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Davide Morrone de Azevedo Muniz

Advisor

Prof. Dr. Aleksandr Nikolaievich Pinzul

Co-Advisor

Prof. Dr. Carolina Matté Gregory

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Davide Morrone de Azevedo Muniz

Prof. Dr. Aleksandr Nikolaievich Pinzul (Advisor)
Institute of Physics, University of Brasília

Prof. Dr. Carolina Matté Gregory (Co-Advisor)
Institute of Physics, University of Brasília

Prof. Dr. Arsen Melikyan
University of Brasília

Prof. Dr. Jerome Gauntlett
Imperial College London

Prof. Dr. Sebastião William da Silva
Coordinator of the Graduate Program in Physics

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Resumo

Nesse trabalho, deduzimos uma fórmula geral para a ação “on-shell” da supergravidade de Romans Euclideana seis-dimensional usando localização equivariante. Começamos revisando a teoria matemática necessária da Cohomologia Equivariante e da Geometria Tórica, então proseguimos introduzindo a supergravidade de Romans Euclidena. Aplicamos a fórmula de localização BV-AB para calcular a ação “on-shell”. Nosso resultado é uma fórmula geral que não necessita do conhecimento explícito de uma solução das equações de movimento, se apoiando quase inteiramente na topologia da solução. A fórmula obtida é aplicada a uma variedade de exemplos que elucidam diferents aspectos. Recuperamos resultados da literatura, incluindo resultados da supergravidade e cálculos da teoria de campos dual holográfica, e obtemos novos estabelescendo previsões para soluções desconhecidas.

Palavras-chave: Ação “on-shell”; Supergravidade de Romans; Holografia; Correspondência AdS/CFT; Geomtria Tórica; Localisação Equivariante.

Abstract

In this work, we derive a general formula for the on-shell action of six-dimensional Euclidean Romans supergravity using equivariant localization. We begin by reviewing the necessary mathematical framework of Equivariant Cohomology and Toric Geometry, then we proceed to introduce the Euclidean Romans supergravity. We then apply the BV-AB localization formula to calculate the on-shell action. Our result is a general formula that does not require an explicit knowledge of a solution of the equations of motion, relying almost entirely on the topology of the solution. The obtained formula is applied to a variety of examples that display different features. We recover previous results found on the literature, including supergravity results and holographical dual field theory calculations, and obtain new ones establishing predictions to unknown solutions.

Keywords: On-shell action; Romans Supergravity; Holography; AdS/CFT Correspondence; Toric Geometry; Equivariant Localization.

Contents

List of Figures

Chapter 1

Introduction

The AdS/CFT correspondence, also known as “Holography", is a conjecture of a duality between gravity and gauge theories defined on the boundary of the manifold where the gravity theory sits. The original proposal of Maldacena [1] was a duality between a supergravity theory that has $AdS_5 \times S^5$ geometry and $\mathcal{N} = 4$ Super Yang-Mills living on four-dimensional Minkowski spacetime, but numerous other examples were proposed for different dimensions. While a proof of the conjecture remains an open problem, the computation and matching of observables on both sides of the duality gives us hints that the conjecture is true and provides insights on how to understand the duality.

Among the interesting observables, there is the on-shell action of the gravity theory which is identified with the free-energy of the dual field theory. The computation of the free-energy on the field theory side has for long been using localization techniques, but the use of such techniques on the gravity side is novel and a general framework is still under development.

The mathematical theory that supports these localization techniques is known as Equivariant Cohomology. This theory studies differential forms on manifolds that are acted by a Lie group and main result of the theory is BV-AB localization formula [2, 3]. If the action of a Lie group on the manifold is not free, there will be fixed points. The BV-AB formula tells us how we can write the integral of a differential form that satisfies certain conditions as a finite sum of contributions over the fixed points of the group action. These conditions are equivalent to a set of differential equations that relate the components of the integrated form to components of forms from lower degree. In this sense, we say that the integral localizes to the fixed points.

In the six-dimensional gauge supergravity known as Romans Supergravity [4], there is a natural $U(1)$ action whose generator is the Killing vector built from the Killing spinor of the theory.[1] The existence of the Killing spinor is implied by supersymmetry and from it we can construct a set of bilinear differential forms that are related to the fields of the theory through a set of differential equations. We can then construct a differential form which integrates to the on-shell action such that the conditions to apply the BV-AB formula are equivalent to the equations of motion and to the supersymmetry conditions of the theory. Finally, we obtain a formula for the action, which is automatically on-shell and supersymmetric. The formula reads as

$$\begin{aligned} I = \Bigg\{ & \sum_{\text{dim } 0} \frac{\chi(\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2 + \sigma^{(3)}\epsilon_3)^3}{\epsilon_1\epsilon_2\epsilon_3} \\ & - \sum_{\text{dim } 2} \frac{\chi(\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2)^2}{\epsilon_1\epsilon_2} \int_{\mathscr{F}_2} 3\,c_1(F) + (\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2)\left(\frac{c_1(L_1)}{\epsilon_1} + \frac{c_1(L_2)}{\epsilon_2}\right) \\ & + \sum_{\text{dim } 4} \chi\sigma^{(1)} \int_{\mathscr{F}_4} 3c_1(F) \wedge c_1(F) + 3\sigma^{(1)} c_1(F) \wedge c_1(L_1) + c_1(L_1) \wedge c_1(L_1) \Bigg\} \frac{F_{S^5}}{27} \,. \end{aligned}$$

Here F_{S^5} is the free-energy of the dual theory on S^5, the ϵ_i's are the weights of the toric action generated by the Killing vector, χ is the chirality of the Killing spinor on the fixed-point set, the $\sigma^{(i)}$'s are signs associated to projection conditions that the Killing spinor satisfies on the fixed-point set, $c_1(F)$ is the first Chern class of the gauge field and $c_1(L_i)$ is the first Chern class of a line bundle that composes the normal bundle that is normal to fixed-point set.

The construction described does not relies on any explicit solution of Romans Supergravity, and hence we obtain a general formula that asks only for the topological information of a given solution. It is then immediate to apply the obtained formula to different classes of examples to obtain expressions for the on-shell action that can be readily compared to the free-energy of five-dimensional field theories that have the same supersymmetry. We consider three classes of examples with topologies given by $\mathbb{R}^6$, $\mathbb{R}^4$ fibered over a two-surface $\mathscr{F}_2$, and $\mathbb{R}^2$ fibered over a four-manifold B_4. In general, there will be a $U(1)$ action only on the $\mathbb{R}^{2k}$ part, which corresponds to the rotation of the $\mathbb{R}^2$ plane, considering a decomposition $\mathbb{R}^{2k} = \mathbb{R}^2 \oplus \ldots \oplus \mathbb{R}^2$. Hence the fixed-point is precisely located at the origin of the planes. In some examples, there will also be a $U(1)$ action on the compact part of the manifold. As a rule, the euclidean conformal boundary of the $\mathbb{R}^{2k}$ part of the manifold is the sphere S^{2k-1}. Thus, the holographical theories will sit on manifolds with topologies S^5, S^3 fibered over $\mathscr{F}_2$, and S^1 fibered over B_4.

This work is organized as follows. We begin by introducing Equivariant Cohomology,

[1] This fact is not exclusive of the Romans Supergravity. In fact, there has been similar constructions on four dimensions. [5]

presenting its most important structures and the BV-AB fixed point formula, which we first give the result for a general compact Lie group and then the $U(1)$ case. We illustrate the theorem with some examples, finishing with the important application to sympletic toric geometry, where we recover the Duistermaat-Heckman formula. In the next chapter, we expand on the topic of toric geometry, giving the general ideas and doing computations which shall be used later on the localization of Romans Supergravity.

Romans Supergravity is then introduced, we make explicit the action, equations of motion, supersymmetry equations and the special truncation that we will consider in the rest of the work. The bilinears are constructed from the spinor giving rise to a $SU(2)$ structure and some differential forms, in special, a one-form dual to the $U(1)$ Killing vector. Then we proceed to localize the theory. We begin by constructing interesting differential forms and analyzing how they behave at the fixed points of the $U(1)$ action. Then we apply the BV-AB formula to obtain an expression to the on-shell action and finish the chapter discussing important additional points. With the expression for the action at hand, we apply it in the next chapter to many examples recovering known supergravity results and obtaining new ones. We finish by comparing some of our results with holographical computations from the literature.

Chapter 2

Equivariant Cohomology

Let M be a differentiable manifold and G a matrix Lie group that acts on M. Let also f be a map on M, $f : M \longrightarrow M$. We say that f is G-equivariant if

$$f(g \cdot x) = g \cdot f(x), \quad \forall x \in M \quad e \quad g \in G, \tag{2.1}$$

where $g \cdot x$ is the action of g on $x \in M$. We want to extend the concept of G-equivariance to differential forms.

First of all, observe that there is a induced action of G on the space of differential forms. It is given by the pull-back where the maps are the elements of G. That is, given a $g \in G$, we can consider it as map from M on M. For every point $x \in M$, g induces a pull-back: $g^* T^*_{g \cdot x} M \longrightarrow T^*_x M$, where $T^*_x M$ is the cotangent space on x. This action naturally extends to the whole exterior algebra ΛM.

Besides being viewed as linear functionals of the vector fields, the differential forms can be regarded as independent objects. We aim to define G-equivariance on the forms in such a manner that we do not break its aspect of differential form by contracting it with some set of vectors. To this end, we shall consider the algebra given by the tensor product of the exterior algebra with the the dual Lie algebra of G.

The Lie algebra of G, $\mathfrak{g}$, is specially important because it can be associated with vector fields on M. Let $\phi \in C^\infty(M)$. The action of G on ϕ is given by

$$(g \cdot \phi)(x) = \phi(g^{-1} \cdot x). \tag{2.2}$$

Now let $X \in \mathfrak{g}$. We have that $\exp(tX)$ is an element of G. Using this exponential map, we can associate the elements of $\mathfrak{g}$ with vector fields on M by the formula:

$$(\tilde{X} \cdot \phi)(x) = \frac{\mathrm{d}}{\mathrm{d}t} \left[\phi(e^{-tX} \cdot x) \right]_{t=0}. \tag{2.3}$$

The map $X \mapsto \tilde{X}$ is a Lie algebra homomorphims, thus, from now on, we will regard $\tilde{X}$ and X as being the same object.

Let $\mathbb{C}[\mathfrak{g}]$ be the complex-valued polynomial algebra over $\mathfrak{g}$[1]. The product $\mathbb{C}[\mathfrak{g}] \otimes \Lambda M$ is the structure that we are after. Note that an element of this space can be seen as a map from the $\mathfrak{g}$ to ΛM. Let $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Lambda M$, the action of G on $\mathbb{C}[\mathfrak{g}] \otimes \Lambda M$ is defined as

$$(g \cdot \alpha)(X) = g \cdot (\alpha(\mathrm{Ad}(g^{-1})X)) = g \cdot \alpha(gXg^{-1}), \tag{2.4}$$

where we used that G is a matrix group to write $\mathrm{Ad}(g)X = g^{-1}Xg$.

Define the set $\Lambda_G M \equiv (\mathbb{C}[\mathfrak{g}] \otimes \Lambda M)^G$ as being the G-invariant subalgebra, *id est*, $(g \cdot \alpha)(X) = \alpha(X)$. But this condition is equivalent to $\alpha(g \cdot X) = g \cdot \alpha(X)$ (the action of G on $\mathfrak{g}$ is the adjoint action), thus $\Lambda_G M$ is really the algebra of G-equivariant forms.

2.1 Derivation and Integration

We can ascribe to the algebra $\mathbb{C}[\mathfrak{g}] \otimes \Lambda M$ a $\mathbb{Z}$-grading. The total degree of a form of $\mathbb{C}[\mathfrak{g}] \otimes \Lambda M$ is two times the degree of the polynomial from $\mathbb{C}[\mathfrak{g}]$ plus the degree of the form from $\Lambda^k M$.

Now let $X \in \mathfrak{g}$ and $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Lambda M$, we define the equivariant exterior derivative by

$$(\mathrm{d}_{\mathfrak{g}}\alpha)(X) \coloneqq \mathrm{d}(\alpha(X)) - X \lrcorner\, \alpha(X), \tag{2.5}$$

where $X \lrcorner$ is the interior product/contraction with the vector field X. The usual exterior derivative increases the degree of the form by one, on the other hand, the interior product decreases by one, but, since we identified the vector fields on M with $\mathfrak{g}$, it will increase by one the degree of the polynomial part of α, thus increasing by two the total degree. That means that the net effect of the interior product on the degree of α is to increase it by one. Henceforth, $\mathrm{d}_{\mathfrak{g}}$ increases by one the total degree of α, like the usual exterior derivative.

Unlike the de Rham operator, d, $\mathrm{d}_{\mathfrak{g}}$ is not nilpotent, actually we have that:

$$\mathrm{d}_{\mathfrak{g}}^2\alpha(X) = X \lrcorner\, \mathrm{d}(\alpha(X)) - \mathrm{d}(X \lrcorner\, \alpha(X)) = -\mathcal{L}_X\alpha(X), \tag{2.6}$$

where $\mathcal{L}_X$ is the Lie derivative along X. But the equivariant forms are the G-invariant elements of $\Lambda_G M$, thus $\mathrm{d}_{\mathfrak{g}}$ is nilpotent in $\Lambda_G M$ and we can write $\mathrm{d}_{\mathfrak{g}} : \Lambda_G^k M \to \Lambda_G^{k+1} M$.

This allows us to define the equivariant cohomology analogously to the usual de Rham cohomology. Indeed, we call the forms of $\Lambda_G M$ that satisfy $\mathrm{d}_{\mathfrak{g}}\alpha = 0$ equivariantly closed

[1] That is, polynomials that take an element of $\mathfrak{g}$ to a complex number.

and the ones with $d_\mathfrak{g}\beta = \alpha$, equivariantly exact. So, the k^{th} **equivariant cohomology** is the quotient space of equivariantly closed forms by the equivariantly exacts, both with degree k.

Recall that we are considering the full exterior algebra, so that each element is a formal sum of various forms of different degrees, hence we may write:

$$\alpha(X) = \sum_{i=0}^{n} \alpha(X)_i, \quad \alpha(X)_i \in \Lambda^i M. \tag{2.7}$$

For equivariantly closed forms, we have a tower of differential equations that imposes relations on the components of the form. These equations are given by

$$X \lrcorner \alpha(X)_i = d\alpha(X)_{i-2}. \tag{2.8}$$

2.1.1 Integration

Let M be a compact oriented manifold, we define the integration of $\alpha \in \Lambda_G M$ over M using the integral of $\alpha(X)$:

$$\left(\int_M \alpha\right)(X) \coloneqq \int_M \alpha(X), \tag{2.9}$$

where, in the right hand side, we are integrating the top component of $\alpha(X)$ which is the one with degree given by the dimension of M. By this definition, the integral is a map from $\Lambda_G M$ to $\mathbb{C}^G$, the algebra of G-invariant polynomials.

2.2 Fiber Bundles and Characteristic Classes

We will need to consider G-equivariant characteristic classes. Those are defined for G-equivariant fiber bundles, which we introduce now. The requirement is that the projection of the total space in the base space needs to be G-equivariant. Let $\pi : E \longrightarrow M$ be a fiber bundle and G a Lie group that acts on both E and M, we say that the bundle is G-equivariant if:

$$g \cdot \pi(x) = \pi(g \cdot x), \ \forall x \in E, \ \forall g \in G. \tag{2.10}$$

Let now E be an equivariant vector bundle, we can define equivariant forms with values on the vector bundle in the same way we defined the ordinary equivariant forms. Indeed, the space of differential forms with values in E is defined by

$$\Lambda^k(M, E) \coloneqq \Lambda^k M \otimes E. \tag{2.11}$$

Thus, the space of equivariant forms with values on E is the G-invariant subspace of $\mathbb{C}[\mathfrak{g}] \otimes \Lambda(M,E)$, which we denote by $\Lambda_G(M,E) \equiv (\mathbb{C}[\mathfrak{g}] \otimes \Lambda(M,E))^G$. This space has the same $\mathbb{Z}$-grading as the ordinary space of equivariant forms.

To define the equivariant covariant derivative, we require that the action of G on $\Lambda(M,E)$ commute with the ordinary covariant derivative, for which the covariant derivative is called G-invariant, namely:

$$[\nabla, \mathcal{L}_X] = 0, \ \forall X \in \mathfrak{g}, \tag{2.12}$$

where $\mathcal{L}_X$ is understood as the extension of the Lie derivative to general sections of the bundle.

Given a G-invariant covariant derivative, we define the G-equivariant covariant derivative by

$$(\nabla_{\mathfrak{g}}\alpha)(X) = (\nabla - X\lrcorner)\alpha(X). \tag{2.13}$$

This definition satisfies the Leibniz rule and preserves $\Lambda^G(M,E)$ since ∇ is taken to be G-invariant.

We can also define the equivariant curvature:

$$\begin{aligned} F_{\mathfrak{g}}(X) &= \nabla^2_{\mathfrak{g}}(X) - \mathrm{d}^2_{\mathfrak{g}}(X), && (2.14)\\ &= \nabla^2_{\mathfrak{g}}(X) + \mathcal{L}_X, && (2.15)\\ &= F - \nabla_X + \mathcal{L}_X && (2.16) \end{aligned}$$

The equivariant curvature is an element of the G-invariant space of differential on M with values on the endomorphisms over E, $\Lambda^G(M, \mathrm{End}(E))$. The equivariant curvature also satisfies the equivariant Bianchi identity:

$$[F_{\mathfrak{g}}(X), \nabla_{\mathfrak{g}}(X)] = 0. \tag{2.17}$$

2.2.1 Characteristic Classes

Having the equivariant covariant derivative and the equivariant curvature, we can define the characteristic classes. They are defined using G-invariant polynomial maps from $\Lambda^G(M, \mathrm{End}(E))$ to $\Lambda^G(M)$. In the equivariant case, the condition on the polynomials is given in terms of the equivariant derivatives, let $P : \Lambda^G(M, \mathrm{End}(E)) \to \Lambda^G(M)$ be an invariant polynomial of degree r, it satisfies:

$$\mathrm{d}_{\mathfrak{g}} P(\alpha) = rP(\nabla_{\mathfrak{g}}\alpha). \tag{2.18}$$

If we take $F_{\mathfrak{g}}$ for α, $\mathrm{d}_{\mathfrak{g}} P(F_{\mathfrak{g}}) = 0$, so that $P(F_{\mathfrak{g}})$ defines an equivariant cohomology. We are mainly interested in the Chern character and in the Euler form which are defined using the trace and the determinant. Both of these polynomial functions are G-invariant and satisfy equation (2.18).

The **equivariant Chern character** is defined by

$$\mathrm{ch}(F_{\mathfrak{g}}) = \mathrm{tr} e^{-F_{\mathfrak{g}}}, \tag{2.19}$$

and, when we have also a G-invariant metric, the **equivariant Euler form** is given by

$$e_{\mathfrak{g}}(F_{\mathfrak{g}}) \coloneqq \mathrm{Pf}(-F_{\mathfrak{g}}) = \sqrt{\det(-F_{\mathfrak{g}})}. \tag{2.20}$$

2.3 BV-AB localization formula

One of the main results of the equivariant cohomology theory is the Berline-Vergne/Atiyah-Bott formula which relates the integral of the top-component of an equivariantly closed form with the integrals of the lower components of the form over the fixed-point surfaces of the action of the compact group G.

The fixed-point set, called $\mathscr{F}$, of the action of G is a submanifold of M. It is given by the set of points $x \in M$ such that $g \cdot x = x$. Regardless from being called "fixed-point", it may be constituted by different components with several dimensions. Recalling that $\exp(tX)$ is an element of G, we may think intuitively that the fixed-point set is the set where $X = 0$, but it is better to take $|X|^2 = 0$ as the definition of the fixed-point set because it is invariant under change of coordinates.

The assumption that G is compact allows us to construct a G-invariant Riemann metric on M. The Riemann metric is used to define the dual form to the vector field which generates the action of G. This form is equivariantly closed and invertible outside the fixed point set. Possessing these properties, it can be used to show that any equivariantly closed form is equivariantly exact outside the fixed point set. This is one of the main facts used in the proof of the localization formula that can be found in [6].

We begin by stating the 0-dimensional version of the localization formula. In this case, there are only isolated fixed points.

If we think the Lie Derivative, with respect to the generators of G, as a linear transformation on the tangent space at each point, it can be shown that this transformation is invertible and anti-symmetric. Let L_p be this transformation at the point p.

If α is an equivariantly closed form on the manifold M of dimension $2n$, and X is a generator of a compact Lie group acting on M, we have that:

$$\int_M \alpha(X) = (2\pi)^n \sum_{p\in\mathscr{F}} \frac{\alpha_0(X)(p)}{\det^{1/2} L_p}, \tag{2.21}$$

where $\mathscr{F}$ is the fixed point set of G. Note that $\alpha_0(X)(p)$ is just a function evaluated at p. The determinant in the denominator can also be thought as the jacobian of the transformation $x \mapsto X(x)$. This shows that the right hand side is related to the integral of $\alpha_0(X)$ times $\delta(X(x))$.

In the general case, we turn our eyes to the normal bundle $\mathcal{N}$ to the fixed point set $\mathscr{F}$. It can be shown that $\mathcal{N}$ is an orientable even dimensional manifold.

Now we state the general result. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\alpha(X)$, $X \in \mathfrak{g}$, be an equivariantly closed form on the $2n$-dimensional manifold M, where G acts. And finally let $\mathscr{F}$ be the fixed point set of the action of G. Then we have that:

$$\int_M \alpha(X) = \int_{\mathscr{F}} (-2\pi)^{\dim(\mathcal{N})/2} \frac{\alpha(X)}{\det^{1/2}\left(R(\mathcal{N}) + \mathcal{L}_X|_{\mathcal{N}}\right)}, \tag{2.22}$$

where R is the curvature two-form. In the normal bundle, the equivariant curvature will simplify to the ordinary curvature plus the lie derivative. Thus (2.22) may also be written using the equivariant Euler form.

2.4 BV-AB for G=$U(1)$

Our main interest is when $G = U(1)$. At first look, this may seem as an almost trivial case, since $U(1)$ is just a one-dimensional Lie Groups. Indeed, there will be some simplifications, many due to the fact that the Lie algebra has only one dimension, and so G have only one generator. On the other hand, the $U(1)$ action is not so restrictive and arises naturally as a symmetry in many geometries, like in the sphere, for example.

We consider the case where the normal bundle can be written as a Whitney sum of complex line bundles: $\mathcal{N} = L_1 \oplus ... \oplus L_k$.

Our generator of the $U(1)$ action is given by

$$\xi = \sum_{i=1}^{k} \epsilon_i \partial_{\varphi_i}, \tag{2.23}$$

here each ∂_{φ_i} rotates L_i.

Like before, the Lie derivative $\mathcal{L}_\xi$ restricted to the normal bundle is a linear transformation which is invertible and skew-symmetric. This implies that the Pfaffian will be just the products of the ϵ_i's. We also write the curvature as the first Chern class. Using that the normal bundle is a Whitney sum, we get the formula:

$$\frac{\det^{1/2}\left(R(\mathcal{N})+\mathcal{L}_X|_{\mathcal{N}}\right)}{(-2\pi)^k}=\prod_{i=1}^{k}\left(c_1(L_i)+\frac{\epsilon_i}{2\pi}\right), \tag{2.24}$$

where we note that $2k=\dim(\mathcal{N})$.

By means of a geometric series trick we can invert both sides of this relation and plug in (2.22). Since we have only one generator, we can omit the argument (ξ) of $\alpha(\xi)$. For $\alpha(\xi)\equiv\Phi$ we finally obtain [7]

$$\begin{aligned}\int_M\Phi=&\sum_{\dim 0}\frac{(2\pi)^n}{\epsilon_1\ldots\epsilon_n}\Phi_0+\sum_{\dim 2}\frac{(2\pi)^{n-1}}{\epsilon_1\ldots\epsilon_{n-1}}\int\left[\Phi_2-\Phi_0\sum_{i=1}^{n-1}\frac{2\pi}{\epsilon_i}c_1(L_i)\right]\\&+\sum_{\dim 4}\frac{(2\pi)^{n-2}}{\epsilon_1\ldots\epsilon_{n-2}}\int\left[\Phi_4-\Phi_2\wedge\sum_{i=1}^{n-2}\frac{2\pi}{\epsilon_i}c_1(L_i)\right.\\&\left.+\Phi_0\sum_{1\leq i\leq j}^{n-2}\frac{(2\pi)^2}{\epsilon_i\epsilon_j}c_1(L_i)\wedge c_1(L_j)\right]+\ldots.\end{aligned} \tag{2.25}$$

2.4.1 Example 1: S^2

Consider the two dimensional sphere S^2. It is naturally described by the usual spherical coordinates θ and φ, with $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi]$. The metric is given by

$$\mathrm{d}s^2=\mathrm{d}\theta^2+\sin\theta^2\mathrm{d}\varphi^2. \tag{2.26}$$

The generator of the $U(1)$ action is simply $\xi=\epsilon\partial_\varphi$. Picturing a rotating sphere, we can see that the fixed points are the poles. Indeed, this is true because the fixed point set equation, $|\xi|^2=0$, gives $\sin\theta^2=0$, thus $\theta=0$ and $\theta=\pi$ are the fixed points.

Let $\Phi=\psi(\theta,\varphi)\mathrm{d}\theta\wedge\mathrm{d}\varphi+\phi(\theta,\varphi)$ be a general polyform which has a 2-form and a 0-form components. Impose that it is equivariantly closed. Applying our main result (2.25) directly gives

$$\int_{S^2}\Phi=\frac{2\pi}{\epsilon}(\phi(0)-\phi(\pi)), \tag{2.27}$$

where the minus sign arises from the ϵ in the denominator which has different signs at each pole because of the orientation of the sphere. This sign difference really comes from the orientation of the normal bundle which must be consistent with the orientation of the sphere. For the sphere, we can actually visualize the normals at the poles and "see" the sign difference, which is carried with ϵ when we evaluated the Lie derivative at the fixed point and got formula (2.24). So saying that ϵ changes sign is not rigorous and this is the reason why the ϵ "inside" the polyform (see below) keeps its sign. This will be a recurrent property in the examples to come and in the application to Romans SUGRA it will be further developed.

The example of the 2d sphere is almost trivial, meaning that we can recover this result easily. The condition that it is equivariantly closed gives

$$\epsilon\psi \mathrm{d}\theta + \mathrm{d}\phi = 0, \tag{2.28}$$

$$\implies \phi = \phi(\theta), \text{ and } \psi = -\tfrac{1}{\epsilon}\partial_\theta\phi. \tag{2.29}$$

Recall that the integration of Φ over S^2 is just the integration of the top form component, hence we have:

$$\int_{S^2}\psi \mathrm{d}\theta \mathrm{d}\varphi = -\frac{2\pi}{\epsilon}\int_0^\pi \partial_\theta\phi \mathrm{d}\theta, \tag{2.30}$$

$$= \frac{2\pi}{\epsilon}(\phi(0) - \phi(\pi)). \tag{2.31}$$

For example, let ψ be $\sin\theta$, so that the integration of Φ gives the volume (the superficial area) of the sphere. It is easy to verify that $\phi = \epsilon\cos\theta$, which, plugging at (2.31), gives the correct result.

2.4.2 Example 2: $S^2 \times S^2 \times S^2$

This next example can also be verified directly, but it illustrates an interesting feature: even if our manifold has a $U(1)^n$ action, we can consider only $U(1)^d$, with $n > d$, to apply the BV-AB formula if we like.

To the metric on $S^2 \times S^2 \times S^2$, we can write just the sum of the metrics on each sphere. Let (θ_i, φ_i) be coordinates of the sphere S_i^2, then we have:

$$\mathrm{d}s^2 = \mathrm{d}\theta_1^2 + \sin\theta_1{}^2\mathrm{d}\varphi_1^2 + \mathrm{d}\theta_2^2 + \sin\theta_2{}^2\mathrm{d}\varphi_2^2 + \mathrm{d}\theta_3^2 + \sin\theta_3{}^2\mathrm{d}\varphi_3^2. \tag{2.32}$$

Our Killing vector is $\xi = \epsilon_1\partial_{\varphi_1} + \epsilon_2\partial_{\varphi_2} + \epsilon_3\partial_{\varphi_3}$ which gives the poles of the spheres as fixed points, $\theta_i = \{0, \pi\}$. For the polyform, we use a generalization of the form used in the previous example:

$$
\begin{aligned}
\Phi &= \psi_1'\psi_2'\psi_3'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&+\epsilon_1\psi_1\psi_2'\psi_3'\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\wedge\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&+\epsilon_2\psi_2\psi_1'\psi_3'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3+\epsilon_3\psi_3\psi_1'\psi_2'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\\
&+\epsilon_2\epsilon_3\psi_2\psi_3\psi_1'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1+\epsilon_1\epsilon_3\psi_1\psi_3\psi_2'\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2+\epsilon_1\epsilon_2\psi_1\psi_2\psi_3'\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&-\epsilon_1\epsilon_2\epsilon_3\psi_1\psi_2\psi_3,
\end{aligned}
\tag{2.33}
$$

where $\psi_i=\psi_i(\theta_i)$ is a smooth function on each sphere and ψ_i' is its derivative with respect to θ.

As mentioned before, now there are $2^3=8$ fixed points corresponding to the poles of the spheres. Applying (2.25) then gives

$$
\int\Phi=-8\pi^3\left(\psi_1(0)-\psi_1(\pi)\right)\left(\psi_2(0)-\psi_2(\pi)\right)\left(\psi_3(0)-\psi_3(\pi)\right). \tag{2.34}
$$

Now we consider that S^2_3 is fixed, thus our generator is $\xi_{12}=\epsilon_1\partial_{\varphi_1}+\epsilon_2\partial_{\varphi_2}$. We can get a equivariantly closed polyform by sending ϵ_3 to zero at (2.33)

$$
\begin{aligned}
\Phi^{12} &= \psi_1'\psi_2'\psi_3'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&-\left(\epsilon_1\psi_1\psi_2'\psi_3'\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\wedge\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3+\epsilon_2\psi_2\psi_1'\psi_3'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\right)\\
&+\epsilon_1\epsilon_2\psi_1\psi_2\psi_3'\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&+0,
\end{aligned}
\tag{2.35}
$$

note that the zero form vanishes, this is not an obstacle to applying the theorem because now the fixed point is 4 copies of S^2_3, each sitting at each pole of the other spheres.

Upon applying (2.25)

$$
\begin{aligned}
\int\Phi^{12} &= 4\pi^2\left(\psi_1(0)-\psi_1(\pi)\right)\left(\psi_2(0)-\psi_2(\pi)\right)\int_{S^2}\psi_3'\mathrm{d}\theta_3\mathrm{d}\varphi_3, && (2.36)\\
&= -8\pi^3\left(\psi_1(0)-\psi_1(\pi)\right)\left(\psi_2(0)-\psi_2(\pi)\right)\left(\psi_3(0)-\psi_3(\pi)\right). && (2.37)
\end{aligned}
$$

We can go even further and set two spheres to be not rotated. This amounts to sending ϵ_2 and ϵ_3 to zero, which leads to the equivariantly closed polyform:

$$
\begin{aligned}
\Phi^{1} &= \psi_1'\psi_2'\psi_3'\mathrm{d}\theta_1\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&-\epsilon_1\psi_1\psi_2'\psi_3'\mathrm{d}\theta_2\wedge\mathrm{d}\varphi_2\wedge\mathrm{d}\theta_3\wedge\mathrm{d}\varphi_3\\
&+0\\
&+0.
\end{aligned}
\tag{2.38}
$$

As expected, the 2-form also vanishes. The fixed point set is now two copies of $S^2 \times S^2$ at each pole of the remaining sphere. Applying (2.25)

$$\begin{aligned}\int \Phi^1 &= -2\pi\left(\psi_1(0)-\psi_1(\pi)\right)\int_{S^2\times S^2}\psi_2'\psi_3'\mathrm{d}\theta_2\mathrm{d}\varphi_2\mathrm{d}\theta_3\mathrm{d}\varphi_3,\\ &= -8\pi^3\left(\psi_1(0)-\psi_1(\pi)\right)\left(\psi_2(0)-\psi_2(\pi)\right)\left(\psi_3(0)-\psi_3(\pi)\right).\end{aligned} \tag{2.39}$$

We recovered the same result in the three different cases. Of course, in general, we want to reduce the integrations as much as possible, specially if we can reduce to a sum over fixed points, but this procedure of sending the rotation parameters to zero is a good way of checking the calculations and will be used latter on. Finally, we note that setting $\psi_i = -\cos(\theta_i)$ makes Φ_6 the volume form of $S^2 \times S^2 \times S^2$ and the BV-AB formula gives the correct result of $(4\pi)^3$.

2.4.3 Example 3: S^4

We present now an example which can also be verified by direct integration, but it has a less trivial equivariantly closed polyform.

The 4-sphere can be embedded in $\mathbb{R}^5$ through the map:

$$(\alpha,\theta,\varphi_1,\varphi_2)\mapsto(\cos\alpha,\sin\alpha\sin\theta\sin\varphi_1,\sin\alpha\sin\theta\cos\varphi_1,\sin\alpha\cos\theta\sin\varphi_2,\sin\alpha\cos\theta\cos\varphi_2),$$

with the angular variables ranging as $\alpha \in [0,\pi]$, $\theta \in [0,\pi/2]$ and $\varphi_i \in [0,2\pi]$. This is not the usual spherical coordinates but rather a mix of spherical and cylindrical coordinate systems. In the chosen coordinates the $U(1)$ action is manifest and at each extreme value of θ there is a 2-sphere, both of these facts can be seen from the line element:

$$ds^2 = \mathrm{d}\alpha^2 + \sin^2\alpha\left(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\varphi_2^2 + \mathrm{d}\varphi_1^2\sin^2\theta\right). \tag{2.40}$$

The generator of the $U(1)$ action is $\xi = \epsilon_1\partial_{\varphi_1} + \epsilon_2\partial_{\varphi_2}$ and the fixed points are given by the equation $|\xi|^2 = \sin^2\alpha\,(\epsilon_1^2\sin^2\theta + \epsilon_2^2\cos^2\theta) = 0$ which, for generic values of the ϵ_i's, has $\alpha = \{0,\pi\}$ as solutions. Thus the fixed points are precisely the poles of S^4.

The polyform is given by

$$\begin{aligned}\Phi =& \sin^3\mathrm{d}\alpha\sin\theta\cos\theta\mathrm{d}\alpha\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi_1\wedge\mathrm{d}\varphi_2\\ &-\frac{1}{2}\sin^3\mathrm{d}\alpha\left(\epsilon_2\mathrm{d}\varphi_1\sin^2\theta+\epsilon_1\mathrm{d}\varphi_2\cos^2\theta\right)\wedge\mathrm{d}\alpha\\ &-\frac{1}{24}\epsilon_1\epsilon_2(\cos(3\alpha)-9\cos\mathrm{d}\alpha).\end{aligned} \tag{2.41}$$

Applying BV-AB (2.25) gives the correct result for the volume of S^4:

$$\int_{S^4} \Phi = \frac{(2\pi)^2}{\epsilon_1 \epsilon_2} \left(\Phi_0(\alpha = 0) - \Phi_0(\alpha = \pi) \right) = \frac{8\pi^2}{3}. \tag{2.42}$$

Where the minus sign comes from the orientation of the normal bundle in the same spirit as was discussed in the S^2 example.

2.4.4 Symplectic Toric Geometry, Duistermaat-Heckman formula and Gaussian Integration

Let M be a symplectic manifold of dimension $2n$. That is, there is a closed non-degenerate 2-form defined on it, this form is called symplectic form and can be written locally as

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i. \tag{2.43}$$

Supposed that there is defined a $U(1)^d$ action on the manifold. Call V^i the d generators of $U(1)^d$ represented at the tangent bundle of M. The action of $U(1)^d$ will be said hamiltonian if there exists a **moment map** μ defined trough:

$$\mu : M \to \mathfrak{u}^*(1)^d \simeq \mathbb{R}^d, \tag{2.44}$$

$$\mathrm{d}\mu^i = V^i \lrcorner\, \omega. \tag{2.45}$$

The existence of the moment map also implies that the action is symplectic in the sense that it preserves the symplectic structure:

$$\mathcal{L}_{V^i}\omega = \mathrm{d}(V^i \lrcorner\, \omega) = \mathrm{d}^2\mu^i = 0, \tag{2.46}$$

where we made use of equation (2.6) and the fact that ω is closed.

From now on, we consider that $d = n$. Our generator of the $U(1)$ action will be $\xi = \epsilon_i V^i = \epsilon_i \partial_{\varphi_i}$ as usual. Thus the polyform $\Phi = \omega + \epsilon_i \mu^i$ will be equivariantly closed. Now we define the exponential of Φ by

$$e^{\Phi} = e^{\epsilon_i \mu^i} \sum_{k=0}^{n} \frac{\omega^k}{k!}, \tag{2.47}$$

where $\omega^2 = \omega \wedge \omega$ and so forth, keep in mind that $\omega^m = 0$ if $m > n$.

It can be verified directly that the exponential of Φ is an equivariantly closed form provided that Φ is equivariantly closed , thus we may apply the BV-AB formula (2.25) to it. The Duistermaat-Heckman formula is the special case when the set of fixed points is discrete:

$$\int_M e^{\Phi} = \frac{1}{n!}\int_M e^{\epsilon_i \mu^i}\omega^n = \sum_{\text{dim }0} \frac{(2\pi)^n}{\epsilon_1 \ldots \epsilon_n} e^{\epsilon_i \mu^i}\Big|_{\text{fixed points}} . \tag{2.48}$$

As an example, consider $M = \mathbb{R}^{2n}$. The symplectic form can be written in polar coordinates as $\omega = r_i \mathrm{d}r_i \wedge \mathrm{d}\varphi_i$. The interior product with ξ is simply $\xi \lrcorner \omega = -\epsilon_i r_i \mathrm{d}r_i$. Then the moment map is minus the Hamiltonian of the harmonic oscillator:

$$\mu = -\frac{\epsilon_i r_i^2}{2} + \mu_0, \tag{2.49}$$

which can be brought to a more familiar form by setting $q_i = \frac{\sqrt{2}}{k_i \sqrt{m_i}} r_i \cos\varphi_i$ and $p_i = \sqrt{2m_i} r_i \sin\varphi_i$.

The fixed point of the $U(1)$ action is just the origin of $\mathbb{R}^{2n}$, *i.e.* $r_i = 0$, thus we can readily apply (2.48)

$$\int_{\mathbb{R}^{2n}} e^{-\frac{\epsilon_i}{2}(x_i^2+y_i^2)+\mu_0} \mathrm{d}^n x \mathrm{d}^n y = n! \frac{(2\pi)^n}{\epsilon_1 \ldots \epsilon_n} e^{\mu_0}. \tag{2.50}$$

Physically, this is the calculation of the partition function of a system of n classical harmonic oscillators. Indeed, the ideas presented in this section have been expanded and generalized to be applied to calculate vacuum-to-vacuum amplitudes in QFT.

Chapter 3

Toric Geometry

Throughout this work, it will be used the mathematical formalism of Toric Geometry. Generally speaking, it studies the situation where a manifold is acted by a torus. It is a very broad subject, but here we will talk about the "Symplectic Toric Geometry" and we shall not worry about the technical and rigorous details of the theory, but rather give a general introduction focused on actual calculations and aimed towards the applications in this work.

We begin by giving a general explanation of the framework and introducing the principal concepts and ideas, then we proceed to explicit examples. First, we deal with the simple $S^2 \times \mathbb{R}^2$ which displays the most important features we want to show. In the next examples, we shall deal with the fibered case and $6d$ manifolds. In principle, there is nothing in our construction that complicates in the fibered case, one just needs to be mindful about the fibration. Additionally, we will not be worried about discussing the moment map in the same way we did here. Our goal will be to calculate the relevant toric data.

3.1 General Theory

Consider the situation described in the end of the last chapter. We had a $2n$-dimensional sympletic manifold with a $U(1)^n$ hamiltonian action defined on it. The group $U(1)^n$ is identified with the real torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$. Wrapping these concepts, we define a **symplectic toric manifold** as a symplectic manifold of dimension $2n$ with a $\mathbb{T}^n$ hamiltonian action.

The symplectic toric manifolds are associated with convex polytopes given by the image of the moment map acting on the symplectic manifold. This polytope is a subset of $\mathbb{R}^n$ which can be characterized as

$$\mathcal{P} = \{x \in \mathbb{R}^n : l_a(x) \equiv x_i v_i^a - \lambda^a \geq 0, \quad \forall a = 1, ..., d\}, \tag{3.1}$$

where $v_a \in \mathbb{Z}^n$ and d is the number of facets of the polytope, where by **facet** we mean a submanifold of $\mathcal{P}$ of codimension 1.

The symplectic manifold is then understood as the total space of the fibration of $\mathbb{T}^n$ over $\mathcal{P}$. This construction is suitable for our work because it naturally encodes the fixed-point sets of the torus action, they are at the boundary of $\mathcal{P}$. In an interior point of the polytope, there is the whole trous fibered over it, but over a facet, there is only $\mathbb{T}^{n-1}$. In general, over a face of codimension m, m 1-torus collapses leaving only $\mathbb{T}^{n-m}$, thus the fixed points of the total torus action are the vertices of $\mathcal{P}$. It should be mentioned that here we make the assumption that exactly n facets meet at each vertex, that is, if p is a vertex point, then $l_a(p) = 0$, for n values of a.

The vectors $v_a \in \mathbb{Z}^n$ are called **toric data** as they encode all the information that we are interested. These vectors are orthogonal to the facets $l_a(x) = 0$ which means that the moment map μ is constant in the direction of v_a when restricted to $\mu^{-1}(l_a = 0)$ and thus specify the torus group that collapses in the facet. We can express this more explicitly, let ∂_{φ_i} be vector fields that generate the torus action on M, then introduce

$$\partial_{\phi_a} = \sum_{i=1}^{n} v_{ai} \partial_{\varphi_i}. \tag{3.2}$$

The vectors ∂_{ϕ_a} are precisely the vectors that fix the facet $l_a(x) = 0$.

Finally, on the vertices we have n vectors v_a which, we assume, constitute a basis for $\mathbb{R}^n$. Note that the v_a's are better seen as elements of $(\mathbb{R}^n)^*$, *id est*, the dual space. Thus it is convenient to introduce a set of vectors $u_a^{(I)} \in \mathbb{Z}^n$ which is a dual basis to the v_a's that are orthogonal to the facets that meet at the vertex I and $a = 1, ..., n$. Hence, if ξ is a toric vector on M represented as

$$\xi = \sum_{i=1}^{n} \epsilon_i \partial_{\varphi_i}, \tag{3.3}$$

then the weights on each vertex I are given by $b_a^{(I)} = u_a^{(I)} \cdot \xi$. Geometrically, the u's may be interpreted as edge vectors that point outwards the vertex I.

3.2 Example 1: $S^2 \times \mathbb{R}^2$

We apply the ideas described above to the simple example of $S^2 \times \mathbb{R}^2$. We take the usual spherical coordinates (θ, φ) to parametrize S^2 and (r, ψ) to be the polar coordinates of $\mathbb{R}^2$. We can take the symplectic form to be

$$\omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi - \mathrm{d}\left(r^2/2\right) \wedge \mathrm{d}\psi, \tag{3.4}$$

where the minus sign is just for convenience.

Given the generators of the toric action ∂_φ and ∂_ψ, the moment map is given by

$$\mu : S^2 \times \mathbb{R}^2 \to \mathcal{P}, \tag{3.5}$$

$$(\theta, \varphi) \times (r, \psi) \mapsto (\cos\theta, r^2/2). \tag{3.6}$$

Thus the image of μ is the polytope $\mathcal{P} = [-1, 1] \times (0, \infty)$. The inverse image of μ of the edge points of $\mathcal{P}$ may take to $\mathbb{R}^2$ sitting at the poles of S^2 or to S^2 sitting at the origin of $\mathbb{R}^2$, depending if we look at $\mu^{-1}(\pm 1, r^2/2)$, with $r \neq 0$, or $\mu^{-1}(\cos\theta, 0)$, with $\theta \neq 0, \pi$, respectively. In the first case, we end up with only ∂_ψ, in the latter one, we get only ∂_φ. In particular, the vertices of $\mathcal{P}$ are the points $(\pm 1, 0)$ which correspond to the poles of the sphere at the origin of the plane, those are exactly the fixed points of the torus action.

This polytope has three facets, thus we have three orthogonal vectors v_a, $a = 1, 2, 3$. We take them to be

$$v_0 = (0, -1); \tag{3.7}$$

$$v_1 = (1, 0); \tag{3.8}$$

$$v_2 = (-1, 0); \tag{3.9}$$

The dual u's are at vertex 1, $u_1^1 = (1, 0)$ and $u_1^2 = (0, -1)$; at vertex 2, $u_2^1 = (-1, 0)$ and $u_2^2 = (0, -1)$. Thus, if $\xi = \epsilon_1 \partial_\varphi + \epsilon_2 \partial_\psi$. Then the weights at vertex 1 are given by $(\epsilon_1, -\epsilon_2)$ and at vertex 2 by $(-\epsilon_1, -\epsilon_2)$.

The important information for us is the weights of ξ at each vertex, considering relative signs. The absolute sign is not important because it is just a matter of convention. For example, we could have taken the polytope facing downward which would be equivalent to take $v_0 = (0, 1)$ and then the weights at each vertex would be (ϵ_1, ϵ_2) and $(-\epsilon_1, \epsilon_2)$. This does not encode any different information from what we had previously done because it came simply from a simple choice. Of course, if we are already working with some specific convention, then we should do our calculations mindful of it so that everything is consistent.

3.3 Toric data for $\mathcal{O}(-p)$ fibered over S^2

First consider a round two-sphere S^2 embedded in $\mathbb{R}^3$ and introduce the two patches $U_{N,S}$ that cover the sphere:

$$
\begin{aligned}
U_N &= \{(x,y,z) \in \mathbb{R}^3 \big| |x|^2 = 1, z > -|\epsilon|\}\,, \\
U_S &= \{(x,y,z) \in \mathbb{R}^3 \big| |x|^2 = 1, z < |\epsilon|\}\,,
\end{aligned}
\tag{3.10}
$$

with ϵ an infinitesimal parameter. The patches intersect at the equator $z = 0$. Let (θ, φ), with $\theta \in [0, \pi]$ and $\varphi \sim \varphi + 2\pi$, be the standard spherical coordinates of S^2. On each patch U_I, $I = N, S$, introduce the coordinates ρ_I and φ_I. They relate to (θ, φ) as $\rho_N = \theta$, $\varphi_N = \varphi$ and $\rho_S = \pi - \theta$, $\varphi_S = -\varphi$. The coordinates (ρ_I, φ_I) are the polar coordinates of $\mathbb{R}^2_I$ and the relations we gave between them and (θ, φ) gives the correct volume form on $\mathbb{R}^2_I$ from the volume form on S^2, namely $\rho_I \mathrm{d}\rho_I \wedge \mathrm{d}\varphi_I$ from $\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi$ which in turn fixes the orientation. Finally, at the equator, we must have:

$$\varphi_N = -\varphi_S = \varphi. \tag{3.11}$$

Now we define the bundle $\mathcal{O}(-p) \to S^2$. The patches of the total space are taken to be $U_I \times \mathbb{C}$ with coordinates $(\rho_I, \varphi_I) \times z_I$, where it is convenient to write $z_I = r_I \mathrm{e}^{\mathrm{i}\psi_I}$. The transition function is $t_{NS} = \mathrm{e}^{-\mathrm{i}p\varphi}$ where p must be an integer[1]. Thus, at the equator, the coordinates of the fiber are related by

$$\psi_N = \psi_S - p\varphi. \tag{3.12}$$

We now proceed to specify the action of $\mathbb{T}^2$ on the bundle. Introduce (ϕ_1, ϕ_2) as the torus coordinates so that the torus action is generated by both ∂_{ϕ_i}. Without loss of generality, we pick $(\phi_1, \phi_2) = (\varphi_N, \psi_N)$. Which gives

$$\partial_{\varphi_N} = \partial_{\phi_1}\,, \qquad \partial_{\psi_N} = \partial_{\phi_2}\,. \tag{3.13}$$

At the south pole, we must have $(\phi_1, \phi_2) = (-\varphi_S, \psi_S + p\varphi_S)$ so everything is consistent with the construction of the bundle. Therefore we have:

$$\partial_{\psi_S} = \partial_{\phi_2}\,, \qquad \partial_{\varphi_S} = -\partial_{\phi_1} + p\partial_{\phi_2}\,. \tag{3.14}$$

[1] In the section 5.4, the number $-p$ will be introduced as the first Chern number of $\mathcal{O}(-p)$, that is, $-p = \int_{S^2} c_1(F)$. It turns out that both definitions are consistent because we can take the curvature to be $F = -p\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi$ [8, 9].

We take the toric data to be minus the vectors ∂_{φ_I} and ∂_{ψ_I} written in the ∂_{ϕ_i} basis. The minus sign is just a convention to agree with the diagram 3.1.

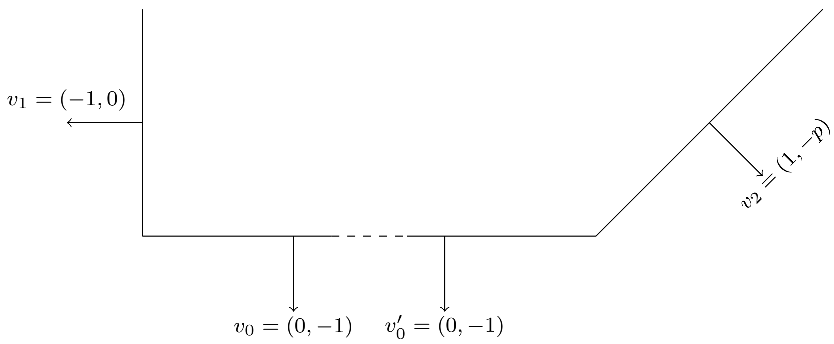

Figure 3.1: The toric diagram for $\mathcal{O}(-p) \to S^2$ with $p > 0$.

Note that for $p < 0$ the diagram looks differently. Moreover for $p = 0$ the toric diagram is simply the open cup, which is the diagram for the first example. The vectors v_0 and v_0' are equal, thus we can disregard v_0' and effectively connect the dashed part of 3.1. Therefore, the toric data is given by

$$\begin{aligned} v_1 &\equiv (-1, 0); \\ v_0 &\equiv (0, -1); \\ v_3 &\equiv (1, -p). \end{aligned} \tag{3.15}$$

On the vertex 1 (north pole), the orthogonal vectors are v_1 and v_0. On the vertex 2 (south pole), they are v_3 and v_0. Thus, the dual basis at vertex 1 is given by $u_1^{(1)} = v_1$ and $u_0^{(1)} = v_0$. At the vertex 2, we have: $u_1^{(2)} = (1, 0)$ and $u_0^{(2)} = (-p, -1)$. Therefore, given a generic toric vector $\xi = (\epsilon_1, \epsilon_2)$, at each vertex the weights will be

$$\begin{aligned} \xi_1 &= (-\epsilon_1, -\epsilon_2); \\ \xi_2 &= (\epsilon_1, -p\epsilon_1 - \epsilon_2). \end{aligned} \tag{3.16}$$

3.4 Toric data for $\mathcal{O}(-p_1, -p_2)$ fibered over $S_1^2 \times S_2^2$

The first example of a 6-dimensional manifold is a complex line bundle fibred over the product of spheres. The arguments of the previous example generalize directly. Consider the patches described in the last section, the patches of $\mathcal{O}(-p_1, -p_2) \to S_1^2 \times S_2^2$ are $U_{IJ} = U_{1,I} \times U_{2,J} \times \mathbb{C}$ where $I, J = N, S$.

There are three relevant intersections of patches. The coordinates of the sphere relate as usual in the equators, $\varphi_{i,N} = -\varphi_{i,S}$. And the transition functions are given by

$$
\begin{aligned}
t_{NN,NS} &= \mathrm{e}^{-\mathrm{i}p_2\varphi_{2,N}}, \\
t_{NN,SN} &= \mathrm{e}^{-\mathrm{i}p_1\varphi_{1,N}}, \\
t_{NN,SS} &= \mathrm{e}^{-\mathrm{i}(p_1\varphi_{1,N}+p_2\varphi_{2N})}.
\end{aligned} \tag{3.17}
$$

Analogously to what we did before, we pick the coordinates of $\mathbb{T}^3$ to be $(\phi_1, \phi_2, \phi_3) = (\varphi_{1,N}, \varphi_{2,N}, \psi_{NN})$. Thus around the north-north pole, the toric data is

$$
\begin{aligned}
\partial_{\varphi_{1,N}} &= \partial_{\phi_1}, \\
\partial_{\varphi_{2,N}} &= \partial_{\phi_2}, \\
\partial_{\varphi_{NN}} &= \partial_{\phi_3}.
\end{aligned} \tag{3.18}
$$

Around the south-south pole, we use $(\phi_1, \phi_2, \phi_3) = (\varphi_{1,S}, \varphi_{2,S}, \psi_{SS})$, which gives

$$
\begin{aligned}
\partial_{\varphi_{1,S}} &= -\partial_{\phi_1} + p_1\partial_{\phi_3}, \\
\partial_{\varphi_{2,S}} &= -\partial_{\phi_2} + p_2\partial_{\phi_3}, \\
\partial_{\varphi_{SS}} &= \partial_{\phi_3}.
\end{aligned} \tag{3.19}
$$

In principle, we would need to analyze the other poles, but it will not yeld any new information. The associated polytope has five facets and four vertices, thus we need only five orthogonal vectors to completely determine the toric data. The relevant data can then be summarized as

$$
\begin{aligned}
v_0 &\equiv \partial_{\psi_{NN}} = (0,0,1), \\
v_1 &\equiv \partial_{\varphi_{1,N}} = (1,0,0), \\
v_2 &\equiv \partial_{\varphi_{2,N}} = (0,1,0), \\
v_3 &\equiv \partial_{\varphi_{1,S}} = (-1,0,p_1), \\
v_4 &\equiv \partial_{\varphi_{2,S}} = (0,-1,p_2).
\end{aligned} \tag{3.20}
$$

The dual basis at each vertex is taken to be

$$ NN: \qquad u_1^{(1)} = v_1, \qquad u_2^{(1)} = v_2, \qquad u_0^{(1)} = (0,0,1); \tag{3.21} $$

$$ SN: \qquad u_1^{(2)} = -v_1, \qquad u_2^{(2)} = v_2, \qquad u_0^{(2)} = (p_1,0,1); \tag{3.22} $$

$$ SS: \qquad u_1^{(3)} = -v_1, \qquad u_2^{(3)} = -v_2, \qquad u_0^{(3)} = (p_1,p_2,1); \tag{3.23} $$

$$ NS: \qquad u_1^{(4)} = v_1, \qquad u_2^{(4)} = -v_2, \qquad u_0^{(4)} = (0,p_2,1). \tag{3.24} $$

From this one can calculate the weights of a generic toric vector $\xi = (\epsilon_1, \epsilon_2, \epsilon_3)$ at each vertex by the formula $\xi_I = (\xi \cdot u_I^1, \xi \cdot u_I^2, \xi \cdot u_I^0)$.

3.5 Toric data for $\mathcal{O}(-p_1) \oplus \mathcal{O}(-p_2)$ fibered over S^2

Our next computation is for $\mathbb{R}^4$ fibered over S^2, where we decompose $\mathbb{R}^4$ into two complex line bundles. Just like before, we use the two patches of S^2 U_N and U_S defined in section 3.3. On each patch, we introduce the coordinates $(\rho_I, \varphi_I) \times (z_{1I}, z_{2I})$, where $I = N, S$ and $z_{iI} = r_{iI} \mathrm{e}^{\mathrm{i}\psi_{iI}}$.

On the overlap of U_N and U_S, the angular coordinates are related as

$$\varphi_N = -\varphi_S, \tag{3.25}$$

$$\psi_{iN} = \psi_{iS} - p_i \varphi_N. \tag{3.26}$$

Now we pick the a basis for the $\mathbb{T}^3$ action:

$$\partial_{\phi_1} \equiv \partial_{\varphi_N}, \tag{3.27}$$

$$\partial_{\phi_2} \equiv \partial_{\psi_{1N}}, \tag{3.28}$$

$$\partial_{\phi_3} \equiv \partial_{\psi_{2N}}. \tag{3.29}$$

This choice is consistent to setting the coordinates of $\mathbb{T}^3$ on the south pole to be $(\phi_1, \phi_2, \phi_3) = (-\varphi_S, \psi_{1S} + p_1\varphi_S, \psi_{2S} + p_2\varphi_S)$. Hence, we obtain:

$$\partial_{\varphi_S} = -\partial_{\phi_1} + p_1 \partial_{\phi_2} + p_2 \partial_{\phi_3}, \tag{3.30}$$

$$\partial_{\psi_{1S}} = \partial_{\phi_2}, \quad \partial_{\psi_{2S}} = \partial_{\phi_3}. \tag{3.31}$$

The polytope associated to $\mathcal{O}(-p_1) \oplus \mathcal{O}(-p_2) \to S^2$ has four faces and two vertices corresponding to the two fixed points. This means that the relevant toric data is given by four vectors which is expected from the computations as $\partial_{\psi_{1S}}$ and $\partial_{\psi_{2S}}$ are the same as their north pole counterparts. Thus, the relevant toric data is given by

$$\begin{aligned} v_1 &\equiv (1, 0, 0); \\ v_2 &\equiv (0, 1, 0); \\ v_0 &\equiv (0, 0, 1); \\ v_3 &\equiv (-1, p_1, p_2). \end{aligned} \tag{3.32}$$

The dual basis at each vertex is then:

$$N:\ u_1^{(1)} = v_1, \qquad u_2^{(1)} = v_2, \qquad u_0^{(1)} = v_0; \tag{3.33}$$

$$S:\ u_1^{(2)} = -v_1, \qquad u_2^{(2)} = (p_1, 1, 0), \qquad u_0^{(2)} = (p_2, 0, 1); \tag{3.34}$$

Similarly, one calculates the weights of $\xi = (\epsilon_1, \epsilon_2, \epsilon_3)$ at both vertexes by the formula $\xi_I = (\xi \cdot u_I^1, \xi \cdot u_I^2, \xi \cdot u_I^0)$.

3.6 Toric data for $\mathcal{O}(-p)$ fibered over $\mathbb{C}P^2$

The computation of the toric data of $\mathbb{C}P^2$ follows the same ideas as the previous ones. Consider $(z_0, z_1, z_3) \in \mathbb{C}^3$, then $\mathbb{C}P^2$ has three charts defined by $U_\mu = \{z_\mu \neq 0\}$. The coordinates on the chart U_μ chart are taken to be $\xi_\mu^\nu \equiv z_\nu / z_\mu$, it is useful to write them as $\xi_\mu^\nu = r_\mu^\nu \mathrm{e}^{\mathrm{i}\varphi_\mu^\nu}$. On the overlap of two charts U_μ and U_ν, we relate the coordinates by

$$\xi_\mu^\lambda = \frac{z_\nu}{z_\mu} \xi_\nu^\lambda. \tag{3.35}$$

Within this construction, we may write the local coordinates of $\mathcal{O}(-p) \to \mathbb{C}P^2$ in the patch U_μ as $\xi_\mu^\nu \times w_\mu$ which we write as $\left(r_\mu^\nu \mathrm{e}^{\mathrm{i}\varphi_\mu^\nu}\right) \times \left(s_\mu \mathrm{e}^{\mathrm{i}\psi_\mu}\right)$. In terms of the angular coordinates, the coordinates of different patches U_μ and U_ν relate as

$$\varphi_\mu^\lambda = \varphi_\mu^\nu + \varphi_\nu^\lambda, \tag{3.36}$$

$$\psi_\mu = \psi_\nu - p\varphi_\mu^\nu. \tag{3.37}$$

Now define the coordinates of the toric action on U_0 to be $\phi_1 \equiv \varphi_0^1$, $\phi_2 \equiv \varphi_0^2$ and $\phi_3 = \psi_0$. Thus the basis of the $\mathbb{T}^3$ on this patch is simply:

$$\frac{\partial}{\partial \varphi_0^1} = \partial_1, \qquad \frac{\partial}{\partial \varphi_0^2} = \partial_2 \qquad \frac{\partial}{\partial \psi_0} = \partial_3, \tag{3.38}$$

where $\partial_i = \partial / \partial \phi_i$.

So that everything is consistent, on U_1, we must have the coordinate change $(\varphi_1^0, \varphi_1^2, \psi_1) \mapsto (\phi_1, \phi_2, \phi_3) = (-\varphi_1^0, \varphi_1^2 - \varphi_1^0, \psi_1 + p\varphi_1^0)$. This leads to the following basis

$$\frac{\partial}{\partial \varphi_1^0} = -\partial_1 - \partial_2 + p\partial_3, \qquad \frac{\partial}{\partial \varphi_1^2} = \partial_2 \qquad \frac{\partial}{\partial \psi_1} = \partial_3, \tag{3.39}$$

By the same reason, on the patch U_2, we must have $(\varphi_2^0, \varphi_2^1, \psi_2) \mapsto (\phi_1, \phi_2, \phi_3) = (\varphi_2^1 - \varphi_2^0, -\varphi_2^0, \psi_2 + p\varphi_2^0)$. This will not lead to any new vector, thus we shall omit the result for the basis.

Collecting the results, the relevant toric data for $\mathcal{O}(-p) \to \mathbb{C}P^2$ is given by

$$\begin{aligned} v_1 &\equiv (1,0,0);\\ v_2 &\equiv (0,1,0);\\ v_3 &\equiv (-1,-1,p);\\ v_4 &\equiv (0,0,1). \end{aligned} \tag{3.40}$$

The toric data for only $\mathbb{C}P^2$ is given by the above v_1, v_2 and v_3 with $p = 0$, thus the toric diagram for $\mathbb{C}P^2$ is a triangle.

The dual basis at each vertex is given by

$$\text{Vertex 1: } u_1^{(1)} = (1,0,0), \qquad u_2^{(1)} = (0,1,0), \qquad u_0^{(1)} = (0,0,1); \tag{3.41}$$
$$\text{Vertex 2: } u_1^{(2)} = (1,-1,0), \qquad u_2^{(2)} = (0,-1,0), \qquad u_0^{(2)} = (0,p,1); \tag{3.42}$$
$$\text{Vertex 3: } u_3^{(3)} = (-1,1,0), \qquad u_2^{(3)} = (-1,0,0), \qquad u_0^{(3)} = (p,0,1). \tag{3.43}$$

3.7 Weights at the vertices from the toric data

In the Toric Geometry framework, the manifold is mapped to a polytope which in our case is a subset of $\mathbb{R}^3$ because we work in six dimensions. This polytope is composed of vertices, edges and faces. The vertices are the fixed points of the $\mathbb{T}^3$ action. On each face, the torus action collapses and this collapse is specified by the toric data, which are the normal vectors to the faces. In our case, each face is associated to a four-dimensional manifold fixed by a $\mathbb{T}^1$ action. The vertices are intersections of three faces and correspond to the fixed points of the total toric action on the manifold.

This framework can be used to directly calculate the weights of the generator of the toric action at the fixed points. Recall the S^2 example, where the generator of rotations is $\xi = \epsilon\partial_\varphi$ and so the weights at the different vertices have opposite signs. This information about the relative signs is what we are looking for. In more complicated geometries, and specially on the fibered cases, it is not easy to obtain this data. Toric geometry then provides a systematic way to do so. In general terms, let there be an isolated fixed point which correspond to a vertex called A on the polytope. Let $u_i^{(A)}$ be the vectors on this vertex such that $u_i^{(A)} \cdot v_j = \delta_{ij}$, where v_j is normal to the face labeled by j, where $i, j = 1, 2, 3$. Of course, there can be more than 3 faces on the polytope, but we consider that the ones that intersect on the vertex A are the faces 1,2,3. Let ξ be the generator of the $U(1)$ action, as usual. Then the weights at this vertex are given by

$$\xi^{(A)} = (\xi \cdot u_1^{(A)}, \xi \cdot u_2^{(A)}, \xi \cdot u_3^{(A)}). \tag{3.44}$$

The vectors $u_i^{(A)}$'s form a dual basis to the v_i's corresponding to the faces that intersect at the vertex A. They are interpreted geometrically as outward pointing vectors coming out of the vertex, but to compute them is an additional step that can be skipped by employing an equivalent method. Let ξ be the R-symmetry vector. We can compute the values of the weights in the direction normal to the face i at the vertex A by the formula:

$$\xi^{(A)i} = \frac{\det(\xi, v_j, v_k)}{\det(v_1, v_2, v_3)} = \frac{\frac{1}{2}\varepsilon^{ijk}\varepsilon_{lmn}v_j^m v_k^n \xi^l}{\det(v_1, v_2, v_3)}, \tag{3.45}$$

where $v_i = v_i^j e_j$ and the indices j, k should be arranged in the correct order, which is given by the orientation of the normal vectors of the polytope inherited from the orientation of the manifold. We will illustrate this with the $\mathbb{R}^2 \times S^2 \times S^2$ example, whose toric diagram is given in figure 6.2.

Let e^i be the dual one-form to the vector v_i. If we ascribe the orientation $e^0 \wedge e^1 \wedge e^2$ to vertex one, we will need vertex 2 to have orientation $e^0 \wedge (-e^1) \wedge e^2 = e^0 \wedge e^4 \wedge e^2$, vertex 3 to have orientation $e^0 \wedge (-e^1) \wedge (-e^2) = e^0 \wedge e^4 \wedge e^3$ and vertex 4 to be oriented as $e^0 \wedge e^1 \wedge e^3$. These flips of sign are a consistency condition that arise from the orientation of the spheres.

Keeping in mind this fixed orientation, if we wish to calculate the weight on the direction of v_0 at each vertex, we can use the formulas

$$\xi \cdot u_0^{(1)} = \frac{\det(\xi, v_1, v_2)}{\det(v_0, v_1, v_2)}, \qquad \xi \cdot u_0^{(2)} = \frac{\det(\xi, v_4, v_2)}{\det(v_0, v_4, v_2)}, \tag{3.46}$$

$$\xi \cdot u_0^{(3)} = \frac{\det(\xi, v_4, v_3)}{\det(v_0, v_4, v_3)}, \qquad \xi \cdot u_0^{(4)} = \frac{\det(\xi, v_1, v_3)}{\det(v_0, v_1, v_3)}. \tag{3.47}$$

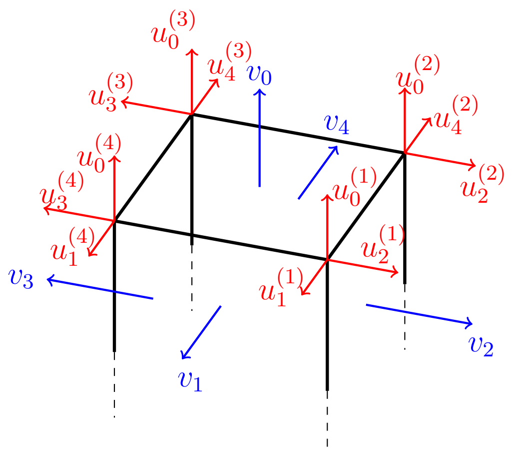

Figure 3.2: Toric diagram for $\mathbb{R}^2 \times S^2 \times S^2$.

To see that this method of calculating the weights by using the determinants yields the same results as the one with the dual vectors is a standard linear algebra calculation. We will omit the superscript (A) on the $u_i^{(A)}$'s for the moment. Write the vectors u_i as $u_i = u_i^j e_j$. The equation $u_i \cdot v_j = \delta_{ij}$ can be seen as a matrix equation where a matrix with entries u_i^j is the inverse of the matrix with entries v_i^j, that is, $u_i^k v_j^k = \delta_{ij}$. We can calculate the inverse of v_i^j by the general formula from the adjugate matrix, the result for u_i^j is

$$u_i^j = \frac{\frac{1}{2}\varepsilon^{imn}\varepsilon_{jkl}v_m^k v_n^l}{\det(v_1, v_2, v_3)}. \tag{3.48}$$

Thus the value of the weight in the direction i is

$$\xi^{(A)i} = \xi \cdot u_i = \frac{\frac{1}{2}\varepsilon^{imn}\varepsilon_{jkl}v_m^k v_n^l \xi^j}{\det(v_1, v_2, v_3)}. \tag{3.49}$$

This is the same as (3.45).

Chapter 4

Euclidean Romans Supergravity

Romans supergravity is a six-dimensional gauged supergravity theory introduced by Romans in the eighties [4]. Latter on, it was recovered as a truncation of type IIA ten-dimensional supergravity on S^4 [10]. Here we work with the euclidean version of the theory [11] which is more suited for holographic purposes.

The bosonic field content of the theory consists of five fields: the metric $g_{\mu\nu}$, the dilaton ϕ, a one-form potential $\mathcal{A}$, a two-form potential B and the R-symmetry $\mathfrak{su}(2)$ gauge field A^i, $i = 1, 2, 3$. The fields strengths are given as $H = \mathrm{d}B$, $\mathcal{F} = \mathrm{d}\mathcal{A} + \frac{2}{3}gB$ and $F^i = \mathrm{d}A^i - \frac{1}{2}g\varepsilon_{ijk}A^j \wedge A^k$, where g is the coupling constant. We also introduce the scalar field $X = \exp\left(-\phi/2\sqrt{2}\right)$.

Exploiting the gauge freedom of the theory, it is possible to annihilate the potential $\mathcal{A}$ [11] such that the B field becomes massive through its relation with $\mathcal{F}$, in fact, we can just forget the $\mathcal{F}$ field and work only with B. Additionally, we can rescale the fields in such way that the coupling constant enters the action as an overall factor and hence can be set to unity. Finally, all the fields are set to be real, with exception of B, which is taken to be purely imaginary.

The bulk action with the mentioned conventions is then [11, 12]

$$\begin{aligned} I_{\text{bulk}} = -\frac{1}{16\pi G_N}\int_{M_6} \Big[& R * 1 - 4X^{-2}\mathrm{d}X \wedge *\mathrm{d}X - \left(\tfrac{2}{9}X^{-6} - \tfrac{8}{3}X^{-2} - 2X^2\right) * 1 \\ & - \tfrac{1}{2}X^{-2}\left(\tfrac{4}{9}B \wedge *B + F^i \wedge *F^i\right) - \tfrac{1}{2}X^4 H \wedge *H \\ & - \mathrm{i}B \wedge \left(\tfrac{2}{27}B \wedge B + \tfrac{1}{2}F^i \wedge F^i\right)\Big] , \end{aligned} \tag{4.1}$$

where G_N is the Newton constant and $*$ denotes Hodge duality and we adopt conventions such that $*1$ is the volume form on M_6. The i factor in the Chern-Simons term is a characteristic of the Eucliden signature.

The $SO(3)$ covariant derivative is given by $D\omega^i \equiv \mathrm{d}\omega^i - \varepsilon_{ijk}A^j \wedge \omega^k$. From the given action, we can derive the equations of motion:

$$\begin{aligned}
\mathrm{d}\left(X^{-1} * \mathrm{d}X\right) =& -\left(\tfrac{1}{6}X^{-6} - \tfrac{2}{3}X^{-2} + \tfrac{1}{2}X^2\right) * 1 \\
& - \tfrac{1}{8}X^{-2}\left(\tfrac{4}{9}B \wedge *B + F^i \wedge *F^i\right) + \tfrac{1}{4}X^4 H \wedge *H \ , \\
\mathrm{d}\left(X^4 * H\right) =& \tfrac{2\mathrm{i}}{9}B \wedge B + \tfrac{\mathrm{i}}{2}F^i \wedge F^i + \tfrac{4}{9}X^{-2} * B \ , \\
D(X^{-2} * F^i) =& -\mathrm{i}F^i \wedge H \ .
\end{aligned} \tag{4.2}$$

And the Einstein equation:

$$\begin{aligned}
R_{\mu\nu} =& 4X^{-2}\partial_\mu X \partial_\nu X + \left(\tfrac{1}{18}X^{-6} - \tfrac{2}{3}X^{-2} - \tfrac{1}{2}X^2\right) g_{\mu\nu} + \tfrac{1}{4}X^4\left(H^2_{\mu\nu} - \tfrac{1}{6}H^2 g_{\mu\nu}\right) \\
& + \tfrac{2}{9}X^{-2}\left(B^2_{\mu\nu} - \tfrac{1}{8}B^2 g_{\mu\nu}\right) + \tfrac{1}{2}X^{-2}\left((F^i)^2_{\mu\nu} - \tfrac{1}{8}(F^i)^2 g_{\mu\nu}\right) \ ,
\end{aligned} \tag{4.3}$$

where $B^2_{\mu\nu} \equiv B_{\mu\rho}B_\nu{}^\rho$ and $H^2_{\mu\nu} \equiv H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}$.

A solution to the equations of motion is supersymmetric provided there exists a non-trivial $SU(2)_R$ doublet of Dirac spinors ϵ_I, $I = 1, 2$, satisfying the Killing spinor and dilatino equations. First, introduce the γ_μ, $\mu = 1, ..., 6$, as Hermitian matrices that generate the Clifford algebra $\mathrm{Cliff}(6, 0)$ in an orthonormal frame as we are working in Euclidean signature. From them, define the chirality operator $\gamma_7 \equiv \mathrm{i}\gamma_{123456}$, which satisfies $\gamma_7^2 = 1$. The $SO(3)$ covariant derivative acting on the spinor is $D_\mu\epsilon_I \equiv (\partial_\mu + \frac{1}{4}\omega_\mu{}^{\nu\rho}\gamma_{\nu\rho})\epsilon_I + \frac{\mathrm{i}}{2}A^i_\mu(\sigma_i)_I{}^J\epsilon_J$, where σ_i are the Pauli matrices and $\omega_\mu{}^{\nu\rho}$ is the Levi-Civita spin connection.

The Killing spinor equation and the dilatino equation are then respectively given by

$$\begin{aligned}
D_\mu\epsilon_I =& \tfrac{\mathrm{i}}{4\sqrt{2}}(X + \tfrac{1}{3}X^{-3})\gamma_\mu\gamma_7\epsilon_I - \tfrac{\mathrm{i}}{24\sqrt{2}}X^{-1}B_{\nu\rho}(\gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\gamma^\rho)\epsilon_I \\
& - \tfrac{1}{48}X^2 H_{\nu\rho\sigma}\gamma^{\nu\rho\sigma}\gamma_\mu\gamma_7\epsilon_I + \tfrac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\gamma^\rho)\gamma_7(\sigma_i)_I{}^J\epsilon_J \ ,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
0 =& -\mathrm{i}X^{-1}\partial_\mu X\gamma^\mu\epsilon_I + \tfrac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\gamma_7\epsilon_I + \tfrac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\gamma_7\epsilon_I \\
& - \tfrac{1}{12\sqrt{2}}X^{-1}B_{\mu\nu}\gamma^{\mu\nu}\epsilon_I - \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\gamma^{\mu\nu}\gamma_7(\sigma_i)_I{}^J\epsilon_J \ .
\end{aligned} \tag{4.5}$$

From now on, we will work on a Abelian truncation of the theory where $A^1 \equiv A^2 \equiv 0$, $A^3 \equiv A$ and $F^3 \equiv F = \mathrm{d}A$. We also impose the symplectic majorana condition: $\varepsilon_I{}^J\epsilon_J = \mathcal{C}\epsilon_I^* \equiv \epsilon_I^c$, where ε_{IJ} is the two-dimensional Levi-Civita symbol and $\mathcal{C}$ denotes the charge conjugation matrix, satisfying $\gamma_\mu^{\mathrm{T}} = \mathcal{C}^{-1}\gamma_\mu\mathcal{C}$. Under these assumptions, the Killing spinor and dilatino equations of ϵ_2 are then the charge conjugated equations for ϵ_1, thus the $SU(2)_R$ doublet becomes $(\epsilon_1, \epsilon_2) = (\epsilon, \epsilon^c)$ and the existence of only ϵ_1 is enough for a solution to have supersymmetry.

4.0.1 AdS_6 vacuum

Setting all the fields, besides the metric, to zero (specially $\phi = 0$ sets $X = 1$) makes all the equations of motion to identically vanish and the Einstein equation becomes:

$$R_{\mu\nu} = -\frac{10}{9} g_{\mu\nu}. \tag{4.6}$$

Recall that a AdS_n space with radius l satisfies:

$$R_{\mu\nu} = \frac{-1}{l^2}(n-1) g_{\mu\nu}. \tag{4.7}$$

Comparing the equations, we conclude that we have an AdS_6 vacuum with AdS radius $l = \frac{3}{\sqrt{2}}$.

To analyze the supersymmetry of the vacuum, we need to look for solutions of the Dilatino and Killing Spinor equations with the values of the fields plugged in. The Dilatino equation is trivially satisfied and the KSE is given simply by

$$\nabla_\mu \epsilon = \frac{\mathrm{i}}{3\sqrt{2}} \gamma_\mu \gamma_7 \epsilon. \tag{4.8}$$

The solution of this equation has 8 integration constants which corresponds to $\mathcal{N} = 4$ supersymmetry [4, 13]

4.1 Bilinears and $SU(2)$ structure

Pick a representation of the γ-matrices such that they are all anti-symmetric and purely imaginary and so the charge-conjugation matrix is given by $\mathcal{C} = -\mathrm{i}\gamma_7$. Using the Killing spinor we may then construct a set of real bilinear differential forms

$$\begin{aligned} &S \equiv \bar{\epsilon}\epsilon\,, \quad P \equiv \bar{\epsilon}\gamma_7\epsilon\,, \quad \xi^\flat \equiv \bar{\epsilon}\gamma_{(1)}\epsilon\,, \quad \tilde{K} \equiv \mathrm{i}\bar{\epsilon}\gamma_{(1)}\gamma_7\epsilon\,, \\ &Y \equiv \mathrm{i}\bar{\epsilon}\gamma_{(2)}\epsilon\,, \quad \tilde{Y} \equiv \mathrm{i}\bar{\epsilon}\gamma_{(2)}\gamma_7\epsilon\,, \quad V \equiv \mathrm{i}\bar{\epsilon}\gamma_{(3)}\epsilon\,, \quad \tilde{V} \equiv \bar{\epsilon}\gamma_{(3)}\gamma_7\epsilon\,, \end{aligned} \tag{4.9}$$

where we have defined $\gamma_{(r)} \equiv \frac{1}{r!}\gamma_{\mu_1\cdots\mu_r}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$, and $\bar{\epsilon} = \epsilon^\dagger$ is the Hermitian conjugate of ϵ.

The one-form $\xi^\flat$ is dual to the Killing vector $\xi^\mu = \bar{\epsilon}\gamma^\mu\epsilon$, furthermore, it was shown in [14] that all the bosonic fields and all the bilinears defined above are annihilated by the Lie derivative in the direction of ξ.

The Killing spinor ϵ defines a $SU(2)$-structure since it is globally defined and $SU(2)$ is its stabilizer. We make a explicit construction of the $SU(2)$-invariant tensors. First decompose ϵ accordingly to its chirality:

$$\epsilon_{\pm} \equiv \tfrac{1}{2}(1 \pm \gamma_7)\epsilon\,, \tag{4.10}$$

and then write

$$\epsilon_+ = \sqrt{S}\sin\vartheta\,\eta_2^*\,, \quad \epsilon_- = \sqrt{S}\cos\vartheta\,\eta_1\,. \tag{4.11}$$

Here ϑ is a function, while η_1, η_2 are two orthogonal unit norm chiral spinors, so that $\bar{\eta}_1\eta_1 = \bar{\eta}_2\eta_2 = 1$ and $\bar{\eta}_2\eta_1 = 0$. To see that $SU(2)$ is the stabilizer of ϵ first note that the stabilizer of each η_i is a different copy of $SU(3)$, to determine the G-structure of the pair (η_1, η_2), we look at the maximal common subgroup of the copies of $SU(3)$ embedded in $SO(6)$, the answer is a $SU(2)$-structure [15]. In other words, the stabilizer of each $\epsilon_\pm$ is $SU(3)$, but the stabilizer of the whole $\epsilon = \epsilon_+ + \epsilon_-$ is $SU(2)$.

The $SU(2)$-invariant set of tensors consist of two real one-forms and three real two-forms given by

$$K_1 - \mathrm{i}K_2 \equiv -\tfrac{1}{2}\varepsilon^{\alpha\beta}\,\eta_\alpha^{\mathrm{T}}\gamma_{(1)}\eta_\beta\,, \quad J_i \equiv -\tfrac{\mathrm{i}}{2}\sigma_i^{\alpha\beta}\bar{\eta}_\alpha\gamma_{(2)}\eta_\beta\,. \tag{4.12}$$

The Riemannian volume form on M_6 is then $\mathrm{vol}_6 = K_1 \wedge K_2 \wedge \frac{1}{2} J_i \wedge J_i$, where here there is no sum on i and this holds for any $i = 1, 2, 3$. We then further distinguish $J \equiv J_3$. The $SU(2)$-structure can also be represented by the K_i's, J and a complex form $\Omega = J_2 + iJ_1$ which are related to the $SU(3)$-structures by [15]

$$J^{\pm} = J \pm K_1 \wedge K_2, \Omega^{\pm} = \Omega \wedge (K_1 \pm iK_2), \tag{4.13}$$

where the superscripts corresponds to each $SU(3)$ associated with ϵ_+ or ϵ_-.One can then verify [14] that the bilinear forms (4.9) may be expressed in terms of this canonically normalized $SU(2)$ structure as

$$\begin{aligned}
&P = -S\cos 2\vartheta\,, \quad \xi^\flat = S\sin 2\vartheta K_1\,, \quad \tilde{K} = -S\sin 2\vartheta K_2\,,\\
&Y = S\left(\cos 2\vartheta K_1 \wedge K_2 - J\right)\,, \quad \tilde{Y} = S\left(-K_1 \wedge K_2 + \cos 2\vartheta J\right)\,,\\
&V = -S\sin 2\vartheta K_1 \wedge J\,, \quad \tilde{V} = -S\sin 2\vartheta K_2 \wedge J\,.
\end{aligned} \tag{4.14}$$

Finally, we record the following differential constraints [14] on the bilinear forms (4.9), which follow from imposing the Supersymmetry conditions (4.4), (4.5)

$$
\begin{aligned}
\mathrm{d}(XS) &= \tfrac{\sqrt{2}}{3}(X^{-2}\tilde{K} - \mathrm{i}\xi \lrcorner B)\,, && (4.15)\\
\mathrm{d}(XP) &= -\tfrac{1}{\sqrt{2}}\xi \lrcorner F\,, && (4.16)\\
\mathrm{d}(X^2\xi^\flat) &= -\tfrac{2\sqrt{2}}{3}X^{-1}\tilde{Y} - \mathrm{i}X^4\xi \lrcorner *H - \sqrt{2}X(PF - \tfrac{2}{3}\mathrm{i}SB)\,, && (4.17)\\
\mathrm{d}(X^{-2}\tilde{K}) &= -\mathrm{i}\xi \lrcorner H\,, && (4.18)\\
\mathrm{d}(X^{-1}Y) &= -\sqrt{2}\tilde{V} + \mathrm{i}(XP)H + \tfrac{1}{\sqrt{2}}X^{-2}(\xi \lrcorner *F + F\wedge\tilde{K})\,, && (4.19)\\
\mathrm{d}(X^{-1}\tilde{Y}) &= \mathrm{i}(XS)H + \mathrm{i}\tfrac{\sqrt{2}}{3}X^{-2}(\xi \lrcorner *B + B\wedge\tilde{K})\,, && (4.20)\\
\mathrm{d}V &= \sqrt{2}(X + \tfrac{1}{3}X^{-3}) * Y + \mathrm{i}\tfrac{\sqrt{2}}{3}X^{-1}(P * B + B\wedge Y)\\
&\quad - \tfrac{1}{\sqrt{2}}X^{-1}(S * F + F\wedge\tilde{Y})\,, && (4.21)
\end{aligned}
$$

where also $\mathrm{d}\tilde{V} = 0$.

Chapter 5

Localizing Romans Supergravity: General Framework

5.1 Equivariantly closed forms

In order to use the fixed point formula (2.25), we need to construct equivariantly closed forms whose top components we want to integrate. It will be presented here 5 of such forms, they will be formed from the fields and bilinears of the theory and equivariantly closedness will then be equivalent to the equations of motion and the supersymmetry conditions.

The first form has the gauge filed strength as top form:

$$\Phi^F \equiv F - \sqrt{2}\,(XP)\,, \tag{5.1}$$

this form is equivariantly closed because of $\mathrm{d}F = 0$ and equation (4.16). Similarly, noting that $\mathrm{d}H = 0$ and using equation (4.18), we conclude that the following form, which has H as top form and is odd degree, is equivariantly closed :

$$\Phi^H \equiv H + \mathrm{i}X^{-2}\tilde{K}\,. \tag{5.2}$$

Now, looking at the equations of motion for F and B, we see that:

$$\mathrm{d}\left(X^{-2} * B + \tfrac{\mathrm{i}}{2} B \wedge B\right) = 0, \tag{5.3}$$

$$\mathrm{d}\left(X^{-2} * F + \mathrm{i}F \wedge B\right) = 0, \tag{5.4}$$

which allows us to construct the following equivariantly closed forms:

$$\Phi^{*B} \equiv \Big[X^{-2} * B + \tfrac{\mathrm{i}}{2} B \wedge B\Big] - \tfrac{3}{\sqrt{2}}[\mathrm{i}X^{-1}\tilde{Y} + (XS)B] - \tfrac{9}{4}\mathrm{i}(XS)^2\,, \tag{5.5}$$

$$\begin{aligned}\Phi^{*F} \equiv\, & [X^{-2} * F + \mathrm{i}F \wedge B] + [\sqrt{2}X^{-1}Y - \sqrt{2}\mathrm{i}(XP)B - \tfrac{3}{\sqrt{2}}(XS)F + 2C] \\ & + 3(XS)(XP)\,,\end{aligned} \tag{5.6}$$

here we have introduced the two-form C via $\tilde{V} = \mathrm{d}_\xi C$, whose (local) existence follows from the equations $\mathrm{d}\tilde{V} = 0 = \xi \lrcorner \tilde{V}$. The equivariant closedness of both of these forms can be checked from the differential constraints on the bilinears.

Finally, we have the equivariantly closed form which calculates the on-shell action. It was presented in [16], but here we adjust the signs as to match our conventions. The mentioned form is given by

$$\Phi^{I_{\text{bulk}}} \equiv \Phi_6^{I_{\text{bulk}}} + \Phi_4^{I_{\text{bulk}}} + \Phi_2^{I_{\text{bulk}}} + \Phi_0^{I_{\text{bulk}}}\,, \tag{5.7}$$

where

$$\begin{aligned}
\Phi_6^{I_{\text{bulk}}} \equiv\, & \tfrac{4}{9}(2+3X^4)X^{-2}\,\mathrm{vol}_6 + \tfrac{1}{3}X^{-2}F \wedge *F + \tfrac{\mathrm{i}}{3} B \wedge F \wedge F\,, \\
\Phi_4^{I_{\text{bulk}}} \equiv\, & -\tfrac{\sqrt{2}}{3}(XP)X^{-2} * F + \tfrac{2\sqrt{2}}{3} X * \tilde{Y} + \tfrac{\sqrt{2}}{3} F \wedge X^{-1} Y \\
& - \tfrac{1}{\sqrt{2}}(XS)F \wedge F - \tfrac{2\sqrt{2}\mathrm{i}}{3}(XP)B \wedge F\,, \\
\Phi_2^{I_{\text{bulk}}} \equiv\, & -\tfrac{2}{3}PY + \tfrac{2\mathrm{i}}{3}(XP)^2 B + 2(XS)(XP)F\,, \\
\Phi_0^{I_{\text{bulk}}} \equiv\, & -\sqrt{2}(XS)(XP)^2\,.
\end{aligned} \tag{5.8}$$

The top component term $\Phi_6^{I_{\text{bulk}}}$ is the action (4.1) evaluated on a solution to the equations of motion and it is identically closed because the manifold is 6-dimensional. The action, including the boundary Gibbons–Hawking–York and holographic counterterms, is

$$I = \frac{\pi^2}{2G_N}\frac{1}{(2\pi)^3}\int_{M_6} \Phi^{I_{\text{bulk}}} + \text{boundary terms}\,. \tag{5.9}$$

5.2 Fixed Point analysis

The $U(1)$ action generated by ξ will have a fixed point set given by the set of points such that $|\xi|^2 = 0$. Denote this fixed point set by $\mathscr{F}$. In principle, $\mathscr{F}$ may have numerous disconnected components, but the dimensions of the components must be 0, 2 or 4. We call $\mathscr{F}_k$ the subset of $\mathscr{F}$ of dimension k.

It can be argued that, since ϵ is a solution of the first order Killing spinor equation, if it vanishes at a point, then it is identically zero in a neighbourhood of that point. This implies that if ϵ is a non-trivial solution of the KSE, then it is nowhere-zero.

In the way that we constructed the $SU(2)$-structure and the bilinears, the norm of the Killing vector may be written as $|\xi| = S|\sin 2\vartheta|$. But since $S = \bar{\epsilon}\epsilon$ is everywhere non-zero, this gives the fixed points as being the points where $\vartheta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\vartheta = 0, \pi$. From (4.11), this shows that $\epsilon = \epsilon_\pm$ on a connected component of the fixed point set. We also get that $S = \pm P$, $\tilde{K} = 0$ and $Y = \pm\tilde{Y}$, when restricted to the fixed point set, from (4.14). Thus we make the partition $\mathscr{F} = \mathscr{F}^+ \cup \mathscr{F}^-$, where $\mathscr{F}^\pm$ is the set where $\epsilon = \epsilon_\pm$.

5.2.1 Action of ξ on the normal bundle and formula for XS

The normal bundle of $\mathscr{F}_{6-2k}$ splits as Whitney sum of complex line bundles: $N\mathscr{F}_{6-2k} = L_1 \oplus ... \oplus L_k$. Let the complex coordinate of L_i be $z_i = |z_i|\mathrm{e}^{\mathrm{i}\varphi_i}$, then the Killing vector may be written as

$$\xi = \sum_{i=1}^{k} \epsilon_i \partial_{\varphi_i}. \tag{5.10}$$

This means that ξ rotates each L_i with the weight ϵ_i. In general, the weights will not be constants and notice that $\mathscr{F}_{6-2k}$ has only k weights associated to it.

Now we look at XS and XP. First notice that the equations (4.16) and (4.15) imply that XP and XS are constant in the fixed point set, in fact, we can obtain a formula for these functions. To get it, we begin by fixing the gauge of A to be the *supersymmetric gauge*, which we define to be $\xi \lrcorner A = \sqrt{2}XP$ [1]. By equation (4.16), in this gauge, the Lie derivative of A in the direction of ξ vanishes: $\mathcal{L}_\xi A = \xi \lrcorner \mathrm{d}A + \mathrm{d}(\xi \lrcorner A) = 0$.

In appendix C it is shown that in this gauge the Killing spinor has charge zero under the R-symmetry vector: $\mathcal{L}_\xi \epsilon = 0$. Here the Lie derivative acting on a spinor is

$$\mathcal{L}_\xi \epsilon = \xi^\mu \nabla_\mu \epsilon + \frac{1}{8}\mathrm{d}\xi^\flat_{\mu\nu}\gamma^{\mu\nu}\epsilon\,, \tag{5.11}$$

$$= \xi^\mu D_\mu \epsilon - \frac{\mathrm{i}}{2}\xi^\mu A_\mu \epsilon + \frac{1}{8}\mathrm{d}\xi^\flat_{\mu\nu}\gamma^{\mu\nu}\epsilon\,. \tag{5.12}$$

To evaluate this expression at the fixed point set, we assume that A is the only diverging field at the fixed points and $\mathrm{d}A$ to be non-singular. Thus, upon using the Killing spinor equation (4.4), which allows us to replace the derivative over ϵ by a linear operator acting on the spinor, the first term is zero at the fixed point set. On the other hand, from (5.10), we get an expression for $\mathrm{d}\xi$ in a local orthonormal frame:

[1] This choice does not fix the gauge completely, but this fact will not be relevant to the rest of the argument.

$$\mathrm{d}\xi^\flat \,|_{\mathcal{N}\mathscr{F}_{6-2k}} = 2 \bigoplus_{i=1}^{k} \begin{pmatrix} 0 & \epsilon_i \\ -\epsilon_i & 0 \end{pmatrix} . \tag{5.13}$$

Hence we obtain:

$$\sum_{i=1}^{k} \epsilon_i \gamma^{(2i-1)2i} \epsilon = \mathrm{i}\sqrt{2}(XP)\epsilon \Big|_{\mathscr{F}_{6-2k}} . \tag{5.14}$$

We can simplify this by noting that the above equation implies that ϵ must be an eigenspinor of each $\gamma^{(2i-1)2i}$. This fact can be checked by picking a representation of the γ-matrices and doing the direct calculation. Now note that $\left[-\mathrm{i}\gamma^{(2i-1)2i}\right]^2 = 1$, thus its eigenvalues are 1 or -1. Hence we conclude that, at a fixed point, we have the projections:

$$-\mathrm{i}\gamma^{(2i-1)2i}\epsilon = \sigma^{(i)}\epsilon\,, \tag{5.15}$$

where $\sigma^{(i)} \in \{\pm 1\}$. Plugging this at (5.14), we get the formula for XS and XP at a fixed point of codimension $2k$:

$$(XS)|_{\mathscr{F}^{\pm}_{6-2k}} = \pm(XP)|_{\mathscr{F}^{\pm}_{6-2k}} = \chi \frac{\sum_{i=1}^{k} \sigma^{(i)}\epsilon_i}{\sqrt{2}}\,, \tag{5.16}$$

where $\chi = \pm$ is the chirality of ϵ.

The $\sigma^{(i)}$'s are deeply related to the chirality of the spinor and to the orientation of the manifold. Recall that $\gamma_7 = \mathrm{i}\gamma^{123456} = (-\mathrm{i}\gamma^{12})(-\mathrm{i}\gamma^{34})(-\mathrm{i}\gamma^{56})$ and that the spinor is necessarily chiral on the fixed point set, that is, $\gamma_7\epsilon = \chi\epsilon$. If the fixed point locus is zero or two dimensional we have that $\chi = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}$. In the first case, equation (5.14) holds for $k = 3$ which implies that we have all three projections (5.15). In the second case, we have two projections (5.15) from equation (5.14) and the other from the fact that the spinor is chiral on the fixed 2-surface. If the fixed point set is a 4-surface, then we have only one projection condition. Suppose without loss of generality that the projection holds for $i = 1$, then we have that $-\gamma^{3456}\epsilon = \eta\epsilon$, with $\eta \in \{\pm 1\}$. Hence we may only write $\chi = \sigma^{(i)}\eta$, where the normal plane lies in the $e^{2i-1} - e^{2i}$ direction. In some particular cases, we may decompose the tangent plane to $\mathscr{F}_4$ in two two-planes, in this cases we may write η as simply the product of the remaining σ's.

The $\sigma^{(i)}$'s also play an important role in ensuring that the overall orientation of the manifold is consistent, given a solution. If there are different connected components of the fixed point set, then the $\sigma^{(i)}$'s specify the relative signs between the normal planes of the different components. This specification is enough to configure different solutions as will be illustrated on the examples.

5.2.2 Form Relations on the Fixed Point Set

The next goal of our analysis is to study how the bilinears and differential forms of the the theory relate at the fixed points, we are specially interested at the form components of the equivariantly closed forms constructed in section 5.1.

First we note that given an equivariantly closed form Φ such that the components Φ_i are non-singular on $\mathscr{F}$, then each component is closed on $\mathscr{F}$. This follows from setting ξ to zero on the equations $\mathrm{d}\Phi_i = \xi \lrcorner \Phi_{i+2}$. This property applies to all the equivariantly closed forms that we have constructed as, like we did in the previous section, we assume all the fields besides A to be non-singular on $\mathscr{F}$.

The zero-form components of all our equivariantly closed forms are composed from XS and XP, thus the analysis of the previous sections fixes the zero-order components. From now on, keep in mind what we have derived before: $S = \pm P$, $\tilde{K} = 0$ and $Y = \pm\tilde{Y}$, when restricted to $\mathscr{F}$. We begin looking to the second-order components. Setting $\xi = 0$ at equation (4.17) gives

$$\left[X^{-1}\tilde{Y} - \mathrm{i}(XS)B + \tfrac{3}{2}(XP)F\right]\Big|_{\mathscr{F}} = \text{exact}\,, \tag{5.17}$$

where the exact form is proportional to $\mathrm{d}(X^2\xi^\flat)$.

Using this relation, we can simplify the 2-form components:

$$\begin{aligned}
\Phi_2^F\Big|_{\mathscr{F}} &= F\,, \quad \Phi_2^{*B}\Big|_{\mathscr{F}} = \tfrac{9}{2\sqrt{2}}\mathrm{i}(XP)F + \text{exact}\,,\\
\Phi_2^{*F}\Big|_{\mathscr{F}} &= \ -3\sqrt{2}(XS)F + 2C + \text{exact}\,, \quad \Phi_2^{I_{\text{bulk}}}\Big|_{\mathscr{F}} = 3(XS)(XP)F + \text{exact}\,,
\end{aligned} \tag{5.18}$$

where the exact part of $\Phi_2^{I_{\text{bulk}}}$ is proportional to $\mathrm{d}(XSX^2\xi^\flat)$ and thus vanishes inside an integral over $\mathscr{F}$ upon using Stoke's Theorem.

The four-form components are more complicated to work with, hence we will focus on the action form $\Phi^{I_{\text{bulk}}}$ and we will derive a weaker expression, in the sense that it is valid only "inside" an integral on $\mathscr{F}$, but this is how we want to use these identities when we apply the fixed point formula (2.25). First note that, on $\mathscr{F}$ we have that:

$$Y \wedge Y|_{\mathscr{F}} = \tilde{Y} \wedge \tilde{Y}\Big|_{\mathscr{F}} = -2\,S * \tilde{Y}\Big|_{\mathscr{F}}\,. \tag{5.19}$$

Now using the identities (4.21) and (5.17) inside a closed integral on $\mathscr{F}$, we get a relation between $\Phi_4^{I_{\text{bulk}}}$, Φ_4^{*B} and $F \wedge F$ in such conditions:

$$\Phi_4^{I_{\text{bulk}}}\Big|_{\mathscr{F}} = -\frac{\mathrm{i}2\sqrt{2}}{9}(XS)\Phi_4^{*B} - \frac{5}{2\sqrt{2}}(XS)F \wedge F\,. \tag{5.20}$$

On the other hand, using the second of the equations of motion in (4.2), one sees that Φ_4^{*B} differs from $F \wedge F$ by an exact form. Thus the final expression for $\Phi_4^{I_{\text{bulk}}}$ on a closed integral on the fixed point set is

$$\Phi_4^{I_{\text{bulk}}}\Big|_{\mathscr{F}} = -\frac{3}{\sqrt{2}}(XS)F \wedge F\,. \tag{5.21}$$

5.3 Localization of the action

We are going to apply the BV-AB localization formula to calculate the on-shell action through the equivariantly closed form $\Phi^{I_{\text{bulk}}}$. But before that, we need to address some points.

The first one is about the boundary of the manifold. M_6 has a UV boundary such that the complete expression of the action will be given by the BV-AB formula plus some boundary terms. However, as argued in [16], assuming that the fixed point set lies in the interior of M_6, these boundary terms will cancel with the boundary terms of the holographical renormalized action. Thus, the formula we get from BV-AB is already the renormalized on-shell action.

Secondly, we use a particular convention for the AdS radius which carries on to a particular normalization of the Newton constant. To make our results more universal, we write everything in terms of the on-shell action of Euclidean AdS_6 which in turn is identified with the free energy of the dual theory on S^5. The expression is taken from [11] and reads as

$$F_{S^5} \equiv I_{AdS_6} = -\frac{27\pi^2}{4G_N}\,. \tag{5.22}$$

Finally, our gauge field is a $SU(2)$ gauge field. The first Chern class of the curvature of such field is well-known to vanish [8]. But, as we are working in the abelian truncation introduced in 4.1, the first Chern class will not be zero. In fact, it will be proportional to F and so we write:

$$c_1(F) \equiv \frac{F}{2\pi}\,. \tag{5.23}$$

Consider a solution to Euclidean Romans Supergravity as outlined in section 4.1 which has a manifold M_6. Then applying the BV-AB localization formula (2.25) with the equivariantly closed form $\Phi^{I_{\text{bulk}}}$ to calculate the on-shell action gives

$$
\begin{aligned}
I = \Bigg\{ & \sum_{\text{dim } 0} \frac{\chi(\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2 + \sigma^{(3)}\epsilon_3)^3}{\epsilon_1\epsilon_2\epsilon_3} \\
& - \sum_{\text{dim } 2} \frac{\chi(\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2)^2}{\epsilon_1\epsilon_2} \int_{\mathscr{F}_2} 3\, c_1(F) + (\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2) \left(\frac{c_1(L_1)}{\epsilon_1} + \frac{c_1(L_2)}{\epsilon_2} \right) \\
& + \sum_{\text{dim } 4} \chi\sigma^{(1)} \int_{\mathscr{F}_4} 3c_1(F) \wedge c_1(F) + 3\sigma^{(1)} c_1(F) \wedge c_1(L_1) + c_1(L_1) \wedge c_1(L_1) \Bigg\} \frac{F_{S^5}}{27} \, .
\end{aligned}
\tag{5.24}
$$

5.4 Gauge Field Flux

In the localized formula for the action (5.24), there appears integrals over $c_1(F)$ and over $c_1(L_i)$. The first kind is physically interpreted as the magnetic fluxes over fixed-point surfaces for the R-symmetry gauge field. It turns out that we can study these integrals to obtain an expression for $\mathscr{F}_2$ and a constraint for $\mathscr{F}_4$.

The Killing spinor is not simple a section of the spin bundle over M_6 because it is charged under the R-symmetry gauge field making it a section of the bundle $\mathcal{S}M_6 \otimes \mathcal{L}^{1/2}$. This can be seen in the local expression of the covariant derivative of the spinor: $D_\mu = \nabla_\mu + \frac{\mathrm{i}}{2}A_\mu$, where A is a connection on the complex line bundle $\mathcal{L}$. The idea then is to use the fact that the Killing spinor is a nowhere-zero section of the bundle $\mathcal{S}M_6 \otimes \mathcal{L}^{1/2}$ and decompose this bundle in fixed-point components and normal bundle components. Having a nowhere-zero section, the top Chern class must vanish. Then we can employ the properties of the Chern class on the decomposed bundle to obtain interesting formulas.

Consider $\mathscr{F}_2$ to be a compact and oriented surface, thus we can take it to be a Riemann surface Σ_g where $g \in \mathbb{Z}_{\geq 0}$ is the genus. Complex line bundles over a Riemann surface are classified by the first Chern number of the bundle which establishes a one-to-one correspondence with $H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}$. Let L be a line bundle over Σ_g such that $\int_{\Sigma_g} c_1(L) = n$, then we may call L as $\mathcal{O}(n)$. In this notation the line bundles that form the normal bundle over Σ_g are written as $L_i = \mathcal{O}(p_i)$, with $p_i = \int_{\Sigma_g} c_1(L_i)$ for $i = 1, 2$. Similarly, the tangent bundle of $\mathscr{F}_2 \cong \Sigma_g$ is $T\Sigma_g = \mathcal{O}(2 - 2g)$, where $2 - 2g$ is the Euler number of Σ_g and we get this result because the Euler class agrees with the top Chern class.

From this it follows that the tangent bundle splits as $TM_6|_{\Sigma_g} \cong \mathcal{O}(2-2g) \oplus \mathcal{O}(p_1) \oplus \mathcal{O}(p_2)$. The chiral spinor bundles $\mathcal{S}^{\pm} \equiv \mathcal{S}^{\pm}M_6|_{\Sigma_g}$ restricted to Σ_g are then

$$
\begin{aligned}
\mathcal{S}^+ \cong \quad & \mathcal{O}(\tfrac{1}{2}p_1 + \tfrac{1}{2}p_2 + (1-g)) \oplus \mathcal{O}(-\tfrac{1}{2}p_1 - \tfrac{1}{2}p_2 + (1-g)) \\
& \oplus \mathcal{O}(-\tfrac{1}{2}p_1 + \tfrac{1}{2}p_2 - (1-g)) \oplus \mathcal{O}(\tfrac{1}{2}p_1 - \tfrac{1}{2}p_2 - (1-g)) \,, \\
\mathcal{S}^- \cong \quad & \mathcal{O}(-\tfrac{1}{2}p_1 + \tfrac{1}{2}p_2 + (1-g)) \oplus \mathcal{O}(\tfrac{1}{2}p_1 - \tfrac{1}{2}p_2 + (1-g)) \\
& \oplus \mathcal{O}(\tfrac{1}{2}p_1 - \tfrac{1}{2}p_2 + (1-g)) \oplus \mathcal{O}(-\tfrac{1}{2}p_1 - \tfrac{1}{2}p_2 - (1-g)) \,.
\end{aligned} \tag{5.25}
$$

We will not prove this formulas here, but rather give a general explanation. Given a Riemann surface and the tangent bundle over it, the spinor bundle is the square-root bundle of the tangent bundle, *i.e.*, $L \otimes L = TM$ [17]. But it follows from (A.45) that $c_1(L \otimes L) = 2c_1(L)$, thus writing $T\Sigma_g = \mathcal{O}(2-2g)$ gives that the chiral spinor bundles will be $\mathcal{O}(\pm(1-g))$ with the sign corresponding to each chirality.

For general dimension manifolds, this construction is not so simple but generalizes in a straight forward manner. If the tangent bundle is decomposable in a direct sum of vector bundles components, then the spinor bundle of a given chirality is a direct sum over the combinations of tensor products of the spinor bundle of each component that give the corresponding chirality. This is precisely what is written in (5.25). Keep in mind that (A.45) implies that $c_1(L_1 \otimes L_2 \otimes L_3) = c_1(L_1) + c_1(L_2) + c_1(L_3)$. Then the first term of (5.25) is really $\mathcal{O}(1-g) \otimes \mathcal{O}(\frac{1}{2}p_1) \otimes \mathcal{O}(\frac{1}{2}p_2)$, but the first Chern number of this line bundle is $\left(\frac{1}{2}p_1 + \frac{1}{2}p_2 + (1-g)\right)$ and so we write it as $\mathcal{O}\left(\frac{1}{2}p_1 + \frac{1}{2}p_2 + (1-g)\right)$. Then each factor on the sums of (5.25) is the square-root/spinor bundle of each factor of $TM_6|_{\Sigma_g} \cong \mathcal{O}(2-2g) \oplus \mathcal{O}(p_1) \oplus \mathcal{O}(p_2)$. Finally, we need to sum over the different combinations that give the positive or negative chiralities as we are constructing chiral spinors on M_6.

The spinor bundle then is decomposed into eight line bundles that correspond to the eight components of a spinor in six dimensions. There are four of each chirality $\chi = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}$ where the sign in each factor can be identified with one of the $\sigma^{(i)}$'s. In special, we claimed in section 5.2.1 that $\sigma^{(3)}$ is effectively the chirality of the spinor along $\mathscr{F}_2 \cong \Sigma_g$ (the two chiral spin bundles on Σ_g being $\mathcal{O}(\pm(1-g)) = \mathcal{O}(\sigma^{(3)}(1-g))$). Therefore, we may write:

$$
\mathcal{O}(\tfrac{1}{2}\sigma^{(1)}p_1 + \tfrac{1}{2}\sigma^{(2)}p_2 + \sigma^{(3)}(1-g)) \,, \tag{5.26}
$$

for all possible choices of $\sigma^{(i)} \in \{\pm 1\}$.

The full Killing spinor ϵ is then a section of $\mathcal{S}M_6 \otimes \mathcal{L}^{1/2}$. Defining the magnetic flux as

$$m \equiv \int_{\Sigma_g} c_1(F) \,, \tag{5.27}$$

this means that the 8 components of ϵ take the form

$$\epsilon : \quad \mathcal{O}(\tfrac{1}{2}\sigma^{(1)}p_1 + \tfrac{1}{2}\sigma^{(2)}p_2 + \sigma^{(3)}(1-g) + \tfrac{1}{2}m) \,. \tag{5.28}$$

On the other hand, the projection conditions (5.15) imply that the spinor has definite $\sigma^{(i)}$'s associated to it, thus it is in precisely one of these 8 components. Finally, recall that ϵ is everywhere nonzero, and if a complex line bundle (5.28) has a nowhere-zero section it must be a trivial line bundle. That is,

$$\mathcal{O}(\tfrac{1}{2}\sigma^{(1)}p_1 + \tfrac{1}{2}\sigma^{(2)}p_2 + \sigma^{(3)}(1-g) + \tfrac{1}{2}m) \cong \mathcal{O}(0) \,, \tag{5.29}$$

which leads to the formula

$$\int_{\Sigma_g} c_1(F) = m = -\sigma^{(1)}p_1 - \sigma^{(2)}p_2 - \sigma^{(3)}(2-2g) \,. \tag{5.30}$$

For a four-dimensional fixed point set, $\mathscr{F}_4$, we can derive a constraint on the integrals of $c_1(F)$ and $c_1(L)$ by a similar argument but cannot fix their expressions. Consider that we have only one connected four-dimensional fixed point set and call it as $\mathscr{F}_4 = B_4$. Now, we have

$$\begin{aligned} \mathcal{S}^+M_6|_{B_4} &\cong \left(\mathcal{S}^+_{B_4} \otimes L_1^{1/2}\right) \oplus \left(\mathcal{S}^-_{B_4} \otimes L_1^{-1/2}\right) \,, \\ \mathcal{S}^-M_6|_{B_4} &\cong \left(\mathcal{S}^+_{B_4} \otimes L_1^{-1/2}\right) \oplus \left(\mathcal{S}^-_{B_4} \otimes L_1^{1/2}\right) \,. \end{aligned} \tag{5.31}$$

Here $\mathcal{S}^\pm_{B_4}$ are the rank two chiral spin bundles of B_4 (which recall we distinguished by $\eta \in \{\pm 1\}$), while the sign of $L^{\pm 1/2}$ is fixed by $\sigma^{(1)} \in \{\pm 1\}$. Again, the analysis at the section 5.2.1 show that ϵ has a fixed $\sigma^{(1)}$ and η due to the projection conditions. Thus, the Killing spinor is a section of precisely one of the $\mathcal{S}^\eta_{B_4} \otimes L_1^{\sigma^{(1)}/2} \otimes \mathcal{L}^{1/2}$, note that this a rank two bundle. Being also nowhere-zero, we have that the second Chern class vanishes:

$$c_2\left(\mathcal{S}^\eta_{B_4} \otimes L_1^{\sigma^{(1)}/2} \otimes \mathcal{L}^{1/2}\right) = 0. \tag{5.32}$$

The constraint will follow from working out this equation. First note that $c_1(\mathcal{S}^\eta_{B_4}) = 0$ because $\mathcal{S}^\eta_{B_4}$ is a $SU(2)$ bundle. Then, using (A.45), we deduce that:

$$c_2\left(\mathcal{S}^\eta_{B_4} \otimes L_1^{\sigma^{(1)}/2} \otimes \mathcal{L}^{1/2}\right) = c_2\left(\mathcal{S}^\eta_{B_4}\right) + \frac{1}{4}\left(\sigma^{(1)} c_1\left(L_1\right) + c_1\left(F\right)\right)^2 = 0, \tag{5.33}$$

where we also used that $c_1\left(L^{\sigma/2}\right) = \frac{\sigma}{2} c_1\left(L\right)$ and $c_1(\mathcal{L}) = c_1(F)$ as F is the curvature of $\mathcal{L}$. To evaluate $c_2\left(\mathcal{S}^\eta_{B_4}\right)$, we make use of the splitting principle and consider $\mathcal{S}^\eta_{B_4}$ to be a sum of line bundles. We write this sum as

$$\mathcal{S}^\eta_{B_4} = \left(M_1^{\sigma_{11}/2} \otimes M_2^{\sigma_{21}/2}\right) \oplus \left(M_1^{\sigma_{12}/2} \otimes M_2^{\sigma_{22}/2}\right), \text{ where} \tag{5.34}$$

$$c\left(B_4^{\mathbb{C}}\right) = (1 + c_1(M_1))(1 + c_1(M_2)), \tag{5.35}$$

and the σ's satisfy: $\sigma_{1i}\sigma_{2i} = \eta$ and $\sigma_{i1} = -\sigma_{i2}$. In the second term, we are considering the Chern class of the complexified tangent bundle of B_4, we have written the decomposition of $\mathcal{S}^\eta_{B_4}$ to be connected explicitly to TB_4. Now we have that $c_2\left(\mathcal{S}^\eta_{B_4}\right) = c_1\left(M_1^{\sigma_{11}/2} \otimes M_2^{\sigma_{21}/2}\right) c_1\left(M_1^{\sigma_{12}/2} \otimes M_2^{\sigma_{22}/2}\right)$, but, just like we did before, $c_1\left(M_1^{\sigma_{1i}/2} \otimes M_2^{\sigma_{2i}/2}\right) = \frac{1}{2}\left(\sigma_{1i} c_1(M_1) + \sigma_{2i} c_1(M_2)\right)$. Then, after simplifying the σ's, we get

$$c_2\left(\mathcal{S}^\eta_{B_4}\right) = \frac{-1}{4}\left(2\eta c_1(M_1) c_1(M_2) + c_1(M_1)^2 + c_1(M_2)^2\right), \tag{5.36}$$

$$= \frac{-1}{4}\left(2\eta e(B_4) + p_1(B_4)\right), \tag{5.37}$$

where $e(B_4)$ and $p_1(B_4)$ are respectively the Euler and first Pontrjagin classes.

Finally, upon integrating equation (5.33) over B_4, we obtain the analogous formula to (5.30)

$$\begin{aligned} 2\eta\chi(B_4) + 3\tau(B_4) &= \int_{B_4} (\sigma^{(1)} c_1(L_1) + c_1(F))^2 \\ &= \int_{B_4} c_1(F) \wedge c_1(F) + 2\sigma^{(1)} c_1(F) \wedge c_1(L_1) + c_1(L_1) \wedge c_1(L_1), \end{aligned} \tag{5.38}$$

where $\chi(B_4), \tau(B_4) \in \mathbb{Z}$ are the Euler number and signature of the oriented four-manifold B_4.

As previously mentioned, this expression may be used to substitute for $\int_{B_4} c_1(F) \wedge c_1(F)$ into the last line of (5.24), but in general still leaves the term with $c_1(F) \wedge c_1(L_1)$, it is better understood as a topological constraint on F. However, it is a useful tool to some special cases of B_4 and will be used in the examples.

5.5 Charge Conjugated Spinor

In the end of the chapter 4.1, we discussed that a solution of Romans SUGRA is supersymmetric if there exists a double of Killing spinors, (ϵ_1, ϵ_2), satisfying the Killing spinor equations (4.4), but we made some assumptions which allows us to write the double as (ϵ, ϵ^c). In the rest of the paper, we used the spinor ϵ, but we could have used ϵ^c to obtain the same results as we are going to show now.

Gauge Field Flux First consider the KSE of ϵ and ϵ^c written schematically as

$$\nabla\epsilon + \tfrac{i}{2}A\epsilon = \mathcal{M}\epsilon + F\mathcal{N}\epsilon, \tag{5.39}$$

$$\nabla\epsilon^c - \tfrac{i}{2}A\epsilon = \mathcal{M}\epsilon^c - F\mathcal{N}\epsilon^c. \tag{5.40}$$

We arrive at these equations by applying the sympletic majorana condition, $\epsilon_1^c = \epsilon_2$ and $\epsilon_2^c = -\epsilon_1$, and the abelian truncation of the gauge field, $A^1 \equiv A^2 \equiv 0$ and $A^3 \equiv A$, to (4.4).

Both equations are the same if we set $A^c \equiv -A$ in the equation for ϵ^c. This means that ϵ^c is a section of $\mathcal{S}M_6 \otimes \mathcal{L}^{-1/2}$ and the consequence is that the flux of F^c has an opposite sign as the flux of F. In the notation introduced in section 5.4, $m^c = -m$.

Bilinears and invariance of the action Now we study how the bilinears (4.9) are different if we construct the using ϵ^c. Without loss of generality, we work on the basis were the γ_μ are anti-symetric and purely imaginary. Thus the charge conjugation matrix is given by $\mathcal{C} = -\mathrm{i}\gamma_7$ and the conjugate spinor is $\epsilon^c = -\mathrm{i}\gamma_7\epsilon^*$. Additionally, we also have that $\bar{\epsilon^c} = \mathrm{i}\epsilon^T\gamma_7$ and $\gamma^*_{\mu_1\dots\mu_n} = (-1)^n\gamma_{\mu_1\dots\mu_n}$.

Consider a general n-form bilinear:

$$\mathcal{B} = \bar{\epsilon}\gamma_{(n)}\epsilon. \tag{5.41}$$

The components of the conjugate bilinear are given by

$$\begin{aligned}
\mathcal{B}^c_{\mu_1\dots\mu_n} &= \bar{\epsilon^c}\gamma_{\mu_1\dots\mu_n}\epsilon^c, && (5.42)\\
&= \mathrm{i}\epsilon^T\gamma_7\gamma_{\mu_1\dots\mu_n}(-\mathrm{i})\gamma_7\epsilon^*, && (5.43)\\
&= (-1)^n\left(\bar{\epsilon}\gamma^*_{\mu_1\dots\mu_n}\epsilon\right)^*, && (5.44)\\
&= \mathcal{B}^*_{\mu_1\dots\mu_n}. && (5.45)
\end{aligned}$$

Employing similar arguments, one can show that for $\mathcal{V} = \bar{\epsilon}\gamma_{(n)}\gamma_7\epsilon$, we have that $\mathcal{V}^c = -\mathcal{V}^*$.

Since the bilinears (4.9) are real, in particular, the above calculations imply that $S^c = S$, $P^c = -P$, $\xi^{\flat c} = \xi^\flat$, $Y^c = -Y$ and $\tilde{Y}^c = \tilde{Y}$. This shows that the polyform (5.7) is the same wether we construct it with (ϵ, A) or (ϵ^c, A^c) and thus the formula for the on-shell action will also be the same. Nevertheless, equation (5.24) can be directly checked by noting that $-\mathrm{i}\gamma^{(2i-1)2i}\epsilon^c = -\sigma^{(i)}\epsilon^c$ and $\gamma_7\epsilon^c = -\chi\epsilon^c$.

Chapter 6

Localizing Romans Supergravity: Examples

Now we are ready to apply the ideas developed so far to different examples. We consider examples where the fixed point set has only one dimension (0, 2 or 4-dimensional). The examples are organized in three groups that have the same $\mathbb{R}^2$ power: $\mathbb{R}^6$, $\mathbb{R}^4 \to \mathscr{F}_2$ and $\mathbb{R}^2 \to \mathscr{F}_4$. We have one of each that has been previously computed in the literature by other methods and thus provide a checking of our formula. In particular, we highlight the hyperbolic black hole discussed in [13], where we have an analytic solution for the fields and for the Killing spinor.

In some examples, we use the formalism of Toric Geometry explained in section 3.7. In this formalism, the manifold is mapped to a polytope which in our case is a subset of $\mathbb{R}^3$ because we work in six dimensions. This polytope is composed of vertices, edges and faces. The vertices are the fixed points of the $\mathbb{T}^3$ action. As explained in section 3.7, on each face, the torus action collapses. In our case, each face is associated to a four-dimensional manifold fixed by a $\mathbb{T}$ action. Thus, on each face, we have one of the projection conditions (5.15). Additionally, we have that at each vertex, there intersects exactly 3 faces. Recall also that, on each face, there is a normal vector, the v's, which specify the collapse of the toric action on the face and, on each vertex, there is a dual basis to the subset of v's, corresponding to the faces that intersect at the given vertex, that we denoted as the u's.

These framework can be used to calculate the action. For example, let there be an isolated fixed point which correspond to a vertex called A on the polytope. Let $u_i^{(A)}$ be the vectors on the vertex such that $u_i^{(A)} \cdot v_j = \delta^i_j$, where v_j is normal to the face labeled by j and thus the projection condition that holds on this face has the sign σ_j, where $i, j = 1, 2, 3$. Of course, there can be more than 3 faces on the polytope, but we consider that the ones that intersect on the vertex A are the faces 1,2,3. Let ξ be the generator of the $U(1)$ action, as usual. Then the contribution of this vertex to the action is

$$\sigma_1\sigma_2\sigma_3 \frac{\left(\sigma_1(\xi \cdot u_1^{(A)}) + \sigma_2(\xi \cdot u_2^{(A)}) + \sigma_3(\xi \cdot u_3^{(A)})\right)^3}{(\xi \cdot u_1^{(A)})(\xi \cdot u_2^{(A)})(\xi \cdot u_3^{(A)})}, \tag{6.1}$$

where we omitted the $F_{S^5}/27$ constant.

The main advantage of using the toric theory is (besides being computationally simple) that we reduce the number of σ's that one would naively expect and we automatically get the relative behavior of the weights at different fixed points. These points will be clear in the relevant examples.

6.1 Hyperbolic Black Hole: $\mathbb{R}^6$ topology

Our first example is the Hyperbolic Black Hole from [13]. This solution is constructed explicitly, we have expressions for the fields and even for the Killing spinor which allows one to check many calculations. Here, we differ from the original paper by the change of variables: $\tau \to n\alpha$ and $\phi \to -\phi$.

- Field Content

  Its metric is given by

$$\mathrm{d}s^2 = \frac{H(r)^{1/2}}{f(r)}\mathrm{d}r^2 + \frac{9f(r)}{2H(r)^{3/2}}n^2\mathrm{d}\alpha^2 + r^2H(r)^{1/2}\mathrm{d}s^2_{\mathbb{H}^4} \tag{6.2}$$

  where

$$H(r) = 1 + \frac{Q}{r^3}, \quad f(r) = -1 - \frac{\gamma}{r^3} + \frac{2}{9}r^2H(r)^2 \tag{6.3}$$

  with Q the charge of the black hole. The scalar field and one-form potential are

$$X(r) = H(r)^{-1/4}, \quad A^3 = 3\sqrt{1 - \frac{\gamma}{Q}}\frac{H(r) - 1}{H(r)}n\mathrm{d}\alpha + \mu n\mathrm{d}\alpha\,. \tag{6.4}$$

  where $\mathbb{H}^4$ is a four-dimensional hyperbolic space with metric

$$\mathrm{d}s^2_{\mathbb{H}^4} = \frac{1}{1+q^2}\mathrm{d}q^2 + q^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\psi^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{6.5}$$

  where the hyperbolic space is realized in a spherical slicing, $q \in [0, \infty]$ and ψ, ϕ, θ are coordinates on S^3.

- Killing Spinor

Supersymmetry require us to set γ to zero in (6.3), it also gives us a Killing spinor which satisfies the dilatino and Killing spinor equations. The Killing spinor have been calculated in the original paper [13], but here we have included a phase $e^{\frac{i}{2}\mu n\alpha}$ to account for the $n\mu d\alpha$ factor in the gauge field.

$$\begin{aligned}\epsilon = \sqrt{\sqrt{q^2+1}+1}&\begin{pmatrix} f_1(r)\left(\kappa_1 e^{-i\phi}+\kappa_2 e^{i\psi}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \\ if_1(r)\left(\kappa_1 e^{-i\phi}-\kappa_2 e^{i\psi}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ f_2(r)\left(\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ if_2(r)\left(-\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \\ -if_2(r)\left(\kappa_1 e^{-i\phi}+\kappa_2 e^{i\psi}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \\ f_2(r)\left(\kappa_1 e^{-i\phi}-\kappa_2 e^{i\psi}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ -if_1(r)\left(\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ f_1(r)\left(-\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \end{pmatrix} \\ +\frac{q}{\sqrt{\sqrt{q^2+1}+1}}&\begin{pmatrix} if_1(r)\left(\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ f_1(r)\left(\kappa_4-\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \\ -if_2(r)\left(\kappa_1 e^{-i\phi}+\kappa_2 e^{i\psi}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \\ f_2(r)\left(\kappa_1 e^{-i\phi}-\kappa_2 e^{i\psi}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ f_2(r)\left(\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \\ if_2(r)\left(-\kappa_4+\kappa_3 e^{i(\psi-\phi)}\right)e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)} \\ -f_1(r)\left(\kappa_1 e^{-i\phi}+\kappa_2 e^{i\psi}\right)\left(e^{-\frac{1}{2}i(\theta+(\mu+1)n\alpha+\psi-\phi)}\right) \\ if_1(r)\left(\kappa_2 e^{i\psi}-\kappa_1 e^{-i\phi}\right)e^{-\frac{1}{2}i(-\theta+(\mu+1)n\alpha+\psi-\phi)} \end{pmatrix},\end{aligned} \tag{6.6}$$

where

$$f_1(r) = \frac{\sqrt[8]{r}\sqrt{2Q+2r^3+3\sqrt{2}r^2}}{\left(Q+r^3\right)^{3/8}}; \tag{6.7}$$

$$f_2(r) = \frac{\sqrt[8]{r}\sqrt{2Q+2r^3-3\sqrt{2}r^2}}{\left(Q+r^3\right)^{3/8}}. \tag{6.8}$$

The 4 κ_a's are integration constants, they show that we have four Killing spinors, which implies that this solution preserve half of the maximal eight supercharges. We studied the four cases where we substitute one of the κ's by the normalization factor 1/4 and set the three others to zero. The spinors with κ_1 or κ_2 different from zero have negative chirality, while for κ_3 or κ_4 non-zero, the spinors have positive chirality.

There are three Killing vectors that can be immediately read off from the metric which we will label by $k_{(1)} := \partial_\alpha$, $k_{(2)} := \partial_\psi$ and $k_{(3)} := \partial_\phi$. The R-symmetry Killing vector is then given as a bilinear in the Killing spinor:

$$\xi = \epsilon^\dagger \Gamma^\mu \epsilon \, \partial_\mu . \tag{6.9}$$

Note that the R-symmetry Killing vector then depends on the choice of integration constant. The expression for all four mentioned κ substitutions is

$$\begin{aligned}\xi =& \epsilon_1 \, k_{(1)} + \epsilon_2 \, k_{(2)} + \epsilon_3 \, k_{(3)} \\ =& \frac{(-1)^{\sin(2\pi(\kappa_3+\kappa_4))}}{n} k_{(1)} + (-1)^{\sin(2\pi(\kappa_2+\kappa_4))} k_{(2)} + (-1)^{\sin(2\pi(\kappa_2+\kappa_3))} k_{(3)} .\end{aligned} \tag{6.10}$$

Note that the coefficients of the R-Symmetry Killing vector precisely correspond to the weights ϵ_i, which we will come back to later. As an example, setting κ_1 to $1/4$ and all the others to zero yields

$$\xi = \frac{1}{n} k_{(1)} + k_{(2)} + k_{(3)} , \tag{6.11}$$

which is equation (2.21) of [13] with ϕ replaced by $-\phi$.

- Fixed Point: Horizon of the Black Hole

  Calculating the norm of the Killing vector

$$\|\xi\|^2 = r^2 H(r)^{1/2} q^2 + \frac{9 f(r)}{2 H(r)^{3/2}} \tag{6.12}$$

and requiring this to be zero to find the fixed point sets suggests that these may only occur for $q = 0$ and r which satisfy $f(r) = 0$, i.e. for r_h the largest root of f, the location of the horizon. This justifies replacing r by a coordinate R depending on the difference $r - r_h$ and rewriting the hyperbolic metric for small q

$$\mathrm{d}s^2_{\mathbb{H}^4} \approx \mathrm{d}q^2 + q^2(\mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\psi^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{6.13}$$

and we introduce $r_1 = q\cos\theta$, $r_2 = q \sin\theta$ and obtain a flat metric around $q = 0$

$$\mathrm{d}s^2_{\mathbb{H}^4} \approx \mathrm{d}r_2^2 + r_2^2 \mathrm{d}\psi^2 + \mathrm{d}r_1^2 + r_1^2 \mathrm{d}\phi^2 . \tag{6.14}$$

We want to write the entire metric in the limit where q is small and r close to r_h and bring the α and r part in a polar coordinates shape. This is achieved by introducing the new radial coordinate

$$R = 2^{3/8}3^{1/4}\frac{r_h^{1/4}}{(\sqrt{2}r_h - 2)^{1/2}}(r - r_h)^{1/2} \tag{6.15}$$

and one obtains equation (2.23) in [13]. Requiring that α smoothly closes off at the horizon, i.e. requiring $f(r_h) = 0$, it must have period β with $\beta = \frac{2\pi}{n(\sqrt{2}r_h - 2)}$ and as the periodicity of α needed to also be 2π, one can rewrite r_h as $r_h = \frac{1+2n}{\sqrt{2}n}$. Note also that the smoothness condition results in a relation between Q and r_h which has been used when rewriting the metric. We can then rewrite the metric (2.23) in [13] of the six-dimensional space as

$$\mathrm{d}s^2 \approx \mathrm{d}R^2 + R^2\mathrm{d}\alpha^2 + H(r_h)^{1/2}r_h^2\mathrm{d}s^2_{\mathbb{H}^4} \tag{6.16}$$

(compare (2.23) in [13] using (2.25) and $\beta = 2\pi n$ and $\tau = n\alpha$).

We also need the gauge field (6.4) to be non-singular at the horizon. Evaluating at $r = r_h$ leads to an expression which fixes the value of μ to be

$$\mu = \frac{1-n}{n}. \tag{6.17}$$

- Charges of Rotations

  In order to find an expression for (XS) at the fixed point, the values for $\sigma^{(i)}$ from the projection conditions (5.15) are required. The direct way to find them is to calculate the projection conditions explicitly by (5.15). Note here that the labeling of the $\sigma^{(i)}$ in (5.15) refers to a particular choice of coordinate basis near the origin. The natural choice of frame for the given metric is the one invariant under the Lie derivative, i.e. $\mathcal{L}_\xi e^j = 0$. In this frame the coordinates q and θ are mixed in such a way that they define radial coordinates for the $\mathbb{C}_i$ planes as laid out in the near-fixed point analysis above. In particular, the ordering of the coordinates does not coincide with the one suggested by the projection conditions (5.15). In our natural frame, the projection conditions do not take the exact form as in (5.15). We can find their form and extract the $\sigma^{(i)}$ by making use of the Lie derivative in the direction of the Killing vectors of our solution evaluated at their fixed point sets.

  The R-Symmetry Killing vector ξ is composed of three individual vectors, each of which is a Killing vector of the solution on its own. Those vectors are $k_{(1)} := \partial_\alpha$, $k_{(2)} := \partial_\psi$ and $k_{(3)} := \partial_\phi$. It is straightforward to verify that each of these has a four-dimensional fixed point set, defined by taking the limits $r \to r_h$, $\theta \to \pi/2$ and $\theta \to 0$ respectively. Note that ∂_ψ and ∂_ϕ also have two-dimensional fixed point sets defined by $q \to 0$. The fixed point set of ξ is then determined by the intersection of the three four-dimensional sets, by taking the simultaneous limit $r \to r_h$ and $q \to 0$ which is the origin of the Hyperbolic Black Hole.

Pick one of the $k_{(i)}$'s defined above and consider its 4-dimensional fixed point set. From the discussion in section 5.2.1, we have that one of the projection conditions (5.15) hold, namely the condition i. On the other hand, by equation (5.31) and the discussion there, the spinor restricted to the 4d fixed point set can be seen as the tensor product between a spinor on the four-dimensional space and a spinor on the normal two-dimensional space which close to the origin is a flat $\mathbb{R}^2$. The $k_{(i)}$ rotates this $\mathbb{R}^2$. As is well known, a spinor on flat $\mathbb{R}^2$ can only have charge under rotation $\pm\frac{1}{2}$ and it is precisely the sign of this charge that corresponds to the $\sigma^{(i)}$ from the projection condition associated to $k_{(i)}$. In other words, we have that

$$\mathcal{L}_{k_{(i)}}\epsilon\Big|_{k_{(i)}=0} = \frac{1}{8}\mathrm{d}(k^\flat_{(i)})_{\mu\nu}\gamma^{\mu\nu}\epsilon = \frac{\mathrm{i}}{2}\sigma^{(i)}\epsilon. \tag{6.18}$$

Having this equation at hand, we can compute the $\sigma^{(i)}$'s by evaluating the Lie derivative of ϵ. But, since we are working in the invariant frame, the Lie derivative acts just as a partial derivative which greatly simplifies the calculations. One can then check that the $\sigma^{(i)}$'s computed in this manner satisfy the above equation even if the frame that we represent the $\gamma^{\mu\nu}$ is not a frame in which the six-dimensional space splits as three two-planes.

The values for $\sigma^{(i)}$ for different choices of Killing spinor are given in the following table:

	$\kappa_1 = 1/4$	$\kappa_2 = 1/4$	$\kappa_3 = 1/4$	$\kappa_4 = 1/4$
$\sigma^{(1)}$	-1	-1	-1	-1
$\sigma^{(2)}$	-1	1	1	-1
$\sigma^{(3)}$	-1	1	-1	1
χ	-1	-1	1	1

Comparing this with the weights we found earlier, it can be observed that the signs of the projection conditions and the signs of the weights for each choice of Killing spinor are directly related by $\frac{\epsilon_i}{|\epsilon_i|} = \chi\,\sigma^{(i)}$. It is this observation which in the end will allow us to conclude that the action for this solution is independent of the choice of Killing spinor. Having the $\sigma^{(i)}$, we can calculate the XS function at the fixed point using (5.16)

$$
\begin{aligned}
XS|_{r=r_h,\, q=0} =& \frac{\chi\sigma^{(1)}\epsilon_1 + \chi\sigma^{(2)}\epsilon_2 + \chi\sigma^{(3)}\epsilon_3}{\sqrt{2}} \\
=& \frac{|\epsilon_1| + |\epsilon_2| + |\epsilon_3|}{\sqrt{2}} \\
=& \frac{1+2n}{n\sqrt{2}},
\end{aligned}
\tag{6.19}
$$

where in the second to last line we have used the relation between the sign of the weights and the σ_i. Note that the expression for XS at the fixed point is independent of the choice of Killing spinor.

- Applying the fixed point formula

  In this example, there is only one isolated fixed point. The weights are the components of (6.10). Note that, for each κ substitution, the product $\epsilon_1\epsilon_2\epsilon_3$ will be $\frac{1}{n}$. In particular, it is always positive as a consequence of the product of the $\sigma^{(i)}$ being equal to the chirality χ. Finally, collecting our results at the formula (5.24), gives

$$
I = \frac{\chi(\sigma^{(1)}\epsilon_1 + \sigma^{(2)}\epsilon_2 + \sigma^{(3)}\epsilon_3)^3}{\epsilon_1\epsilon_2\epsilon_3}\frac{F_{S^5}}{27} = \frac{(2n+1)^3}{27n^2}F_{S^5}. \tag{6.20}
$$

  This result agrees with the free energy given by equation (2.37) of [13].

6.1.1 Wilson Loop

Introduce the polyform

$$
\Phi^{WL} = X^{-2}K_1 \wedge K_2 + iB - \tfrac{3}{\sqrt{2}}XS. \tag{6.21}
$$

This polyform is not equivariantly closed because the top-component is not closed. Nevertheless, when restricted to a 2-dimensional surface, it is closed.

We want to apply BV-AB to integrate it over a surface Σ that is a submanifold of M_6. Consider the case where M_6 has topology $\mathbb{R}^6$. The fixed point is just the origin of $\mathbb{R}^6$. Construct the surface Σ_i as having the $\mathbb{R}^2_i$ plane as the tangent plane and being located at the origin of the remaining $\mathbb{R}^4$. The Killing vector restricted to Σ_i reduces to $\xi|_{\Sigma_i} = \epsilon_i\partial_{\varphi_i}$ (no sum). To apply BV-AB, we need to consider the normal bundle to the fixed point inside only Σ_i, not the entire M_6, thus the normal plane is just $\mathbb{R}^2_i$, not $\mathbb{R}^6$. Applying (2.22), we get

$$\int_{\Sigma_i}\Big(X^{-2}K_1\wedge K_2+iB\Big)=-\frac{2\pi}{\epsilon_i}.\frac{3}{\sqrt{2}}\chi\frac{\sigma^{(1)}\epsilon_1+\sigma^{(2)}\epsilon_2+\sigma^{(3)}\epsilon_3}{\sqrt{2}}$$
$$=-\frac{3\pi\chi(\sigma^{(1)}\epsilon_1+\sigma^{(2)}\epsilon_2+\sigma^{(3)}\epsilon_3)}{\epsilon_i}, \tag{6.22}$$

where we used (5.16) because Σ_i was constructed to be in the origin $\mathbb{R}^2_j\oplus\mathbb{R}^2_k,\ j,k\neq i$. This agrees with formula (3.19) of [14].

Now we check for the Hyperbolic Black Hole discussed in the text. Plugging for XS given by (6.19) gives

$$\int_{\Sigma_i}\Big(X^{-2}K_1\wedge K_2+iB\Big)=-\frac{3\pi(1+2n)}{n\epsilon_i}. \tag{6.23}$$

The labelling of the the planes is analogous to that in section 4.1.1. So, for the first surface, the one which wraps the r and α directions, we have:

$$\int_{\Sigma_1}\Big(X^{-2}K_1\wedge K_2+iB\Big)=(-1)^{1+\sin(2\pi(\kappa_1+\kappa_2))}3\pi(1+2n), \tag{6.24}$$

which agrees with equation (2.39) of [13]. For the others, the calculation does not seems to make sense. The point is that the conformal boundary is at $r\to\infty$, but the surfaces we used are defined at $r=r_h$, so we are not calculating the WL of the dual theory. Also, the auxiliary radial coordinates of the Hyperbolic BH does not go to the conformal boundary. Perhaps, it is a matter of introducing a frame to the solution which decomposes the space in three 2-planes and such that we can get to the conformal boundary by taking any radial coordinate of any of the 2-planes to infinity.

6.2 $\mathcal{O}(-p_1)\oplus\mathcal{O}(-p_2)$ fibered over Σ_g

Our first example of fibration is $\mathbb{R}^4\to\Sigma_g$. This example follows closely the discussion of section 5.4 and we refer to it for more details. The fixed point set is the Riemann surface Σ_g and the normal bundle is decomposed in a sum of two complex line bundles $\mathcal{O}(-p_1)$ and $\mathcal{O}(-p_2)$. In this case, we have precisely an exact formula for the flux of F trough Σ_g given by equation (5.30) which pluggin at (5.24) gives the on-shell action:

$$I=\frac{F_{S^5}\sigma^{(3)}\chi_{\mathbb{R}^4}(\epsilon_1+\chi_{\mathbb{R}^4}\epsilon_2)^2}{27\epsilon_1^2\epsilon_2^2}\Big[6\sigma^{(3)}(1-g)\epsilon_1\epsilon_2$$
$$+p_1\sigma^{(1)}\epsilon_2(\chi_{\mathbb{R}^4}\epsilon_2-2\epsilon_1)+p_2\sigma^{(1)}\epsilon_1(\epsilon_1-2\chi_{\mathbb{R}^4}\epsilon_2)\Big], \tag{6.25}$$

where $\chi_{\mathbb{R}^4} = \sigma^{(1)}\sigma^{(2)}$.

For the case of a trivial fibration, we can calculate the on-shell action by simply setting $p_1 = p_2 = 0$ in the above formula. The result is

$$I = \frac{2}{9}(1-g)F_{S^5}\chi_{\mathbb{R}^4}\frac{(\epsilon_1 + \chi_{\mathbb{R}^4}\epsilon_2)^2}{\epsilon_1\epsilon_2}. \tag{6.26}$$

This formula is then very reminiscent of the known one in four dimensions which should be expected. In particular, if $\sigma^{(1)} = \sigma^{(2)}$ and $\epsilon_1 = \epsilon_2$, then:

$$I = -\frac{8}{9}(1-g)F_{S^5}\sigma^{(3)}\chi, \tag{6.27}$$

which should be compared with equation (4.31) from [18].

6.2.1 $O(-p_1) \oplus \mathcal{O}(-p_2)$ fibered over S^2

We consider now the case $O(-p_1) \oplus \mathcal{O}(-p_2) \to S^2$. Besides the $U(1)$ action on the line bundles, we have a natural $U(1)$ action on the sphere corresponding to a rotation around its axis. The R-symmetry vector field is taken to be

$$\xi = \epsilon\partial_\varphi + \epsilon_1\partial_{\varphi_1} + \epsilon_2\partial_{\varphi_2}\,, \tag{6.28}$$

where the φ_i rotate the $\mathbb{R}^2_i$ plane and φ rotates S^2. The fixed point set consists of two isolated points located at the intersection of the planes and the two poles of the sphere.

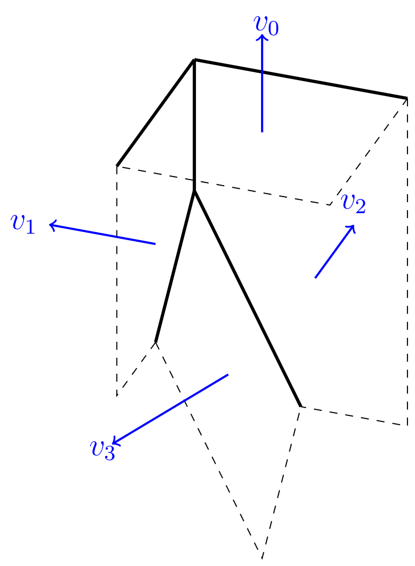

Figure 6.1: Toric diagram for $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to S^2$.

We calculate the weights of the Killing vector ξ at each pole using the toric data of section 3.5 and the simple method mentioned in the very end of that section. The results are

$$\xi_N = (\epsilon, \epsilon_1, \epsilon_2), \qquad\qquad \xi_S = (-\epsilon, \epsilon_1 + p_1\epsilon, \epsilon_2 + p_2\epsilon). \tag{6.29}$$

Having the weights at hand, we can naively write down the on-shell action:

$$I = \frac{F_{S^5}}{27}\left\{\frac{\chi_N(\sigma_N^{(1)}\epsilon + \sigma_N^{(2)}\epsilon_1 + \sigma_N^{(3)}\epsilon_2)^3}{\epsilon\epsilon_1\epsilon_2} - \frac{\chi_S(-\sigma_S^{(1)}\epsilon + \sigma_S^{(2)}(\epsilon_1 + p_1\epsilon) + \sigma_S^{(3)}(\epsilon_2 + p_2\epsilon))^3}{\epsilon(\epsilon_1 + p_1\epsilon)(\epsilon_2 + p_2\epsilon)}\right\}. \tag{6.30}$$

It turns out that the σ's are not all different. As explained in the beginning of this chapter, we have only four different ones in this case, because the toric polytope has only four facets. The σ's that are identified are the ones that give the projections on the planes. Thus, $\sigma_N^{(2)} = \sigma_S^{(2)} \equiv \sigma^{(2)}$ and $\sigma_N^{(3)} = \sigma_S^{(3)} \equiv \sigma^{(3)}$. The action then becomes

$$\begin{aligned} I = \frac{F_{S^5}}{27\epsilon}\sigma^{(2)}\sigma^{(3)}\Bigg(&\frac{\sigma_N^{(1)}(\epsilon_1\sigma^{(2)} + \epsilon_2\sigma^{(3)} + \sigma_N^{(1)}\epsilon)^3}{\epsilon_1\epsilon_2} \\ &- \frac{\sigma_S^{(1)}(\epsilon_1\sigma^{(2)} + \sigma^{(3)}(\epsilon_2 + p_2\epsilon) + p_1\sigma^{(2)}\epsilon - \sigma_S^{(1)}\epsilon)^3}{(\epsilon_1 + p_1\epsilon)(\epsilon_2 + p_2\epsilon)}\Bigg). \end{aligned} \tag{6.31}$$

On the remaining σ, $\sigma^{(1)}$, there are two possibilities, $\sigma_N^{(1)} = \pm\sigma_S^{(1)}$. In the positive one, the chirality of the spinor is the same on the whole sphere while in the other it is reversed at the poles. We want to take the limit $\epsilon \to 0$ to compare with (6.25), but this limit is only well-defined if $\sigma_N^{(1)} = \sigma_S^{(1)}$. This fact is in accordance with the result that the Killing spinor has definite chirality at the connected components of the fixed point set.

Then setting $\sigma_N^{(1)} = \sigma_S^{(1)}$ and taking $\epsilon \to 0$, we get

$$\begin{aligned} I = \frac{F_{S^5}\sigma_N^{(1)}\chi_{\mathbb{R}^4}(+\epsilon_1 + \chi_{\mathbb{R}^4}\epsilon_2)^2}{27\epsilon_1^2\epsilon_2^2}&\Big[6\sigma_N^{(1)}\epsilon_1\epsilon_2 \\ &+\epsilon_2 p_1\sigma_N^{(2)}(-2\epsilon_1 + \chi_{\mathbb{R}^4}\epsilon_2) + p_2\sigma_N^{(2)}\epsilon_1(\epsilon_1 - 2\chi_{\mathbb{R}^4}\epsilon_2)\Big]. \end{aligned} \tag{6.32}$$

This result matches with (6.25) upon setting $g = 0$ there and identifying $\sigma_N^{(1)} \equiv \sigma^{(3)}$, $\sigma_N^{(2)} \equiv \sigma^{(1)}$ and $\sigma_N^{(3)} \equiv \sigma^{(2)}$.

We can also compute the magnetic charge threading through the two-sphere using the polyform in (5.1). We find

$$\begin{aligned} m &= \frac{1}{2\pi}\int_{S^2}\Phi^F \\ &= -\frac{\sigma_N^{(1)}\epsilon+\sigma_N^{(2)}\epsilon_1+\sigma_N^{(3)}\epsilon_2}{\epsilon}+\frac{-\sigma_S^{(1)}\epsilon+\sigma_S^{(2)}(\epsilon_1+p_1\epsilon)+\sigma_S^{(3)}(\epsilon_2+p_2\epsilon)}{\epsilon} \\ &= -\chi(S^2)\sigma^{(1)}+p_1\sigma^{(2)}+p_2\sigma^{(3)}, \end{aligned} \tag{6.33}$$

where in last line we already considered the condition that the chirality of the sphere is fixed in order to compare with the formula for the flux (5.30), where there is a precise agreement.

The results for the trivial fibration case are obtained by setting $p_1 = p_2 = 0$, but here we will give only the one where the chirality of the spinor is the same along the sphere, that is, $\sigma_N^{(1)} = \sigma_S^{(1)}$. The result is

$$I = \frac{2F_{S^5}\chi_{\mathbb{R}^4}}{27\epsilon_1\epsilon_2}\Big(3(\epsilon_1+\chi_{\mathbb{R}^4}\epsilon_2)^2+\epsilon^2\Big)\,. \tag{6.34}$$

6.3 $\mathbb{R}^2$ fibered over B_4

Unlike in the case of a fixed 2-dimensional surface, we do not have a closed formula for the integrals of $c_1(F)$, just the constraint (5.38). It is not worthy to use this constraint to eliminate the integral of $c_1(F)^2$ because it still leaves the integral of $c_1(F)\wedge c_1(L_1)$ and does not give any interesting insight, unless we are in the special case of a trivial fibration. If the fibration is trivial, we have that $c_1(L_1) = 0$. Thus the identity (5.38) becomes a completely topological formula for the integral of $c_1(F)^2$ which upon plugging in the action gives

$$I = \frac{1}{9}F_{S^5}\chi\sigma^{(1)}(2\eta\chi(B_4)+3\tau(B_4)). \tag{6.35}$$

6.3.1 $\mathcal{O}(-p_1,-p_2)$ fibered over $\Sigma_{g_1}\times\Sigma_{g_2}$

The next example is $\mathcal{O}(-p_1,-p_2)\to\Sigma_{g_1}\times\Sigma_{g_2}$, that is a line bundle fibred over $B_4 = \Sigma_{g_1}\times\Sigma_{g_2}$ where $-p_i$ is the integral of $c_1(L)$ over Σ_{g_i}.[1] We take the $\mathbb{R}^2$ part of the manifold to be in the directions e^1-e^2 and Σ_{g_1} and Σ_{g_2} to lie in the directions e^3-e^4 and e^5-e^6 respectively.

[1]Note that this definition of the p_i's is different from the one in the section 5.4. There we had one Riemann surface and two line bundles over it, here it is two Riemann surfaces and one line bundle.

In the section 5.4, it was derived an expression for the flux of F over Σ_g considering that the fixed point set was Σ_g. In the present case, the fixed point set is a 4-manifold and thus we do not have an exact formula for the fluxes over the Riemann surfaces, but we have a constraint over them that comes from the topological identity (5.38). Let m_i be the magnetic charge over Σ_{g_i}, then the constraint is given by

$$\sigma^{(2)}\sigma^{(3)}\chi_1\chi_2 = m_1m_2 - \sigma^{(1)}(m_1p_2 + m_2p_1) + p_1p_2, \tag{6.36}$$

where we used that $\tau(B_4) = 0$, $\chi(\Sigma_{g_1} \times \Sigma_{g_2}) = \chi(\Sigma_{g_1})\chi(\Sigma_{g_2}) \equiv \chi_1\chi_2$ and $\eta = \sigma^{(2)}\sigma^{(3)}$ because it is possible to decompose the tangent planes accordingly with the Riemann surfaces.

Inspired by (5.30) we redefine the magnetic charges to be

$$\begin{aligned} m_1 =& \sigma^{(1)}p_1 - \sigma^{(2)}\chi_1 l_1\,, \\ m_2 =& \sigma^{(1)}p_2 - \sigma^{(3)}\chi_2 l_2\,, \end{aligned} \tag{6.37}$$

which upon inputting into the constraint equation (6.36), gives the simple condition:

$$1 - l_1l_2 = 0\,. \tag{6.38}$$

Clearly a solution is $l_1 = l_2 = 1$. One would now like to solve this subject to the constraint that the magnetic charges are integer. This is equivalent to solving the constraint subject to $y_i \equiv \chi_i l_i \in \mathbb{Z}$. In terms of y_i the constraint is

$$\chi_1\chi_2 = y_1y_2\,, \tag{6.39}$$

and should be solved for integer y_i. An obvious solution, valid for all choices of Riemann surfaces is $y_i = \chi_i \Leftrightarrow l_i = 1$, however this is not the only choice one can take.

Since the chirality on the fixed 4-surface is written as $\eta = \sigma^{(2)}\sigma^{(3)}$, the chirality of the spinor will be given by $\chi = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}$. We can now insert the solutions for the magnetic charges (subject to the constraint) into the localization formula (5.24) which gives

$$\begin{aligned} I = \frac{F_{S^5}}{27}\Big[&6\chi(\Sigma_{g_1})\chi(\Sigma_{g_2})l_1l_2 \\ &- 3\sigma^{(1)}\sigma^{(3)}\chi(\Sigma_{g_1})l_1p_2 - 3\sigma^{(1)}\sigma^{(2)}\chi(\Sigma_{g_2})l_2p_1 + 2\sigma^{(2)}\sigma^{(3)}p_1p_2\Big] \end{aligned} \tag{6.40}$$

For the universal solution $l_i = 1$ we find

$$I = \frac{2F_{S^5}}{27}\Big[12(1-g_1)(1-g_2) - 3\sigma^{(1)}\sigma^{(3)}(1-g_1)p_2 - 3\sigma^{(1)}\sigma^{(2)}(1-g_2)p_1 + \sigma^{(2)}\sigma^{(3)}p_1p_2\Big] \tag{6.41}$$

If we are interested in the trivial fibration case, we can just take $p_1 = p_2 = 0$. This solution is the euclidean Suh Black-Hole. The result is

$$I = \frac{8}{9}(1-g_1)(1-g_2)F_{S^5}. \tag{6.42}$$

This result agrees with minus the black hole entropy $I = -S_{\mathrm{BH}}$, computed in Suh [19, 20]. This could also have been calculated from equation (6.35) by using $\tau(B_4) = 0$ and $\chi(B_4) = \chi(\Sigma_{g_1})\chi(\Sigma_{g_2}) = 4(1-g_1)(1-g_2)$.

6.3.2 $\mathcal{O}(-p_1, -p_2)$ fibered over $S^2_{\epsilon_1} \times \Sigma_g$

Consider now the case where one of the Riemann surfaces is a sphere with an equivariant parameter of rotations turned on. While in general there is no toric action on the Riemann surface, we can use the toric geometry of $\mathcal{O}(-p_1) \to S^2_{\epsilon_1}$, detailed at the section 3.3, to study this case and obtain a formula for the on-shell action. Just like in the other examples, we will use the toric data to calculate the weights of the Killing spinor at each fixed point, but now "glued" to each fixed point there is a Riemann surface. That is, the fixed surfaces are two copies of the Riemann surface sitting at the two poles of the sphere and at the origin of the plane. The action is readily written as

$$\begin{aligned} I = -\frac{F_{S^5}}{27}\Bigg(& \frac{\chi_N(\sigma_N\epsilon_N^{(1)} + \sigma^{(\mathbb{R}^2)}\epsilon_N)^2\left(3m_N - \frac{p_2(\sigma_N\epsilon_N^{(1)}+\sigma^{(\mathbb{R}^2)}\epsilon_N)}{\epsilon_N}\right)}{\epsilon_N^{(1)}\epsilon_N} \\ & + \frac{\chi_S(\sigma_S\epsilon_S^{(1)} + \sigma^{(\mathbb{R}^2)}\epsilon_S)^2\left(3m_S - \frac{p_2(\sigma_S\epsilon_S^{(1)}+\sigma^{(\mathbb{R}^2)}\epsilon_S)}{\epsilon_S}\right)}{\epsilon_S^{(1)}\epsilon_S}\Bigg), \end{aligned} \tag{6.43}$$

where we identified $\sigma_N^{(\mathbb{R}^2)} = \sigma_S^{(\mathbb{R}^2)} \equiv \sigma^{(\mathbb{R}^2)}$, as justified by the toric diagram.

The conventions we used for this particular calculation are slightly different from the ones adopted at the section 3.3. Our toric data reads

$$v_0 = (1,0), \qquad v_1 = (0,1), \qquad v_2 = (p_1, -1). \tag{6.44}$$

Take φ to be the angular coordinate on the line bundle and φ_1 to rotate $S^2_{\epsilon_1}$ around its axis, then the Killing vector on M_6 is

$$\xi = \epsilon\partial_\varphi + \epsilon_1\partial_{\varphi_1}. \tag{6.45}$$

The weights at each vertex are calculated to be

$$\begin{cases} (\epsilon_N, \epsilon_N^{(1)}) = (\epsilon, -\epsilon_1), & \text{North pole;} \\ (\epsilon_S, \epsilon_S^{(1)}) = (\epsilon + p_1\epsilon_1, \epsilon_1), & \text{South pole.} \end{cases} \tag{6.46}$$

Using these expressions, we obtain for the on-shell action

$$\begin{aligned} I = -\frac{F_{S^5}}{27\epsilon_1} \Bigg(& \frac{3m_S\chi_S(\epsilon_1(p_1\sigma^{(\mathbb{R}^2)} + \sigma_S) + \sigma^{(\mathbb{R}^2)}\epsilon)^2}{p_1\epsilon_1 + \epsilon} \\ & - \frac{3m_N\chi_N(\sigma_N\epsilon_1 - \sigma^{(\mathbb{R}^2)}\epsilon)^2}{\epsilon} \\ & + \frac{p_2\chi_N(\sigma^{(\mathbb{R}^2)}\epsilon - \sigma_N\epsilon_1)^3}{\epsilon^2} \\ & - \frac{p_2\chi_S(\epsilon_1(p_1\sigma^{(\mathbb{R}^2)} + \sigma_S) + \sigma^{(\mathbb{R}^2)}\epsilon)^3}{(p_1\epsilon_1 + \epsilon)^2} \Bigg) . \end{aligned} \tag{6.47}$$

The formula for the trivial fibration case is obtained by setting $p_1 = p_2 = 0$. In this case, the fluxes m_N and m_S are given by $m_N = -\sigma_N^{(\Sigma_g)}(2-2g)$ and $m_S = -\sigma_S^{(\Sigma_g)}(2-2g)$ by formula (5.30). The action then simplifies to

$$I = -\frac{F_{S^5}}{9\epsilon\epsilon_1}(2-2g)\left(\sigma_N^{(\Sigma_g)}\chi_N(\sigma_N\epsilon_1 - \sigma^{(\mathbb{R}^2)}\epsilon)^2 - \sigma_S^{(\Sigma_g)}\chi_S(\sigma^{(\mathbb{R}^2)}\epsilon + \sigma_S\epsilon_1)^2\right) . \tag{6.48}$$

Setting the chiralities to match at both poles, we recover equation (6.42) with one of the g's being equal to zero. This is expected because, to take the limit where the sphere is also fixed, there can only be one chirality on the fixed surface.

Another interesting case is anti-twist where $\sigma^N = -\sigma^S$. The general formula for the magnetic charge threading on the sphere is $m = -\sigma^N - \sigma^S$, so that the twist corresponds to a magnetically charged black hole, while the anti-twist doesn't have magnetic charge. For the anti-twist we get

$$I = -\frac{4F_{S^5}(1-g)\sigma^N\sigma^{(\mathbb{R}^2)}(\sigma^N\epsilon_1 + \sigma^{(\mathbb{R}^2)}\epsilon)^2}{9\epsilon\epsilon_1} . \tag{6.49}$$

6.3.3 $\mathcal{O}(-p_1, -p_2)$ fibered over $S^2_{\epsilon_1} \times S^2_{\epsilon_2}$

In this example we will proceed similarly to what we did in section 6.2.1. Now the Torus action is the rotation of the normal plane and the rotations of the spheres. There are four isolated fixed points corresponding to the intersection of the origin of the plane and the two poles of the two spheres. The Killing vector is

$$\xi = \epsilon_1 \partial_{\varphi_1} + \epsilon_2 \partial_{\varphi_2} + \epsilon \partial_{\varphi}, \tag{6.50}$$

where ∂_{φ_i} rotates the sphere $S^2_{\epsilon_i}$ and ∂_φ rotates the plane of the normal bundle. Using the toric data described at section 3.4, we can calculate the weights at the four fixed points:

$$\xi^{(IJ)} = \begin{cases} (\epsilon, \epsilon_1, \epsilon_2) & NN; \\ (\epsilon + p_1\epsilon_1 + p_2\epsilon_2, -\epsilon_1, -\epsilon_2) & SS; \\ (\epsilon + p_1\epsilon_2, -\epsilon_1, \epsilon_2) & SN; \\ (\epsilon + p_2\epsilon_2, \epsilon_1, -\epsilon_2,) & NS; \end{cases} \tag{6.51}$$

where $I, J = N, S$.

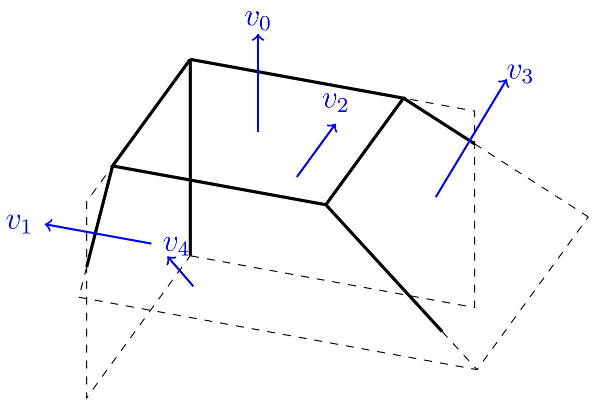

Figure 6.2: Toric diagram for $\mathcal{O}(-1,-2) \to S^2 \times S^2$.

From here we can compute the on-shell action, finding

$$\begin{aligned} I =& \frac{F_{S^5}\sigma^{(\mathbb{R}^2)}}{27\epsilon_1\epsilon_2}\Bigg[\frac{\sigma_N^{(1)}\sigma_N^{(2)}(\epsilon_1\sigma_N^{(1)} + \epsilon_2\sigma_N^{(2)} + \epsilon\sigma^{(\mathbb{R}^2)})^3}{\epsilon} \\ &- \frac{\sigma_S^{(1)}\sigma_N^{(2)}(-\epsilon_1\sigma_S^{(1)} + \epsilon_2\sigma_N^{(2)} + (\epsilon + p_1\epsilon_1)\sigma^{(\mathbb{R}^2)})^3}{\epsilon + p_1\epsilon_1} - \frac{\sigma_N^{(1)}\sigma_S^{(2)}(\epsilon_1\sigma_N^{(1)} - \epsilon_2\sigma_S^{(2)} + (\epsilon + p_2\epsilon_2)\sigma^{(\mathbb{R}^2)})^3}{\epsilon + p_2\epsilon_2} \\ &+ \frac{\sigma_S^{(1)}\sigma_S^{(2)}(-\epsilon_1\sigma_S^{(1)} - \epsilon_2\sigma_S^{(2)} + (\epsilon + p_1\epsilon_1 + p_2\epsilon_2)\sigma^{(\mathbb{R}^2)})^3}{\epsilon + p_1\epsilon_1 + p_2\epsilon_2}\Bigg]. \end{aligned} \tag{6.52}$$

In this expression, we already identified the relevant σ's. The toric polytope has five facets now, thus there are only five relevant σ's. In particular, note that $\sigma^{(\mathbb{R}^2)}$ here is associated with the $\mathbb{R}^2$ direction.

Like we discussed on the example $\mathbb{R}^4 \to S^2$, we are interested in taking the limit where the spheres stop rotating. This limt is only well-defined if the Killing spinor has a definite chirality on each sphere, that is, we must require that $\sigma_N^{(1)} = \sigma_S^{(1)}$ and $\sigma_N^{(2)} = \sigma_S^{(2)}$. Applying this constriant and taking $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, gives

$$I = \frac{2F_{S^5}}{27}\left(12 - 3\sigma_N^{(2)}\sigma^{(\mathbb{R}^2)}p_2 + -3\sigma^{(\mathbb{R}^2)}\sigma_N^{(1)}p_1 + \sigma_N^{(1)}\sigma_N^{(2)}p_1p_2\right) \tag{6.53}$$

which precisely matches with the result of $O(-p_1, -p_2) \to \Sigma_{g_1} \times \Sigma_{g_2}$ for $g_1 = g_2 = 0$, equation (6.41) when identifying the projection values there as $\sigma^{(1)} = \sigma^{(\mathbb{R}^2)}$, $\sigma^{(2)} = \sigma_N^{(1)}$ and $\sigma^{(3)} = \sigma_N^{(2)}$.

We can also compute the magnetic charge over the spheres. The interesting two-cycle is one of the spheres sitting at the origin of the plane and on one of the poles of the other sphere. The result is the same for both poles, considering the case when $\sigma_N^{(1)} = \sigma_S^{(1)}$ and $\sigma_N^{(2)} = \sigma_S^{(2)}$, they are given by

$$m_1 = -\sigma_N^{(1)} - \sigma_S^{(1)} + p_1\sigma^{(\mathbb{R}^2)}\,, \quad m_2 = -\sigma_N^{(2)} - \sigma_S^{(2)} + p_2\sigma^{(\mathbb{R}^2)}\,. \tag{6.54}$$

This is in agreement with formula (5.30).

6.3.4 Complex Projective Plane

The topological constraint (5.38) is enough to give us an expression to the flux of $c_1(F)$ if the second cohomology of B_4 is one dimensional. We illustrate this with the $\mathcal{O}(-p) \to \mathbb{C}P^2$ example. It is known that $H^2(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}$ [21], let this cohomology class be generated by the 2-form H which satisfies $\int_{\mathbb{C}P^2} H \wedge H = 1$ and it is usually called the Hyperplane Class. Then we may write $c_1(F) = mH$ and $c_1(\mathcal{O}(-p)) = -pH$.

On the other hand, the total Chern class of $\mathbb{C}P^2$ is given by $c(\mathbb{C}P^2) = (1 + H)^3$ [9]. This gives us the Euler and first Pontrjagin classes to be $e(\mathbb{C}P^2) = p_1(\mathbb{C}P^2) = 3H^2$. Hence we get that $\chi(\mathbb{C}P^2) = 3$ and $\tau(\mathbb{C}P^2) = 1$. Then the identity (5.38) leads to

$$6\eta + 3 = m^2 - 2\sigma^{(1)}mp + p^2, \tag{6.55}$$

$$\implies m = p\sigma^{(1)} \pm \sqrt{6\eta + 3}. \tag{6.56}$$

This solution for m is only real when $\eta = 1$, hence the flux is given by $m = p\sigma^{(1)} \pm 3$. We can plug this expression for m on (5.24) to obtain the on-shell action:

$$I = \frac{F_{S^5}}{27}\left(27 + p^2 \pm 9p\sigma^{(1)}\right). \tag{6.57}$$

6.3.5 Complex Projective Plane with rotation

We can consider $\mathbb{C}P^2$ to be "rotating" as well, that is, we consider a $\mathbb{T}^2$ action on $\mathbb{C}P^2$ additionally to the action on the fiber. Now there are three fixed points. From now on, we refer to the toric data computed at the section 3.6. At each vertex, meets three faces

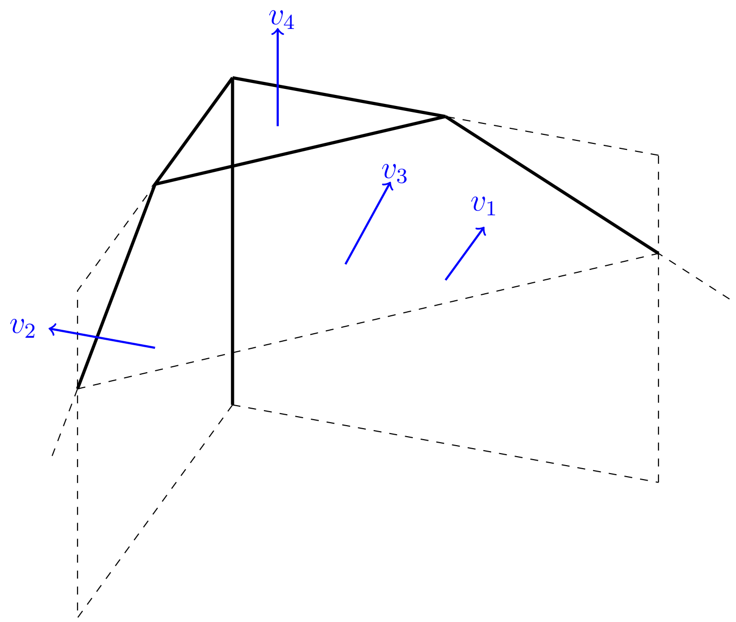

Figure 6.3: Toric diagram for $\mathcal{O}(-2) \to \mathbb{C}P^2$.

which have the same label as the normal vectors. At vertex 1, intersects faces 1, 2, 4; at vertex 2, faces 1, 3, 4; and at vertex 3, faces 2, 3, 4. Thus, using the toric data computed on 3.6 and the formulas of section (3.7) for the weights, we obtain at each vertex:

$$(\epsilon_1, \epsilon_2, \epsilon_3) = \begin{cases} (\epsilon_1, \epsilon_2, \epsilon_3), & \text{at vertex 1;} \\ (\epsilon_1 - \epsilon_2, -\epsilon_2, \epsilon_3 - pb_2), & \text{at vertex 2;} \\ (\epsilon_2 - \epsilon_1, -\epsilon_1, \epsilon_3 - pb_1), & \text{at vertex 3.} \end{cases} \tag{6.58}$$

The action then is readily written as

$$\begin{aligned} I = \frac{\sigma^{(\mathbb{R}^2)} F_{S^5}}{27} \Bigg[& \frac{\sigma_1\sigma_2(\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma^{(\mathbb{R}^2)})^3}{\epsilon_1\epsilon_2\epsilon_3} \\ & - \frac{\sigma_1\sigma_3(\sigma_1(\epsilon_1 - \epsilon_2) + \sigma^{(\mathbb{R}^2)}(\epsilon_3 + \epsilon_2 p) - \epsilon_2\sigma_3)^3}{\epsilon_2(\epsilon_1 - \epsilon_2)(\epsilon_3 + \epsilon_2 p)} \\ & - \frac{\sigma_2\sigma_3(\sigma_2(\epsilon_2 - \epsilon_1) + \sigma^{(\mathbb{R}^2)}(\epsilon_3 + \epsilon_1 p) - \epsilon_1\sigma_3)^3}{\epsilon_1(\epsilon_2 - \epsilon_1)(\epsilon_3 + \epsilon_1 p)} \Bigg], \end{aligned} \tag{6.59}$$

where we identified the same σ's at different vertices. That is, in principle, we could have distinguished between σ_1 at vertex 1 and σ_1 at vertex 2, but this factors come from the projection conditions (5.15) holding at the same face (which correspond to a 4-dimensional fixed-point set) and thus they must be equal.

To compare with the calculation where the whole $\mathbb{C}P^2$ is fixed, we must have only one definite chirality on it. Thus we need that $\sigma_1\sigma_2 = \sigma_1\sigma_3 = \sigma_2\sigma_3$, which amounts to $\sigma_1 = \sigma_2 = \sigma_3$. Additionally, the Killing vector must rotate only the $\mathbb{R}^2$ part and so we must send ϵ_1 and ϵ_2 to zero. One can check that, applying these conditions on the above formula for the action, gives the expression (6.57) upon identifying $\sigma^{(\mathbb{R}^2)}$ here with $\sigma^{(1)}$ there.

6.3.6 Holography and Field Theory Comparison

Many results obtained in this section have been calculated previously in the relevant holographically dual field theory. As a general rule, the $\mathbb{R}^n$ part of the gravity side will be compared to S^{n-1} on the field theory side. In our context, this means that the solutions $\mathbb{R}^6$ topology will be compared to field theories on S^5, $\mathbb{R}^4 \times \mathscr{F}_2$ to $S^3 \times \mathscr{F}_2$, and $\mathbb{R}^2 \times B_4$ to $S^1 \times B_4$. The result for the Hyperbolic Black-Hole was calculated in the original paper [13]. To compare what we got for the examples of the kind $\mathbb{R}^4 \times \mathscr{F}_2$, we need to relate our weights of the Killing vector (5.10) with the squashing parameter, b, of the squashed 3-sphere, S^3_b. As can be seen explicitly at [22] (cf. (2.17), (5.3) and (5.19) there), they are related by

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}. \tag{6.60}$$

It is useful to also write the expression for the squared Q that appears in the formulas of the field theory partition function:

$$Q = \tfrac{1}{2}(b + b^{-1}), \tag{6.61}$$

$$Q^2 = \frac{(\epsilon_1 + \epsilon_2)^2}{4\epsilon_1\epsilon_2}. \tag{6.62}$$

The results we obtained for $\mathbb{R}^4 \times \Sigma_g$ match with equation (3.2) of [23] which reads as

$$F^{univ}_{S^3_b \times \Sigma_g} = -\frac{8}{9}(g-1)Q^2 F_{S^5}. \tag{6.63}$$

The superscript "*univ*" here stands for universal twist. Upon using expression (6.62), we recover the result (6.26) with $\chi_{\mathbb{R}^4} = 1$. Equation (6.27) is simply the case of the round S^3_b where $Q = 1$.

The result for $\mathbb{R}^4 \times S^2$, where there is a $U(1)$ action on S^2, equation (6.34), can be compared to equation (3.23) of [24]

$$F_{S^3_b \times S^2_\epsilon}(\Delta_i, \mathfrak{t}_i, \epsilon|b) = \frac{8}{27}\frac{Q^2}{\epsilon}\left[F_{S^5}\left(\Delta_i + \frac{\epsilon}{2}\mathfrak{t}_i\right) - F_{S^5}\left(\Delta_i - \frac{\epsilon}{2}\mathfrak{t}_i\right)\right], \tag{6.64}$$

$$F_{S^5}(\Delta_i) = F_{S^5}\left(\Delta_1\Delta_2\right)^{\frac{3}{2}}, \tag{6.65}$$

where the Δ_i's are the chemical potentials and the $\mathfrak{t}_i$'s are the fluxes for the flavor symmetry. In our work, we are considering a theory without matter, thus we need to set $\Delta_1 = \Delta_2$ and $\mathfrak{t}_\mathfrak{i} = 1$. Additionally, the parameter ϵ there relates to the weight ϵ_3 here by

$$\epsilon^{\text{there}} = \frac{2\epsilon_3}{\epsilon_1 + \epsilon_2}. \tag{6.66}$$

One then matches the results by plugging these values for the parameters and using (6.62).

The result for $\mathbb{R}^2 \times S^2_{b_2} \times \Sigma_g$ in the anti-twist case can also be compared to a result in [24]. Equation (5.101) there can be compared to our (6.49). Their formulas reads as

$$\log Z_{(S^2_\epsilon \times S^1)\times\Sigma_g} = -\frac{\pi}{2\epsilon}\left[F_{S^3\times\Sigma_g}\left(\Delta_i + \frac{\epsilon}{2}\mathfrak{t}_i, \mathfrak{s}_i\right) + F_{S^3\times\Sigma_g}\left(\Delta_i - \frac{\epsilon}{2}\mathfrak{t}_i, \mathfrak{s}_i\right)\right], \tag{5.101}$$

$$F_{S^3\times\Sigma_g}(\Delta_i, \mathfrak{s}_i) = \frac{8}{27\pi^2}F_{S^5}\sum_{i=1}^{2}\mathfrak{s}_i\frac{\partial(\Delta_1\Delta_2)^{\frac{3}{2}}}{\partial\Delta_i},$$

where $\mathfrak{s}_1 + \mathfrak{s}_2 = 2 - 2g$, $\Delta_1 + \Delta_2 = 2\pi + \epsilon$ and $\mathfrak{t}_1 + \mathfrak{t}_2 = 0$. Like before, setting $\Delta_1 = \Delta_2 = \pi + \frac{1}{2}\epsilon$ and $\mathfrak{t}_1 = \mathfrak{t}_2 = 0$, we obtain

$$\log Z_{(S^2_\epsilon \times S^1)\times\Sigma_g} = \frac{2F_{S^5}(g-1)(\epsilon + 2\pi)^2}{9\pi\epsilon} \tag{6.67}$$

This result also matches with our result (6.49) upon taking ϵ or ϵ_1 to be 2π and setting $\sigma^N\sigma^{(\mathbb{R}^2)} = 1$.

Finally, the result obtained for the Black-Hole with horizon $\Sigma_{g_1} \times \Sigma_{g_2}$, equation (6.42), matches with the calculation of [25], equation (3.107).

Chapter 7

Conclusion

In this work, we studied how the localization formula from Equivariant Cohomology can be applied to calculate the on-shell action of six-dimensional Romans Supergravity. We showed how to construct an equivariantly closed form using the fields of the theory and bilinears constructed from the Killing spinor, which is equivariantly closed by means of the equations of motion and supersymmetry equations. The integral of this form gives the on-shell action and was localized to the fixed points of the toric action generated by the Killing vector constructed as a bilinear of the Killing spinor. The result is a formula that relies heavily on the topological information of a solution to Romans supergravity, but can be used without the explicit knowledge of the solution.

The developed formula was applied to many examples that display different features. Besides the Hyperbolic Black-Hole, the examples were considered without the knowledge of the metric or the other fields of the theory. The identification of the fixed-point locus was immediate because we considered examples that split in a part that admits a toric action and a part that does not, generally taken as $\mathbb{R}^{2k}$ fibered over $\mathscr{F}_{6-2k}$. This was anticipated in our construction as the separation of the manifold between the fixed-point locus and its normal bundle was present in our analysis since the beginning. We also considered examples in which $\mathscr{F}_{6-2k}$ admits a toric action. In this cases, the fixed-point locus was just a collection of isolated points and we obtained formulas for them using the Toric Geometry formalism. But, imposing the condition that the spinor has a definite chirality on connected components of the fixed-point set, we took the limit were the toric action on $\mathscr{F}_{6-2k}$ vanished and we recovered the result for the case where $\mathscr{F}_{6-2k}$ does not have a toric action. This provides a consistent check on our results.

Many of the results obtained for the product space examples (geometries of the kind $\mathbb{R}^{2k}\times\mathscr{F}_{6-2k}$) were compared to the results for the free-energy on the relevant holographical theory, which is a theory that has the same kind of supersymmetry living on a $S^{2k-1}\times\mathscr{F}_{6-2k}$ manifold. Many of these results had been obtained previously on the gravity side by different calculation methods, but some of them are a new matching. The expressions obtained for the fibration cases and for some product space cases that does not have a holographical calculation make interesting new predictions. In particular, we note the expression for $\mathbb{R}^2 \times B_4$ (6.35), which is neat for being expressed in terms of topological invariants of B_4, and the expression for $\mathbb{R}^2 \times \mathbb{C}P^2$, which is just the free energy on S^5.

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Appendix A

General Geometry

A.1 Fibre Bundles

A fiber bundle is a structure that locally looks like a direct product of two spaces. Here we focus on the case of differentiable fiber bundles and only of some kinds of bundle that are relevant to us.

A **fiber bundle** is a quintuple (E, π, M, F, G), where E, M and F are manifolds respectively called **total space**, **base space** and **fiber**; $\pi : E \to M$ is a surjective map called **projection** which satisfies $\pi^{-1}(x) = F$, for all $x \in M$; and G is a Lie group called **structure group**. It is also imposed that E is locally trivial through the condition that there exists a covering of M with charts U_i and a set of diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$, called **local trivializations**, such that $\pi(\phi(x, f)) = x$, where $f \in F$. Finally, we also require that, when two charts U_i and U_j overlap, the **transition function** $t_{ij} U_i \cap U_j \to G$ that relates the two charts be an element of G for each point $x \in U_i \cap U_j$, where t_{ij} is defined trough:

$$\phi_i \circ \phi_j^{-1} : (U_i \times U_j) \times F \to (U_i \times U_j) \times F \tag{A.1}$$

$$(\phi_i \circ \phi_j^{-1})(x, f) = (x, t_{ij}(x) f). \tag{A.2}$$

On a fiber bundle, there is the important notion of a section. A **section** is a smooth map $s : E \to M$ such that $\pi \circ s = 1$. The sections generalize the vector fields over a manifold. Note that a section may be defined only on an open chart U of M. The collection of all sections defined on U is $\Gamma(U, F)$.

A.1.1 Vector Bundles

When the fiber of the fiber bundle is a vector space it is called a **vector bundle**. The vector bundles may be either real or complex, whether the fiber is $\mathbb{R}^k$ or $\mathbb{C}^k$. The structure group will be $GL(k,\mathbb{R})$ or $GL(k,\mathbb{C})$ for each case.

If the vector space is one-dimensional (wether it be over $\mathbb{R}$ or over $\mathbb{C}$), the bundle is called a **line bundle**. In particular, we have the result that if a line bundle admits a nowhere zero section, then it is a trivial bundle.

Special cases of vector bundles are the (co)tangent bundle $TM(T^*M)$, whose typical fiber is the (co)tangent space at each point of M. Thus, $\Gamma(M,TM)$ is the space of vector fields on M and $\Gamma(M,T^*M)$ is the space of one-forms. It is also possible to take tensor products of vector bundles, those are associated with different representations of $GL(n,\mathbb{R})$ [6, 8], but essentially they are vector bundles whose typical fiber is a tensor product. On these bundles, we define tensor fields as the sections. For example, we have the exterior algebra bundle ΛM whose sections are the differential forms.[1]

The **normal bundle** is also an example of vector bundle. Consider a manifold M with a Riemannian metric g and S a submanifold. For $y \in S$, the normal space N_yS is the set of all vectors v from T_yM, such that $g(v,w) = 0$ for all $w \in T_yS$. The normal bundle is then the collection of normal spaces over S. Note that the base space is S.

Another important vector bundle is the **conjugate bundle**. Let E be a complex vector bundle, the conjugate bundle $\bar{E}$ is the bundle E but with the multiplication by a scalar defined taking the conjugate of the scalar. Let v be an element of the vector space that is a typical fiber of E and $z \in \mathbb{C}$, then, on $\bar{E}$, $z \cdot v = \bar{z}v$.

Consider two vector bundles over the same base space: $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$. We can construct the **Whitney sum bundle** $E_1 \oplus E_2$ which has projection $\pi_{12}(u_1, u_2) = p$. Thus the fiber of the Whitney sum bundle is the direct sum of the fibers: $\pi_{12}^{-1}(p) = \pi_1^{-1}(p) \oplus \pi_2^{-1}(p)$.

A.1.2 Principal and Associated Bundles

A fiber bundle whose fiber is the structure group is called a **principal bundle**. It is usually denoted $P(M,G)$. The action of G is defined on the right such that $\pi(fg) = \pi(f)$, for $f \in \pi^{-1}(U_i)$ and $g \in G$. The action of G is free and transitive.

Given a principal bundle $P(M,G)$ and a G-invariant vector space V, we can construct a vector bundle over M with fiber V called the **associated bundle**. This bundle is constructed by taking the direct product $P \times V$ and identifying the point $(p \cdot g, v)$ with $(p, \rho(g)v)$, where $\rho(g)$ is the representation of g on V. The associated bundle is denoted

[1] We abuse the notation and also call ΛM the space of differential forms.

by $E = P \times_G V$ and the identification is done so that the identified points have the same fiber and thus the projection $\pi_E(p, v) = \pi(p)$ is well defined.

Consider a vector bundle E with fiber $\mathbb{R}^k(\mathbb{C}^k)$. This vector bundle is an associated bundle to the principal bundle with structure group $GL(k, \mathbb{R})(GL(k, \mathbb{C}))$, that is, the principal bundle $P(M, GL(k, \mathbb{R}))(P(M, GL(k, \mathbb{C}))$. This principal bundle is called the frame bundle of E because specifying a frame is the same as specifying a linear invertible map from the canonical base of $\mathbb{R}^k(\mathbb{C}^k)$ to the frame. The **frame bundle** of M is the frame bundle of the tangent bundle which may have fiber $\mathbb{R}^n$ or $\mathbb{C}^n$, where n is the dimension of M.

Let G be a Lie subgroup of $GL(n, \mathbb{R})$. A G**-structure** on M is a subbundle of the frame bundle such that it is a principal bundle over M with structure group G. In many cases, defining a G-structure is equivalent to establishing one or more globally defined G-invariant tensors. For example, let $G = O(n)$, then the $O(n)$-structure is the principal bundle $P(M, O(n))$ which corresponds to picking only the orthogonal frames. Since we can always define a metric on $\mathbb{R}^n$, we can always reduce $GL(n, \mathbb{R})$ to $O(n)$, that is, to give an $O(n)$-structure is the same as to define a Riemannian metric, which is a $O(n)$ invariant tensor [9]. In general, given G, it is not always possible to establish the G-structure on M. For example, a $SL(n)$-structure is the same as to orient the manifold but this is only possible if M is orientable and in this case the invariant tensor is the volume form.

A.2 Differential Forms and de Rham Cohomology

Consider the exterior bundle over M. The sections of this bundle are the differential forms, they form a graded algebra under the wedge product. An element of this space is often denoted in the text as a "polyform" because it is a sum of forms of different degrees, but here we use simply "form" or "differential form".

Denote the space of forms as ΛM. The grading is given by the tensor rank (keep in mind that ΛM is a tensor product bundle), for example, a function is a 0-form, a differential, *i.e.* a dual vector, is a 1-form and so on. In general, the space of k-forms is denoted $\Lambda^k M$. The highest grading is given by the dimension of the manifold.

In the exterior algebra, there is a natural operation called the **wedge product**. Let $\mathrm{gr}(\alpha)$ be the grade of a $\mathrm{gr}(\alpha)$-form, the wedge product is characterized by the following property:

$$\alpha \wedge \beta = (-1)^{\mathrm{gr}(\alpha)\mathrm{gr}(\beta)} \beta \wedge \alpha, \tag{A.3}$$

along with the usual properties of the product of an algebra (bilinearity and associativity).

In the space of differential forms, there exists a unique anti-derivation called **exterior derivative** [26]. It can be defined by the following properties:

- let $f \in C^\infty(M)$, then $\mathrm{d}f\,[X] = X\,[f]$, for all vector fields X;
- $\mathrm{d}^2 = 0$;
- $\mathrm{d} : \Lambda^k M \to \Lambda^{k+1} M$;
- let α be a $\mathrm{gr}(\alpha)$-form, while β may be ungraded, then we have the Leibniz's rule:

$$\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^{\mathrm{gr}(\alpha)} \alpha \wedge (\mathrm{d}\beta). \tag{A.4}$$

Another important operator used in this work is the **interior multiplication**, also called contraction. We denote it here by "$X \lrcorner$", but is often found in the literature as $i(X)$ or i_X. Given a vector field, it is a unique operator defined by

- $X \lrcorner\, \alpha = \alpha[X]$, for α a 1-form;
- $X \lrcorner : \Lambda^k M \to \Lambda^{k-1} M$;
- $X \lrcorner\, (\alpha \wedge \beta) = (X \lrcorner\, \alpha) \wedge \beta + (-1)^{\mathrm{gr}(\alpha)} \alpha \wedge (X \lrcorner\, \beta)$.

If a k-form α is such that $\mathrm{d}\alpha = 0$, we say that it is **closed**; if it can be written as $\alpha = \mathrm{d}\beta$, then it is called **exact**. It follows that every exact form is closed because $\mathrm{d}^2 = 0$. There is an important result called Poincare's Lemma that says that if M is simply connected and path connected, then every closed form is exact, which implies that locally the result holds [8, 9, 26].

From the Poincare's Lemma, for example, we see that the relationship between closed and exact forms over a manifold is closely related to its topology. The object that encodes the topological information from the differential forms is the **de Rham Cohomology**. Let α and β be closed k-forms, we say that they are equivalent with they differ by an exact form: $\alpha - \beta = \mathrm{d}\theta$. The k-cohomology is then quotient of closed k-forms by exact k-forms.

We want to define an integration of forms over a manifold, but to do this we need to address the concept of orientation. We say that a n-dimensional manifold is orientable if there is a nowhere vanishing n-form defined on it, this is equivalent to saying that the bundle $\Lambda^n M - \{0\}$ has exactly two components, where $\omega' = f\omega$, f everywhere positive, and ω have been identified. Picking an orientation then means picking one of the two components of $\Lambda^n M - \{0\}$, the positive component or the negative component. It can be shown that this construction is equivalent to requiring that the jacobian of coordinate transformations is positive.

Assume that M is paracompact, which means that it can be covered by a finite collection of sets U_i. Define the partition of unity $\{\varepsilon_i\}$ by the conditions

$$0 \leq \varepsilon_i(x) \geq 1; \tag{A.5}$$

$$x \notin U_i \implies \varepsilon_i(x) = 0; \tag{A.6}$$

$$\sum_i \varepsilon_i(x) = 1. \tag{A.7}$$

Then we can finally define integration over a manifold M as

$$\int_M \omega \equiv \sum_i \int_{U_i} \varepsilon_i \omega = \sum_i \int_{\mathbb{R}^n} (\phi_i^{-1})^* \varepsilon_i \omega, \tag{A.8}$$

where $\phi_i : U_i \to \mathbb{R}^n$ is a coordinate map.

To close this section, we will present the Stokes Theorem.

Consider the half-space $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_n \geq 0\}$. The boundary of a manifold, denoted ∂M, is the set of points of M such that the neighborhoods that cover them are diffeomorphic to $\mathbb{H}^n$. The boundary ∂M is a submanifold of dimension $n-1$ which does not have a boundary, $\partial\partial M = \varnothing$. The boundary of M inherits the orientation of M. Note that not all manifolds have boundaries. Now we can write the Stokes' Theorem [9, 26]

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{A.9}$$

A.3 Connections

Let f be a smooth function on M, then $\pi^* f = f \circ \pi$ is a smooth function on the fiber. Then a vector which has $X(\pi^* f) = 0$, for any f, will be tangent to the fiber, such a vector is called a **vertical vector**. We denote by VE the bundle of vertical vectors.

A **connection one-form** A is a one-form which takes values on VE and obey $X \lrcorner A = X$ if $X \in VE$. The **horizontal bundle** is defined as the kernel of A.

In the case of a principal bundle $P(M, G)$, VE is identified with the Lie algebra of G, $\mathfrak{g}$ [6]. Hence A becomes a Lie algebra-valued one-form. It is further imposed that A be G-invariant

$$\mathcal{L}_X A + [X, A] = 0, \tag{A.10}$$

where this equation is equivalent to saying that the right action of G on A is given by the adjoint action (of the inverse element).

Let E be a vector bundle over M. We define the **covariant derivative** ∇ as an operator specified by

1) Domain and image:
$$\nabla : \Gamma(M, E) \to \Gamma(M, T^* \otimes E); \tag{A.11}$$

2) Leibniz rule, let $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$, then:
$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s; \tag{A.12}$$

3) The difference of covariant derivatives is an one-form which takes value on the endomorphisms of E, $\mathrm{End}(E)$.[2]

On a trivial vector bundle, the exterior derivative is an example of covariant derivative. It follows then that any covariant derivative can be represented locally as
$$\nabla s = ds + A \cdot s, \tag{A.13}$$
where A is an one-form which takes value in $\mathrm{End}(E)$.

Note that we have used A as a connection on a principal bundle and now discussing the covariant derivative on a vector bundle. Indeed the two concepts are closely related as one can always construct a covariant derivative on the associated bundle of a principal bundle from the connection on the principal bundle. In particular, there is a one-to-one correspondence between covariant derivatives on E and connections on the frame bundle $GL(E)$ [6]. Doing actual calculations, we use equation (A.13) with A being the appropriate representation of the connection on the associated bundle.

Finally, we introduce the **curvature** F considering that the covariant derivative comes from a connection on a principal bundle. The curvature is a $\mathfrak{g}$-valued two-form defined by
$$F = \nabla^2 \tag{A.14}$$
$$= \mathrm{d}A + \tfrac{1}{2}\left[A, A\right]. \tag{A.15}$$

Let $\nabla_X \equiv X \lrcorner \nabla$ be the covariant derivative w.r.t. the vector field X, then it can be shown that the above definition for F is equivalent to
$$F(X, Y) = [\nabla_X, \nabla_Y] + \nabla_{[X,Y]}. \tag{A.16}$$

[2]An endomorphism of E is a map from E to E. For example, operators are endomorphisms of a vector space.

The curvature also satisfies the Bianchi identity:

$$\nabla F = 0. \tag{A.17}$$

A.3.1 Compatibility conditions

Recall that for a principal bundle the structure group is identified with the fibers. Consider $A_{i(j)}$, a one-form connection defined locally on the chart $U_{i(j)}$, with $U_i \cap U_j \neq \varnothing$. We want to know how A_i and A_j are related. Since A_i is $\mathfrak{g}$-valued and $t_{ij} \in G$, we expect A_i to be acted with the adjoint action of t_{ij}. On the other hand, it is also a one-form and thus transforms by means of a pull-back. The final answer is a mixture of both [8]

$$A_i = t_{ij}^{-1} A_j t_{ij} + t_{ij}^{-1} \mathrm{d} t_{ij}, \tag{A.18}$$

which yields for the curvature:

$$F_i = t_{ij}^{-1} F_j t_{ij}. \tag{A.19}$$

These are the familiar gauge transformations from physics.

A.4 Riemannian Geometry

Like we mentioned previously, a $O(n)$-structure on M is equivalent to endowing M with a riemannian metric $g(\cdot,\cdot)$ which satisfies: (i) $g(X,Y) = g(X,Y)$; (ii)$g(X,X) \geq 0$ with the equalty holding only if $u = 0$. Consider additionally that M is orientable. Thus the tangent bundle of M is an associated bundle to $SO(M)$, the bundle of oriented orthonormal frames on M.

$$TM \simeq SO(M) \times_{SO(M)} \mathbb{R}^n. \tag{A.20}$$

The covariant derivative on TM will be formed from the connection on $SO(n)$ which is $\mathfrak{so}(n)$-valued in the $\mathbb{R}^n$ representation. We look at the **Levi-Civita connection** which preserves the metric and is torsion-free, meaning

$$\mathrm{d}(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y), \tag{A.21}$$

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0. \tag{A.22}$$

The curvature is called **Riemannian Curvature**, it is a $\mathfrak{so}(n)$-valued 2-form which we denote by $\mathcal{R}$. We use the conventional notation $\mathcal{R}(X,Y)Z$, where $\mathcal{R}(X,Y)$ is a representation of $\mathfrak{so}(n)$ which acts on $\mathbb{R}^n$. The explicit expression of $\mathcal{R}$ is

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{A.23}$$

The Riemannian curvature satisfies some properties:

$$\begin{align}
\mathcal{R}(X,Y) + \mathcal{R}(Y,X) &= 0; \tag{A.24}\\
g(\mathcal{R}(W,X)Y,Z) + g(Y,\mathcal{R}(W,X)Z) &= 0; \tag{A.25}\\
\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y &= 0; \tag{A.26}\\
g(\mathcal{R}(W,X)Y,Z) &= g(\mathcal{R}(Y,Z)W,X); \tag{A.27}\\
(\nabla_X\mathcal{R})(Y,Z)W + (\nabla_Z\mathcal{R})(X,Y)W + (\nabla_Y\mathcal{R})(Z,X)W &= 0, \tag{A.28}
\end{align}$$

where (A.26) and (A.27) hold only for torsion-free connections and the last one is the Bianchi identity.

A.4.1 Local Expressions

Let $\{e_\alpha\}$ be an orthonormal frame of the tangent bundle at some chart, that is, they satisfy:

$$g(e_\alpha, e_\beta) = \delta_{\alpha\beta}. \tag{A.29}$$

Let $\{e^\alpha\}$ be the dual frame and $\omega = {\omega_\alpha}^\beta = {(\omega_\gamma)_\alpha}^\beta e^\gamma$ be the one-form connection, note that α and β in the connection are indices of a $\mathfrak{so}(n)$ matrix, so ${\omega_\alpha}^\beta = -{\omega_\beta}^\alpha$. The covariant derivative is then

$$\nabla_\beta X = \left(\partial_\beta X^\alpha + {(\omega_\beta)_\gamma}^\alpha X^\gamma\right) e_\alpha. \tag{A.30}$$

We can also get local expressions for the curvature which we use to define the **Ricci curvature** and the **scalar curvature**

$$\begin{align}
{\mathcal{R}^\alpha}_{\beta\gamma\lambda} &= e^\alpha\left[\mathcal{R}(e_\gamma, e_\lambda)e_\beta\right]; \tag{A.31}\\
\mathcal{R}_{\alpha\beta\gamma\lambda} &= g(\mathcal{R}(e_\gamma, e_\lambda)e_\beta, e_\alpha); \tag{A.32}\\
\mathrm{Ric}_{\alpha\beta} &= \sum_\gamma \mathcal{R}_{\alpha\gamma\beta\gamma}; \tag{A.33}\\
R &= \sum_{\alpha,\beta} \mathcal{R}_{\alpha\beta\alpha\beta}. \tag{A.34}
\end{align}$$

Note that in the first expression the "α" and "β" indices are matrix indices like on the connection one-form, thus we can construct the two-form ${\mathcal{R}^\alpha}_\beta = \frac{1}{2}{\mathcal{R}^\alpha}_{\beta\gamma\lambda} e^\gamma \wedge e^\lambda$. It is worth mentioning that we then have Cartan's structure equations which are essentially the covariant derivative of the dual frame and a local expression of the curvature, the first one is zero because of the torsion-free condition [8]

$$\mathrm{d}e^\alpha + {\omega_\beta}^\alpha \wedge e^\beta = 0; \tag{A.35}$$

$$\mathrm{d}{\omega_\alpha}^\beta + {\omega_\alpha}^\gamma \wedge {\omega_\gamma}^\beta = {\mathcal{R}^\alpha}_\beta. \tag{A.36}$$

A.4.2 Arbitrary Frame

Let $\{\partial_\mu\}$ be a non-orthonormal frame, we can transition from the orthonormal one by a $GL(\mathbb{R}, n)$ matrix: $e_\alpha = {e_\alpha}^\mu \partial_\mu$; and using the inverse matrix for the dual frame:$e^\alpha = {e^\alpha}_\mu \mathrm{d}x^\mu$, where by inverse we mean that ${e_\alpha}^\mu {e^\beta}_\mu = \delta^\beta_\alpha$ and ${e_\alpha}^\mu {e^\alpha}_\nu = \delta^\mu_\nu$. The local components of the metric on this other frame are then given by

$$g_{\mu\nu} = {e^\alpha}_\mu {e^\beta}_\nu \delta_{\alpha\beta}. \tag{A.37}$$

For the connection, the transformation is not so simply. We calculate it by plugging $e_\alpha = {e_\alpha}^\mu \partial_\mu$ at $\nabla_\gamma e_\alpha = {(\omega_\gamma)_\alpha}^\beta e_\beta$. The result is

$${(\omega_\gamma)_\alpha}^\beta = {e^\beta}_\nu {e_\gamma}^\mu \left(\partial_\mu {e_\alpha}^\nu + {\Gamma^\nu}_{\mu\lambda} {e_\alpha}^\lambda \right). \tag{A.38}$$

This is essentially equation (A.18), but with the form index explicit.

The frame $\{e^\alpha\}$ defines a volume form on M. It is simply $e^1 \wedge ... \wedge e^n$, which in a arbitrary frame is $\det({e^\alpha}_\mu)\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n = \sqrt{g}\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n$, where $g = \det(g_{\mu\nu})$.

Finally, note that everything was done for a Riemannian metric. If we wish to work at a semi-Riemannian structure, there will need to be appropriate adaptations. For example, for the Lorentzian case, we should consider a $SO(n-1, 1)$ structure such that $\delta_{\alpha\beta}$ is replaced by the Minkowski metric at (A.29).

A.4.3 Hodge Duality

The k-th exterior bundle over a n-dimensional manifold M, $\Lambda^k M$, has dimension $\frac{n!}{k!(n-k)!}$. The $(n-k)$-th exterior bundle has the same dimension which hints us that both spaces are isomorphic. In fact they are and the isomorphism is done by the **Hodge star** $* : \Lambda^k M \to \Lambda^{n-k} M$. Let $\omega \in \Lambda^k M$, then, locally:

$$*\omega = \frac{\sqrt{g}}{r!(n-r)!} \omega_{\mu_1 \cdots \mu_k} {\varepsilon^{\mu_1 \cdots \mu_k}}_{\mu_{k+1} \cdots \mu_n} \mathrm{d}x^{\mu_{k+1}} \wedge ... \wedge \mathrm{d}x^{\mu_n}, \tag{A.39}$$

where g is the determinant of the metric on M.

In principle, one can define the Hodge star without any respect to the metric, but here we are not interested in this case. This convention for $*$ gives $*1 = \sqrt{g}\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_n}$ which is the volume element of M. Additionally, given η another k-form, we have the important expression:

$$\eta \wedge *\omega = \frac{1}{k!}\eta_{\mu_1 \dots \mu_k}\omega^{\mu_1 \dots \mu_k} * 1. \tag{A.40}$$

A.5 Characteristic Classes

Characteristic classes are essentially polynomials of the curvature. These polynomials are closed differential forms and thus are interesting from the cohomological point-of-view.

Let f be a complex polynomial and ∇ the covariant derivative with curvature F. Then we define:

$$f(F) := \sum_k \frac{f^k(0)}{k!}F^k, \tag{A.41}$$

this is a finite sum because F^k is nilpotent. Since F is a $\mathfrak{g}$-valued form, we take the trace of $f(F)$ to obtain an ordinary differential form: $P(F) = \mathrm{Tr} f(F)$.

Before continuing, note that, locally, $\mathrm{Tr}(\nabla\alpha) = \mathrm{Tr}(\mathrm{d}\alpha) + \mathrm{Tr}(\mathrm{ad}(A)\alpha) = \mathrm{d}\mathrm{Tr}(\alpha)$, for α a $\mathfrak{g}$-valued k-form. Building on this, it can be shown that $P(F)$ is a closed differential form. It can also be shown that if ∇_1 and ∇_2 are two different covariant derivatives on the vector bundle, then $P(F_1) - P(F_2)$ is exact. These properties tell us that $P(F)$ defines a cohomology class. [6, 8]

A.5.1 Chern Classes

Our first characteristic class is the **Chern class**, defined as

$$c(F) \equiv \det\left(1 + \frac{i}{2\pi}F\right), \tag{A.42}$$

$$= \sum_k c_k(F), \tag{A.43}$$

where $c_0 = 1$ and $c_k(F)$ is a $2k$-form and each c_k is closed and thus defines a class of the $2k$-cohomology $H^{2k}(M)$. It is also established that if the vector bundle E has rank n (as a complex vector bundle), then $c_{m>n}(F) = 0$. In particular, we have that $c_1(F) = \frac{\mathrm{i}}{2\pi}\mathrm{Tr}F$. The Chern class satisfies some interesting properties [8, 9]

- If the bundle is trivial, then $c(F) = 1$;
- The Chern class of the Whitney sum bundle is the product of classes: $c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$;

- If E has a non-vanishing section, then $c_n(F) = 0$.
- Splitting principle: If a polynomial identity holds for E being a Whitney sum of line bundles, then it holds in general.

The Splitting Principle is a good tool to calculate Chern classes because it allows to calculate considering that the vector bundle is a sum of complex line bundles and the result will be correct even if the vector bundle can not be decomposed as the sum of line bundles. Hence, using the splitting principle, we can write the Chern class of a vector bundle as

$$c(E) = \prod_{i=1}^{n}(1 + c_1(L_i)), \tag{A.44}$$

where L_i are line bundles and the "wedge" product is implicit. Now we state some formulas for the Chern class of a tensor product. Let E and H be vector bundles of rank n and m respectively and L_i and M_j be the line bundles used to calculate $c(E)$ and $c(H)$, then we have [9]

$$c(E \otimes H) = \prod_{i,j=1}^{i=n,j=m} (1 + c_1(L_i) + c_1(M_j)), \tag{A.45}$$

$$c(E \otimes L) = \sum_{i=0}^{n} c_i(E)(1 + c_1(L))^{n-i}, \text{ if } L \text{ is a line bundle.} \tag{A.46}$$

A.5.2 Euler and Pontrjagin classes

The Euler and the Pontrjagin classes are closely related to the Chern class, but they are defined for real vector bundles over orientable manifolds. In such cases, we can always take the curvature to be skew-symmetric, because we can endow the vector bundle with an orthogonal structure.

The **Pontrjagin class** is defined by

$$p(F) \equiv \det\left(1 + \frac{1}{2\pi}F\right), \tag{A.47}$$

$$= \sum_k p_k(F). \tag{A.48}$$

This is very similar to the Chern class, actually, it can be shown that $p_i(F) = (-1)^i c_{2i}(F_{\mathbb{C}})$, where $F_{\mathbb{C}}$ is the curvature of the complexified vector bundle. It follows then that the Pontrjagin class has many properties akin to the Chern class. In particular, it can be calculated trough the splitting principle by the formula:

$$p(E) = \prod_{i=1}^{n}(1 + c_1(L_i)^2), \tag{A.49}$$

where E here is a real oriented vector bundle of dimension $2n$ such that, to calculate the characteristic, we can consider its complexification to split as a sum of line bundles: $E \otimes \mathbb{C} = L_1 \oplus \bar{L}_1 \oplus ... \oplus L_n \oplus \bar{L}_n$.

An important characteristic of the Pontrjagin class is that each p_i is an element of $H^{4i}(M)$. This can be seen by noting that $p(F)$ is an even polynomial of F, since F is antisymmetric. This fact hints that the Pontrjagin classes are related to the signature of manifold. The **signature of a manifold**, $\tau(M)$, is defined only for manifolds with dimension $n = 4k$, here we define it using the Hirzebruch Signature Theorem [6, 8, 9, 21]

$$\tau(M) = \int_M L(TM), \tag{A.50}$$

where $L(F)$ is the L**-genus** given by

$$L(F) = \sqrt{\det\left(\frac{F/2\pi}{\tanh\left(F/2\pi\right)}\right)}. \tag{A.51}$$

The L-genus can be written using Pontrjagin classes [8, 21], the first terms are $L = 1 + \frac{1}{3}p_1$. Thus, for a four dimensional manifold B_4, $\tau(B_4) = \int \frac{1}{3}p_1(TB_4)$. We also have the Whitney sum formula for L: $L(E \oplus H) = L(E) \wedge L(H)$. Using this, we can see that if a 4-dim manifold is a direct product of two 2-manifolds, then $L(M_2 \times M_2) = 0 \implies \tau(M_2 \times M_2) = 0$.

Now we finally proceed to the **Euler class**. The Euler class is defined as the Pfaffian of the curvature:

$$e(F) \equiv \mathrm{Pf}(F/2\pi) = \sqrt{\det(F/2\pi)}. \tag{A.52}$$

The Euler class can also be calculated by employing the splitting principle and the Chern class. Let E be as in the Pontrjagin class case (below equation (A.49)). Then we can calculate its Euler class by

$$e(E) = \prod_{i=1}^{n} c_1(L_i). \tag{A.53}$$

Note that, from this equation, it follows that the Euler class agrees with the top Chern class.

An important case of the Euler class is the **Euler form** which is the Euler class of the tangent bundle, $e(TM)$, also denoted $e(M)$. We set $e(M) = 0$ if $\dim(M)$ is odd in accordance with the fact that the Pfaffian vanishes in odd dimensions. The integral of the Euler form over the manifold is the Euler number, this is a theorem that generalizes the famous Gauss-Bonet theorem, mathematically, we have:

$$\chi(M) = \int_M e(M). \tag{A.54}$$

A.6 Spin Bundles

Now we proceed to introduce spinors as sections over a manifold. Like the tangent bundle can be regarded as an associated bundle to the principal bundle $SO(n)$, the spinor bundle will be constructed in association to the $SPIN(n)$ group. To study the $SPIN(n)$ group we first need the Clifford Algebra.

Let V be a n-dimensional vector space with a quadratic form Q. The Clifford Algebra $C(V)$ is given by

$$vw + wv = -2Q(v, w). \tag{A.55}$$

Here we are interested in representing the Clifford algebra with gamma matrices and quadratic form given by a inner product. This means that we are interested in structures like:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}. \tag{A.56}$$

We can connect the gamma representation with the first one by writing a vector as $v = \gamma^\mu v_\mu$. Anyway, from now on we denote the elements of $C(V)$ of degree 1 by c_i which are associated with the element e_i of a frame of V.

The Clifford algebra is closely related to the exterior algebra, note that both have dimension 2^n. Map the latter into the first by the quantization map c. It is reminiscent of the quantization of fermionic fields where the classical fermion anti-commutes but the quantized satisfies anti-commutation relations. Mathematically,

$$c : \Lambda V \to C(V), \tag{A.57}$$

$$e_i \wedge ... \wedge e_k \mapsto c_i ... c_k. \tag{A.58}$$

Define a grading of $C(V)$ by $C^k(V) \equiv c(\Lambda^k M)$. It can be shown that $C^2(V)$ is a Lie algebra isomorphic to $\mathfrak{so}(V)$ with the isomorphism defined by

$$\tau : C^2(V) \to \mathfrak{so}(V), \tag{A.59}$$

$$\tau(a) := [a, v], \tag{A.60}$$

where $v \in C^1(V)$.

To represent a $\mathfrak{so}(V)$ element, say A_i, as a Clifford algebra element we use the following formula:

$$\tau^{-1}(A_i) = \frac{1}{2}\sum_{j<k} g(A_i e_j, e_k) c_j c_k, \tag{A.61}$$

$$= \frac{1}{4}\left(\omega_i\right)_{jk} c_j c_k, \tag{A.62}$$

where there is a sum in the last line and $\left(\omega_i\right)_{jk} \equiv g(A_i e_j, e_k)$ is known as the **spin connection**.

Finally, define the $SPIN(V)$ group as the exponential of $C^2(V)$. So if $g \in SPIN(V)$, then $g = \exp(t^{ij} c_i c_j)$. An important property of the spin group is that it is a double cover of $SO(V)$, if $n > 1$. Hence it carries the interesting properties of the action of $SO(V)$ on V, namely, preserves the inner-product and the orientation.

A.6.1 Spinor bundle

To construct the spinor representation of $SPIN(V)$, we define the chirality operator. Let c_i be an oriented orthonormal basis of $C^1(V)$, then define the **chirality operator** as

$$c_{n+1} = i^p c_1 ... c_n, \tag{A.63}$$

where $p = n/2$ if n is even and $p = (n+1)/2$ if n is odd. The chirality operator satisfies some important properties: $c_{n+1} \in C(V) \otimes \mathbb{C}$; $c_{n+1}^2 = 1$; and it anti-commutes with all the c_i if n is even.

Consider the case where n is even. It is possible to construct a unique representation of the spin group called the **spinor module** S such that the endomorphisms of S are given by $C(V) \otimes \mathbb{C}$ and S is written as $S = S^+ \oplus S^-$, where $S^\pm$ are the half-spinor representations [3] given by

[3] Both $S^\pm$ are representations of the spin group.

$$S^{\pm} = \{\epsilon \in S | c_{n+1} \cdot \epsilon = \pm\epsilon\}. \tag{A.64}$$

Like we did with other G-structures, define the **spin-structure** on M as a $SPIN(n)$-principal bundle over M denoted as $SPIN(M)$. The question wether it is possible to define a spin-structure on M have been investigated, but we will not discuss it here. On the other hand, it is important to note that if M admits a spin-structure then it is a Riemannian oriented manifold and $SPIN(M)$ inherits the Levi-Civita connection from $SO(M)$.

Having at hands the spin-structure, we can define the **spinor bundle** $\mathcal{S}$ as the associated bundle of $SPIN(M)$ over S. Mathematically, this is written as

$$\mathcal{S} = SPIN(M) \times_{SPIN(n)} S. \tag{A.65}$$

Lastly we give the local expression of the Levi-Civita covariant derivative on the spinor bundle. Recall that a local expression of the covariant derivative is given by equation (A.13) with the appropriate representation of the connection. In the spinor case, the representation of $\mathfrak{so}(n)$ is given in the Clifford algebra by (A.61). Thus, we have

$$\nabla_i = \partial_i + \frac{1}{4}\left(\omega_i\right)_{jk} c_j c_k, \tag{A.66}$$

where we have a summation on the indices j and k.

A.7 Summary of Conventions of Differential Forms

Here we give the conventions for forms and the operators that was used throughout this work. Let ω be a k-form:

$$\omega = \frac{1}{k!}\omega_{\mu_1\ldots\mu_k}\mathrm{d}x^{\mu_1} \wedge \ldots \wedge x^{\mu_k}; \tag{A.67}$$

$$\mathrm{d}\omega = \frac{1}{k!}\frac{\partial}{\partial x^{\nu}}\omega_{\mu_1\ldots\mu_k}\mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge x^{\mu_k}; \tag{A.68}$$

$$X \lrcorner\, \omega = \frac{1}{k!}X^{\nu}\omega_{\nu\mu_1\ldots\mu_{k-1}}\mathrm{d}x^{\mu_1} \wedge \ldots \wedge x^{\mu_{k-1}}; \tag{A.69}$$

$$*\omega = \frac{\sqrt{g}}{r!(n-r)!}\omega_{\mu_1\ldots\mu_k}\varepsilon^{\mu_1\ldots\mu_k}{}_{\mu_{k+1}\ldots\mu_n}\mathrm{d}x^{\mu_{k+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}; \tag{A.70}$$

$$\eta \wedge *\omega = \frac{1}{k!}\eta_{\mu_1\ldots\mu_k}\omega^{\mu_1\ldots\mu_k} * 1. \tag{A.71}$$

Appendix B

Lie derivative of a spinor and $U(1)$ charge on $\mathbb{R}^2$

It is not so immediate to define and calculate the Lie derivative of a spinor as it is for tensors. While for vectors we have push-forwards and for differential forms we have pull-backs, in general, there is no natural differential map for spinors. Nevertheless, a construction can be found on [27], there it is derived a formula equivalent to the following

$$\mathcal{L}_\xi \epsilon = \xi^\mu \nabla_\mu \epsilon + \frac{1}{8}\left(\mathrm{d}\xi^\flat\right)_{\mu\nu} \gamma^{\mu\nu}\epsilon. \tag{B.1}$$

We now want to consider spinors with a definite charge under rotations near the fixed point of the rotations. For simplicity, we look at spinors on $\mathbb{R}^2$. Mathematically that means that we are interested in spinors that satisfy:

$$\mathcal{L}_\xi \epsilon = \mathrm{i} q \epsilon, \tag{B.2}$$

the constant q is the charge. On $\mathbb{R}^2$, the generator of rotations is $\xi = x\partial_y - y\partial_x$. Thus we have:

$$\xi^\flat = g_{\mu\nu}\xi^\nu \mathrm{d}x^\mu = -y\mathrm{d}x + x\mathrm{d}y, \tag{B.3}$$

$$\implies \mathrm{d}\xi^\flat = 2\mathrm{d}x \wedge \mathrm{d}y. \tag{B.4}$$

We take the first two Pauli matrices to be our gamma-matrices of $\mathbb{R}^2$. That is, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^2$. Hence, $\gamma^{12} = -\gamma^{21} = \mathrm{i}\sigma^3$. Contracting with $\mathrm{d}\xi^\flat$, we have $\left(\mathrm{d}\xi^\flat\right)_{\mu\nu}\gamma^{\mu\nu} = 4\mathrm{i}\sigma^3$.

To analyze near the fixed point, we assume that the spinor satisfies a differential equation like $\nabla_\mu \epsilon = (\mathcal{M}\cdot\gamma)\,\epsilon$[1], where $\mathcal{M}$ is a combination of differential forms non-

[1]The Killing spinor equation that appeared in the main text is an example of such condition.

singular at the fixed point set of ξ. With this assumption, we have that $\xi^\mu \nabla_\mu \epsilon|_{\mathscr{F}} = 0$, where $\mathscr{F}$ is the fixed point set of ξ.

Collecting the above considerations, we get

$$\mathcal{L}_\xi|_{\mathscr{F}} = \frac{\mathrm{i}}{2}\sigma^3\epsilon = \mathrm{i}q\epsilon, \tag{B.5}$$

from which we deduce that $|q| = \frac{1}{2}$. The sign of q determines the chirality of ϵ because $\sigma^3 = -\mathrm{i}\sigma^1\sigma^2$ can be taken to be the chirality operator on $\mathbb{R}^2$. This result is not surprising because, naively speaking, the spinor rotates by "half the angle". In fact, we could deduce the same thing by looking at the rotation of a constant spinor around the origin of $\mathbb{R}^2$.

Appendix C

Charge of the Killing spinor

We want to show that the Killing spinor has zero charge under the action of the Killing vector built from it, $\xi^\mu = \bar{\epsilon}\gamma^\mu\epsilon$, i.e. $\mathcal{L}_\xi\epsilon = 0$ in the supersymmetric gauge, that is for the choice of gauge field A such that $\xi \lrcorner A = \sqrt{2}XP$.

$$\begin{aligned} \bar{\epsilon}\mathcal{L}_\xi\epsilon &= \bar{\epsilon}(\xi^\mu\nabla_\mu\epsilon + \tfrac{1}{8}\mathrm{d}\xi^\flat_{\mu\nu}\gamma^{\mu\nu}\epsilon) \\ &= \bar{\epsilon}(\xi^\mu(D_\mu - \tfrac{\mathrm{i}}{2}A_\mu)\epsilon + \tfrac{1}{8}\mathrm{d}\xi^\flat_{\mu\nu}\gamma^{\mu\nu}\epsilon). \end{aligned} \tag{C.1}$$

One can use the right hand side of the Killing spinor equation (4.4) to substitute $D_\mu\epsilon$ and equation (4.17) of the differential conditions on the bilinears to obtain

$$\begin{aligned} = &-\tfrac{\mathrm{i}}{\sqrt{2}}(XP)S - \tfrac{1}{24\sqrt{2}}X^{-1}B_{\nu\rho}(\xi \lrcorner V)^{\nu\rho} + \tfrac{3\mathrm{i}X^{-1}}{8\sqrt{2}}(\xi \lrcorner F) \lrcorner \tilde{K} \\ &+ \tfrac{\mathrm{i}}{8X^2}\left(\tfrac{2\sqrt{2}}{3}X^{-1}\tilde{Y} + \mathrm{i}X^4\xi \lrcorner {*H} + \sqrt{2}X(PF - \tfrac{2}{3}\mathrm{i}SB)\right)_{\mu\nu} Y^{\mu\nu}\epsilon \\ &- \tfrac{1}{4X}(\mathrm{d}X \wedge \xi^\flat)_{\mu\nu}\bar{\epsilon}\gamma^{\mu\nu}\epsilon \\ &- \tfrac{X^2\xi^\mu}{48}(3({*H})_\mu{}^{\tau\sigma}\mathrm{i}\bar{\epsilon}\gamma_{\tau\sigma}\epsilon + 3\bar{\epsilon}H_\mu{}^{\rho\sigma}\gamma_{\rho\sigma}\gamma_7\epsilon), \end{aligned} \tag{C.2}$$

where we used that some terms of the KSE vanish immediately when we employ the bilinears and the $SU(2)$ structure. Namely, the first two vanish because $\xi^\mu\bar{\epsilon}\gamma_\mu\gamma_7\epsilon = 0$, the fourth vanishes as $B_{\mu\nu}\xi^\mu\xi^\nu = 0$ and the sixth vanishes because we end up with something proportional to $F_{\nu\rho}\left(\xi \lrcorner \tilde{K} \wedge J\right)^{\nu\rho}$, which is zero.

Furthermore, the term with $H_{\nu\rho\sigma}\gamma^{\nu\rho\sigma}\gamma_\mu\gamma_7$ was replaced by terms containing H and $*H$. To see this, first note that $[\gamma^{\nu\rho\sigma}, \gamma_\mu] = 2\gamma^{\nu\rho\sigma}{}_\mu$ and $\{\gamma^{\nu\rho\sigma}, \gamma_\mu\} = 6\delta_\mu{}^{[\nu}\gamma^{\rho\sigma]}$, from which follows that $\gamma^{\nu\rho\sigma}\gamma_\mu = \gamma^{\nu\rho\sigma}{}_\mu + 3\delta_\mu{}^{[\nu}\gamma^{\rho\sigma]}$. Then writing $\gamma^{\nu\rho\sigma}{}_\mu = \frac{\mathrm{i}}{2}\varepsilon^{\nu\rho\sigma}{}_\mu{}^{\tau_1\tau_2}\gamma_{\tau_1\tau_2}\gamma_7$ leads to the desired result.

Writing everything with the bilinears (4.9), we get

$$\begin{aligned} =& -\tfrac{\mathrm{i}}{\sqrt{2}}(XP)S - \tfrac{1}{24\sqrt{2}}X^{-1}B_{\nu\rho}(\xi \lrcorner V)^{\nu\rho} + \tfrac{3\mathrm{i}X^{-1}}{8\sqrt{2}}(\xi \lrcorner F)\lrcorner \tilde{K} + \tfrac{\mathrm{i}\sqrt{2}}{12X^3}\tilde{Y}_{\mu\nu}Y^{\mu\nu} \\ &- \tfrac{X^2}{8}(\xi \lrcorner *H)_{\mu\nu}Y^{\mu\nu} + \tfrac{\sqrt{2}\mathrm{i}}{8X}PF_{\mu\nu}Y^{\mu\nu} + \tfrac{\sqrt{2}}{12X}SB_{\mu\nu}Y^{\mu\nu} + \tfrac{\mathrm{i}}{4X}(\mathrm{d}X\wedge\xi^\flat)_{\mu\nu}Y^{\mu\nu} \\ &- \tfrac{X^2}{16}(\xi \lrcorner *H)^{\tau\sigma}Y_{\tau\sigma} + \tfrac{\mathrm{i}X^2}{16}(\xi \lrcorner H)^{\rho\sigma}\tilde{Y}_{\rho\sigma}\,. \end{aligned} \tag{C.3}$$

Writing equation (A.1) in Appendix A of [14] with $\mathbb{A} = 1$ yields

$$\tfrac{\sqrt{2}S}{12X}B^{\mu\nu}Y_{\mu\nu} = \tfrac{\mathrm{i}S}{\sqrt{2}}(X - X^{-3})P - \tfrac{SX^2}{12}H^{\mu\nu\rho}\tilde{V}_{\mu\nu\rho} - \tfrac{\mathrm{i}\sqrt{2}S}{8X}F^{\mu\nu}\tilde{Y}_{\mu\nu}, \tag{C.4}$$

using again equation (A.1), this time with $\mathbb{A} = \gamma_\alpha$ and contracting the whole equation with ξ^α gives a further relation

$$\begin{aligned} -\tfrac{1}{24\sqrt{2}X}B^{\mu\nu}(\xi \lrcorner V)_{\mu\nu} =& -\tfrac{X^2}{16}(\xi \lrcorner *H)^{\tau\sigma}Y_{\tau\sigma} + \tfrac{\mathrm{i}}{8\sqrt{2}X}(\xi \lrcorner F)\lrcorner \tilde{K} \\ &+ \tfrac{\mathrm{i}}{4X}(\xi^\flat \wedge \mathrm{d}X)^{\alpha\mu}Y_{\alpha\mu}\,. \end{aligned} \tag{C.5}$$

We can insert both into the expression for $\bar{\epsilon}\mathcal{L}_\xi\epsilon$ to be left with

$$\begin{aligned} \bar{\epsilon}\mathcal{L}_\xi\epsilon =& \Big(-\tfrac{X^2}{16}(\xi \lrcorner *H)^{\tau\sigma}Y_{\tau\sigma} - \tfrac{\mathrm{i}}{8\sqrt{2}X}(\xi \lrcorner F)\lrcorner \tilde{K}\Big) \\ &+ \tfrac{3\mathrm{i}X^{-1}}{8\sqrt{2}}(\xi \lrcorner F)\lrcorner \tilde{K} \\ &+ \tfrac{\mathrm{i}\sqrt{2}}{12X^3}\tilde{Y}_{\mu\nu}Y^{\mu\nu} - \tfrac{X^2}{8}(\xi \lrcorner *H)_{\mu\nu}Y^{\mu\nu} + \tfrac{\sqrt{2}\mathrm{i}}{8X}PF_{\mu\nu}Y^{\mu\nu} \\ &+ \Big(-\tfrac{\mathrm{i}S}{\sqrt{2}}X^{-3}P - \tfrac{SX^2}{12}H^{\mu\nu\rho}\tilde{V}_{\mu\nu\rho} - \tfrac{\mathrm{i}\sqrt{2}}{8X}SF^{\mu\nu}\tilde{Y}_{\mu\nu}\Big) \\ &- \tfrac{X^2}{16}(\xi \lrcorner *H)^{\tau\sigma}Y_{\tau\sigma} + \tfrac{\mathrm{i}X^2}{16}(\xi \lrcorner H)^{\rho\sigma}\tilde{Y}_{\rho\sigma}, \end{aligned} \tag{C.6}$$

where the terms

$$-\tfrac{X^2}{16}(\xi \lrcorner *H)^{\tau\sigma}Y_{\tau\sigma} + -\tfrac{X^2}{8}(\xi \lrcorner *H)_{\mu\nu}Y^{\mu\nu} - \tfrac{SX^2}{12}H^{\mu\nu\rho}\tilde{V}_{\mu\nu\rho} - \tfrac{X^2}{16}(\xi \lrcorner *H)^{\tau\sigma}Y_{\tau\sigma} \tag{C.7}$$

are canceling one another (using that $H \lrcorner *V = -(*H)\lrcorner V$, $\tilde{V} = *V$ and $\xi^\flat \wedge Y = SV$) and we are left with only

$$\begin{aligned} \bar{\epsilon}\mathcal{L}_\xi\epsilon =& \tfrac{\mathrm{i}}{2\sqrt{2}X}(\xi \lrcorner F)\lrcorner \tilde{K} + \tfrac{\mathrm{i}}{4\sqrt{2}X}PF_{\mu\nu}Y^{\mu\nu} - \tfrac{\mathrm{i}}{4\sqrt{2}X}SF^{\mu\nu}\tilde{Y}_{\mu\nu} \\ &+ \tfrac{\mathrm{i}\sqrt{2}}{12X^3}\tilde{Y}_{\mu\nu}Y^{\mu\nu} - \tfrac{\mathrm{i}S}{\sqrt{2}}X^{-3}P \\ &+ \tfrac{\mathrm{i}X^2}{16}(\xi \lrcorner H)^{\rho\sigma}\tilde{Y}_{\rho\sigma}\,. \end{aligned} \tag{C.8}$$

Each line of this equation cancel on its own. The cancellation of the first two lines follows directly from the relations of the bilinears in terms of the $SU(2)$-structure (4.14).

For the last term, we need to use again equation (A.1) from the appendix of [14] with the choice $\mathbb{A} = \gamma_\alpha$ and contracting ξ^α into it, this time using the lower choice of sign

$$\begin{aligned}(\xi \lrcorner \mathrm{d}X)S = &- \tfrac{1}{\sqrt{2}2}(X^2 - X^{-2})\xi \lrcorner \tilde{K} - \mathrm{i}\tfrac{X^3}{4}(\xi \lrcorner H)^{\nu\rho}\tilde{Y}_{\nu\rho} \\ &+ \tfrac{\mathrm{i}}{6\sqrt{2}}(\xi \lrcorner B) \lrcorner \xi - \tfrac{1}{8\sqrt{2}}F^{\mu\nu}(\xi \lrcorner \tilde{V})_{\mu\nu}\end{aligned} \tag{C.9}$$

where all terms apart from the third line of (C.8) immediately vanish, and hence so does this one. Finally, we are left with

$$\bar{\epsilon}\mathcal{L}_\xi\epsilon = 0\,. \tag{C.10}$$

Appendix D

Conventions for γ-matrices

Here we give an explicit representation of the γ-matrices that form $\mathrm{Cliff}(6,0)$ and that was used in the calculations of the Hyperbolic Black Hole, they all are 8×8 hermitian matrices. We also give $\gamma^7 = \mathrm{i}\gamma^1 ... \gamma^6$, which satisfies $(\gamma^7)^2 = 1$ and is used to build the chirality operator $\mathrm{i}\gamma^7$. In this basis, the matrices $\gamma^{(2i-1)2i}$ are all diagonal, which is useful to verify (5.14).

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \otimes \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{D.1}$$

$$\gamma^3 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathrm{i}\sigma^3 \\ -\mathrm{i}\sigma^3 & 0 \end{pmatrix}, \tag{D.2}$$

$$\gamma^5 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix}, \qquad \gamma^6 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathrm{i}\sigma^1 \\ -\mathrm{i}\sigma^1 & 0 \end{pmatrix}, \tag{D.3}$$

$$\gamma^7 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \gamma^{12} = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \otimes \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{D.4}$$

$$\gamma^{34} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \otimes \begin{pmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \qquad \gamma^{56} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \otimes \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}. \tag{D.5}$$