Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Floquet Theory on isolated time scales and applications

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Brasília

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Dissertation submitted to the Department of Mathematics of the University of Brasília, as part of the requirements for obtaining a Master's degree in Mathematics.

Advisor:
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"Always remember who you are."

Avatar: The Last Airbender,
2005

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Resumo

Teoria de Floquet em escalas temporais isoladas e aplicações

Neste trabalho, estudamos a ω-periodicidade para funções definidas em escalas temporais isoladas, bem como a equação dinâmica linear de primeira ordem em escalas temporais isoladas, para a qual sua função de coeficietnes de matriz é ω-periódica e regressiva. Apresentamos a teoria de Floquet em escalas de tempo isoladas, baseada na nova definição de funções ω-periódicas, e algumas aplicações de nossos resultados. Os resultados referentes à teoria de Floquet são originais e podem ser encontrados em [6].

Palavras-chave: Teoria de Floquet; escalas temporais isoladas; periodicidade

Abstract

In this work, we study ω-periodicity for functions defined on isolated time scales, as well as the first-order linear dynamic equation on isolated time scales for which its coefficient matrix function is ω-periodic and regressive. We provide Floquet theory on isolated time scales, based on the new definition of ω-periodic functions and present some applications of our results. The results concerning Floquet theory are original and can be found in [6].

Keywords: Floquet theory; isolated time scales; periodicity

Notation

$\mathbb{T}$	Time scale
σ	Forward jump operator
μ	Graininess function
$\mathbb{T}^\kappa$	$\mathbb{T}^\kappa$ scale
$x^\Delta(t)$	Delta derivative of x on t
C_{rd}	Set of all rd-continuous functions
$\int_a^b x(t)\Delta t$	Delta integral of x from a to b
$\mathcal{R}$	Set of regressive and rd-continuous functions
$\mathcal{R}^+$	Set of all positively regressive functions and rd-continuous functions
$\xi_h(z)$	Cylinder transformation
$\oplus$	"Circle plus addition"
$\ominus$	"Circle minus subtraction"
$e_p(t,s)$	Exponential function
$\nu(t)$	Iterated shift operator
$\mathcal{P}$	Set of all periodic functions
$\Phi(t)$	Fundamental matrix of Floquet equation

Contents

Introduction **3**

1 Basic on the theory of time scales **8**

1.1 Basic Definitions 8
1.2 Delta-derivatives 10
1.3 Integration 22
1.4 Chain Rules 32
1.5 The Regressive Group 39
1.6 Regressive Matrices 45

2 Periodicity on Isolated Time Scales **49**

2.1 Iterated Shifts 49
2.2 Periodicity 53
2.3 Examples 56
2.4 Homogeneous Linear Dynamic Equation 59
2.5 Remarks on other concepts of periodicity for general time scales 62

3 Floquet theory on isolated time scales **65**

3.1 Periodic Functions 66
3.2 Floquet Theory 68
3.3 Examples 73

References **75**

Introduction

The theory of time scales was introduced by Stefan Hilger in his PhD thesis (see [10]), in 1988. This theory, that can unify discrete and continuous analysis, as well the cases in "between", which means the hybrid cases, has been attracting the attention of many researchers, due to its power of unification, extension and discretization since then (see [8, 9]). The basis of this theory will be explored in the first chapter. Although this theory has the goal of unification discrete, hybrid and continuous theory, avoiding that one needs to prove twice or even more times the analogue results, there are some open questions related on how to generalize some fundamental concepts for all time scales. One example is the concept of periodicity, which was completely open during many years how to generalize for any time scale. The difficulty behind comes from the fact that the classical definition of ω-periodicity for a function is given by the following property below:

$$f(t+\omega) = f(t)$$

which should be fulfilled for all t in the domain of f. However, clearly, this property is not well-defined for every time scale, since it requires the additive property of the time scale to ensure it makes sense. More precisely, one needs to ensure that $t+\omega \in \mathbb{T}$ for all $t \in \mathbb{T}$. However, in the framework of time scales, it is quite restrictive and excludes many interesting time scales. For instance, the quantum scale

$$q^{\mathbb{N}_0} = \{q^n \colon n \in \mathbb{N}_0\}$$

for $q > 1$ does not satisfy such property. On the other hand, it is a known fact that this scale plays a crucial role for quantum calculus which has several applications in quantum physics.

Therefore, motivated by this gap in the classical definition of periodicity, M. Bohner and R. Chieochan [4] introduced for the first time the concept of ω-periodicity for quantum time scale, which can be translated by:

$$q^{\omega} f(q^{\omega} t) = f(t)$$

for all $t \in q^{\mathbb{N}_0}$. One first look at this definition may be very different from the one that would be expected in this case, specially because it appears an extra factor multiplying the function f. However, this definition is motivated by the fact that this definition preserves a very important property of ω-periodic functions stated below:

$$\int_{t}^{q^{\omega} t} f(s)\Delta s = \int_{t_0}^{q^{\omega} t_0} f(s)\Delta s$$

for all $t \in q^{\mathbb{N}_0}$. This property can be described below for the case $q = 2$ and for the case that $\int_t^{2t} f(s)\Delta s = c$.

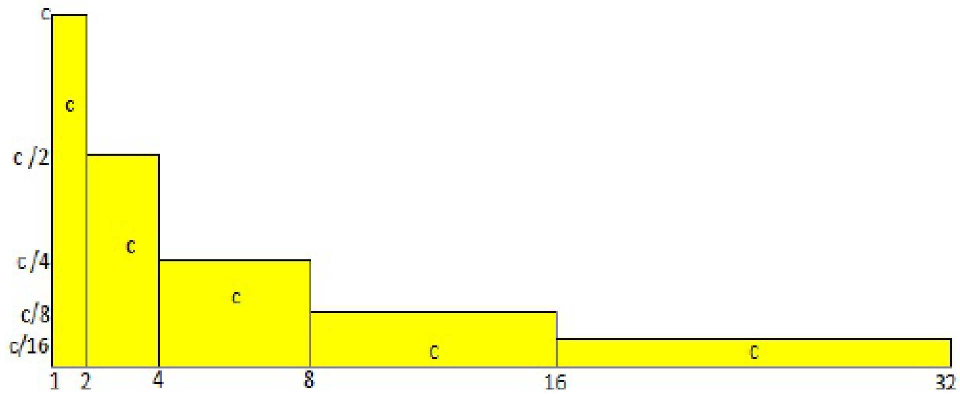

Figure 1: The constant area of the rectangle corresponding to the 1-periodic function f on the intervals $[2^n, 2^{n+1}],\ n \in \{0, 1, 2, 3, 4\}$. This picture was borrowed from reference [4].

The analogue of this property for the classical case can be read as follows:

$$\int_{t}^{t+\omega} f(s)\Delta s = \int_{t_0}^{t_0+\omega} f(s)\Delta s,$$

for all $t \in \mathbb{T}$. This property represents a key for the investigations of periodic functions, specially in the study of differential equations. Hence, a general definition of periodicity needs to keep this property preserved. However, during many years, the question on how to define the concept of periodicity for any time scales was open in the literature and many researchers in the field tried to present some suitable definition.

In 2012, M. Adivar [1] made a very important progress in this direction. He presented the so-called periodicity in shifts, which can be stated as below (see [1] for more details).

Definition 0.0.1 (Periodicity in shifts)**.** *Let $\mathbb{T}$ be a time scale with the shift operators $\delta_{\pm}$ associated with the initial point $t_0 \in \mathbb{T}^*$, which is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_{\pm}\colon [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ exist. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_{\pm}$ if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_{\pm}$ for all $t \in \mathbb{T}^*$, where $D_{\pm} := \{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^*\colon \delta_{\pm}(s,t) \in \mathbb{T}^*\}$. Futhermore, if*

$$P := \inf\{p \in (t_0,\infty)_{\mathbb{T}^*}\colon (p,t) \in D_{\pm} \text{ for all } t \in \mathbb{T}^*\} \neq t_0,$$

then P is called the period of time scale $\mathbb{T}$.

With this definition in hand, M. Adivar [1] introduced a general definition of periodic functions in the context of time scales described below:

Definition 0.0.2 (Periodic function in shifts $\delta_{\pm}$)**.** *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. We say that a real valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_{\pm}$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in D_{\pm} \text{ and } f(\delta_{\pm}^{T}(t)) = f(t) \text{ for all } t \in \mathbb{T}^*, \text{ where } \delta_{\pm}^{T} := \delta_{\pm}(T,t). \tag{1}$$

The smallest number $T \in [P,\infty)_{\mathbb{T}^*}$ such that (1) holds is called the period of f.

From the definition of Adivar, we get that the quantum scale can be considered a periodic time scale in shifts. However, the concept presented by Adivar in [1] for periodic functions in shifts does not recover the important property that we remarked to be essential in the investigations of periodic functions, specially in the context of dynamic equations on time scales.

On the other hand, in the same article of Adivar, it is possible to find another concept that fits better with the central role of periodic functions, they are so-called Δ-periodic function in shifts $\delta_{\pm}$. The definition can be stated as follows (see [1]).

Definition 0.0.3 (Δ-periodic functions in shift $\delta_{\pm}$)**.** *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period P. We say that a real valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_{\pm}$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in D_{\pm} \text{ for all } t \in \mathbb{T}^*, \tag{2}$$

the shifts $\delta_{\pm}^{T}$ are Δ-differentiable with rd-continuous derivatives and

$$f(\delta_{\pm}^{T}(t))\delta_{\pm}^{T}(t) = f(t) \text{ for all } t \in \mathbb{T}^{*}, \tag{3}$$

where $\delta_{\pm}^{T}(t) := \delta_{\pm}(T, t)$.

The smallest number $T \in [P, \infty)_{\mathbb{T}^{*}}$ *such that the properties above (2)-(3) hold is called the period of* f.

This definition is consistent to the one presented by M. Bohner and R. Chieochan in [4] for quantum case.

Later, in [8], M. Bohner, J. G. Mesquita and S. Streipert presented a concept of ω-periodicity for all isolated time scales only using iterated shifts, which is simply the composition of the forward jump operator ω times. Also, in [8], the authors investigated the existence of periodic solutions for the first order linear dynamic equation on time scales, employing this new concept.

Also, in the same article, the authors comparised their concept and the concept of Δ-periodicity in shifts presented in Adivar's paper. We also bring some comments concerning it in this work, to elucidate the reader about both definitions.

After all these advances in the construction of the definition of periodicity on isolated time scales, many results have been proved, and interesting models were investigated. See [7], [8], [9].

Although all these developments, some important questions remain completely open. One of them is concerning the Floquet theory for all isolated time scales. Considering the importance and relevance of this theory for the investigations for first-order equations, our goal in this dissertation is to fulfill this gap and present a version of Floquet theory in this context, employing the definition introduced by M. Bohner, J. G. Mesquita and S. Streipert in [7].

On the other hand, the Floquet theory plays an important role in many applications, such as in linear dynamic systems with periodic coefficient matrix functions, and in many physical and technical situations, such as chaos and population growth. It gives a cannonical form for each fundamental matrix solution of the common linear system, and also, provides Floquet multipliers of

$$\frac{dx}{dt} = A(t)x, \tag{4}$$

whose distribution in the complex plane gives us valuable information about the solvability and stability of periodic solutions to (non-)homogeneous systems of differential equations. The study of classical Floquet theory can be found in Hartman [3], and Ahlbrandt and Ridenhour have studied Floquet theory on periodic time scales [5], with the additive property.

The study of Floquet theory for quantum calculus have been developed in a separated way by M. Bohner and R. Chieochan [4]. This theory was used as basis to investigate a version for Floquet theory for any isolated time scales. More precisely, we work with the following equation

$$x^{\Delta} = A(t)x, \tag{5}$$

where

$$x^{\Delta} := \frac{x(\sigma(t)) - x(t)}{\mu(t)}, \text{ for all } t \in \mathbb{T}, \tag{6}$$

since we are dealing with isolated time scales. Assume A is an ω-periodic matrix function and is also regressive, i.e., $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, where I is the identity matrix. All the results of the Floquet theory on isolated time scales are original and have not been investigated in the literature so far. The new results presented here can be found in the paper [6].

This work is organized as follows: Chapter 1 is devoted to present auxiliary results of time scales, presenting the necessary tools to understand this work. In Chapter 2, we introduce some results of iterated shifts and the concept of periodicity on isolated time scales used here. Also, we bring a discussion concerning different definitions which appear in the literature for periodic functions making a comparasion. Finally, in Chapter 3, we present our main result, that is, a version of Floquet theory on isolated time scales, some important properties and examples are given to illustrate our new definition.

Chapter

1

Basic on the theory of time scales

In this chapter, we will introduce some basic definitions and fundamental results about time scales that will be useful for the comprehension of this work.

1.1 Basic Definitions

A time scale, denoted by $\mathbb{T}$, is a closed and nonempty subset of the real numbers. Some examples of time scales are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, although $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, and $\mathbb{C}$, are not time scales.

In this section, we introduce some fundamentals concepts in the theory of time scales.

We start with the definition of forward and backward jump operators, which play an essential role in this work.

Definition 1.1.1 (See [2, Definition 1.1]). *Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ and the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$, respectively, by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \textit{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In addition, in this definition, denoting $\emptyset$ as the empty set, we adopt the following convention:

i) If $\mathbb{T}$ has a maximum t, $\inf \emptyset = \sup \mathbb{T}$, i. e., $\sigma(t) = t$;

ii) If $\mathbb{T}$ has a minimum t, $\sup \emptyset = \inf \mathbb{T}$, i. e., $\rho(t) = t$.

The value of $\sigma(t)$ and $\rho(t)$ play a central role for a classification of the points belonging to $\mathbb{T}$:

1. If $t < \sigma(t)$, t is called right-scattered;
2. If $t < \sup \mathbb{T}$ and $t = \sigma(t)$, t is called right-dense;

3. If $\rho(t) < t$, t is called left-scattered;
4. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, t is called left-dense;
5. If $\rho(t) < t < \sigma(t)$, t is called isolated;
6. If $\rho(t) = t = \sigma(t)$, t is called dense.

Now, we are ready to define the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) := \sigma(t) - t$$

for all $t \in \mathbb{T}$, that is, the *graininess function* is the distance from a point to the closest point on the right and plays a central role in the analysis on time scales.

If $\mathbb{T}$ has a left-scattered maximum m, we define $\mathbb{T}^\kappa = \mathbb{T} - m$, otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. In summary, we can write:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a function, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)), \text{ for all } t \in \mathbb{T},$$

i.e. $f^\sigma = f \circ \sigma$.

Below, we present some examples to illustrate our definition:

Example 1.1.2 (See [2, Example 1.2]). *If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$*

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t,$$

and

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) = 0, \textit{ for all } t \in \mathbb{T}.$$

Example 1.1.3 (See [2, Example 1.2]). *If $\mathbb{T} = \mathbb{Z}$, then for every $t \in \mathbb{Z}$*

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, ...\} = t+1,$$

and

$$\rho(t) = \sup\{s \in \mathbb{Z} : s < t\} = \sup\{..., t-3, t-2, t-1\} = t-1.$$

Hence, every point $t \in \mathbb{Z}$ *is isolated. The graininess function* μ *turns out to be*

$$\mu(t) = 1, \text{ for all } t \in \mathbb{T}.$$

Example 1.1.4. *If* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, *then for every* $t \in \mathbb{N}_0$

$$\sigma(t) = \inf\{s \in q^{\mathbb{N}_0} : s > t\} = \inf\{qt, q^2t, q^3t, ...\} = qt,$$

and

$$\rho(t) = \sup\{s \in q^{\mathbb{N}_0} : s < t\} = \sup\left\{..., \frac{t}{q^3}, \frac{t}{q^2}, \frac{t}{q}\right\} = \frac{t}{q}.$$

Hence, every point $t \in \mathbb{N}_0$ *is isolated. The graininess function* μ *turns out to be*

$$\mu(t) = t(q-1), \text{ for all } t \in q^{\mathbb{N}_0}.$$

Below we state the induction principle on time scales, which is different from the classical one, and it is very useful to prove some fundamental results. However, we omit its proof here. The reader may consult [2] for details.

Theorem 1.1.5 (Induction Principle, see [2, Theorem 1.7]). *Let* $t_0 \in \mathbb{T}$ *and assume that*

$$\{S(t) : t \in [t_0, \infty)\}_{\mathbb{T}}$$

is a family of statements satisfying:

I. The statement $S(t_0)$ *is true.*

II. If $t \in [t_0, \infty) \in \mathbb{T}$ *is right-scattered and* $S(t)$ *is true, then* $S(\sigma(t))$ *is also true.*

III. If $t \in [t_0, \infty) \in \mathbb{T}$ *is right-dense and* $S(t)$ *is true, then there is a neighborhood* U *of* t *such that* $S(s)$ *is true for all* $s \in U \cap (t, \infty) \in \mathbb{T}$.

IV. If $t \in (t_0, \infty) \in \mathbb{T}$ *is left-dense and* $S(s)$ *is true for all* $s \in [t_0, t)$, *then* $S(t)$ *is true.*

Then $S(t)$ *is true for all* $t \in [t_0, \infty) \in \mathbb{T}$.

1.2 Delta-derivatives

In this section, our goal is to present the definition of Δ-derivative and the main results and properties.

Consider a function $f\colon \mathbb{T} \to \mathbb{R}$ and define the *delta (or Hilger) derivative* of f at a point $t \in \mathbb{T}^\kappa$.

Definition 1.2.1 (See [2, Definition 1.10]). *Assume $f\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$, define $f^\Delta(t)$ to be the number with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e. $U = (t-\delta,\ t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s| \text{ for all } s \in U.$$

*We call $f^\Delta(t)$ the **delta (or Hilger) derivative** of f at t.*

We say that f is **delta (or Hilger) differentiable** on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$, and the function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the **(delta) derivative of** f on $\mathbb{T}^\kappa$.

Remark 1.2.2. *Although the definition above is stated to functions taking value in $\mathbb{R}$, this definition can be generalized analogously to functions taking value in $\mathbb{R}^n$ or in any arbitrary Banach space X.*

Example 1.2.3 (See [2, Example 1.13]).

(i) If $f\colon \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is a constant, $f^\Delta(t) \equiv 0$. Indeed, for any $\epsilon > 0$,

$$|[f(\sigma(t)) - f(s)] - 0.[\sigma(t) - s]| = |\alpha - \alpha| = 0 \le \epsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

(ii) If $f\colon \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^\Delta \equiv 1$. Indeed, for every $\epsilon > 0$,

$$|[f(\sigma(t)) - f(s)] - 1.[\sigma(t) - s]| = |\sigma(t) - s - (\sigma(t) - s)| = 0 \le \epsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

(iii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t^2$ for all $t \in \mathbb{T}$, then $f^\Delta(t) = \sigma(t) + t$, for all $t \in \mathbb{T}$. In fact, by the definition, given $\epsilon > 0$, there exists a neighborhood $U = (t-\epsilon, t+\epsilon) \cap \mathbb{T}$ such that

$$|(\sigma(t))^2 - s^2 - (\sigma(t) + t)(\sigma(t) - s)| = |(s - t) \cdot (\sigma(t) - s)| < \epsilon|(\sigma(t) - s)|$$

holds for all $s \in U$.

Below, we present some useful properties concerning the delta derivative.

Theorem 1.2.4 (See [2, Theorem 1.16]). *Assume $f\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. The following statements hold:*

(i) If f is Δ-differentiable at t, then f is continuous at t.
(ii) If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is Δ-differentiable at t iff the limit

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is Δ-differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Proof. (i) Assume f is Δ-differentiable at t, then given $\epsilon^* > 0$, there is a neighborhood U of t, i.e. $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that $|t - s| < \delta$ and

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon^* |\sigma(t) - s|$$

for all $s \in U$. Note that

$$\begin{aligned} |f(t) - f(s)| &= |f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] - f(\sigma(t)) + f(t) + f^{\Delta}(t)[\sigma(t) - s]| \\ &= |f(t) - f(s) + f(\sigma(t)) - f(\sigma(t)) + f^{\Delta}(t)[\sigma(t) - s] - f^{\Delta}(t)[\sigma(t) - s]| \\ &\leq \epsilon^* |\sigma(t) - s| + \epsilon^* |\mu(t)| + |f^{\Delta}(t)||t - s|. \end{aligned}$$

Since

$$|\sigma(t) - s| = |\sigma(t) - s + t - t| \leq |\sigma(t) - t| + |t - s| = |\mu(t)| + |t - s|,$$

it implies that

$$|f(t) - f(s)| \leq \epsilon^* |\sigma(t) - s| + \epsilon^* |\mu(t)| + |f^{\Delta}(t)||t - s|.$$

If $\delta = \epsilon^*$ and $|t - s| < \delta$, then

$$|f(t) - f(s)| \leq \epsilon^* (|\mu(t)| + |t - s|) + \epsilon^* |\mu(t)| + |f^{\Delta}(t)||t - s|$$

$$\leq \epsilon^*(2|\mu(t)| + 1) + \epsilon^*|f^{\Delta}(t)|.$$

For $\epsilon^* \in (0,1)$, define:

$$\epsilon \colon = \epsilon^* \left(|2|\mu(t)| + 1 + |f^{\Delta}(t)||\right)$$

Then

$$|f(t) - f(s)| \leq \epsilon^* \left(|2|\mu(t)| + 1 + |f^{\Delta}(t)||\right) = \epsilon.$$

It follows that f is continuous at t.

(ii) Assume f is continuous at t and is right-scattered. By continuity,

$$\lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Using the definition of limit, given $\epsilon > 0$, there is a neighborhood U of t ($U = (t - \delta, t + \delta) \cap \mathbb{T}$) such that

$$|t - s| < \delta \implies \left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - \frac{f(\sigma(t) - f(t)}{\mu(t)} \right| < \epsilon.$$

Multiplying all by $|\sigma(t) - s|$, we get

$$\left| f(\sigma(t)) - f(s) - \frac{f(\sigma(t)) - f(t)}{\mu(t)} |\sigma(t) - s| \right| \leq \epsilon |\sigma(t) - s|.$$

Using Definition 1.2.1, we conclude that

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

proving the desired result.

(iii) Assume t is right-dense and f is Δ-differentiable at t. Let $\epsilon > 0$, there is a neighborhood U of t ($U = (t - \delta, t + \delta) \cap \mathbb{T}$) for $\delta > 0$ such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)|\sigma(t) - s| \right| \leq \epsilon |\sigma(t) - s|.$$

Since $\sigma(t) = t$,

$$\left| f(t) - f(s) - f^{\Delta}(t)|t - s| \right| \leq \epsilon |t - s|$$

for all $s \in U$. Therefore,

$$\left| \frac{f(t) - f(s)}{|t - s|} - f^{\Delta}(t) \right| \leq \epsilon$$

for all $s \in U$, $s \neq t$. It implies that

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

proving the result. Reciprocally, assuming

$$\lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s}$$

exists and t is right-dense, then f is Δ-differentiable at t, which means that given $\epsilon > 0$, there exists a neighborhood U of t such that:

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)|\sigma(t) - s| \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. That implies that

$$\left| \frac{f(\sigma(t)) - f(s)}{|t - s|} |t - s| - \frac{f^{\Delta}(t)|\sigma(t) - s||t - s|}{|t - s|} \right| \leq \epsilon \frac{|\sigma(t) - s||t - s|}{|t - s|}.$$

Since $\sigma(t) = t$, we get

$$\left| \frac{f(t) - f(s)}{|t - s|} - f^{\Delta}(t) \right| |t - s| \leq \epsilon |t - s|.$$

Hence the limits exists, and

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|.$$

It implies that

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If t is right-dense, then $\mu(t) = \sigma(t) - t = t - t = 0$ and, clearly,

$$f(\sigma(t)) = f(t) + 0.f^{\Delta}(t) = f(t) + \mu(t) f^{\Delta}(t).$$

If t is right-scattered, then using (ii),

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t).\left(\frac{f(\sigma(t)) - f(t)}{\mu(t)} \right) \\ &= f(t) + \mu(t) f^{\Delta}(t), \end{aligned}$$

getting the desired result. □

Remark 1.2.5. *The formula (iv) is called the "simple useful formula", since it can be used in many different situations and it can be applied for any points in the time scale.*

Example 1.2.6 (See [2, Example 1.18]).
(i) If $\mathbb{T} = \mathbb{R}$, then Theorem 1.2 (iii) yields that $f : \mathbb{R} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{R}$ iff

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \text{ exists},$$

i.e., iff f is differentiable at t.
(ii) If $\mathbb{T} = \mathbb{Z}$, then Theorem 1.2 (ii) yields that $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \Delta f(t),$$

where Δ is the usual forward difference operator.

With that in hands, we can find the derivatives of sums, products, and quotients of Δ-differentiable functions. It is the content of the next result.

Theorem 1.2.7 (See [2, Theorem 1.20]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^{\kappa}$. Then:*
(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is Δ-differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Proof. Assume f and g are Δ-differentiable at $t \in \mathbb{T}^{\kappa}$.

(i) Let $\epsilon > 0$, then there exist neighborhoods U_1 and U_2 of t with

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \frac{\epsilon}{2}\left|[\sigma(t) - s]\right| \tag{1.1}$$

for all $s \in U_1$ and

$$\left|g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]\right| \leq \frac{\epsilon}{2}\left|[\sigma(t) - s]\right| \tag{1.2}$$

for all $s \in U_2$. Let $U = U_1 \cap U_2$, then for all $s \in U$,

$$\begin{aligned}
&\left|(f+g)(\sigma(t)) - (f+g)(s) - (f+g)^{\Delta}(t)[\sigma(t) - s]\right| \\
&= \left|f(\sigma(t)) + g(\sigma(t)) - f(s) - g(s) - f^{\Delta}(t)[\sigma(t) - s] - g^{\Delta}(t)][\sigma(t) - s]\right| \\
&\leq \left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| + \left|g(\sigma(t)) - g(s) - g^{\Delta}(t)][\sigma(t) - s]\right| \\
&\overset{(1.1)-(1.2)}{\leq} \frac{\epsilon}{2}|\sigma(t) - s| + \frac{\epsilon}{2}|\sigma(t) - s| \\
&= \epsilon|\sigma(t) - s|.
\end{aligned}$$

Therefore, $f + g$ is Δ-differentiable at t and $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$ holds for every $t \in \mathbb{T}^{\kappa}$.

(ii) Let $\epsilon > 0$, then there exists a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \frac{\epsilon}{|\alpha|}|\sigma(t) - s|$$

for all $s \in U$. With that, we have two cases:

- $|\alpha| \geq 1$
  Therefore $(\alpha f)(t) = \alpha f(t)$:

$$\begin{aligned}
\left|(\alpha f)(\sigma(t)) - (\alpha f)(s) - (\alpha f^{\Delta}(t))[\sigma(t) - s]\right| &= |\alpha|\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| \\
&\leq |\alpha|\frac{\epsilon}{\alpha}|\sigma(t) - s| \\
&= \epsilon|\sigma(t) - s|,
\end{aligned}$$

getting that (αf) is Δ-differentiable for $t \in \mathbb{T}^{\kappa}$.

- $|\alpha| < 1$
  Using $0 \leq |\alpha| < 1$, we get

$$\left|(\alpha f)(\sigma(t)) - (\alpha f)(s) - (\alpha f^{\Delta}(t))[\sigma(t) - s]\right| \leq |\alpha|\epsilon|\sigma(t) - s| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$. Then, αf is Δ-differentiable at t and $(\alpha f)^{\Delta} = \alpha f^{\Delta}$.

(iii) Let $\epsilon > 0$, there exist neighborhoods U_1 and U_2 of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U_1$, and

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U_2$. Using Theorem 1.2.4 (i), f and g are continuous, i.e., there exists a neighborhood U_3 of t such that

$$|f(t) - f(s)| \leq \epsilon$$

for all $s \in U_3$. Define $U := U_1 \cap U_2 \cap U_3$ and let $s \in U$, then

$$\begin{aligned}
&\left|(fg)(\sigma(t)) - (fg)(s) - [f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t)][\sigma(t) - s]\right| \\
&= \left|f(\sigma(t))g(\sigma(t)) - f(s)g(s) - f^{\Delta}(t)[\sigma(t) - s]g(\sigma(t)) - f(t)[\sigma(t) - s]g^{\Delta}(t)\right|.
\end{aligned}$$

Adding and subtracting the terms $f(s)g(\sigma(t))$, $g(\sigma(t))f(t)$, $g(s)f(t)$, and $g^{\Delta}(t)[\sigma(t) - s][f(s) - f(t)]$, we get

$$\begin{aligned}
&|f(\sigma(t))g(\sigma(t)) - f(s)g(s) - f^{\Delta}(t)[\sigma(t) - s]g(\sigma(t)) - f(t)[\sigma(t) - s]g^{\Delta}(t) \\
&+ f(s)g(\sigma(t)) - f(s)g(\sigma(t)) + g(\sigma(t))f(t) - g(\sigma(t))f(t) + g(s)f(t) \\
&- g(s)f(t) + g^{\Delta}(t)[\sigma(t) - s][f(s) - f(t)] - g^{\Delta}(t)[\sigma(t) - s][f(s) - f(t)]| \\
&= |g(\sigma(t))[f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] + f(t)[g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]] \\
&+ [f(s) - f(t)][g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]] + [\sigma(t) - s]g^{\Delta}(t)[f(s) - f(t)]| \\
&\leq \left|g(\sigma(t))[f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right| + \left|f(t)[g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]\right| \\
&+ \left|[f(s) - f(t)][g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]\right| + \left|[\sigma(t) - s]g^{\Delta}(t)[f(s) - f(t)]\right| \\
&\leq \epsilon|\sigma(t) - s||g(\sigma(t))| + \epsilon|\sigma(t) - s||f(t)| + \epsilon|\sigma(t) - s|\epsilon + \epsilon|[\sigma(t) - s]g^{\Delta}(t)|
\end{aligned}$$

$\leq \epsilon|\sigma(t) - s|[|g(\sigma(t))| + |f(t)| + 1 + |g^{\Delta}(t)|]$.

Since $\epsilon > 0$, $|g(\sigma(t))| \geq 0$, $|f(t)| \geq 0$, and $|g^{\Delta}(t)| \geq 0$, we define

$$\epsilon^* = \epsilon[|g(\sigma(t))| + |f(t)| + 1 + |g^{\Delta}(t)|] > 0.$$

Then, we have

$$\big|(fg)(\sigma(t)) - (fg)(s) - [f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t)][\sigma(t) - s]\big| \leq \epsilon^*|\sigma(t) - s|.$$

Therefore, fg is Δ-differentiable at t and $(fg)^{\Delta} = f^{\Delta}g^{\sigma} + fg^{\Delta}$. Analogously, one can prove that $(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}$.

(iv) Let $\epsilon > 0$, there exists a neighborhood U of t such that

$$\big|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\big| \leq \frac{\epsilon}{2}|\sigma(t) - s|$$

for all $s \in U$. Hence,

$$\begin{aligned}
&\left[\frac{1}{f(\sigma(t))} - \frac{1}{f(s)} + \frac{f^{\Delta}(t)[\sigma(t) - s]}{f(t)f(\sigma(t))}\right]\left|\frac{f(s)f(\sigma(t))}{f(s)f(\sigma(t))}\right| = \\
&= \left[f(s) - f(\sigma(t)) + \frac{f(s)f^{\Delta}(t)[\sigma(t) - s]}{f(t)}\right]\left|\frac{1}{f(s)f(\sigma(t))}\right| \\
&= \left[f(s) - f(\sigma(t)) + \frac{f(s)f^{\Delta}(t)[\sigma(t) - s]}{f(t)} + f^{\Delta}(t)\left([\sigma(t) - s] - [\sigma(t) - s]\right)\right]\left|\frac{1}{f(s)f(\sigma(t))}\right| \\
&\leq \left[\big|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\big| + \left|f^{\Delta}(t)[\sigma(t) - s]\left[\frac{f(s) - f(t)}{f(t)}\right]\right|\right]\left|\frac{1}{f(s)f(\sigma(t))}\right| \\
&\leq \left[\frac{\epsilon}{2}|\sigma(t) - s||f(s)f(\sigma(t))| + \frac{\epsilon}{2}|\sigma(t) - s||f(s)f(\sigma(t))|\right]\left|\frac{1}{f(s)f(\sigma(t))}\right| \\
&= [\epsilon|\sigma(t) - s||f(s)f(\sigma(t))|]\left|\frac{1}{f(s)f(\sigma(t))}\right|.
\end{aligned}$$

Therefore, $\dfrac{1}{f}$ is Δ-differentiable at t and

$$\left(\frac{1}{f}\right)^{\Delta} = -\frac{f^{\Delta}}{ff^{\sigma}}.$$

(v) Using Theorem 1.2.7 (iii) and (iv) and having $g(t)g(\sigma(t)) \neq 0$

$$\begin{aligned}\left(\frac{f}{g}\right)^{\Delta}(t) &= f(t)\left(-\frac{g^{\Delta}(t)}{g(t)g(\sigma(t))}\right) + \frac{f^{\Delta}(t)}{g(\sigma(t))} \\ &= \frac{-f(t)g^{\Delta}(t)g(\sigma(t)) + f^{\Delta}(t)g(t)g(\sigma(t))}{g(t)g(\sigma(t))g(\sigma(t))} \\ &= -\frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.\end{aligned}$$

Therefore, $\dfrac{f}{g}$ is Δ-differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}},$$

getting the desired result. □

Example 1.2.8 (See [2, Exercise 1.23]). *By Theorem 1.2.7 (iii), we have*

$$(f^2)^{\Delta} = (f.f)^{\Delta} = f^{\Delta}f + f^{\sigma}f^{\Delta} = (f + f^{\sigma})f^{\Delta}.$$

Let us find the generalization of this formula of the derivative of the f^{n+1}. We start by checking the formula for f^3:

$$(f^3)^{\Delta} = (f^2f)^{\Delta} = (f^2)^{\Delta}f + (f^2)^{\sigma}f^{\Delta} = f^{\Delta}(f + f^{\sigma})f + (f^2)^{\sigma}f^{\Delta} = f^{\Delta}[f^2 + ff^{\sigma} + (f^2)^{\sigma}].$$

Analogously, we get for f^4:

$$(f^4)^{\Delta} = (f^3f)^{\Delta} = (f^3)^{\Delta}f + (f^3)^{\sigma}f^{\Delta} = f^{\Delta}[f^3 + f^2f^{\sigma} + f(f^2)^{\sigma} + (f^3)^{\sigma}].$$

A careful examination leads us to

$$(f^{n+1})^{\Delta} = f^{\Delta}\sum_{k=0}^{n} f^k(f^{\sigma})^{n-k},$$

for all $n \in \mathbb{N}$. Now, we prove this formula is fulfilled. In order to do this, we use induction. For it, consider $n = 1$, then:

$$\begin{aligned}f^{\Delta}\sum_{k=0}^{n} f^k(f^{\sigma})^{n-k} &= f^{\Delta}\sum_{k=0}^{1} f^k(f^{\sigma})^{1-k} \\ &= f^{\Delta}\left[f^0(f^{\sigma})^1 + f^1(f^{\sigma})^0\right]\end{aligned}$$

$$= f^{\Delta}(f + f^{\sigma})$$
$$= (f^2)^{\Delta}.$$

Suppose this holds for $p = n+1$, that means,

$$(f^p)^{\Delta} = f^{\Delta}\sum_{k=0}^{p-1} f^k (f^{\sigma})^{p-1-k}$$

Now, let us prove that it holds for $p+1$

$$\begin{aligned}
(f^{p+1})^{\Delta} &= (f^p f)^{\Delta} = (f^p)^{\Delta} f + (f^{\sigma})\sigma f^{\Delta} \\
&= \left[f^{\Delta}\sum_{k=0}^{p-1} f^k (f^{\sigma})^{p-1-k} \right] f + (f^{\sigma})^p f^{\Delta} \\
&= \left[f^{\Delta}\sum_{k=0}^{p-1} f^k (f^{\sigma})^{p-1-k} f \right] + (f^{\sigma})^p f^{\Delta} \\
&= f^{\Delta} \left[\left[(f^{\sigma})^{p-1} f + f^1 (f^{\sigma})^{p-2} f + f^2 (f^{\sigma})^{p-3} f + \dots + f^{p-1} f \right] + (f^{\sigma})^p \right] \\
&= f^{\Delta} \sum_{k=0}^{p} f^k (f^{\sigma})^{p-k}.
\end{aligned}$$

Using the example above, we can obtain the next theorem.

Theorem 1.2.9 (See [2, Theorem 1.24])**.** *Let α be a constant and $m \in \mathbb{N}$.*

(i) For f defined by $f(t) = (t-\alpha)^m$, we have

$$f^{\Delta}(t) = \sum_{v=0}^{m-1} (\sigma(t) - \alpha)^v (t-\alpha)^{m-1-v}, \ t \in \mathbb{T}.$$

(ii) For g defined by $g(t) = \dfrac{1}{(t-\alpha)^m}$, we have

$$g^{\Delta}(t) = -\sum_{v=0}^{m-1} \frac{1}{(\sigma(t)-\alpha)^{m-v}(t-\alpha)^{v+1}}, \ t \in \mathbb{T},$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$.

Proof. (i) If $m = 1$, $f(t) = t - \alpha$, and $f^{\Delta}(t) = 1$. Assume

$$f^{\Delta}(t) = \sum_{v=0}^{m-1} (\sigma(t) - \alpha)^v (t-\alpha)^{m-1-v}$$

holds for $f(t) = (t-\alpha)^m$. Let $F(t) = (t-\alpha)^{m+1} = (t-\alpha)^m(t-\alpha) = f(t)(t-\alpha)$. Using the product rule:

$$\begin{aligned} F^\Delta(t) &= f^\Delta(t)(t-\alpha) + f(\sigma(t))(t-\alpha)^\Delta \\ &= \left(\sum_{v=0}^{m-1}(\sigma(t)-\alpha)^v(t-\alpha)^{m-1-v}\right)(t-\alpha) + (f(\sigma(t)) \\ &= \sum_{v=0}^{m-1}(\sigma(t)-\alpha)^v(t-\alpha)^{m-v} + (\sigma(t)-\alpha)^m \\ &= \sum_{v=0}^{m}(\sigma(t)-\alpha)^v(t-\alpha)^{m-v}. \end{aligned}$$

(ii) To prove this part, we will use Theorem 1.2.4 (iv). Let $g(t) = \dfrac{1}{(t-\alpha)^m} = \dfrac{1}{f(t)}$,

$$g^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$

This implies that

$$\begin{aligned} g^\Delta(t) &= -\frac{\sum_{v=0}^{m-1}(\sigma(t)-\alpha)^v(t-\alpha)^{m-1-v}}{(t-\alpha)^m(\sigma(t)-\alpha)^m} \\ &= -\frac{\sum_{v=0}^{m-1}(t-\alpha)^{m-1-v}(\sigma(t)-\alpha)^v}{(t-\alpha)^m(\sigma(t)-\alpha)^m} - \sum_{v=0}^{m-1}\frac{1}{(t-\alpha)^{v+1}(\sigma(t)-\alpha)^{m-v}}, \end{aligned}$$

getting the desired result. □

The higher order derivatives of functions on time scales is defined in the usual way, as one can check in the following definition.

Definition 1.2.10 (See [2, Definition 1.27]). *For a function $f\colon \mathbb{T} \to \mathbb{R}$, the second derivative $f^{\Delta\Delta}$ exists and is well-defined provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly, we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$ and $\rho^2(t) = \rho(\rho(t))$, and $\sigma^n(t)$ and $\rho^n(t)$ for $n \in \mathbb{N}$ are defined accordingly. For convenience, we also put*

$$\rho^0(t) = \sigma^0(t) = t, \quad f^{\Delta^0} = f, \quad \textit{and} \quad \mathbb{T}^{\kappa^0} = \mathbb{T}.$$

Below, let us show an example to illustrate this definition.

Example 1.2.11 (See [2, Exercise 1.29]). *Let us find the second derivative of f on an arbitrary time scale:*

(i) $f(t) \equiv 1$

$$f^{\Delta}(t) = 0 \ \ and \ \ f^{\Delta\Delta}(t) = 0$$

(ii) $f(t) \equiv t$

$$f^{\Delta}(t) = 1 \ \ and \ \ f^{\Delta\Delta}(t) = 0$$

(iii) $f(t) \equiv t^2$

$$f^{\Delta}(t) = \sigma(t) + t$$

For the second derivative, we need to apply Theorem 1.2.4 (ii) and (iii). Hence, if $f^{\Delta}(t) = \sigma(t) + t$ is continuous at t and t is right-scattered, then f^{Δ} is Δ-differentiable at t with

$$f^{\Delta\Delta} = (f^{\Delta}(t))^{\Delta} = \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\mu(t)} = \frac{\sigma(t) + \sigma(\sigma(t)) - t - \sigma(t)}{\mu(t)} = \frac{\sigma^2(t) - t}{\mu(t)}.$$

On the other hand, if t is right-dense, then f^{Δ} is Δ-differentiable at t with

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} = \lim_{s\to t} \frac{t + \sigma(t) - s - \sigma(s)}{t - s} = \lim_{s\to t} \frac{2t - 2s}{t - s} = 2.$$

Example 1.2.12 (See [2, Example 1.31]). *In general, fg is not twice differentiable even if both f and g are twice differentiable, since*

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}.$$

If f and g are twice differentiable and if f^{σ} is differentiable, then

$$\begin{aligned}(fg)^{\Delta} &= (f^{\Delta} g + f^{\sigma} g^{\Delta})^{\Delta} \\ &= f^{\Delta\Delta} g + f^{\Delta^{\sigma}} g^{\Delta} + f^{\sigma^{\Delta}} g^{\Delta} + f^{\sigma^{\sigma}} g^{\Delta\Delta} \\ &= f^{\Delta\Delta} g + (f^{\Delta\sigma} + f^{\sigma\Delta}) g^{\Delta} + + f^{\sigma\sigma} g^{\Delta\Delta},\end{aligned}$$

where we write $f^{\Delta\sigma}$ to denote $f^{\Delta^{\sigma}}$.

1.3 Integration

The definitions below will help us to describe classes of functions that are integrable.

Definition 1.3.1 (See [2, Definition 1.57]). *A function $f\colon \mathbb{T} \to \mathbb{R}$ is called **regulated** provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 1.3.2 (See [2, Definition 1.58]). *A function $f\colon \mathbb{T} \to \mathbb{R}$ is called **rd-continuous** provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted here by*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are Δ-differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

The next result will bring important properties from those types of functions. These properties will be important for our purposes.

Theorem 1.3.3 (See [2, Theorem 1.60]). *Assume $f : \mathbb{T} \to \mathbb{R}$, then the following statements hold.*

(i) If f is continuous, then f is rd-continuous.

(ii) If f is rd-continuous, then f is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If f is regulated or rd-continuous, then so is f^σ.

(v) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Proof.

(i) The item (i) follows directly from the definition of rd-continuous functions.

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, for all left-dense points at $\mathbb{T}$ its left-sided limits exist, since f is continuous for all right-dense points at $\mathbb{T}$, its right-sided limits exist, and we conclude f is regulated.

(iii) Let $t \in \mathbb{T}$ be a right-dense point. Let $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that if $s \in (t - \delta, t + \delta) \cap \mathbb{T}$, then $|\sigma(s) - \sigma(t)| < \epsilon$. Hence, for $s \in [t, t + \delta]$, $\sigma(s) \geq \sigma(t) = t$, and

$$\sigma(s) - \sigma(t) = \sigma(s) - t = s - t < \delta < \epsilon.$$

Also, for $s \in (t - \delta, t)$, we have that

$$-\delta < s - t < 0$$

Then

$$-\epsilon < -\delta < s - t = s - \sigma(t) \leq \sigma(s) - \sigma(t).$$

Therefore, for $s \in (t-\delta, t+\delta)$, we get $|\sigma(s) - \sigma(t)| < \epsilon$, showing that σ is continuous at right-dense points.

(iv) Assume f is regulated and $t \in \mathbb{T}$ is right-dense, then

$$\lim_{s\to t^+} f^\sigma(s) = \lim_{s\to t^+} f(\sigma(s)) = \lim_{u\to t^+} f(u)$$

exists. Also, if $t \in \mathbb{T}$ is left-dense, since σ is rd-continuous, the limit $\lim_{s\to t^-} \sigma(s)$ exists. Defining $v\colon = \lim_{s\to t^-} \sigma(s)$, we get

$$\lim_{s\to t^-} f^\sigma(s) = \lim_{s\to t^-} f(\sigma(s)) = \lim_{u\to v} f(u)$$

also exists, which implies that f^σ is regulated. Now, assume f is rd-continuous and $t \in \mathbb{T}$ is right-dense, we have the limit

$$\lim_{s\to t} f^\sigma(s) = \lim_{s\to t} f(\sigma(s)) = \lim_{u\to t} f(u)$$

exists. Then f^σ is continuous at t. Also, for all left-dense points $t \in \mathbb{T}$, the left-sided limits exist for f^σ, concluding that f^σ is rd-continuous.

(v) Assuming f is continuous and g is rd-continuous, then clearly $f \circ g$ is rd-continuous. The same happens for the regulated case. □

Definition 1.3.4 (See [2, Definition 1.62]). *A continuous function $f\colon \mathbb{T} \to \mathbb{R}$ is called **pre-differentiable with region of differentiation** D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.*

Below, we state the Mean Value Theorem for time scales, which is analogue from the classical one, and which corollary is very useful. However, we omit its proof here. The reader may consult [2] for details.

Theorem 1.3.5 (Mean Value Theorem, see [2, Theorem 1.67]).

Let f and g be real-valued functions defined on $\mathbb{T}$, both pre-differentiable with D. Then

$$|f^\Delta(t)| \leq g^\Delta(t)$$

for all $t \in D$ implies

$$|f(s) - f(r)| \leq g(t) - g(r)$$

for all $r, s \in \mathbb{T},\ r \leq s$.

Corollary 1.3.6 (See [2, Corollary 1.68]). *Suppose f and g are pre-differentiable with the region of differentiation D. Then the following statements hold.*

(i) If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^\kappa \cap D} |f^\Delta(t)| \right\} |s - r|.$$

(ii) If $f^\Delta(t) = 0$ for all $t \in D$, then f is a constant function.

(iii) If $f^\Delta(t) = g^\Delta(t)$ for all $t \in D$, then

$$g(t) = f(t) + C$$

for all $t \in \mathbb{T}$, where C is a constant.

Proof. (i) Suppose f is pre-differentiable with the region of differentiation D and let $r, s \in \mathbb{T}$ with $r \leq s$. Define

$$g(t) := \left\{ \sup_{\tau \in [r,s]^\kappa \cap D} |f^\Delta(t)| \right\} (t - r)$$

for $t \in \mathbb{T}$. Considering $g^\Delta(t) \geq |f^\Delta(t)|$ and using the Mean Value Theorem, we get

$$|f(t) - f(r)| \leq g(s) - g(r) = g(s) = \left\{ \sup_{r \in [r,s]^\kappa \cap D} |f^\Delta(\tau)| \right\} (s - r),$$

proving the result.

(ii) Considering $f^\Delta(t) = 0$ and using (i), we have

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^\kappa \cap D} |0| \right\} |r - s| = 0.$$

It follows that $|f(s) - f(r)| = 0$ for $r, s \in \mathbb{T}$, and thus, f is constant.

(iii) Define $h(t) \colon = f(t) - g(t)$, for each $t \in \mathbb{T}$. Using (ii),

$$h^\Delta(t) = |f - g|^\Delta(t) = f^\Delta(t) - g^\Delta(t) = C,$$

which implies $f^\Delta(t) = g^\Delta(t) + C$. □

Theorem 1.3.7 (Existence of Pre-Antiderivatives, see [2, Theorem 1.70]). *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D*

such that

$$F^{\Delta}(t) = f(t)$$

hold for all $t \in D$.

Definition 1.3.8 (See [2, Definition 1.71]). *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a regulated function. Any function* F *defined as in Theorem 1.3.5 is called a* ***pre-antiderivative of*** f. *We define the* ***indefinite integral of a regulated function*** f *by*

$$\int f(t)\Delta t = F(t) + C,$$

where C *is an arbitrary constant and* F *is a pre-antiderivative of* f. *We define the* ***Cauchy*** Δ***-integral*** *by*

$$\int_r^s f(t)\Delta t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}$. *A function* F *is called an* ***antiderivative*** *of* $f : \mathbb{T} \to \mathbb{R}$ *provided*

$$F^{\Delta}(t) = f(t)$$

holds for all $t \in \mathbb{T}^{\kappa}$.

Example 1.3.9 (See [2, Example 1.72]). *If* $\mathbb{T} = \mathbb{Z}$, *evaluate the indefinite integral*

$$\int a^t \Delta t,$$

where $a \neq 1$ *is a constant. Since*

$$\left(\frac{a^t}{a-1}\right)^{\Delta} = \Delta\left(\frac{a^t}{a-1}\right) = \frac{a^{t+1} - a^t}{a-1} = a^t,$$

we get that

$$\int a^t \Delta t = \frac{a^t}{a-1} + C,$$

where C *is an arbitrary constant.*

Theorem 1.3.10 (Existence of Antiderivatives, see [2, Theorem 1.74]). *Every rd-continuous function has an antiderivative. In particular if* $t_0 \in \mathbb{T}$, *then* F *defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau$$

for $t \in \mathbb{T}$, *is an antiderivative of* f.

Proof. Suppose f is an rd-continuous, by Theorem 1.3.3 (ii), f is regulated. Let F be a function, whose the existence is guaranteed by Theorem 1.3.5 with D satisfying

$$F^{\Delta}(t) = f(t)$$

for all $t \in D$. We have that F is pre-differentiable with D. Let $t \in \mathbb{T}^{\kappa}\backslash D$, then t is right-dense because $\mathbb{T}^{\kappa}\backslash D$ cannot contain any right-scattered points. Let $\epsilon > 0$, then there exists a neighborhood U of t with

$$|f(s) - f(t)| \leq \epsilon$$

for all $s \in U$. Define

$$h(\tau) := f(\tau) - f(t)(\tau - t_0)$$

for $\tau \in \mathbb{T}$. Then h is pre-differentiable with D and we have

$$h^{\Delta}(\tau) = F^{\Delta}(t) - f(t) = f(\tau) - f(t)$$

for all $\tau \in D$. Hence,

$$|h^{\Delta}(s)| = |f(s) - f(t)| \leq \epsilon$$

for all $s \in D \cap U$. Therefore,

$$\sup_{s \in D\cap U} |h^{\Delta}(s)| \leq \epsilon.$$

Thus, by Corollary 1.3.6, for $r \in U$,

$$\begin{aligned} |F(t) - F(r) - f(t)(t-r)| &= |h(t) + f(t)(t - t_0) - [h(r) + f(t)(r - t_0)] - f(t)(t - r)| \\ &= |h(t) - h(r)| \\ &\leq \left\{ \sup_{s \in D\cap U} |h^{\Delta}(s)| \right\} |t - s| \leq \epsilon|t - s| \end{aligned}$$

This implies that F is Δ-differentiable at t with $F^{\Delta}(t) = f(t)$, concluding the proof. □

Theorem 1.3.11 (See [2, Theorem 1.75]). *If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Proof. Since $f \in C_{\rm rd}$, using Theorem 1.3.10, there exists an antiderivative F of f,

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = F(\sigma(t)) - F(t) = \mu(t)F^{\Delta}(t) = \mu(t)f(t)$$

getting the result. □

Theorem 1.3.12 (See [2, Theorem 1.76]). *If $f \in C_{rd}$ and $f^{\Delta} \geq 0$, then f is nondecreasing.*

Proof. Let $f^{\Delta} \geq 0$ on $[a,b] \cap \mathbb{T}$ and let $s, t \in \mathbb{T}$, with $a \leq s \leq t \leq b$. Then

$$f(t) = f(s) + \int_s^t f^{\Delta}(\tau)\Delta\tau \geq f(s),$$

proving the result. □

Theorem 1.3.13 (See [2, Theorem 1.77]). *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$
(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
(v) $\int_a^b f(\sigma(t))g^{\Delta}t\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$
(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$
(vii) $\int_a^a f(t)\Delta t = 0;$
(viii) If $|f(t)| \leq g(t)$ on $[a,b)$, then

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t$$

(ix) If $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$.

Proof. Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, $f, g \in C_{\rm rd}$, and since $f, g \in C_{\rm rd}$, by Theorem 1.3.10, f and g have antiderivatives F and G, respectively. This implies that:

(i) $F + G$ is an antiderivative of $f + g$, so that

$$\begin{aligned}\int_a^b [f(t) + g(t)]\Delta t &= (F+G)(b) - (F+G)(a) \\ &= F(b) - F(a) + G(b) - G(a) \\ &= \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.\end{aligned}$$

(ii) Having $\alpha f(t) = (\alpha f)(t)$, we have that

$$\begin{aligned}\int_a^b |(\alpha f)(t)|\,\Delta t &= (\alpha F)(b) - (\alpha F)(a)\\ &= \alpha F(b) - \alpha F(a)\\ &= \alpha \int_a^b f(t)\Delta t.\end{aligned}$$

(iii) Let

$$\int_a^b f(t)\Delta t = F(b) - F(a)$$

Therefore,

$$|F(b) - F(a)| = -\,|F(a) - F(b)| = -\int_b^a f(t)\Delta t.$$

(iv) Let

$$\int_a^b f(t)\Delta t = F(b) - F(a) = F(c) - F(a) + F(b) - F(c) = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$$

proving the result.

(v) By Theorem 1.2.7 (iii),

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta.$$

Integrating, we have

$$\int_a^b (fg)^\Delta(t)\Delta t = \int_a^b f^\Delta(t)g(t)\Delta t + \int_a^b f^\sigma(t)g^\Delta(t)\Delta t.$$

Since fg is antiderivative of $f^\sigma g^\Delta + f^\Delta g$,

$$\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t.$$

(vi) The proof of this item is very similar to the proof of the previous one. Thus, we omit it here.

(vii) Note that

$$\int_a^a f(t)\Delta t = F(a) - F(a) = 0,$$

proving the result.

(viii) Since $|f(t)| \leq g(t)$ on $[a,b)$, integrating both sides, we get

$$\int_a^b |f(t)|\Delta t \leq \int_a^b g(t)\Delta t$$

On the other hand, for $t \in [a,b)$, we have that $-|f(t)| \leq f(t) \leq |f(t)|$, integrating all the sides,

$$\int_a^b -|f(t)|\Delta t \leq \int_a^b f(t)\Delta t \leq \int_a^b |f(t)|\Delta t.$$

Therefore,

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\Delta t \leq \int_a^b g(t)\Delta t.$$

Hence,

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t.$$

(ix) Since $f(t) \geq 0$, $F^{\Delta}(t) = f(t) \geq 0$. Using Theorem 1.3.12, $F(t)$ is nondecreasing, then for $b > a$, we get:

$$F(b) - F(a) \geq 0 \iff \int_a^b f(t)\Delta t \geq 0,$$

getting the desired result. □

Theorem 1.3.14 (See [2, Theorem 1.79]). *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right hand side is the usual Riemann integral from calculus.

(ii) If $[a,b] \cap \mathbb{T}$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)\cap\mathbb{T}} \mu(t)f(t) & \text{if} \quad a < b; \\ 0 & \text{if} \quad a = b; \\ -\sum_{t\in[b,a)\cap\mathbb{T}} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(t) & \text{if} \quad a < b; \\ 0 & \text{if} \quad a = b; \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(t) & \text{if} \quad a > b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$*, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if} \quad a < b; \\ 0 & \text{if} \quad a = b; \\ -\sum_{t=b}^{a-1} f(t) & \text{if} \quad a > b. \end{cases}$$

Proof. (i) The proof of part (i) follows from Example 1.2.6 and the standard Fundamental Theorem of Calculus.

(ii) In this case, we can write $[a, b] \cap \mathbb{T} = \{a = t_0, t_1, ..., t_n = b\}$, where $t_0, ..., t_n \in \mathbb{T}$, since $[a, b] \cap \mathbb{T}$ consists of only isolated points. Using Theorem 1.3.13 (iv),

$$\int_a^b f(t)\Delta t = \int_{t_0}^{t_1} f(t)\Delta t + \int_{t_1}^{t_2} f(t)\Delta t + ... + \int_{t_{n-1}}^{t_n} f(t)\Delta t.$$

Using Theorem 1.3.11, we have:

$$\int_a^b f(t)\Delta t = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(t)\Delta t = \sum_{k=0}^{n-1} \int_{t_\kappa}^{\sigma(t_\kappa)} f(t)\Delta t = \sum_{k=0}^{n-1} \mu(t_k) f(t_k) = \sum_{t \in [a,b) \cap \mathbb{T}} \mu(t) f(t).$$

Analogously, we can prove the other cases.

(iii) If $a < b$, using item (ii),

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b) \cap \mathbb{T}} \mu(t) f(t) = \sum_{t \in [a,b) \cap \mathbb{T}} f(t)h = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h$$

The other cases follow analogously.

(iv) This is a particular case of (iii) when $h = 1$. □

Definition 1.3.15 (See [2, Definition 1.82]). *If* $a \in \mathbb{T}$, $\sup\ \mathbb{T} = \infty$*, and* f *is rd-continuous on* $[a, \infty)$*, then we define the* ***improper integral*** *by*

$$\int_a^\infty f(t)\Delta t := \lim_{b \to \infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

1.4 Chain Rules

If $f, g : \mathbb{R} \to \mathbb{R}$, then the chain rule from calculus states that if g is differentiable at t and if f is differentiable at $g(t)$, then

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

The analogue chain rule from calculus does not hold for all time scales, as you can check on the next example:

Example 1.4.1 (See [2, Example 1.85]). *Let $\mathbb{T} = \mathbb{Z}$, and assume $f, g : \mathbb{Z} \to \mathbb{Z}$ are defined by $f(t) = t^2$ and $g(t) = 2t$. Thus,*

$$(f \circ g)^{\Delta} = f^{\Delta}(g(t)) = f^{\Delta}(2t) = (4t^2)^{\Delta} = 4(\sigma(t) + t) = 4(2t + 1) = 8t + 4$$

for all $t \in \mathbb{Z}$, and

$$\begin{aligned} f^{\Delta}(g(t))g^{\Delta}(t) &= [\sigma(g(t)) + g(t)][g(t + 1) - g(t)] \\ &= [2g(t) + 1] \cdot 2 = [2(2t) + 1] \cdot 2 = [4t + 1] \cdot 2 = 8t + 2 \end{aligned}$$

for all $t \in \mathbb{Z}$. Then, we have

$$(f \circ g)^{\Delta} = 8t + 4 \neq 8t + 2 = f^{\Delta}(g(t))g^{\Delta}(t).$$

Although the chain rule does not work here, we will present some alternative properties that hold to any time scale and may be useful.

Theorem 1.4.2 (Chain Rule, see [2, Theorem 1.87]). *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t), \text{ for all } t \in \mathbb{T}^{\kappa}.$$

Proof. Fix $t \in \mathbb{T}^{\kappa}$. Assume that t is right-scattered. Then,

$$(f \circ g)^{\Delta}(t) = \frac{f(g(\sigma(t))) - f(g(t))}{\mu(t)}.$$

If $g(\sigma(t)) = g(t)$, then we get $(f \circ g)^{\Delta}(t) = 0$, and $g^{\Delta} = 0$. This implies that

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t)$$

holds for any c in the real interval $[t, \sigma(t)]$.

Assuming $g(\sigma(t)) \neq g(t)$, then

$$(f \circ g)^{\Delta}(t) = \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} . \frac{g(\sigma(t)) - g(t)}{\mu(t)} = f'(\xi)g^{\Delta}(t),$$

where ξ is between $g(t)$ and $g(\sigma(t))$, by Mean Value Theorem (Theorem 1.3.5). Since $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, there exists $c \in [t, \sigma(t)]$ such that $g(c) = \xi$, getting the desired result.

Now, suppose t is right-dense. Then we have

$$\begin{aligned}(f \circ g)^{\Delta}(t) &= \lim_{s \to t} \frac{f(g(t)) - f(g(s))}{t - s} \\ &= \lim_{s \to t} \left\{ f'(\xi_s) . \frac{g(t) - g(s)}{t - s} \right\},\end{aligned}$$

where ξ_s is between $g(s)$ and $g(t)$, by Mean Value Theorem (Theorem 1.3.5). By the continuity of g, we get

$$\lim_{s \to t} \xi_s = g(t)$$

which gives the desired result. □

Below, we present an example to illustrate the result above.

Example 1.4.3 (See [2, Example 1.88]). *Given $\mathbb{T} = \mathbb{Z}$, $f(t) = t^2$, $g(t) = 2t$, let us find directly the value c guaranteed by Theorem 1.4.2 so that*

$$(f \circ g)^{\Delta}(3) = f'(g(c))g^{\Delta}(3)$$

and show that c belongs to the interval guaranteed by Theorem 1.4.2. Using Example 1.4.1, we have

$$(f \circ g)^{\Delta}(3) = 8.3 + 4 = 24 + 4 = 28$$

for all $t \in \mathbb{Z}$, and

$$f'(g(t))g^{\Delta}(t) = f'(2c)2 = [((2c)^2)'] \cdot 2 = (4c)' \cdot 2 = (4c) \cdot 2 = 8c$$

for all $t \in \mathbb{Z}$. *Then, we have*

$$(f \circ g)^{\Delta}(3) = f'(g(c))g^{\Delta}(3).$$

Solving for c, *we get*

$$c = \frac{28}{8} = \frac{7}{2}$$

which is in the real interval $[3, \sigma(3)] = [3, 4]$ *as we guaranteed by Theorem 1.4.2.*

Now, we present a version of chain rule which calculates $(f \circ g)^{\Delta}$, where $g\colon \mathbb{T} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$.

Theorem 1.4.4 (Chain Rule, see [2, Theorem 1.90]). *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $g\colon \mathbb{T} \to \mathbb{R}$ *is delta differentiable. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t)$$

holds.

Proof. First, of all we apply the Substitution rule from calculus to find

$$\begin{aligned} f(g(\sigma(t))) - f(g(t)) &= \int_{g(t)}^{g(\sigma(t))} f'(\tau)d\tau \\ &= [g(\sigma(t)) - g(t)] \int_0^1 f'(hg(\sigma(t)) + (1-h)g(t))dh. \end{aligned}$$

Let $t \in \mathbb{T}^{\kappa}$ and $\epsilon > 0$ be given. Since g is Δ-differentiable at t, there exists a neighborhood U_1 of t such that

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon^* |\sigma(t) - s| \quad \text{for all} \quad s \in U_1.$$

where

$$\epsilon^* = \frac{\epsilon}{1 + 2\int_0^1 |f'(hg(\sigma(t)) + (1-h)g(t))|dh}.$$

Furthermore, since f' is continuous on $\mathbb{R}$, so it is uniformly continuous on closed subsets of $\mathbb{R}$, and hence there exists a neighborhood U_2 of t such that

$$|f'(hg(\sigma(t)) + (1-h)g(s)) - f'(hg(\sigma(t)) + (1-h)g(t))| \leq \frac{\epsilon}{2(\epsilon^* + |g^{\Delta}(t)|)} \text{ for all } s \in U_2.$$

Also, note that

$$|hg(\sigma(t)) + (1-h)g(s) - (hg(\sigma(t)) + (1-h)g(t))| = (1-h)|g(s) - g(t)| \leq |g(s) - g(t)|$$

holds for all $0 \leq h \leq 1$. Defining $U = U_1 \cap U_2$ and leting $s \in U$. We put

$$\alpha = hg(\sigma(t)) + (1-h)g(s) \quad \text{and} \quad \beta = hg(\sigma(t)) + (1-h)g(t).$$

Then, we have

$$\begin{aligned}
&\left|(f \circ g)(\sigma(t)) - (f \circ g)(s) - (\sigma(t) - s)g^{\Delta}(t)\int_0^1 f'(\beta)dh\right| \\
&= \left|[g(\sigma(t)) - g(s)]\int_0^1 f'(\alpha)dh - (\sigma(t) - s)g^{\Delta}(t)\int_0^1 f'(\beta)dh\right| \\
&= \left|[g(\sigma(t)) - g(s) - (\sigma(t) - s)g^{\Delta}(t)]\int_0^1 f'(\alpha)dh + (\sigma(t) - s)g^{\Delta}(t)\int_0^1 (f'(\alpha) - f'(\beta))dh\right| \\
&\leq |g(\sigma(t)) - g(s) - (\sigma(t) - s)g^{\Delta}(t)|\int_0^1 |f'(\alpha)|dh + |\sigma(t) - s||g^{\Delta}(t)|\int_0^1 |f'(\alpha) - f'(\beta)|dh \\
&\leq \epsilon^*|\sigma(t) - s|\int_0^1 |f'(\alpha)|dh + |\sigma(t) - s||g^{\Delta}(t)|\int_0^1 |f'(\alpha) - f'(\beta)|dh \\
&\leq \epsilon^*|\sigma(t) - s|\int_0^1 |f'(\beta)|dh + |\epsilon^* + |g^{\Delta}(t)||\sigma(t) - s|\int_0^1 |f'(\alpha) - f'(\beta)|dh \\
&\leq \frac{\epsilon}{2}|\sigma(t) - s| + \frac{\epsilon}{2}|\sigma(t) - s| \\
&= \epsilon|\sigma(t) - s|.
\end{aligned}$$

Thus, $f \circ g$ is Δ-differentiable at t and the derivative is as claimed above. □

Example 1.4.5 (See [2, Example 1.91]). *We define $g : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ by*

$$g(t) = t^2 \quad \textit{and} \quad f(x) = \exp(x)$$

Then

$$g^{\Delta}(t) = (t+1)^2 - t^2 = 2t + 1 \quad \textit{and} \quad f'(x) = \exp(x).$$

Hence, by Theorem 1.4.4,

$$\begin{aligned}
(f \circ g)^{\Delta} &= \left\{\int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh\right\} g^{\Delta}(t) \\
&= (2t+1)\int_0^1 \exp(t^2 + h(2t+1))dh
\end{aligned}$$

$$
\begin{aligned}
&= (2t+1)\exp(t^2)\int_0^1 \exp(h(2t+1))dh \\
&= (2t+1)\exp(t^2)\frac{1}{2t+1}[\exp(h(2t+1))]_{h=0}^{h=1} \\
&= (2t+1)\exp(t^2)\frac{1}{2t+1}(\exp(2t+1)-1) \\
&= \exp(t^2)(\exp(2t+1)-1).
\end{aligned}
$$

Also, it is easy to check that

$$
\begin{aligned}
\Delta f(g(t)) &= f(g(t+1)) - f(g(t)) \\
&= \exp((t+1)^2) - \exp(t^2) \\
&= \exp(t^2+2t+1) - \exp(t^2) \\
&= \exp(t^2+2t+1) - \exp(t^2) \\
&= \exp(t^2)(\exp(2t+1)-1).
\end{aligned}
$$

Let $\mathbb{T}$ be a time scale and $\nu : \mathbb{T} \to \mathbb{R}$ be an strictly increasing function such that $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale. By $\tilde{\sigma}$ we denote the jump function on $\tilde{\mathbb{T}}$ and by $\tilde{\Delta}$ we denote the derivative on $\tilde{\mathbb{T}}$. Then $\nu \circ \sigma = \tilde{\sigma} \circ \nu$.

Theorem 1.4.6 (Chain rule, see [2, Theorem 1.93]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$
(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}.
$$

Proof. Let $0 < \epsilon < 1$ be given and define

$$
\epsilon^* : = \frac{\epsilon}{[1 + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|]}.
$$

Note that $0 < \epsilon^* < 1$. According to the assumptions, there exist neighborhoods U_1 of t and U_2 of $\nu(t)$ such that

$$
|\nu(\sigma(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)| \le \epsilon^* |\sigma(t) - s| \text{ for all } s \in U_1
$$

and

$$
|w(\sigma(\nu(t))) - w(r) - (\sigma(\nu(t)) - r)w^{\tilde{\Delta}}(\nu(t))| \le \epsilon^* |\sigma(\nu(t)) - r|, \text{ for all } r \in U_2.
$$

Writing $U = U_1 \cap \nu^{-1}(U_2)$ and letting $s \in U$. Then, if $s \in U_1$, $\nu(s) \in U_2$ we obtain

$$
\begin{aligned}
&|w(\nu(\sigma(t))) - w(\nu(s)) - (\sigma(t) - s)[w^{\Delta}(\nu(t))\nu^{\Delta}(t)]| \\
&= |w(\nu(\sigma(t))) - w(\nu(s)) - (\sigma(\nu(t)) - \nu(s))w^{\Delta}(\nu(t)) + [\sigma(\nu(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)]w^{\Delta}(\nu(t))| \\
&\leq \epsilon^*|\sigma(\nu(t)) - \nu(s)| + \epsilon^*|\sigma(t) - s||w^{\Delta}(\nu(t))| \\
&\leq \epsilon^*\{|\sigma(\nu(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)| + |\sigma(t) - s||\nu^{\Delta}(t)| + |\sigma(t) - s||w^{\Delta}(\nu(t))|\} \\
&\leq \epsilon^*\{\varepsilon^*|\sigma(t) - s| + |\sigma(t) - s||\nu^{\Delta}(t)| + |\sigma(t) - s||w^{\Delta}(\nu(t))|\} \\
&= \epsilon^*|\sigma(t) - s| \left\{\epsilon^* + |\nu^{\Delta}(t)| + |w^{\Delta}(\nu(t))|\right\} \\
&\leq \epsilon^*\{1 + |\nu^{\Delta}(t)| + |w^{\Delta}(\nu(t))|\}|\sigma(t) - s| \\
&= \epsilon|\sigma(t) - s|.
\end{aligned}
$$

getting the desired result. □

Example 1.4.7 (See [2, Example 1.94]). *Let $\mathbb{T} = \mathbb{N}_0$ and $\nu(t) = 4t + 1$. Hence*

$$\tilde{\mathbb{T}} = \nu(\mathbb{T}) = \{4n + 1 : n \in \mathbb{N}_0\} = \{1, 5, 9, 13, ...\}.$$

Moreover, let $w : \tilde{\mathbb{T}} \to \mathbb{R}$ be defined by $w(t) = t^2$. Then

$$
\begin{aligned}
(w \circ \nu)^{\Delta}(t) &= [4(t + 1) + 1]^2 - (4t + 1)^2 \\
&= (4t + 5)^2 - (4t + 1)^2 \\
&= 16t^2 + 40t + 25 - 16t^2 - 8t - 1 \\
&= 32t + 24.
\end{aligned}
$$

Applying Theorem 1.4.6 to obtain the Δ-derivative of the composite function:

$$\nu^{\Delta}(t) = (4t + 1)^{\Delta} = 4$$

and then

$$w^{\tilde{\Delta}}(t) = \frac{w(\tilde{\sigma}(t)) - w(t)}{\tilde{\sigma}(t) - t} = \frac{(t + 4)^2 - t^2}{t + 4 - t} = \frac{8t + 16}{4} = 2t + 4$$

therefore,

$$(w^{\tilde{\Delta}} \circ \nu)(t) = w^{\tilde{\Delta}}(\nu(t)) = w^{\tilde{\Delta}}(4t + 1) = 2(4t + 1) + 4 = 8t + 6$$

Thus, we obtain

$$[(w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}](t) = (8t + 6).4 = 32t + 24 = (w \circ \nu)^{\Delta}(t).$$

As consequence of Theorem 1.4.6, we can write the formula for the derivative of the inverse function and the substitution rule for integrals.

Theorem 1.4.8 (Derivative of the Inverse, see [2, Theorem 1.97])**.** *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Then*

$$\frac{1}{\nu^{\Delta}} = (\nu^{-1})^{\tilde{\Delta}} \circ \nu$$

at points where ν^{Δ} *is different from zero.*

Theorem 1.4.9 (Substitution, see [2, Theorem 1.98])**.** *Assume* $\nu\colon \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is an rd-continuous function and* ν *is* Δ*-differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$*,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s.$$

Proof. Since f is rd-continuous and ν is Δ-differentiable with rd-continuous derivative, $f\nu^{\Delta}$ is a rd-continuous function. It has antiderivative F, i.e.,

$$F^{\Delta} = f\nu^{\Delta}$$

and

$$\begin{aligned}\int_a^b f(t)\nu^{\Delta}(t)\Delta t &= \int_a^b F^{\Delta}(t)\Delta t = F(b) - F(a) \\ &= (F \circ \nu^{-1})(\nu(b)) - (F \circ \nu^{-1})(\nu(a)) = \int_{\nu(a)}^{\nu(b)} (F \circ \nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta} s.\end{aligned}$$

By Theorem 1.3.10,

$$\begin{aligned}\int_{\nu(a)}^{\nu(b)} (F^{\Delta} \circ \nu^{-1})(s)(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta} s &= \int_{\nu(a)}^{\nu(b)} \left((f\nu^{\Delta}) \circ \nu^{-1}\right)(s)(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta} s \\ &= \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)[(\nu^{\Delta} \circ \nu^{-1})(\nu^{-1})^{\tilde{\Delta}}](s)\tilde{\Delta} s.\end{aligned}$$

Then, using Theorem 1.3.11,

$$\int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s,$$

we get the desired result. □

Example 1.4.10 (See [2, Example 1.99]). *Let us apply the method of substitution given by Theorem 1.3.12 to evaluate the integral*

$$\int_0^t \left(\sqrt{\tau^2+1}+\tau\right) 3^{\tau^2} \Delta\tau$$

for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}_0\}$. *Define*

$$\nu(t) = t^2$$

for $t \in \mathbb{N}_0^{\frac{1}{2}}$. *Then* $\nu : \mathbb{N}_0^{\frac{1}{2}} \to \mathbb{R}$ *is strictly increasing and* $\nu\left(\mathbb{N}_0^{\frac{1}{2}}\right) = \mathbb{N}_0$ *is a time scale. Also,*

$$\nu^\Delta(t) = \sqrt{t^2+1} + t.$$

Hence if $f(t) := 3^{\tau^2}$, *from Theorem 1.19, then*

$$\int_0^1 \left(\sqrt{\tau^2+1}+\tau\right) 3^{\tau^2} \Delta\tau = \int_0^1 f(\tau)\nu^\Delta(\tau)\Delta\tau$$
$$= \int_0^{t^2} f(\sqrt{s})\tilde{\Delta}s = \int_0^{t^2} 3^s \tilde{\Delta}s = \left[\frac{1}{2}3^s\right]_{s=0}^{s=t^2} = \frac{1}{2}(3^{t^2}-1).$$

1.5 The Regressive Group

In this section, we will define the generalized exponential function for an arbitrary time scale $\mathbb{T}$.

Definition 1.5.1 (See [2, Definition 2.25]). *A function* $p\colon \mathbb{T} \to \mathbb{R}$ *is called* ***regressive*** *provided*

$$1 + \mu(t)p(t) \neq 0, \text{ for all } t \in \mathbb{T}^\kappa$$

We will denote the set of regressive and rd-continuous functions $p\colon \mathbb{T} \to \mathbb{R}$ *by*

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Definition 1.5.2 (See [2, Theorem 2.7] and [2, Definition 2.13]). *The* ***"circle plus" addition*** *and* ***"circle minus" subtraction*** *on* $\mathcal{R}$ *are defined, respectively, by*

$$p \oplus q = p + q + \mu pq \quad \text{and} \quad p \ominus q = \frac{p-q}{1+\mu q}.$$

We can also define the "circle minus" subtraction on $\mathcal{R}$ by

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t) \text{ for all } t \in \mathbb{T}^{\kappa},$$

where $\ominus p$ is defined by

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \text{ for all } t \in \mathbb{T}^{\kappa}.$$

It is not difficult to see that circle minus and circle plus operations can be transformed into the usual subtraction and additivity when $\mathbb{T} = \mathbb{R}$. However, depending on the time scale, we may have different operations, but quite useful ones. The following theorem describes many properties that are consequence of these operations, which will be useful.

Theorem 1.5.3 (See [2, Exercise 2.28]). *Suppose* $p, q \in \mathcal{R}$, *then*
(i) $p \ominus p = 0$;
(ii) $\ominus(\ominus p) = p$;
(iii) $p \ominus q \in \mathcal{R}$;
(iv) $p \ominus q = \frac{p-q}{1+\mu q}$;
(v) $\ominus(p \ominus q) = q \ominus p$;
(vi) $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Proof. (i) Note that

$$\begin{aligned} p \ominus p &= p \oplus (\ominus p) \\ &= p + (\ominus p) + \mu p(\ominus p) \\ &= p + \left(-\frac{p}{1+\mu p}\right) + \mu p \left(-\frac{p}{1+\mu p}\right) \\ &= p + (1+\mu p)\left(\frac{-p}{1+\mu p}\right) \\ &= 0 \end{aligned}$$

(ii) Notice that:

$$\begin{aligned} \ominus(\ominus p) &= -\frac{(\ominus p)}{1+\mu(\ominus p)} \\ &= -\frac{-\frac{p}{1+\mu p}}{1+\mu\left(-\frac{p}{1+\mu p}\right)} \end{aligned}$$

$$= -\left(\frac{-p}{1+\mu p} \cdot \frac{1+\mu p}{1+\mu p-\mu p}\right)$$
$$= p.$$

(iii) We need to show that $1+\mu(p \ominus q) \neq 0$ for all $t \in \mathbb{T}^\kappa$. For $t \in \mathbb{T}^\kappa$, we have

$$\begin{aligned}1+\mu(t)(p \ominus q) &= q+\mu(t)\left(\frac{p-q}{1+\mu(t)q(t)}\right)\\ &= \frac{1+\mu(t)q(t)+\mu(t)p(t)-\mu(t)q(t)}{1+\mu(t)q(t)}\\ &= \frac{1+\mu(t)p(t)}{1+\mu(t)q(t)}\\ &\neq 0.\end{aligned}$$

We omit the proofs of (vi), (v) and (vi), since they follow as immediate consequence of the properties of circle minus and circle plus. □

As a consequence, we obtain the following property which ensures that we can work with this operation in the set of regressive functions.

Theorem 1.5.4 (See [2, Theorem 2.7]). *If we define the "circle plus" addition $\oplus$ on $\mathcal{R}$ by*

$$p(t) \oplus q(t) := p(t)+q(t)+\mu(t)p(t)q(t),$$

*with $p, q \in \mathcal{R}$, then $(\mathcal{R}, \oplus)$ is an abelian group called the **regressive group**.*

Now we define the generalized exponential function $e_p(t,s)$ on time scale.

Definition 1.5.5 (See [2, Definition 2.21]). *For $h>0$, let $\xi_h \colon \mathbb{C}_h \to \mathbb{Z}_h$ be given by*

$$\xi_h(z) = \frac{1}{h} Log(1+zh),$$

and for $h=0$, we define $\xi_0(z)=z$, for all $z \in \mathbb{C}$. Then if $p \in \mathcal{R}$, then we define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad for \quad s,t \in \mathbb{T}$$

Theorem 1.5.6 (See [2, Theorem 2.36]). *If $p, q \in \mathcal{R}$, then*
(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;
(viii) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Definition 1.5.7 (See [2, Definition 2.32]). *If $p \in \mathcal{R}$, then the first order linear dynamic equation on time scale*

$$y^{\Delta} = p(t)y$$

*is called **regressive**.*

The next result brings an important property of the exponential function, which relates it to theory of dynamic equations on time scales, and this is also expected.

Theorem 1.5.8 (See [2, Theorem 2.33]). *Suppose $y^{\Delta} = p(t)y$ is regressive and fix $t_0 \in \mathbb{T}$. Then $e_p(\cdot,t_0)$ is a solution of the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1$$

on $\mathbb{T}$.

Now, we are ready to state and prove the uniqueness of solution of the equation:

$$\begin{cases} y^{\Delta} = p(t)y \\ y(t_0) = 1. \end{cases} \tag{1.3}$$

Theorem 1.5.9 (See [2, Theorem 2.35]). *If $y^{\Delta} = p(t)y$ is regressive, the only solution of* (1.3) *is given by $e_p(\cdot,t_0)$.*

Proof. Assuming y is a solution and consider the quotient $\frac{y}{e_p(\cdot,t_0)}$. Using Theorem 1.5.9,

$$\begin{aligned}\left(\frac{y}{e_p(\cdot,t_0)}\right)^{\Delta}(t) &= \frac{y^{\Delta}(t)e_p(t,t_0) - y(t)e_p^{\Delta}(t,t_0)}{e_p(t,t_0)e_p(\sigma(t),t_0)} \\ &= \frac{p(t)y(t)e_p(t,t_0) - y(t)p(t)e_p(t,t_0)}{e_p(t,t_0)e_p(\sigma(t),t_0)} \\ &= 0.\end{aligned}$$

Thus, $\frac{y(t)}{e_p(t,t_0)}$ is constant and

$$\frac{y(t)}{e_p(t,t_0)} = \frac{y(t_0)}{e_p(t_0,t_0)} = 1.$$

This implies that $y(t) = e_p(t,t_0)$, for every $t \in \mathbb{T}$, getting the desired result. □

To conclude this section, we study the first order nonhomogeneous linear equation.

Definition 1.5.10 (See [2, p. 75]). *Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function, we call the* ***first order nonhomogeneous linear equation*** *by*

$$y^{\Delta} = p(t)y + f(t)$$

on a time scale $\mathbb{T}$.

Theorem 1.5.11 (Variation of Constants Formula, see [2, Theorem 2.74]). *Suppose $y^{\Delta} = p(t)y + f(t)$ is regressive. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^{\Delta} = -p(t)x^{\sigma} + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t,\tau)f(\tau)\Delta\tau.$$

Proof. Let us consider that

$$x^{\Delta} = -p(t)x^{\sigma}(t) + f(t), \quad x(t_0) = x_0$$

has a solution $x(t)$. Hence,

$$\begin{aligned} e_p(t,t_0)f(t) &= e_p(t,t_0)x^{\Delta} + e_p(t,t_0)p(t)x^{\sigma}(t) \\ &= [e_p(\cdot,t_0)x]^{\Delta}(t). \end{aligned}$$

Integrating from t_0 to t and applying Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_{t_0}^{t} [e_p(\tau,t_0)f(\tau)]\Delta\tau &= \int_{t_0}^{t} [e_p(\tau,t_0)x(\tau)]^{\Delta}\Delta\tau \\ &= e_p(t,t_0)x(t) - e_p(t_0,t_0)x(t_0) \\ &= e_p(t,t_0)x(t) - x_0. \end{aligned}$$

Thus, we write

$$x(t) = \frac{x_0}{e_p(t,t_0)} + \frac{\int_{t_0}^{t}[e_p(\tau,t_0)f(\tau)]\Delta\tau}{e_p(t,t_0)}$$
$$= e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t,\tau)f(\tau)\Delta\tau,$$

proving the result. □

Now, we finish this section with a more general result, which in the first order dynamic equation on time scale does not appear the composition y and σ in the right hand side.

Theorem 1.5.12 (Variation of Constants, see [2, Theorem 2.77]). *Suppose $y^{\Delta} = p(t)y + f(t)$ is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial problem*

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau. \tag{1.4}$$

Proof. Since $y^{\sigma}(t) = y(t) + \mu(t)y^{\Delta}(t)$, it follows

$$y^{\Delta}(t) = p(t)\left[y^{\sigma}(t) - \mu(t)y^{\Delta}(t)\right] + f(t).$$

It implies

$$(1 + p(t)\mu(t))y^{\Delta}(t) = p(t)y^{\sigma}(t) + f(t).$$

Hence,

$$y^{\Delta}(t) = \frac{p(t)}{1 + p(t)\mu(t)}y^{\sigma}(t) + \frac{f(t)}{1 + p(t)\mu(t)}.$$

Hence,

$$y(t) = - \ominus p y^{\sigma}(t) + \frac{f(t)}{1 + \mu(t)p(t)}.$$

Applying Theorem 1.5.11, and using $\ominus(\ominus p)(t) = p(t)$, we have

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^{t} e_p(t,\tau)\frac{f(\tau)}{1 + \mu(\tau)p(\tau)}\Delta\tau$$
$$= e_p(t,t_0)y_0 + \int_{t_0}^{t} \frac{f(\tau)}{e_p(\tau,t)[1 + \mu(\tau)p(\tau)]}\Delta\tau$$

$$= e_p(t, t_0)y_0 + \int_{t_0}^{t} \frac{f(\tau)}{e_p(\sigma(\tau), t)} \Delta\tau$$
$$= e_p(t, t_0)y_0 + \int_{t_0}^{t} f(\tau)e_p(t, \sigma(\tau))\Delta\tau,$$

obtaining the expression (1.4). □

1.6 Regressive Matrices

In this section, we will define matrices for an arbitrary time scale $\mathbb{T}$, and give some important properties which will be very useful for our purposes. We omit the proofs here, since they will follow very similarly from the results presented previously in this chapter.

Definition 1.6.1 (See [2, Definition 5.1]). *Let A be an $m \times n$-matrix-valued function on $\mathbb{T}$. We say that A is* ***rd-continuous on*** $\mathbb{T}$ *if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $m \times n$–matrix-valued functions on $\mathbb{T}$ is denoted, similar to the scalar case, by*

$$\mathbb{C}_{rd} = \mathbb{C}_{rd}(\mathbb{T}) = \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R}^{m\times n}).$$

We say that A is Δ-differentiable on $\mathbb{T}$ provided each entry of A is Δ-differentiable on $\mathbb{T}$, and in this case, we put

$$A^{\Delta} = (a_{ij}^{\Delta})_{1\le i\le m, 1\le j\le n}, \quad \textit{where} \quad A = (a_{ij})_{1\le i\le m, 1\le j\le n}.$$

Theorem 1.6.2 (See [2, Theorem 5.2]). *If A is Δ-differentiable at $t \in \mathbb{T}^{\kappa}$, then $A^{\sigma}(t) = A(t) + \mu(t)A^{\Delta}(t)$.*

Theorem 1.6.3 (See [2, Theorem 5.3]). *Suppose A and B are Δ-differentiable $n \times n$-matrix-valued functions. Then*

(i) $(A + B)^{\Delta} = A^{\Delta} + B^{\Delta}$;

(ii) $(\alpha A)^{\Delta} = \alpha A^{\Delta}$ if α is constant;

(iii) $(AB)^{\Delta} = A^{\Delta}B^{\sigma} + AB^{\Delta} = A^{\sigma}B^{\Delta} + A^{\Delta}B$

(iv) $(A^{-1})^{\Delta} = -(A^{\sigma})^{-1}A^{\Delta}A^{-1} = -A^{-1}A^{\Delta}(A^{\sigma})^{-1}$ if AA^{σ} is invertible;

(v) $(AB^{-1})^{\Delta} = (A^{\Delta} - AB^{-1}B^{\Delta})(B^{\sigma})^{-1} = (A^{\Delta} - (AB^{-1})^{\sigma}B^{\Delta})B^{-1}$ if BB^{σ} is invertible.

We consider the linear system of dynamic equations on time scale

$$y^{\Delta} = A(t)y, \tag{1.5}$$

where A is an $n \times n$-matrix-valued function on $\mathbb{T}$. A vector-valued $y\colon \mathbb{T} \to \mathbb{R}^n$ is said to be a solution of (1.5) on $\mathbb{T}$ provided $y^{\Delta}(t) = A(t)y(t)$ holds for each $t \in \mathbb{T}^{\kappa}$. To state the main theorem on solvability of initial value problems involving equation (1.5), we introduce the following definition, which is very similar to the scalar case.

Definition 1.6.4 (Regressivity, see [2, Definition 5.5]). *An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called **regressive** provided*

$$I + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T}^{\kappa},$$

and the class of all such regressive and rd-continuous matrices functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathcal{R}^{n\times n}).$$

*We say that the system (1.5) is **regressive** provided $A \in \mathcal{R}$.*

The existence and uniqueness theorem for initial value problems of equation (1.5) reads as follows.

Theorem 1.6.5 (Existence and Uniqueness Theorem, see [2, Theorem 5.8]). *Let $A \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta} = A(t)y + f(t), \quad y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$.

The same way as before, we can define the operations in the context of regressive functions the "circle plus" addition $\oplus$ and the "circle minus" subtraction $\ominus$, as follows.

Definition 1.6.6 (See [2, Definition 5.10] and [2, Definition 5.14]). *Assume A and B are regressive $n \times n$-matrix-valued functions on $\mathbb{T}$. Then we define $A \oplus B$ by*

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa},$$

we define $\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}.$$

and we define $A \ominus B$ by

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t) \text{ for all } t \in \mathbb{T}^{\kappa}.$$

Definition 1.6.7 (Matrix Exponential Function, see [2, Definition 5.18]). *Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$-matrix-valued function. The unique matrix-valued solution of the IVP*

$$Y^{\Delta} = A(t)Y, \quad Y(t_0) = I,$$

where I denotes as usual the $n \times n$-identity matrix, is called the ***matrix exponential function*** *(at t_0), and it is denoted by $e_A(\cdot, t_0)$.*

Example 1.6.8 (See [2, Example 5.19]). *Assume A is a constant $n \times n$-matrix. If $\mathbb{T} = \mathbb{R}$, then*

$$e_A(t, t_0) = e^{A(t-t_0)},$$

while if $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then

$$e_A(t, t_0) = (I + A)^{(t-t_0)}.$$

Theorem 1.6.9 (See [2, Theorem 5.21]). *If $A, B \in \mathcal{R}$ are matrix-valued functions on $\mathbb{T}$, then*

(i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$;
(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;
(iii) $e_A^{-1}(t, s) = e_{\ominus A^*}(t, s)$;
(iv) $e_A(t, s) = e_A^{-1}(s, t) = e_{\ominus A^*}(s, t)$;
(v) $e_A(t, s)e_A(s, r) = e_A(t, r)$;
(vi) $e_A(t, s)e_B(t, s) = e_{A \oplus B}(t, s)$ if $e_A(t, s)$ and $B(t)$ commute.

Theorem 1.6.10 (Variation of Constants Formula, see [2, Definition 5.24]). *Let $A \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta} = A(t)y + f(t), \; y(t_0) = y_0 \tag{1.6}$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}$. Moreover, this solution is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau. \tag{1.7}$$

As in the scalar case, we consider the analogue for its adjoint equation

$$x^{\Delta} = -A^*(t)x^{\sigma}. \tag{1.8}$$

Theorem 1.6.11 (Variation of Constants Formula, see [2, Theorem 5.27]). *Let $A \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let*

$t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the initial value problem

$$x^{\Delta} = -A^*(t)x^{\sigma} + f(t),\ x(t_0) = x_0 \tag{1.9}$$

has a unique solution $x : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by

$$x(t) = e_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus A^*}(t, \tau)f(\tau)\Delta\tau. \tag{1.10}$$

To finish this section, we present Liouville's formula in the context of time scale.

Theorem 1.6.12 (Liouville's Formula, see [2, Theorem 5.28])**.** *Let A be a 2×2 regressive matrix-valued function on $\mathbb{T}$. Assume X is a matrix-valued solution of*

$$X^{\Delta} = A(t)X,\ t \in \mathbb{T}.$$

Then X satisfies

$$detX(t) = e_{trA+\mu detA}(t, t_0)detX(t_0),\ \ t \in \mathbb{T}, \tag{1.11}$$

where trA and $detA$ denote the trace and the determinant of A, respectively.

Chapter 2

Periodicity on Isolated Time Scales

In this chapter, our goal is to introduce some concepts about iterated shifts and the definition of ω-periodic functions on isolated time scales. The main references here are [1], [8]. In this entire chapter, $\mathbb{T}$ will denote an isolated time scale, that is, all points in $\mathbb{T}$ are right-scattered and left-scattered, except when the time scale has a minimum or maximum (or both). In this case, the minimum point must be right-scattered and the maximum point must be left-scattered. .

2.1 Iterated Shifts

Definition 2.1.1 (See [8, p. 262]). *Let $\omega \in \mathbb{N}$, we define the iterated shift $\nu : \mathbb{T} \to \mathbb{T}$ by*

$$\nu := \sigma^{\omega} := \underbrace{\sigma \circ \cdots \circ \sigma}_{\omega\text{-times}}.$$

Definition 2.1.2 (See [8, p. 262]). *Let $f : \mathbb{T} \to \mathbb{R}$, we use the following notation to simplify*

$$f^{\nu} = f \circ \nu.$$

This implies that

$$f^{\nu\sigma} := (f \circ \nu)^{\sigma} = f \circ \nu \circ \sigma = (f \circ \sigma)^{\nu} =: f^{\sigma\nu},$$

which gives us that σ and ν commute, i.e.,

$$\sigma \circ \nu = \nu \circ \sigma, \text{ i.e., } \sigma^{\nu} = \nu^{\sigma}.$$

Lemma 2.1.3 (See [8, Lemma 3.1]). *We have*

$$\nu^{\Delta} = \frac{\mu^{\nu}}{\mu}. \tag{2.1}$$

Proof. Let $t \in \mathbb{T}$. Since $\mathbb{T}$ is an isolated time scale, we get

$$\nu^{\Delta}(t) = \frac{\nu(\sigma(t)) - \nu(t)}{\mu(t)} = \frac{\mu(\nu(t))}{\mu(t)} = \frac{\mu^{\nu}(t)}{\mu(t)},$$

getting the desired result. □

Lemma 2.1.4 (See [8, Lemma 3.2]). *For $f : \mathbb{T} \to \mathbb{R}$, we have*

$$f^{\nu\Delta} = \nu^{\Delta} f^{\Delta\nu}. \tag{2.2}$$

Proof. Using Lemma 2.1.3, we have

$$f^{\nu\Delta} = \frac{f^{\nu}(\sigma(t)) - f^{\nu}}{\mu} = \frac{\mu^{\nu}}{\mu^{\nu}} \cdot \frac{f^{\sigma\nu} - f^{\nu}}{\mu} = \nu^{\Delta} \cdot \frac{f^{\sigma\nu} - f^{\nu}}{\mu^{\nu}} = \left(\frac{f^{\sigma} - f}{\mu}\right)^{\nu} = \nu^{\Delta} f^{\Delta\nu},$$

which gives us the desire result. □

Lemma 2.1.5 (See [8, Lemma 3.3]). *We have*

$$\nu^{\Delta\Delta} = \nu^{\Delta} \frac{\sigma^{\Delta\nu} - \sigma^{\Delta}}{\mu^{\sigma}} \tag{2.3}$$

Proof. By Lemma 2.1.3,

$$\nu^{\Delta\Delta} = \left(\nu^{\Delta}\right)^{\Delta} = \left(\frac{\mu^{\nu}}{\mu}\right)^{\Delta}.$$

By the quotient rule, we have

$$\left(\frac{\mu^{\nu}}{\mu}\right)^{\Delta} = \frac{\mu^{\nu\Delta}\mu - \mu^{\nu}\mu^{\Delta}}{\mu\mu^{\sigma}}.$$

Using Lemmas 2.1.3 and 2.1.4, we get $\mu^{\nu} = \nu^{\Delta}\mu$. With that

$$\begin{aligned} \left(\frac{\mu^{\nu}}{\mu}\right)^{\Delta} &= \frac{\nu^{\Delta}\mu^{\Delta\nu}\mu - \nu^{\Delta}\mu\mu^{\Delta}}{\mu\mu^{\sigma}} \\ &= \frac{\nu^{\Delta}\left(\mu\mu^{\Delta\nu} - \mu\mu^{\Delta}\right)}{\mu\mu^{\sigma}} \\ &= \frac{\nu^{\Delta}\mu\left(\mu^{\Delta\nu} - \mu^{\Delta}\right)}{\mu\mu^{\sigma}} \end{aligned}$$

$$= \frac{\nu^\Delta \left(\mu^{\Delta\nu} - \mu^\Delta\right)}{\mu^\sigma}.$$

Since $\sigma(t) - t = \mu(t)$, $\mu^\Delta(t) = \sigma^\Delta(t) - t^\Delta = \sigma^\Delta - 1$. Applying it to the equation:

$$\begin{aligned}\frac{\nu^\Delta \left(\sigma^{\Delta\nu} - 1^\nu\right) - \left(\sigma^\Delta - 1\right)}{\mu^\sigma} &= \frac{\nu^\Delta \left(\sigma^{\Delta\nu} - \sigma^\Delta - 1 + 1\right)}{\mu^\sigma} \\ &= \nu^\Delta \frac{\sigma^{\Delta\nu} - \sigma^\Delta}{\mu^\sigma},\end{aligned}$$

concluding the proof. □

Remark 2.1.6 (See [8, Remark 3.4]). *Using the "simple useful formula",*

$$\mu^\sigma = \mu + \mu\mu^\Delta = \mu(1 + \mu^\Delta) = \mu(1 + \sigma^\Delta - 1) = \mu\sigma^\Delta.$$

This implies that

$$\nu^{\Delta\Delta} = \nu^\Delta \frac{\sigma^{\Delta\nu} - \sigma^\Delta}{\mu\sigma^\Delta}.$$

The examples below will show some properties of the iterated shifts for the cases $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}_0}$.

Example 2.1.7 (See [8, Example 3.5]). *If $\mathbb{T} = \mathbb{R}$, then $\nu(t) = t$ and $\nu^\Delta(t) = 1$.*

Example 2.1.8 (See [8, Example 3.6]). *If $\mathbb{T} = \mathbb{Z}$, then $\nu(t) = t + \omega$ and $\nu^\Delta(t) = 1$, for all $t \in \mathbb{T}$.*

Example 2.1.9 (See [8, Example 3.7]). *If $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$, then $\nu(t) = q^\omega t$ and $\nu^\Delta(t) = q^\omega$, for every $t \in \mathbb{T}$.*

Lemma 2.1.10 (See [8, Lemma 3.8]). *For $f \colon \mathbb{T} \to \mathbb{R}$, define*

$$F_\nu(t) := \int_t^{\nu(t)} f(\tau)\Delta\tau. \tag{2.4}$$

Then

$$F_\nu^\Delta = \nu^\Delta f^\nu - f. \tag{2.5}$$

Proof. Let $t_0 \in \mathbb{T}$ and define $F(t) := \int_{t_0}^t f(\tau)\Delta\tau$. We obtain $F^\Delta = f$. Hence,

$$F_\nu(t) + F(t) = \int_t^{\nu(t)} f(\tau)\Delta\tau + \int_{t_0}^t f(\tau)\Delta\tau = \int_{t_0}^{\nu(t)} f(\tau)\Delta\tau = F^\nu(t)$$

Thus, $F^\nu - F = F_\nu$. Using Lemma 2.1.4, we get

$$F_\nu^\Delta = (F^\nu)^\Delta(t) - F^\Delta(t) = \nu^\Delta(t)(F^\Delta)^\nu(t) - F^\Delta(t).$$

Since $F^\Delta(t) = f(t)$, we get

$$\nu^\Delta(t)(F^\Delta)^\nu(t) - F^\Delta(t) = \nu^\Delta(t)f^\nu(t) - f(t),$$

finishing the proof. □

Lemma 2.1.11 (See [8, Lemma 3.9]). *Let $t_0 \in \mathbb{T}$. For $f \in \mathcal{R}$, we have*

$$h(t) := e_f(\nu(t), t) \tag{2.6}$$

then

$$h^\Delta(t) = ((\nu^\Delta f^\nu) \ominus f)h(t)$$

and

$$e_f(\nu(t), t) = e_f(\nu(t_0), t_0)\frac{e_{\nu^\Delta f^\nu}(t, t_0)}{e_f(t, t_0)}, \quad \forall t \in \mathbb{T}. \tag{2.7}$$

Proof. Let

$$h(t) := e_f(\nu(t), t).$$

By the semigroup property,

$$\begin{aligned} h(t) &= e_f(\nu(t), t_0)e_f(t_0, t) \\ &= e_f(\nu(t), t_0)e_{\ominus f}(t, t_0). \end{aligned}$$

Using the product and chain rules, we get

$$\begin{aligned} h^\Delta(t) &= \left[(\nu(t), t_0)^\Delta e_f(\nu(t), t_0)\right]\left[(1 + \mu(t))\left(\ominus f\right)(t)\right] e_{\ominus f}(t, t_0) + e_f(\nu(t), t_0)\left[(\ominus f)(t)e_{\ominus f}(t, t_0)\right] \\ &= \nu^\Delta(t)f(\nu(t))e_f(\nu(t), t_0)\left(1 + \mu(t)(\ominus f)(t)\right) e_{\ominus f}(t, t_0) + e_f(\nu(t), t_0)(\ominus f)(t)e_{\ominus f}(t, t_0) \\ &= e_f(\nu(t), t_0)(e_{\ominus f}(t, t_0)\left[\nu^\Delta(t)f(\nu(t))\left(1 + \mu(t)(\ominus f)(t)\right)\right] + e_f(\nu(t), t_0)e_{\ominus f}(t, t_0)(\ominus f)(t) \\ &= h(t)\left(\nu^\Delta(t)f(\nu(t))(1 + \mu(t)(\ominus f)(t)\right) + h(t)(\ominus f)(t) \\ &= h(t)\left[\nu^\Delta(t)f(\nu(t))(1 + \mu(t)(\ominus f)(t)) + (\ominus f)(t)\right] \\ &= h(t)\frac{\nu^\Delta(t)f(\nu(t)) - f(t)}{1 + \mu(t)f(t)} \\ &= h(t)\left(\nu^\Delta f(\nu(t)) \ominus f\right)(t) \\ &= h(t)(\nu^\Delta f^\nu \ominus f)(t), \end{aligned}$$

proving (2.6). To see (2.7), we get from (2.6) that

$$h(t) = e_{\nu^\Delta f^\nu \ominus f}(t, t_0)h(t_0) = e_{\nu^\Delta f^\nu \ominus f}(t, t_0) \cdot e_f(\nu(t_0), t_0).$$

Hence,

$$e_f(\nu(t), t) = e_f(\nu(t_0), t_0) \cdot \frac{e_{\nu^\Delta f}(t, t_0)}{e_f(t, t_0)},$$

proving (2.7). □

Lemma 2.1.12 (See [8, Lemma 3.10]). *For $f \in \mathcal{R}$, we have*

$$e_f(\nu(t), \nu(s)) = e_{\nu^\Delta f^\nu}(t, s), \quad \forall s, t \in \mathbb{T}. \tag{2.8}$$

Proof. By Lemma 2.1.11 and using the semigroup property,

$$\begin{aligned} e_f(\nu(t), \nu(s)) &= e_f(\nu(t), t)e_f(t, s)e_f(s, \nu(t)) \\ &= e_f(\nu(t), t)e_f(t, s)e_{\ominus f}(\nu(s), s) \\ &= \frac{e_f(\nu(t), t)}{e_f(\nu(s), s)} e_f(t, s) \\ &= \frac{e_{\nu^\Delta f^\nu}(t, s)}{e_f(t, s)} e_f(t, s) \\ &= e_{\nu^\Delta f^\nu}(t, s), \end{aligned}$$

fulfilling (2.8). □

2.2 Periodicity

We start this section by recalling the concept of ω-**periodic** functions on isolated time scales. This definition will be essential to our purposes.

Definition 2.2.1 (See [8, Definition 4.1]). *A function $p\colon \mathbb{T} \to \mathbb{R}$ is called ω-periodic provided*

$$\nu^\Delta(t)p^\nu(t) = p(t) \tag{2.9}$$

The set of all ω-periodic functions $p\colon \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{P}_\omega = \mathcal{P} = \mathcal{P}(\mathbb{T}, \mathbb{R})$.

Remark 2.2.2 (See [8, Remark 4.2]). *Since (2.1) holds, $p \in \mathcal{P}$ if, and only if,*

$$(\mu p)^\nu = \mu p. \tag{2.10}$$

Below, we show some examples which ensure that this definition is consistent with the analogue known ones.

Example 2.2.3 (See [8, Example 4.3]). *If $\mathbb{T} = \mathbb{Z}$, then $p \in \mathcal{P}$ provided*

$$p(t) = \nu^{\Delta}(t)p(\nu(t)) = 1 \cdot p(t+\omega) = p(t+\omega), \text{ for all } t \in \mathbb{T},$$

which is the usual definition of ω-periodicity.

Example 2.2.4 (See [8, Example 4.5]). *If $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$, then $p \in \mathcal{P}$ provided*

$$p(t) = \nu^{\Delta}(t)p(\nu(t)) = q^{\omega} \cdot p(q^{\omega}t) \text{ for all } t \in \mathbb{T},$$

which is the periodicity condition from quantum calculus.

Lemma 2.2.5 (See [8, Lemma 4.6]). *We have $\mathcal{P}_{\omega} \subset \mathcal{P}_{2\omega}$.*

Proof. Define $\tilde{\nu} \colon \mathbb{T} \to \mathbb{T}$ by

$$\tilde{\nu}(t) = \sigma^{2\omega} = \nu(\nu(t))$$

Assuming $p \colon \mathbb{T} \to \mathbb{R}$ is ω-periodic and using the Chain rule (2.2) applied to ν, we get

$$\begin{aligned}
\tilde{\nu}^{\Delta}(t)p(\tilde{\nu}(t)) &= \nu^{\Delta}(\nu(t))p(\nu(\nu(t))) \\
&= \nu^{\Delta}(t)\nu^{\Delta}(\nu(t))p(\nu(\nu(t))) \\
&= \nu^{\Delta}(t)[\nu^{\Delta}(\nu(t))p^{\nu}(\nu(t))] \\
&\overset{(2.9)}{=} \nu^{\Delta}(t)p(\nu(t)) \\
&\overset{(2.9)}{=} p(t),
\end{aligned}$$

which shows that p is also 2ω-periodic. □

Some important properties of periodicity are obtained next.

Theorem 2.2.6 (See [8, Theorem 4.7]). *If $p \in \mathcal{P}$, then the integral*

$$\int_t^{\nu(t)} p(\tau)\Delta\tau \text{ is independent of } t \in \mathbb{T}.$$

Proof. Define $F_{\nu}(t) := \int_t^{\nu(t)} p(\tau)\Delta\tau$. By (2.4) and (2.5), and since p is ω-periodic, we get

$$F_{\nu}^{\Delta}(t) = \nu^{\Delta}(t)p(\nu(t)) - p(t) = p(t) - p(t) = 0.$$

Hence, F_{ν} is constant, proving the desired result. □

Theorem 2.2.7 (See [8, Theorem 4.8]). *If $p \in \mathcal{P}$, then*

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_{s}^{t} p(\tau)\Delta\tau \text{ for all } s,t \in \mathbb{T}. \tag{2.11}$$

Proof. Let $p \in \mathcal{P}$. Hence,

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_{\nu(s)}^{s} p(\tau)\Delta\tau + \int_{s}^{t} p(\tau)\Delta\tau + \int_{t}^{\nu(t)} p(\tau)\Delta\tau.$$

By Theorem 2.2.6,

$$\int_{t}^{\nu(t)} p(\tau)\Delta\tau = C \text{ and } \int_{\nu(s)}^{s} p(\tau)\Delta\tau = -\int_{s}^{\nu(s)} p(\tau)\Delta\tau = -C.$$

Then,

$$\int_{\nu(s)}^{\nu(t)} p(\tau)\Delta\tau = \int_{s}^{t} p(\tau)\Delta\tau,$$

getting the property (2.11). □

To finish, we obtain that the exponential function is ω-periodic in both variables. This is a quite interesting property.

Theorem 2.2.8 (See [8, Theorem 4.9]). *If $p \in \mathcal{P} \cap \mathcal{R}$, then*

$$e_p(\nu(t),t) \text{ is independent of } t \in \mathbb{T} \tag{2.12}$$

and

$$e_p(\nu(t),\nu(s)) = e_p(t,s) \text{ for all } s,t, \in \mathbb{T}. \tag{2.13}$$

Proof. To prove (2.12), using Lemma 2.1.11, motice that defining

$$h(t) := e_p(\nu(t),t)$$

implies

$$h^{\Delta}(t) = ((\nu^{\Delta}p^{\nu}) \ominus p)h(t).$$

If $p \in \mathcal{P}$,

$$h^{\Delta}(t) = ((\nu^{\Delta}p^{\nu}) \ominus p)h(t) = (p \ominus p)h(t) = 0 \cdot h(t) = 0.$$

Hence $h(t)$ is constant, getting (2.13), using Lemma 2.1.12, and the fact that $p \in \mathcal{R}$, we get

$$e_p(\nu(t),\nu(s)) = e_{\nu^{\Delta}p^{\nu}}(t,s) = e_p(t,s)$$

for all $s, t \in \mathbb{T}$. □

2.3 Examples

In this section, our goal is to present examples of periodic functions. A quite interesting property that comes from directly to the definition of periodic functions in this way is the possibility of a characterization of 1-periodic function on an arbitrary isolated time scale. See the result below.

Theorem 2.3.1 (See [8, Theorem 5.1]). *Let $f\colon \mathbb{T} \to \mathbb{R}$. Then f is 1-periodic if, and only if, there exists $c \in \mathbb{R}$ such that*

$$f(t) = \frac{c}{\mu(t)} \text{ for all } t \in \mathbb{T}. \tag{2.14}$$

Proof. Suppose there exists $c \in \mathbb{R}$ such that $f\colon \mathbb{T} \to \mathbb{R}$ is given by (2.14). Then

$$\sigma^{\Delta}(t)f(\sigma(t)) = \frac{\mu(\sigma(t))}{\mu(t)} \cdot \frac{c}{\mu(\sigma(t))} = \frac{c}{\mu(t)} = f(t).$$

Now, suppose f is 1-periodic, by Definition 2.2.1,

$$\sigma^{\Delta}(t)f(\sigma(t)) = f(t).$$

By Remark 2.2.2,

$$(\mu f)^{\sigma} = \mu f.$$

Hence, $(\mu f)(t)$ is independent of $t \in \mathbb{T}$, and equal to a constant c, which implies

$$f(t) = \frac{c}{\mu(t)},$$

for all $t \in \mathbb{T}$. □

Remark 2.3.2 (See [8, Remark 5.2]). *As a consequence of Theorem 2.3.1, any 1-periodic function $f\colon \mathbb{T} \to \mathbb{R}$ for a given isolated time scale $\mathbb{T}$ can be described uniquely by the area between two consecutive time points, since*

$$\int_{t}^{\sigma(t)} f(\tau)\Delta\tau = f(t)\mu(t) = \frac{c}{\mu(t)}\mu(t) = c.$$

In the sequence, we present some examples of ω-periodic functions on isolated time scales.

Example 2.3.3 (See [8, Example 5.3]). *Consider any time scale*

$$\mathbb{T} = \{t_i : i \in \mathbb{Z}\} \text{ with } \sigma(t_i) = t_{i+1} > t_i \text{ for all } t \in \mathbb{Z}.$$

Define $f \colon \mathbb{T} \to \mathbb{R}$ *by*

$$f(t_i) = \frac{(-1)^i}{\mu(t_i)} \text{ for all } i \in \mathbb{Z}.$$

Then

$$\begin{aligned}(\sigma^2)^\Delta(t_i) f(\sigma^2(t_i)) &\overset{(3.3)}{=} \frac{\mu(\sigma^2(t_i))}{\mu(t_i)} f(\sigma^2(t_i)) \\ &= \frac{\mu(t_{i+2})}{\mu(t_i)} f(t_{i+2}) \\ &= \frac{\mu(t_{i+2})}{\mu(t_i)} \cdot \frac{(-1)^{i+2}}{\mu(t_{i+2})} \\ &= \frac{(-1)^i}{\mu(t_i)} = f(t_i)\end{aligned}$$

Hence, f *is* 2*-periodic on* $\mathbb{T}$.

Example 2.3.4. *Let* $t_0 \in \mathbb{T}$, *with* $\mathbb{T} = \bigcup_{n \in \mathbb{Z}} \nu_n(t_0)$. *Define* $f : \mathbb{T} \to \mathbb{R}$ *by*

$$f(t) = \begin{cases} \dfrac{1}{\mu(t)} & \text{if } n \text{ is odd} \\[2ex] \dfrac{2}{\mu(t)} & \text{if } n \text{ is even.} \end{cases}$$

Then, clearly, we get

$$\nu^\Delta(t) f(\nu(t)) = f(t),$$

Hence, by Definition 2.2.1, f *is* ω*-periodic.*

To finish, we present a result such that enables us with given periodic functions to construct more examples of ω-periodic functions.

Theorem 2.3.5 (See [8, Theorem 5.6]). *Assume* $p, q \in \mathcal{P}$ *and* $\alpha, \beta \in \mathbb{R}$. *Then*

$$\alpha p + \beta q \in \mathcal{P} \text{ and } \mu p q \in \mathcal{P}.$$

Moreover, if $\alpha + \mu(t) q(t) \neq 0$ *for all* $t \in \mathbb{T}$, *then*

$$\frac{p}{\alpha + \mu p} \in \mathcal{P}.$$

Proof. Suppose $p, q \in \mathcal{P}_\omega$, and $\alpha, \beta \in \mathbb{R}$, then

$$[\mu(\alpha p + \beta q)]^\nu = \mu^\nu(\alpha p + \beta q)^\nu = \alpha\mu^\nu p^\nu + \beta\mu^\nu q^\nu = \alpha\mu p + \beta\mu q = \mu(\alpha p + \beta q)$$

and

$$[\mu(\mu pq)]^\nu = \mu^\nu(\mu pq)^\nu = (\mu p)^\nu(\mu q)^\nu = (\mu p)(\mu q) = \mu(\mu pq),$$

that is, $\mu(\alpha p + \beta q) \in \mathcal{P}_\omega$ and $\mu(\mu pq) \in \mathcal{P}_\omega$, by Remark 2.2.2. If $\alpha + \mu(t)q(t) \neq 0$ for all $t \in \mathbb{T}$, we get

$$\left[\mu\frac{p}{\alpha + \mu q}\right]^\nu = \mu^\nu\left(\frac{p}{\alpha + \mu q}\right)^\nu = \mu^\nu\frac{p^\nu}{\alpha + (\mu q)^\nu} = \frac{(\mu p)^\nu}{\alpha + (\mu q)^\nu} = \frac{\mu p}{\alpha + \mu q} = \mu\frac{p}{\alpha + \mu q},$$

implying that $\dfrac{p}{\alpha + \mu q} \in \mathcal{P}$. □

Remark 2.3.6 (See [8, Remark 5.7]). *Theorem 2.3.5 together with Theorem 2.3.1 shows that $p(t) \neq 0$ for all $t \in \mathbb{T}$ implies that*

$$p \in \mathcal{P} \text{ if, and only if, } \frac{1}{\mu^2 p} \in \mathcal{P}.$$

Hence, if $p \in \mathcal{P}$ and $p \neq 0$,

$$\left(\mu\frac{1}{\mu^2 p}\right)^\nu = \mu^\nu\frac{1}{(\mu^\nu)^2 p^\nu} = \frac{1}{\mu^\nu p^\nu} = \frac{1}{(\mu p)^\nu} = \frac{1}{\mu p} = \mu\frac{1}{\mu^2 p} \in \mathcal{P}.$$

Reciprocally, if $\dfrac{1}{\mu^2 p}$ is 1-periodic, then

$$\frac{1}{\mu p} = \mu\frac{1}{\mu^2 p} = \mu^\nu\frac{1}{(\mu^\nu)^2 p^\nu} = \frac{1}{\mu^\nu p^\nu} \in \mathcal{P}.$$

As an immediate consequence, it follows that the set of all ω-periodic and regressive functions is a subgroup of the set of regressive functions, which is expected in the theory of time scales, showing consistency of the definition.

Corollary 2.3.7 (See [8, Corollary 5.8]). *If $p \in \mathcal{P} \cap \mathcal{R}$, then $\ominus p \in \mathcal{P} \cap \mathcal{R}$. If $p, q \in \mathcal{P} \cap \mathcal{R}$, then $p \oplus q \in \mathcal{P} \cap \mathcal{R}$.*

Proof. Let $p \in \mathcal{P} \cap \mathcal{R}$. Then, $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. By Theorem 2.3.5, for $\alpha = 1$,

$$\frac{p}{1 + \mu p} \in \mathcal{P}.$$

Again by Theorem 2.3.5, we get

$$-\frac{p}{1+\mu p} = \ominus p \in \mathcal{P},$$

proving the first part. By the definition of ″circle plus" and by Theorem 2.3.5, for $p, q \in \mathcal{P}$, we get:

$$p + q \in \mathcal{P} \text{ and } \mu pq \in \mathcal{P},$$

which imply that $p + q \in \mathcal{P}$ and $\mu pq \in \mathcal{P}$,

$$(p + q) + (\mu pq) = p \oplus q \in \mathcal{P},$$

proving the desired result. □

2.4 Homogeneous Linear Dynamic Equation

In this section, we apply the definition of ω-periodicity to investigate the properties of a homogeneous linear dynamic equations on isolated time scales.

Theorem 2.4.1 (See [8, Theorem 6.1]). *Let $a \in \mathcal{R}$. If*

$$x^{\Delta} = a(t)x \tag{2.15}$$

has a nontrivial ω-periodic solution, then

$$\left(a + \frac{1}{\mu}\right)\sigma^{\Delta} \in \mathcal{P}. \tag{2.16}$$

Proof. Suppose (2.15) has a nontrivial ω-periodic solution $\bar{x}$, then by Variation of Constants Formula (Theorem 1.5.11), we get

$$\bar{x}(\nu(t)) = e_a(\nu(t), t)\bar{x}(t) \text{ for all } t \in \mathbb{T}.$$

Since $\bar{x}$ is ω-periodic, we get

$$\bar{x}(t) = \nu^{\Delta}(t)\bar{x}(\nu(t)) = \nu^{\Delta}(t)e_a(\nu(t), t)\bar{x}(t), \text{ for all } t \in \mathbb{T}. \tag{2.17}$$

Since $x \not\equiv 0$, then $x(t) \neq 0$ for all $t \in \mathbb{T}$. From (2.17), we get

$$\nu^{\Delta}(t)e_a(\nu(t), t) = 1. \tag{2.18}$$

Applying the Δ-derivative in (2.18), using the product rule and Lemma 2.1.11, we obtain

$$\nu^{\Delta\Delta}(t)e_a(\nu(t),t) + \nu^{\Delta\sigma}[((\nu^{\Delta}a^{\nu}) \ominus a)e_a(\nu(t),t)] = 0.$$

This implies

$$\nu^{\Delta\Delta}(t) + \nu^{\Delta\sigma}((\nu^{\Delta}a^{\nu}) \ominus a) = 0. \tag{2.19}$$

By Lemma 2.4.2 below, (2.16) and (2.19) are equivalent, proving the desired result. □

The next lemma gives two equivalent conditions to (2.16).

Lemma 2.4.2 (See [8, Lemma 6.2])**.** *If $a \in \mathcal{R}$, then a satisfies (2.16) if, and only if,*

$$\nu^{\Delta\Delta} + \nu^{\Delta}\nu^{\Delta\sigma}a^{\nu} = \nu^{\Delta}a \tag{2.20}$$

holds, and (2.16) is equivalent to

$$(\nu^{\Delta}a^{\nu}) \ominus a = -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}}. \tag{2.21}$$

Proof. Let $a \in \mathcal{R}$, by Remark 2.2.2, (2.16) is equivalent to

$$\left[\mu\left(a + \frac{1}{\mu}\right)\sigma^{\Delta}\right]^{\nu} = \mu\left(a + \frac{1}{\mu}\right)\sigma^{\Delta}. \tag{2.22}$$

Using Remark 2.1.6, we get

$$\mu\left(a + \frac{1}{\mu}\right)\sigma^{\Delta} = (\mu a + 1)\sigma^{\Delta} = \mu\sigma^{\Delta}a + \sigma^{\Delta} = \mu^{\sigma}a + \sigma^{\Delta}.$$

Therefore,

$$(\mu^{\sigma}a)^{\nu} + \sigma^{\Delta\nu} = a\mu^{\sigma} + \sigma^{\Delta},$$

which implies by Lemma 2.1.5,

$$\begin{aligned}
\nu^{\Delta}a &= \nu^{\Delta}\frac{\mu^{\sigma}}{\mu^{\sigma}}a \\
&= \frac{\nu^{\Delta}}{\mu^{\sigma}}\left(\sigma^{\Delta\nu} - \sigma^{\Delta} + (\mu^{\sigma}a)^{\nu}\right) \\
&= \nu^{\Delta}\left(\frac{\sigma^{\Delta\nu} - \sigma^{\Delta}}{\mu^{\sigma}} + \frac{\mu^{\nu\sigma}}{\mu^{\sigma}}a^{\nu}\right) \\
&= \nu^{\Delta}\left(\nu^{\Delta} + \nu^{\Delta\sigma}a^{\nu}\right)
\end{aligned}$$

$$= \nu^{\Delta\Delta} + \nu^{\Delta}\nu^{\Delta\sigma}a^{\nu},$$

which is (2.20). Reciprocally by Lemma 2.1.5, we get

$$\nu^{\Delta} = \nu^{\Delta\sigma} - \nu^{\Delta\Delta}\mu,$$

then we conclude that (2.20) is equivalent to

$$\begin{aligned}\nu^{\Delta}a^{\nu} &= \frac{\nu^{\Delta}a - \nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} \\ &= \frac{(\nu^{\Delta\sigma} - \mu\nu^{\Delta\Delta})a - \nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} \\ &= -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} + a - \mu\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}}\nu^{\Delta\sigma}a \\ &= \left(-\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} \oplus a\right),\end{aligned}$$

proving the desired result. □

The next theorem describes two interesting properties for the exponential function.

Theorem 2.4.3 (See [8, Theorem 6.3]). *Let $a \in \mathcal{R}$ and assume* (2.15). *For $t_0 \in \mathbb{T}$, we have*

$$e_a(\nu(t), t) = e_a(\nu(t_0), t_0)\frac{\nu^{\Delta}(t_0)}{\nu^{\Delta}(t)} \text{ for all } t \in \mathbb{T}. \tag{2.23}$$

Moreover, we have

$$e_a(\nu(t), \nu(s)) = e_a(t, s)\frac{\nu^{\Delta}(s)}{\nu^{\Delta}(t)} \text{ for all } s, t \in \mathbb{T}. \tag{2.24}$$

Proof. The proof of this resut can be found in [8, Theorem 6.3]. □

To finish, the theorem below supplements Theorem 2.4.1 to describe a complete characterization of periodic solution of (2.14).

Theorem 2.4.4 (See [8, Theorem 6.4]). *Let $a \in \mathcal{R}$ and assume* (2.15). *If*

$$\nu^{\Delta}(t_0)e_a(\nu(t_0), t_0) = 1,$$

then all solutions of (2.14) *are ω-periodic. Otherwise, no nontrivial solution of* (2.14) *is ω-periodic.*

2.5 Remarks on other concepts of periodicity for general time scales

In this section, our goal is to discuss other concepts of ω-periodicity which appeared in the literature, in order to generalize this concept for any time scale. In 2013, Adivar [1] presented a unified concept of periodicity for any time scale, based on the definition of shifts operators.

We start this section by presenting this definition.

Definition 2.5.1 (See [1, Definition 3])**.** *Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm\colon [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:*

P1) The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0,t),(T_0,u) \in D_\pm := \{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^*\colon \delta_\pm(s,t) \in \mathbb{T}^*\},$$

then

$$T_0 \le t < u$$

implies

$$\delta_\pm(T_0,t) < \delta_\pm(T_0,u).$$

P2) If $(T_1,u),(T_2,u) \in D_-$ with $T_1 < T_2$, then

$$\delta_-(T_1,u) > \delta_-(T_2,u)$$

and if $(T_1,u),\ (T_2,u) \in D_+$ with $T_1 < T_2$, then

$$\delta_+(T_1,u) < \delta_+(T_2,u).$$

P3) If $t \in [t_0,\infty)_\mathbb{T}$, then $(t,t_0) \in D_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in D_+$ and $\delta_+(t_0,t) = t$ holds.

P4) If $(s,t) \in D_\pm$ then $(s,\delta_\pm(s,t)) = t$, respectively.

P5) If $(s,t) \in D_\pm$ and $(u,\delta_\pm(s,t)) \in D_\pm$, then $(s,\delta_\pm(u,t)) \in D_\pm$ and $\delta_\pm(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\pm(u,t))$ respectively.

*Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be **backward** and **forward shift operators** on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0,\infty)_\mathbb{T}$ in $\delta_\pm(s,t)$ is called the shift size. The values $\delta_+(s,t)$ and $\delta-(s,t)$ in $\mathbb{T}^*$ indicate s*

units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $D_\pm$ are the domains of the shift operators respectively.

As an immediate consequence of this definition, one can prove the following properties of the shift operators.

Lemma 2.5.2 (See [1, Lemma 1]). *Let δ_- and δ_+ be the shift operators associated with the initial point t_0. We have:*

i) $\delta_-(t,t) = t_0$ for all $t \in [t_0,\infty)_\mathbb{T}$.

ii) $\delta_-(t_0,t) = t$ for all $t \in \mathbb{T}^*$.

iii) If $(s,t) \in D_+$, then $\delta_+(s,t) = u$ implies $\delta_-(s,u) = t$. Conversely, if $(s,u) \in D_-$, then $\delta_-(s,u) = t$ implies $\delta_+(s,t) = u$.

iv) $\delta_+(t,\delta_-(s,t_0)) = \delta_-(s,t)$ for all $(s,t) \in D_+$ with $t \geq t_0$.

v) $\delta_+(u,t) = \delta_+(t,u)$ for all $(u,t) \in ([t_0,\infty)_\mathbb{T} \times [t_0,\infty)_\mathbb{T}) \cap D_+$.

vi) $\delta_+(s,t) \in [t_0,\infty)_\mathbb{T}$ for all $(s,t) \in D_+$ with $t \geq t_0$.

vii) $\delta_-(s,t) \in [t_0,\infty)_\mathbb{T}$ for all $(s,t) \in ([t_0,\infty)_\mathbb{T} \times [s,\infty)_\mathbb{T}) \cap D_-$.

viii) If $\delta_+(s,\cdot)$ id Δ-differentiable in its second variable, then $\delta_+^{\Delta t}(s,\cdot) > 0$.

ix) $\delta_+(\delta_-(u,s),\delta_-(s,v)) = \delta_-(u,v)$ for all $(s,v) \in ([t_0,\infty)_\mathbb{T} \times [s,\infty)_\mathbb{T}) \cap D_-$ and $(u,s) \in ([t_0,\infty)_\mathbb{T} \times [u,\infty)_\mathbb{T}) \cap D_-$.

x) If $(s,t) \in D_-$ and $\delta_-(s,t) = t_0$, then $s = t$.

With these notions and properties in hand, we are ready to present the definition of periodicity in shifts.

Definition 2.5.3 (See [1, Definition 4]). *Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be **periodic in shifts** $\delta_\pm$ if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_\pm$ for all $t \in \mathbb{T}^*$. Futhermore, if*

$$P := \inf\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in D_\pm \text{ for all } t \in \mathbb{T}^*\} \neq t_0,$$

*then P is called the **period** of the time scale $\mathbb{T}$.*

Although this definition seems to be quite general, it does not colapses in the case of quantum scale with the expected definition for periodic functions, since that by this definition, the periodicity in such case may be read as follows

$$f(q^{\omega}t) = f(t)$$

for all $t \in q^{\mathbb{N}_0}$. Therefore, with this definition, the important property of periodicity for areas does not keep preserved. However, Adivar [1] presented another concept that is called textitΔ-periodic functions in shifts $\delta_{\pm}$ in his paper, which keeps such property preserved. Below, we state this definition.

Definition 2.5.4 (See [1, Definition 6])**.** *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with period P. We say that a real valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_{\pm}$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that*

$$(T, t) \in D_{\pm} \textit{ for all } t \in \mathbb{T}^*,$$

the shifts $\delta_{\pm}^{T}$ are Δ-differentiable with rd-continuous derivatives and

$$f(\delta_{\pm}^{T}(t))\delta_{\pm}^{\Delta T}(t) = f(t)$$

for all $t \in \mathbb{T}^*$, where

$$\delta_{\pm}^{T}(t) := \delta_{\pm}(T, t).$$

*The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that the definition holds is called the **period** of f.*

This definition brings a more general concept and could also be used in our treatment here. However, we choose in this work to deal with the definition presented in [8] by simplicity, since for this last concept one does not need to deal with the shifts operators. Moreover, although the definition from Adivar includes also the continuous case, there are some isolated time scales that are not so clear if they are included by those shifts operators. For more details about it, see [8], Appendix.

Chapter

3

Floquet theory on isolated time scales

This chapter is the most important of this work, since it brings original contributions for the investigations of ω-periodic functions on isolated time scales. More precisely, here we are interested to study Floquet theory for isolated time scales, using the new definition (Definition 2.2.1) of periodic function on the isolated time scales. In this entire chapter, we consider $\mathbb{T}$ as an isolated time scale. We focus on the first-order linear equation, called Floquet isolated equation,

$$x^{\Delta}(t) = A(t)x(t), \ t \in \mathbb{T}, \tag{3.1}$$

where

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}, \text{ for all } t \in \mathbb{T}, \tag{3.2}$$

A is an ω-periodic matrix function defined below, and is also regressive, i.e., $I+\mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, where I is the identity matrix.

The chapter is organized as follows: Section 1 is devoted to present some auxiliary results involving periodicity on any isolated time scale. In Section 2, we investigate Floquet Theory and prove several important properties. In Section 3, examples are presented to illustrate our results. The results presented here, are completely original and can be found in [6].

3.1 Periodic Functions

Lemma 3.1.1. *If B is an ω-periodic and regressive matrix-valued function on $\mathbb{T}$, then*

$$e_B(t,s) = e_B(\nu(t),\nu(s)) \text{ for all } t,s \in \mathbb{T}.$$

Proof. Using the semigroup property, exponential properties and Lemma 2.1.11, we get

$$\begin{aligned} e_B(\nu(t),\nu(s)) &= e_B(\nu(t),t)e_B(t,s)e_B(s,\nu(s)) \\ &= \frac{e_B(\nu(t),t)}{e_B(\nu(s),s)}e_B(t,s) \\ &= \frac{e_{\nu^{\Delta}B^{\nu}}(t,s)}{e_B(t,s)}e_B(t,s) \\ &= e_{\nu^{\Delta}B^{\nu}}(t,s) \\ &= e_B(t,s), \end{aligned}$$

since B is ω-periodic. □

Theorem 3.1.2. *Let $t_0 \in \mathbb{T}$ and $\omega \in \mathbb{N}$. If C is a nonsingular $k \times k$ constant matrix, then there exists an ω-periodic regressive matrix-valued function B on $\mathbb{T}$ such that*

$$e_B(\nu(t_0),t_0) = C.$$

Proof. Let μ_i be the eigenvalues of C, $1 \le i \le k$. For $p \in \{0,1,2,...,\omega-2\}$, define

$$R_p := \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_n \end{pmatrix}$$

where either J_i is the 1×1 matrix given by $J_i = \mu_i$ or

$$J_i := \begin{pmatrix} \mu_i & 1 & 0 & \dots & 0 \\ 0 & \mu_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_i & 1 \\ 0 & \dots & 0 & 0 & \mu_i \end{pmatrix}$$

with $1 \le i \le k$.

Now, define

$$R_{\omega-1} := \frac{1}{\mu(\sigma^{\omega-1}(t_0))} \left\{ \prod_{k=0}^{\omega-2} \left(I + \mu(\sigma^k(t_0))R_k\right)^{-1} C - I \right\}.$$

Hence, we get:

$$\mu(\sigma^{\omega-1}(t_0))R_{\omega-1} + I = \prod_{k=0}^{\omega-2} \left(I + \mu(\sigma^k(t_0))R_k\right)^{-1} C.$$

This implies that

$$\prod_{k=0}^{\omega-1} \left(I + \mu(\sigma^k(t_0))R_k\right) = C, \tag{3.3}$$

where $\prod_{k=0}^{\omega-1} \left(I + \mu(\sigma^k(t_0))R_k\right)$ is the product starting from the left to the right. Since C is nonsingular, from (3.3), it follows that R_p is regressive for all $p \in \{0, 1, 2, ..., \omega - 1\}$.

We define:

$$B(\nu^m(\sigma^j(t_0))) : = \frac{R_j \mu(\sigma^j(t_0))}{\mu(\nu^m(\sigma^j(t_0)))}$$

for all $j \in \{0, 1, 2, ..., \omega - 1\}$ and all $m \in \mathbb{N}_0$.

Let $t \in \mathbb{T}$ be such that $t \ge t_0$, then there exists $m \in \mathbb{N}_0$ and $j \in \{0, 1, 2, ..., \omega - 1\}$ such that $t = \nu^{m-1}(\sigma^j(t_0))$. Therefore, we get

$$\begin{aligned}
\nu^\Delta(t)B(\nu(t)) &= \nu^\Delta(\nu^{m-1}(\sigma^j(t_0)))B(\nu^m(\sigma^j(t_0))) \\
&= \nu^\Delta(\nu^{m-1}(\sigma^j(t_0))) \cdot \frac{R_j \mu(\sigma^j(t_0))}{\mu(\nu^m(\sigma^j(t_0)))} \\
&= \frac{\mu(\nu^m(\sigma^j(t_0)))}{\mu(\nu^{m-1}(\sigma^j(t_0)))} \cdot \frac{R_j \mu(\sigma^j(t_0))}{\mu(\nu^m(\sigma^j(t_0)))} \\
&= B(\nu^{m-1}(\sigma^j(t_0)) \\
&= B(t),
\end{aligned}$$

proving that B is ω-periodic. Also, for $t = \nu^{m-1}(\sigma^j(t_0))$, we get

$$\begin{aligned}
I + \mu(t)B(t) &= I + \mu(\nu^{m-1}(\sigma^j(t_0))) \cdot B(\nu^{m-1}(\sigma^j(t_0))) \\
&= I + \mu(\nu^m(\sigma^j(t_0)))B(\nu^m(\sigma^j(t_0))) \\
&= I + \mu(\sigma^j(t_0))R_j \\
&\ne 0,
\end{aligned}$$

since R_j is regressive for each $j \in \{0, ..., \omega - 1\}$ and using the ω-periodicity of B. On the other hand,

$$\begin{aligned} e_B(\nu(t_0), t_0) &= \prod_{k=0}^{\omega-1} \{I + \mu(\sigma^k(t_0))B(\sigma^k(t_0))\} \\ &= \prod_{k=0}^{\omega-1} \left\{ (I + \mu(\sigma^k(t_0)) \frac{\mu(\nu(\sigma^k(t_0)))}{\mu(\sigma^k(t_0))} \cdot B(\nu(\sigma^k(t_0))) \right\} \\ &= \prod_{k=0}^{\omega-1} \left\{ I + \mu(\sigma^k(t_0)) R_k \right\} \\ &\overset{(3.3)}{=} C, \end{aligned}$$

proving the desired result.

□

3.2 Floquet Theory

In this section, our goal is to prove the Floquet theory on isolated time scales.

Lemma 3.2.1. *Let $t_0 \in \mathbb{T}$, suppose $x : \mathbb{T} \to \mathbb{R}^n$ is a solution of* (3.1) *satisfying*

$$x(t_0) = \nu^{\Delta}(t_0) x(\nu(t_0)). \tag{3.4}$$

If (2.19) *is satisfied, then x is ω-periodic.*

Proof. Define a function $f : \mathbb{T} \to \mathbb{R}^n$ by

$$f(t) := \nu^{\Delta}(t) x(\nu(t)) - x(t), \text{ for all } t \in \mathbb{T}. \tag{3.5}$$

Then, by (3.5)

$$\begin{aligned} f^{\Delta}(t) &= \nu^{\Delta\Delta}(t) x(\nu(t)) + \nu^{\Delta}(t) x^{\Delta}(\nu(t)) \nu^{\Delta\sigma}(t) - x^{\Delta}(t) \\ &\overset{(3.1)}{=} \nu^{\Delta\Delta}(t) x(\nu(t)) + \nu^{\Delta}(t) \nu^{\Delta\sigma}(t) \left[A(\nu(t)) x(\nu(t))\right] - \left[A(t) x(t)\right] \\ &= \left[\nu^{\Delta\Delta}(t) + \nu^{\Delta}(t) \nu^{\Delta\sigma}(t) A(\nu(t))\right] x(\nu(t)) - \left[A(t) x(t)\right] \\ &\overset{(2.19)}{=} \nu^{\Delta}(t) A(t) x(\nu(t)) - A(t) x(t) \\ &= A(t) \left[\nu^{\Delta}(t) x(\nu(t)) - x(t)\right] \\ &\overset{(3.4)}{=} A(t) f(t), \end{aligned}$$

and since $f(t_0) = 0$, we get the desired result. □

As usual, a matrix-valued function Φ is called a *fundamental matrix of the Floquet equation* (3.1) provided it solves (3.1) and is nonsingular for all $t \in \mathbb{T}$. The following results gives a representation for any fundamental matrix (3.1).

Theorem 3.2.2. *Suppose $\Phi : \mathbb{T} \to \mathbb{R}^{n\times n}$ is a fundamental matrix for equation* (3.1)*. Define the matrix-valued function $\Psi : \mathbb{T} \to \mathbb{R}^{n\times n}$ by*

$$\Psi(t) := \nu^{\Delta}(t)\Phi(\nu(t)), \text{ for all } t \in \mathbb{T}. \tag{3.6}$$

Then Ψ is also a fundamental matrix of (3.1)*. Futhermore, there exist a regressive matrix-valued function B and an ω-periodic matrix-valued function $P : \mathbb{T} \to \mathbb{R}^{n\times n}$ such that*

$$\Phi(t) = P(t)e_B(t, t_0), \text{ for all } t \in \mathbb{T}. \tag{3.7}$$

Proof. We have

$$\begin{aligned}
\Psi^{\Delta}(t) &= \nu^{\Delta\Delta}(t)\Phi(\nu(t)) + \nu^{\Delta}(t)\Phi^{\Delta}(\nu(t))\nu^{\Delta\sigma}(t) \\
&= \left[\nu^{\Delta\Delta}(t) + \nu^{\Delta}(t)\nu^{\Delta\sigma}(t)A(\nu(t))\right]\Phi(\nu(t)) \\
&\overset{(2.19)}{=})\nu^{\Delta}(t)A(t)\Phi(\nu(t)) \\
&\overset{(3.4)}{=} A(t)\Psi(t).
\end{aligned}$$

Then, Ψ is a fundamental matrix of (2.19). Futhermore, define the nonsingular constant matrix C by

$$C := \Phi^{-1}(t_0)\Psi(t_0).$$

Hence, the function $D : \mathbb{T} \to \mathbb{R}^{n\times n}$ defined by

$$D(t) = \Psi(t) - \Phi(t)C, \text{ for all } t \in \mathbb{T}$$

satisfies $D(t_0) = 0$ and

$$\begin{aligned}
D^{\Delta}(t) &= \Psi^{\Delta}(t) - \Phi^{\Delta}(t)C \\
&= A(t)\Psi(t) - A(t)\Phi(t)C \\
&= A(t)D(t).
\end{aligned}$$

It implies that

$$\Psi(t) = \Phi(t)C, \text{ for all } t \in \mathbb{T}.$$

Thus by Theorem 3.1.2, there exists an ω-periodic and regressive matrix-valued function B such that

$$e_B(\nu(t_0), t_0) = C$$

Defining the matrix-valued function $P : \mathbb{T} \to \mathbb{R}^{n\times n}$ by

$$P(t) := \Phi(t)e_B^{-1}(t, t_0),\ t \in \mathbb{T}.$$

we get that P is a nonsingular matrix-valued function on $\mathbb{T}$. Using (3.1), (3.7) and (3.4), we obtain

$$\begin{aligned}
\nu^\Delta(t)P(\nu(t)) &= \nu^\Delta(t)\left[\Phi(\nu(t))e_B^{-1}(\nu(t), t_0)\right] \\
&= \Psi(t)e_B^{-1}(\nu(t), t_0) \\
&= \Phi(t)Ce_B(t_0, \nu(t)) \\
&= \Phi(t)Ce_B(t_0, \nu(t_0))e_B(\nu(t_0), \nu(t)) \\
&= \Phi(t)e_B(\nu(t_0), t_0)e_B(t_0, \nu(t_0))e_B(\nu(t_0), \nu(t)) \\
&= \Phi(t)e_B(\nu(t_0), \nu(t)) \\
&= \Phi(t)e_B(t_0, t) \\
&= P(t)
\end{aligned}$$

for all $t \in \mathbb{T}$, i.e., P is ω-periodic, proving the desired result. □

Theorem 3.2.3. *Suppose Φ, P, and B satisfy all the hypotheses of Theorem 3.2.2. Then x solves* (3.1) *if, and only if, y given by $y(t) = P^{-1}(t)x(t)$, $t \in \mathbb{T}$, solves $y^\Delta(t) = B(t)y(t)$.*

Proof. Let $t_0 \in \mathbb{T}$. Assume x solves (3.1), by uniqueness of solution,

$$x(t) = \Phi(t)x(t_0),\ \text{for all } t \in \mathbb{T}.$$

Defining $y(t) := P^{-1}(t)x(t)$, $t \in \mathbb{T}$, we have

$$y(t) = P^{-1}(t)\Phi(t)x(t_0) = e_B(t, t_0)x(t_0)$$

which solves $y^\Delta(t) = B(t)y(t)$.

Reciprocally, suppose y solves $y^\Delta(t) = B(t)y(t)$ and define $x : \mathbb{T} \to \mathbb{R}^n$ by $x(t) = P(t)y(t)$, for all $t \in \mathbb{T}$. By the uniqueness of solutions, we have

$$y(t) = e_B(t, t_0)y(t_0),\ \text{for all } t \in \mathbb{T}.$$

Therefore,

$$x(t) = P(t)e_B(t,t_0)y(t_0) = \Phi(t)y(t_0),$$

proving the desired result. □

Definition 3.2.4. *Let Φ be a fundamental matrix for* (3.1) *and $t_0 \in \mathbb{T}$. The eigenvalues of*

$$\nu^{\Delta}(t_0)\Phi^{-1}(t_0)\Phi(\nu(t_0))$$

are called the ***Floquet multipliers*** *of* (3.4).

Remark 3.2.5. *Since fundamental matrices for* (3.1) *are not unique, let us show that the Floquet multipliers are well defined. Let Φ and Ψ be any fundamental matrices for* (3.1) *and let*

$$C := \nu^{\Delta}(t_0)\Phi^{-1}(t_0)\Phi(\nu(t_0)) \text{ and } D := \nu^{\Delta}(t_0)\Psi^{-1}(t_0)\Psi(\nu(t_0)).$$

We affirm that C and D have the same eigenvalues. By Theorem 3.2.2, there exists a nonsingular constant matrix M such that

$$\Psi(t) = \Phi(t)M, \text{ for all } t \in \mathbb{T}.$$

Hence,

$$\begin{aligned} D &= \nu^{\Delta}(t_0)M^{-1}\Phi^{-1}(t_0)\Phi(\nu(t_0))M \\ &= M^{-1}\Phi^{-1}(t_0)\nu^{\Delta}(t_0)\Phi(\nu(t_0))M \\ &= M^{-1}CM. \end{aligned}$$

Then, C and D are similar matrices, which implies that they have the same eigenvalues. Therefore, the Floquet multipliers are well defined.

Remark 3.2.6. *Note that the proof of Theorem 3.2.2 shows that the matrix-valued function satisfies*

$$\nu^{\Delta}(t)\Phi^{-1}(t)\Phi(\nu(t)) = \Phi^{-1}(t)\Psi(t). \tag{3.8}$$

On the other hand, notice that

$$\begin{aligned} (\Phi^{-1}(t)\Psi(t))^{\Delta} &= (\Phi^{-1}(t))^{\Delta} \cdot \Psi(t) + \Phi^{-1}(\sigma(t))\Psi^{\Delta}(t) \\ &= -\Phi^{-1}(\sigma(t))\Phi^{\Delta}(t)\Phi^{-1}(t)\Psi(t) + \Phi^{-1}(\sigma(t))A(t)\Psi(t) \\ &= \Phi^{-1}(\sigma(t))[-A(t)\Phi(t)\Phi^{-1}(t)\Psi(t) + A(t)\Psi(t)] \\ &= 0. \end{aligned}$$

It implies that

$$\Phi^{-1}(t)\Psi(t) = \Phi^{-1}(t_0)\Psi(t_0)$$

for all $t \in \mathbb{T}$*. Hence by* (3.8)*, we have,*

$$\begin{aligned}\nu^{\Delta}(t)\Phi^{-1}(t)\Phi(\nu(t)) &= \Phi^{-1}(t)\Psi(t)\\ &= \Phi^{-1}(t_0)\Psi(t_0)\\ &= \nu^{\Delta}(t_0)\Phi^{-1}(t_0)\Phi(\nu(t_0))\end{aligned}$$

which implies that

$$\nu^{\Delta}(t)\Phi^{-1}(t)\Phi(\nu(t))$$

does not depend on $t \in \mathbb{T}$*. Thus, the Floquet multipliers of* (3.1) *are also equal to the eigenvalues of* $\nu^{\Delta}(t)\Phi^{-1}(t)\Phi(\nu(t))$*, for* $t \in \mathbb{T}$*.*

Theorem 3.2.7. *The number* μ_0 *is a Floquet multiplier of* (3.1) *if, and only if, there exists a nontrivial solution* x *of* (3.1) *such that*

$$\nu^{\Delta}(t)x(\nu(t)) = \mu_0 x(t), \ \forall t \in \mathbb{T}.$$

Proof. Assume μ_0 is a *Floquet multiplier* of (3.1). Let $t \in \mathbb{T}$, by Remark 3.2.6, μ_0 is an eigenvalue of $C := \Phi^{-1}(t_0)\Psi(t_0) = \Phi^{-1}(t)\Psi(t)$, where Φ is a fundamental matrix of (3.1).

Let x_0 be the eigenvector corresponding to μ_0, i.e., $Cx_0 = \mu_0 x_0$. Define $x(t) := \Phi(t)x_0$ for all $t \in \mathbb{T}$, then x is nontrivial solution of (3.1) and

$$\begin{aligned}\nu^{\Delta}(t)x(\nu(t)) &= \nu^{\Delta}(t)\Phi(\nu(t))x_0\\ &= \Psi(t)x_0\\ &= \Phi(t)Cx_0\\ &= \Phi(t)\mu_0 x_0\\ &= \mu_0 x(t),\end{aligned}$$

proving the desired result. Reciprocally, assume that there exists a nontrivial solution x of (3.1), such that $\nu^{\Delta}(t)x(\nu(t)) = \mu_0 x(t)$ for all $t \in \mathbb{T}$. Let Ψ be a fundamental matrix of (3.1), then $x(t) = \Psi(t)y_0$ for all $t \in \mathbb{T}$ and some nonzero constant vector y_0. Hence,

$$\nu^{\Delta}(t)\Psi(\nu(t))y_0 = \nu^{\Delta}(t)x(\nu(t)) = \mu_0 x(t) = \mu_0\Psi(t)y_0$$

Let $t = t_0$,

$$\nu^{\Delta}(t_0)\Psi(\nu(t_0))y_0 = \mu_0\Psi(t_0)y_0,$$

it implies that

$$\nu^{\Delta}(t_0)\Psi^{-1}(t_0)\Psi(\nu(t_0))y_0 = \mu_0 y_0.$$

Therefore,

$$Dy_0 = \mu_0 y_0.$$

where $D := \nu^{\Delta}(t_0)\Psi^{-1}(t_0)\Psi(\nu(t_0))$. Hence, μ_0 is an eigenvalue of D, proving the desired result. $\square$

Remark 3.2.8. *By Theorem 3.2.7, the equation* (3.1) *has a ω-periodic solution if and only if $\mu_0 = 1$ is a Floquet multiplier. The case $x \equiv 0$ follows directly. Theferore, we assume, without loss of generality, that x is a nontrivial solution, which is ω-periodic. Then, $\nu^{\Delta}(t)x(\nu(t)) = x(t)$ for all $t \in \mathbb{T}$. From Theorem 3.2.7, $\mu_0 = 1$. Reciprocally, if $\mu_0 = 1$ is a Floquet multiplier, by Theorem 3.2.7,*

$$\nu^{\Delta}(t)x(\nu(t)) = \mu_0 x(t) = x(t)$$

which imples that x is ω-periodic.

3.3 Examples

In this section, we present some examplse to illustrate our results.

Example 3.3.1. *Let p be the ω-periodic and regressive function on $\mathbb{T}$. Define*

$$A(t) := \begin{pmatrix} 0 & \dfrac{1}{\mu(t)}\cosh_p(\nu(t), t) \\ \dfrac{1}{\mu(t)}\sinh_p(\nu(t), t) & 0 \end{pmatrix}, \quad \textit{for all } t \in \mathbb{T}. \tag{3.9}$$

Then, we show that the coefficient matrix-valued function A is ω-periodic. Indeed,

$$\begin{aligned}\nu^{\Delta}(t)A(\nu(t)) &= \frac{\mu(\nu(t))}{\mu(t)}\begin{pmatrix} 0 & \dfrac{1}{\mu(\nu(t))}\cosh_p(\nu(\nu(t)), \nu(t)) \\ \dfrac{1}{\mu(\nu(t))}\sinh_p(\nu(\nu(t)), \nu(t)) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \dfrac{1}{\mu(t)}\cosh_p(\nu(\nu(t)), \nu(t)) \\ \dfrac{1}{\mu(t)}\sinh_p(\nu(\nu(t)), \nu(t)) & 0 \end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & \dfrac{1}{\mu(t)} \cosh_p(\nu(t), t) \\ \dfrac{1}{\mu(t)} \sinh_p(\nu(t), t) & 0 \end{pmatrix} \\
&= A(t),
\end{aligned}
$$

from the definition of $\cosh_p$ *and* $\sinh_p$*, and the exponential function. The solution of* (3.1)*, where* A *is defined as in* (3.9)*, satisfying the initial condition* $x(t_0) = x_0$*, is* $x(t) = e_A(t, t_0)x_0$*,* $t \in \mathbb{T}$*. Assume also that* μ_1 *and* μ_2 *are eigenvalues corresponding to the constant matrix*

$$C := \nu^{\Delta}(t_0) e_A^{-1}(t_0, t_0) e_A(\nu(t_0), t_0) = \nu^{\Delta}(t_0) e_A(\nu(t_0), t_0) = \nu^{\Delta}(t_0) e_A(\nu(t), t)$$

where this last equality follows from Theorem 2.2.8. Applying Liouville's formula (Theorem 1.6.12), we get

$$
\begin{aligned}
\mu_1\mu_2 = \det C &= \det\left(\nu^{\Delta}(t_0) e_A(\nu(t), t)\right) \\
&= \left(\nu^{\Delta}(t_0)\right)^2 \det e_A(\nu(t), t) \\
&= \left(\nu^{\Delta}(t_0)\right)^2 e_{trA + \mu \det A}(\nu(t), t_0) \det e_A(t_0, t_0) \\
&= \left(\nu^{\Delta}(t_0)\right)^2 e_f(\nu(t), t_0)
\end{aligned}
$$

where $f\colon \mathbb{T} \to \mathbb{R}$ *is defined by*

$$f(t) = -\frac{1}{\mu(t)} \cosh_p(\nu(t), t) \sinh_p(\nu(t), t), \text{ for all } t \in \mathbb{T}.$$

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