). *Let $z, w \in \mathbb{C}_h$. We define the **circle minus subtraction** $\ominus$ on $\mathbb{C}_h$ by*

$$z \ominus w := z \oplus (\ominus w).$$

Definition 1.2.4 (See [10, Page 57]). *For $h > 0$, we define the set $\mathbb{Z}_h$ as*

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} \, : \, -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\}.$$

Finally, we consider the cylinder transformation defined as follow.

Definition 1.2.5 (See [10, Definition 2.21]). *For $h > 0$, we define the **cylinder transformation** $\xi_h \colon \mathbb{C}_h \to \mathbb{Z}_h$ by*

$$\xi_h(z) := \frac{1}{h} Log(1 + zh), \tag{1.12}$$

where Log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) := z$ for all $z \in \mathbb{C}$.

The inverse of the cylinder transformation is given next and it will be useful later.

Example 1.2.6. *It is easy to verify that the inverse of the cylinder transformation is given by*

$$\xi_h^{-1}(z) = \frac{1}{h}(e^{zh} - 1). \tag{1.13}$$

1.2.2 Regressive functions and its properties

In this section, we define the regressive functions and the analogue "circle plus addition" and "circle minus subtraction" operations for this class of functions. This class of functions will be really importante to introduce the concept of exponential function on time scales.

Definition 1.2.7 (See [10, Definition 2.25]). *We call a function* $p\colon \mathbb{T} \to \mathbb{R}$ ***regressive*** *provided*

$$1 + \mu(t)p(t) \neq 0 \ \textit{for all} \ \ t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ *or simply* $\mathcal{R}$*.*

Theorem 1.2.8. *Let* $p, q \in \mathcal{R}$ *and define the* ***circle plus addition*** *on* $\mathcal{R}$ *by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \ \textit{for all} \ \ t \in \mathbb{T}^\kappa.$$

Then, $(\mathcal{R}, \oplus)$ *is an Abelian group called the* ***regressive group*** *.*

Proof. Let $p, q \in \mathcal{R}$ and note that

$$\begin{aligned} 1 + \mu(t)(p \oplus q)(t) &= 1 + \mu(t)[p(t) + q(t) + \mu(t)p(t)q(t)] \\ &= [1 + \mu(t)q(t)][1 + \mu(t)p(t)] \neq 0, \end{aligned}$$

i.e., $p \oplus q \in \mathcal{R}$. Moreover, the associative and commutative properties follows directly from the definition of the circle plus addition and the element 0 satisfies $(p \oplus 0)(t) = p(t) = (0 \oplus p)(t)$ for all $t \in \mathbb{T}$. Also, note that

$$(p \oplus (\ominus p))(t) = p(t) + (\ominus p)(t) + \mu(t)p(t)(\ominus p)(t) = p(t) + (\ominus p)(t)[1 + \mu(t)p(t)],$$

which implies that $(p \oplus (\ominus p))(t) = 0 = ((\ominus p) \oplus p)(t)$ if, and only if,

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)}$$

and $(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$ ($\ominus p \in \mathcal{R}$ by the following proposition) is the element such that $(p \oplus (\ominus p))(t) = 0 = ((\ominus p) \oplus p)(t)$ for all $t \in \mathbb{T}^\kappa$. □

Definition 1.2.9. *Let* $p, q \in \mathcal{R}$*. We define the* ***circle minus subtraction*** *on* $\mathcal{R}$ *by*

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t) \ \textit{for all} \ \ t \in \mathbb{T}^\kappa.$$

The next result shows some elementary properties for the group of regressive functions.

Proposition 1.2.10. *Let* $p, q \in \mathcal{R}$*. Then:*

(i) $p \ominus p = 0$;

(ii) $\ominus(\ominus p) = p$;

(iii) $\ominus p \in \mathcal{R}$;

(iv) $p \ominus q \in \mathcal{R}$;

(v) $p \ominus q = \dfrac{p-q}{1+\mu q}$;

(vi) $\ominus(p \ominus q) = q \ominus p$;

(vii) $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Proof. Let $p, q \in \mathcal{R}$ and $t \in \mathbb{T}^\kappa$. Then:

(i) By direct computation, we have

$$\begin{aligned}(p \ominus p)(t) &= (p \oplus (\ominus p))(t) = p(t) + (\ominus p)(t) + \mu(t)p(t)(\ominus p)(t)\\ &= p(t) - \frac{p(t)}{1+\mu(t)p(t)} - \mu(t)p(t)\frac{p(t)}{1+\mu(t)p(t)}\\ &= p(t) - \left[\frac{p(t)}{1+\mu(t)p(t)}\right][1+\mu(t)p(t)] = p(t) - p(t) = 0.\end{aligned}$$

(ii) Again with direct computation

$$(\ominus(\ominus p))(t) = \left(\ominus\left(-\frac{p(t)}{1+\mu(t)p(t)}\right)\right)(t) = -\left(\frac{-\frac{p(t)}{1+\mu(t)p(t)}}{1+\mu(t)\left[-\frac{p(t)}{1+\mu(t)p(t)}\right]}\right)$$

$$= \frac{\frac{p(t)}{1+\mu(t)p(t)}}{\frac{1+\mu(t)p(t)-\mu(t)p(t)}{1+\mu(t)p(t)}} = p(t).$$

(iii) We need to show that $1 + \mu(t)(\ominus p)(t) \neq 0$. Indeed, we have

$$\begin{aligned}1 + \mu(t)(\ominus p)(t) &= 1 + \mu(t)\left[-\frac{p(t)}{1+\mu(t)p(t)}\right] = 1 - \frac{\mu(t)p(t)}{1+\mu(t)p(t)}\\ &= \frac{1+\mu(t)p(t)-\mu(t)p(t)}{1+\mu(t)p(t)} = \frac{1}{1+\mu(t)p(t)} \neq 0.\end{aligned}$$

(iv) Analogously to item (iii), we get

$$\begin{aligned}1 + \mu(t)(p \ominus q)(t) &= 1 + \mu(t)[p(t) + (\ominus q)(t) + \mu(t)p(t)(\ominus q)(t)]\\ &= [1 + \mu(t)(\ominus q)(t)][1 + \mu(t)p(t)] \neq 0.\end{aligned}$$

(v) By direct computation, we obtain

$$(p \ominus q)(t) = p(t) + (\ominus q)(t) + \mu(t)p(t)(\ominus q)(t)$$

$$
\begin{aligned}
&= p(t) - \frac{q(t)}{1+\mu(t)q(t)} - \mu(t)p(t)\frac{q(t)}{1+\mu(t)q(t)} \\
&= \frac{p(t)[1+\mu(t)q(t)] - q(t) - \mu(t)p(t)q(t)}{1+\mu(t)q(t)} \\
&= \frac{p(t)+\mu(t)p(t)q(t) - q(t) - \mu(t)p(t)q(t)}{1+\mu(t)q(t)} \\
&= \frac{p(t)-q(t)}{1+\mu(t)q(t)}.
\end{aligned}
$$

(vi) From item (v), note that

$$
\begin{aligned}
\ominus(p\ominus q)(t) &= -\frac{(p\ominus q)(t)}{1+\mu(t)(p\ominus q)(t)} = -\frac{p(t)-q(t)}{1+\mu(t)p(t)} \\
&= \frac{q(t)-p(t)}{1+\mu(t)p(t)} = \frac{q(t)[1+\mu(t)p(t)] - p(t) - \mu(t)q(t)p(t)}{1+\mu(t)p(t)} \\
&= q(t) - \frac{p(t)}{1+\mu(t)p(t)} - \frac{\mu(t)q(t)p(t)}{1+\mu(t)p(t)} \\
&= q(t) + (\ominus p)(t) + \mu(t)q(t)(\ominus p)(t) = (q\ominus p)(t).
\end{aligned}
$$

(vii) Finally, we get

$$
\begin{aligned}
(\ominus(p\oplus q))(t) &= -\frac{(p\oplus q)(t)}{1+\mu(t)(p\oplus q)(t)} \\
&= \frac{-p(t)-q(t)-\mu(t)p(t)q(t)}{[1+\mu(t)q(t)][1+\mu(t)p(t)]} \\
&= \frac{-p(t)[1+\mu(t)q(t)] - q(t)[1+\mu(t)p(t)] + \mu(t)p(t)q(t)}{[1+\mu(t)q(t)][1+\mu(t)p(t)]} \\
&= -\frac{p(t)}{1+\mu(t)p(t)} - \frac{q(t)}{1+\mu(t)q(t)} + \frac{\mu(t)p(t)q(t)}{[1+\mu(t)p(t)][1+\mu(t)q(t)]} \\
&= (\ominus p)(t) + (\ominus q)(t) + \mu(t)(\ominus p)(t)(\ominus q)(t) \\
&= [(\ominus p)\oplus(\ominus q)](t).
\end{aligned}
$$

□

We can also consider the set of positively regressive functions and define a "circle product" between an scalar and a regressive or positively regressive function, depending if the scalar is natural or not.

Definition 1.2.11 (See [9, Page 34]). *The set of all rd-continuous functions $p\colon \mathbb{T}\to\mathbb{R}$ that satisfies*

$$1+\mu(t)p(t) > 0 \ \text{ for all } \ t\in\mathbb{T}$$

will be called the set of ***positively regressive functions*** *and will be denoted by* $\mathcal{R}^+$. *With this definition, we can introduce the notation*

$$\mathcal{R}(\alpha) = \begin{cases} \mathcal{R} & \text{if } \alpha \in \mathbb{N}, \\ \mathcal{R}^+ & \text{if } \alpha \in \mathbb{R} \setminus \mathbb{N}. \end{cases}$$

Definition 1.2.12 (See [9, Definition 2.35]). *For* $\alpha \in \mathbb{R}$ *and* $p \in \mathcal{R}(\alpha)$, *we define*

$$(\alpha \odot p)(t) := \alpha p(t) \int_0^1 (1 + \mu(t)p(t)h)^{\alpha - 1} dh.$$

The following example will be useful later in chapter 2 to exemplify some periodic functions on isolated time scales.

Example 1.2.13. *For* $p \in \mathcal{R}(\alpha)$ *we have*

$$2 \odot p = p \oplus p \ \text{ and } \ \frac{1}{2} \odot p = \frac{p}{1 + \sqrt{1 + \mu p}}.$$

Indeed, by Definition 1.2.12, note that

$$\begin{aligned}(2 \odot p)(t) &= 2p(t) \int_0^1 [1 + \mu(t)p(t)h]^{2-1} dh = 2p(t) \left[h|_0^1 + \mu(t)p(t) \frac{h^2}{2} \Big|_0^1 \right] \\ &= 2p(t) + \mu(t)(p(t))^2 = (p \oplus p)(t),\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{2} \odot p\right)(t) &= \frac{1}{2} p(t) \int_0^1 [1 + \mu(t)p(t)h]^{\frac{1}{2} - 1} dh = \frac{1}{2} p(t) \int_0^1 \frac{1}{\sqrt{1 + \mu(t)p(t)h}} dh \\ &= \frac{1}{2} p(t) \int_1^{1 + \mu(t)p(t)} \frac{1}{\sqrt{u}} \frac{du}{\mu(t)p(t)} = \frac{1}{2\mu(t)} \int_1^{1 + \mu(t)p(t)} \frac{1}{\sqrt{u}} du \\ &= \frac{1}{\mu(t)} \left(\sqrt{1 + \mu(t)p(t)} - 1 \right) = \frac{1 + \mu(t)p(t) - 1}{\mu(t) \left(\sqrt{1 + \mu(t)p(t)} + 1 \right)} \\ &= \frac{p(t)}{1 + \sqrt{1 + \mu(t)p(t)}}.\end{aligned}$$

1.2.3 The exponential function and homogeneous dynamic equations

Firstly, we define the concept of first-order dynamic equations on time scales.

Definition 1.2.14 (See [10, Definition 2.1]). *Suppose $f\colon \mathbb{T}\times\mathbb{R}^2 \to \mathbb{R}$. Then, the equation $y^\Delta = f(t,y,y^\sigma)$ is called a first order* ***dynamic equation****. If*

$$f(t,y,y^\sigma) = f_1(t)y + f_2(t) \text{ or } f(t,y,y^\sigma) = f_1(t)y^\sigma + f_2(t),$$

for functions f_1 and f_2, then $y^\Delta = f(t,y,y^\sigma)$ is called a ***linear dynamic equation*** *. Moreover, for $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem*

$$\begin{cases} y^\Delta = f(t,y,y^\sigma), \\ y(t_0) = y_0 \end{cases}$$

is called initial value problem (IVP) and a solution y of this equation with $y(t_0) = y_0$ is called the solution of this IVP.

Definition 1.2.15 (See [10, Definition 2.32]). *Let $p \in \mathcal{R}$. Then, the first order dynamic equation*

$$y^\Delta = p(t)y$$

is called ***regressive*** *.*

Now, we define the exponential function on time scales.

Definition 1.2.16 (See [10, Definition 2.30]). *Let $s,t \in \mathbb{T}$ and $p \in \mathcal{R}$, then we define the* ***exponential function*** *by*

$$e_p(t,s) := \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right). \tag{1.14}$$

The next example is an specific case for the exponential function for the quantum calculus.

Example 1.2.17. *Consider $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n \,:\, n \in \mathbb{N}_0\}$ for $q > 1$. It is easy to verify that*

$$\sigma(t) = qt \text{ and } \mu(t) = t(q-1) \text{ for all } t \in \mathbb{T}.$$

Thus, for $t,s \in \mathbb{T}$ with $t > s$ and $p \in \mathcal{R}$, we have

$$\begin{aligned} e_p(t,s) &\overset{(1.14)}{=} \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \overset{(1.8)}{=} \exp\left(\sum_{\tau\in[s,t)_\mathbb{T}} \mu(\tau)\xi_{\mu(\tau)}(p(\tau))\right) \\ &\overset{(1.12)}{=} \exp\left(\sum_{t\in[s,t)_\mathbb{T}} \mu(\tau)\frac{1}{\mu(\tau)}\ln\left(1+\mu(\tau)p(\tau)\right)\right) = \exp\left(\sum_{\tau\in[s,t)_\mathbb{T}} \ln(1+\mu(\tau)p(\tau))\right) \end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\sum_{\tau\in[s,t)_{\mathbb{T}}} \ln(1+(q-1)\tau p(\tau))\right) = \exp\left(\sum_{\tau\in[s,t)_{\mathbb{T}}} \ln(1+(q-1)q^{\log_q \tau} p(q^{\log_q \tau}))\right)\\
&= \exp\left(\sum_{k=\log_q s}^{\log_q t-1} \ln(1+(q-1)q^k p(q^k))\right) = \prod_{k=\log_q s}^{\log_q t-1} \exp(\ln(1+(q-1)q^k p(q^k)))\\
&= \prod_{k=\log_q s}^{\log_q t-1} [1+(q-1)q^k p(q^k)].
\end{aligned}
$$

Once defined the concept of exponential function in this context, we can also define the analogue $\cosh$ and $\sinh$ functions on time scales. It is also possible to define the $\sin$ and $\cos$ functions in this setting, but there will be no use for those later in this text.

Definition 1.2.18 (See [10, Definition 3.17]). *(Hyperbolic Functions). Let $p \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $-\mu p^2 \in \mathcal{R}$ (i.e., $p, -p \in \mathcal{R}$). The* ***hyperbolic functions*** $\cosh_p$ *and* $\sinh_p$ *are defined as*

$$\cosh_p := \frac{e_p + e_{-p}}{2} \quad \textit{and} \quad \sinh_p := \frac{e_p - e_{-p}}{2}.$$

The next lemma is one of the most important property of the exponential function, the semigroup property.

Lemma 1.2.19 (See [10, Lemma 2.31]). *If $p \in \mathcal{R}$, then the* semigroup property

$$e_p(t,r)e_p(r,s) = e_p(t,s) \quad \textit{for all} \quad r,s,t \in \mathbb{T} \tag{1.15}$$

is satisfied.

Proof. Let $r, s, t \in \mathbb{T}$ then, we have

$$e_p(t,r)e_p(r,s) \overset{(1.14)}{=} e^{\int_r^t \xi_{\mu(\tau)}\Delta\tau} e^{\int_s^r \xi_{\mu(\tau)}\Delta\tau} = e^{\int_r^t \xi_{\mu(\tau)}\Delta\tau + \int_s^r \xi_{\mu(\tau)}\Delta\tau} = e^{\int_s^t \xi_{\mu(\tau)}\Delta\tau} = e_p(t,s). \quad \square$$

In sequel, we prove that the exponential function on time scales has the expected property to be the derivative of itself, up to a power.

Theorem 1.2.20 (See [10, Theorem 2.62]). *Let $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then, $y(t) = y_0 e_p(t, t_0)$ is the solution of the following IVP*

$$\begin{cases} y^{\Delta}(t) = p(t)y(t), \\ y(t_0) = y_0. \end{cases}$$

Proof. Consider $y(t) = y_0 e_p(t,t_0)$. If $y_0 = 0$, then it follows trivially, thus suppose $y_0 \neq 0$ in the entire proof. It is easy to see that $y(t_0) = y_0$. Thus, we just need to show that $y(t) = y_0 e_p(t,t_0)$ satisfies $y^\Delta(t) = p(t)y(t)$ for all $t \in \mathbb{T}^\kappa$. Indeed, let $t \in \mathbb{T}^\kappa$. If $\sigma(t) > t$, then

$$\begin{aligned}
[y_0 e_p(\cdot,t_0)]^\Delta(t) &\overset{(1.1)}{=} y_0 \frac{e_p(\sigma(t),t_0) - e_p(t,t_0)}{\mu(t)} \overset{(1.15)}{=} y_0 \frac{e_p(\sigma(t),t)e_p(t,t_0) - e_p(t,t_0)}{\mu(t)} \\
&= y_0 \frac{e_p(\sigma(t),t) - 1}{\mu(t)} e_p(t,t_0) = y_0 \frac{\exp\left(\int_t^{\sigma(t)} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) - 1}{\mu(t)} e_p(t,t_0) \\
&\overset{(1.13)}{=} y_0 \xi^{-1}_{\mu(t)}(\xi_{\mu(t)}(p(t))) \cdot e_p(t,t_0) = p(t) y_0 e_p(t,t_0) = p(t)y(t).
\end{aligned}$$

Now, if $\sigma(t) = t$, then

$$\begin{aligned}
|y(\sigma(t)) - y(s) - p(t)y(t)[\sigma(t) - s]| &= |y(t) - y(s) - p(t)y(t)[t-s]| \\
&= |y_0 e_p(t,t_0) - y_0 e_p(s,t_0) - p(t) y_0 e_p(t,t_0)[t-s]| \\
&\overset{(1.15)}{=} |y_0 e_p(t,t_0) - y_0 e_p(s,t) e_p(t,t_0) - p(t) y_0 e_p(t,t_0)[t-s]| \\
&= |y_0 e_p(t,t_0)| \cdot |1 - e_p(s,t) - p(t)[t-s]| \\
&\le |y_0 e_p(t,t_0)| \cdot \left[\left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| \right. \\
&\quad \left. + \left|\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)[t-s]\right|\right] \\
&= |y_0 e_p(t,t_0)| \cdot \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| \\
&\quad + |y_0 e_p(t,t_0)| \cdot \left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right|. \qquad (1.16)
\end{aligned}$$

Thus, since $\sigma(t) = t$ and $p \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, we have

$$\lim_{r\to t} \xi_{\mu(r)}(p(r)) = \xi_0(p(t)),$$

i.e., given $\varepsilon > 0$, there exists a neighborhood U_1 of t such that

$$|\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))| < \frac{\varepsilon}{3|y_0 e_p(t,t_0)|} \text{ for all } \tau \in U_1. \qquad (1.17)$$

Thus, by Theorem 1.1.25 (viii) and the inequality (1.17), for $s \in U_1$ the following holds

$$\left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right| \le \int_s^t |\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))|\Delta\tau$$

$$\begin{aligned}
&< \int_s^t \frac{\varepsilon}{3|y_0 e_p(t,t_0)|}\Delta\tau \\
&= \frac{\varepsilon(t-s)}{3|y_0 e_p(t,t_0)|}, \\
&\leq \frac{\varepsilon|t-s|}{3|y_0 e_p(t,t_0)|},
\end{aligned}$$

from where, we have

$$|y_0 e_p(t,t_0)| \left| \int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau \right| \leq \frac{\varepsilon}{3}|t-s|. \tag{1.18}$$

Moreover, by L'Hôpital's rule

$$\lim_{z\to 0} \frac{1-z-e^{-z}}{z} = \lim_{z\to 0}(-1+e^{-z}) = 0.$$

Thus, taking $z = \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau$ and applying the limit definition it follows that, for every $\varepsilon^* > 0$, there exists a neighborhood U_2 of t such that if $s \in U_2$, then

$$\begin{aligned}
\left| \frac{1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e^{-\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau}}{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \right| &= \left| \frac{1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e^{\int_t^s \xi_{\mu(\tau)}(p(\tau))\Delta\tau}}{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \right| \\
&= \left| \frac{1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)}{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \right| < \varepsilon^*, \quad (1.19)
\end{aligned}$$

where

$$\varepsilon^* = \min\left\{1, \frac{\varepsilon}{1+3|p(t)y_0 e_p(t,t_0)|}\right\}.$$

Hence, from (1.19), we have

$$\begin{aligned}
\left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| &< \varepsilon^* \left| \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right| \\
&\leq \varepsilon^* \left| \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - \int_s^t \xi_0(p(t))\Delta\tau + \int_s^t \xi_0(p(t))\Delta\tau \right| \\
&= \varepsilon^* \left| \int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau + \xi_0(p(t)) \int_s^t \Delta\tau \right| \\
&= \varepsilon^* \left| \int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau + p(t)[t-s] \right| \\
&\leq \varepsilon^* \left| \int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau \right| + \varepsilon^* |p(t)| \cdot |t-s|.
\end{aligned} \tag{1.20}$$

Thus, from (1.20), we get

$$\begin{aligned}
&|y_0 e_p(t,t_0)| \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| \\
&\le |y_0 e_p(t,t_0)| \left[\varepsilon^* \left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right| + \varepsilon^* |p(t)| \cdot |t-s|\right] \\
&\le |y_0 e_p(t,t_0)| \left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right| + |y_0 e_p(t,t_0)|\varepsilon^* |p(t)| \cdot |t-s] \\
&\le |y_0 e_p(t,t_0)| \left|\int_s^t [\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))]\Delta\tau\right| + \frac{\varepsilon}{3}|t-s] \\
&\le \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s] \\
&= \frac{2\varepsilon}{3} \qquad (1.21)
\end{aligned}$$

where the last inequality holds for $s \in U_1$ by (1.18). Thus, substituting (1.18) and (1.21) in (1.16), we get

$$|y(\sigma(t)) - y(s) - p(t)y(t)[t-s]| \le \frac{2\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s| = \varepsilon|t-s|,$$

for all $s \in U = U_1 \cap U_2$, obtaining the desired result. □

Theorem 1.2.21 (See [10, Theorem 2.62]). *Let $t_0 \in \mathbb{T}$, $p \in \mathcal{R}$ and $y_0 \in \mathbb{R}$. Then, $y(t) = e_p(t,t_0)y_0$ is the unique solution of the IVP*

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ y(t_0) = y_0. \end{cases}$$

Proof. By Theorem 1.2.20, we know that $y(t) = e_p(t,t_0)y_0$ is indeed a solution of the IVP. Thus, assuming x is any solution of the IVP, consider $\frac{x}{e_p(t,t_0)}$ and note that

$$\begin{aligned}
\left(\frac{x}{e_p(t,t_0)}\right)^\Delta &= \frac{x^\Delta(t)e_p(t,t_0) - x(t)e_p^\Delta(t,t_0)}{e_p(t,t_0)e_p^\sigma(t,t_0)} \\
&= \frac{p(t)x(t)e_p(t,t_0) - p(t)x(t)e_p(t,t_0)}{e_p(t,t_0)e_p^\sigma(t,t_0)} = 0,
\end{aligned}$$

and, therefore, $\frac{x}{e_p(t,t_0)}$ is constant. On the other hand, since $\frac{x(t_0)}{e_p(t_0,t_0)} = y_0$, we get $\frac{x(t)}{e_p(t,t_0)} \equiv y_0$, which implies that x is given by $x(t) = e_p(t,t_0)y_0$, which verifies that $y(t) = e_p(t,t_0)y_0$, showing that $y(t)$ is the unique solution. □

Some useful properties of the exponential function are given next.

Theorem 1.2.22 (See [10, Theorem 2.36]). *Let $p, q \in \mathcal{R}$. Then:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = [1+\mu(t)p(t)]e_p(t,s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

(viii) $\left(\frac{1}{e_p(t,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(t,s)}$.

Proof. Let $p, q \in \mathcal{R}$. Then:

(i) By the definition of cylinder transformation and the exponential function, we have

$$e_0(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(0)\Delta\tau\right) = \exp\left(\int_s^t \frac{1}{\mu(\tau)}\ln(1+0\mu(\tau))\Delta\tau)\right) = 1$$

and

$$e_p(t,t) = \exp\left(\int_t^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) = 1.$$

(ii) By Theorem 1.2.20, we have

$$\begin{aligned} e_p(\sigma(t),s) &\overset{(1.3)}{=} e_p(t,s) + \mu(t)e_p^{\Delta}(t,s) \\ &= e_p(t,s) + \mu(t)p(t)e_p(t,s) \\ &= [1+\mu(t)p(t)]e_p(t,s). \end{aligned}$$

(iii) Consider the following IVP

$$\begin{cases} y^{\Delta}(t) = (\ominus p)(t)y(t), \\ y(s) = 1. \end{cases} \tag{1.22}$$

Thus, by Proposition 1.2.10 (iii), we know that $\ominus p \in \mathcal{R}$. Then, the equation in (1.22) is regressive. So, $y(t) = \frac{1}{e_p(t,s)}$ satisfies the IVP. Indeed, by item (i), $y(s) = \frac{1}{e_p(s,s)} = 1$. Furthermore, by Theorem 1.2.20 and item (ii), we have

$$\begin{aligned} y^{\Delta}(t) = \left(\frac{1}{e_p(t,s)}\right)^{\Delta}(t) &\overset{(1.5)}{=} -\frac{e_p^{\Delta}(t,s)}{e_p(t,s)e_p(\sigma(t),s)} \\ = -\frac{p(t)e_p(t,s)}{e_p(t,s)e_p(\sigma(t),s)} &= -\frac{p(t)}{e_p(\sigma(t),s)} \end{aligned}$$

$$= -\frac{p(t)}{[1+\mu(t)p(t)]e_p(t,s)} = -\frac{p(t)}{1+\mu(t)p(t)} \cdot \frac{1}{e_p(t,s)}$$
$$= (\ominus p)(t)y(t)$$

and $y(t) = \frac{1}{e_p(t,s)}$ is indeed a solution of the IVP (1.22). However, by Theorem 1.2.21, the unique solution of the IVP (1.22) is given by $y(t) = e_{\ominus p}(t,s)$. Hence, since $y(t) = \frac{1}{e_p(t,s)}$ is also a solution, $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$.

(iv) From Definition 1.2.16 and by item (iii), we have

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) = \exp\left(-\int_t^s \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$
$$= \frac{1}{exp\left(\int_t^s \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)} = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t).$$

(v) It has been already shown in Lemma 1.2.19.

(vi) Consider the IVP

$$\begin{cases} y^{\Delta}(t) = (p \oplus q)(t)y(t), \\ y(s) = 1. \end{cases} \tag{1.23}$$

Thus, the equation in (1.23) is regressive and $y(t) = e_p(t,s)e_q(t,s)$ is a solution of the IVP. Indeed, $y(s) = e_p(s,s)e_q(s,s) = 1$. Moreover, by Theorem 1.2.20 and item (ii), we have

$$\begin{aligned} y^{\Delta}(t) &= [e_p(\cdot,s)e_q(\cdot,s)]^{\Delta}(t) \\ &\overset{(1.4)}{=} e_p^{\Delta}(t,s)e_q(t,s) + e_p(\sigma(t),s)e_q^{\Delta}(t,s) \\ &= p(t)e_p(t,s)e_q(t,s) + e_p(\sigma(t),s)q(t)e_q(t,s) \\ &= p(t)e_p(t,s)e_q(t,s) + [1+\mu(t)p(t)]e_p(t,s)q(t)e_q(t,s) \\ &= e_p(t,s)e_q(t,s)[p(t)+q(t)+\mu(t)p(t)q(t)]] \\ &= e_p(t,s)e_q(t,s)(p \oplus q)(t) \\ &= (p \oplus q)(t)y(t). \end{aligned}$$

Therefore, $e_{p\oplus q}(t,s) = e_p(t,s)e_q(t,s)$, since the unique solution of the IVP (1.23) is $e_{p\oplus q}(t,s)$ by Theorem 1.2.21.

(vii) By items (iii) and (vi), we have that

$$\frac{1}{e_q(t,s)} = e_{\ominus q}(t,s)$$

implies

$$\frac{e_p(t,s)}{e_q(t,s)} = e_p(t,s)e_{\ominus q}(t,s) = e_{p\oplus(\ominus q)}(t,s) = e_{p\ominus q}(t,s).$$

(viii) By Theorem 1.2.20 and items (ii) and (iii), we have

$$\begin{aligned}\left(\frac{1}{e_p(t,s)}\right)^{\Delta} &= [e_{\ominus p}(t,s)]^{\Delta} = (\ominus p)(t)e_{\ominus p}(t,s)\\ &= -\frac{p(t)}{1+\mu(t)p(t)}e_{\ominus p}(t,s) = -\frac{p(t)}{[1+\mu(t)p(t)]e_p(t,s)}\\ &= -\frac{p(t)}{e_p(\sigma(t),s)} = -\frac{p(t)}{e_p^{\sigma}(t,s)}.\end{aligned}$$

□

1.2.4 Inhomogeneous dynamic equations

Next, we define a first order inhomogeneous linear dynamic equation and we solve two different types of IVP, both by the Variation of Constants Formula.

Definition 1.2.23 (See [10, Page 75])**.** *Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function. We call a **first order inhomogeneous linear dynamic equation** by equations of type*

$$y^{\Delta} = p(t)y + f(t).$$

First, let us consider the following IVP

$$\begin{cases} x^{\Delta} = -p(t)x^{\sigma}(t) + f(t), \\ x(t_0) = x_0. \end{cases} \tag{1.24}$$

Assume (1.24) has a solution $x(t)$. Then, multiplying the equation in (1.24) by $e_p(t,t_0)$, we have

$$\begin{aligned}e_p(t,t_0)f(t) &= e_p(t,t_0)[x^{\Delta}(t) + p(t)x^{\sigma}(t)]\\ &= e_p(t,t_0)x^{\Delta}(t) + e_p(t,t_0)p(t)x^{\sigma}(t)\\ &= [e_p(\cdot,t_0)x]^{\Delta}(t).\end{aligned}$$

Thus, integrating this inequality from t_0 to t, we obtain

$$\int_{t_0}^{t} [e_p(\cdot, t_0)x]^{\Delta}(\tau)\Delta\tau = \int_{t_0}^{t} e_p(\tau, t_0)f(\tau)\Delta\tau,$$

i.e.,

$$e_p(t, t_0)x(t) - e_p(t_0, t_0)x(t_0) = \int_{t_0}^{t} e_p(\tau, t_0)f(\tau)\Delta\tau,$$

which implies

$$e_p(t, t_0)x(t) - x_0 = \int_{t_0}^{t} e_p(\tau, t_0)f(\tau)\Delta\tau.$$

Therefore, we can write

$$x(t) = \frac{x_0}{e_p(t, t_0)} + \frac{1}{e_p(t, t_0)} \int_{t_0}^{t} e_p(\tau, t_0)f(\tau) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

This method is called the *Variation of Constants Formula*. This is very useful in the study of inhomogeneous equation.

Theorem 1.2.24 (See [10, Theorem 2.74])**.** *(Variation of Constants Formula). Let $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. Then,*

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau$$

is the unique solution of the IVP (1.24).

Proof. From the calculations above, we already know that the solutions of (1.24) have the form

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

Moreover, for the initial value, note that

$$x(t_0) = e_{\ominus p}(t_0, t_0)x_0 + \int_{t_0}^{t_0} e_{\ominus p}(t_0, \tau)f(\tau)\Delta\tau = 1 \cdot x_0 + 0 = x_0.$$ □

For the formulation of the IVP without the composition with σ, we have a similar solution and it can be seen as an immediate consequence of Theorem 1.2.24, since the same construction can be made in order to get a solution by the Variation of Constants Formula.

Theorem 1.2.25 (See [10, Theorem 2.77]). *(Variation of Constants Formula). Let $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then,*

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau \tag{1.25}$$

is the unique solution of the following IVP

$$\begin{cases} y^{\Delta}(t) = p(t)y(t) + f(t), \\ y(t_0) = y_0. \end{cases} \tag{1.26}$$

Proof. Firstly, we have

$$y^{\sigma}(t) \overset{(1.3)}{=} y(t) + \mu(t)y^{\Delta}(t),$$

i.e., $y(t) = y^{\sigma}(t) - \mu(t)y^{\Delta}(t)$. Thus we can rewrite the equation in (1.26) as

$$y^{\Delta}(t) = p(t)[y^{\sigma}(t) - \mu(t)y^{\Delta}(t)] + f(t),$$

i.e., $y^{\Delta}(t)[1 + p(t)\mu(t)] = p(t)y^{\sigma}(t) + f(t)$ and we conclude

$$\begin{aligned} y^{\Delta}(t) &= \frac{p(t)}{1 + p(t)\mu(t)}y^{\sigma}(t) + \frac{f(t)}{1 + p(t)\mu(t)} \\ &= -(\ominus p)(t)y^{\sigma}(t) + \frac{f(t)}{1 + p(t)\mu(t)}. \end{aligned} \tag{1.27}$$

Thus, equation (1.27) satisfies the hypothesis on Theorem 1.2.24. Therefore, we have the following unique solution

$$\begin{aligned} y(t) &= e_{\ominus(\ominus p)}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus(\ominus p)}(t, \tau)\frac{f(\tau)}{1 + \mu(\tau)p(\tau)}\Delta\tau \\ &= e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \tau)\frac{f(\tau)}{1 + \mu(\tau)p(\tau)}\Delta\tau \\ &= e_p(t, t_0)y_0 + \int_{t_0}^{t} \frac{f(\tau)}{e_p(\tau, t)[1 + \mu(\tau)p(\tau)]}\Delta\tau. \\ &= e_p(t, t_0)y_0 + \int_{t_0}^{t} \frac{f(\tau)}{e_p(\sigma(\tau), t)}\Delta\tau \\ &= e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau, \end{aligned}$$

concluding the desired result. □

1.3 Regressive Matrices

In this section we discuss some fundamental definitions and results concerning regressive matrices, which will be useful later to give sufficient conditions to the first order linear dynamic equation to have an almost periodic solution.

Definition 1.3.1 (See [10, Definition 5.1]). *An $m\times n$-matrix-valued function $A\colon \mathbb{T}\to\mathbb{R}^{n\times m}$ is called **rd-continuous** provided each entry of $A(t)$ is rd-continuous on $\mathbb{T}$. The set of all rd-continuous $m\times n$-matrix-valued functions $A\colon \mathbb{T}\to\mathbb{R}^{n\times m}$ will be denoted by $C_{rd}(\mathbb{T},\mathbb{R}^{n\times m})$.*

Definition 1.3.2 (See [10, Definition 5.1]). *An $m\times n$-matrix-valued function $A\colon \mathbb{T}\to\mathbb{R}^{n\times m}$ is Δ-**differentiable** on $\mathbb{T}$ provided each entry of $A(t)$ is Δ-differentiable on $\mathbb{T}$. In this case, we put*

$$A^{\Delta}(t)=(a_{ij}^{\Delta}(t))_{1\le i\le m,1\le j\le n},\ \ \textit{where}\ \ A(t)=(a_{ij}(t))_{1\le i\le m,1\le j\le n}.$$

Theorem 1.3.3 (See [10, Theorem 5.2]). *If A is Δ-differentiable at $t\in\mathbb{T}^{\kappa}$, then*

$$A^{\sigma}(t)=A(t)+\mu(t)A^{\Delta}(t).$$

Proof. With direct computation, we have

$$\begin{aligned}A^{\sigma}&=(a_{ij}^{\sigma})\overset{(1.3)}{=}(a_{ij}+\mu a_{ij}^{\Delta})\\&=(a_{ij})+\mu(a_{ij}^{\Delta})=A+\mu A^{\Delta}.\end{aligned}$$

□

Definition 1.3.4 (See [10, Definition 5.5]). *We call an $m\times n$-matrix-valued function $A\colon \mathbb{T}\to\mathbb{R}^{n\times m}$ **regressive** provided*

$$I+\mu(t)A(t)\ \ \textit{is invertible for all}\ \ t\in\mathbb{T}^{\kappa}.$$

The set of all regressive and rd-continuous $m\times n$-matrix-valued functions will be denoted by $\mathcal{R}(\mathbb{T},\mathbb{R}^{n\times m})$ or simply $\mathcal{R}$.

Definition 1.3.5 (See [10, Definition 5.10]). *Assume A and B are regressive $n\ timesn$-matrix-valued functions on $\mathbb{T}$. Then we define $A\oplus B$ by*

$$(A\oplus B)(t)=A(t)+B(t)+\mu(t)A(t)B(t)\ \ \textit{for all}\ \ t\in\mathbb{T}^{\kappa},$$

and we define $\ominus A$ by

$$(\ominus A)(t)=-[I+\mu(t)A(t)]^{-1}A(t)\ \ \textit{for all}\ \ t\in\mathbb{T}^{\kappa}.$$

Proposition 1.3.6 (See [10, Lemma 5.12]). *($\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $\oplus$) is a group.*

Theorem 1.3.7 (See [10, Theorem 5.8]). *(Existence and Uniqueness Theorem). Let $A \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f\colon \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$\begin{cases} y^{\Delta}(t) = A(t)y(t) + f(t), \\ y(t_0) = y_0 \end{cases}$$

has a unique solution $y\colon \mathbb{T} \to \mathbb{R}^n$.

Proof. The proof can be found in [10, Theorem 5.8]. □

Remark 1.3.8. *It follows from Theorem 1.3.7 that the matrix initial value problem*

$$\begin{cases} Y^{\Delta}(t) = A(t)Y(t), \\ Y(t_0) = Y_0, \end{cases}$$

where Y_0 is a constant $n \times n$-matrix, has a unique matrix-valued solution. In particular, for $Y_0 = I$, we also have a unique solution and it motivates the next definition.

Definition 1.3.9 (See [10, Definition 5.18]). *(Matrix Exponential Function). Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$-matrix-valued function. The unique matrix-valued solution of the IVP*

$$\begin{cases} Y^{\Delta}(t) = A(t)Y(t), \\ Y(t_0) = I \end{cases}$$

*is called the **matrix exponential function** and it is denoted by $e_A(\cdot, t_0)$.*

Remark 1.3.10 (See [10, Exercise 5.17]). *It is not difficult to show that, using the notation A^* to denote its conjugate transpose, A^* is regressive and $\ominus A^* = (\ominus A)^*$.*

Remark 1.3.11 (See [10, Theorem 5.3]). *Let $A, B \in \mathcal{R}$ be Δ-differentiable $n \times n$-matrix-valued functions on $\mathbb{T}$. Then*

$$(AB)^{\Delta} = A^{\Delta}B^{\sigma} + AB^{\Delta} = A^{\sigma}B^{\Delta} + A^{\Delta}B.$$

Indeed, if $A = (a_{ij})$ and $B = (b_{ij})$, then

$$AB = \left(\sum_{k=1}^{n} a_{ik}b_{kj} \right)_{1\le i,j\le n}$$

and it follows that the ijth entry of $(AB)^\Delta$ is

$$\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)^\Delta = \sum_{k=1}^{n}(a_{ik}b_{kj})^\Delta \overset{(1.4)}{=} \sum_{k=1}^{n}(a_{ik}^\Delta b_{kj}^\sigma + a_{ik}b_{kj}^\Delta)$$
$$= \sum_{k=1}^{n} a_{ik}^\Delta b_{kj}^\sigma + \sum_{k=1}^{n} a_{ik}b_{kj}^\Delta,$$

which is the ijth entry of the matrix $A^\Delta B^\sigma + AB^\Delta$. Analogously, we also obtain that $(AB)^\Delta = A^\sigma B^\Delta + A^\Delta B$.

Theorem 1.3.12 (See [10, Theorem 5.21])**.** *Let $A, B \in \mathcal{R}$ be matrix-valued functions on $\mathbb{T}$. Then:*

(i) $e_0(t,s) \equiv I$ *and* $e_A(t,t) \equiv I$;

(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t,s)$;

(iii) $e_A^{-1}(t,s) = e_{\ominus A^*}^*(t,s)$;

(iv) $e_A(t,s) = e_A^{-1}(s,t) = e_{\ominus A^*}^*(s,t)$;

(v) $e_A(t,s)e_A(s,r) = e_A(t,r)$;

(vi) $e_A(t,s)e_B(t,s) = e_{A\oplus B}(t,s)$ *if* $e_A(t,s)$ *and* $B(t)$ *commute.*

Proof. The proof is analogous to Theorem 1.2.22, but making use of Theorem 1.3.7, Theorem 1.3.3, Remark 1.3.10 and Remark 1.3.11 □

Theorem 1.3.13 (See [10, Theorem 5.24])**.** *(Variation of Constants). Let $a \in \mathcal{R}$ be an $n \times n$-matrix-valued-funciton on $\mathbb{T}$ and suppose that $f\colon \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the IVP*

$$\begin{cases} y^\Delta(t) = A(t)y(t) + f(t), \\ y(t_0) = y_0 \end{cases} \tag{1.28}$$

has a unique solution $y\colon \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by

$$y(t) = e_A(t,t_0)y_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau. \tag{1.29}$$

Proof. Using Theorem 1.3.12 (v) and Remark 1.3.11, it is not difficult to verify that (1.29) is indeed a solution of (1.28). Moreover, for the uniqueness of solution, it is just another application of 1.3.12 (v). □

Chapter 2

Preliminaries on periodicity and almost periodicity

In this chapter, we give some fundamental definitions and results concerning periodicity on isolated time scales, following the main reference [7]. In special, we introduce the *iterated shifts*, some useful formulas for the iterated shift operator and its properties for compositions with the exponential function on isolated time scales.

Furthermore, we present the Bochner's and Bohr's definitions of almost periodicity for quantum calculus, along with several results. All the proofs for the quantum case can be found in [4] and it will not be explored in this chapter, since the next chapter we give a generalization of almost periodicity on any isolated time scale and we prove those results in a more general context.

Those two concepts will be inspiring to generalize the almost periodicity from the quantum calculus to any isolated time scale.

2.1 Periodicity on isolated time scales

This section develops the basic results around the theory of periodicity on isolated time scales (i.e., time scales which all of its points are right-scattered and left-scattered, except when the time scale has a minimum or maximum (or both). In this case, the minimum point must be right-scattered and the maximum point must be left-scattered) and it can be summarized as follows: Subsection 1 introduces the iterated shifts as the fundamental tool for generalizing periodicity on isolated time scales and subsection 2 defines the periodicity on the isolated case. In this whole section, consider $\mathbb{T}$ an isolated time scale.

2.1.1 Iterated shifts

We start with the definition of the iterated shift operator.

Definition 2.1.1 (See [7, Page 262]). *Let $\omega \in \mathbb{N}$. We define the* ***iterated shift operator*** *$\nu\colon \mathbb{T} \to \mathbb{T}$ by*

$$\nu := \sigma^{\omega} := \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{\omega \text{ times}}.$$

Notation 2.1.2 (See [7, Page 262]). *Let $f\colon \mathbb{T} \to \mathbb{R}$. Analogously to the notation $f^{\sigma} = f \circ \sigma$, we will use the notation*

$$f^{\nu} = f \circ \nu.$$

Remark 2.1.3 (See [7, Page 262]). *With the previous notation, we have that*

$$f^{\nu\sigma} = f^{\nu\circ\sigma} = f^{\sigma^{\omega}\circ\sigma} = f^{\sigma\circ\sigma^{\omega}} = f^{\sigma\circ\nu} = f^{\sigma\nu}. \tag{2.1}$$

Moreover, note that

$$\sigma \circ \nu = \sigma \circ \sigma^{\omega} = \sigma^{\omega} \circ \sigma = \nu \circ \sigma, \tag{2.2}$$

i.e., $\sigma^{\nu} = \nu^{\sigma}$ with the previous notation. Thus, σ and ν commute.

Remark 2.1.4. *Since we are dealing with isolated time scales, we have that every function defined on these time scales is continuous. Also, since all the points are right-scattered, it follows that μ is never zero.*

The following lemma gives an useful expression of the derivative of the iterated shift ν.

Lemma 2.1.5 (See [7, Lemma 3.1]). *We have*

$$\nu^{\Delta} = \frac{\mu^{\nu}}{\mu}. \tag{2.3}$$

Proof. Let $t \in \mathbb{T}$. By direct computation, we have

$$\nu^{\Delta}(t) \overset{(1.1)}{=} \frac{\nu(\sigma(t)) - \nu(t)}{\mu(t)} \overset{(2.2)}{=} \frac{\sigma(\nu(t)) - \nu(t)}{\mu(t)} = \frac{\mu(\nu(t))}{\mu(t)},$$

as desired. □

The next lemma is a chain rule for compositions between any real valued function defined on a time scale and the iterated shift ν. We call the attention of the reader that in the case of isolated time scales, such property is valid.

Lemma 2.1.6 (See [7, Lemma 3.2]). *Let* $f\colon \mathbb{T} \to \mathbb{R}$. *We have*

$$f^{\nu\Delta} = \nu^{\Delta} f^{\Delta\nu}. \tag{2.4}$$

Proof. Let $t \in \mathbb{T}$. Then,

$$\begin{aligned}(f^{\nu})^{\Delta}(t) &\overset{(1.1)}{=} \frac{(f^{\nu})^{\sigma}(t) - f^{\nu}(t)}{\mu(t)} \overset{(2.1)}{=} \frac{(f^{\sigma})^{\nu}(t) - f^{\nu}(t)}{\mu(t)} = \frac{\mu^{\nu}(t)}{\mu^{\nu}(t)} \frac{(f^{\sigma})^{\nu}(t) - f^{\nu}(t)}{\mu(t)} \\ &\overset{(2.3)}{=} \nu^{\Delta}(t) \frac{(f^{\sigma})^{\nu}(t) - f^{\nu}(t)}{\mu^{\nu}(t)} = \nu^{\Delta}(t) \left(\frac{f^{\sigma} - f}{\mu}\right)^{\nu}(t) \overset{(1.1)}{=} \nu^{\Delta}(t)(f^{\Delta})^{\nu}(t),\end{aligned}$$

confirming the result. □

An useful expression for the second derivative of the iterated shift ν is given next.

Lemma 2.1.7 (See [7, Lemma 3.3]). *We have*

$$\nu^{\Delta\Delta} = \nu^{\Delta} \frac{\sigma^{\Delta\nu} - \sigma^{\Delta}}{\mu^{\sigma}}.$$

Proof. Let $t \in \mathbb{T}$. Firstly, since $\mu(t) = \sigma(t) - t$, it follows

$$\mu^{\Delta}(t) = \sigma^{\Delta}(t) - 1. \tag{2.5}$$

Thus, by Theorem 1.1.11 (v), we find

$$\begin{aligned}\nu^{\Delta\Delta}(t) &\overset{(2.3)}{=} \left(\frac{\mu^{\nu}}{\mu}\right)^{\Delta}(t) \overset{(1.6)}{=} \frac{(\mu^{\nu})^{\Delta}(t)\mu(t) - \mu^{\nu}(t)\mu^{\Delta}(t)}{\mu(t)\mu^{\sigma}(t)} \overset{(2.4)}{=} \frac{\nu^{\Delta}(t)(\mu^{\Delta})^{\nu}(t)\mu(t) - \mu^{\nu}(t)\mu^{\Delta}(t)}{\mu(t)\mu^{\sigma}(t)} \\ &\overset{(2.3)}{=} \frac{\nu^{\Delta}(t)(\mu^{\Delta})^{\nu}(t)\mu(t) - \nu^{\Delta}(t)\mu(t)\mu^{\Delta}(t)}{\mu(t)\mu^{\sigma}(t)} = \nu^{\Delta}(t)\frac{(\mu^{\Delta})^{\nu}(t) - \mu^{\Delta}(t)}{\mu^{\sigma}(t)} \\ &\overset{(2.5)}{=} \nu^{\Delta}(t)\frac{\sigma^{\Delta}(\nu(t)) - 1 - [\sigma^{\Delta}(t) - 1]}{\mu^{\sigma}(t)} = \nu^{\Delta}(t)\frac{(\sigma^{\Delta})^{\nu}(t) - \sigma^{\Delta}(t)}{\mu^{\sigma}(t)},\end{aligned}$$

as desired. □

Remark 2.1.8 (See [7, Remark 3.4]). *Note that*

$$\mu(\sigma(t)) \overset{(1.3)}{=} \mu(t) + \mu(t)\mu^{\Delta}(t) = \mu(t)[1 + \mu^{\Delta}(t)] = \mu(t)\sigma^{\Delta}(t) \ \textit{for all} \ \ t \in \mathbb{T},$$

i.e., $\mu^{\sigma} = \mu\sigma^{\Delta}$.

Next, we show three simple examples of isolated time scales followed by the computation of ν^{Δ} and $\nu^{\Delta\Delta}$.

Example 2.1.9 (See [7, Example 3.5]). *If $\mathbb{T} = \mathbb{Z}$, then*

$$\sigma(t) = t + 1,\ \mu(t) = 1 \text{ and } \nu(t) = t + \omega \text{ for all } t \in \mathbb{T}.$$

Hence, $\nu^{\Delta}(t) = 1$ and $\nu^{\Delta\Delta}(t) = 0$ for all $t \in \mathbb{T}$.

Example 2.1.10 (See [7, Example 3.6]). *If $\mathbb{T} = h\mathbb{Z}$, with $h > 0$, then*

$$\sigma(t) = t + h,\ \mu(t) = h \text{ and } \nu(t) = t + h\omega \text{ for all } t \in \mathbb{T}.$$

Thus, $\nu^{\Delta}(t) = 1$ and $\nu^{\Delta\Delta}(t) = 0$ for all $t \in \mathbb{T}$.

Example 2.1.11 (See [7, Example 3.7]). *If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then*

$$\sigma(t) = qt, \mu(t) = (q-1)t \text{ and } \nu(t) = q^{\omega}t \text{ for all } t \in \mathbb{T}.$$

Therefore, $\nu^{\Delta}(t) = q^{\omega}$ and $\nu^{\Delta\Delta}(t) = 0$ for all $t \in \mathbb{T}$.

Remark 2.1.12. *Considering $q > 1$, the time scale $q^{\mathbb{N}_0}$ is not isolated, since $1 \in q^{\mathbb{N}_0}$ is not left-scattered. But, it does not matter for our calculations, since we only use the forward jump operator in this whole chapter and all the results holds for $q^{\mathbb{N}_0}$*

Remark 2.1.13. *Note that in all of these examples, we have ν^{Δ} as a constant, implying $\nu^{\Delta\Delta}$ being zero. It is not a general rule (e.g., $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$), but in most cases this holds.*

The next lemma shows a useful result about the derivative of an integral from t to $\nu(t)$.

Lemma 2.1.14 (See [7, Lemma 3.8]). *Let $f\colon \mathbb{T} \to \mathbb{R}$ and define*

$$F_{\nu}(t) := \int_{t}^{\nu(t)} f(\tau)\Delta\tau.$$

Then

$$F_{\nu}^{\Delta} = \nu^{\Delta} f^{\nu} - f. \tag{2.6}$$

Proof. Let $t_0 \in \mathbb{T}$ and define $F\colon \mathbb{T} \to \mathbb{R}$ by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}.$$

Let $t \in \mathbb{T}$. By Theorem 1.1.22, we have $F^{\Delta}(t) = f(t)$ and note that

$$F^{\nu}(t) - F(t) = \int_{t_0}^{\nu(t)} f(\tau)\Delta\tau - \int_{t_0}^{t} f(\tau)\Delta\tau = \int_{t_0}^{\nu(t)} f(\tau)\Delta\tau + \int_{t}^{t_0} f(\tau)\Delta\tau$$

$$= \int_t^{\nu(t)} f(\tau)\Delta\tau = F_\nu(t).$$

Hence, we conclude

$$F_\nu^\Delta(t) = (F^\nu)^\Delta(t) - F^\Delta(t) \overset{(2.4)}{=} \nu^\Delta(t)(F^\Delta)^\nu(t) - F^\Delta(t) = \nu^\Delta(t)f^\nu(t) - f(t). \qquad \square$$

The next two following lemmas brings two helpful formulas for composition of the exponential function and the iterated shift.

Lemma 2.1.15 (See [7, Lemma 3.9])**.** *Let $t_0 \in \mathbb{T}$. For $f \in \mathcal{R}$, we have*

$$h(t) := e_f(\nu(t), t) \quad \textit{implies} \quad h^\Delta(t) = ((\nu^\Delta f^\nu) \ominus f)(t)h(t) \tag{2.7}$$

and

$$e_f(\nu(t), t) = e_f(\nu(t_0), t_0)\frac{e_{\nu^\Delta f^\nu}(t, t_0)}{e_f(t, t_0)} \quad \textit{for all} \quad t \in \mathbb{T}. \tag{2.8}$$

Proof. Defining $h\colon \mathbb{T} \to \mathbb{R}$ by

$$h(t) := e_f(\nu(t), t) \quad \text{for all} \quad t \in \mathbb{T},$$

we have

$$h(t) = e_f(\nu(t), t) \overset{(1.15)}{=} e_f(\nu(t), t_0)e_f(t_0, t),$$

from where, since f is regressive, we get

$$h(t) = e_f(\nu(t), t_0)e_{\ominus f}(t, t_0).$$

Thus, we have

$$\begin{aligned} h^\Delta(t) &\overset{(1.4)}{=} [e_f(\nu(t), t_0)]^\Delta e_{\ominus f}(\sigma(t), t_0) + e_f(\nu(t), t_0)[e_{\ominus f}(t, t_0)]^\Delta \\ &\overset{(1.9)}{=} \nu^\Delta(t)f(\nu(t))e_f(\nu(t), t_0)e_{\ominus f}(\sigma(t), t_0) + e_f(\nu(t), t_0)(\ominus f)(t)e_{\ominus f}(t, t_0), \end{aligned}$$

where we could apply Theorem 1.1.29 because ν is strictly increasing and $\nu(\mathbb{T}) = \mathbb{T}$ is also a time scale. Hence, we obtain

$$\begin{aligned} h^\Delta(t) &= \nu^\Delta(t)f(\nu(t))e_f(\nu(t), t_0)e_{\ominus f}(\sigma(t), t_0) + e_f(\nu(t), t_0)(\ominus f)(t)e_{\ominus f}(t, t_0) \\ &= \nu^\Delta(t)f(\nu(t))e_f(\nu(t), t_0)[1 + \mu(t)(\ominus f)(t)]e_{\ominus f}(t, t_0) + e_f(\nu(t), t_0)(\ominus f)(t)e_{\ominus f}(t, t_0) \\ &= [\nu^\Delta(t)f(\nu(t))[1 + \mu(t)(\ominus f)(t)] + (\ominus f)(t)]e_f(\nu(t), t_0)e_{\ominus f}(t, t_0) \\ &= [\nu^\Delta(t)f(\nu(t))[1 + \mu(t)(\ominus f)(t)] + (\ominus f)(t)]h(t) \end{aligned}$$

$$= [\nu^\Delta(t)f(\nu(t)) + (\ominus f)(t) + \mu(t)\nu^\Delta(t)f(\nu(t))(\ominus f)(t)]h(t)$$
$$= [\nu^\Delta(t)f(\nu(t)) \oplus (\ominus f)(t)]h(t)$$
$$= [\nu^\Delta(t)f(\nu(t)) \ominus f(t)]h(t)$$
$$= ((\nu^\Delta f^\nu) \ominus f)(t)h(t),$$

concluding (2.7).
Furthermore, consider the following IVP

$$\begin{cases} g^\Delta(t) = ((\nu^\Delta f^\nu) \ominus f)(t)g(t), \\ g(t_0) = h(t_0) \end{cases} \tag{2.9}$$

and notice that $g(t) = h(t)$ is a solution of (2.9). Moreover, since $f \in \mathcal{R}$, we have that $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$ and hence

$$1 + \nu^\Delta(t)f(\nu(t)) \overset{(2.3)}{=} 1 + \mu(\nu(t))f(\nu(t)) \neq 0 \text{ for all } t \in \mathbb{T},$$

i.e., $\nu^\Delta f^\nu \in \mathcal{R}$, which implies that $(\nu^\Delta f^\nu) \ominus f \in \mathcal{R}$ and the equation (2.9) is regressive. Thus, by Theorem 1.2.20, $g(t) = e_{((\nu^\Delta f^\nu)\ominus f)}(t, t_0)h(t_0)$ satisfies (2.9) and since the solution is unique, we have that

$$h(t) = e_{((\nu^\Delta f^\nu)\ominus f)}(t, t_0)h(t_0),$$

i.e.,

$$e_f(\nu(t), t) = e_f(\nu(t_0), t_0)\frac{e_{\nu^\Delta f^\nu}(t, t_0)}{e_f(t, t_0)} \text{ for all } t \in \mathbb{T},$$

proving the desired result. □

Lemma 2.1.16 (See [7, Lemma 3.10])**.** *For $f \in \mathcal{R}$, we have*

$$e_f(\nu(t), \nu(s)) = e_{\nu^\Delta f^\nu}(t, s) \textit{ for all } s, t \in \mathbb{T}. \tag{2.10}$$

Proof. As an immediate consequence of Lemma 2.1.15, we have

$$e_f(\nu(t), \nu(s)) \overset{(1.15)}{=} e_f(\nu(t), t)e_f(t, s)e_f(s, \nu(s)) = \frac{e_f(\nu(t), t)}{e_f(\nu(s), s)}e_f(t, s)$$
$$\overset{(2.8)}{=} \frac{e_{\nu^\Delta f^\nu}(t, s)}{e_f(t, s)}e_f(t, s) = e_{v^\Delta f^\nu}(t, s) \text{ for all } t, s \in \mathbb{T}.$$ □

2.1.2 Periodicity

In this subsection, the definition of periodic functions on isolated time scales will be given, together with some simple examples. Moreover, the preservation of area property for periodic functions and some properties for the exponential function when it is composed with the iterated shift ν and the exponent is periodic (and also regressive) will be seen.

The next definition provides the so-expected definition of periodic functions on isolated time scales.

Definition 2.1.17 (See [7, Definition 4.1]). *A function $p\colon \mathbb{T} \to \mathbb{R}$ is called ω-**periodic** provided*

$$\nu^{\Delta} p^{\nu} = p. \tag{2.11}$$

*The set of all ω-*periodic *functions $p\colon \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{P}_{\omega} = \mathcal{P} = \mathcal{P}(\mathbb{T}, \mathbb{R})$.*

Remark 2.1.18 (See [7, Remark 4.2]). *By Lemma 2.1.5 we have that $\nu^{\Delta} = \frac{\mu^{\nu}}{\mu}$. Thus, $p \in \mathcal{P}$ (i.e., p is ω-*periodic*) if and only if $\mu^{\nu} p^{\nu} = \mu p$, i.e., if and only if*

$$(\mu p)^{\nu} = \mu p.$$

Now, we give three examples expressing the definition of periodic functions for $\mathbb{Z}$, $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$.

Example 2.1.19 (See [7, Example 4.3]). *If $\mathbb{T} = \mathbb{Z}$, then $\mu \equiv 1$ and $\nu(t) = t + \omega$ for all $t \in \mathbb{T}$. Thus, $p \in \mathcal{P}$ provided*

$$p(t) = p(t + \omega) \ \text{for all} \ t \in \mathbb{T},$$

which is the usual definition of ω-periodicity.

Example 2.1.20 (See [7, Example 4.4]). *If $\mathbb{T} = h\mathbb{Z}$, then $\mu \equiv h$ and $\nu(t) = t + h\omega, \forall t \in \mathbb{T}$. Thus, $p \in \mathcal{P}$ provided*

$$p(t) = p(t + h\omega) \ \text{for all} \ t \in \mathbb{T}.$$

Example 2.1.21 (See [7, Example 4.5]). *If $\mathbb{T} = q^{\mathbb{N}_0}$, then $\mu(t) = (q-1)t$ and $\nu(t) = q^{\omega} t, \forall t \in \mathbb{T}$. Thus, $p \in \mathcal{P}$ provided*

$$(q-1)\nu(t) p(\nu(t)) = (q-1) t p(t),$$

i.e., when

$$p(t) = q^{\omega} p(q^{\omega} t) \ \text{for all} \ t \in \mathbb{T},$$

which is the definition of periodicity from quantum calculus (see [3, Definition 3.1]).

The next lemma expresses the expected property for periodic functions.

Lemma 2.1.22 (See [7, Lemma 4.6])**.** *We have* $\mathcal{P}_\omega \subset \mathcal{P}_{2\omega}$.

Proof. Let $\hat{\nu}\colon \mathbb{T} \to \mathbb{T}$ defined by

$$\hat{\nu}(t) = \sigma^{2\omega}(t) = (\underbrace{\sigma \circ \cdots \circ \sigma}_{2\omega \text{ times}})(t) = \nu(\nu(t))$$

and let $p\colon \mathbb{T} \to \mathbb{R}$ be a ω-periodic function. It is sufficiently to show that $\hat{\nu}^\Delta p^{\hat{\nu}} = p$. More precisely, let $t \in \mathbb{T}$ and note that

$$\begin{aligned}\hat{\nu}^\Delta(t)p(\hat{\nu}(t)) &= (\nu^\nu)^\Delta(t)p(\hat{\nu}(t)) \overset{(2.4)}{=} \nu^\Delta(t)\nu^\Delta(\nu(t))p(\nu(\nu(t))) \\ &\overset{(2.11)}{=} \nu^\Delta(t)p(\nu(t)) \overset{(2.11)}{=} p(t),\end{aligned}$$

showing the desired result. □

The next two theorems show the preservation of area, also an expected property for periodic functions.

Theorem 2.1.23 (See [7, Theorem 4.7])**.** *If* $p \in \mathcal{P}$*, then the integral*

$$\int_t^{\nu(t)} p(\tau)\Delta\tau$$

is independent of $t \in \mathbb{T}$.

Proof. Define $h\colon \mathbb{T} \to \mathbb{R}$ by

$$h(t) = \int_t^{\nu(t)} p(\tau)\Delta\tau.$$

From Corollary 1.1.17 (ii), it is sufficient to show that h is constant. Indeed,

$$h^\Delta(t) \overset{(2.6)}{=} \nu^\Delta(t)p(\nu(t)) - p(t) \overset{(2.11)}{=} 0.$$ □

Theorem 2.1.24 (See [7, Theorem 4.8])**.** *If* $p \in \mathcal{P}$*, then*

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_s^t p(\tau)\Delta\tau \;\; \textit{for all} \;\; s, t \in \mathbb{T}.$$

Proof. Consider $h\colon \mathbb{T} \to \mathbb{R}$ given by

$$h(t) = \int_t^{\nu(t)} p(\tau)\Delta\tau$$

and note that

$$\begin{aligned}\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau &= \int_{\nu(s)}^{s} p(\tau)\Delta\tau + \int_s^t p(\tau)\Delta\tau + \int_t^{\nu(t)} p(\tau)\Delta\tau \\ &= -\int_s^{\nu(s)} p(\tau)\Delta\tau + \int_s^t p(\tau)\Delta\tau + \int_t^{\nu(t)} p(\tau)\Delta\tau \\ &= -h(s) + \int_s^t p(\tau)\Delta\tau + h(t),\end{aligned}$$

from where, since h is constant by Theorem 2.1.23, we conclude

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_s^t p(\tau)\Delta\tau. \qquad \square$$

Finally, two important properties for the exponential function follows.

Theorem 2.1.25 (See [7, Theorem 4.9]). *If $p \in \mathcal{P} \cap \mathcal{R}$, then*

$$e_p(\nu(t), t) \text{ is independent of } t \in \mathbb{T} \tag{2.12}$$

and

$$e_p(\nu(t), \nu(s)) = e_p(t, s) \text{ for all } s, t \in \mathbb{T}. \tag{2.13}$$

Proof. Since $p \in \mathcal{R}$, for any $t_0 \in \mathbb{T}$, we have

$$e_p(\nu(t), t) \overset{(2.8)}{=} e_p(\nu(t_0), t_0)\frac{e_{\nu^\Delta p^v}(t, t_0)}{e_p(t, t_0)} \text{ for all } t \in \mathbb{T},$$

and since $p \in \mathcal{P}$ we conclude that

$$e_p(\nu(t_0), t_0)\frac{e_{\nu^\Delta p^v}(t, t_0)}{e_p(t, t_0)} \overset{(2.11)}{=} e_p(\nu(t_0), t_0)\frac{e_p(t, t_0)}{e_p(t, t_0)} = e_p(\nu(t_0), t_0) \text{ for all } t \in \mathbb{T},$$

i.e., $e_p(\nu(t), t) = e_p(\nu(t_0), t_0)$ for all $t \in \mathbb{T}$ and (2.12) holds.
Now, since $p \in \mathcal{R}$, by Lemma 2.1.16, we get

$$e_p(\nu(t), \nu(s)) \overset{(2.10)}{=} e_{\nu^\Delta p^\nu}(t, s) \overset{(2.11)}{=} e_p(t, s) \text{ for all } t \in \mathbb{T}$$

and (2.13) holds. □

2.2 Almost periodicity in quantum calculus

This section develops the theory of almost periodicity for the quantum time scale, following the main reference [4]. Thus, consider $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$ in this whole section.

The first subsection establishes the Bochner definition of almost periodicity in quantum calculus along with some expected properties. Next, it examines some results surrounding this concept, along with its relation with the exponential function and a condition for a linear type of dynamic equation to has a Bochner almost periodic solution. For the second subsection, a more natural definition of almost periodicity will be seen, i.e., the Bohr definition, along with some characterizations for this class of Bohr almost periodic functions. Then, the equivalence between those two definitions will be proved, as the main result of this chapter. There will be no examples in this chapter, but several examples will be seen in the next chapter, when we provide a generalization of almost periodicity for isolated time scales.

2.2.1 Bochner almost periodicity in quantum calculus

We start with the Bochner definition for the quantum time scale.

Definition 2.2.1 (See [4, Definition 3.1]). *The function $f\colon \mathbb{T} \to \mathbb{R}$ is called **Bochner almost periodic** on $\mathbb{T}$ if for every sequence $\{t'_n\} \subset \mathbb{T}$, there exists a subsequence $\{t_n\} \subset \{t'_n\}$ such that the limit*

$$\lim_{n\to\infty} t_n f(tt_n)$$

exists uniformly on $\mathbb{T}$.

We state some elementary properties surrounding this definition.

Theorem 2.2.2 (See [4, Theorem 3.2]). *If $f, g\colon \mathbb{T} \to \mathbb{R}$ are Bochner almost periodic on $\mathbb{T}$, then*

(i) $f + g$ is Bochner almost periodic on $\mathbb{T}$;

(ii) cf is Bochner almost periodic on $\mathbb{T}$, for every $c \in \mathbb{R}$;

(iii) $f_k\colon \mathbb{T} \to \mathbb{R}$ defined by $f_k(t) := f(tq^k)$ is Bochner almost periodic on $\mathbb{T}$, for each $k \in \mathbb{N}_0$.

Next, we provide a definition of a q-bounded function.

Definition 2.2.3 (See [4, Definition 3.3]). *A function $f\colon \mathbb{T} \to \mathbb{R}$ is called q-**bounded** if*

$$t|f(t)| \leq K \ \textit{for all} \ \ t \in \mathbb{T},$$

for some $K > 0$.

The next result shows that any Bochner almost periodic function is q-bounded, an expected property.

Theorem 2.2.4 (See [4, Theorem 3.4]). *If $f\colon \mathbb{T} \to \mathbb{R}$ is Bochner almost periodic on $\mathbb{T}$, then f is q-bounded.*

Remark 2.2.5 (See [4, Remark 3.5]). *We will denote $T_{t_n} f = \overline{f}$ to represent that*

$$\lim_{n\to\infty} t_n f(tt_n) = \overline{f}(t) \ \textit{for every} \ \ t \in \mathbb{T}.$$

Definition 2.2.6 (See [4, Definition 3.6]). *The set*

$$H(f) := \{g\colon \mathbb{T} \to \mathbb{R} \ : \ \textit{there exists } \{t_n\} \subset \mathbb{T} \textit{ with } T_{t_n} f = g \textit{ uniformly}\}$$

*is called the **hull** of $f\colon \mathbb{T} \to \mathbb{R}$.*

The following theorem states a similar property of Bochner almost periodicity for the exponential function. Moreover, it is possible to determine the convergence of its subsequence in terms of the regressivity of $\overline{f}$ on a particular set.

Theorem 2.2.7 (See [4, Theorem 3.7]). *Assume $f\colon \mathbb{T} \to \mathbb{R}$ is regressive and Bochner almost periodic on $\mathbb{T}$. Then, for every sequence $\{t'_n\} \subset \mathbb{T}$, there exists a subsequence $\{t_n\} \subset \{t'_n\}$ such that for all $t, s \in \mathbb{T}$, denoting $T_{t_n} f = \overline{f}$, we have*

$$\lim_{n\to\infty} e_f(tt_n, st_n) = \begin{cases} e_{\overline{f}}(t,s), & \textit{if } \overline{f} \textit{ is regressive on } K, \\ 0, & \textit{otherwise}, \end{cases}$$

where $K = \{\min\{t, s\}\} \cup [\min\{t, s\}, \max\{t, s\})_{\mathbb{T}}$.

Remark 2.2.8 (See [4, Remark 3.8]). *If we assume that $f\colon \mathbb{T} \to \mathbb{R}$ is Bochner almost periodic on $\mathbb{T}$ and positive, then $\overline{f} = T_{t_n} f$ is also a positive function. Thus, if f is also regressive, the same will happen to $\overline{f}$.*

As an immediately consequence of the previous theorem, but with also asking $-f$ to be regressive, we have the same results for the $\cosh_f$ and $\sinh_f$ depending on two parameters.

Corollary 2.2.9 (See [4, Corollary 3.9]). *Let $f\colon \mathbb{T} \to \mathbb{R}$ be regressive with $-f$ also being regressive. If f is Bochner almost periodic on $\mathbb{T}$, then for every sequence $\{t'_n\} \subset \mathbb{T}$, there exists a subsequence $\{t_n\} \subset \{t'_n\}$ such that*

$$\lim_{n\to\infty} \cosh_f(tt_n, st_n) \ \textit{and} \ \lim_{n\to\infty} \sinh_f(tt_n, st_n)$$

exist uniformly on $\mathbb{T}$.

The next theorem states a condition for a solution of the following linear dynamic equation to be Bochner almost periodic, depending on the almost periodicity of the initial value of the problem.

Theorem 2.2.10 (See [4, Theorem 3.10]). *Let $a, b\colon \mathbb{T} \to \mathbb{R}$ be Bochner almost periodic functions on $\mathbb{T}$ with a regressive. If $x\colon \mathbb{T} \to \mathbb{R}$ solves*

$$x^{\Delta}(t) = a(t)x(t) + \frac{b(t)}{t}$$

and if for every $\{t'_n\} \subset \mathbb{T}$, there exists a subsequence $\{t_n\} \subset \{t'_n\}$ such that

$$\lim_{n\to\infty} t_n x(t_0 t_n) = x(t_0)$$

holds uniformly on $\mathbb{T}$, then x is Bochner almost periodic on $\mathbb{T}$.

Proof. Let $\{t'_n\} \subset \mathbb{T}$ be an arbitrary sequence. Since a and b are Bochner almost periodic on $\mathbb{T}$, there exists a subsequence $\{t_n\} \subset \{t'_n\}$ such that

$$\lim_{n\to\infty} t_n a(tt_n) = \overline{a}(t) \ \text{and} \ \lim_{n\to\infty} t_n b(tt_n) = \overline{b}(t)$$

exist uniformly on $\mathbb{T}$. Moreover, by Theorem 1.2.25, we have

$$x(t) = e_a(t, t_0 t_n)x(t_0 t_n) + \int_{t_0 t_n}^{t} e_a(t, \sigma(s))\frac{b(s)}{s}\Delta s.$$

Therefore, note that

$$\begin{aligned} t_n x(tt_n) &= t_n \left[e_a(tt_n, t_0 t_n)x(t_0 t_n) + \int_{t_0 t_n}^{tt_n} e_a(tt_n, \sigma(s))\frac{b(s)}{s}\Delta s \right] \\ &= e_a(tt_n, t_0 t_n)t_n x(t_0 t_n) + t_n \int_{t_0 t_n}^{tt_n} e_a(tt_n, \sigma(s))\frac{b(s)}{s}\Delta s \\ &\overset{(1.11)}{=} e_a(tt_n, t_0 t_n)t_n x(t_0 t_n) + t_n \int_{t_0}^{t} t_n e_a(tt_n, \sigma(st_n))\frac{b(st_n)}{st_n}\Delta s \end{aligned}$$

$$
\begin{aligned}
&= e_a(tt_n, t_0t_n)t_nx(t_0t_n) + \int_{t_0}^{t} e_a(tt_n, \sigma(st_n))\frac{t_nb(st_n)}{s}\Delta s \\
&= e_a(tt_n, t_0t_n)t_nx(t_0t_n) + \int_{t_0}^{t} e_a(tt_n, \sigma(s)t_n)\frac{t_nb(st_n)}{s}\Delta s
\end{aligned}
$$

and then

$$
\begin{aligned}
\lim_{n\to\infty} t_nx(tt_n) &= \lim_{n\to\infty}\left[e_a(tt_n, t_0t_n)t_nx(t_0t_n) + \int_{t_0}^{t} e_a(tt_n, t_n\sigma(s))\frac{t_nb(st_n)}{s}\Delta s\right] \\
&= e_{\overline{a}}(t, t_0)x(t_0) + \int_{t_0}^{t} e_{\overline{a}}(t, \sigma(s))\frac{\overline{b}(s)}{s}\Delta s,
\end{aligned}
$$

by Theorem 2.2.7, i.e., x is Bochner almost periodic on $\mathbb{T}$. □

For functions with two variables, we give the definition of Bochner almost periodicity depending on one variable. Similarly, it is possible to state this same definition (also with the following two results) for functions with more variables, with the periodicity depending on a certain variable.

Definition 2.2.11 (See [4, Definition 3.12]). *The function $f\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is called **Bochner almost periodic** on t for each $x\in\mathbb{R}$ if for every sequence $\{t'_n\}\subset\mathbb{T}$, there exists a subsequence $\{t_n\}\subset\{t'_n\}$ such that the limit*

$$
\lim_{n\to\infty} t_nf(tt_n, x)
$$

exists uniformly on $\mathbb{T}$ for each $x\in\mathbb{R}$.

Remark 2.2.12 (See [4, Remark 3.13]). *We will use the same notation $T_{t_n}f=\overline{f}$ to represent that*

$$
\lim_{n\to\infty} t_nf(tt_n, x) = \overline{f}(t, x) \ \text{for each}\ \ x\in\mathbb{R}.
$$

The elementary properties of Bochner almost periodicity also holds with this definition.

Theorem 2.2.13 (See [4, Theorem 3.14]). *If $f, g\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ are Bochner almost periodic with respect to the first variable for each $x\in\mathbb{R}$, then*

(i) $f+g$ is Bochner almost periodic with respect to the first variable, for each $x\in\mathbb{R}$;

(ii) cf is Bochner almost periodic with respect to the first variable, for each $x\in\mathbb{R}$ and for every $c\in\mathbb{R}$.

The next result shows that for a Bochner almost periodic function, as in the previous definition, which satisfies the Lipschitz condition in respect to a Bochner almost periodic funciton L (with one variable), the Lipschitz condition also holds for $\overline{f}$ in respect to $\overline{L}$.

Theorem 2.2.14 (See [4, Theorem 3.15]). *Assume $f\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is Bochner almost periodic for each $x\in\mathbb{R}$ and suppose that f satisfies Lipschitz condition*

$$|f(t,x)-f(t,y)|\leq L(t)|x-y| \text{ for all } t\in\mathbb{T} \text{ and } x,y\in\mathbb{R},$$

where $L\colon \mathbb{T}\to(0,\infty)$ is Bochner almost periodic, i.e., for every sequence $\{t'_n\}\subset\mathbb{T}$, there exists a subsequence $\{t_n\}\subset\{t'_n\}$ such that

$$\lim_{n\to\infty} t_n L(tt_n)=\overline{L}(t)$$

exists uniformly on $\mathbb{T}$. Then, $\overline{f}$ given by $T_{\alpha_n}f=\overline{f}$ satisfies the Lipschitz condition with the function $\overline{L}$.

2.2.2 Bohr almost periodicity in quantum calculus

We start with the Bohr's definition of almost periodicity, which is a more natural generalization of the periodicity concept.

Definition 2.2.15 (See [4, Definition 4.1]). *We say that $f\colon\mathbb{T}\to\mathbb{R}$ is **Bohr almost periodic** if for every $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that any N_ε consecutive elements of $\mathbb{T}$ contain at least one s with*

$$|sf(ts)-f(t)|<\varepsilon \text{ for all } t\in\mathbb{T}.$$

With this definition in hands, it is simple to show that any periodic function is Bohr almost periodic.

Remark 2.2.16 (See [4, Remark 4.2]). *If $f\colon\mathbb{T}\to\mathbb{R}$ is ω-periodic, that is,*

$$q^\omega f(q^\omega t)=f(t) \text{ for all } t\in\mathbb{T},$$

then f is Bohr almost periodic. Indeed, let $t\in\mathbb{T}$ and note that for every $\varepsilon>0$, there exists $N_\varepsilon=\omega+1\in\mathbb{N}$ such that any N_ε consecutive elements of $\mathbb{T}$ contain at least one s with

$$|sf(ts)-f(t)|=0<\varepsilon,$$

i.e., for any N_ε consecutive numbers, it is possible to find a multiple of ω, $k\omega$, such that $s = q^{k\omega}$ and thus

$$\begin{aligned} sf(ts) &= q^{k\omega} f(tq^{k\omega}) = q^{(k-1)\omega} f(q^{\omega}(q^{(k-1)\omega}t)) \\ &= q^{(k-1)\omega} f(q^{(k-1)\omega}t), \end{aligned}$$

from where, proceeding with this, we have that $sf(ts) = f(t)$.

Now, we start to show some results in order to show the equivalence between Bohr and Bochner definitions. The next result shows a characterization of Bohr almost periodic functions defined on $\mathbb{T}$ with some particular ones defined on $\mathbb{N}_0$.

Theorem 2.2.17 (See [4, Theorem 4.3]). *A necessary and sufficient condition for a function $g\colon \mathbb{T} \to \mathbb{R}$ to be Bohr almost periodic on $\mathbb{T}$ is the existence of a Bohr almost periodic function $f\colon \mathbb{N}_0 \to \mathbb{R}$ such that $g(t) = \frac{f(\log_q t)}{t}$ for every $t \in \mathbb{T}$.*

Theorem 2.2.18 (See [14, Theorem 1.27]). *A necessary and sufficient condition for a function $g\colon \mathbb{Z} \to \mathbb{R}$ to be Bohr almost periodic is the existence of a Bohr almost periodic $f\colon \mathbb{R} \to \mathbb{R}$ such that $g(n) = f(n)$ for all $n \in \mathbb{Z}$.*

The following result shows a characterization of Bohr almost periodic functions defined on $\mathbb{T}$ with some particular ones defined on $\mathbb{R}$.

Theorem 2.2.19 (See [4, Theorem 4.5]). *A necessary and sufficient condition for $g\colon \mathbb{T} \to \mathbb{R}$ to be Bohr almost periodic on $\mathbb{T}$ is the existence of a Bohr almost periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that $g(t) = \frac{f(\log_q t)}{t}$ for every $t \in \mathbb{T}$.*

Before proceeding to the next result, let us define the Bochner and Bohr almost periodicity concept for functions defined on $\mathbb{Z}$. Also, we state a result showing their equivalence in this case.

Definition 2.2.20 (See [14, Page 45]). *A function $f\colon \mathbb{Z} \to \mathbb{R}$ is **Bohr almost periodic** if for every $\varepsilon > 0$, there exists $N_\varepsilon \in\ N$ such that any N_ε consecutive elements of $\mathbb{Z}$ contain at least one s with*

$$|f(t+s) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{Z}.$$

*Moreover, f is **Bochner almost periodic** if for every sequence $\{t_n\} \subset \mathbb{Z}$, there exists a subsequence $\{t_n\} \subset \{t'_n\}$ such that the limit*

$$\lim_{n\to\infty} f(t+t_n)$$

exists uniformly on $\mathbb{Z}$.

Theorem 2.2.21 (See [14, Theorem 1.26]). *A function $f\colon \mathbb{Z} \to \mathbb{R}$ is Bohr almost periodic if, and only if, f is Bochner almost periodic.*

Remark 2.2.22. *Note that if a function f is defined on a subset of $\mathbb{Z}$, then Theorem 2.2.21 also holds, where the Bochenr's and Bohr's definitions are analogue.*

The next result shows the so expected equivalence between the two definitions of almost periodicity and it is the main result of the paper [4].

Theorem 2.2.23 (See [4, Theorem 4.6]). *A function $f\colon \mathbb{T} \to \mathbb{R}$ is Bochner almost periodic if, and only if, f is Bohr almost periodic.*

Proof. Suppose f is Bochner almost periodic on $\mathbb{T}$, but f is not Bohr almost periodic. Therefore, there exists at least one $\varepsilon > 0$ such that for any $N_\varepsilon \in \mathbb{N}$, the set of N_ε consecutive numbers in $\mathbb{T}$ does not contain any element t satisfying

$$|sf(ts) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{T}. \tag{2.14}$$

Let $\tau \in \mathbb{T}\backslash\{1, q\}$ and any arbitrary number $\alpha_1 \in \mathbb{N}$, then there are no elements satisfying (2.14) on $[\tau, \tau q^{\alpha_1}) \cap \mathbb{T}$. Also, for $\alpha_2 = \log_q(\tau)\alpha_1$, there are no elements satisfying (2.14) on $[\tau q^{\alpha_1}, \tau q^{\alpha_1+\alpha_2}) \cap \mathbb{T}$. Proceeding with this, we can construct a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_k = q^{\alpha_k}$ such that $\{t_k\}$ is strictly increasing and $t_k \to \infty$ when $k \to \infty$. Therefore, for any $i > j > 1$, we have

$$\begin{aligned}
\sup_{t\in\mathbb{T}} |t_i f(tt_i) - t_j f(tt_j)| &= \sup_{t\in\mathbb{T}} \left| \frac{t_j}{t_j} t_i f(tt_i) - t_j f(tt_j) \right| \\
&= t_j \sup_{t\in\mathbb{T}} \left| \frac{t_i}{t_j} f\left(\frac{t}{t_j} t_i\right) - f\left(\frac{t}{t_j} t_j\right) \right| \\
&= t_j \sup_{t\in\mathbb{T}} \left| \frac{t_i}{t_j} f\left(t\frac{t_i}{t_j}\right) - f(t) \right| \\
&\geq \sup_{t\in\mathbb{T}} \left| \frac{t_i}{t_j} f\left(t\frac{t_i}{t_j}\right) - f(t) \right| \geq \varepsilon.
\end{aligned}$$

Thus, $\{t_n f(tt_n)\}$ cannot contain any uniformly convergent sequence, which is a contradiction for the fact that f is Bochner almost periodic on $\mathbb{T}$. Then, f is Bohr almost periodic.

Reciprocally, if f is Bohr almost periodic, then, by Theorem 1.1.16, the function $g\colon \mathbb{N}_0 \to \mathbb{R}$ given by

$$g(n) = q^n f(q^n) \text{ for } n \in \mathbb{N}_0$$

is Bohr almost periodic. Hence, by Theorem 2.2.21, g is Bochner almost periodic, i.e., for every sequence $\{n'_k\} \subset \mathbb{N}_0$, there exists a subsequence $\{n_k\}$ such that the limit

$$\lim_{k\to\infty} g(n+n_k)$$

exists uniformly for every $n \in \mathbb{N}_0$. Thus, since

$$g(n+n_k) = q^{n+n_k} f(q^{n+n_k}) = q^n q^{n_k} f(q^n q^{n_k}) \text{ for all } n \in \mathbb{N}_0,$$

the limit $\lim_{k\to\infty} q^n q^{n_k} f(q^n q^{n_k})$ also exists uniformly. Now, let $\{t'_n\} \subset \mathbb{T}$ be a sequence. Then, $t'_k = q^{n'_k}$ for some $n'_k \in \mathbb{N}_0$ and we can construct a sequence $\{n'_k\} \subset \mathbb{N}_0$. Hence, there exists a subsequence $\{n_k\} \subset \{n'_k\}$ such that the limit $\lim_{k\to\infty} q^n q^{n_k} f(q^n q^{n_k})$ exists uniformly for every $n \in \mathbb{N}_0$, i.e., since $q^n \geq 1$, the limit

$$\lim_{k\to\infty} q^{n_k} f(q^n q^{n_k})$$

also exists uniformly for every $n \in \mathbb{N}_0$. Therefore, considering the subsequence $\{t_k\} \subset \{t'_k\}$ given by $t_k = q^{n_k}$ and denoting $t = q^n$, we obtain that the limit

$$\lim_{k\to\infty} t_k f(tt_k)$$

exists uniformly for every $t \in \mathbb{T}$ and f is Bochner almost periodic. □

Chapter 3

Almost periodicity on isolated time scales

This chapter is the most important of this work and generalize the theory of almost periodicity from the quantum time scale to any isolated time scale. All the results found here are completely new in the literature and they can be found in [5]. Section 1 introduces a new definition of iterated shift. Section 2 gives the Bochner's definition of almost periodic functions, we prove several properties and we discuss Bochner almost periodic solutions of a first order linear dynamic equation. In Section 3, we provide the Bohr's definition of the almost periodicity, we prove that any ω-periodic function (in the sense of [7]) is also Bohr almost periodic, we state some equivalences regarding this definition and clarify its relation with the almost periodicity from Bochner. Section 4 exemplifies those concepts for different types of functions and for some of the most classical isolated time scales, e.g., $\mathbb{Z}$, $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$. In this entire chapter, $\mathbb{T}$ will be considered an isolated time scale (i.e., all points in $\mathbb{T}$ are right-scattered and left-scattered, except when the time scale has a minimum or maximum (or both). In this case, the minimum point must be right-scattered and the maximum point must be left-scattered).

3.1 Iterated shifts

In order to be able to present the definition of an almost periodic function on isolated time scales, we need to introduce a new iterated shift operator.

Definition 3.1.1. *Let $\alpha \in \mathbb{Z}$ and define the* ***iterated shift operator*** $\nu_\alpha \colon \mathbb{T} \to \mathbb{T}$ *by*

$$\nu_\alpha(t) = \begin{cases} \sigma^\alpha(t), & \alpha > 0, \\ t, & \alpha = 0, \\ \rho^\alpha(t), & \alpha < 0. \end{cases}$$

For $f\colon \mathbb{T}\to\mathbb{R}$, we use the notation

$$f^{\nu_\alpha} = f\circ\nu_\alpha.$$

To this new iterated shift operator, the following lemmas also extend the analogue ones for the similar shift operator presented in the previous chapter. Therefore, we omit their proofs here.

Lemma 3.1.2. *For all $\alpha\in\mathbb{Z}$, we have*

$$\nu_\alpha^\Delta = \frac{\mu^{\nu_\alpha}}{\mu}.$$

Proof. The proof is analogous to the proof of Lemma 2.1.5. □

Lemma 3.1.3. *For $f\colon \mathbb{T}\to\mathbb{R}$, we have*

$$f^{\nu_\alpha\Delta} = \nu_\alpha^\Delta f^{\Delta\nu_\alpha}. \tag{3.1}$$

Proof. The proof is analogous to the proof of Lemma 2.1.6. □

Lemma 3.1.4. *For $f\in\mathcal{R}$, we have*

$$e_f(\nu_\alpha(t),\nu_\alpha(s)) = e_{\nu_\alpha^\Delta f^{\nu_\alpha}}(t,s) \text{ for all } s,t\in\mathbb{T}. \tag{3.2}$$

Proof. The proof is analogous to the proof of Lemma 2.1.16. □

Remark 3.1.5. *Analogously, it is possible to prove the Lemma 3.1.4 changing f by a regressive $n\times n$-matrix-valued-function A defined on $\mathbb{T}$, but using the properties from Theorem 1.3.12.*

3.2 Bochner almost periodicity on isolated time scales

In this section, our goal is to investigate the Bochner almost periodic functions in the framework of isolated time scales.

Definition 3.2.1. *A function $f\colon\mathbb{T}\to\mathbb{R}$ is called **Bochner almost periodic** on $\mathbb{T}$ if for every sequence $\{\alpha_n'\}\subset D$, there exists a subsequence $\{\alpha_n\}\subset\{\alpha_n'\}$ such that $\lim_{n\to\infty}\nu_{\alpha_n}^\Delta(t)f(\nu_{\alpha_n}(t))$ exists uniformly on $\mathbb{T}$, for D defined by*

$$D := \begin{cases} \mathbb{Z}, & \textit{if } \sup\mathbb{T}=+\infty \textit{ and } \inf\mathbb{T}=-\infty,\\ \mathbb{N}_0, & \textit{if } \sup\mathbb{T}=+\infty \textit{ and } \inf\mathbb{T}>-\infty,\\ -\mathbb{N}_0, & \textit{if } \sup\mathbb{T}<+\infty \textit{ and } \inf\mathbb{T}=-\infty,\end{cases}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $-\mathbb{N}_0 = \{-n \ : \ n \in \mathbb{N}_0\}$. *We will denote the set of all Bochner almost periodic and regressive functions* $f \colon \mathbb{T} \to \mathbb{R}$ *by* $AP_B(\mathbb{T}, \mathbb{R})$ *or simply* AP_B.

Remark 3.2.2. *The set* D *was constructed in order to admit a directly bijection with the respective time scale, which further we will state an equivalence between* D *and* $\mathbb{T}$ *within Bohr's framework of almost periodicity.*

Remark 3.2.3. *The set* D *in Definition 3.2.1 does not consider the case when* $\sup \mathbb{T} < +\infty$ *and* $\inf \mathbb{T} > -\infty$ *at the same time because in this case* D *should take values in a finite subset of* $\mathbb{Z}$, *from which any sequence in* D *will have at least one of its values repeating infinitely many times (by the Dirichlet's principle), and, hence, any function in this case will be Bochner almost periodic. Thus, from now on, we will only consider isolated time scales* $\mathbb{T}$ *with* $\sup \mathbb{T} = +\infty$ *or* $\inf \mathbb{T} = -\infty$ *(or both).*

Remark 3.2.4. *It is also possible to define a Bochner almost periodic* $n \times n$*-matrix-valued-function defined on* $\mathbb{T}$ *in analogy, i.e., an* $n \times n$*-matrix-valued-function* $A(t)$ *defined on* $\mathbb{T}$ *is* ***Bochner almost periodic*** *on* $\mathbb{T}$ *if for every sequence* $\{\alpha'_n\} \subset D$, *there exists a subsequence* $\{\alpha_n\} \subset \{\alpha'_n\}$ *such that* $\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) A(\nu_{\alpha_n}(t))$ *exists uniformly on* $\mathbb{T}$.

Remark 3.2.5. *By Lemma 3.1.2, the definition above can be rewritten as follows: A function* $f \colon \mathbb{T} \to \mathbb{R}$ *is called* ***Bochner almost periodic*** *on* $\mathbb{T}$ *if for every sequence* $\{\alpha'_n\} \subset D$, *there exists a subsequence* $\{\alpha_n\} \subset \{\alpha'_n\}$ *such that* $\lim_{n\to\infty} \frac{\mu^{\nu_{\alpha_n}(t)}}{\mu(t)} f(\nu_{\alpha_n}(t))$ *exists uniformly on* $\mathbb{T}$.

Remark 3.2.6. *Although the Definition 3.2.1 is presented for functions taking value in* $\mathbb{R}$, *it is possible to extend this notion for functions taking value in any arbitrary Banach space* X. *The definition follows similarly. The same happens to the other properties.*

A first look at this definition may be different from the known ones for the discrete cases, but below we bring some examples which show that our definition is consistent with the eixsting ones.

Example 3.2.7. *Let* $\mathbb{T} = \mathbb{Z}$. *Then Definition 3.2.1 can be read as follows: A function* $f \colon \mathbb{Z} \to \mathbb{R}$ *is called* ***Bochner almost periodic*** *if for every sequence* $\{\alpha'_n\} \subset \mathbb{Z}$, *there exists a subsequence* $\{\alpha_n\} \subset \{\alpha'_n\}$ *such that*

$$\lim_{n\to\infty} f(t + \alpha_n)$$

exists uniformly on $\mathbb{Z}$. *This happens since* $\nu_{\alpha_n}(t) = t + \alpha_n$ *for* $t \in \mathbb{Z}$ *by definition of shifted operator. This definition is consistent with the definition for discrete case (see [14, Page 45]).*

Example 3.2.8. *Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. The Definition 3.2.1 can be read as follows: A function $f\colon q^{\mathbb{N}_0} \to \mathbb{R}$ is called **Bochner almost periodic** on $q^{\mathbb{N}_0}$ if for every sequence $\{\alpha'_n\} \subset \mathbb{N}_0$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that*

$$\lim_{n\to\infty} q^{\alpha_n} f(q^{\alpha_n} t)$$

exists uniformly on $q^{\mathbb{N}_0}$. Again, it is consistent with the known one (see [4, Definition 3.1]).

Remark 3.2.9. *A careful examination at the Definition 3.2.1 shows that the changing on D does not affect significantly the proofs of the results. Therefore, in order to avoid repetitions, we will state and prove the results for the case when $D = \mathbb{Z}$, since the other ones follow analogously.*

In the sequel, we prove some elementary properties of Bochner almost periodicity.

Theorem 3.2.10. *If $f, g\colon \mathbb{T} \to \mathbb{R}$ are Bochner almost periodic functions, then*

(i) $f + g$ is Bochner almost periodic;

(ii) cf is Bochner almost periodic for every $c \in \mathbb{R}$;

(iii) Let $f_\omega\colon \mathbb{T} \to \mathbb{R}$ be defined by $f_\omega(t) := f(\nu_\omega(t))$. If for every sequence $\{\alpha'_n\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that the limit

$$\lim_{n\to\infty} \frac{\nu^{\Delta}_{\alpha_n}(t)}{\nu^{\Delta}_{\alpha_n}(\nu_\omega(t))}$$

exists uniformly, then f_ω is Bochner almost periodic for each $\omega \in \mathbb{Z}$.

Proof. (i) Let $\{\alpha'_n\} \subset \mathbb{Z}$ be an arbitrary sequence. Since f and g are Bochner almost periodic functions, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) f(\nu_{\alpha_n}(t)) \quad \text{and} \quad \lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) g(\nu_{\alpha_n}(t))$$

exist uniformly. Therefore,

$$\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t)(f+g)(\nu_{\alpha_n}(t)) = \lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) f(\nu_{\alpha_n}(t)) + \lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) g(\nu_{\alpha_n}(t))$$

also exists uniformly and $f + g$ is Bochner almost periodic.

(ii) Let $\{\alpha'_n\} \subset \mathbb{Z}$ be an arbitrary sequence. Since f is Bochner almost periodic, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) f(\nu_{\alpha_n}(t))$$

exists uniformly. Therefore,

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t)(cf)(\nu_{\alpha_n}(t)) = c \lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t)),$$

also exists uniformly and cf is Bochner almost periodic.

(iii) Let $\{\alpha_n'\} \subset \mathbb{Z}$ be an arbitrary sequence. Since f is Bochner almost periodic function, there exists a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t))$$

exists uniformly. Therefore, for each $\omega \in \mathbb{Z}$, we have

$$\begin{aligned}\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t)f_{\omega}(\nu_{\alpha_n}(t)) &= \lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(\nu_{\omega}(t)))\\ &= \lim_{n\to\infty} \frac{\nu_{\alpha_n}^{\Delta}(t)}{\nu_{\alpha_n}^{\Delta}(\nu_{\omega}(t))}\nu_{\alpha_n}^{\Delta}(\nu_{\omega}(t))f(\nu_{\alpha_n}(\nu_{\omega}(t)))\end{aligned}$$

exists uniformly. Thus, f_{ω} is Bochner almost periodic for each $\omega \in \mathbb{Z}$. □

We provide a definition of a ν-bounded function, which is similar to the definition of a q-bounded function, but here it is in a more general setting.

Definition 3.2.11. *For each $t \in \mathbb{T}$, a function $f\colon \mathbb{T} \to \mathbb{R}$ is called **ν-bounded** if there exists $K > 0$ such that*

$$\nu_{\alpha}^{\Delta}(t)|f(\nu_{\alpha}(t))| < K \text{ for all } \alpha \in \mathbb{Z}.$$

Remark 3.2.12. *Notice that if $\mathbb{T} = \mathbb{Z}$, then it coincides to the concept of bounded functions. The next result shows that the same way as the boundedness is a consequence of the classical almost periodic functions, the ν-boundedness is a consequence of almost periodicity for any isolated time scale. It is the content of the next result.*

Theorem 3.2.13. *Bochner almost periodic functions $f\colon \mathbb{T} \to \mathbb{R}$ are ν-bounded.*

Proof. Let $t \in \mathbb{T}$ and $f\colon \mathbb{T} \to \mathbb{R}$ be a Bochner almost periodic function which is not ν-bounded. Then, there exists a sequence $\{\alpha_n'\} \subset \mathbb{Z}$ such that for every subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$, we have

$$\nu_{\alpha_n}^{\Delta}(t)|f(\nu_{\alpha_n}(t))| \to \infty.$$

Thus, the limit

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t))$$

does not converges for any subsequence of $\{\alpha_n'\}$, which contradicts the fact that f is Bochner almost periodic. Therefore, f should be ν-bounded, proving the desired result. □

Example 3.2.14. *Let $\mathbb{T} = \mathbb{Z}$. Then the function $f\colon \mathbb{Z} \to \mathbb{R}$ given by*

$$f(n) = \cos(n\pi) + \sin(\sqrt{n}\pi)$$

is Bochner almost periodic and, by Theorem 3.2.13, f is also ν-bounded.

Remark 3.2.15. *We will denote $T_{\alpha_n} f = \overline{f}$ to represent*

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t)) = \overline{f}(t) \ \text{for every} \ \ t \in \mathbb{T}.$$

Definition 3.2.16. *The set*

$$H(f) \coloneqq \{g\colon \mathbb{T} \to \mathbb{R} \mid \text{there exists } \{\alpha_n\} \subset D \text{ with } T_{\alpha_n} f = g \text{ uniformly}\}$$

*is called the **hull** of $f\colon \mathbb{T} \to \mathbb{R}$.*

Below, we prove some properties of Bochner almost periodicity related to the circle minus and circle plus operations on time scales. This result shows consistency of the definition introduced in the framework of isolated time scales.

Theorem 3.2.17. *If $f, g\colon \mathbb{T} \to \mathbb{R}$ are Bochner almost periodic and regressive functions, then*

(i) $f \oplus g$ is Bochner almost periodic;

(ii) $f \ominus g$ is Bochner almost periodic;

(iii) $\ominus f$ is Bochner almost periodic.

Proof. Let $\{\alpha_n'\} \subset \mathbb{Z}$ be an arbitrary sequence. Since f and g are Bochner almost periodic functions, there exists a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t)) = \overline{f}(t) \ \text{ and } \ \lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) g(\nu_{\alpha_n}(t)) = \overline{g}(t) \tag{3.3}$$

exist uniformly. Using this, let us prove each item.

(i) For $f \oplus g$, applying Lemma 3.1.2, we get

$$\nu_{\alpha_n}^{\Delta}(t)(f \oplus g)(\nu_{\alpha_n}(t)) = \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t)) + \nu_{\alpha_n}^{\Delta}(t) g(\nu_{\alpha_n}(t))$$

$$+ \mu(t)\nu^{\Delta}_{\alpha_n}(t)f(\nu_{\alpha_n}(t))\nu^{\Delta}_{\alpha_n}(t)g(\nu_{\alpha_n}(t)),$$

and, hence, by (3.3),

$$\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t)(f \oplus g)(\nu_{\alpha_n}(t)) = \overline{f}(t) + \overline{g}(t) + \mu(t)\overline{f}(t)\overline{g}(t) = (\overline{f} \oplus \overline{g})(t).$$

Therefore, $f \oplus g$ is Bochner almost periodic.

(ii) For $f \ominus g$, by Lemma 3.1.2, it follows that

$$\nu^{\Delta}_{\alpha_n}(t)(f \ominus g)(\nu_{\alpha_n}(t)) = \frac{\nu^{\Delta}_{\alpha_n}(t)f(\nu_{\alpha_n}(t)) - \nu^{\Delta}_{\alpha_n}(t)g(\nu_{\alpha_n}(t))}{1 + \mu(t)\nu^{\Delta}_{\alpha_n}(t)g(\nu_{\alpha_n}(t))},$$

and, thus, again by (3.3),

$$\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t)(f \ominus g)(\nu_{\alpha_n}(t)) = \frac{\overline{f}(t) - \overline{g}(t)}{1 + \mu(t)\overline{g}(t)} = (\overline{f} \ominus \overline{g}).$$

Therefore, $f \ominus g$ is Bochner almost periodic.

(iii) By Lemma 3.1.2, we have

$$\nu^{\Delta}_{\alpha_n}(t)(\ominus f)(\nu_{\alpha_n}(t)) = -\frac{\nu^{\Delta}_{\alpha_n}(t)f(\nu_{\alpha_n}(t))}{1 + \mu(t)\nu^{\Delta}_{\alpha_n}(t)f(\nu_{\alpha_n}(t))},$$

and, then,

$$\lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t)(\ominus f)(\nu_{\alpha_n}(t)) = -\frac{\overline{f}(t)}{1 + \mu(t)\overline{f}(t)} = (\ominus \overline{f})(t).$$ □

As an immediate consequence, we get the following result.

Corollary 3.2.18. $(AP_B, \oplus)$ *is a subgroup of* $(\mathcal{R}, \oplus)$.

Proof. It is clear that AP_B is nonempty and $AP_B \subset \mathcal{R}$. Also, from Theorem 3.2.17, AP_B is closed under the given operation. Thus, since any function $f \in AP_B$ has a symmetric element $\ominus f \in AP_B$, the zero function is clearly the identity element for this operation and the associativity holds for this operation, then AP_B is a group and, hence, a subgroup of $\mathcal{R}$. □

The next lemma provides a different way to write the definition of Bochner almost periodicity.

Lemma 3.2.19. *A function $f\colon \mathbb{T} \to \mathbb{R}$ is Bochner almost periodic if, and only if, for every sequence $\{\alpha'_n\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that*

$$\lim_{n\to\infty} \mu(\nu_{\alpha_n}(t))f(\nu_{\alpha_n}(t)) = \mu(t)\overline{f}(t)$$

uniformly for all $t \in \mathbb{T}$.

Proof. It follows directly by Lemma 3.1.2. □

The next result shows that linear combinations of Bochner almost periodic functions and some other expressions involving Bochner almost periodic functions are also Bochner almost periodic.

Theorem 3.2.20. *Under the same assumptions of Theorem 3.2.17, one has:*

$$af + bg \in AP_B \text{ and } \mu fg \in AP_B, \text{ for all } a, b \in \mathbb{R}.$$

Also, if $a + \mu(t)g(t) \neq 0$ and $a + \mu(t)\overline{g}(t) \neq 0$ for all $t \in \mathbb{T}$, then

$$\frac{f}{a+\mu g} \in AP_B.$$

Proof. Assuming $a, b \in \mathbb{R}$ and $f, g \in AP_B$, for every sequence $\{\alpha'_n\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$[\mu(af+bg)]^{\nu_{\alpha_n}} = a(\mu f)^{\nu_{\alpha_n}} + b(\mu g)^{\nu_{\alpha_n}} \xrightarrow{n\to\infty} a(\mu\overline{f}) + b(\mu\overline{g})$$

uniformly. Also,

$$[\mu(\mu fg)]^{\nu_{\alpha_n}} = (\mu f)^{\nu_{\alpha_n}}(\mu g)^{\nu_{\alpha_n}} \xrightarrow{n\to\infty} \mu\overline{f}\cdot\mu\overline{g}$$

uniformly. Finally, if $a + \mu(t)g(t) \neq 0$ and $a + \mu(t)\overline{g}(t) \neq 0$ for all $t \in \mathbb{T}$, then

$$\left[\mu \cdot \frac{f}{a+\mu g}\right]^{\nu_{\alpha_n}} = \frac{(\mu f)^{\nu_{\alpha_n}}}{a + (\mu g)^{\nu_{\alpha_n}}} \xrightarrow{n\to\infty} \frac{\mu\overline{f}}{a+\mu\overline{g}}$$

uniformly, proving the desired result. □

In what follows, we show a very interesting property for exponential function.

Theorem 3.2.21. *Assume $f \in AP_B(\mathbb{T}, \mathbb{R})$. Then for every sequence $\{\alpha'_n\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that for all $t, s \in \mathbb{T}$, denoting $T_{\alpha_n} f = \overline{f}$, we have*

$$\lim_{n\to\infty} e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) = \begin{cases} e_{\overline{f}}(t,s), & \text{if } \overline{f} \text{ is regressive on } R, \\ 0, & \text{otherwise,} \end{cases}$$

where $R = \{\min\{t,s\}\} \cup [\min\{t,s\}, \max\{t,s\})_{\mathbb{T}}$.

Proof. If $f\colon \mathbb{T} \to \mathbb{R}$ is Bochner almost periodic, then for every sequence $\{\alpha'_n\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t)) = \overline{f}(t) \text{ for every } t \in \mathbb{T}$$

uniformly, i.e., $T_{\alpha_n} f = \overline{f}$. Therefore, for $s < t$,

$$\begin{aligned} e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) &\overset{(3.2)}{=} e_{\nu_{\alpha_n}^{\Delta} f^{\nu_{\alpha_n}}}(t,s) \\ &= \exp\left(\sum_{\tau\in[s,t)} \mu(\tau) \frac{1}{\mu(\tau)} \ln\left(1 + \mu(\tau)\nu_{\alpha_n}^{\Delta}(\tau) f(\nu_{\alpha_n}(\tau))\right) \right) \\ &= \prod_{\tau\in[s,t)} \left(1 + \mu(\tau)\nu_{\alpha_n}^{\Delta}(\tau) f(\nu_{\alpha_n}(\tau))\right) \end{aligned}$$

which implies

$$\begin{aligned} \lim_{n\to\infty} e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) &= \lim_{n\to\infty} \prod_{\tau\in[s,t)} \left(1 + \mu(\tau)\nu_{\alpha_n}^{\Delta}(\tau) f(\nu_{\alpha_n}(\tau))\right) \\ &= \prod_{\tau\in[s,t)} \left(1 + \mu(\tau)\overline{f}(\tau)\right) \\ &= e_{\overline{f}}(t,s), \end{aligned}$$

if $\overline{f}$ is regressive on R. Otherwise, we obtain

$$\lim_{n\to\infty} e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) = \prod_{\tau\in[s,t)} [1 + \mu(\tau)\overline{f}(\tau)] = 0.$$

Now, if $t = s$, then

$$e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(t)) = 1 = e_{\overline{f}}(t,t),$$

where $\overline{f}$ is regressive at t by hypothesis. Finally, if $t < s$, then

$$\lim_{n\to\infty} e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) = \lim_{n\to\infty} \frac{1}{e_f(\nu_{\alpha_n}(s), \nu_{\alpha_n}(t))} = \frac{1}{e_{\overline{f}}(s,t)} = e_{\overline{f}}(t,s),$$

if $\overline{f}$ is regressive on R. □

Remark 3.2.22. *It is possible to prove the Theorem 3.2.21 changing f by an regressive and Bochner almost periodic $n \times n$-matrix-valued-function A defined on $\mathbb{T}$ (Remark 3.2.4), by using Remark 3.1.5.*

Remark 3.2.23. *Assuming that $f\colon \mathbb{T} \to \mathbb{R}$ is a nonnegative Bochner almost periodic function, then $\overline{f} = T_{\alpha_n} f$ is also a nonnegative function. Thus, if f is also regressive, the same happen to $\overline{f}$, because*

$$1 + \mu(t)\overline{f}(t) > 0 \ \textit{for all} \ \ t \in \mathbb{T},$$

since $\mu(t) > 0$ and $\overline{f}(t) \geq 0$ for all $t \in \mathbb{T}$.

As an immediately consequence of the previous theorem, the following holds.

Corollary 3.2.24. *Let $f\colon \mathbb{T} \to \mathbb{R}$ be regressive with $-f$ being also regressive. If f is Bochner almost periodic, then for every sequence $\{\alpha'_n\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that*

$$\lim_{n\to\infty} \cosh_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) \ \textit{and} \ \lim_{n\to\infty} \sinh_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s))$$

exist uniformly on $\mathbb{T}$.

Proof. Let $\{\alpha'_n\} \subset \mathbb{Z}$ be an arbitrary sequence. Thus, by Theorem 3.2.21, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that $\lim_{n\to\infty} e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s))$ and $\lim_{n\to\infty} e_{-f}(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s))$ both exist uniformly for every $t, s \in \mathbb{T}$. Therefore, since

$$\cosh_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) = \frac{e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) + e_{-f}(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s))}{2}$$

and

$$\sinh_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) = \frac{e_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) - e_{-f}(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s))}{2},$$

we conclude that both

$$\lim_{n\to\infty} \cosh_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s)) \ \text{and} \ \lim_{n\to\infty} \sinh_f(\nu_{\alpha_n}(t), \nu_{\alpha_n}(s))$$

exist uniformly for each $t, s \in \mathbb{T}$. □

We now establish the definition of exponential dichotomy and some preliminary results in order to derive the sufficient conditions to the first order linear dynamic equation to have a Bochner almost periodic solution.

Definition 3.2.25. *Let $A(t)$ be a $n \times n$ rd-continuous and regressive matrix-valued function on $\mathbb{T}$. We say that the linear system*

$$X^{\Delta}(t) = A(t)X(t) \tag{3.4}$$

has an ***exponential dichotomy*** *on $\mathbb{T}$ if there exist positive constants K and γ, and a projection*

P, which commutes with $X(t)$, $t \in \mathbb{T}$, where $X(t)$ is a fundamental matrix of (3.4) *satisfying*

$$|X(t)PX^{-1}(s)| \leq Ke_{\ominus\frac{\gamma}{\mu}}(t,s) \text{ for all } s,t \in \mathbb{T}, t \geq s,$$

and

$$|X(t)(I-P)X^{-1}(s)| \leq Ke_{\ominus\frac{\gamma}{\mu}}(s,t) \text{ for all } s,t \in \mathbb{T}, s \geq t.$$

Remark 3.2.26. *A first look for the definition of exponential dichotomy may be different, since it appears $\frac{\gamma}{\mu}$ instead of γ. However, since we are dealing with isolated time scales, we have $\frac{\gamma}{\mu} > 0$ whenever $\gamma > 0$. Therefore, here we have a generalization in the sense that we have a positive functions appearing instead of a positive constant. This will be necessary to our purposes. Notice that if μ is constant, as in the case of $\mathbb{Z}, h\mathbb{Z}$ and $\mathbb{N}$, this definition colapses with the classical one, bringing novelty only in the cases that μ is not constant.*

The next result describes a solution of (3.4) when it admits an exponential dichotomy.

Theorem 3.2.27 (See [26, Lemma 2.13])**.** *If the linear system* (3.4) *admits an exponential dichotomy, then the system*

$$X^{\Delta}(t) = A(t)X(t) + f(t)$$

has a bounded solution $x(t)$ as follows:

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s,$$

where $X(t)$ is the fundamental matrix of (3.4).

Remark 3.2.28. *Notice that the fundamental matrix $X(t)$ is a solution of*

$$\begin{cases} X^{\Delta}(t) = A(t)X(t), \\ X(t_0) = X_0. \end{cases}$$

Therefore, by the uniqueness of solution

$$X(t)X^{-1}(t_0) = e_A(t,t_0).$$

Lemma 3.2.29. *If $A\colon \mathbb{T} \to \mathbb{R}^{n\times n}$ is an almost periodic function, then for every sequence $\{\alpha'_n\} \subset D$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that*

$$\lim_{n\to\infty} X(\nu_{\alpha_n}(t))X^{-1}(\nu_{\alpha_n}(t_0))$$

exists uniformly on $\mathbb{T}$.

Proof. It follows directly from Remark 3.2.28 and Remark 3.2.22. □

Finally, we give sufficient conditions to the first order linear dynamic equation to have a Bochner almost periodic solution.

Theorem 3.2.30. *Let* $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ *be Bochner almost periodic and nonsingular on* $\mathbb{T}$. *Suppose* (3.4) *admits an exponential dichotomy with positive constants* K *and* γ *and for every sequence* $\{\alpha'_n\} \subset \mathbb{Z}$, *there exists a subsequence* $\{\alpha_n\} \subset \{\alpha'_n\}$ *such that*

$$\lim_{n\to+\infty} \nu^{\Delta}_{\alpha_n}(t)\nu^{\Delta\sigma}_{\alpha_n}(t)f(\nu_{\alpha_n}(t)) = \overline{f}(t) \tag{3.5}$$

exists uniformly for all $t \in \mathbb{T}$. *Also, assume that* $\frac{\nu^{\Delta}_{\alpha_n}(t)}{\nu^{\Delta}_{\alpha_n}(\sigma(s))}$, *for* $t, s \in \mathbb{T}$, *is bounded for each* $n \in \mathbb{N}$. *Then, the equation*

$$X^{\Delta}(t) = A(t)X(t) + f(t) \tag{3.6}$$

has an almost periodic solution.

Proof. By Theorem 3.2.27, the following function

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s$$

is a bounded solution of (3.6). It remains to show that the solution is almost periodic. Since A is almost periodic and (3.5) holds, for every $\{\alpha'_n\} \subset D$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$X(\nu_{\alpha_n}(t))X^{-1}(\nu_{\alpha_n}(\sigma(s)))$$

and

$$\nu^{\Delta}_{\alpha_n}(t)\nu^{\Delta\sigma}_{\alpha_n}(t)f(\nu_{\alpha_n}(t))$$

converges uniformly as $n \to \infty$ for every $t, s \in \mathbb{T}$. The first convergence follows as a consequence of Lemma 3.2.29. Therefore, we get

$$\begin{aligned}
&\nu^{\Delta}_{\alpha_n}(t)x(\nu_{\alpha_n}(t)) \\
&\quad = \int_{-\infty}^{\nu_{\alpha_n}(t)} \nu^{\Delta}_{\alpha_n}(t)X(\nu_{\alpha_n}(t))PX^{-1}(\sigma(s))f(s)\Delta s \\
&\qquad - \int_{\nu_{\alpha_n}(t)}^{+\infty} \nu^{\Delta}_{\alpha_n}(t)X(\nu_{\alpha_n}(t))(I-P)X^{-1}(\sigma(s))f(s)\Delta s
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{t} \nu_{\alpha_n}^{\Delta}(t) X(\nu_{\alpha_n}(t)) P X^{-1}(\sigma(\nu_{\alpha_n}(s))) \nu_{\alpha_n}^{\Delta}(s) f(\nu_{\alpha_n}(s)) \Delta s \\
&\quad - \int_{t}^{+\infty} \nu_{\alpha_n}^{\Delta}(t) X(\nu_{\alpha_n}(t)) (I-P) X^{-1}(\sigma(\nu_{\alpha_n}(s))) \nu_{\alpha_n}^{\Delta}(s) f(\nu_{\alpha_n}(s)) \Delta s \\
&= \int_{-\infty}^{t} \frac{\nu_{\alpha_n}^{\Delta}(t)}{\nu_{\alpha_n}^{\Delta}(\sigma(s))} X(\nu_{\alpha_n}(t)) X^{-1}(\nu_{\alpha_n}(\sigma(s))) P \nu_{\alpha_n}^{\Delta}(s) \nu_{\alpha_n}^{\Delta\sigma}(s) f(\nu_{\alpha_n}(s)) \Delta s \\
&\quad - \int_{t}^{+\infty} \frac{\nu_{\alpha_n}^{\Delta}(t)}{\nu_{\alpha_n}^{\Delta}(\sigma(s))} X(\nu_{\alpha_n}(t)) X^{-1}(\nu_{\alpha_n}(\sigma(s))) (I-P) \nu_{\alpha_n}^{\Delta}(s) \nu_{\alpha_n}^{\Delta\sigma}(s) f(\nu_{\alpha_n}(s)) \Delta s.
\end{aligned}
$$

Applying the limit from both sides when $n \to \infty$, we get that since the integrals from the right hand side converges duo to the exponential dichotomy property and the functions are bounded for each $n \in \mathbb{N}$, the Dominated Convergence Theorem for Δ-integrals implies that

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) x(\nu_{\alpha_n}(t))$$

converges uniformly for each $t \in \mathbb{T}$. It implies that x is almost periodic solution, proving the desired result. □

For functions with two variables, we give the definition of Bochner almost periodicity depending on one variable analogously.

Definition 3.2.31. *A function $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is called **Bochner almost periodic** on $t \in \mathbb{T}$ for each $x \in \mathbb{R}$, if for every sequence $\{\alpha'_n\} \subset D$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that*

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t), x)$$

exists uniformly on $\mathbb{T}$ for each $x \in \mathbb{R}$.

Remark 3.2.32. *As before, we use the notation $T_{\alpha_n} f = \overline{f}$ to represent*

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t), x) = \overline{f}(t, x) \text{ for each } x \in \mathbb{R}.$$

Now, we present some similar properties of Bochner almost periodic functions for this type of functions. Since the proofs follow almost identical, we omit it here.

Theorem 3.2.33. *If $f, g\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are Bochner almost periodic functions with respect to the first variable for each $x \in \mathbb{R}$, then*

(i) $f + g$ is Bochner almost periodic with respect to the first variable, for each $x \in \mathbb{R}$;

(ii) cf is Bochner almost periodic with respect to the first variable, for each $x \in \mathbb{R}$, where $c \in \mathbb{R}$;

(iii) *Let $f_\omega\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ defined by $f_\omega(t,x) := f(\nu_\omega(t),x)$. If for every sequence $\{\alpha_n'\}\subset\mathbb{Z}$, there exists a subsequence $\{\alpha_n\}\subset\{\alpha_n'\}$ such that the limit*

$$\lim_{n\to\infty}\frac{\nu^{\Delta}_{\alpha_n}(t)}{\nu^{\Delta}_{\alpha_n}(\nu_\omega(t))}$$

exists uniformly for each $t\in\mathbb{T}$, then f_ω is Bochner almost periodic with respect to the first variable, for each $\omega\in\mathbb{Z}$.

Theorem 3.2.34. *If $f,g\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ are Bochner almost periodic and regressive functions with respect to the first variable for each $x\in\mathbb{R}$, then*

(i) *$f\oplus g$ is Bochner almost periodic with respect to the first variable, for each $x\in\mathbb{R}$;*

(ii) *$f\ominus g$ is Bochner almost periodic with respect to the first variable, for each $x\in\mathbb{R}$;*

(iii) *$\ominus f$ is Bochner almost periodic with respect to the first variable, for each $x\in\mathbb{R}$.*

In sequel, we state a result concerning Bochner almost periodic functions, which satisfy the Lipschitz condition.

Theorem 3.2.35. *Let $f\colon \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ be Bochner almost periodic for each $x\in\mathbb{R}$ and suppose f satisfies Lipschitz condition*

$$|f(t,x)-f(t,y)|\leq L(t)|x-y| \text{ for all } t\in\mathbb{T} \text{ and } x,y\in\mathbb{R},$$

where $L\colon \mathbb{T}\to(0,\infty)$ is Bochner almost periodic, i.e., for every sequence $\{\alpha_n'\}\subset\mathbb{Z}$, there exists a subsequence $\{\alpha_n\}\subset\{\alpha_n'\}$ such that

$$\lim_{n\to\infty}\nu^{\Delta}_{\alpha_n}(t)L(\nu_{\alpha_n}(t))=\overline{L}(t)$$

exists uniformly on $\mathbb{T}$. Then, $\overline{f}$ given by $T_{\alpha_n}f=\overline{f}$ satisfies the Lipschitz condition with the function $\overline{L}$.

Proof. Let $t\in\mathbb{T}$ and $x,y\in\mathbb{R}$ such that $x\neq y$. Let $\varepsilon>0$. Since f and L are Bochner almost periodic, for every sequence $\{\alpha_n'\}\subset\mathbb{Z}$, there exists a subsequence $\{\alpha_n\}\subset\{\alpha_n'\}$ such that

$$|\overline{f}(t,x)-\nu^{\Delta}_{\alpha_n}(t)f(\nu_{\alpha_n}(t),x)|\leqslant\frac{\varepsilon}{3},$$
$$|\overline{f}(t,y)-\nu^{\Delta}_{\alpha_n}(t)f(\nu_{\alpha_n}(t),y)|\leqslant\frac{\varepsilon}{3},$$

$$|\overline{L}(t) - \nu_{\alpha_n}^{\Delta}(t)L(\nu_{\alpha_n}(t))| \leqslant \frac{\varepsilon}{3|x-y|}$$

for n sufficiently large and for each $t \in \mathbb{T}$ and $x, y \in \mathbb{R}$. Therefore, we obtain by the Lipschitz assumption of f

$$\begin{aligned}
|\overline{f}(t,x) - \overline{f}(t,y)| &\leq |\overline{f}(t,x) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),x)| + |\overline{f}(t,y) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),y)| \\
&\quad + |\nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),x) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),y)| \\
&\leq |\overline{f}(t,x) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),x)| + |\overline{f}(t,y) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),y)| \\
&\quad + \nu_{\alpha_n}^{\Delta}(t)L(\nu_{\alpha_n}(t))|x-y| \\
&= |\overline{f}(t,x) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),x)| + |\overline{f}(t,y) - \nu_{\alpha_n}^{\Delta}(t)f(\nu_{\alpha_n}(t),y)| \\
&\quad + \nu_{\alpha_n}^{\Delta}(t)|f(\nu_{\alpha_n}(t),x) - f(\nu_{\alpha_n}(t),y)| \\
&\leq \frac{2\varepsilon}{3} + |\nu_{\alpha_n}^{\Delta}(t)L(\nu_{\alpha_n}(t))[x-y]| \\
&= \frac{2\varepsilon}{3} + |[\nu_{\alpha_n}^{\Delta}(t)L(\nu_{\alpha_n}(t)) + \overline{L}(t) - \overline{L}(t)][x-y]| \\
&\leq \frac{2\varepsilon}{3} + |\overline{L}(t) - \nu_{\alpha_n}^{\Delta}(t)L(\nu_{\alpha_n}(t))| \cdot |x-y| + \overline{L}(t)|x-y| \\
&\leq \varepsilon + \overline{L}(t)|x-y|,
\end{aligned}$$

from where letting $\varepsilon \to 0^+$, we conclude the desired. □

3.3 Bohr almost periodicity on isolated time scales

In this section, we aim to study the Bohr's definition of almost periodic functions on the isolated time scales context, and provide a relation between this concept with the Bochner almost periodicity definition.

Remark 3.3.1. *Let $t_0 \in \mathbb{T}$. We can write*

$$\mathbb{T} = \bigcup_{n \in \mathbb{Z}} \nu_n(t_0).$$

Note that if $\mathbb{T}$ has a minimum or maximum (or both), we can also write $\mathbb{T}$ as above, since $\rho(t) = t$ if $t = \inf \mathbb{T} > -\infty$ and $\sigma(t) = t$ if $t = \sup \mathbb{T} < +\infty$.

We start with the Bohr's definition of almost periodicity.

Definition 3.3.2. *A function $f\colon \mathbb{T} \to \mathbb{R}$ is **Bohr almost periodic** if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that any N_ε consecutive elements of D contain at least one $\omega \in D$ such* *that*

$$|\nu_\omega^\Delta(t) f(\nu_\omega(t)) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{T}.$$

Remark 3.3.3. *Again, we will only state and prove the results for the case when $D = \mathbb{Z}$, since the other cases follow analogous.*

Remark 3.3.4. *If $f\colon \mathbb{T} \to \mathbb{R}$ is ω-periodic, in the sense of*

$$\nu_\omega^\Delta(t) f(\nu_\omega(t)) = f(t) \text{ for all } t \in \mathbb{T},$$

for some $\omega \in \mathbb{N}$, then f is Bohr almost periodic. Indeed, let $t \in \mathbb{T}$ and note that for every $\varepsilon > 0$, there exists $N_\varepsilon = \omega + 1 \in \mathbb{N}$ such that any N_ε consecutive elements of $\mathbb{Z}$ contain at least one $s \in \mathbb{Z}$ with

$$|\nu_s^\Delta(t) f(\nu_s(t)) - f(t)| = 0 < \varepsilon,$$

i.e., for any N_ε consecutive numbers, it is possible to find a multiple of ω, $s = k\omega$, and thus

$$\begin{aligned}
\nu_s^\Delta(t) f(\nu_s(t)) &= \nu_{k\omega}^\Delta(t) f(\nu_{k\omega}(t)) \\
&\overset{(3.1)}{=} \nu_{(k-1)\omega}^\Delta(t) \nu_\omega^\Delta(\nu_{(k-1)\omega}(t)) f(\nu_\omega(\nu_{(k-1)\omega}(t))) \\
&= \nu_{(k-1)\omega}^\Delta(t) f(\nu_{(k-1)\omega}(t)),
\end{aligned}$$

from where proceeding this way, we have that $\nu_s^\Delta(t) f(\nu_s(t)) = f(t)$, getting the desired result.

The next result shows a strong correspondence between Bohr almost periodic functions defined on $\mathbb{T}$ and $\mathbb{Z}$. We recall the definition of Bohr almost periodic functions in this context: a function $f\colon \mathbb{Z} \to \mathbb{R}$ is called **Bohr almost periodic**, if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that any N_ε consecutive elements of $\mathbb{Z}$ contains at least one an integer ω with the property

$$|f(t + \omega) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{Z}.$$

Theorem 3.3.5. *A necessary and sufficient condition for a function $g\colon \mathbb{T} \to \mathbb{R}$ to be Bohr almost periodic on $\mathbb{T}$ is the existence of a Bohr almost periodic function $f\colon \mathbb{Z} \to \mathbb{R}$ such that $\nu_n^\Delta(t_0) g(\nu_n(t_0)) = f(n)$ for every $n \in \mathbb{Z}$.*

Proof. Suppose $g\colon \mathbb{T} \to \mathbb{R}$ is Bohr almost periodic on $\mathbb{T}$ and define $f\colon \mathbb{Z} \to \mathbb{R}$ by $f(n) = \nu_n^\Delta(t_0) g(\nu_n(t_0))$ for $n \in \mathbb{Z}$. Let $\varepsilon > 0$. Given $n \in \mathbb{Z}$, since g is Bohr almost periodic, there exists $N_\varepsilon \in \mathbb{N}$ such that any N_ε consecutive elements of $\mathbb{Z}$ contain at least one ω with

$$|\nu_\omega^\Delta(t) g(\nu_\omega(t)) - g(t)| < \frac{\varepsilon}{\nu_n^\Delta(t_0)} \text{ for all } t \in \mathbb{T}.$$

Let $t \in \mathbb{T}$ be such that $t = \nu_n(t_0)$, we have

$$\begin{aligned}
|f(n+\omega) - f(n)| &= |\nu^{\Delta}_{n+\omega}(t_0) g(\nu_{n+\omega}(t_0)) - \nu^{\Delta}_{n}(t_0) g(\nu_n(t_0))| \\
&\overset{(3.1)}{=} |\nu^{\Delta}_{n}(t_0)\nu^{\Delta}_{\omega}(\nu_n(t_0)) g(\nu_\omega(\nu_n(t_0))) - \nu^{\Delta}_{n}(t_0) g(\nu_n(t_0))| \\
&= |\nu^{\Delta}_{n}(t_0)\nu^{\Delta}_{\omega}(t) g(\nu_\omega(t)) - \nu^{\Delta}_{n}(t_0) g(t)| \\
&< \nu^{\Delta}_{n}(t_0) \frac{\varepsilon}{\nu^{\Delta}_{n}(t_0)} = \varepsilon.
\end{aligned}$$

obtaining that f is Bohr almost periodic on $\mathbb{Z}$.
On the other hand, suppose f is Bohr almost periodic. Let $\varepsilon > 0$ and $t \in \mathbb{T}$. Then, there exists $n_0 \in \mathbb{Z}$ such that $t = \nu_{n_0}(t_0)$ and also, there exists $N_\varepsilon \in \mathbb{N}$ such that among any N_ε consecutive integers, there exists $\omega \in \mathbb{Z}$ such that

$$|f(n+\omega) - f(n)| < \varepsilon \nu^{\Delta}_{n_0}(t_0) \text{ for all } n \in \mathbb{Z}.$$

Thus, we have

$$\begin{aligned}
|f(n_0+\omega) - f(n_0)| &= |\nu^{\Delta}_{n_0+\omega}(t_0) g(\nu_{n_0+\omega}(t_0)) - \nu^{\Delta}_{n_0}(t_0) g(\nu_{n_0}(t_0))| \\
&\overset{(3.1)}{=} |\nu^{\Delta}_{n_0}(t_0)\nu^{\Delta}_{\omega}(\nu_{n_0}(t_0)) g(\nu_\omega(\nu_{n_0}(t_0))) - \nu^{\Delta}_{n}(t_0) g(\nu_{n_0}(t_0))| \\
&= \nu^{\Delta}_{n_0}(t_0) |\nu^{\Delta}_{\omega}(t) g(\nu_\omega(t)) - g(t)|,
\end{aligned}$$

i.e.,

$$|\nu^{\Delta}_{\omega}(t) g(\nu_\omega(t)) - g(t)| = \frac{|f(n_0+\omega) - f(n_0)|}{\nu^{\Delta}_{n_0}(t_0)} < \frac{\varepsilon \nu^{\Delta}_{n_0}(t_0)}{\nu^{\Delta}_{n_0}(t_0)} = \varepsilon,$$

that is, g is Bohr almost periodic. □

Remark 3.3.6. *Theorem 3.3.5 can be extended for other cases when $D \neq \mathbb{Z}$ for f also defined on D, obtaining a direct bijection to $\mathbb{T}$.*

We also have a correspondence between Bohr almost periodic functions defined on $\mathbb{Z}$ and $\mathbb{R}$.

Theorem 3.3.7 (See [14, Theorem 1.27])**.** *A necessary and sufficient condition for a function $g\colon \mathbb{Z} \to \mathbb{R}$ to be Bohr almost periodic is the existence of a Bohr almost periodic $f : \mathbb{R} \to \mathbb{R}$ such that $g(n) = f(n)$ for all $n \in \mathbb{Z}$.*

The next result is a consequence of Theorems 3.3.5 and 3.3.7.

Theorem 3.3.8. *A necessary and sufficient condition for $g\colon \mathbb{T} \to \mathbb{R}$ to be Bohr almost periodic on $\mathbb{T}$ is the existence of a Bohr almost periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\nu_n^\Delta(t_0)g(\nu_n(t_0)) = f(n)$ for every $n \in \mathbb{Z}$.*

Proof. If $g\colon \mathbb{T} \to \mathbb{R}$ is Bohr almost periodic, then Theorem 3.3.5 implies that there exists a Bohr almost periodic function $\varphi\colon \mathbb{Z} \to \mathbb{R}$ such that $\nu_n^\Delta(t_0)g(\nu_n(t_0)) = \varphi(n)$ for every $n \in \mathbb{Z}$. Thus, by Theorem 3.3.7, there exists a Bohr almost periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(n) = f(n)$. Therefore, there exists $f\colon \mathbb{R} \to \mathbb{R}$ such that $\nu_n^\Delta(t_0)g(\nu_n(t_0)) = f(n)$ for every $n \in \mathbb{Z}$.
On the other hand, if there exists a Bohr almost periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\nu_n^\Delta(t_0)g(\nu_n(t_0)) = f(n)$ for every $n \in \mathbb{Z}$, we can consider $\varphi\colon \mathbb{Z} \to \mathbb{R}$ given by $\varphi = f|_{\mathbb{Z}}$ and by Theorem 3.3.5, it follows the desired result. $\square$

The next result states an equivalence of Bochner almost periodic functions and Bohr almost periodic functions on isolated time scales.

Theorem 3.3.9. *Suppose $f\colon \mathbb{T} \to \mathbb{R}$ is Bochner almost periodic and there exists $M > 0$ such that*

$$\frac{\mu^{\nu_k}(t)}{\mu(t)} > M \ \text{ for all } \ t \in \mathbb{T} \ \text{ and } \ k \in \mathbb{Z}. \tag{3.7}$$

Then f is Bohr almost periodic. Conversely, if f is Bohr almost periodic, then f is Bochner almost periodic.

Proof. Suppose f is Bochner almost periodic, but f is not Bohr almost periodic. Therefore, there exists at least one $\varepsilon > 0$ such that for any $N_\varepsilon \in \mathbb{N}$, the set of N_ε consecutive numbers in $\mathbb{Z}$ does not contain any element ω satisfying

$$|\nu_\omega^\Delta(t)f(\nu_\omega(t)) - f(t)| < \frac{\varepsilon}{M} \ \text{ for all } \ t \in \mathbb{T}. \tag{3.8}$$

Consider an arbitrary number $\alpha_1 \in \mathbb{N}$, then (3.8) is not satisfied for all $t \in B := [\nu_{-\alpha_1}(t_0), t_0)_{\mathbb{T}} \cup [t_0, \nu_{\alpha_1}(t_0))_{\mathbb{T}}$. Take $\alpha_2 = n \cdot \alpha_1$ (for some $n \in \mathbb{N} \setminus \{1\}$), then (3.8) is not satisfied for all $t \in [\nu_{-(\alpha_1+\alpha_2)}(t_0), \nu_{-\alpha_1}(t_0))_{\mathbb{T}} \cup [\nu_{\alpha_1}(t_0), \nu_{\alpha_1+\alpha_2}(t_0))_{\mathbb{T}}$. Proceeding in this way, we can construct a sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ where $\alpha_k = n \cdot \alpha_{k-1}$ ($k = 2, 3, \ldots$) and such that the set

$$B \cup \left(\bigcup_{n=2}^{\infty} ([\nu_{-\sum_{i=1}^{n} \alpha_i}(t_0), \nu_{-\sum_{i=1}^{n-1} \alpha_i}(t_0))_{\mathbb{T}} \cup [\nu_{\sum_{i=1}^{n-1} \alpha_i}(t_0), \nu_{\sum_{i=1}^{n} \alpha_i}(t_0))_{\mathbb{T}}) \right)$$

covers all $\mathbb{T}$. Therefore, for any $i, j \in \mathbb{N}$ such that $i > j > 1$ and considering $i = j + h$

($h \in \mathbb{N}$), we obtain

$$\begin{aligned}\sup_{t\in\mathbb{T}} &|\nu^{\Delta}_{\alpha_i}(t) f(\nu_{\alpha_i}(t)) - \nu^{\Delta}_{\alpha_j}(t) f(\nu_{\alpha_j}(t))| \qquad (3.9)\\ &= \sup_{t\in\mathbb{T}} |\nu^{\Delta}_{\alpha_{j+h}}(t) f(\nu_{\alpha_{j+h}}(t)) - \nu^{\Delta}_{\alpha_j}(t) f(\nu_{\alpha_j}(t))|\\ &= \sup_{t\in\mathbb{T}} |\nu^{\Delta}_{n^h\alpha_j}(t) f(\nu_{n^h\alpha_j}(t)) - \nu^{\Delta}_{\alpha_j}(t) f(\nu_{\alpha_j}(t))|\\ &= \sup_{t\in\mathbb{T}} \nu^{\Delta}_{\alpha_j}(t) |\nu^{\Delta}_{(n^h-1)\alpha_j}(\nu_{\alpha_j}(t)) f(\nu_{(n^h-1)\alpha_j}(\nu_{\alpha_j}(t))) - f(\nu_{\alpha_j}(t))|\\ &\geq \sup_{t\in\mathbb{T}} \frac{\mu^{\nu_{\alpha_j}}(t)}{\mu(t)} \frac{\varepsilon}{M} > \varepsilon,\end{aligned}$$

where the second equality follows from the definition of the sequence $\{\alpha_k\}$. Hence, the sequence $\{\nu^{\Delta}_{\alpha_k}(t) f(\nu_{\alpha_k}(t))\}$ cannot contain any uniformly convergent subsequence, which contradicts the fact that f is Bochner almost periodic. Reciprocally, if f is Bohr almost periodic, then given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that any N_ε consecutive elements of $\mathbb{Z}$ contain at least one ω with

$$|\nu^{\Delta}_{\omega}(t) f(\nu_{\omega}(t)) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{T}.$$

Let $t \in \mathbb{T}$ and $i \in \mathbb{Z}$ be such that $t = \nu_i(t_0)$. Defining $g\colon \mathbb{Z} \to \mathbb{R}$ by

$$g(n) := \nu^{\Delta}_{n}(t_0) f(\nu_n(t_0)), \quad n \in \mathbb{Z},$$

we obtain from Theorem 3.3.5 that g is Bohr almost periodic, and hence, by Theorem 2.2.21, g is Bochner almost periodic, i.e., for every sequence $\{\alpha'_k\} \subset \mathbb{Z}$, there exists a subsequence $\{\alpha_k\} \subset \{\alpha'_k\}$ such that

$$\lim_{k\to\infty} g(n + \alpha_k)$$

exists uniformly for every $n \in \mathbb{Z}$. Hence,

$$\begin{aligned}\lim_{n\to\infty} g(i + \alpha_n) &= \lim_{n\to\infty} \nu^{\Delta}_{i+\alpha_n}(t_0) f(\nu_{i+\alpha_n}(t_0))\\ &= \lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(\nu_i(t_0)) \nu^{\Delta}_{i}(t_0) f(\nu_{\alpha_n}(\nu_i(t_0)))\\ &= \lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) \nu^{\Delta}_{i}(t_0) f(\nu_{\alpha_n}(t)),\\ &= \nu^{\Delta}_{i}(t_0) \lim_{n\to\infty} \nu^{\Delta}_{\alpha_n}(t) f(\nu_{\alpha_n}(t)),\end{aligned}$$

exists uniformly, i.e., the limit

$$\lim_{n\to\infty} \nu_{\alpha_n}^{\Delta}(t) f(\nu_{\alpha_n}(t))$$

exists uniformly for all $t \in \mathbb{T}$, obtaining the desired result. □

Remark 3.3.10. *The hypothesis* (3.7) *for a Bochner almost periodic function to be a Bohr almost periodic function is reasonable. Indeed, for $\mathbb{Z}$, $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$ this condition is trivially satisfied for $M < 1$. Thus, all the previous results show that all properties for Bochner almost periodic functions that satisfies* (3.7) *also holds for Bohr almost periodic functions. This motivates us to call just **almost periodic functions** all the ones that are Bohr/Bochner almost periodic functions.*

3.4 Examples and Applications

In this section, we provide some examples of almost periodic functions on isolated time scales. Also, we present an application of Theorem 3.2.30.

Example 3.4.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be any almost periodic function on $\mathbb{R}$. For $t \in \mathbb{T}$, there exists $n \in \mathbb{Z}$ such that $t = \nu_n(t_0)$. Thus, by Theorem 3.3.8, the function $F\colon \mathbb{T} \to \mathbb{R}$ given by*

$$F(t) = \frac{f(n)}{\nu_n^{\Delta}(t_0)},$$

is also an almost periodic function on $\mathbb{T}$.

Example 3.4.2. *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = \cos t + \cos(\sqrt{2}t)$ is almost periodic on $\mathbb{R}$ (see [17, Page 3]). Therefore, by Example 3.4.1, we have*

(i) If $\mathbb{T} = \mathbb{Z}$, let $t \in \mathbb{T}$. Then, if $t > t_0$, there exists $n \in \mathbb{N}$ such that $t = \nu_n(t_0) = t_0 + n$ for $n = t - t_0$ and $\nu_n^{\Delta}(t_0) = 1$ (the cases when $t < t_0$ or $t = t_0$ follow similarly). Thus, the function $F\colon \mathbb{T} \to \mathbb{R}$ given by

$$F(t) = \begin{cases} f(t - t_0) & \text{if } t > t_0, \\ f(t_0 - t) & \text{if } t < t_0, \\ f(0) & \text{if } t = t_0, \end{cases}$$

i.e.,

$$F(t) = \begin{cases} \cos(t - t_0) + \cos(\sqrt{2}(t - t_0)) & \text{if } t > t_0, \\ \cos(t_0 - t) + \cos(\sqrt{2}(t_0 - t)) & \text{if } t < t_0, \\ 2 & \text{if } t = t_0 \end{cases}$$

is almost periodic on $\mathbb{Z}$.

(ii) If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, let $t \in \mathbb{T}$. Hence,

$$F(t) = \begin{cases} \cos\left(\frac{t-t_0}{h}\right) + \cos\left(\sqrt{2}\left(\frac{t-t_0}{h}\right)\right) & \text{if } t > t_0, \\ \cos\left(\frac{t_0-t}{h}\right) + \cos\left(\sqrt{2}\left(\frac{t_0-t}{h}\right)\right) & \text{if } t < t_0, \\ 2 & \text{if } t = t_0 \end{cases}$$

is almost periodic on $h\mathbb{Z}$.

(iii) If $\mathbb{T} = q^{\mathbb{N}_0}$, then given $t \in \mathbb{T}$ and considering $t_0 = 1$, there exists $n \in \mathbb{N}_0$ such that $t = q^n$. Hence, the function $F\colon \mathbb{T} \to \mathbb{R}$ given by

$$F(t) = \frac{f(\log_q t)}{t} = \frac{\cos(\log_q t) + \cos(\sqrt{2}\log_q t)}{t}$$

is almost periodic on $q^{\mathbb{N}_0}$, since $\nu_n^{\Delta}(t_0) = q^n = t$.

Example 3.4.3. *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = \sin t + \sin(\pi t)$ is almost periodic on $\mathbb{R}$ (see [14, Page 107]). Therefore, analogously, we have*

(i) If $\mathbb{T} = \mathbb{Z}$, the function $F\colon \mathbb{T} \to \mathbb{R}$ given by

$$F(t) = \begin{cases} \sin(t - t_0) + \sin(\pi(t - t_0)) & \text{if } t > t_0, \\ \sin(t_0 - t) + \sin(\pi(t_0 - t)) & \text{if } t < t_0, \\ 2 & \text{if } t = t_0 \end{cases}$$

is almost periodic on $\mathbb{Z}$.

(ii) If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, the function $F\colon \mathbb{T} \to \mathbb{R}$ given by

$$F(t) = \begin{cases} \sin\left(\frac{t-t_0}{h}\right) + \sin\left(\pi\left(\frac{t-t_0}{h}\right)\right) & \text{if } t > t_0, \\ \sin\left(\frac{t_0-t}{h}\right) + \sin\left(\pi\left(\frac{t_0-t}{h}\right)\right) & \text{if } t < t_0, \\ 2 & \text{if } t = t_0 \end{cases}$$

is almost periodic on $h\mathbb{Z}$.

(iii) If $\mathbb{T} = q^{\mathbb{N}_0}$, for $t \in \mathbb{T}$ and considering $t_0 = 1$, the function $F\colon \mathbb{T} \to \mathbb{R}$ given by

$$F(t) = \frac{f(\log_q t)}{t} = \frac{\sin(\log_q t) + \sin(\pi \log_q t)}{t}$$

is almost periodic on $q^{\mathbb{N}_0}$.

Example 3.4.4. *Consider* $\mathbb{T} = q^{\mathbb{N}_0}$ *and* $t \in \mathbb{T}$. *By Examples 3.4.2 and 3.4.3 and Theorem 3.2.10 (i), the function* $F\colon \mathbb{T} \to \mathbb{R}$ *given by*

$$F(t) = \frac{\cos(\log_q t) + \cos(\sqrt{2}\log_q t) + \sin(\log_q t) + \sin(\pi \log_q t)}{t}.$$

is almost periodic on $q^{\mathbb{N}_0}$. *Thus, since for any sequence* $\{\alpha_n\} \subset \mathbb{Z}$ *the limit*

$$\lim_{n\to\infty} \frac{\nu^{\Delta}_{\alpha_n}(t)}{\nu^{\Delta}_{\alpha_n}(\nu_\omega(t))} = \lim_{n\to\infty} \frac{q^{\alpha_n}}{q^{\alpha_n}} = 1$$

always exists uniformly on $\mathbb{T}$, *by Theorem 3.2.10 (iii), the function* $f_\omega\colon \mathbb{T} \to \mathbb{R}$ *defined by*

$$f_\omega(t) = F(\nu_\omega(t)) = \frac{1}{\nu_\omega(t)}[\cos(\log_q \nu_\omega(t)) \cos(\sqrt{2}\log_q \nu_\omega(t)) + \\ + \sin(\log_q \nu_\omega(t)) + \sin(\pi \log_q \nu_\omega(t))],$$

i.e.,

$$f_\omega(t) = \frac{1}{q^\omega t}[\cos(\omega + \log_q t) + \cos(\sqrt{2}(\omega + \log_q t)) \\ + \sin(\omega + \log_q t) + \sin(\pi(\omega + \log_q t))]$$

is almost periodic on $q^{\mathbb{N}_0}$.

Example 3.4.5. *Let* $g\colon \mathbb{Z} \to \mathbb{R}$ *be a normal* ω*-periodic function. Then, for* $\mathbb{T} = \{\sqrt{n} \ : \ n \in \mathbb{N}_0\}$, *the function* $f\colon \mathbb{T} \to \mathbb{R}$

$$f(t) = \frac{g(t^2)}{\sqrt{t^2+1} - t}$$

is ω*-periodic [7, See Example 5.5]. Thus, by Remark 3.3.4,* f *is almost periodic.*

Example 3.4.6. *Consider the following equation:*

$$x^{\Delta}(t) = Ax(t) + f(t) \tag{3.10}$$

on $\mathbb{T} = 2\mathbb{Z}$. *Consider*

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

and $f(t) = \cos(\pi t) + \sin\left(\frac{\pi}{2}t\right)$. *Then, clearly* f *satisfies the hypothesis of Theorem 3.2.30. Also,*

$$I + \mu(t)A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

is invertible. Using the fact that the eigenvalues of the coefficient matrix are $\lambda_1 = \lambda_2 = -4$ *and applying Putzer Algorithm, we obtain that the* P*-matrices are given by*

$$P_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $P_1 = (A - \lambda_1 I)P_0 = (A + 4I)P_0$. *Again, by Putzer Algorithm, we get*

$$\begin{aligned} r_1^{\Delta}(t) &= -4r_1(t), \;\; r_1(t_0) = 1, \\ r_2^{\Delta}(t) &= r_1(t) - 4r_2(t), \;\; r_2(t_0) = 0. \end{aligned}$$

From this, we obtain

$$r_1(t) = e_{\ominus 4}(t, t_0) = e_{\ominus \frac{8}{\mu}}(t, t_0).$$

Now, applying the Variation Constant Formula, we get the following expression to r_2

$$\begin{aligned} r_2(t) &= e_{\ominus 4}(t, t_0)r_2(t_0) + \int_{t_0}^{t} e_{\ominus 4}(t, \sigma(s))4r_1(s)\Delta s \\ &= \int_{t_0}^{t} e_{\ominus 4}(t, \sigma(s))4e_{\ominus 4}(s, t_0)\Delta s \\ &= \int_{t_0}^{t} e_{\ominus 4}(t, s)e_{\ominus 4}(s, \sigma(s))4e_{\ominus 4}(s, t_0)\Delta s \\ &= e_{\ominus 4}(t, t_0)\int_{t_0}^{t} 4\left[1 - \frac{4\mu(s)}{1 + 4\mu(s)}\right]\Delta s \\ &= e_{\ominus 4}(t, t_0)\int_{t_0}^{t} 4\frac{1}{1 + 4\mu(s)}\Delta s. \end{aligned}$$

Finally, applying Putzer Algorithm again, we get

$$\begin{aligned} e_A(t, t_0) &= r_1(t)P_0 + r_2(t)P_1 \\ &= e_{\ominus 4}(t, t_0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(4e_{\ominus 4}(t, t_0)\int_{t_0}^{t} \frac{\Delta s}{1 + 4\mu(s)}\right)(A + 4I) \\ &= \begin{bmatrix} e_{\ominus 4}(t, t_0) & 0 \\ 0 & e_{\ominus 4}(t, t_0) \end{bmatrix}. \end{aligned}$$

Hence, for $t \geq s$, we obtain

$$\begin{aligned}
\left\|X(t)P_0X^{-1}(s)\right\| &= \left\| e_{\ominus 4}(t,t_0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e_4(s,t_0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\| \\
&= \left\| \begin{bmatrix} e_{\ominus 4}(t,t_0) & 0 \\ 0 & e_{\ominus 4}(t,t_0) \end{bmatrix}\begin{bmatrix} e_4(s,t_0) & 0 \\ 0 & e_4(s,t_0) \end{bmatrix}\right\| \\
&= \sqrt{2}e_{\ominus 4}(t,s).
\end{aligned}$$

Taking $K = \sqrt{2}$ and $\gamma = 8$, we obtain that the equation (3.4) *admits exponential dichotomy. Therefore, we can check that all the conditions of Theorem 3.2.30 are satisfied, then the equation* (3.10) *has an almost periodic solution.*

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