University of Brasília - UnB
Institute of Exact Sciences - IE
Academic Master's in Mathematics

On Some Residual Properties in Groups

by

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Brasília/DF

2024

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Master's Dissertation presented to the Graduate Program in Mathematics at the University of Brasília as part of the requirements for obtaining the degree of Master in Mathematics.

Advisor: Prof. Dr. Igor dos Santos Lima

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Approved on: 29/11/2024

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ACKNOWLEDGMENTS

I would like to thank my parents Marco Fernandes, Patrícia Fernandes and my brother, Lucas Fernandes, who, at every lunch, would tease me by asking how my master's studies were going, yet on the other hand, provided all the support I needed to complete this work. I thank my girlfriend, Larissa Vidal, who gave me all the support, care, and strength during the moments I needed it most.

I would like to extend a special thank you to my father, Dr. Marco Fernandes, who supported me with his knowledge and references for the study of some topics addressed in this dissertation, often during coffee breaks at Araújo's ice cream shop, enriching my experience at UnB.

I am grateful to my old friends from school and childhood, who, in my free time, helped me unwind and relax; to my fellow master's colleagues, and to my guidance siblings of Dr. Igor Lima.

I wish to express my gratitude to my advisor, Professor Dr. Igor dos Santos Lima, who has been with me for several years, supervising two PIBIC projects and this master's thesis, always ready to answer my questions and support my academic journey.

To Professors Dr. Yuri Santos Rego, Dr. Kisnney Emiliano de Almeida, and Dr. Sheila Campos Chagas, who kindly agreed to serve on my committee.

Finally, I thank CNPq for the financial support.

"Mathematical achievements, whatever their value, are the most enduring."
-G. H. Hardy

ABSTRACT

We say that a group G is LERF if every finitely generated subgroup of G is closed in the profinite topology of G. We say that G satisfies (LR) if every finitely generated subgroup is a retract of a subgroup of finite index of G. In this work, we study residual properties such as LERF and (LR), exploring whether these properties are preserved under free constructions, direct products, semidirect products, and amalgamated products. The three main articles studied were written by A. Minasyan, by R. Gitik, S. Margolis, and B. Steinberg together, and by N. Andrew.

Keywords: Virtual retracts, residual properties, LERF, LR.

Sobre Algumas Propriedades Residuais em Grupos

RESUMO

Dizemos que um grupo G é LERF se todo subgrupo finitamente gerado de G é fechado na topologia profinita de G. E dizemos que G satisfaz (LR) se todo subgrupo finitamente gerado é retrato de um subgrupo de índice finito de G. Neste trabalho abordamos o estudo de propriedades residuais, como LERF e (LR), exploramos se essas propriedades são preservadas por construções livres, produtos direto, semidireto e entrelaçado. Os principais artigos estudados foram de A. Minasyan, de R. Gitik, S. Margolis, B. Steinberg e de N. Andrew.

Palavras-chave: Retração virtual, propriedades residuais, LERF, LR.

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Introduction

We say that a group G is residually finite (RF) if the intersection of all normal subgroups of finite index in G is trivial. We say that G is LERF if every finitely generated subgroup of G is closed in the profinite topology of G. Additionally, we say that G satisfies (LR), or is (LR), if every finitely generated subgroup is a virtual retract of G. These properties described are examples of residual properties. The study of these properties has shown that they are quite relevant in Group Theory, and one of the greatest motivations for studying these properties is their relation to the well-known Dehn Problems (see 2.5). In 1940, A. Mal'cev showed that finitely presented, RF groups have a decidable word problem. Years later, in 1958, A. Mal'cev also showed that finitely presented groups satisfying the LERF property have decidable generalized word problem. Furthermore, he also showed in [12] that these two residual properties are related: LERF implies residual finiteness.

Moreover, the (LR) property implies LERF, which implies residual finiteness. The converse does not hold in general; for instance, it is well known that the group F_2 is RF, (LR), and LERF, yet $F_2 \times F_2$ is RF but not LERF and, therefore, not (LR).

In this work, we study how these residual properties are preserved for certain free products/factorizations/constructions. For this purpose, the main articles studied were [1], [38], and [7]. We address the following free constructions: direct product, semidirect product, wreath product, free product, amalgamated free product, and HNN extensions. Inspired by the work of A. Elsawy, with the creation of tables to relate some residual properties and free constructions in [50], we organize the results studied in this dissertation in the following table:

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	not necessarily	not necessarily
$A \times B$	yes	not necessarily	not necessarily
$A \rtimes B$	not necessarily	not necessarily	not necessarily
$A * B$	yes	yes	yes
HNN	not necessarily	not necessarily	not necessarily
$A *_H B$	not necessarily	not necessarily	not necessarily

Table 1: Table of Residual Properties

We will see that (LR) groups are RF and imply LERF (see 3.28). We will also see that the direct product of two groups with the (LR) property will not necessarily itself be a group with this property. On the other hand, we will see that A. Minasyan, in [1], establishes a condition for this product to preserve the (LR) property (see 3.37). Additionally, we will see that groups containing some subgroup with the (LR) property do not necessarily inherit this property, but through certain finite kernel quotients and extensions, this property is preserved (see 3.39).

In [38], several results stemming from the Burns-Romanovskii Theorem were studied, which states that the free product of LERF groups will be a LERF group. Since the theory surrounding this study is Bass-Serre Theory, two chapters were dedicated to understanding this theory. However, most proofs of these results fall outside the scope of this work (including the proofs of the main theorems asserting that the free product preserves both the (LR) and LERF properties 3.51); therefore, some proofs were omitted, and these chapters were placed in the appendix.

In Chapter 1, we address free constructions, specifically defining a free group, free product, amalgamated free product, and HNN extensions. For each free construction, we show the existence and uniqueness of its construction. We also study how groups can be represented through generators and relations. All these definitions are necessary for studying the preservation of residual properties under these constructions. Finally, we also address various topological concepts, as the LERF property has a strong topological definition: we can define a group with this property if and only if every finitely generated subgroup of this group is closed in its profinite topology. Additionally, we can establish an equivalence by showing that a group is RF if and only if its profinite topology is Hausdorff (see 3.8).

In Chapter 2, we study the residual finiteness property, as well as examples of groups that satisfy this property, namely RF groups. We see that this property has various equivalences and verify that the free groups defined in Chapter 1 are RF 2.6. We observe that this property is closed under subgroups 3.8, which means that a subgroup of a RF group will also be RF. Finally, we present results supporting our table by examining whether residual finiteness is preserved under free constructions. In 2.12, we see that the residual finiteness property is preserved under direct products. Gruenberg's Theorem provides a necessary and sufficient condition for the wreath product to preserve residual finiteness. (see 2.19).

We show that the preservation of the RF property by free products follows from Theorem 3.8. If the free product is amalgamated, however, this property is no longer preserved, as seen in 2.22. Lastly, in Example 2.22, we demonstrate that HNN extensions do not necessarily preserve the RF property.

In Chapter 3, we define LERF groups and their equivalences and also introduce virtual retracts, since a group satisfying the (LR) property is one in which every finitely generated subgroup is a virtual retract. We present examples and equivalences of the LERF property. We show that (LR) implies LERF, which in turn implies residual finiteness (see 3.28). We also show that every free group satisfies (LR) (see 3.29), and therefore satisfies LERF. We then present results and examples that support Table 3.4. In 3.37, we show a sufficient condition for the direct product of groups to be (LR). In 3.9, we show that the direct product does not preserve the LERF property; note that this example also demonstrates that the direct product does not preserve (LR). Example 3.40 shows that the amalgamated free product of groups satisfying (LR) is not necessarily (LR). Note that this example also shows that the LERF property is not preserved by amalgamated free products. Theorem 3.42 provides an example in which both (LR) and LERF properties are not preserved by wreath products.

Finally, at the end of the chapter, we present Theorem 3.51, concluding that the free product preserves both (LR) and LERF properties. The chapter concludes with Table 3.4, which displays the collection of results obtained from studying the properties under the free constructions presented.

Chapter 1

Preliminaries

In this chapter, we cover basic and fundamental results in Combinatorial Group Theory. Through generators and relations, we will study the following concepts: free groups, presentations, free product, amalgamated free product, and HNN extensions. Some theorems on normal form will be demonstrated using van der Waerden's Method. This chapter on free constructions is based on comprehensive research, with emphasis on the classical works of Daniel E. Cohen in [3], Magnus, Karras, and Solitar in [6], and Lyndon and Schupp in [4], which served as pillars for our understanding of the techniques and concepts discussed.

1.1 Free Constructions

We can say that a group G is **free** over a set of generators if there are no relations between these generators other than those directly implied by the group operations in G. Formally, free groups are defined through a Universal Property. The Universal Property is a fundamental concept in Category Theory.

Definition 1.1 (**Universal Property of Free Groups**). *Let X be a set, G a group, and $i : X \longrightarrow G$ a function. We say that the pair (G, i) is **free** on X if for every group H and function $f : X \longrightarrow H$, there exists a unique homomorphism $\phi : G \longrightarrow H$ such that $\phi \circ i = f$.*

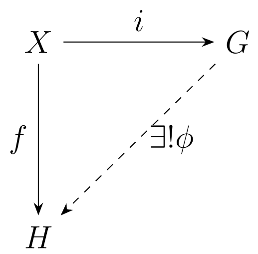


Figure 1.1: Diagram of the Universal Property of Free Groups.

Example 1.2. ***The trivial group is free on the empty set.***

Let $X = \emptyset$, $G = 1$, and $i : \emptyset \longrightarrow 1$ the unique function. For any group H and function $f : \emptyset \longrightarrow H$, f is also unique, as $\emptyset$ admits exactly one function to any set. By the Universal Property, there exists a unique homomorphism $\phi : 1 \longrightarrow H$ such that $\phi \circ i = f$.

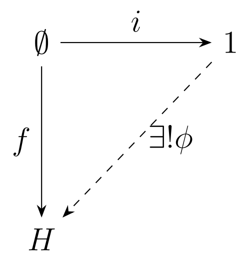


Figure 1.2: Universal Property applied to $G = 1$ with $X = \emptyset$.

Thus, $G = 1$ satisfies the Universal Property of the free group on $\emptyset$, and we conclude that the trivial group is free on the empty set.

Example 1.3. ***The additive group of integers is free on the set $\{x\}$.***

Let $X = \{x\}$, $G = \mathbb{Z}$, and $i : \{x\} \longrightarrow \mathbb{Z}$ defined by $i(x) = 1$. For any group H and function $f : \{x\} \longrightarrow H$, there exists a unique homomorphism $\phi : \mathbb{Z} \longrightarrow H$ such that $\phi(1) = f(x)$. This ϕ is determined by the rule $\phi(n) = f(x)^n$ for all $n \in \mathbb{Z}$.

Thus, $\mathbb{Z}$ satisfies the Universal Property of the free group on $\{x\}$, and we conclude that $\mathbb{Z}$ is free on one generator.

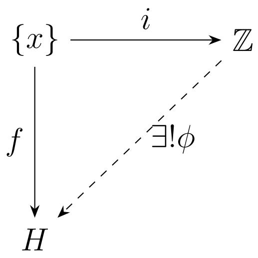


Figure 1.3: Universal Property applied to $G = \mathbb{Z}$ with $X = \{x\}$.

We define a free group by a Universal Property, so we must verify the existence and uniqueness related to this definition. Moreover, we will check the injectivity of i in Definition 1.1. We will start with uniqueness:

Proposition 1.4. *Let (G_1, i_1) and (G_2, i_2) be free on a set X. Then there exists an isomorphism $\phi : G_1 \longrightarrow G_2$ such that $\phi \circ i_1 = i_2$.*

Proof. We will use the Universal Property of Free Groups twice: first, to induce the homomorphism $\phi : G_1 \longrightarrow G_2$, and then to induce the homomorphism $\tilde{\phi} : G_2 \longrightarrow G_1$. We have the following diagrams:

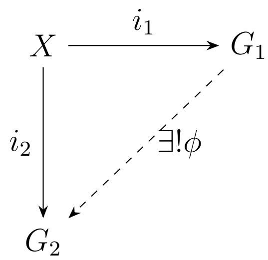


Figure 1.4: Universal Property applied to G_1 with $H = G_2$.

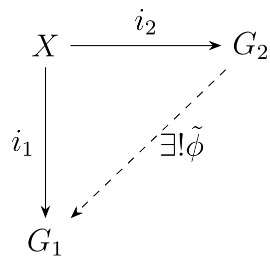


Figure 1.5: Universal Property applied to G_2 with $H = G_1$.

Thus, by the uniqueness in Definition 1.1, we have that $\phi \circ i_1 = i_2$ and $\phi \circ i_2 = i_1$. Therefore, $\phi \circ \tilde{\phi} \circ i_2 = i_2 = id_{G_2} \circ i_2$ and $\tilde{\phi} \circ \phi \circ i_1 = i_1 = id_{G_1} \circ i_1$. By the uniqueness of the Universal Property, we conclude that $\phi \circ \tilde{\phi} = id_{G_2}$ and $\tilde{\phi} \circ \phi = id_{G_1}$. By the definition of inverse function, we can assert that $\tilde{\phi} = \phi^{-1}$. Thus, ϕ is an isomorphism.

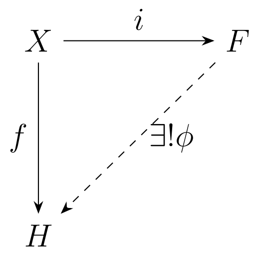
□

Next, given that (G, i) is free on X we will check the injectivity of i, as we will use it to prove the existence of free groups.

Proposition 1.5. *Let F be a free group on the set X, and let i be the function from X to F. Then i is injective.*

To prove this result, we will show that there exists a group H in which X will be embedded. That is, we will demonstrate the existence of an injective map from X to H. Consequently, we will conclude that i is injective.

Proof. As F is free on X, by the Universal Property of free groups, for any group H and a map $f : X \longrightarrow H$, there exists a unique homomorphism $\phi : F \longrightarrow H$ such that $\phi \circ i = f$.



In particular, let

$$H = \mathbb{Z}^X := \{\text{functions from } X \text{ to } \mathbb{Z}\} = \{\gamma_y : X \longrightarrow \mathbb{Z},\ y \in X\},$$

where

$$\gamma_y(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

We define $f : X \longrightarrow H$ such that $f(x) = \gamma_x$. By construction, f is injective. For $x, y \in X$,

$$f(x) = f(y) \implies \gamma_x = \gamma_y \implies x = y.$$

Now, we will show that the map i is injective. Indeed, if $x, y \in X$ and $\phi(x) = \phi(y)$, we have

$$\phi \circ i(x) = \phi \circ i(y) \implies f(x) = f(y) \implies x = y.$$

Hence, i is injective.

□

To show the existence of free groups, as we will see next, we will construct an equivalence class and define the operations between the elements of these classes.

1.1.1 Existence of Free Groups

Consider a set X formed by elements that we will call **letters**, similar to the letters of a dictionary. In addition, we will consider another set $\overline{X}$ disjoint from X, such that there is a bijection where each $x \in X$ has a corresponding element denoted by x^{-1} in $\overline{X}$. We will consider elements of the union $X \cup \overline{X}$ of these sets, such that the elements can be written in the following form: $x_{i_1}^{\epsilon_1}, \ldots, x_{i_n}^{\epsilon_n}$, with $\epsilon_i = \pm 1$, for $i = 1, \ldots, n$. Thus, by concatenating letters $x_{i_1}^{\epsilon_1} \ldots x_{i_n}^{\epsilon_n}$, a **word** is formed. Later, we will see that any non-trivial element of a free group can be written uniquely as a word of a specific kind.

We will denote the set of all words with letters in $X \cup \overline{X}$ by $X^{\pm}$.

Let w be a word in $X^{\pm}$. We say that w is a **reducible word** if there exists j such that $x_{i_{j+1}}^{\epsilon_{j+1}} = x_{i_j}^{-\epsilon_j}$. If w is not reducible, we say that w is a **reduced word**. We can reduce w if it is a reducible word. To do this, simply take $w' = x_{i_1}^{\epsilon_1} \dots x_{i_{j-1}}^{\epsilon_{j-1}} x_{i_{j+2}}^{\epsilon_{j+2}} \dots x_{i_n}^{\epsilon_n}$. This reduction of w to w' is called an **elementary reduction**.

If, after this reduction, the word w' is still not reduced, we can repeat the process of elementary reduction successively until the word becomes reduced.

Example 1.6. *Let w and w' be words in $X^{\pm}$ given by $w = bb^{-1}aabcc^{-1}$ and $w' = b^{-1}baab$. Note that through an elementary reduction on w we obtain the word $aabcc^{-1}$ by canceling bb^{-1}. We can apply another elementary reduction on $aabcc^{-1}$, canceling cc^{-1}, obtaining the reduced word aab. Note also that we can reduce w' to aab by canceling $b^{-1}b$.*

We say that $w \sim w'$ whenever:

(i) $w = w'$ or

(ii) There exists a sequence of words $w_1, \dots, w_k$ for some k, such that $w_1 = w, w_k = w'$ and w_i, w_{i+1} with $i = 1, \dots, k-1$ differ by an elementary reduction.

In the previous example, we have that w is equivalent to w', since the sequence:

$$bb^{-1}aabcc^{-1}, aabcc^{-1}, aab, b^{-1}baab$$

satisfies (ii) of the relation seen above.

By definition, $xx^{-1} \sim$ **empty word**.

We will show next that $\sim$ is an equivalence relation. To do this, we must satisfy the following properties:

Let u, v, w be words in $X^{\pm}$.

(a) $u \sim u$;

(b) $u \sim v \implies v \sim u$;

(c) $u \sim v$ and $v \sim w \implies u \sim w$.

Indeed, we have that:

(a) $u \sim u$, since $u = u$.

(b) If $u = v$, then $v = u$ is direct. Now, if there exists a sequence $w_1, \dots, w_k$ of words such that $w_1 = u$ and $w_k = v$ and w_i, w_{i+1} with $i = 1, \dots, k-1$ differ by an elementary reduction, we will have a new sequence of words $u_j = w_{k-(j-1)}$ with

$j = 1, \ldots, k$ such that two consecutive words differ by an elementary reduction and $u_1 = v$ and $u_k = u$.

(c) If $u_1, \ldots, u_k$ and $v_1, \ldots, v_l$ are two sequences of words such that $u_1 = u, u_k = v, v_1 = v, v_l = w$ and consecutive terms differ by an elementary reduction, then $u_1, \ldots, u_k, v_2, \ldots, v_l$ is a sequence of words whose consecutive terms also differ by an elementary reduction; if $u = v$ or $v = w$, the proof is straightforward.

Notation 1.7. *We will denote the set of all equivalence classes $[w]$ of words w in $X^{\pm}$ by* $\mathbf{F(X)}$.

Notation 1.8. *The class of the empty word will be denoted by $[\ ]$.*

We define on $F(X)$ the product $[w]\cdot[v] = [wv]$ such that wv is a simple juxtaposition of the words w and v.

- Let v, v', w, w' be words in $X^{\pm}$; the product is well-defined: $w' \sim w$ and $v' \sim v \implies w'v' \sim wv$.

  We will show that $F(X)$ with the product $\cdot$ is a group.

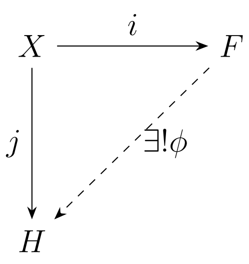
- If $w = x_{i_1}^{\epsilon_1} \ldots x_{i_n}^{\epsilon_n}$ is any word in $F(X)$, then we define $w^{-1} = x_{i_n}^{-\epsilon_n} \ldots x_{i_1}^{-\epsilon_1}$. We have that $[ww^{-1}] = [\ ] = [w^{-1}w]$. Thus, $[w]^{-1} = [w^{-1}]$. Note that $[\ ]$ is the identity element of $F(X)$ and w^{-1} is the inverse of w.

- Let u, v, w be words in $F(X)$; we have that $(wv)u = w(vu)$. That is, $F(X)$ is associative.

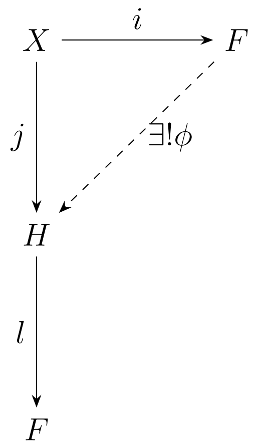
Therefore, $(F(X), \cdot)$ is a group.

Lemma 1.9. *If F is free in X, then X generates F.*

Proof. Let $H = \langle X \rangle := \bigcap\{K \leq F \mid K \supseteq X\}$, and let $j : X \longrightarrow H$ be an inclusion. Since F is the set of all equivalence classes of words in X, we have that $X \subseteq F$, hence $i : X \longrightarrow F$ is an inclusion of X in F. By the Universal Property of Free Groups, there exists a unique homomorphism $\phi : F \longrightarrow H$



such that $\phi \circ i = j$. Note that $\langle X \rangle$ contains X and $X^{-1} = \{x^{-1} \mid x \in X\}$. Thus, there exists an inclusion $l : H \longrightarrow F$.



Therefore, consider the map $l \circ \phi : F \longrightarrow H \longrightarrow F$. We have that

$$(l \circ \phi) \circ i = l \circ j = i = \mathrm{Id}_F \circ i.$$

By the uniqueness of the Universal Property of Free Groups, we have that $\mathrm{id}_F = l \circ \phi$. It follows that l has a right inverse, hence it is surjective, and ϕ has a left inverse, hence it is injective. However, $l : H \longrightarrow F$ is the inclusion map of the subgroup, so l was already injective. Therefore, $H = F$.

□

Notation 1.10. *We say that X is the **basis** of F and $|X|$ is the **rank** of F, denoted by $r(F)$.*

The following theorem is one of the most important results in this chapter, and we will see that the technique used in its proof will appear several times throughout this chapter.

Theorem 1.11. *(**Normal Form Theorem for Free Groups**) Every equivalence class of $F(X)$ has only one reduced word.*

Proof. (**Van der Waerden's Method**): Let S be the set of all reduced words in $X^{\pm}$ and $G = \mathrm{Perm}(S)$ the group of permutations of the set S. Let $w = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ be a reduced word, where $\epsilon_k = \pm 1$. We define $f : X \longrightarrow G$ as follows:

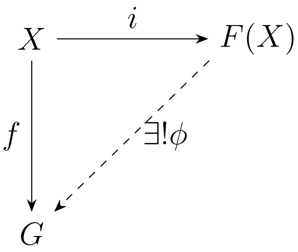
$$f(x)(x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}) = \begin{cases} x_{i_2}^{\epsilon_2} \dots x_{i_n}^{\epsilon_n} & \text{if } x_{i_1}^{\epsilon_1} = x^{-1}, \\ x x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \dots x_{i_n}^{\epsilon_n} & \text{if } x_{i_1}^{\epsilon_1} \neq x^{-1}. \end{cases}$$

In this way, we also have that the function $f(x)^{-1}$ defined by:

$$f(x)^{-1}(x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}) = \begin{cases} x_{i_2}^{\epsilon_2} \dots x_{i_n}^{\epsilon_n} & \text{if } x_{i_1}^{\epsilon_1} = x, \\ x^{-1} x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \dots x_{i_n}^{\epsilon_n} & \text{otherwise.} \end{cases}$$

is the inverse of $f(x)$. Thus, $f(x) \in G$.

As we saw in Lemma 1.1.1, the set X generates the free group $F(X)$. By the Universal Property of Free Groups, there exists a unique homomorphism $\phi : F(X) \longrightarrow G$ such that $\phi([x]) = f(x)$, $\forall x \in X$, and $\phi([w]) = f(x_{i_1})^{\epsilon_1} \dots f(x_{i_n})^{\epsilon_n}$.



We want to show that if there is another reduced word in the equivalence class, besides w, it must be equal to w. Thus, if we have $w \sim w'$ where w and w' are both reduced words, then

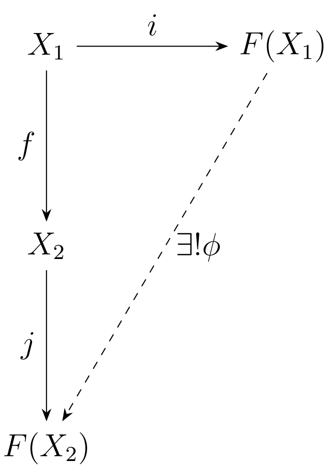
$$[w] = [w'] \text{ and } \phi([w])[\ ] = \phi([w'])[\ ] \implies w = w'.$$

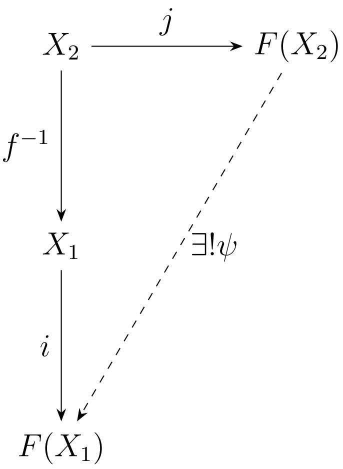
$\square$

Proposition 1.12. *F_i is free on $X_i (i = 1, 2)$ and $F_1 \cong F_2 \iff |X_1| = |X_2|$.*

Proof. Assuming that $G = \mathbb{Z}_2$ is the cyclic group of order 2, since $F_1 \cong F_2$, we have $|Hom(F_1, \mathbb{Z}_2)| = |Hom(F_2, \mathbb{Z}_2)|$, thus $|Map(X_1, \mathbb{Z}_2)| = |Map(X_2, \mathbb{Z}_2)|$. For any sets B and C with cardinalities b and c, respectively, we know $|Map(B, C)| = c^b$. Therefore, we have $2^{|X_1|} = 2^{|X_2|}$, and by taking the base-2 logarithm, we obtain $|X_1| = |X_2|$.

Conversely, assuming $|X_1| = |X_2|$, let f be a bijection from X_1 to X_2. Then, there exists a unique homomorphism $\phi : F(X_2) \longrightarrow F(X_1)$ extending $j \circ f$. Additionally, let f^{-1} be the bijection from X_2 to X_1, so there exists a unique homomorphism $\psi : F(X_1) \longrightarrow F(X_2)$ extending $i \circ f^{-1}$.





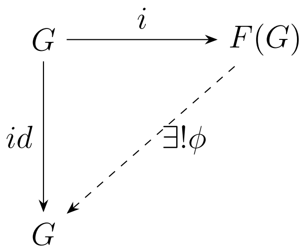
Note that both $\psi \circ \phi$ and the identity function $Id_{F(X_1)}$ extend the identity function on X_1, and by the uniqueness of the universal property of the free group, we have $\psi \circ \phi = Id_{F(X_1)}$. Similarly, we have $\phi \circ \psi = Id_{F(X_2)}$. Therefore, $\phi = \psi^{-1}$, from which we conclude that ϕ is an isomorphism.

□

Now, we present propositions and equivalences to characterize free groups.

Proposition 1.13. *Every group is isomorphic to a quotient of some free group.*

Proof. Consider the identity map $id : G \longrightarrow G$ and the monomorphism $i : G \longrightarrow F(G)$ such that $i(g) = [g]$ for any $g \in G$. Since $F(G)$ is free, by the Universal Property, there exists a unique homomorphism $\phi : F(G) \longrightarrow G$ such that, for all $g \in G$, we have $\phi([g]) = g$.



We see that ϕ is surjective, and by the Isomorphism Theorem, $G \cong \frac{F(G)}{ker(\phi)}$.

□

Proposition 1.14. *Let X be a subset of a group G. Then, the following are equivalent:*

i) G is free with basis X;

ii) Any element of G can be uniquely written as $x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ for $n \geq 0$, $x_{i_k} \in X$, $\epsilon_k = \pm 1$, where $\epsilon_{k+1} \neq -\epsilon_k$ if $i_{k+1} = i_k$;

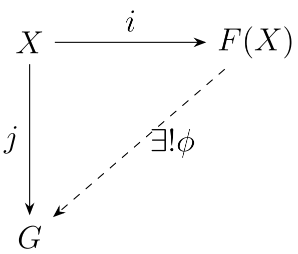
iii) X generates G, and 1 cannot be written as $g = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$, with $n > 0$, $x_{i_k} \in X$, $\epsilon_k = \pm 1$, and $\epsilon_{k+1} \neq -\epsilon_k$ if $i_{k+1} = i_k$.

Proof. $(ii) \implies (iii)$: This follows directly.

$(iii) \implies (ii)$: Suppose X generates G, and let g be an arbitrary element of G. Then $g = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ with $n \geq 0$, $x_{i_k} \in X$, $\epsilon_k = \pm 1$, and $\epsilon_{k+1} \neq -\epsilon_k$ if $i_{k+1} = i_k$. If g has another distinct decomposition, $g = x_{j_1}^{\lambda_1} \dots x_{j_m}^{\lambda_m}$, then we have $1 = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n} x_{j_m}^{-\lambda_m} \dots x_{j_1}^{-\lambda_1}$, which represents a non-trivial product of elements in X, contradicting (iii).

$(i) \implies (ii)$ and (iii): If $G \cong F(X)$, then X generates G. If $1 = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ with the conditions of (ii) and $n > 0$, then $[\ ] = [x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}]$. Since the word $g = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$ is reduced, by Theorem 1.11, it must be unique.

(ii) and $(iii) \implies (i)$: By the Universal Property of Free Groups, the inclusion of X in G induces a homomorphism ϕ from $F(X)$ to G given by $\phi([x]) = x$ for every $x \in X$.



Since G is generated by X, we have that ϕ is surjective. To show that ϕ is an isomorphism, it remains to demonstrate injectivity. Let g and $g' \in G$. Then, by (ii), both can be written uniquely. If $g'g^{-1} = 1$, then by (iii), we have that $g'g^{-1}$ can be reduced to the empty word. Thus, $[g'g^{-1}] = [\ ]$. Therefore, since $ker(\phi)$ is trivial, the injectivity of ϕ guarantees that $G \cong F(X)$. □

1.1.2 Generators and Relations

Before discussing generators and relations, it is important to recall the definition of normal closure, as it will be essential for understanding the definitions that follow.

Definition 1.15 (Normal Closure of a Subset)**.** *The normal closure of a subset S of a group G, denoted by $\langle S \rangle^G$, is the smallest normal subgroup of G that contains S.*

When S is non-empty, then,

$$\langle S \rangle^G = \left\{ \prod_{i=1}^{k} g_i^{-1} s_i^{\epsilon_i} g_i \;\middle|\; g_i \in G,\ s_i \in S,\ \epsilon_i = \pm 1,\ k \geq 0 \right\}.$$

Let G be a group, X a subset, and $\phi : F(X) \longrightarrow G$ an injective homomorphism. It follows that $G = \langle \phi([x]) \mid x \in X \rangle$. Since we have an embedding of X into $F(X)$, we can use $\phi(x)$ instead of $\phi([x])$. Thus, we have that $G = \langle \phi(X) \rangle$. We will then call X the set of **generators** of the group G.

Let R be a subset of $F(X)$ that satisfies the equality $\langle R \rangle^{F(X)} = ker(\phi)$. Then we will call R the set of **relations** of G. Thus, we can say that G has the **presentation** $\langle X \mid R \rangle^{\phi}$. Often denoted simply as $\langle X \mid R \rangle$. We can have multiple presentations for the same group. However, such presentations differ only by manipulations of the equalities of the relations.

When there is a presentation for G such that X is finite, G is said to be **finitely generated**. When there is a presentation for G such that both G and R are finite, G is called **finitely presented**. We will use these concepts later in the definition of groups that have the $LERF$ property.

Example 1.16. *We can see that the group $\mathbb{Z}_n$ generated by x has the following presentation: $\langle x \mid x^n \rangle$ where $\phi : F(\{x\}) \longrightarrow \mathbb{Z}_n$ is the homomorphism given by $\phi([x]) = x$. We can see that $R = \{[x^n]\}$.*

In the previous case, the relation $x^n = 1$ was denoted simply by x^n. It will be common to denote it this way when one side of the relation equals 1.

We say that a group G is n-generated if there is a presentation for G such that G is generated by n elements. We denote by D_{2n} the dihedral group of order $2n$. Below are examples of some presentations of the dihedral group D_8:

Example 1.17.

$D_8 = \langle a, b \mid a^2, b^2, abab^{-1} \rangle = \langle a, b \mid a^2 = 1, b^2 = 1, abab^{-1} = 1 \rangle = \langle a, b \mid a^2 = 1, b^2 = 1, ab = ba^{-1} \rangle$

Note that a group of order $2n$ will have the following presentation:

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle.$$

We can think of the dihedral group as a group where 2 types of objects interact: a are the rotations and b are the reflections. The relation that defines the rotation is $a^n = 1$ since after n rotations, we return to the identity. The relation that defines the reflection is $b^2 = 1$ since 2 reflections are sufficient to return to the identity. Finally, the last relation $ab = ba^{-1}$ defines how the rotation and reflection interact.

Furthermore, the infinite dihedral group D_∞ has the presentation:

$$D_\infty = \langle x, y \mid y^2, (xy)^2 \rangle.$$

Example 1.18. *The free group generated by X has the presentation $\langle X \mid \rangle$*

Since free groups do not have any relations, we can show that there is no element of finite order in a group G.

Proposition 1.19. *Free groups are torsion-free.*

Proof. Let $w \in F$ be a non-trivial element of a free group F. By taking conjugates, if necessary, we have that if $w^n = 1$ for some integer $n \neq 0$, then $w = 1$, otherwise, w would have relations. □

As we have seen, a group can have multiple presentations, and the **Tietze Transformations** play a fundamental role in manipulating and simplifying group presentations, as well as in showing that two distinct presentations of the same group are indeed equivalent.

Definition 1.20 (Tietze Transformations). *Let $S = \{a_1, \ldots, a_n\}$ and $R = \{r_1, \ldots, r_m\}$, and let $G = \langle S \mid R \rangle$. The* ***Tietze Transformations*** *are as follows:*

T1: Add a relation $w = 1$ that is contained in $[w] \in R$;

T2: Delete a relation that is a consequence of the others;

T3: Simultaneously, add a new generator a_{n+1} to S and new relations of the form $a_{n+1} = w \in F_n$ to R;

T4: Remove a generator $a_i = w$ and replace a_i for w in the relations.

Theorem 1.21 (H. Tietze, 1908). *$\langle S_1 \mid R_1 \rangle \cong \langle S_2 \mid R_2 \rangle$ if, and only if, there exists a finite sequence of Tietze transformations leading $\langle S_1 \mid R_1 \rangle$ to $\langle S_2 \mid R_2 \rangle$.*

Example 1.22. *Let $G = \langle a, b, c, d \mid ab = c, bc = d, cd = a, da = b \rangle$. We want to show that $G \cong \mathbb{Z}/5\mathbb{Z}$.*

$$
\begin{aligned}
G &= \langle a, b, c, d \mid ab = c, bc = d, cd = a, da = b \rangle \text{ (Apply T1)} \\
&= \langle a, b, c, d \mid ab = c, bc = d, cd = a, da = b, cbc = a \rangle \text{ (Apply T2)} \\
&= \langle a, b, c, d \mid ab = c, bc = d, da = b, cbc = a \rangle \text{ (Apply T1)} \\
&= \langle a, b, c, d \mid ab = c, bc = d, da = b, cbc = a, ca = 1 \rangle \text{ (Apply T2)} \\
&= \langle a, b, c, d \mid ab = c, bc = d, cbc = a, ca = 1 \rangle \text{ (Apply T4)} \\
&= \langle a, b, c \mid ab = c, bbab = 1, aba = 1 \rangle \text{ (Apply T4)} \\
&= \langle a, b \mid a = b^{-3}, aba = 1 \rangle \text{ (Apply T1)} \\
&= \langle a, b \mid a = b^{-3}, b^{-5} = 1 \rangle \text{ (Apply T4)} \\
&\cong \langle b \mid b^{-5} = 1, b^{-5} = 1 \rangle \text{ (Apply T1 and T2)}
\end{aligned}
$$

$$\cong \langle b \mid b^5 = 1, b^{-5} = 1 \rangle \text{ (Apply T2)}$$

$$\cong \mathbb{Z}/5\mathbb{Z}.$$

The following theorem allows us to infer properties of groups and relate different algebraic structures through homomorphisms, establishing a relationship between the kernel and the image of a group homomorphism.

Theorem 1.23. ***(von Dyck's Theorem)*** *Let $G = \langle X \mid R \rangle^{\phi}$, $f : X \longrightarrow H$ a function from X to any group H, and $\varphi : F(X) \longrightarrow H$ the corresponding extension in the universal diagram of $F(X)$. Then, there exists a homomorphism $\psi : G \longrightarrow H$ such that $f(x) = \psi \circ \phi$, $\forall x \in X$, if $R \subseteq \ker(\phi)$. Moreover, ψ is an epimorphism if $f(X)$ generates H.*

Proof. The proof of this result can be found in [4] (Theorem 14, p. 19). □

1.2 Free Product

Definition 1.24 (Universal Property of the Free Product). *Let G_1, G_2 and G be groups, and $i_1 : G_1 \longrightarrow G$, $i_2 : G_2 \longrightarrow G$ homomorphisms that satisfy the following Universal Property: for any group H and any homomorphisms $f_1 : G_1 \longrightarrow H$ and $f_2 : G_2 \longrightarrow H$, there exists a unique homomorphism $\phi : G \longrightarrow H$ such that $\phi \circ i_1 = f_1$ and $\phi \circ i_2 = f_2$. Then, G is called the* ***free product*** *of G_1 and G_2 and is denoted by $G_1 * G_2$.*

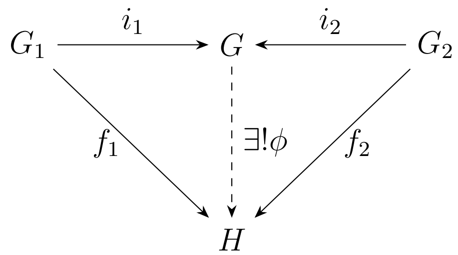


Figure 1.6: Diagram of the Universal Property of the Free Product.

We are considering only the free product of two groups. Note that this definition can be extended to the free product of a family of groups.

The free product can also be defined in terms of group presentations: Let A and B be groups with presentations $A = \langle X_1 \mid R_1 \rangle$ and $B = \langle X_2 \mid R_2 \rangle$, respectively, where $X_1 \cap X_2 = \emptyset$. Then,

$$A * B = \langle X_1 \cup X_2 | \; R_1 \cup R_2 \rangle^{\phi}$$

where ϕ is the natural homomorphism from $F(X_1 \cup X_2)$ to $\frac{F(X_1 \cup X_2)}{\langle R_1 \cup R_2 \rangle^{F(X_1 \cup X_2)}}$.

A and B are called free factors. The free product $A * B$ is independent of the presentation of A and B.

As demonstrated for free groups, we will investigate results regarding the existence and uniqueness of free products. Additionally, we will examine whether i_α, where $\alpha \in \{1, 2\}$, is a monomorphism.

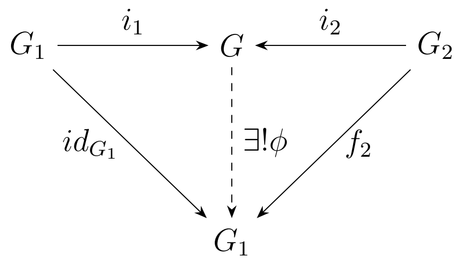
Proposition 1.25. *If G and H are free products of the groups G_1 and G_2, then there exists a unique isomorphism $\phi : G \longrightarrow H$ such that $\phi \circ i_k = j_k$, $k = 1, 2$, where i_k are homomorphisms from G_k to G and j_k are homomorphisms from G_k to H.*

Proof. By the Universal Property of the Free Product, there exists a unique homomorphism $\phi : G \longrightarrow H$ such that $\phi \circ i_k = j_k$. Similarly, there exists a unique $\phi' : H \longrightarrow G$ such that $\phi' \circ j_k = i_k$. Thus, $\phi' \circ \phi \circ i_k = i_k$ and $\phi \circ \phi' \circ j_k = j_k$. By the uniqueness of homomorphisms in the Universal Property, $\phi \circ \phi' = id_H$ and $\phi' \circ \phi = id_G$. Therefore, we have $\phi' = \phi^{-1}$ and ϕ is an isomorphism. □

Proposition 1.26. *If $G_1 * G_2$ is the free product with $i_1 : G_1 \longrightarrow G_1 * G_2$, $i_2 : G_2 \longrightarrow$*

$G_1 * G_2$, then i_1 and i_2 are monomorphisms.

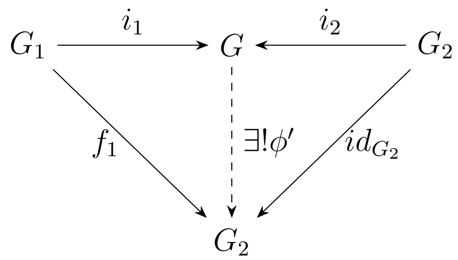
Proof. Consider the following diagram, where we will take the group H to be G_1, $f_1 = id_{G_1}$, and $f_2 : G_2 \longrightarrow G_1$ to be any homomorphism.



By the Universal Property, we have $\phi \circ i_1 = id_{G_1}$ and $\phi \circ i_2 = f_2$. Now we will show that i_1 is injective. Let $a, b \in G$ such that $i_1(a) = i_1(b)$. Then,

$$\phi \circ i_1(a) = \phi \circ i_1(b) \implies id_{G_1}(a) = id_{G_1}(b) \implies a = b.$$

Similarly, we show that i_2 is injective by taking $H = G_2$, $f_2 = id_{G_2}$, and f_1 to be any homomorphism.



By the Universal Property, we have $\phi' \circ i_2 = id_{G_2}$ and $\phi' \circ i_1 = f_1$. Let $a, b \in G$ such that $i_2(a) = i_2(b)$. Then,

$$\phi' \circ i_2(a) = \phi' \circ i_2(b) \implies id_{G_2}(a) = id_{G_2}(b) \implies a = b.$$

Therefore, we conclude that i_1 and i_2 are monomorphisms. □

1.2.1 Existence of Free Product

Theorem 1.27. *Let G_1 and G_2 be groups. Then, the free product $G_1 * G_2$ exists.*

Proof. As we have seen earlier, we can visualize the free product of two groups through group presentations. Thus, let $G_1 = \langle X_1 \mid R_1 \rangle^{\phi_1}$ and $G_2 = \langle X_2 \mid R_2 \rangle^{\phi_2}$ with $X_1 \cap X_2 = \emptyset$

as the presentations of G_1 and G_2, respectively. Then,

$$G = \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle^{\phi},$$

where ϕ is the natural homomorphism from $F(X_1 \cup X_2)$ to $\frac{F(X_1 \cup X_2)}{\langle R_1 \cup R_2 \rangle^{F(X_1 \cup X_2)}}$. By Theorem 1.23, there exists a homomorphism $i_k : G_k \longrightarrow G$ such that $i_k \circ \phi_k = \phi$, for $k = 1, 2$. □

Example 1.28. *The free group $F(X)$ is the free product of the infinite cyclic groups $\langle x \rangle$, $x \in X$.*

To verify this, consider $\iota_x : \langle x \rangle \to F(X)$ that sends x to the corresponding element in $F(X)$. This is a natural inclusion of $\langle x \rangle$ into $F(X)$, which is guaranteed by the definition of $F(X)$ as the free group generated by X.

Now, for each $x \in X$, consider $\phi_x : \langle x \rangle \to G$, where G is an arbitrary group. By the Universal Property of the Free Product, there exists a unique homomorphism $\phi : F(X) \longrightarrow G$ such that $\phi \circ \iota_x = \phi_x$ for all $x \in X$.

Thus, $F(X)$ satisfies the Universal Property of the Free Product of the cyclic groups $\langle x \rangle$, which implies that $F(X)$ is indeed the free product of these groups.

Theorem 1.29 (Normal Form Theorem for Free Products). *Let $G = G_1 * G_2$ be a free product. Then,*

i) $i_k : G_k \longrightarrow G_1 * G_2$ is a monomorphism, where $k = 1, 2$;

ii) Taking i_1 and i_2 as inclusions, every element of G can be uniquely expressed as $g_1 \dots g_n$, where $n \geq 0$, $g_i \in G_1 \cup G_2$ and g_i, g_{i+1} do not belong to the same group, for $i < n$.

The proof of this theorem also utilizes the van der Waerden Method and is similar to the proof of the normal form for free groups (See 1.11).

Proof. Denote $i_\alpha(g_\alpha)$ by $\overline{g_\alpha}$ with $k = 1, 2$ and $g_\alpha \in G_k$. By Proposition 1.26, item (i) follows. Any $u \in G$ can be expressed as $\overline{g_1} \dots \overline{g_n}$, with $n \geq 0$, $g_i \in G_1 \cup G_2$, $g_i \neq 1$ since $G_1 \cup G_2$ generates G. If $g_{i+1} \neq {g_i}^{-1}$, for $i < n$, $g_i, g_{i+1} \in G_k$, then $g = g_1 \dots g_{i-1}(g_i g_{i+1}) g_{i+2} \dots g_n$. On the other hand, if $g_{i+1} = {g_i}^{-1}$, then g will be reduced to $g_1 \dots g_{i-1} g_{i+2} \dots g_n$ since the term $g_i g_{i+1}$ is canceled. We want to show that g can be uniquely expressed as $g_1 \dots g_n$. To do this, we will utilize the van der Waerden Method.

Let S be the set of all sequences $(g_1, \dots, g_n)$ with $n \geq 0$, such that g_i, g_{i+1} do not belong to the same group. In particular, $(\ ) \in S$. Let h_k be a nontrivial element of G_k. Then, a mapping $f_k : G_k \longrightarrow Perm(S)$ is defined as follows:

$$f_k(h_k)(g_1,\ldots,g_n) = \begin{cases} (h_k, g_1, \ldots, g_n) & \text{if } g_1 \notin G_k, \\ (h_k g_1, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h_k g_1 \neq 1, \\ (g_2, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h_k g_1 = 1. \end{cases}$$

and $f_k(1) = id_S$.

We want to show that f_k is a homomorphism, and furthermore, that f_k has an inverse. Let $h'_k \in G_k, h'_k h_k \neq 1$. Then,

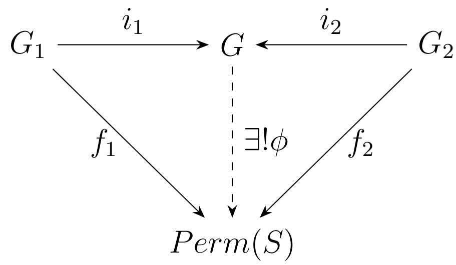
$$f_k(h_k h'_k)(g_1, \ldots g_n) = \begin{cases} (h_k h'_k, g_1, \ldots, g_n) & \text{if } g_1 \notin G_k, \\ (h_k h'_k g_1, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h_k h'_k g_1 \neq 1, \\ (g_2, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h_k h'_k g_1 = 1. \end{cases}$$

and

$$f_k(h_k) \circ f_k(h'_k)(g_1, \ldots g_n) = \begin{cases} (h_k h'_k, g_1, \ldots, g_n) & \text{if } g_1 \notin G_k, \\ (h_k h'_k g_1, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h'_k g_1 \neq 1 \text{ and } h_k h'_k g_1 \neq 1, \\ (g_2, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h'_k g_1 \neq 1 \text{ and } h_k h'_k g_1 = 1, \\ (h_k, \ldots, g_n) & \text{if } g_1 \in G_k \text{ and } h'_k g_1 = 1. \end{cases}$$

Thus, $f_k(h_k h'_k) = f_k(h_k) \circ f_k(h'_k)$, and f_k is a homomorphism. Since $f_k(h_k^{-1}) = f_k(h_k)^{-1} \; \forall h_k \in G_k$, we have that $f_k(h_k)$ has an inverse, and f_k is well-defined.

Since G is a free product, there exists a unique homomorphism ϕ that makes the diagram below commute.



Suppose that an element $g \in G$ can be written in two different forms as $g_1 \ldots g_n$, with $g_i \in G_1 \cup G_2,\ g_i \neq 1$ and $g_i,\ g_{i+1}$ not belonging to the same group, for $i < n$, and also as $h_1 \ldots h_m$ under the same conditions for $h_j \in G_1 \cup G_2,\ h_j \neq 1$ and $h_j,\ h_{j+1}$ not belonging to the same group, for $j < n$. Thus, we have that

$$\phi(g)(\ ) = f_{k_1}(g_1) \circ \cdots \circ f_{k_n}(g_n)(\ ) = f_{k_1}(g_1) \circ \cdots \circ f_{k_{n-1}}(g_n) = (g_1, \ldots, g_n) = (h_1, \ldots, h_m),$$

with $k_1, \ldots, k_n \in \{1, 2\}$.

Therefore, we have $m = n$ and $g_i = h_i,\ \forall i \leq n$. □

The following theorem provides a **characterization of free products**, and its proof is analogous to the characterization of free groups.

Theorem 1.30. *Let G_1 and G_2 be subgroups of a group G. The following are equivalent:*

(i) $G = G_1 * G_2$*;*

(ii) Every element of G can be uniquely written as $g_1 \ldots g_n$ where $n > 0$, $g_i \in G_1 \cup G_2$, $g_i \neq 1$, and g_i, g_{i+1} do not belong to the same group;

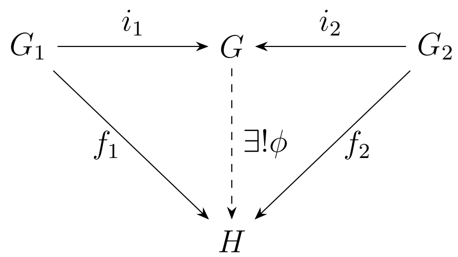
(iii) G is generated by G_1 and G_2, and 1 cannot be written as a product $g_1 \ldots g_n$ with $n > 0$, $g_i \in G_1 \cup G_2$, $g_i \neq 1$, and g_i, g_{i+1} not belonging to the same group for $i < n$.

Proof. $(ii) \implies (iii)$: Since every element of G can be uniquely written as $g_1 \ldots g_n$, with $n \geq 0$, $g_i \in G_1 \cup G_2$, $g_i \neq 1$, and g_i, g_{i+1} not belonging to the same group for $i < n$, we have that every element of G is indeed a product of elements in $G_1 \cup G_2$, so G is generated by the subgroups G_1 and G_2. Moreover, 1 is the identity element of G with $n = 0$.

$(iii) \implies (ii)$: Since G_1 and G_2 generate G, every element of G can be written as $g_1 \ldots g_n$, with $n \geq 0$, $g_i \in G_1 \cup G_2$, $g_i \neq 1$, and g_i, g_{i+1} not belonging to the same group for $i < n$. We will prove by contradiction that this representation is unique. Suppose that $g \in G$ has two distinct representations: $g = g_1 \ldots g_n$ and $g = h_1 \ldots h_m$. Then, $1 = g_1 \ldots g_n h_m^{-1} \ldots h_1^{-1}$, implying that 1 can be expressed as a product of elements in G, a contradiction.

$(i) \implies (ii)$: Follows from Theorem 1.29 part (ii).

(ii) and $(iii) \implies (i)$: By the Universal Property of Free Products, there exists a unique homomorphism $\phi : G_1 * G_2 \to G$ such that the diagram



commutes, and $\phi|_{G_k}$ is the inclusion of G_k in H for $k \in \{1,2\}$. Thus, g is written as in (ii), and the inclusion $\phi|_{G_k}$ also gives us the equality $g_1 \dots g_n = \phi(g_1 \dots g_n)$. Therefore, ϕ is an isomorphism, as given two elements $g_1 \dots g_n, h_1 \dots h_m \in G$ with $\phi(g_1 \dots g_n) = \phi(h_1 \dots h_m)$, we have $g_1 \dots g_n = h_1 \dots h_m$. Hence, $G = G_1 * G_2$. □

To introduce the amalgamated product of groups, we will first define what a **push-out** of groups is.

1.3 Push-Outs and Amalgamated Product

Definition 1.31 (**Universal Property of Push-Outs**). *Let G_0, G_1, and G_2 be groups, and let $i_k : G_0 \to G_k$, $k = 1,2$, be homomorphisms. Let G be a group, and $j_k : G_k \to G$ be homomorphisms. We denote G as the* ***push-out*** *of i_1 and i_2 if:*

1) $j_1 i_1 = j_2 i_2$;

2) For any group H and homomorphisms $f_r : G_r \to H$ with $r = 1,2$ and $f_1 \circ i_1 = f_2 \circ i_2$, there exists a unique homomorphism $\phi : G \to H$ such that $f_r = \phi \circ j_r$, $r = 1,2$.

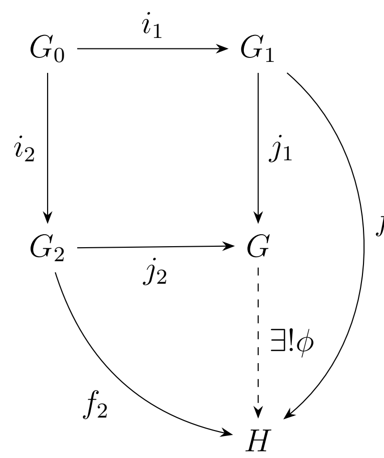
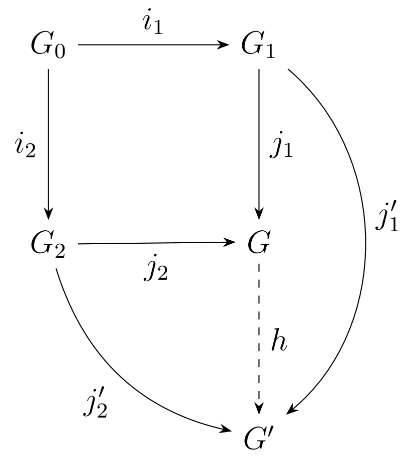


Figure 1.7: Diagram of the Universal Property of Push-Outs of groups.

As with all constructions using the Universal Property, the push-out is unique up to isomorphism.

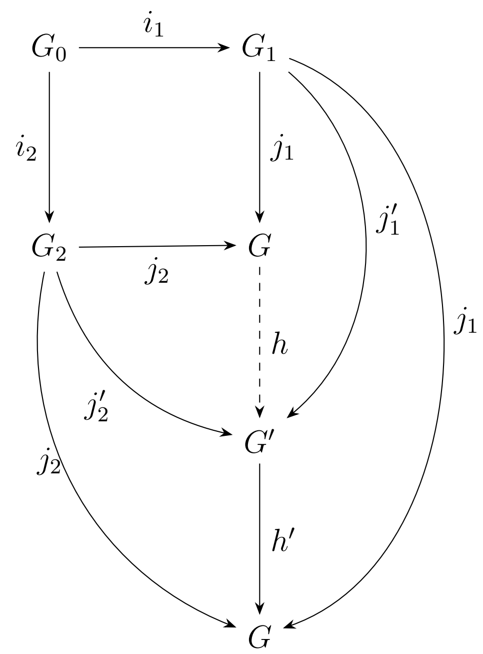
Proposition 1.32. *Given groups G_0, G_1, and G_2, with homomorphisms $i_k : G_0 \to G_k$ for $k = 1,2$, the push-out G of i_1 and i_2 is unique.*

Proof. We will show that if G' is another push-out of i_1 and i_2, with corresponding homomorphisms $j'_1 : G_1 \to G'$ and $j'_2 : G_2 \to G'$, then there exists a unique isomorphism $h : G \to G'$ such that the diagram



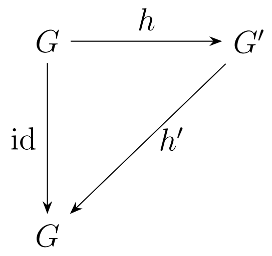
commutes.

By the Universal Property of G, there exists a unique homomorphism $h : G \to G'$. Applying the Universal Property again, this time for G', we find that the diagram



also commutes.

From this, we can deduce that the identity map $\mathrm{id} : G \to G$ makes the following diagram commute:



Thus, by the uniqueness of the Universal Property, we have $h' \circ h = \text{id}$. Reversing the roles of G and G' in a similar construction, we obtain $h \circ h' = \text{id}$. Therefore, by the definition of an inverse function, we conclude that h is an isomorphism with $h^{-1} = h'$. $\square$

Now, with the following theorem, we will prove the existence of the push-out given any pair of homomorphisms.

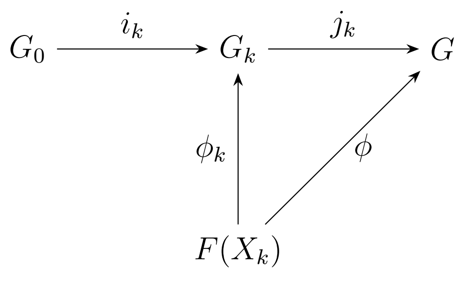
Theorem 1.33. *Let G_0, G_1, G_2 be groups, and $i_1 : G_0 \to G_1$ and $i_2 : G_0 \to G_2$ be homomorphisms. Then, the pair (i_1, i_2) has a push-out.*

Proof. Let $G_k = \langle X_k \mid R_k \rangle^{\phi_k}$ for $k = 1, 2$, with $X_1 \cap X_2 = \emptyset$. For each $x_0 \in G_0$, define elements $\alpha_{x_0,1} \in F(X_1)$ and $\alpha_{x_0,2} \in F(X_2)$ such that $i_1(x_0) = \phi_1(\alpha_{x_0,1})$ and $i_2(x_0) = \phi_2(\alpha_{x_0,2})$.

Define the group

$$G = \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\alpha_{x_0,1}\alpha_{x_0,2}^{-1}\} \rangle^{\phi}$$

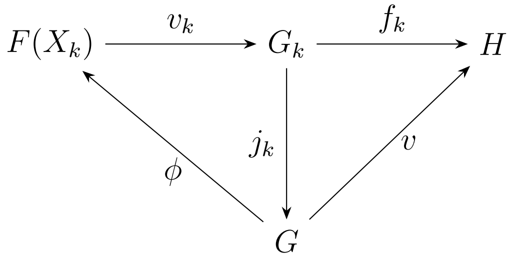
with homomorphisms $j_k : G_k \to G$ such that $\phi(x_k) = j_k \circ \phi_k(x_k)$ for $k = 1, 2$.



For each $x_0 \in G_0$,

$$[j_1 \circ i_1(x_0)][j_2 \circ i_2(x_0)]^{-1} = j_1(\phi_1(\alpha_{x_0,1}))j_2(\phi_2(\alpha_{x_0,2}))^{-1} = 1 \Rightarrow j_1 \circ i_1 = j_2 \circ i_2.$$

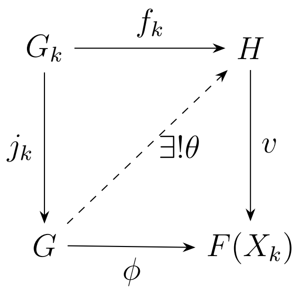
Let H be a group and $f_k : G_k \to H$ be homomorphisms such that $f_1 \circ i_1 = f_2 \circ i_2$. Define the homomorphisms $v_k = f_k \circ \phi_k$, which are trivial on R_1 and R_2, respectively. These induce a homomorphism $v : F(X_1 \cup X_2) \to H$ such that $v(x_k) = v_k(x_k)\ \forall x_k \in X_k$ for $k = 1, 2$.



Furthermore,

$$v(\alpha_{x_0,1}\alpha_{x_0,2}^{-1}) = v_1(\alpha_{x_0,1})v_2(\alpha_{x_0,2})^{-1} = [f_1 \circ \phi_1(\alpha_{x_0,1})][f_2 \circ \phi_2(\alpha_{x_0,2})]^{-1} = [f_1 \circ i_1(x_0)][f_2 \circ i_2(x_0)]^{-1} = 1.$$

Therefore, $\ker(\phi) \subseteq \ker(v)$. By Theorem 1.23, there exists a homomorphism $\theta : G \to H$ such that for $k = 1, 2$, $\theta \circ \phi(x_k) = v_k(x_k)$ for all $x_k \in X_k$.



Thus,

$$f_k(\phi_k(x_k)) = v_k(x_k) = v(x_k) = \theta \circ \phi(x_k) = \theta \circ j_k(\phi_k(x_k)) \Rightarrow \theta \circ j_k = f_k.$$

Therefore, the diagram commutes, and the pair (i_1, i_2) has a push-out. □

When i_1 and i_2 are injections, that is, injective homomorphisms, the push-out of G is called the **amalgamated free product** of G_1 and G_2 with G_0 amalgamated. We denote this specific product by $G = G_1 *_{G_0} G_2$. In this case, G_0 is a subgroup of both G_1 and G_2.

Below, we present another definition of an amalgamated free product via its presentation.

Definition 1.34. *Let $X = \langle G_1 \mid R_1 \rangle$ and $Y = \langle G_2 \mid R_2 \rangle$. The amalgamated free product $X *_H Y$ is then:*

$$X *_H Y = \langle G_1, G_2 \mid R_1, R_2, \varphi_1(h) = \varphi_2(h) \textit{ for all } h \in H \rangle$$

where φ_1 *and* φ_2 *are the inclusions of* H *in* X *and* Y*, respectively.*

As an example of an amalgamated free product, we have:

Example 1.35. *The braid group* B_3*: let*

$$G = \langle a \mid \rangle,\ H = \langle b \mid \rangle,\ K = \langle a^3 \rangle,\ \phi(a^3) = b^2,$$

such that $G, H,$ *and* K *are all infinite cyclic. The presentation is then given by*

$$G *_K H = \langle a, b \mid a^3 = b^2 \rangle,$$

which is isomorphic to the braid group B_3*.*

For the reader interested in studying braid groups, see [8].

To define the Normal Form Theorem for amalgamated free products, it is necessary to recall the definition of a right transversal of a subgroup.

Definition 1.36 (Right Transversal)**.** *Let* A *be a group and* $B \leq A$ *a subgroup of* A*. Then, a subset* C *of* A *is a* ***right transversal*** *of* B *in* A *if*

$$A = \coprod_{c \in C} Bc.$$

Let $G = A *_C B$ be a group, and let S, T be right transversals of C in A and B, respectively, with $1 \in S \cap T$, that is, S contains a representative of each coset Ca. We then obtain a normal form theorem using the transversals S and T.

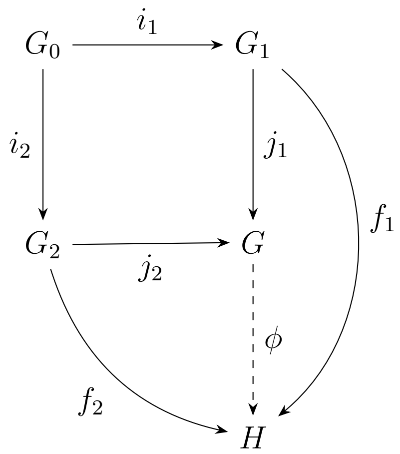
Theorem 1.37 (**Normal Form Theorem for Amalgamated Free Products**)**.** *Let* $G = G_1 *_{G_0} G_2$*. Then:*

i) j_1 *and* j_2 *are monomorphisms;*

ii) $j_1(G_1) \cap j_2(G_2) = j_1(G_0) = j_2(G_0)$;

iii) Considering j_1 *and* j_2 *as inclusions, any element of* G *can be uniquely expressed as* $g_0 u_1 \dots u_n$*, where* $n \geq 0$*,* $g_0 \in G_0$ *and* $u_1, \dots, u_n$ *alternate between* $S - \{1\}$ *and* $T - \{1\}$*, with* S *being a right transversal of* G_0 *in* G_1 *and* T *being a right transversal of* G_0 *in* G_2*, and* $1 \in S \cup T$*.*

Proof. (i) Suppose there exists a group H and monomorphisms $f_1 : G_1 \longrightarrow H$, $f_2 : G_2 \longrightarrow H$, such that $f_1 \circ i_1 = f_2 \circ i_2$. By the Universal Property of Push-Outs, there exists a unique homomorphism $\phi : G \longrightarrow H$ such that the diagram



commutes. That is, $\phi \circ j_k = f_k, k = 1, 2$. Therefore, $\phi \circ j_1$ and $\phi \circ j_2$ are monomorphisms. We want to show the injectivity of j_k: let $x, y \in G_1$ with $j_1(x) = j_1(y)$, then we have:

$$j_1(x) = j_1(y) \implies \phi \circ j_1(x) = \phi \circ j_1(y) \implies x = y$$

which implies that j_1, j_2 are monomorphisms. Thus, to prove (i) it is sufficient to find H, f_1, and f_2 that satisfy these hypotheses. Define $H = Perm(G_0 \times S \times T)$ and $f_1 : G_1 \longrightarrow H$, defined as follows:

$$f_1(g_1)(g_0, s, t) = (\tilde{g}_0, \tilde{s}, t), \text{ where } g_1 g_0 s = \tilde{g}_0 \tilde{s} \text{ (since } G_1 = \coprod_{s \in S} G_0 s).$$

We have that $f_1(g_1^{-1})(\tilde{g}_0, \tilde{s}, t) = (g_0', s', t)$ such that

$$g_1^{-1} \tilde{g}_0 \tilde{s} = g_0' s' \implies g_0 s = g_0' s' \implies g_0' = g_0 \text{ and } s' = s,$$

since $G_1 = \coprod_{s \in S} G_0 s$. Thus,

$$f_1(g_1^{-1}) f_1(g_1)(g_0, s, t) = f_1(g_1^{-1})(\tilde{g}_0, \tilde{s}, t) = (g_0', s', t) = (g_0, s, t) = f_1(g_1) f_1(g_1^{-1})(g_0, s, t).$$

Therefore, $f_1(g_1^{-1}) f_1(g_1) = f_1(g_1) f_1(g_1^{-1}) = id_{G_0 \times S \times T}$. Hence, $f_1(g_1^{-1}) = f_1(g_1)^{-1}$ for each $g_1 \in G_1$. Thus, $f_1(g_1)$ is a permutation for all $g_1 \in G_1$. Now we want to show that f_1 is a homomorphism. Hence, for all $(g_0, s, t) \in G_0 \times S \times T$ and $g_1, g_1' \in G_1$, we have that

$$f_1(g_1) f_1(g_1')(g_0, s, t) = f_1(g_1)(g_0', s', t) = (\tilde{g}_0, \tilde{s}, t),$$

where

$$g_1' g_0 s = g_0' s' \text{ and } g_1 g_0' s' = \tilde{g}_0 \tilde{s}, \text{ that is, } (g_1 g_1') g_0 s = \tilde{g}_0 \tilde{s}.$$

Thus, $f_1(g_1)f_1(g_1') = f_1(g_1g_1'), \forall g_1, g_1' \in G_1$. That is, f_1 is a homomorphism. Constructing f_2 in a similar way, we will have two homomorphisms f_1 and f_2. Thus, it remains to show that f_1 and f_2 are injective. We will first demonstrate the injectivity of f_1 by showing that the kernel $ker(f_1)$ is trivial. Indeed, if $g_1 \in ker(f_1)$, then,

$$f_1(g_1)(g_0, s, t) = (g_0, s, t) \iff g_1g_0s = g_0s \iff g_1 = 1.$$

Similarly, we show that f_2 is injective. It remains to verify that $f_1|_{G_0} = f_2|_{G_0}$. However, for any $g_0 \in G_0$ and $(g_0', s, t) \in G_0 \times S \times T$, we have that

$$f_1(g_0)(g_0', s, t) = (g_0g_0', s, t) \text{ and } f_2(g_0)(g_0', s, t) = (g_0g_0', s, t),$$

since $g_0g_0' \in G_0$.

(iii) Taking j_1 and j_2 as inclusions, we have that $G_1 \cup G_2$ generates G. Thus, for any element $g \in G$, we have $g = g_1 \dots g_n, g_i \in G_1 \cup G_2, i \in \{1, \dots, n\}$.

We will show, by induction on n, that g can be written as in (iii).

(Induction Base): If $n = 1$, then $g = g_0s$, or $g = g_0t$, or $g = g_0, g_0 \in G_0, s \in S - \{1\}, t \in T - \{1\}$.

(Induction Hypothesis): Now, for $n > 1$, suppose that $g_2 \dots g_n = g_0u_1 \dots u_m$, with $u_1, \dots, u_m$ alternating between $S - \{1\}$ and $T - \{1\}, m > 0$.

Now, we will analyze the cases where $g_1 \in G_0$, $g_1 \in G_1 - G_0$, and $g_1 \in G_2 - G_0$:

- If $g_1 \in G_0$, then $g_1g_0 \in G_0$, so g can be written as in (iii).
- If $g_1 \in G_1 - G_0$, we have the following cases:
  1) If $u_1 \in S$, then $g_1g_0u_1 \in G_1 \implies g_1g_0u_1 = g_0's, g_0' \in G_0, s \in S - \{1\}$; therefore, $g = g_0'su_2 \dots u_m$, just like in (iii);
  2) If $u_1 \in T$, then $g_1g_0 \in G_1 - G_0 \implies g_1g_0 = g_0's, g_0' \in G_0, s \in S - \{1\}$; therefore, $g = g_0'su_1u_2 \dots u_m$, just like in (iii).
- Similarly, the case where $g_1 \in G_1 - G_0$ also allows g to be written as in (iii).

Finally, suppose that $m = 0$. Then:

- If $g_1 \in G_0$, item (iii) follows.
- If $g_1 \in G_1 - G_0$, then:

$$g_1g_0 \in G_1 - G_0 \implies g_1g_0 = g_0's, g_0' \in G_0, s \in S - \{1\}.$$

- Similarly, if $g_1 \in G_2$, then:

$$g_1g_0 \in G_2 - G_0 \implies g_1g_0 = g_0't, g_0' \in G_0, t \in T - \{1\}.$$

It remains to show the uniqueness of this representation. We will use van der Waerden's method, as seen in Theorem 1.1. Let W be the set of all sequences $(g_0, u_1, \ldots, u_n)$, with $g_0 \in G_0, n \geq 0$, and $u_1, \ldots, u_n$ alternating between $S - \{1\}$ and $T - \{1\}$. Define $f_1 : G_1 \longrightarrow Perm(W)$ such that for each $g_1 \in G_1$ and $(g_0, u_1, \ldots, u_n) \in W$, we have:

$$f_1(g_1)(g_0, u_1, \ldots, u_n) = \begin{cases} (g_0', s, u_1, \ldots, u_n) & \text{if } u_1 \in T, g_1g_0 = g_0's, g_0' \in G_0, s \in S - \{1\}; \\ (g_0', u_1, \ldots, u_n) & \text{if } u_1 \in T, g_1g_0 = g_0', g_0' \in G_0; \\ (g_0', s, u_2, \ldots, u_n) & \text{if } u_1 \in S, g_1g_0u_1 = g_0's, g_0' \in G_0, s \in S - \{1\}; \\ (g_0', u_2, \ldots, u_n) & \text{if } u_1 \in S, g_1g_0u_1 = g_0', g_0' \in G_0. \end{cases}$$

Now, we will determine the values of $f_1(g_1^{-1})f_1(g_1)(g_0, u_1, \ldots, u_n)$:

$$\begin{cases} g_1^{-1}g_0's = g_0 \in G_0 \implies f_1(g_1^{-1})(g_0', s, u_1, \ldots, u_n) = (g_0, u_1, \ldots, u_n); \\ u_1 \in T, g_1^{-1}g_0' = g_0 \in G_0 \implies f_1(g_1^{-1})(g_0', u_1, \ldots, u_n) = (g_0, u_1, \ldots, u_n); \\ g_1^{-1}g_0's = g_0u_1, g_0' \in G_0, u_1 \in S - \{1\} \implies f_1(g_1^{-1})(g_0', s, u_2, \ldots, u_n) = (g_0, u_1, \ldots, u_n); \\ u_2 \in T, g_1^{-1}g_0' = g_0u_1, g_0 \in G_0, u_1 \in S - \{1\} \implies f_1(g_1^{-1})(g_0', u_2, \ldots, u_n) = (g_0, u_1, \ldots, u_n). \end{cases}$$

Note that when we apply $f_1(g_1)$ to $f_1(g_1^{-1})(g_0, u_1, \ldots, u_n)$, we will obtain the same values. Therefore, since $f_1(g_1^{-1})f_1(g_1) = f_1(g_1)f_1(g_1^{-1})$, we conclude that $f_1(g_1)$ has an inverse $f_1(g_1^{-1})$. Thus, $f_1(g_1)$ is a permutation for any $g_1 \in G$.

We want to show that f_1 is a homomorphism. That is, we will show that $f_1(g)f_1(g_1) = f_1(gg_1)$. Let $g, g_1 \in G$. Then $f_1(g)f_1(g_1)$ will take the following values:

$$\begin{cases} f_1(g)(g'_0, s, u_1, \ldots, u_n) = (g''_0, u_1, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0 = g''_0 \in G_0; \\ f_1(g)(g'_0, s, u_1, \ldots, u_n) = (g''_0, s', u_1, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0 = g''_0s',\ g''_0 \in G_0, s' \in S - \{1\}; \\ f_1(g)(g'_0, u_1, \ldots, u_n) = (g''_0, u_1, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0 = g''_0 \in G_0; \\ f_1(g)(g'_0, u_1, \ldots, u_n) = (g''_0, s'u_1, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0 = g''_0s',\ g''_0 \in G_0, s' \in S - \{1\}; \\ f_1(g)(g'_0, s, u_2, \ldots, u_n) = (g''_0, u_2, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0 = g''_0s' \in G_0; \\ f_1(g)(g'_0, s, u_2, \ldots, u_n) = (g''_0, s, u_2, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0u_1 = g''_0s',\ g''_0 \in G_0, s' \in S - \{1\}; \\ f_1(g)(g'_0, u_2, \ldots, u_n) = (g''_0, s', u_2, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0u_1 = g''_0s',\ g''_0 \in G_0, s' \in S - \{1\}; \\ f_1(g)(g'_0, u_2, \ldots, u_n) = (g''_0, u_2, \ldots, u_n) & \text{if } gg'_0s = gg_1g_0u_1 = g''_0 \in G_0. \end{cases}$$

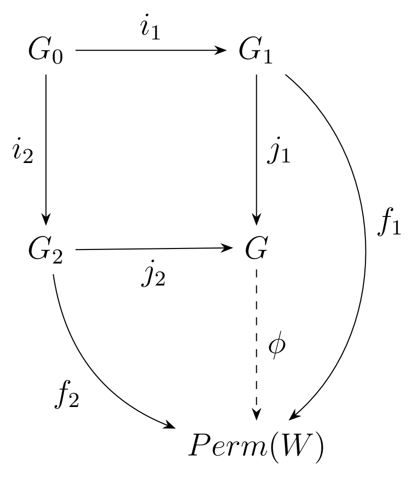
Additionally, $f_1(gg_1)$ will take the following values:

$$\begin{cases} f_1(gg_1)(g_0, u_1, \ldots, u_n) = (g''_0, u_1, \ldots, u_n) & \text{if } u_1 \in T,\ gg_1g_0 = g''_0 \in G_0; \\ f_1(gg_1)(g_0, u_1, \ldots, u_n) = (g''_0, s', u_1, \ldots, u_n) & \text{if } u_1 \in T,\ gg_1g_0 = g''_0s',\ g''_0 \in G_0,\ s' \in S - \{1\}; \\ f_1(gg_1)(g_0, u_1, \ldots, u_n) = (g''_0, u_2, \ldots, u_n) & \text{if } u_1 \in S,\ gg_1g_0u_1 = g''_0 \in G_0; \\ f_1(gg_1)(g_0, u_1, \ldots, u_n) = (g''_0, s', u_2, \ldots, u_n) & \text{if } u_1 \in S,\ gg_1g_0u_1 = g''_0s',\ g''_0 \in G_0,\ s' \in S - \{1\}. \end{cases}$$

In a similar way, we can define $f_2 : G_2 \longrightarrow Perm(W)$. That is, we define the homomorphisms f_1 and f_2. Let $g \in G_0$. We have that

$$f_1(g) = (g_0, u_1, \ldots, u_n) = (gg_0, u_1, \ldots, u_n) = f_2(g)(g_0, u_1, \ldots, u_n).$$

We have the following diagram:



Therefore, by the Universal Property, there exists a unique homomorphism $\phi : G \longrightarrow$

$Perm(W)$ such that $\phi|_{G_1} = f_1$ and $\phi|_{G_2} = f_2$. We want to show the uniqueness of the normal form. To do this, assume that $g_0 u_1 \dots u_n$ and $g_0' u_1' \dots u_m'$ are two distinct normal forms of $g \in G$. Then,

$$\begin{aligned}
\phi(g)(1) &= \phi(g_0)\phi(u_1)\dots\phi(u_n)(1) \\
&= \phi(g_0)\phi(u_1)\dots\phi(u_{n-1})(1, u_n) \\
&= (g_0, u_1, \dots, u_n) \\
&= \phi(g_0')\phi(u_1')\dots\phi(u_m')(1) \\
&= (g_0', u_1', \dots u_m') \\
&\implies g_0 = g_0',\ m = n \text{ and } u_i = u_i',\ i = 1, \dots, n.
\end{aligned}$$

Therefore, we have the uniqueness of the normal form.

(ii): As we saw in (iii), we have that $f_1|_{G_0} = f_2|_{G_0}$, thus $j_1(G_0) = j_2(G_0) \subseteq j_1(G_1) \cap j_2(G_2)$. It is enough to prove that $j_1(G_0) \cap j_2(G_0) \subseteq j_1(G_1) = j_2(G_2)$. In fact, let $g \in j_1(G_1) \cup j_2(G_2)$. We have that

$$g = j_1(g_0 s) = j_2(g_0' t),\ g_0,\ g_0' \in G_0,\ s \in S,\ t \in T.$$

Therefore, by the uniqueness of the normal form, we have that $s = t = 1$ and $g = j_1(g_0) = j_2(g_0') \in j_1(G_0) = j_2(G_0)$. Thus, if j_1, j_2 are inclusions, then $G_1 \cap G_2 = G_0$.

□

1.4 HNN Extension

The **HNN (Higman-Neumann-Neumann) extension**, or **HNN extension**, was introduced by mathematicians G. Higman, B. Neumann, and H. Neumann in 1949 as an attempt to generalize the notion of amalgamated free products of groups. HNN extensions allow us to extend a group G via an isomorphism between a subgroup H and a conjugate subgroup in G, preserving the structure of both.

Before presenting the formal definition of HNN extensions, it is important to note that we will initially define it in a more direct way, without using the Universal Property that characterizes this type of extension. Later, we will provide an equivalent formulation using the Universal Property of HNN Extensions.

Definition 1.38. *Let G and A be groups, i_0 and i_1 monomorphisms from A into G, and let P be an infinite cyclic group generated by p. Define $N = \langle \{p^{-1} i_0(a) p i_1(a)^{-1} \mid a \in A\} \rangle^{G*P}$.*

*Then $H = (G * P)/N$ is called an **HNN extension of the base group** G **with stable letter** p **and associated subgroups** $\mathbf{i_0(A)}$ **and** $\mathbf{i_1(A)}$.*

It is common to consider A as a subgroup of G with i_0 denoting an inclusion. Let $G = \langle X \mid R \rangle$, $B = i_1(A)$, and $\phi : A \longrightarrow B$ be given by $\phi(a) = i_1(a)$ as an isomorphism. Then an HNN extension H is given by $\langle X,\ p|\ p^{-1}Ap = B\rangle$, or alternatively, $\langle X,\ p \mid R,\ p^{-1}ap = \phi(a)\rangle$.

More generally, we can consider a family of groups A_α with monomorphisms $i_{0\alpha}$ and $i_{1\alpha}$ from A_α into G. Let P be free on the set $\{p_\alpha\}$. The following is a more general definition of HNN extensions:

Definition 1.39. *Let G and A be groups, i_0 and i_1 monomorphisms from A into G. Let P be free on $\{p_\alpha\}$, and let N be the normal subgroup of $G * P$ generated by:*

$$\{p_\alpha^{-1} i_{0\alpha}(a_\alpha) p_\alpha i_{1\alpha}(a_\alpha)^{-1} \mid \alpha, a_\alpha \in A_\alpha\}.$$

*Then $H = (G * P)/N$ is called an **HNN extension of the base group** $\mathbf{G}$ **with stable letters** $\{\mathbf{p}_\alpha\}$ **and associated pairs of subgroups** $\mathbf{A}_\alpha\mathbf{i}_{0\alpha}$ **and** $\mathbf{A}_\alpha\mathbf{i}_{1\alpha}$.*

Notation 1.40. *$HNN(G,\ A,\ p,\ \phi)$ denotes an HNN extension of the base group G, associated subgroup A, stable letter p, and isomorphism ϕ.*

Example 1.41. *$F(X)$ is an HNN extension of the trivial group with stable letter $x \in X$.*

Example 1.42. *Consider the groups $G = \mathbb{Z}$ and $A = \langle a \rangle$, both isomorphic to $\mathbb{Z}$, and $B = 2\mathbb{Z}$, isomorphic to $\langle a^2 \rangle$. Suppose the isomorphism $\phi : A \to B$ is given by $\phi(a) = a^2$. The extension $HNN(G, A, p, \phi)$ has a presentation given by $H = \langle a, p \mid p^{-1}a^n p = a^{2n}\rangle$, where n is an integer. Notably, $p^{-1}a^n p = (p^{-1}ap)^n$, resulting in $H = \langle a, p \mid p^{-1}ap = a^2\rangle$, known as the **Baumslag-Solitar group** $BS(1, 2)$.*

Now, to state the Universal Property of HNN Extensions, consider

$$H = HNN(G, A, p, \phi),$$

with $j : G \longrightarrow H$ the homomorphism induced by the inclusion of G in $G * P$ and

$$N = \left\langle \{p^{-1}ap\phi(a)^{-1} \mid a \in A\} \right\rangle^{G*P}.$$

Definition 1.43 (Universal Property of HNN Extensions). *Let θ be a homomorphism from G into K, where K is a group such that there exists $k \in K$ with $k^{-1}\theta(a)k = \theta(\varphi(a))$ for all $a \in A$. Then, there exists a unique homomorphism $\phi : H \longrightarrow K$ such that $\phi \circ j = \theta$ and $\phi(p) = k$.*

We will now state the Normal Form Theorem for HNN Extensions, which will be used to obtain a characterization of HNN extensions, providing a specific structure for

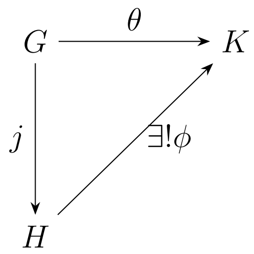


Figure 1.8: Diagram of the Universal Property of HNN Extensions

their elements.

Theorem 1.44 (**Normal Form Theorem for HNN Extensions**). *Let*

$$H = HNN(G, A, p, \phi),$$

and $j : G \to H$ the homomorphism induced by the inclusion of G in $G * P$, and S, T be right transversals of A and B in G respectively, with $1 \in S \cap T$. Then,

i) j is a monomorphism;

ii) Any element $h \in H$ can be uniquely written as $h = g_0 p^{\epsilon_1} g_1 p^{\epsilon_2} \dots p^{\epsilon_n} g_n$, where $n \geq 0$, $\epsilon_i = \pm 1$, $g_0 \in G$, for $i \geq 1$, $g_i \in S$ if $\epsilon_i = -1$, $g_i \in T$ if $\epsilon_i = 1$, and if $\epsilon_i = -\epsilon_{i+1}$, then $g_i \neq 1$.

Proof. The proof of this theorem, as with normal forms for free groups and free products, follows by using the van der Waerden method and can be found in [4] (Theorem 1.5.1, p. 31). □

Given the arbitrariness in choosing transversal sets, the following theorem provides an alternative form of the normal form theorem for HNN extensions without the use of these sets.

Theorem 1.45 (**Reduced Form Theorem or Britton's Lemma**). *Let*

$$H = HNN(G, A, p, \phi).$$

Then:

i) Any element $h \in H$ can be written as $g_0 p^{\epsilon_1} g_1 \dots p^{\epsilon_n} g_n$, where $n \geq 0$, $\epsilon_i = \pm 1$, $g_i \in G$, and h has no subword $p^{-1}ap$, $a \in A$, or pbp^{-1}, $b \in B$. This is called the ***reduced form*** *of h;*

ii) If h has another reduced form $h_0 p^{\delta_1} h_1 \dots p^{\delta_m} h_m$, then $m = n$ and $\epsilon_i = \delta_i$ for each $i = 1, \dots, n$. Additionally, if $\epsilon_1 = 1$, then $h_0 A = g_0 A$. If $\epsilon_1 = -1$, then $h_0 B = g_0 B$;

iii) If h has reduced form and $n > 0$, then $h \notin G$;

iv) If $h = g_0 p^{\epsilon_1} g_1 \dots p^{\epsilon_n} g_n \in G$, with $n > 0$, $g_i \in G$, $\epsilon_i = \pm 1$, then h has a subword $p^{-1}ap$, $a \in A$ or pbp^{-1}, $b \in B$.

Proof. (i) We will prove by contradiction, assuming that h is in its normal form but not in its reduced form. Then there exists a subword of the form $p^{-1}ap$, $a \in A$, or pbp^{-1}, $b \in B$ in h. Assuming it is the subword $p^{-1}ap$, if $a \in S$ then $a = 1$, since $A \cap S = 1$, but this would imply that the subword $p^{-1}ap$ could be reduced to the empty word, which contradicts the assumption that h is in normal form. Now, if the subword in h is of the form pbp^{-1}, then if $b \in T$ then $b = 1$, leading to a similar contradiction.

(ii) Suppose h has another reduced form $h_0 p^{\delta_1} h_1 \dots p^{\delta_m} h_m$. We want to show that $m = n$ and $\epsilon_i = \delta_i$ for $i = 1, \dots, n$.

We proceed by induction on the length of the reduced form of h in H.

(Base case): When the length of h's reduced form is 0, we have $h = h_0 \in G$, which is the normal form of h. Thus, the statement is trivially true in this case.

(Induction hypothesis): Assume the statement holds for all elements h' whose reduced forms have length less than n, where $n \geq 1$.

Now, let $h = g_0 p^{\epsilon_1} g_1 \dots p^{\epsilon_n} g_n$ with $n \geq 1$ be a reduced form of h. This implies that $p^{\epsilon_2} g_2 \dots p^{\epsilon_n} g_n$ is in reduced form and has a normal form $g'_0 p^{\delta_2} g'_2 \dots p^{\delta_m} g'_m$, where $m < n$.

By the induction hypothesis, we know $m = n$ and $\epsilon_i = \delta_i$ for each $i = 2, \dots, n$. Additionally, if $\epsilon_2 = 1$, then $g'_0 \in A$; otherwise, if $\epsilon_2 = -1$, then $g'_0 \in B$.

Now, we can write h as:

$$h = g_0 p^{\epsilon_1} g_1 p^{\epsilon_2} g_2 \dots p^{\epsilon_n} g_n = g_0 p^{\epsilon_1} g_1 g'_0 p^{\epsilon_2} g'_2 \dots p^{\epsilon_n g'_n}.$$

Considering the case $\epsilon_1 = 1$, we have $pg_1 g'_0 = apt$, where $b \in B$ and $t \in T$, with $a = \phi^{-1}(b)$. If $t \neq 1$, then $(g_0 a)ptp^{\epsilon_2} g'_2 \dots p^{\epsilon_n} g'_n$ is the normal form of h, so $g_0 aA = g_0 A$. If $t = 1$ and $\epsilon_2 = -1$, then h has reduced form $g_0 p g_1 p^{-1} \dots p^{\epsilon_n} g_n$. Since $g'_0 \in B$, this implies $g_1 \in B$, so pbp^{-1} is a subword of h, a contradiction. The case where $\epsilon_1 = -1$ is analogous.

(iii) If $h = g_0 p^{\epsilon_1} g_1 p^{\epsilon_2} g_2 \dots p^{\epsilon_n} g_n \in G$ is in reduced form, $n > 0$, then $h = g'_0 \in G$ is also in reduced form, with length 0. This contradicts item (ii), so h must contain a word of the form $p^{-1}ap$ or pbp^{-1}.

(iv) The conclusion follows directly from the contrapositive of statement (iii).

□

1.5 Topology

Topology is a fascinating field of study not only on its own but it also plays a crucial role in providing the groundwork for further exploration in analysis, geometry, and algebraic topology. In particular, the study of the profinite topology 1.93 is of great interest, as it aids in understanding fundamental definitions and results in the study of residual properties discussed in Chapters 2 and 3. In this chapter, the primary references include the book "Topology" by J. Hocking and G. Young [24], the book "Topology, a First Course" by J. Munkres [26], and "Profinite Groups" by J. Wilson [25].

Definition 1.46. *A **topological space** is a set X with a family of subsets, called **open sets**, satisfying:*

(O_1) The empty set and X are open sets;

(O_2) The intersection of any two open sets is open;

(O_3) An arbitrary union of open sets is open.

The collection of open sets is called the **topology** on X.

Given a set X, we can assign different topologies to X by selecting its open sets. Topology, therefore, is a particular choice of open sets for a given set. However, there are two extreme cases when choosing open sets to form a topology, as shown in the following examples:

Example 1.47. *Given a set X, the topology $\mathcal{T} = \{\emptyset,\ X\}$, which consists only of the set X and the empty set, is called the **trivial topology**. In this topology, only the empty set and the set X are considered open.*

Example 1.48. *The topology on X that consists of all subsets of X is called the **discrete topology**.*

Example 1.49. *If Y is a subset of X, then the collection of all subsets of the form $Y \cap U$, with U open in X, is a topology on Y; this topology is called the **induced topology** or the subspace topology of X.*

Definition 1.50. *A subset F is said to be **closed** in X if its complement $X - F$ is open.*

Definition 1.51. *If Y is a subset of X, the **closure** $\overline{Y}$ of Y is the intersection of all closed sets containing Y.*

Theorem 1.52. *If X is any subset of S, then X is closed if and only if $X = \overline{X}$.*

Proof. Suppose that $X = \overline{X}$. Then, for any point $s \in S - X$, there exists an open set disjoint from X. The arbitrary union of these open sets forms $S - X$, and this union is open by Definition 1.46. Since the complement of X is open, X is closed.

Conversely, suppose X is closed, so $S - X$ is open. If $p \in S - X$, then the set $S - X$ is already an open set containing p but containing no points of X. This implies that p does not belong to the closure of X, or $p \notin \overline{X}$. Therefore, no point of X can belong to $S - X$, which implies $X \subseteq \overline{X}$.

Now, assume that $x \in \overline{X}$. This means that x belongs to every closed set that contains X. Since X is closed, it is one of these closed sets containing X. Therefore, $x \in X$, which implies $\overline{X} \subseteq X$.

Thus, we have $X \subseteq \overline{X}$ and $\overline{X} \subseteq X$, so $X = \overline{X}$. □

The following theorem establishes a formal duality between the axioms (O_1), (O_2), and (O_3), and an equivalent definition of these axioms by using closed sets instead of open sets.

Theorem 1.53. *The closed sets $\{C_\lambda\}$ of a topological space S satisfy the following properties:*

(C_1) The intersection of any number of closed sets is closed;

(C_2) The union of a finite number of closed sets is closed;

(C_3) The sets S and $\emptyset$ are closed.

From now on, we denote λ as an element of some arbitrary index set Λ. The notion of covering will become important for understanding the following concepts.

A proof of Theorem 1.53 can be found in [24] (Theorem 1-2, p. 6), where a well-known tool from Set Theory called De Morgan's Law is used.

Remark 1.54 (De Morgan's Law). *Let S be any set and $\{X_\lambda\}$ any collection of subsets of S. Then, $\bigcap_\lambda X_\lambda = S - \bigcup_\lambda (S - X_\lambda)$.*

Note that through Theorem 1.53, we could define a topological space by taking closed sets in place of open sets; for this, it would suffice to replace the words "open" with "closed" and "union" with "intersection" in Definition 1.46. The same can be done for any true statement about open sets.

It is natural to question whether a choice of subsets for a topology is the minimal possible to define the opens of that topology. The following definition can provide an answer to this question.

Definition 1.55. *A* ***basis*** *for a topology on X is a collection of open sets $\{X_\lambda\}$ such that every open set is a union of some of the sets X_λ.*

An interesting topology on the real line $\mathbb{R}$ can be described in terms of its basis:

Definition 1.56. *If $\mathcal{B}$ is a collection of open intervals on the real line*

$$(a,b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the ***usual topology on the real line****.*

Note that the usual topology is the topology generated by the metric of the space. Generally, open sets in the usual topology are defined in terms of open balls in the metric. For readers interested in learning more about this topic, see [24].

Definition 1.57. *A subset Y of X is said to be* ***dense*** *in X if $\overline{Y} = X$.*

Example 1.58. *$\mathbb{Q}$ is dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$.*

The most important theorems in differential calculus, such as the Intermediate Value Theorem and the Maximum Value Theorem, depend not only on the continuity of the considered function but also on the connectedness and compactness properties of the topological space $[a, b]$. Our next goal is to define these concepts for arbitrary topological spaces.

Definition 1.59. *A topological space X is said to be* ***connected*** *if it cannot be written as the disjoint union of two nonempty open sets.*

Example 1.60. *Let X be a space consisting of a set of cardinality 2 with the trivial topology. Then, X cannot be written as the disjoint union of open sets. Therefore, X is connected.*

The following theorem provides an equivalent definition of a connected space.

Theorem 1.61. *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.*

Proof. If A is a nonempty proper subset of X that is both open and closed, then the disjoint union of the open sets $U = A$ and $V = X - A$ equals X. Conversely, if U and V are disjoint open sets whose union is X, then U is nonempty and different from X. Moreover, U is both open and closed in X. □

For the opposite extreme with respect to connectedness, we have the following definition:

Definition 1.62. *A topological space X is* ***totally disconnected*** *if each connected subspace has at most one element. We also say that X is totally disconnected if each connected component[1] has only one element.*

Below, we present some examples of connected and totally disconnected spaces.

[1] A connected component of X is a subset $C \subseteq X$ that is connected and maximal with respect to inclusion.

Example 1.63. *Let* Y *be the subspace* $[-1,0)\cup(0,1]$ *of the real line. Each of the sets* $[-1,0)$ *and* $(0,1]$ *are nonempty and open in* Y*. Therefore,* X *is not connected.*

Example 1.64. *The set of rational numbers* $\mathbb{Q}$ *is not connected. Indeed, if* Y *is a subspace of* $\mathbb{Q}$ *containing two points* p *and* q*, we can choose an irrational number* a *with* $p<a<q$*, and* Y *will be the union of the open sets* $Y\cap(-\infty,a)$ *and* $Y\cap(a,+\infty)$*. By the previous argument and since* $\mathbb{Q}$ *is dense in* $\mathbb{R}$*, no open interval (containing more than one point) can be contained in* $\mathbb{Q}$*. Hence,* $\mathbb{Q}$ *is totally disconnected.*

We will see that the most important topological space presented in this chapter is totally disconnected, compact, and Hausdorff. We will define these latter two properties. First, we define compact spaces, but before that, we need to define the concept of covering.

Definition 1.65. *Let* X *be a set in a topological space.*

(i) A family of sets $\{X_\lambda\}_{\lambda\in\Lambda}$ *is said to* ***cover*** *the set* X*, or is called a* ***covering*** *of* X*, if the union* $\cup_\lambda X_\lambda$ *contains* X*.*

(ii) If each X_λ *is open, then* $\{X_\lambda\}_{\lambda\in\Lambda}$ *is an* ***open cover*** *of* X*.*

(iii) If $\{X_\lambda\}_{\lambda\in\Lambda}$ *has a finite subcollection of sets that also covers* X*, then that finite subcolletion is a* ***finite subcover*** *of* X*.*

The following are examples of coverings and subcoverings of sets.

Example 1.66. *The set of intervals* $\{[0,2],\ (1,4),\ (5,7)\}$ *is a covering of the interval* $[0,3]$*.*

Example 1.67. *The set* $\{(\frac{1}{k},\ 2-\frac{1}{k})\}_{k=1}^{\infty}$ *is an open cover of* $(0,2)$*.*

Example 1.68. *The set* $\{(-\infty,\ 0),\ (-1,\ \frac{1}{2}),\ (0,\ 1),\ (0.1,\ 9)\}$ *is an open cover of* $(-2,\ 1)$*.*

Example 1.69. *The collection of unit balls with integer coordinates:*

$$\{B((M,\ N),\ 1),\ M,N\in\mathbb{Z}\}$$

forms an open cover of $\mathbb{R}^2$*.*

Example 1.70. *Returning to Example 1.66, the set of intervals* $\{[0,2],\ (1,4),\ (5,7)\}$ *is a cover of the interval* $[0,3]$*, and the set* $\{[0,2],\ (1,4)\}$ *is a finite subcover of* $(0,3)$*. Note that any cover of* A *can serve as a finite subcover of* A*.*

Definition 1.71. *A topological space* X *is said to be* ***compact*** *if every open cover* $\{X_\lambda\}$ *of* X *contains a finite subcover of* X*.*

Example 1.72. *The set* $E=\mathbb{R}$ *with the usual topology is not compact. Consider the cover* $\mathcal{C}=\{(-n,\ n)\mid n\in\mathbb{N}\}$*. Suppose that* $\mathcal{V}=\{(-n_1,\ n_1),\ldots,\ (-n_k,\ n_k)\}$ *is a finite subcover of* $\mathbb{R}$*. Let* $n=\max\{n_1,\ldots,\ n_k\}$ *be the largest among them, so the union of* $\mathcal{V}$ *is* $(-n,\ n)$*. Since* $\mathcal{V}$ *is a cover of* $\mathbb{R}$*, we have that* $\mathbb{R}\subset(-n,\ n)$*, which is a contradiction, as* $n+1\in\mathbb{R}$ *but* $n+1\notin(-n,\ n)$*.*

Example 1.73. *The subspace X of the real line given by*

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$$

is compact. Indeed, let $\{X_\lambda\}$ be a cover of X. Then, there exists an element U in $\{X_\lambda\}$ containing $\{0\}$. The set U contains all points except for a finite number of points $1/n$. For each point in $X - U$, consider an element in $\{X_\lambda\}$ containing it. The collection of these elements in $\{X_\lambda\}$, together with U, forms a finite subcover of $\{X_\lambda\}$.

Definition 1.74. *A topological space X is said to be **Hausdorff** if, for every pair of distinct elements $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Theorem 1.75. *Any subset of a Hausdorff space of cardinality 1[2] is closed.*

Proof. Let $\{x\}$ be an arbitrary singleton in a Hausdorff space. Consider $y \in X - \{x\}$ an arbitrary point of $X - \{x\}$. There exist open sets U_x and V_y such that $x \in U_x$ and $y \in V_y$. By considering all open sets of points in $X - \{x\}$, we have

$$X - \{x\} = \bigcup_{y \in X - \{x\}} V_y.$$

Since the union of open sets is an open set, the complement of $\{x\}$ is open, and therefore $\{x\}$ is closed. □

An interesting example of a topological space is given by a set X of at least 2 elements, with a topology:

$$\mathcal{T} = \{X - F \mid F \subset X \text{ is finite}\} \cup \{\emptyset\}.$$

This topology is called the **cofinite topology** or the **finite-complement topology**. We will see that if X has an infinite cardinality, with this topology, it provides an example of a non-Hausdorff space.

Example 1.76. *The topological space S consisting of a set X of infinite cardinality with the cofinite topology is non-Hausdorff. In fact, let $U = X - F_1$ and $V = X - F_2$ be open subsets of X, then*

$$U \cap V = (X - F_1) \cap (X - F_2) = X - (F_1 \cup F_2) \neq \emptyset.$$

Therefore, the space S is not Hausdorff.

The following example will help us understand how the properties of compactness,

[2] A space of cardinality $\{1\}$ is also called a **singleton**.

connectedness, and Hausdorffness behave for finite sets. These properties are of interest as we will later see that a profinite group is a compact, totally disconnected, and Hausdorff group.

Example 1.77. *Consider the symmetric group on 6 elements S_3. Suppose S_3 has a open cover $\{X_\lambda\}$, namely F. For each point $x \in F$, as $\bigcup X_\lambda$ covers F, take λ_x such that $x \in \{X_{\lambda_x} : x \in F\}$ is a finite subcover. Hence F is compact.*

Furthermore, since $|S_3| < \infty$ and we equipped it with the discrete topology, for any distinct elements $x, y \in S_3$, the sets $U_x = \{x\}$ and $U_y = \{y\}$ are open and disjoint, implying that S_3 is Hausdorff.

Finally, we will show that S_3, with the discrete topology, is totally disconnected, meaning that the only connected subspaces have cardinality 1. By Theorem 1.75, all singletons in S_3 are closed, and hence any singleton is clopen (open and closed). By Theorem 1.61, such subsets are connected. Any set with a cardinality greater than 1 cannot be connected, as it can be written as a disjoint union of its elements. Therefore, S_3 is totally disconnected.

Of course, this example can be generalized to any finite set, as shown in the example below.

Example 1.78. *Any space X containing a finite number of points, with the discrete topology, is necessarily compact, totally disconnected, and Hausdorff.*

Lemma 1.79. *Let X be a compact and Hausdorff space.*

(a) If C, D are closed subsets such that $C \cap D = \emptyset$, then there exist open subsets U, V such that $C \subseteq U$, $D \subseteq V$, and $U \cap V = \emptyset$;

(b) Let $x \in X$, and let A be the intersection of all subsets of X containing x that are both open and closed. Then A is connected;

(c) If X is totally disconnected, then every open set is a union of subsets that are both open and closed.

Proof. The proof of this result can be found in [25] (Lemma 0.1.1, p. 2). □

Another fundamental concept in the study of topology is the continuity of functions defined on topological spaces.

Definition 1.80. *Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is called* ***continuous*** *if for every open set U in Y, the set $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is open in X. We say that f is a* ***homeomorphism*** *if f is bijective and both f and f^{-1} are continuous.*

Lemma 1.81. *Let X, Y be topological spaces, then* *(a) Every closed subset of a compact space is compact;*

(b) Every compact subset of a Hausdorff space is closed;

(c) If $f : X \longrightarrow Y$ is continuous and X is compact, then $f(X)$ is compact;

(d) If $f : X \longrightarrow Y$ is continuous and X is connected, then $f(X)$ is connected;

(e) If $f : X \longrightarrow Y$ is continuous and bijective, X is Hausdorff and Y is Hausdorff, then f is a homeomorphism;

(f) If $f, g : X \longrightarrow Y$ are continuous and Y is Hausdorff, then $\{x \in X \mid f(x) = g(x)\}$ is closed.

Proof. The proof of this lemma can be found in [25] (Lemma 0.1.2, p. 3). □

1.5.1 Cartesian Product

The **Cartesian product** of a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of sets is a set

$$C = \prod_{\lambda \in \Lambda} X_\lambda$$

such that its elements are functions x from Λ to $\bigcup_{\lambda \in \Lambda} X_\lambda$ such that $x(\lambda) \in X_\lambda$ for each λ. We can think of the elements of C as vectors with entries indexed by elements of Λ. An arbitrary element of C can be written as (x_λ). This element is the function that maps λ to x_λ.

The function

$$\pi_\lambda : \prod_{\lambda \in \Lambda} X_\lambda \longrightarrow X_\lambda$$

that takes each element of C to its α-th coordinate, defined by

$$\pi_\lambda(x_\lambda) = x_\lambda,$$

is called the **projection map** associated with the index α.

Definition 1.82. *Let $C = \prod_{\lambda \in \Lambda} X_\lambda$ be the Cartesian product of topological spaces. The* ***product topology*** *on C has as open sets the union of sets of the form*

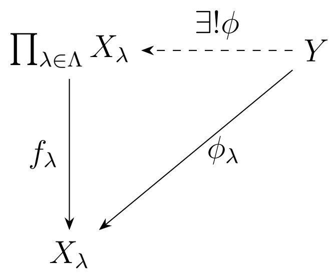
$$\pi_{\lambda_1}^{-1}(U_1) \cap \cdots \cap \pi_{\lambda_n}^{-1}(U_n)$$

with finite n, where each $\lambda_i \in \Lambda$ and U_i are open in X_{λ_i}. The product topology on C is the weakest topology such that π_λ is continuous for each $\lambda \in \Lambda$.

Notation 1.83. *The Cartesian product of a finite family $X_1, \ldots, X_n$ of sets is denoted* *by* $X_1 \times \cdots \times X_n$.

We can also use the universal property to define the Cartesian product of topological spaces.

Definition 1.84. *The Cartesian product of topological spaces is a topological space with a collection of continuous maps* $f_\lambda : \prod_{\lambda \in \Lambda} X_\lambda \longrightarrow X_\lambda$ *such that for any topological space and a family of continuous maps* $\phi_\lambda : Y \longrightarrow X_\lambda$, *there exists a unique continuous map* ϕ *such that the diagram*



commutes, that is, $\phi_\lambda = \phi \circ f_\lambda$.

Another important example of topology that we will use is the quotient topology, but before that, we need to define what a quotient map is.

Definition 1.85. *Let* X *and* Y *be topological spaces, and let* $p : X \longrightarrow Y$ *be a surjective map. We say that* p *is a* ***quotient map*** *if the following statement holds: A subset* U *of* Y *is open in* Y *if and only if* $p^{-1}(U)$ *is open in* X.

While a continuous map only guarantees that pre-images of open sets are open, a quotient map requires that the pre-images of open sets are open and only those.

Open maps, which map open sets to open sets, and **closed maps**, which map closed sets to closed sets, are examples of quotient maps.

Definition 1.86. *If* X *is a space,* A *a set, and* $p : X \longrightarrow A$ *a surjective map, then there exists exactly one topology* $\mathcal{T}$ *on* A *such that* p *is a quotient map; it is called the* ***quotient topology*** *induced by* p.

Example 1.87. *Let* $\pi_1 : X \times Y \longrightarrow X$ *be a projection from the direct product of the topological spaces* X *and* Y *onto* X. *We have that* π_1 *is continuous and surjective. If* $U \times V$ *is a basis element of* $X \times Y$, *its image* $\pi_1(U \times V) = U$ *is open in* X. *It follows that* π_1 *is an open map. However, in general,* π_1 *is not a closed map. Consider* $\pi_1 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, *a projection from the direct product of* $\mathbb{R}$ *and* $\mathbb{R}$ *onto* $\mathbb{R}$. *The image of the closed set* $\{(x, y) \mid xy = 1\}$ *is the non-closed set* $\mathbb{R} - \{0\}$.

The following result is fundamental for understanding how the product preserves important topological properties; the same result will be used later to show that the inverse limit preserves these properties.

Theorem 1.88. *Let* $\{X_\lambda\}_{\lambda \in \Lambda}$ *be a family of topological spaces. Then:*

(a) If each X_λ is Hausdorff, then $\Pi_{\lambda\in\Lambda}X_\lambda$ is Hausdorff;

(b) If each X_λ is totally disconnected, then $\Pi_{\lambda\in\Lambda}X_\lambda$ is totally disconnected;

(c) If each X_λ is compact, then $\Pi_{\lambda\in\Lambda}X_\lambda$ is compact.

Proof. (a): Let $x = (x_\lambda)$ and $y = (y_\lambda)$ with $x \neq y$ in $\Pi_{\lambda\in\Lambda}X_\lambda$. Then there exists some $\lambda_0 \in \Lambda$ such that $x_{\lambda_0} \neq y_{\lambda_0}$ in X_{λ_0}. Since X_{λ_0} is Hausdorff, there exist open sets $U_{\lambda_0}, V_{\lambda_0} \subset X_{\lambda_0}$ such that $x_{\lambda_0} \in U_{\lambda_0}$, $y_{\lambda_0} \in V_{\lambda_0}$, and $U_{\lambda_0} \cap V_{\lambda_0} = \emptyset$. Hence, $x \in \pi_{\lambda_0}^{-1}(U_{\lambda_0})$ and $y \in \pi_{\lambda_0}^{-1}(V_{\lambda_0})$. Moreover, we have

$$\begin{cases} \pi_{\lambda_0}^{-1}(U_{\lambda_0}) = \Pi_{\lambda\in\Lambda}X_\lambda \times U_{\lambda_0}, \text{ with } \lambda \neq \lambda_0, \\ \pi_{\lambda_0}^{-1}(V_{\lambda_0}) = \Pi_{\lambda\in\Lambda}X_\lambda \times V_{\lambda_0}, \text{ with } \lambda \neq \lambda_0. \end{cases}$$

Thus, we have

$$\pi_{\lambda_0}^{-1}(U_{\lambda_0}) \cap \pi_{\lambda_0}^{-1}(V_{\lambda_0}) = \pi_{\lambda_0}^{-1}(U_{\lambda_0} \cap V_{\lambda_0}) = \pi_{\lambda_0}^{-1}(\emptyset) = \emptyset.$$

Therefore, $\pi_{\lambda_0}^{-1}(U_{\lambda_0})$ and $\pi_{\lambda_0}^{-1}(V_{\lambda_0})$ are disjoint open sets in $\Pi_{\lambda\in\Lambda}X_\lambda$, implying that $\Pi_{\lambda\in\Lambda}X_\lambda$ is Hausdorff.

(b): Before proving this item, we use the following fact: if $f : X \longrightarrow Y$ is a continuous map from a topological space X to another topological space Y with $U \subset X$ connected, then $f(U)$ is connected. In fact, we only need to show that $U = X$ and $f(U) = Y$. Suppose that $f(U)$ is not connected. Then there exists a set B that is open and closed, different from Y and $\emptyset$. Consider $A = f^{-1}(B)$, which is a subset of X that is both open and closed since f is continuous. Moreover, $A \neq X$ since $Y \neq B$, and $A \neq \emptyset$ since f is surjective. Thus, A is an open and closed proper subset of X, contradicting the assumption that X is connected.

Now, we use this fact to prove item (b). We want to show that $X = \Pi_{\lambda\in\Lambda}X_\lambda$ is totally disconnected, so suppose that X is not. Let C be a connected component of X containing elements $x = (x_\lambda)$ and $y = (y_\lambda)$, with $x \neq y$. Let λ be such that $x_\lambda \neq y_\lambda$ but x_λ and $y_\lambda \in \pi_\lambda(C)$, which is connected by the previous fact. Thus, we would have $\pi_\lambda(C) \subset X_\lambda$, where X_λ is totally disconnected, contradicting the assumption that X is not totally disconnected. Hence, item (b) follows.

The proof of item (c) follows from Zorn's Lemma and can be found in [25], Theorem 0.2.1, p. 04. □

Definition 1.89. *Given a group G with a topology, if the operations $m : G \times G \longrightarrow G$ and $i : G \longrightarrow G$, defined by $m(x, y) = xy$ and $i(x) = x^{-1}$, are continuous, we say that G is a* ***topological group****. Note that G is also a topological space.*

Examples of topological groups include:

Example 1.90. *The groups:* $(\mathbb{R},\ +)$, $(\mathbb{R}^n,\ +)$, $(\mathbb{R}^*,\ \cdot)$, $(\mathbb{C},\ +)$, $(\mathbb{C}^*,\ \cdot)$, *are topological groups with the usual topology.*

Example 1.91. *The group* $GL(n)$*, the general linear group of all invertible matrices of order* n*, is a topological group. The topology usually considered on* $GL(n)$ *is the usual topology, induced by the Euclidean metric on* $\mathbb{R}^{n^2}$*, where each matrix is viewed as a vector of* n^2 *entries.*

The following example will be the most important instance of a topological group for studying the residual properties considered in this dissertation. For this reason, we will show that it satisfies the necessary conditions of Definition 1.89. However, to do so, we first need the following result:

Lemma 1.92. *Let* G *be a group and* $H \leq G$*. Then it holds: if* H *has finite index, then* $\mathrm{Core}_G(H) := \bigcap_{g\in G} g^{-1}Hg$[3] *has finite index.*

Proof. Suppose $|G : H| = n$. The group G acts on the left cosets of H by left multiplication, inducing a homomorphism ϕ from G to S_n. Let $\{H, g_2H, \ldots, g_nH\}$ denote the set of left cosets of H, where $n = |G : H|$. We define the action of $g \in G$ as $g(g_iH) = (gg_i)H$.

The kernel of this action is

$$\begin{aligned}
\ker(\phi) &= \{g \in G \mid gxH = xH,\ \forall x \in G\},\\
&= \{g \in G \mid x^{-1}gxH = H,\ \forall x \in G\},\\
&= \{g \in G \mid x^{-1}gx \in H,\ \forall x \in G\},\\
&= \{g \in G \mid g \in H^x,\ \forall x \in G\},\\
&= \mathrm{Core}_G(H).
\end{aligned}$$

Finally, we have

$$|G/\mathrm{Core}_G(H)| = |G/\ker(\phi)| = |\mathrm{Im}(\phi)| < n!$$

(by the First Isomorphism Theorem). □

Example 1.93. *The* ***profinite topology*** *is a topology that is compact, totally disconnected, and Hausdorff. If* G *is a group with the profinite topology on* G*, given by the basis* $B = \{gH \mid H \leq_{f.i.} G, g \in G\}$*, then* G *with the profinite topology is a topological group. Indeed, let*

$$B = \{gH \mid H \leq_{f.i.} G,\ g \in G\},\ \textit{and}$$

$$B' = \{gN \mid N \trianglelefteq_{f.i.} G,\ g \in G\}$$

[3]The **core** of $H \leq G$ is the largest normal subgroup of G contained in H.

We will show that every open set in the profinite topology is an element of the topology generated by B'. Each element of B' belongs to B, thus the open sets generated by B' are included in the profinite topology. If $H \leq_f G$, then there exists $N \trianglelefteq_{f.i.} G$ such that $N \leq H$. We define $N := \cap_{g \in G} H^g$. By Lemma 1.92, we have:

$$|G : H| = n \implies |G : H^g| \mid n!$$

That is, N has finite index in G, say r. Hence,

$$H = \underbrace{N \cup g_1 N \cup \cdots \cup g_{r-1} N}_{r \text{ cosets}} \implies gH = gN \cup gg_1 N \cup \cdots \cup gg_{r-1} N.$$

Note that $gN, gg_1N, \ldots, gg_{r-1}N \in B'$. Thus, any element of B can be expressed as a union of elements from B'. That is, B' is also a basis for the same topology. Now we will show that the operations of multiplication and inversion are continuous in the basis B':

(i) ***Inversion in the basis*** *B': Consider the map:*

$$i : G \longrightarrow G \text{ such that } g \mapsto g^{-1}.$$

Then $i^{-1}(gN) = g^{-1}N$, which is open since it belongs to the basis B'. Therefore, i is continuous.

(ii) ***Multiplication in the basis*** *B': Consider the multiplication map $m : G \times G \to G$ defined by $m(g, h) = gh$. To prove that m is continuous, let $U \subseteq G$ be an open set, and consider $m^{-1}(U) = \{(g, h) \in G \times G \mid gh \in U\}$. We need to verify that $m^{-1}(U)$ is open in $G \times G$.*

Let U be an open set in G. Since U is open, it can be written as a union of cosets $U = \bigcup_{g' \in G} g'N$, where $N \trianglelefteq_{f.i.} G$ and $g' \in G$.

Now, fix $(g, h) \in G \times G$ such that $gh \in g'N$ for some coset $g'N \subseteq U$. The coset $g'N$ can be written as $g'N = \{g'n \mid n \in N\}$, and thus $gh = g'n$ for some $n \in N$. Rearranging, we have $h = g^{-1}g'n$, which implies $h \in g^{-1}g'N$.

Define the sets:

$$V_1 = gN \quad \text{and} \quad V_2 = g^{-1}g'N.$$

Here, V_1 and V_2 are cosets in G and belong to the basis B', making them open in the topology of G. Then, the product $V_1 \times V_2 \subseteq G \times G$ is open in the product topology, and $(g, h) \in V_1 \times V_2$.

Since $m(V_1 \times V_2) \subseteq g'N$, and $m^{-1}(g'N)$ is a union of such open sets, it follows that $m^{-1}(g'N)$ is open. Finally, $m^{-1}(U) = \bigcup_{g'N \subseteq U} m^{-1}(g'N)$, which is also open as a *union of open sets.*

Therefore, the multiplication map m is continuous.

Thus, we conclude that G with the profinite topology is indeed a topological group.

The results of the following lemma are fundamental for the study of topological groups.

Lemma 1.94. *Let G be a topological group. Then:*

(a) The map $(x, y) \mapsto xy$ from $G \times G$ to G is continuous, and the map $x \mapsto x^{-1}$ from G to G is a homeomorphism. For each $g \in G$, the maps $x \mapsto xg$ and $x \mapsto gx$ from G to G are homeomorphisms;

(b) If H is an open subgroup of G, then the cosets Hg and gH are open in G;

(c) Every open subgroup of G is closed. If H is a closed subgroup of finite index in G, then H is open;

(d) If G is compact, every open subgroup of G has finite index in G;

(e) If U is a nontrivial open subgroup of the subgroup $H \leq G$, then H is open;

(f) If $H \leq G$, then H is a topological group with the induced topology. If K is a normal subgroup of G, then G/K is a topological group, and the homomorphism $q : G \longrightarrow G/K$ is an open map;

(g) G is Hausdorff if and only if every singleton is closed in G; if K is normal in G, then G/K is Hausdorff if and only if K is closed in G;

(h) If G is totally disconnected, then G is Hausdorff;

(i) If G is compact and Hausdorff and C, D are closed subsets of G, then CD is closed.

Proof. (a): The map $(x, y) \mapsto xy$ from $G \times G$ is continuous if and only if its product with each projection is continuous (since $G \times G$ has the product topology). Thus, if $\theta : G \longrightarrow G$ and $\varphi : G \longrightarrow G$ are continuous, the map $x \mapsto (\theta(x), \varphi(x))$ from G to $G \times G$ will be continuous. First, we apply this by taking θ as the constant map $x \mapsto 1$ and $\varphi = \mathrm{id}_G$, and now we can compose the resulting map with the continuous map $c : (x, y) \mapsto xy^{-1}$ from $G \times G$ to G. That is, $x \mapsto (q, x)$ and $(1, x) \mapsto x^{-1}$ by c. Therefore, the map i defined by $i(x) = x^{-1}$ is continuous, since i is a composition of continuous maps. Since $i = i^{-1}$, we have that i is a homeomorphism. Thus, the map $(x, y) \mapsto (x, y^{-1})$ is continuous and its product with c is continuous: $(x, y) \mapsto (x, y^{-1})$ and $(x, y^{-1}) \mapsto xy$ by c. That is, the map $(x, y) \mapsto xy$, resulting from this composition, is continuous.

Now, we take $\theta = \mathrm{id}_G$ and φ as the constant map $x \mapsto g^{-1}$, and we now take the product of the resulting map with c:

$$x \mapsto (x, g^{-1}) \mapsto xg.$$

Thus, the map $x \mapsto xg$ is continuous, and its inverse $x \mapsto xg^{-1}$ is also continuous. Similarly, it can be shown that the map $x \mapsto gx$ is a homeomorphism.

(b): Follows from (a).

(c): We have that $G \backslash H = \bigcup (Hg \mid g \notin H)$. Thus, if H is open, then $G\backslash H$ is open (by (b)), and H is closed. If H has finite index, then $G\backslash H$ is a finite union of its cosets. Therefore, if H is also closed, then $G\backslash H$ is closed, and H will be open.

(d): Follows from (c) and the definition of compactness.

(e): $H = \bigcup (Uh \mid h \in H)$ thus, by (b), H is open.

(f): Follows from (a) and the universal property of the product of topological spaces. For more details, see [25] (Lemma 0.3.1 (e), p. 06).

(g): By Theorem 1.75, we have that $\{1\}$ is closed in a Hausdorff space. On the other hand, suppose that $\{1\}$ is closed. Let $a, b \in G$ be distinct elements of G. By (a), we have that $\{a^{-1}b\}$ is closed. Therefore, $G - \{ab^{-1}\}$ is an open set that contains 1. Since the map $(x, y) \mapsto xy^{-1}$ is continuous, the preimage of $G - \{a^{-1}b\}$ must be open in $G \times G$. By the definition of the product topology, there exist open sets V, W that contain 1 with $VW^{-1} \subseteq U$. Thus, $a^{-1}b \notin VW^{-1}$, hence we have $aV \cap bW = \emptyset$. Since aV and bW are open. The remaining statements follow from this fact.

(h): Follows from (g).

(i): Follows from Lemma 1.81. □

The following lemma will be used in the proof of important results regarding compact topological groups.

Lemma 1.95 ([25], Lemma 0.3.2, p. 07)**.** *If C is a clopen subset containing 1 of a compact topological group G, then C contains a normal open subgroup.*

The next result uses Lemma 1.95 in its proof and will contribute to our understanding of the properties and tools that permeate the study of an important topological group that we will define, called a profinite group.

Notation 1.96. *To indicate that a normal subgroup N is open in G, we will use the notation $N \triangleleft_O G$. In the case where F is a closed normal subgroup of a group G, we will denote it by $N \triangleleft_f G$.*

The following proposition is an important result about compact totally disconnected topological groups and follows from Lemma 1.95 and Lemma 1.79.

Proposition 1.97. *Let G be a compact totally disconnected topological group. Then*

(a) Every open subset in G is a union of normal open subgroups;

(b) A subset is clopen if and only if it is a union of a finite number of cosets of normal open subgroups;

(c) If X is a closed subset of G, then $X = \bigcap_{N \triangleleft_O G} NX$. In particular, $\bigcap_{N \triangleleft_O G} N = 1$.

Proof. (a) Since G is a totally disconnected topological group, by Lemma 1.94 (h), we have that G is Hausdorff. Let U be a non-trivial open set in G and $x \in U$. The set Ux^{-1} is an open set that contains 1_G. To verify this, we could define a function $f : U \longrightarrow Ux^{-1}$ such that $u \mapsto ux^{-1}$. By Lemma 1.94 (a), we have that f is a homeomorphism. That is, the function $f^{-1} : Ux^{-1} \longrightarrow U$ is continuous, so since Ux^{-1} is in the pre-image of the open set U, we have that Ux^{-1} is open. Moreover,

$$x \in U \implies 1_G = xx^{-1} \in Ux^{-1},$$

hence Ux^{-1} is indeed an open neighborhood[4] of 1_G. Since G is Hausdorff, then by Lemma 1.46(c), we have that Ux^{-1} is a union of subsets that are clopen, and by Lemma 1.95, we have that each of these clopen subsets will contain a normal open subgroup K_x of G, so $U = \bigcup_{x \in U} K_x x$.

(b) If P is an clopen set, since P is a closed subset of a compact space, we have that P is compact (Lemma 1.81 (a)). Furthermore, since P is a closed subset of a totally disconnected topological group, P is also totally disconnected. Thus, by (a), we have that $P = \bigcup_{x \in P} K_x x$. But since P is compact, the cover $\bigcup_{x \in P} K_x x$ admits a finite subcover $P = \bigcup_{i=1}^{n} K_{x_i} x_i$. On the other hand, let P be a union of a finite number of cosets of normal open subgroups $P = \bigcup_{i=1}^{n} K_{x_i} x_i$. By Lemma 1.94 (c), each open subgroup is closed, and by Lemma 1.94 (b), the coset $K_{x_i} x_i$ is open, and the finite union of clopen subsets is clopen.

(c) Let X be a closed subset of G, then $G - X$ is an open set. By item (a) we have that

$$G - X = \bigcup_{x \in G - X} K_x x,$$

and by De Morgan's Law,

$$X = \bigcap_{x \in G - X} (G - K_x x) \supseteq \bigcap_{N \triangleleft_O G} NX \supseteq X.$$

[4]An open neighborhood of an element X is an open set that contains X.

□

Definition 1.98. *Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a family of topological groups, then $\Pi_{\lambda\in\Lambda}G_\lambda$ is also a topological group with respect to the product topology. The multiplication in $\Pi_{\lambda\in\Lambda}G_\lambda$ is defined coordinate-wise, that is, $(x_\lambda)(y_\lambda) = (x_\lambda y_\lambda)$ for any $(x_\lambda), (y_\lambda) \in \Pi_{\lambda\in\Lambda}G_\lambda$.*

Notation 1.99. *We denote a subgroup H of finite index in G by $H \leq_{f.i.} G$.*

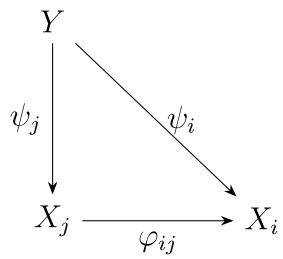
Definition 1.100. *A **directed set** is a partially ordered set I such that for any $i_1,\ i_2 \in I$, there exists an element $j \in I$ such that $i_1 \leq j$ and $i_2 \leq j$.*

Definition 1.101. *An **inverse system** (X_i, φ_{ij}) of topological spaces indexed by a directed set I consists of a family $\{X_i\}_{i\in I}$ of topological spaces and a family $\{\varphi_{ij} : X_j \longrightarrow X_i \mid i, j \in I,\ i \leq j\}$ of continuous maps such that φ_{ii} is the identity map id_{X_i} for each i and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ whenever $i \leq j \leq k$.*

If each X_i is a topological group and each φ_{ij} is a continuous homomorphism, then $(X_i,\ \varphi_{ij})$ is an inverse system of topological groups.

Example 1.102. *Let $I = \mathbb{N}$ ordered in the usual way. Let $\{X_i\}_{i\in\mathbb{N}}$ be finite sets, and $\varphi_{i,i+1} :\ X_{i+1} \longrightarrow X_i$ be arbitrary maps for each i. We define $\varphi_{ii} = id_{X_i}$ for each i and $\varphi_{ij} = \varphi_{i,i+1} \ldots \varphi_{j-1,j}$ for $j < i$. Thus, $(X_i,\ \varphi_{ij})$ is an inverse system of finite sets.*

Definition 1.103. *Now, let $(X_i,\ \varphi_{ij})$ be an inverse system of topological spaces, and let Y be a topological space. We say that a family $(\psi_i :\ Y \longrightarrow X_i \mid i \in I)$ of continuous maps is **compatible** if $\varphi_{ij}\psi_j = \psi_i$ whenever $i \leq j$. That is, the diagram*



commutes.

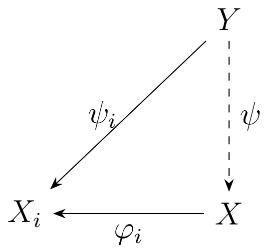
Definition 1.104 (Universal Property of Inverse Limit). *An **inverse limit**, or **projective limit**, $(X,\ \varphi_i)$, of an inverse system $(X_i,\ \varphi_{ij})$ of topological spaces is a topological space X with compatible applications $(\varphi_i : X \longrightarrow X_i)$ of continuous mappings with the following universal property: whenever $(\psi_i : Y \longrightarrow X_i)$ is a family of compatible continuous mappings from a space Y, there exists a unique continuous mapping $\psi : Y \longrightarrow X$ such that $\varphi_i\psi = \psi_i$ for each i. That is, the diagram*

commutes.

The following result shows the uniqueness and existence of the inverse limit.

Proposition 1.105. *Let $(X_i,\ \varphi_{ij})$ be an inverse system indexed by I. Then*

(a) If $(X^{(1)},\ \varphi_i^{(1)})$ and $(X^{(2)},\ \varphi_i^{(2)})$ are inverse limits of an inverse system, then there



exists an isomorphism $\overline{\varphi} : X^{(1)} \longrightarrow X^{(2)}$ *such that* $\varphi_i^{(2)}\overline{\varphi} = \varphi_i^{(1)}$ *for each* i*;*

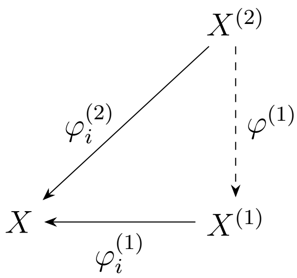
(b) Let $C = \Pi\{X_i\}_{i\in I}$ *and for each* i*,* π_i *be the projection from* C *to* X_i*. We define*

$$X = \{c \in C \mid \varphi_{ij}\pi_j(c) = \pi_i(c) \text{ for each } i,\ j \text{ with } j \geq i\} \leq \Pi_{i\in I}X_i$$

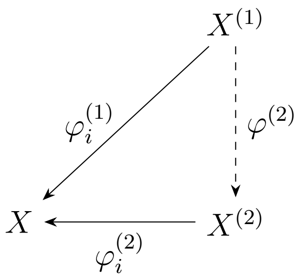
and $\varphi_i = \pi_i|_X$ *for each* i*. Thus* $(X,\ \varphi_i)$ *is the inverse limit of* $(X_i,\ \varphi_{ij})$*;*

(c) If $(X_i,\ \varphi_{ij})$ *represents the inverse limit of topological groups and continuous mappings, then* X *is a topological group and the mappings* φ_i *are continuous homomorphisms.*

Proof. (a): The Universal Property of the Inverse Limit of $(X^{(1)},\ \varphi_i^{(1)})$ applied to the family $\{\varphi_i^{(2)}\}$ of compatible mappings induces the mapping $\varphi^{(1)} : X^{(2)} \longrightarrow X^{(1)}$ and the diagram:



commutes. That is, $\varphi_i^{(1)}\varphi^{(1)} = \varphi_i^{(2)}$ for each i. Similarly, using the Universal Property of $(X^{(2)}, \varphi_i^{(2)})$, we can obtain the mapping $\varphi^{(2)} : X^{(1)} \longrightarrow X^{(2)}$ such that the diagram:



commutes. That is, $\phi_i^{(2)}\varphi^{(2)} = \varphi_i^{(1)}$ for each i. By the Universal Property of $(X^{(1)}, \varphi_i^{(1)})$, there exists a unique mapping $\psi : X^{(1)} \longrightarrow X^{(1)}$ such that $\varphi_i^{(1)}\psi = \varphi_i^{(1)}$ for each i. However, both mappings $\varphi^{(1)}\varphi^{(2)}$ and $id_{X^{(1)}}$ extend this morphism. Therefore, we must have $\varphi^{(1)}\varphi^{(2)} = id_{X^{(1)}}$. Similarly, $\varphi^{(2)}\varphi^{(1)} = id_{X^{(2)}}$, thus $\varphi^{(2)}$ is an isomorphism.

(b), (c): To prove (b) we want to define X as the set of elements of C that satisfy the compatibility conditions, thus ensuring that X will be a subgroup of C. In fact, endowing C with the product topology and X with the subspace topology, we have that the mappings φ_i are necessarily continuous, and the Universal Property of X guarantees that $\varphi_{ij}\varphi_j = \varphi_i$ whenever $j \geq i$. Therefore, if $(X_i,\ \varphi_{ij})$ represents the inverse limit of topological groups, then X is a topological group and each mapping φ_i is a homomorphism.

In summary, the general idea of the proof of (c) will show that the inverse limit of topological groups and continuous mappings results in a topological group, and the mappings φ_i that define it are continuous homomorphisms. In fact, suppose $(\psi_i : Y \longrightarrow X_i)$ is a compatible family of mappings. We want to show that there exists a unique continuous mapping $\psi : Y \longrightarrow X$ such that $\varphi_i\psi = \psi_i$ for each i. Let $\overline{\psi}$ be the mapping from Y to C taking each element y to the vector $(\psi_i(y))$. Then $\pi_i\overline{\psi} = \psi_i$ for each i, and $\overline{\psi}$ is continuous (since its product with each projection is continuous). If $j \geq i$, we have that

$$\pi_i\overline{\psi} = \psi_i = \varphi_{ij}\psi_j = \varphi_{ij}\pi_j\overline{\psi}$$

and it follows that $\overline{\psi}$ takes Y into X. Now define $\psi : Y \longrightarrow X$ by $\psi(y) = \overline{\psi}(y)$ for each y. Thus ψ is continuous, and $\varphi_i\psi = \psi_i$ for each i. If $\psi' : Y \longrightarrow X$ is a mapping satisfying $\varphi_i\psi' = \psi_i$ for each i and $y \in Y$, then the image in X_i of $\psi'(y)$ is an inverse system of groups and homomorphisms, and the mappings $\psi_i : Y \longrightarrow X_i$ are group homomorphisms, as is ψ.

□

Notation 1.106. *From now on, we will denote the inverse limit of an inverse system* $(X_i,\ \varphi_{ij})$ *by* $\varprojlim(X_i,\ \varphi_{ij})$.

Proposition 1.107. *Let* $(X_i,\ \varphi_{ij})$ *be an inverse system indexed by* I*, and write* $X = \varprojlim_{i \in I} X_i$*. Then*

(a) If each X_i *is Hausdorff, then* X *is Hausdorff;*

(b) If each X_i *is totally disconnected, then* X *is totally disconnected;*

(c) If each X_i *is a Hausdorff topological group, then* X *is closed in* $C = \Pi\{X_i\}_{i\in I}$*;*

(d) If each X_i *is compact and Hausdorff, then* X *is also compact;*

(e) If each X_i *is a non-empty compact Hausdorff space, then* X *is non-empty.*

Proof. Since we are more interested in the properties of Hausdorff, totally disconnected, and compact, we will demonstrate items (a), (b), and (d):

(a): If each X_i is Hausdorff, then, by Theorem 1.88 (a), we have that $\Pi_{i\in I}X_i$ is Hausdorff. Now, since $X \subset \Pi_{i\in I}X_i$, it follows that X is Hausdorff.

(b): If X_i is totally disconnected, then, by Theorem 1.88 (b), we have that $\Pi_{i\in I}X_i$ is totally disconnected. Now, since $X \subset \Pi_{i\in I}X_i$, it follows that X is totally disconnected.

(d): If X_i is compact and Hausdorff, then, by (a), $X = \varprojlim_{i\in I} X_i$ is compact. Since X_i is compact, by (c), we have that X is closed in $\Pi_{i\in I}X_i$. Since X_i is compact, by Theorem 1.88 (c), we have that $\Pi_{i\in I}X_i$ is compact. Now, since X is a closed set contained in $\Pi_{i\in I}X_i$, which is compact, we have that X is compact (by Lemma 1.81 (a)).

The proofs of the remaining items can be found in [25], Proposition 1.1.5, p. 14. □

Proposition 1.108 ([25], Proposition 1.1.7, p. 16)**.** *Let X be a compact Hausdorff totally disconnected topological space. Then, X is the inverse limit of its discrete quotient spaces.*

Notation 1.109. *We say that $\mathcal{C}$ is a **class** if we have*

$$F_1 \in \mathcal{C}, \text{ and } F_2 \cong F_1 \implies F_2 \in \mathcal{C}.$$

Let $\mathcal{C}$ be a class of finite groups. We say that a group F is a $\mathcal{C}$-group if $F \in \mathcal{C}$, and we say that G is a pro-$\mathcal{C}$ group if G is the inverse limit of $\mathcal{C}$-groups.

For the case in which the class $\mathcal{C}$ we are referring to is the one of finite groups, we have the following definition of profinite groups:

Definition 1.110. *A **profinite group** is the inverse limit of an inverse system of finite groups.*

From this definition, we obtain the following characterization of profinite groups.

Theorem 1.111. *Let G be a topological group. Then the following are equivalent:*

(i) G is profinite;

(ii) G is isomorphic to a closed subgroup of the Cartesian product of finite groups;

(iii) G is compact and $\bigcap_{N \triangleleft_O G} N = 1$;

(iv) G is compact and totally disconnected.

Proof. (i) $\implies$ (ii): If G is profinite, then $G = \varprojlim_{i\in I} G_i$ with each $|G_i| < \infty$. Since each G_i is Hausdorff, by Proposition 1.107 (c), we have that G is closed in $\prod_{i\in I} G_i$.

(ii) $\implies$ (iii): Suppose that G is a closed subgroup of $\prod_{i\in I} G_i$, where each G_i is finite. Since each G_i is compact (being finite), the Cartesian product $\prod_{i\in I} G_i$ is also compact by Theorem 1.88 (c). As G is a closed subgroup of a compact space, G is also compact by Lemma 1.81 (a).

Define $K_i = \ker(\pi_i)$, where $\pi_i : \prod_{i\in I} G_i \to G_i$ is the canonical projection. Note that K_i is an open normal subgroup of $\prod_{i\in I} G_i$. Considering $N_i = G \cap K_i$, we have that $N_i \lhd_O G$.

The intersection of all K_i's is trivial, as it is the intersection of the kernels of all projections, i.e.,

$$\bigcap_{i\in I} K_i = \bigcap_{i\in I} \ker(\pi_i) = \{1\}.$$

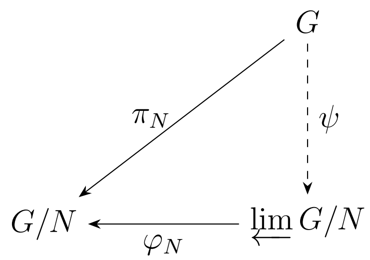
Thus,

$$\bigcap_{i\in I} N_i = G \cap \left(\bigcap_{i\in I} K_i\right) = G \cap \{1\} = \{1\}.$$

Therefore,

$$\bigcap_{N \lhd_O G} N = 1.$$

$(iii) \implies (iv)$: Consider the inverse system $\{G/N \mid N \lhd_O G\}$. By the Universal Property, we have a map $\psi : G \longrightarrow \varprojlim G/N$ as per the diagram below:



where $g \mapsto (gN)$ and $\ker(\psi) = \bigcap_{N\lhd_O G} N$. For each $N \lhd_O G$, we have

$$\varphi_N(\psi(G)) = \pi_N(G) = G/N.$$

To show that ψ is an isomorphism, we first note that ψ is injective, as $\ker(\psi) = \bigcap_{N\lhd_O G} N = 1$.

Now we verify that ψ is surjective. Let $(g_N) \in \varprojlim G/N$. This means that $g_N \in G/N$ and the projections are compatible; that is, if $N \subseteq M$, then the image of g_M in G/N is g_N.

Define $g \in G$ such that for each $N \lhd_O G$, the class of g in G/N is g_N. Such g exists because the g_N are compatible and G is the original group whose quotients we are considering. Thus, $\psi(g) = (g_N)$, showing that ψ is surjective. Therefore, ψ is a topological group isomorphism.

Finally, to show that G is totally disconnected, observe that $\varprojlim G/N$ is a totally disconnected topological group since it is the inverse limit of finite groups, which are totally disconnected.

$(iv) \implies (i)$: Suppose that G is compact and totally disconnected. By Proposition 1.97 (a), we can consider a basis of open normal subgroups $\{N \triangleleft_O G\}$ and form the inverse system $\{G/N \mid N \triangleleft_O G\}$.

Since G is compact, there exists a natural morphism

$$\psi : G \to \varprojlim G/N$$

which is injective because $\bigcap_{N \triangleleft_O G} N = 1$.

Therefore, by the same argument as before, we have $G = \varprojlim G/N$, i.e., G is profinite. □

From a methodological perspective, studying profinite groups provides a technique to replace infinite hypotheses about small and diverse objects with a single hypothesis about a large object.

Example 1.112. *Let G be any group, $\{G/U \mid U \triangleleft_f G\}$ a projective system, and I a family of subgroups with the following property:*

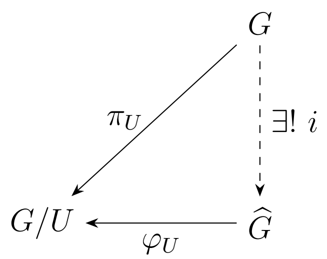
$$U_1, U_2 \in I,\ \exists U \subset I,\ \textit{such that } U \subset U_1 \cap U_2.$$

For $U \leq V$, we define

$$\varphi_{UV} : G/U \longrightarrow G/V, \quad gU \mapsto gV.$$

Then $\{G/U, \varphi_{UV}\}$ is a projective system, and $\widehat{G} = \varprojlim_{U \triangleleft_f G} G/U$ is a profinite group called the ***profinite completion*** *of G.*

Note that, by the Universal Property:



there exists a homomorphism

$$i : G \longrightarrow \widehat{G}, \quad g \mapsto (gU).$$

The map i is not necessarily injective, as its kernel is given by $\ker(i) = \bigcap U$. *When i is injective, meaning that* $\ker(i)$ *is trivial, we say that G is residually finite (see 2.2).*

Example 1.113. *The profinite completion of $\mathbb{Z}$, denoted by $\widehat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, is a profinite ring*[5]*. This profinite ring is a procyclic additive group.*

Certain classes of profinite groups have special names: as we saw earlier, if all the finite groups G_λ belong to some class of groups C (and the inverse system is surjective), then $G = \varprojlim G_\lambda$ is called a pro-C group. When C is the class of finite p-groups for some prime p, we call G a pro-p group. We then define the pro-p completion through the following example.

Example 1.114. *In Example 1.112, if for every $U \triangleleft_f G$ we have $|G : U| = p^n$ for some $n \in \mathbb{Z}$, then the profinite group $\widehat{G} = \varprojlim G/U$ is called the pro-p completion of G. If the map $i : G \longrightarrow \widehat{G}_p$ is a monomorphism, then G is said to be residually p.*

Example 1.115. *The pro-p completion of $\mathbb{Z}$, denoted by $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, is the ring of p-adic integers. This ring is also an additive group.*

[5] A profinite ring is the inverse limit of finite rings.

Chapter 2

Residually Finite Groups

Definition 2.1. *Let $\mathscr{P}$ be an arbitrary property of a group. A group G is said to be* ***residually*** $\mathscr{P}$ *if, for any nontrivial element $g \in G$, there exists a quotient group $\overline{G}$ satisfying $\mathscr{P}$ such that the element $\overline{g}$, the image of g in $\overline{G}$, is nontrivial. By considering $\mathscr{P}$ as the property of finiteness, we obtain residually finite groups.*

The study of residually finite groups began to gain prominence in the 20th century, mainly due to P. Hall. In [9], the mathematician provided the following definition:

Definition 2.2. *A group G is said to be* ***residually finite (RF)*** *if, and only if, for each element $x \neq 1$ in G, there exists at least one normal subgroup $K = K(x)$ of finite index in G such that $x \notin K$. In other words, RF groups are those that can be embedded in a Cartesian product of finite groups.*

RF groups are of great importance in Combinatorial Group Theory. In particular, we have the following examples:

(a) Finite groups;

(b) Free groups (F. Levi, [10]);

(c) Finitely generated nilpotent groups (K. Hirsch, [11]);

(d) Every finitely generated linear group[1] over a field of characteristic 0 (A. Mal'cev, [17]);

(e) Ascending HNN extensions of linear groups are RF (Borisov and Sapir, [21]);

(f) The automorphism group $Aut(G)$, where G is RF and finitely generated (G. Baumslag, [18]);

[1]The term "linear group" refers to a group that can be represented as a subgroup of a group of linear matrices over a field.

One of the important results in this section on RF groups, establishing criteria to obtain significant counterexamples of groups that are not RF, will be presented below.
Theorem 2.3. *(S. Meskin, [13]) The Baumslag-Solitar group:*

$$BS(n,m) = \langle a, b \mid a^{-1}b^{n}a = b^{m} \rangle, \text{ where } n, m \in \mathbb{Z},$$

is RF if, and only if, at least one of the following cases is satisfied:

- $|n| = 1,$
- $|m| = 1,$
- *or* $|n| = |m|.$

The following theorem presents equivalences for the definition of residual finiteness.
Theorem 2.4. *A group G is said to be RF if any of the following equivalences is satisfied:*

(a) For any distinct elements $x, y \in G$, there exists a homomorphism $\phi : G \longrightarrow \overline{G}$ such that $\phi(x) \neq \phi(y)$ in $\overline{G}$;

(b) For every $g \neq 1$ in G, there exists a (normal) subgroup of finite index that does not contain g;

(c) The intersection of all normal subgroups of finite index in G is trivial;

(d) The intersection of all subgroups of finite index in G is trivial.

Proof. $(a) \implies (b)$: Let g be a nontrivial element of G. By (a), there exists a homomorphism $\phi : G \longrightarrow \overline{G}$ such that $\phi(g)$ is nontrivial. Taking g and the identity element in G, we apply condition (a). Denote $D = \ker(\phi)$; thus, D is a (normal) subgroup of finite index in G such that $g \notin D$.

$(b) \implies (c)$: Let H be the intersection of all subgroups of finite index in G. We will show that H is trivial, meaning that H contains only the identity element of G. Suppose $g \in H$ is a nontrivial element of H. Then g belongs to all normal subgroups of finite index in G. On the other hand, by (b), there exists a normal subgroup D of finite index in G that does not contain g. This contradicts our assumption that g belongs to all normal subgroups of finite index in G.

$(c) \implies (d)$: The intersection of all subgroups of finite index in G is a subgroup of the intersection of all normal subgroups of finite index in G.

$(d) \implies (a)$: Let $x, y \in G$ such that $x \neq y$, and let $g = xy^{-1}$, a nontrivial element of G. By (d), we have $\cap_{H \leq G} H = 1$ where H has finite index in G, so g does not belong to at least one subgroup K of finite index in G. The subgroup $N := \cap_{z \in G} K^{z}$, is the largest normal subgroup of G contained in K and therefore has finite index in G. Since $g \notin N \leq K$, we have $xN \neq yN$. To prove this implication, we can show that $xN = yN \implies g \in N$. Indeed,

$$xN = yN \iff y^{-1}xN = N \iff y^{-1}x \in N \iff xy^{-1} \in N \implies g \in N.$$

Now consider ϕ as the natural homomorphism from G to G/N. That is, ϕ is the homomorphism that maps G to a finite group $\overline{G} := G/N$ such that $\phi(x) \neq \phi(y)$. □

One of the primary motivations for studying RF groups lies in their connection to the **Dehn problems** introduced by M. Dehn in 1912.

Definition 2.5. *For a given group G with a specified presentation:*

(a) For an arbitrary word w in the generators, decide in a finite number of steps whether W represents the identity element of G. ***(Word Problem)***

(b) For every element $g \in G$ and every finitely generated subgroup $H \leq G$ decide in a finite number of steps whetherg is in H. ***(Generalized Word Problem)***[2]

(c) For two arbitrary words w_1 and w_2 in the generators, decide in a finite number of steps whether w_1 and w_2 represent conjugate elements of G. ***(Conjugacy Problem)***

(d) For an arbitrary group H defined by a different presentation, decide in a finite number of steps whether G is isomorphic to H. ***(Isomorphism Problem)***

These problems are decidability problems, in the sense that, if there exists, for example, an algorithm that, through a number of steps, decides whether a word in G represents the identity element of G or not, we say that G has a solvable/decidable word problem.

The first connection established between the Dehn problems and residual finiteness is attributed to O. Schreier, who in 1927, while solving the word problem for free groups, proved that free groups are RF (see [49]).

In the 1930s, Levi proved that free groups are RF. A proof of this important result is given below.

Theorem 2.6. *If G is a free group, then G is RF.*

Proof. Let $w = x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$, with $\epsilon_j \in \{\pm 1\}$ and $x_{i_1}, \dots, x_{i_k} \in X$, be a nontrivial reduced word in $F(X)$.

---

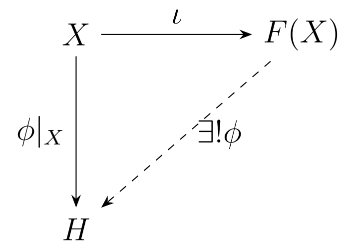
[2]The term generalized word problem is due to W. Magnus but it is also known as the occurrence problem.

We will show the equivalence (b) of Theorem 2.4. That is, we want to show that there exists $N \lhd F(X)$, $|F(X)/N| < \infty$ such that $w \notin N$. Let $H = S_{k+1}$. If the homomorphism $\phi : F(X) \longrightarrow H$ is such that $\phi(w) \neq 1_H$, then $N := \ker(\phi)$. Thus, by the Isomorphism Theorem, $|F(X)/N| = |\text{Im } \phi| \leq |H| = (k+1)! < \infty$.

We want $\phi(x_{i_k}^{\epsilon_k})$ to belong to H. For this, we define $\phi(x_{i_1}) \dots \phi(x_{i_k})$ as follows:

$$\phi(x_{i_j}) := (j \ j+1) \implies \phi(x_{i_j}^{\epsilon_j}) = (j \ j+1)$$

By the Universal Property of Free Groups, we have:



$$\phi(w)(k+1) = \underbrace{\phi(x_{i_1}^{\epsilon_1}) \, \underbrace{\phi(x_{i_2}^{\epsilon_2}) \dots \underbrace{\phi(x_{i_k}^{\epsilon_k})(k+1)}_{k}}_{2}}_{1} = 1 \neq k+1.$$

Thus, $\phi(w) \neq \text{id} = 1_H$.

□

The following example helps illustrate how we can use the homomorphism discussed in the proof of Theorem 2.6, showing that the image of a nontrivial reduced word in $F(X)$ is nontrivial.

Example 2.7. *The word $w = cca^{-1}bc^{-1}a$, which has length 6, will be mapped to a nontrivial element of the group S_7.*

First, we define the homomorphism $\phi : F(X) \to S_7$. For this, we need to map the generators of $F(X)$ to elements of S_7. Suppose $X = \{a, b, c\}$ for this example.

Let's define:

$$\phi(a) = (1 \ 2), \ \phi(b) = (2 \ 3), \ \phi(c) = (3 \ 4) :$$

Now, we apply ϕ to the word $w = cca^{-1}bc^{-1}a$. We have:

$$\begin{aligned}\phi(w) &= \phi(c)\phi(c)\phi(a^{-1})\phi(b)\phi(c^{-1})\phi(a)\\ &= (3\ 4)(3\ 4)(1\ 2)^{-1}(2\ 3)(3\ 4)^{-1}(1\ 2)\\ &= (3\ 4)(3\ 4)(1\ 2)(2\ 3)(3\ 4)(1\ 2)\\ &= (1\ 2)(2\ 3)(1\ 2)\\ &= (1\ 2\ 3).\end{aligned}$$

Since $(1\ 2\ 3)$ *is a cycle of length 3 in* S_7*, we conclude that the image of the word* w *is nontrivial in* S_7*, as expected.*

The following example presents a group that is not RF.

Example 2.8. *G. Higman, in [11], showed that the group:*

$$G = \langle a_1, a_2, \ldots, a_n \mid a_i^{-1}a_{i+1}a_i = a_{i+1}^2,\ a_n^{-1}a_1a_n = a_i^2,\ i = 1, 2, \ldots, n-1,\ n \geq 4\rangle,$$

is not RF because G *lacks a normal subgroup of finite index. Thus, it does not satisfy equivalence* (b) *of Theorem 2.4.*

In 1940, A. Mal'cev demonstrated that finitely presented RF groups have a decidable word problem.

Theorem 2.9. *If* G *is a finitely presented RF group, then* G *has a decidable word problem (see [3], Theorem 4.6, p. 195).*

In the same year, A. Mal'cev contributed with a another important result.

Theorem 2.10. *If a finitely generated group* G *is RF, then* G *is Hopfian. That is, every epimorphism of* G *is an automorphism.*

Proof. The proof of this theorem can also be found in [6], p. 415. Note that in this case, G must be finitely generated; otherwise, consider $G = \langle x_1, x_2, \ldots\rangle$, a group with infinitely many generators. Define the homomorphism $\phi : G \longrightarrow G$ as follows:

$$\begin{aligned}x_1 &\mapsto 1,\\ x_2 &\mapsto x_1,\\ x_3 &\mapsto x_2,\\ &\vdots\end{aligned}$$

Since ϕ is surjective but the kernel $\ker(\phi) = \langle x_1\rangle$, ϕ is not an isomorphism. Thus, G is not Hopfian in this case. $\square$

Another interesting result connecting concepts from the previous chapter on profinite topology and residual finiteness is the following:

Theorem 2.11. *A group G is RF if and only if its profinite topology is Hausdorff.*

Proof. Suppose G is a group with a profinite topology that is Hausdorff. Take a nontrivial element $g \in G$ such that $g = \bigcap_{N \triangleleft G} N$ with $|G : N| < \infty$. Then g and e (the identity element) belong to the same open sets in the profinite topology of G, which leads to a contradiction.

Conversely, assume G is RF. Suppose $\bigcap_{N \triangleleft G} N = \{e\}$ where $|G : N| < \infty$. Then there exists a normal subgroup N of finite index such that $xy^{-1} \notin N$, where $x \neq y$. Let $A = xN$ and $B = yN$. Thus $A \cap B = \emptyset$, indicating that the topology is Hausdorff. □

Now, let's explore how residual finiteness is preserved in various group constructions, such as the wreath product, direct product, semidirect product, free product, and HNN extension. Our focus will be to investigate whether these constructions retain residual finiteness when applied to groups that already have this characteristic. The main results on this topic are due to K. Gruenberg, with further work by G. Baumslag, who extended Gruenberg's findings.

Theorem 2.12. *Let $\{G_i\}_{i \in I}$ be a family of RF groups. The direct product $G = \prod_{i \in I} G_i$ is RF.*

Proof. To show that the direct product $G = \prod_{i \in I} G_i$ of RF groups is RF, we need to show that G satisfies one of the equivalences in Theorem 2.4.

For each $g = (g_i) \neq 1$ in G, there exists an index $k \in I$ such that $g_k \neq 1$ in G_k. Since G_k is RF, there exists a normal subgroup of finite index N_k in G_k that does not contain g_k. We define $N = \prod_{i \in I} N_i$, where $N_i = G_i$ if $i \neq k$ and $N_i = N_k$ otherwise. Thus, N is a normal subgroup of finite index in G that does not contain g.

Therefore, G satisfies condition **(b)** of Theorem 2.4, implying that G is RF. □

For each result presented that supports filling in any gap of Table 3.4, we will update it with the current progress.

A and B	RF	LERF	(LR)
$A \wr B$	?	?	?
$A \times B$	yes	?	?
$A \rtimes B$	?	?	?
$A * B$	?	?	?
HNN	?	?	?
$A *_H B$	?	?	?

Table 2.1: **Table of Residual Properties**

2.1 Wreath Product of Groups and Gruenberg's Theorem

We now define what is meant by the wreath product of groups, to then study how residual finiteness behaves under this product. The wreath product is defined as a product of two groups, and there are two versions of it: the restricted wreath product and the complete (or unrestricted) wreath product. The observations and definitions of this product were inspired by and derived from the book by J. Meldrum [32].

The wreath product arises in the context of permutation groups. Let (A, X) and (B, Y) be permutation groups, where the group A acts on the set X and the group B acts on the set Y. Define $Z := X \times Y$ as the Cartesian product of X and Y. We denote by A^Y the power of copies of A indexed by Y. Thus, an element of A^Y is written as a map from Y to A. Moreover, we denote by a_y the element $f \in A^Y$ such that $f(y) = a$ and $f(y') = e$ for all $y' \neq y \in Y$. We define an action of B on A^Y by

$$f^b(y) = f(yb^{-1}), \quad y \in Y, \, b \in B.$$

That is, for $a \in A$ and $y \in Y$, we have

$$(a_y)^b = a_{yb}.$$

Both formulas above provide a definition of B as a group of automorphisms of A^Y. Additionally, these formulas define B as a group of automorphisms of $A^{(Y)}$, the restricted direct power of A indexed by elements of Y. This allows us to define the semidirect product of A^Y (or $A^{(Y)}$) by B.

Definition 2.13. *Let (A, X) and (B, Y) be two permutation groups. We define the **complete (permutation) wreath product** of A and B, denoted $A\, Wr_Y B$, as a permutation* *group acting on* Z*, by*

$$(x, y)(fb) = (xf(y), yb), \quad f \in A^Y, b \in B.$$

The ***restricted (permutation) wreath product*** $A\ wr_Y B$ *of* A *and* B*, defined as a permutation group on* Z*, is given in the same form, but with* $f \in A^{(Y)}$*. If either* A *or* B *is trivial, we will refer to the wreath product as the trivial permutation wreath product.*

It is worth noting that the groups we wish to operate on are not always permutation groups, but by using permutation representations we can define the wreath product of these groups as follows:

Definition 2.14. *The* ***standard wreath product*** *of two groups* A *and* B *is the wreath product* $A\ Wr_Y B$ *(or* $A\ wr_Y B$*) where* A *and* B *are considered as permutation groups in their right regular representations. The standard wreath product is usually denoted by* $A\ WrB$ *or* $A\ wrB$*.*

To better understand this definition, we will define what is meant by a right regular representation of a group. First, we will define what a representation is:

Definition 2.15. *Given a group* G*, a* ***representation*** *of* G *is a homomorphism* ρ *from* G *into some group* $\mathcal{G}$*.*

A **regular representation** is defined via the "translation" map $T : g \mapsto T_g$.

Definition 2.16. *Let* G *be a group,* G_0 *the set* G*, and* $\mathcal{P}(G_0)$ *the group of permutations of* G_0*. A* ***right regular representation*** *is a homomorphism:*

$$\begin{aligned} T : G &\longrightarrow \mathcal{P}(G_0) \\ g &\longmapsto T_g : G_0 \to G_0 \\ &\qquad\qquad a \mapsto ga. \end{aligned}$$

Notation 2.17. *From now on, we will denote the restricted wreath product of the groups* A *and* B *by* $A \wr B$*.*

As mentioned, we want to investigate how residual finiteness behaves under the wreath product of groups. Gruenberg's Theorem is fundamental for understanding when the wreath product of RF groups will also be RF. This theorem is used for various properties of a group, not only for residual finiteness. In particular, it is a theorem of groups that are residually $\mathscr{P}$, a broader property defined in 2.1. In 1957, Gruenberg introduced the following concept:

Definition 2.18 (Root Property)**.** *Let* $\mathscr{P}$ *be a property that satisfies:*

- *If a group* G *has the property* $\mathscr{P}$*, then any subgroup of* G *also has the property* $\mathscr{P}$*;*

- *If G and H are groups with property $\mathscr{P}$, then $G \times H$ has the property $\mathscr{P}$;*
- *For any chain $K \trianglelefteq H \trianglelefteq G$ such that G/H and H/K satisfy $\mathscr{P}$, there exists $L \trianglelefteq G$ such that L is contained in K and G/L satisfies $\mathscr{P}$.*

Then, we say that $\mathscr{P}$ is a ***root property****.*

Examples of root properties include: solvability, finiteness, and having order as a power of a prime p. Now we have what we need to state **Gruenberg's Theorem**.

Theorem 2.19 ([16], Theorem 3.2, p. 42)**.** *Let $\mathscr{P}$ be a root property and suppose G and Γ are RF. If Γ is given in its regular representation, then $G \wr \Gamma$ is residually $\mathscr{P}$ if and only if Γ satisfies $\mathscr{P}$, or G is abelian.*

Example 2.20. *Let G and Γ be RF groups, with $|\Gamma| = \infty$ and $Z(G) \neq G$. The theorem 2.19 shows that the group $G \wr \Gamma$ is not RF. In particular, we have that the product $S_3 \wr \mathbb{Z}$ is not RF, since $|\mathbb{Z}| = \infty$ and $Z(S_3) \neq S_3$. Note that, by Theorem 2.6, $\mathbb{Z}$ is RF, and since $|S_3| < \infty$, we also see that S_3 is RF.*

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	?	?
$A \times B$	yes	?	?
$A \rtimes B$	not necessarily	?	?
$A * B$	?	?	?
HNN	?	?	?
$A *_H B$	?	?	?

Table 2.2: **Table of Residual Properties**

Note that, since the wreath product is a particular case of a semidirect product, we also fill the gap for the semidirect product with Example 2.20.

Now, we are interested in determining whether the free product of two RF groups is, in itself, RF. In [16], Gruenberg showed this through the following theorem:

Theorem 2.21 ([16], Theorem 4.1, p. 43)**.** *If $\mathscr{P}$ is a root property, then every free product of residually $\mathscr{P}$ groups is residually $\mathscr{P}$ if and only if every free group is residually $\mathscr{P}$.*

Note that, as we saw earlier, by Theorem 2.6, every free group is RF. Therefore, we can use Theorem 2.21 to conclude that the free product of RF groups is also RF.

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	?	?
$A \times B$	yes	?	?
$A \rtimes B$	not necessarily	?	?
$A * B$	yes	?	?
HNN	?	?	?
$A *_H B$	?	?	?

Table 2.3: **Table of Residual Properties**

With the aim of generalizing K. Gruenberg's Theorem 2.21 on free products, G. Baumslag showed, in [19], that the free product of two RF groups with a finite amalgamated subgroup is RF. Later, in 1973, M. Tretkoff, in [20], provided a topological proof of this same result.

In [23], G. Higman showed that the amalgamated free product of RF groups is not always RF. Following this reasoning, in [22], B. Evans showed that we can obtain examples of non-residually finite groups through the following theorem:

Theorem 2.22 ([22], B. Evans)**.** *Let A be a RF group with an element a of infinite order. There exists a RF group B with an element b of infinite order such that the group G, which is the free product of A and B, amalgamating $a = b$, is not RF.*

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	?	?
$A \times B$	yes	?	?
$A \rtimes B$	not necessarily	?	?
$A * B$	yes	?	?
HNN	?	?	?
$A *_H B$	not necessarily	?	?

Table 2.4: **Table of Residual Properties**

Now, we need to understand the behavior of an HNN extension of a RF group. The main result on this topic is also due to Baumslag, who presented the following theorem in [19]:

Theorem 2.23 ([19], Theorem 3.1, p. 184)**.** *Let A be a RF group, H, K finite subgroups of A, and φ an isomorphism from H to K. Then the HNN extension*

$$G = \langle t, A \mid t^{-1}ht = \varphi(h),\ h \in H \rangle$$

is a RF group.

Proof. Essentially, the idea of the proof is to show that G is RF by demonstrating that it is residually Y, where Y is the set of groups of the form

$$\langle t, \overline{A} \mid t^{-1}ht = \varphi(h),\ h \in H\rangle,$$

where $\overline{A}$ is the finite image A/N of the homomorphism from A, with $H \cap N = K \cap N = 1$. To verify this, we manipulate the words in G, using Theorem 1.45 (Britton's Lemma). For readers interested in seeing the complete proof of this theorem, see [19]. □

In general, an HNN extension with a RF base group may not be a RF group. In particular, we have the following example:

Example 2.24. *The Baumslag-Solitar group $BS(2,3)$ is an HNN extension of $\mathbb{Z}$ with respect to the isomorphism $f : \langle a^2\rangle \longrightarrow \langle a^3\rangle$ and is not RF, since it does not satisfy the necessary and sufficient conditions of Theorem 2.3.*

Gathering the previously exhibited results about the behavior of RF groups under different group products, we have the following table:

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	?	?
$A \times B$	yes	?	?
$A \rtimes B$	not necessarily	?	?
$A * B$	yes	?	?
HNN	not necessarily	?	?
$A *_H B$	not necessarily	?	?

Table 2.5: **Table of Residual Properties**

Chapter 3

Properties of Virtual Retractions in Groups

This chapter will address the important concepts of virtual retractions and groups with the properties LERF and (LR). The main reference used in the study of this topic was the article [1] by A. Minasyan. In 1949, M. Hall introduced the LERF property [29]. This property gained greater significance in 1958 when A. Mal'cev demonstrated that LERF groups have a decidable generalized word problem.

Definition 3.1. *A group G is called* ***locally extended residually finite (LERF)*** *or* ***subgroup separable*** *if for every finitely generated subgroup $H \leq G$ and $g \in G \setminus H$ there exists a subgroup of finite index, $K_g \leq_{f.i.} G$, such that $H \leq K_g$ and $g \notin K_g$.*

The next result will present some equivalences of the LERF property.

Proposition 3.2. *The following conditions are equivalent:*

(a) G is LERF;

(b) For any two finitely generated subgroups $H_1 \neq H_2$ in G, there exists a homomorphism ϕ from G to some finite group $\overline{G}$ such that $\phi(H_1) \neq \phi(H_2)$;

(c) For every finitely generated subgroup H of G and every $g \in G \setminus H$, there exists a homomorphism ϕ from G to some finite group $\overline{G}$ such that $\phi(g) \notin \phi(H)$;

Proof. $(a) \implies (b)$: Let $H_1 \neq H_2$ be finitely generated subgroups of G. Without loss of generality, take $g \in H_2 \setminus H_1$. Since G is LERF, there exists a subgroup of finite index, $D \leq G$, such that $H_1 \leq D$ and $g \notin D$. Taking the core of D, we have a normal subgroup N of finite index in G such that $N := \bigcap_{x \in D} D^x$. Thus, the quotient group $\overline{G} := G/N$ is finite. Let $\phi : G \longrightarrow \overline{G}$ be the natural epimorphism from G to $\overline{G}$. We have that

$\phi(g) \notin \phi(H_1)$, because, otherwise, we would have

$$gN \subseteq H_1N \implies g \in H_1N \subseteq D \cdot D = D,$$

which would be a contradiction since D was assumed to be a subgroup of G that does not contain g. Therefore, we have that

$$\phi(g) \notin \phi(H_1) \implies \phi(H_1) \neq \phi(H_2).$$

$(b) \implies (c)$: Let H_1 be a finitely generated subgroup of G, and let $g \in G \setminus H_1$. Let H_2 be the subgroup of G defined by $H_2 = \langle H_1, g \rangle$, then $H_1 \neq H_2$. Thus, by (b), there exists a homomorphism $\phi : G \longrightarrow \overline{G}$ from G to a finite group $\overline{G}$ such that $\phi(H_1) \neq \phi(H_2)$ in $\overline{G}$. Since $\phi(H_2) = \langle \phi(H_1), \phi(g) \rangle$ and $\phi(H_1) \neq \phi(H_2)$, we have that $\phi(g) \notin \phi(H_1)$.

$(c) \implies (a)$ Let H be a finitely generated subgroup of G and $g \in G \setminus H$. By (c), there exists a homomorphism $\phi : G \longrightarrow \overline{G}$ from G to a finite group $\overline{G}$ such that $\phi(g) \notin \phi(H)$ in $\overline{G}$. Therefore, we have that $N := \ker(\phi)$ is a normal subgroup of finite index in G since

$$|G_1/\ker(\phi)| \cong |\phi(G_1)| < \infty.$$

Moreover, N satisfies

$$gN \notin \{hN : h \in H\} = HN.$$

Define $K = HN$, then K is a subgroup of finite index in G (Correspondence Theorem), such that $H \leq K$ and $g \notin K$. □

In 1949, M. Hall, after defining the LERF property, proved that free groups satisfy it. In 3.29, we show that free groups satisfy an even stronger property, which implies LERF. In 1958, A. Mal'cev [12] showed that polycyclic groups are LERF . R. Burns, A. Karrass, and D. Solitar showed in [51], that the group $K = \langle y, \alpha, \beta \mid \alpha^y = \beta\alpha,\ \beta^y = \beta \rangle$ is not LERF. In [52] V. Metaftsis and E. Raptis established in 2008 the following condition: A **R**ight-**A**ngled **A**rtin **G**roup (**RAAG**[1]) $A(\Gamma)$ is LERF if, and only if, Γ contains neither a path of length three nor a square as full subgraphs. Following the study of Artin groups, in 2021, K. Almeida and I. Lima identified a condition on the underlying graph of an Artin group that completely determines whether it is LERF (see [45], Theorem A).

Some important examples relating subgroup separability to braid groups will be presented in the following theorems. The demonstrations of these results have been omitted as they fall outside the scope of this dissertation.

Theorem 3.3 ([53])**.** *Let S be the disk. The braid group $B_n(S)$ is LERF if, and only if,*

[1]A *RAAG* is defined as a free product where $[u, v] = 1$ if and only if u, v are connected by an edge in the graph.

$n \leq 3$.

Theorem 3.4 ([53])**.** *Let S be the disk. The pure braid group $P_n(S)$ is LERF if, and only if, $n \leq 3$.*

Theorem 3.5 (see [47] Corollary 4)**.** *The surface braid group $B_n(S)$ (resp. the virtual braid group VB_n) is LERF if, and only if, the surface pure braid group $P_n(S)$ (resp. the pure virtual braid group VP_n) is also LERF.*

Theorem 3.6 ([47])**.** *Let S be a large surface*[2] *and let $n \geq 2$. Then $P_n(S)$ is not LERF.*

Theorem 3.7 ([47])**.** *Let S be a non-large surface, then:*

- *$P_2(S)$ is LERF if and only if S is not the Klein bottle.*
- *$P_3(S)$ is LERF if and only if S is the disk, the sphere, or the projective plane.*
- *$P_4(S)$ is LERF if and only if S is the sphere.*
- *$P_n(S)$ is not LERF for all $n \geq 5$.*

Given a finitely presented RF group G, the following result by P. Scott [15] is fundamental for the study of RF groups and for studying groups with the LERF property, as it shows how the properties of LERF and residual finiteness pass to subgroups and are inherited by overgroups.

Lemma 3.8 ([15], Lemma 1.1)**.** *If G is RF or LERF, then any subgroup of G has the same property, as does any group K containing G as a finite index subgroup.*

Proof. By Theorem 2.4 (a), G is RF if and only if for any distinct elements $g_1, g_2 \in G$, there exists a homomorphism $\phi : G \longrightarrow \overline{G}$ from G to a finite group $\overline{G}$ such that $\phi(g_1) = \phi(g_2)$ in $\overline{G}$. In particular, let $H \leq G$. For any two distinct elements $h_1, h_2 \in H$, there exists a homomorphism $\phi|_H : H \longrightarrow \overline{G}$ such that the images of h_1 and h_2 are distinct in $\overline{G}$. Thus, H is RF.

Now, by Theorem 3.2, G is LERF if and only if for every two distinct subgroups $G_1, G_2 \leq G$, there exists a homomorphism $\phi : G \longrightarrow \overline{G}$ from G to a finite group $\overline{G}$ such that $\phi(G_1) \neq \phi(G_2)$. Again, let $H \leq G$. For any two distinct subgroups H_1, H_2 of H, there exists a homomorphism $\phi|_H : H \longrightarrow \overline{G}$, such that the images of H_1 and H_2 under $\phi|_H$ are distinct in $\overline{G}$. Thus, H is LERF.

Now, let K be a group such that $G \leq_{i.f.} K$. If G is not normal in K, take the group $\text{Core}_K(G) = G_0$. Thus, $G_0 \leq G$ and $G_0 \trianglelefteq_{i.f.} K$ (Lemma 1.92). Now, without loss of generality, assume G is normal in K. Let F be the finite quotient group K/G, and let $\rho : K \longrightarrow F$ be the projection map. We will consider the following cases:

[2] A compact surface is said to be **large** if it is different from the following surfaces: sphere, projective plane, disk, annulus, torus, Möbius strip, or Klein bottle.

(i) If G is RF, then for any non-trivial element $k \in K$, if $k \in G$, then G has a subgroup G_1, of finite index, that does not contain k. Now G_1 also has finite index in K:

$$|K : G_1| = |K : G| \cdot |G : G_1| < \infty.$$

Therefore, for any $1 \neq g \in K$, there exists a subgroup $G_1 \leq_{i.f.} K$ that does not contain g. Now, if $k \notin G$, then G itself is the finite index subgroup in K that does not contain k. Hence, K is RF.

(ii) Now, suppose G is LERF. Given a finitely generated subgroup $S \leq K$ and some element $k \in K \setminus S$, we have $S \cap G \triangleleft S$ (since $G \triangleleft K$) with quotient some subgroup F_1 of F, let $K_1 = \rho^{-1}(F_1)$. Note that F_1 is finite, as it is a subgroup of F. Furthermore, $S \cap G$ is finitely generated, as it has finite index in S. If $k \notin K_1$, then K_1 is the desired subgroup, as K_1 contains S and has finite index in K:

$$|K : K_1| = |F : F_1| < \infty.$$

Now, if $k \in K_1$, we proceed as follows:

Write $k = gs$, where $g \in G$ and $s \in S$. Since $k \notin S$, we know that $g \notin S \cap G$. Since G is LERF, we obtain a subgroup G_2, of finite index in G, that contains $S \cap G$ but does not contain g. Let $G_3 = \text{Core}_S(G_2)$. Thus, G_3 is also a finite index subgroup of G that contains $S \cap G$ but does not contain g, and G_3 is normalized by S. Let K_3 be the subgroup of K_1 generated by G_3 and S. Then G_3 is a normal subgroup of K_3 with quotient F_1. Clearly, K_3 has finite index in K and contains S. Furthermore, k cannot belong to K_3, as K_3 contains S but not g. Hence, K_3 is the desired subgroup of K. Therefore, K is LERF.

□

Allenby and Gregorac in [35] showed that the LERF property is not preserved by direct products. We will show that the direct product of two free groups of rank 2 is not LERF.

Example 3.9 (Allenby, Gregorac, [35]). *Let $F_2 = \langle a, b \mid \rangle$, the free group of rank 2. Define a surjective homomorphism $\varphi : F_2 \to G$, where G is not RF, for instance take:*

$$G = \langle a, b \mid a^{-1}b^3a = b^2 \rangle,$$

By Theorem 2.3 This group is not residually finite. Define the diagonal subgroup:

$$\Delta = \{(g, g) \mid g \in G\} \leq G \times G.$$

This subgroup is not separable in $G \times G$. Specifically, there exist elements $(g,h) \notin \Delta$ that cannot be distinguished from Δ by any finite-index subgroup of $G \times G$. Using the surjective map $\varphi : F_2 \to G$, construct the map $\varphi \times \varphi : F_2 \times F_2 \to G \times G$. Then define:

$$Q = (\varphi \times \varphi)^{-1}(\Delta).$$

Moreover, the subgroup Q consists of pairs $(u,v) \in F_2 \times F_2$ such that $(\varphi(u), \varphi(v)) \in \Delta$, i.e., $\varphi(u) = \varphi(v)$. The subgroup Q is finitely generated because Δ is finitely generated in $G \times G$, and the preimage of a finitely generated subgroup under a homomorphism is finitely generated. Specifically, Q is generated by:

$$\{(a,a), (b,b), (a^{-1}b^3ab^{-2}, 1), (1, a^{-1}b^3ab^{-2})\}.$$

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	?	?
$A \times B$	yes	not necessarily	?
$A \rtimes B$	not necessarily	?	?
$A * B$	yes	?	?
HNN	not necessarily	?	?
$A *_H B$	not necessarily	?	?

Table 3.1: **Table of Residual Properties**

The LERF property is a stronger version of residual finiteness. This property can also be defined topologically as per the following lemma.

Lemma 3.10. *G is LERF if and only if every finitely generated subgroup is closed in the profinite topology.*

Proof. Suppose that G is LERF. Let $H \leq_{f.g.} G$ be a finitely generated subgroup of G such that $H = \bigcap_{g \notin H} K_g$, as in Definition 3.1 . We can assume, without loss of generality, that K_g is normal for each $g \notin H$. Otherwise, we would only need to consider the core of K_g. This will have finite index by Lemma 1.92. Therefore, since the basis of the profinite topology is given by the cosets of normal subgroups of finite index in G (Example 1.93), we have that the subgroups K_g are open, normal, and of finite index in G. Hence, they are also closed subgroups in the profinite topology (Lemma 1.94 (c)). Therefore H is closed.

On the other hand, suppose that every finitely generated subgroup of a group G

is closed in the profinite topology. Let $H \leq_{f.g.} G$; we have that $G \setminus H$ is open. Thus, for any $g \in G \setminus H$, since the opens in the profinite topology have finite index in G, and are given by the basis

$$\mathcal{B} = \{gN \mid g \in G, N \lhd G \text{ with } [G : N] < \infty\},$$

there exists a subgroup $N \trianglelefteq_{f.i.} G$ such that $gN \subseteq G \setminus H$.

Let $\phi : G \longrightarrow G/N$ be the natural homomorphism from G to G/N, then $\phi(g) \notin \phi(H)$. Otherwise, gN would be equal to hN for some $h \in H$, so $h \in gN \cap H$, which would contradict the fact that $gN \subseteq G \setminus H$. Given that $g \in G \setminus H$, we have $gN \subseteq G \setminus H$, ensuring that the element g is not in H.

□

Definition 3.11. *We say that a subgroup H of a group K is a* ***retract*** *if there exists a homomorphism $\rho : K \longrightarrow H$ such that $\rho|_H = id_H$. That is, K decomposes as a semidirect product $ker(\rho) \rtimes H$. The map ρ is called a* ***retraction****.*

In other words, we can think of a retraction as a homomorphism that maps elements of a group to elements of a subgroup of that group, thereby preserving its structure. In some cases, when H is a retract of G, we will denote it by $H \leq_r G$.

Here are examples of retractions:

Proposition 3.12. *Let G be a group.*

(a) If G is a direct product of subgroups, then every direct factor of G is a retract (R. Baer, [43]).

(b) If G is a free product of subgroups, then every free factor of G is a retract.

Proof. (a) Let $G = H \times K$, where H and K are subgroups of G. Define the projection $\pi_H : G \longrightarrow H$ by

$$\pi_H(h, k) = h, \quad \text{for all } h \in H \text{ and } k \in K.$$

It is straightforward to verify that π_H is a homomorphism and that $\pi_H|_H = \text{id}_H$. Thus, π_H is a retract of G onto H.

(b) Let $G = G_1 * G_2$, where G_1 and G_2 are subgroups of G. Define a map $\rho : G \longrightarrow G_1$ as follows. For any element $g \in G$ expressed in its reduced form $g = g_1 g_2 \cdots g_n$, where $g_i \in G_1 \cup G_2 \setminus \{1\}$ and consecutive terms belong to different subgroups, set:

$$\rho(g_i) = \begin{cases} g_i, & \text{if } g_i \in G_1, \\ 1, & \text{if } g_i \in G_2. \end{cases}$$

To show that ρ is a homomorphism, let $g, h \in G$ be two reduced words:

$$g = g_1 g_2 \cdots g_n, \quad h = h_1 h_2 \cdots h_m,$$

where consecutive terms alternate between G_1 and G_2. The product gh is obtained by concatenating the reduced forms of g and h, followed by a possible reduction. Applying ρ to gh, we have:

$$\rho(gh) = \rho(g_1 g_2 \cdots g_n h_1 h_2 \cdots h_m).$$

Since ρ maps all elements of G_2 to 1, it follows that:

$$\rho(gh) = g_1 g_2 \cdots g_n h_1 h_2 \cdots h_m,$$

which simplifies to:

$$\rho(gh) = \rho(g)\rho(h).$$

Finally, observe that $\rho|_{G_1} = \mathrm{id}_{G_1}$. Thus, ρ is a retract of G onto G_1. □

In [33], J. Boler and B. Evans showed that the free product of RF groups amalgamated over a retract of one of the factors is RF. The following result will be fundamental throughout this chapter:

Lemma 3.13 ([27], Lemma 3.9, p. 255)**.** *Let G be a RF group and $\varphi : G \longrightarrow G$ be a retraction of G. Then:*

(a) $\varphi(G)$ is closed in the profinite topology;

(b) Any closed subgroup of $\varphi(G)$ in the profinite topology of $\varphi(G)$ is also closed as a subgroup of G.

Proof. (a): Since $H = \varphi(G)$ is a retract of G, if $N = ker(\varphi)$, then $G = NH$ and $N \cap H = \{1\}$. Using the fact that G is RF, consider G_i a sequence of subgroups of finite index, normal in G, such that

$$\bigcap_{G_i \triangleleft_{f.i.} G} G_i = \{1_G\}.$$

Consider $N_i = N \cap G_i$. Then, applying Lagrange's theorem we can note that

$$|G : N_i H| = |NH : N_i H| = |N : N_i| \leq |G : G_i|.$$

Hence, $N_i H$ is a sequence of subgroups of finite index in G. On the other hand, since any element of G can be uniquely written as a product nh, where $n \in N,\ h \in H$, we have

$$\bigcap N_i H = H.$$

Thus, item (a) follows.

(b) Let K be the subgroup $\varphi(G)$ endowed with the profinite topology, and let L be a closed subgroup of K. Since $\varphi : G \longrightarrow G$ is continuous, we have that $\varphi^{-1}(L)$ is closed in K. Therefore, it will also be closed in G. Thus, since L is the intersection of the closed subgroups $\varphi^{-1}(L)$ and K, closed in G, we have that L must also be closed in G. $\square$

Definition 3.14. *We say that a subgroup $H \leq G$ of a group G is a **virtual retract** of G if there exists a subgroup $K \leq_{f.i.} G$ (of finite index in G) that contains the subgroup H, where H is a retraction of K. We denote this by $H \leq_{vr} G$.*

Example 3.15. *Every finite subgroup of a RF group is a virtual retract.*

First, we need to show the following statement, which is an equivalence of the definition of RF groups:

Lemma 3.16. *If G is RF and $H \leq G$ is a finite subgroup of G, then there exists a normal subgroup N, of finite index in G, such that $N \cap H = \{1_G\}$.*

Proof. If $H \leq G$ with $|H| < \infty$, then by Definition 2.2, we have that for each $h_i \in H \backslash \{1_G\}$, where $H = \{h_1, h_2, \ldots, h_k\}$, there exists a normal subgroup, of finite index $N_i \lhd_{f.i.} G$ such that $h_i \notin N_i$. Consider N as the intersection of these N_i's. Since each N_i is normal in G and has finite index, the finite intersection N will also be normal, with finite index in G. If $h \in H$ is non-trivial, then $h = h_i$ for some i. Since $h_i \notin N_i$, it follows that $h \notin N$. Thus, $N \cap H = \{1_G\}$. $\square$

Returning to Example 3.15, since G is RF, we have that $\exists N \lhd_{f.i.} G$ such that $N \cap H = \{1_G\}$. Therefore, $K = HN \leq_{f.i.} G$, hence we have that $K \cong N \rtimes H$.

The argument in Example 3.15 is used many times when we want to show that a subgroup $H \leq G$ is indeed a virtual retract of G. That is, suppose that $H \leq_{vr} G$ and $K \leq G$ is a subgroup of finite index containing H such that there exists a retraction ρ from K to H. Then, $N = ker(\rho) \lhd K$, $HN = K$, and $H \cap N = \{1\}$. In other words, K decomposes as the semidirect product $K = N \rtimes H$. Thus, to show that a subgroup $H \leq G$ is, in fact, a virtual retract of G, it suffices to demonstrate the existence of N with the described properties.

Studying virtual retractions is very useful in the study of profinite topology, as suggested by the following lemma.

Lemma 3.17 ([1], Lemma 2.2, p. 07)**.** *If G is RF and $H \leq_{vr} G$, then H is closed in the profinite topology.*

Proof. Since $H \leq_{vr} G$, there exists a finite index subgroup $K \leq_{i.f.} G$ such that H is a retract of K. Since G is RF, we have that K is also RF (Lemma 3.8). By Lemma 3.13, we know that H is closed in the profinite topology of K. Since $|G : K| < \infty$, we will show that any closed subset of K is also closed in G. Let C be a closed subset of K, that is, $K \setminus C$ is open. Since K has finite index in G, K is closed in the profinite topology of G, by Lemma 1.94 (c). Thus, we have:

$$G \setminus C = (G \setminus K) \cup (K \setminus C).$$

That is, $G \setminus C$ is a union of open sets, hence it is an open set, and consequently, C is a closed set in G. Therefore, H is closed in the profinite topology of G.

□

Definition 3.18. *If all finitely generated subgroups of a group G are virtual retracts of G, we say that G has the* ***(LR) property****.*

Notation 3.19. *From now on, when we say that G has the (LR) property, we will say that G satisfies (LR), or simply that G is (LR).*

Every finite group satisfies the (LR) property. This is due to the fact that every subgroup of a finite group G will be a virtual retract, since every subgroup of G has finite index in G. In this case, the virtual retraction will be the identity map of the subgroup onto itself. P. Scott [15] proved that all surface groups[3] satisfies property (LR). M. Hall showed that every free group is (LR) while demonstrating that free groups are LERF [29]. H. Wilton, in [30] showed that limit groups[4] satisfies (LR). Following the study of K. Almeida, I. Lima, and O. Ocampo on braid groups and residual properties, the following results illustrate some examples of braid groups that satisfy, or do not satisfy, the (LR) property. Again, the proofs of these results will be omitted as they are beyond the scope of this dissertation.

Theorem 3.20 ([47]). *Let S be a compact surface. If S is large, then $P_n(S)$ is not (LR) for all $n \geq 2$. If S is not large, then:*

(a) $P_2(S)$ is (LR) if S is the disk, sphere, projective plane, annulus, or torus. $P_2(S)$ is not (LR) if S is the Klein bottle;

(b) $P_3(S)$ is (LR) if S is the sphere or the disk, and is not (LR) if S is the annulus, torus, Möbius strip, or Klein bottle;

(c) $P_4(S)$ is (LR) if, and only if, S is the sphere;

(d) $P_n(S)$ is not (LR) for all $n \geq 5$.

Theorem 3.21 ([47]). *Let S be a compact surface. If S is large, then $B_n(S)$ is not (LR) for all $n \geq 2$. If S is not large, then:*

[3]A surface group is the fundamental group of a compact and orientable surface.

[4]Limit groups are finitely generated residually free groups.

(a) *If S is the Klein bottle, then $B_2(S)$ is not (LR);*

(b) *If S is the annulus, Möbius strip, Klein bottle, or torus, then $B_3(S)$ is not (LR);*

(c) *If S is not the sphere, then $B_4(S)$ is not (LR);*

(d) *$B_n(S)$ is not (LR) for all $n \geq 5$.*

The next proposition provides us with a useful equivalence for virtual retracts that will be utilized throughout this chapter.

Lemma 3.22 ([1], Remark 3.1, p.8)**.** *Let H be a subgroup of a group G. Then H is a virtual retract of G if and only if there exists a subgroup $N \leq G$, normalized by H, such that $|G : HN| < \infty$ and $H \cap N = \{1\}$.*

Proof. Let $H \leq_{vr} G$. By definition, there exists a finite index subgroup $K \leq_{f.i.} G$ such that $H \leq K$ and H is a retract of K. That is, there exists a retraction $\rho : K \longrightarrow H$ such that $\rho(h) = h$ for every $h \in H$.

Define $N = \ker(\rho)$. Note that $N \leq K$. By the Isomorphism Theorem, $K/N \cong H$. Thus, K is a semidirect product of H and N, i.e., $K = HN$ and $H \cap N = \{1\}$. It is clear that N is normalized by H, since $N \lhd G \geq H$. Moreover, we have

$$|G : HN| = |G : K| < \infty.$$

Conversely, suppose there exists a subgroup $N \leq G$, normalized by H, such that $|G : HN| < \infty$ and $H \cap N = \{1\}$. Define $K = HN$. Note that $K \leq_{f.i.} G$ and $H \leq K$. We want to show that H is a retract of K. Define the following map $\rho : K \longrightarrow H$ by $\rho(hn) = h$, where $h \in H$ and $n \in N$.

To verify that ρ is well-defined, consider $hn = h'n'$ for $h, h' \in H$ and $n, n' \in N$. This implies that

$$h^{-1}h' = nn'^{-1}.$$

Since $H \cap N = \{1\}$, we have $h^{-1}h' = 1$ and $nn'^{-1} = 1$, thus $h = h'$ and $n = n'$.

Next, we verify that ρ is a homomorphism:

If $h_1n_1, h_2n_2 \in K$, then

$$\rho((h_1n_1)(h_2n_2)) = \rho(h_1(n_1h_2)n_2) = \rho(h_1h_2(n_1^{h_2}n_2)),$$

where $n_1^{h_2} = h_2^{-1}n_1h_2 \in N$ since N is normalized by H. Since ρ only takes the part in H, we have

$$\rho(h_1n_1)\rho(h_2n_2) = h_1h_2 = \rho(h_1n_1 \cdot h_2n_2).$$

Finally, for $h \in H$, we have $\rho(h) = h$, so ρ restricts to the identity on H.

Thus, H is a retract of K, and K has finite index in G. Therefore, $H \leq_{vr} G$. $\square$

Lemma 3.23 ([1], Lemma 3.2, p.8)**.** *Suppose G_1 and G_2 are groups.*

(i) Let $H \leq_{vr} G_1$ and $A \leq G_1$ be any subgroup containing H. Then $H \leq_{vr} A$;

(ii) Suppose $H \leq G_1$ and $\varphi : G_1 \longrightarrow G_2$ is a homomorphism whose restriction to H is injective and $\varphi(H) \leq_{vr} G_2$. Then $H \leq_{vr} G_1$;

(iii) If $H \leq_{vr} G$ and $\alpha : G_1 \longrightarrow G_1$ is an automorphism, then $\alpha(H) \leq_{vr} G_1$. In particular, $gHg^{-1} \leq_{vr} G_1$ for any $g \in G_1$;

(iv) If $H \leq_{vr} G_1$ and $F \leq_{vr} H$, then $F \leq_{vr} G_1$. In particular, if $H \leq_{vr} G_1$ and $F \leq_{f.i.} H$, then $F \leq_{vr} G_1$;

(v) If $H_1 \leq_{vr} G_1$ and $H_2 \leq_{vr} G_2$, then $H_1 \times H_2 \leq_{vr} G_1 \times G_2$.

Proof. (i) To prove that $H \leq_{vr} A$, we need to show that there exists a subgroup of finite index in A for which H is a retract.

We know that $H \leq_{vr} G_1$. This means there exists a subgroup of finite index $K \leq_{f.i.} G_1$ such that H is a retract of K. That is, there exists a retraction $r : K \to H$ such that $r(h) = h$ for all $h \in H$.

Now, consider the subgroup $A \leq G_1$ that contains H. Our goal is to show that $H \leq_{vr} A$. We want to restrict the retraction from K to the subgroup $A \cap K$. For H to be a virtual retract of A, we need $A \cap K$ to have finite index in A.

Indeed, since K has finite index in G_1, there are finitely many cosets of K in G_1. In other words, we can write G_1 as a union of cosets: $G_1 = \bigcup_{i=1}^{n} g_i K$, where $n = [G_1 : K]$.

Now, we consider the intersections of these cosets with A. The cosets of $A \cap K$ in A are given by $a(A \cap K)$ for $a \in A$. Each $a(A \cap K)$ is contained in a coset of K, since $A \subseteq G_1$. Thus, the intersections $aK \cap A$ correspond to a finite number of cosets of $A \cap K$.

The retraction homomorphism $r : K \to H$ can be restricted to the subgroup $A \cap K$, which means that H is a retract of $A \cap K$. More explicitly, the restriction $r|_{A\cap K} : A \cap K \to H$ still satisfies $r(h) = h$ for all $h \in H$, ensuring that H is a retract of $A \cap K$.

Therefore, $H \leq_{vr} A$, as desired.

(ii) First, we want to show that without loss of generality, we can assume that φ is surjective. Indeed, let $\varphi(G_1) = \tilde{G_1}$. Note that $\tilde{G_1} \leq G_2$. Consider the restriction of φ to its image, which is a surjective homomorphism: $\varphi|_{\tilde{G_1}} : G_1 \longrightarrow \tilde{G_1}$. Now we have

$H \leq G_1$ and $\varphi : G_1 \longrightarrow \tilde{G_1}$, where $\tilde{G_1} \leq G_2$. By item (i), we know that if $H \leq_{vr} \tilde{G_1}$, then $H \leq_{vr} G_1$ since $G_1 \leq G_2$.

If $\varphi(H) \leq_{vr} G_2$, by Proposition 3.22, there must exist $N_2 \leq G_2$ normalized by $\varphi(H)$, such that the subgroup $\varphi(H)N_2 \leq_{f.i.} G_2$ and $\varphi(H) \cap N_2 = \{1\}$. Let $N_1 = \varphi^{-1}(N_2)$. Thus, N_1 will be normalized by H, since

$$\varphi(h^{-1})N_2\varphi(h) = N_2 \implies \varphi^{-1}(\varphi(h^{-1})N_2\varphi(h)) = \varphi^{-1}(N_1) \implies h^{-1}N_1h = N_1.$$

Moreover, since

$$G_2/\varphi(H)N_2 = \varphi(G_1/HN_1)$$

by the Correspondence Theorem (Isomorphism), we have that

$$|G_1 : HN_1| = |G_2 : \varphi(H)N_2| < \infty.$$

Finally, $H \cap N_1 = \{1\}$ in G_1, because $\varphi(H) \cap \varphi(N) = \{1\}$ and $H \cap \ker(\varphi) = \{1\}$, since φ is injective on H. Thus, $H \leq_{vr} G_1$ by Proposition 3.22.

(iii) Since $H \leq_{vr} G_1$, by Proposition 3.22, there exists a subgroup $N_1 \leq G_1$, normalized by H, such that $H \cap N_1 = \{1\}$ and $|G_1 : HN_1| < \infty$. Consider $N_2 = \alpha^{-1}(N_1)$. Since α is an automorphism, we have that

$$|G_1 : \alpha(H)N_2| = |\alpha(G_1) : \alpha(HN_1)| = |G_1 : HN_1| < \infty.$$

Now we want to verify that N_2 is normalized by $\alpha(H)$. Indeed, since N_1 is normalized by H, for any $h \in H$, we have

$$h^{-1}N_1h = N_1.$$

Note that

$$\alpha(h)^{-1}N_2\alpha(h) = \alpha(h^{-1})\alpha(N_1)\alpha(h) = \alpha(h^{-1}N_1h) = \alpha(N_1) = N_2.$$

Thus, N_2 is normalized by $\alpha(H)$. Finally, since

$$\alpha(H) \cap N_2 = \alpha(H) \cap \alpha(N_1) = \alpha(H \cap N_1) = \{1\},$$

we conclude that N_2 satisfies the hypotheses of Proposition 3.22, and hence $\alpha(H) \leq_{vr} G_1$.

In particular, for any $g \in G_1$, considering the automorphism $c_g : G_1 \longrightarrow G_1$ given by

$$c_g(h) = ghg^{-1},$$

we obtain:

$$c_g(H) = gHg^{-1}.$$

By applying the general result above, we have that $gHg^{-1} \leq_{vr} G_1$.

(iv) Suppose that $H \leq_{vr} G_1$ and $F \leq_{vr} H$. Then, there exists a subgroup of finite index $K \leq G_1$ and a homomorphism $\rho : K \longrightarrow H$ such that $H \subseteq K$ and $\rho(h) = h$ for $\rho|_H$. Since $F \subseteq H$, we can deduce that $\rho|_F$ is injective and $\rho(F) = F \leq_{vr} H$. Therefore, $F \leq_{vr} K$ by (ii), and this implies that $F \leq_{vr} G_1$, since $|G_1 : K| < \infty$.

(v) Assuming that $H_1 \leq_{vr} G_1$ and $H_2 \leq_{vr} G_2$, we can find subgroups of finite index $K_1 \leq G_1$ and $K_2 \leq G_2$ such that $H_1 \subseteq K_1$, $H_2 \subseteq K_2$, and there are retractions $\rho_1 : K_1 \longrightarrow H_1$ and $\rho_2 : K_2 \longrightarrow H_2$. We want to show that the map $\varphi = (\rho_1, \rho_2) : K_1 \times K_2 \longrightarrow H_1 \times H_2$, given by

$$\varphi(k_1, k_2) = (\rho_1(k_1), \rho_2(k_2)), \text{ where } k_1 \in K_1,\ k_2 \in K_2,$$

is a retraction from $K_1 \times K_2$ to $H_1 \times H_2$. Indeed, we have

$$|G_1 \times G_2 : K_1 \times K_2| = |G_1 : K_1| \cdot |G_2 : K_2| < \infty.$$

That is, $K_1 \times K_2 \leq_{f.i.} G_1 \times G_2$.

Additionally,

$$\varphi(h_1, h_2) = (\rho_1(h_1), \rho_2(h_2)) = (h_1, h_2),$$

for all $(h_1, h_2) \in H_1 \times H_2$, since ρ_1 and ρ_2 are retractions in their respective groups. Therefore, $H_1 \times H_2 \leq_{vr} G_1 \times G_2$.

□

Another important result that motivates the study of virtual retracts is the following.

Lemma 3.24. *If G is a finitely generated group and $H \leq_{vr} G$ is a virtual retract of G, then H is finitely generated.*

Proof. Since H is a virtual retract of G, there exists a finite index subgroup $K \leq_{f.i.} G$ such that $H \subseteq K$ and H is a retract of K. That is, there exists a retraction $\varphi : K \longrightarrow H$ such that $\varphi|_H = \mathrm{id}_H$. Since φ is surjective, we have $H \cong K/\ker\varphi$, and a quotient of a finitely generated group is finitely generated.

□

The next result shows that whenever we have a RF group G, any subgroup containing a retract of this group as a finite index subgroup will be a virtual retract of G.

Lemma 3.25 ([1], Lemma 3.4, p. 10)**.** *Let G be a RF group. Suppose that H is a retract of G and $A \leq G$ is a subgroup satisfying $H \subseteq A$ and $|A : H| < \infty$. Then $A \leq_{vr} G$. In particular, $B \leq_{vr} G$ for every finite subgroup B of G.*

Proof. Let $\rho : G \longrightarrow H$ be a retraction and $N = \ker \rho \lhd G$. Then $G = HN$ and $H \cap N = \{1\}$. Therefore, the intersection $A \cap N$ must be finite, since $|A : H| < \infty$.

To justify this last assertion, consider

$$A = \bigcup_{i=1}^{k} Hx_i$$

for some $x_i \in A$. It is known that $H \cap N = \{1\}$, and any coset $Hx_i \cap N$ consists of elements hx_i, where $h \in H$ and $x_i \in N$ because if x_i were in H, the intersection would not be trivial. Since $H \cap N = \{1\}$, each $Hx_i \cap N$ contains only x_i. Thus, as there are a finite number of cosets, we have $A \cap N < \infty$.

Since G is RF, there exists a normal subgroup of finite index $L \lhd_{f.i.} G$ such that $L \cap (A \cap N) = \{1\}$.

Since G is RF, for each non-trivial element $x \in A \cap N$, there exists a normal subgroup of finite index $L_x \lhd_{f.i.} G$ such that $x \notin L_x$. To cover all elements of $A \cap N$, we will consider the normal subgroup L, which is the intersection of all L_x for $x \in A \cap N$. Since $A \cap N$ is finite, we need a finite number of subgroups L_x, and the intersection of a finite number of subgroups of finite index also has finite index in G. Therefore, by construction, we have $L \cap (A \cap N) = \{1\}$.

Let $\tilde{N} = L \cap N$, note that $\tilde{N} \lhd G$ since $\tilde{N}$ is the intersection of normal subgroups and $|N : \tilde{N}| < \infty$ because:

L has finite index in G, so

$$G = \bigcup_{i=1}^{m} g_i L$$

for some $m \in \mathbb{Z}$. For each $n \in N$, we can consider a decomposition in terms of the cosets of L:

$$N = \bigcup_{i=1}^{j} (N \cap g_i L).$$

Therefore, it follows that $H\tilde{N}$ has finite index in $HN = G$, and by the previous assertion and the fact that H has finite index in A, we have $|G : A\tilde{N}| < \infty$.

Finally, $A \cap \tilde{N} = A \cap (L \cap N) = \{1\}$ by the choice of L, thus $A \leq_{vr} G$ by Proposition

3.22.

The second statement of the lemma follows from the first by considering the case where H is trivial. □

The next lemma will be important for us to demonstrate that every subgroup of a finitely generated virtually abelian group will be a virtual retract.

Lemma 3.26 ([1], Lemma 4.2, p. 12)**.** *Let L be a finitely generated virtually abelian group*[5] *and let $S \lhd L$ be a normal subgroup. Then there exists a normal, torsion-free subgroup $R \lhd L$ such that $|L : SR| < \infty$ and $S \cap R$ is trivial.*

The proof of the result above relies on Representation Theory and is therefore beyond the scope of this dissertation.

Lemma 3.27 ([1], Corollary 4.3, p. 13)**.** *If P is a finitely generated, virtually abelian group then every subgroup $Q \leq P$ is a virtual retract.*

Proof. Since P is virtually abelian, P contains a normal abelian subgroup $F \lhd_{f.i.} P$. We will show that $L = QF \leq_{f.i.} P$. Indeed,

$$|P : F| < \infty, \text{ and } F \subseteq QF = L \implies |P : L| < \infty.$$

Furthermore, we have that $S = Q \cap F$ is normal in L because

$$Q \cap F \lhd Q \text{ and } Q \cap F \lhd F \implies Q \cap F \lhd QF = L.$$

Note that $q \in Q \subseteq P$, hence

$$F \lhd P \implies pfp^{-1} \in F, \ \forall p \in P,$$

particularly, for $q \in P$, we have $qfq^{-1} \in F$. This means that F is normalized by Q. The group L is finitely generated since $|P : L| < \infty$ and P is finitely generated. Additionally, L is virtually abelian since it contains $F \leq_{f.i.} L$. By Lemma 3.26, there exists a normal, torsion-free subgroup $R \lhd L$ such that $R \cap S = 1$ and $|L : SR| < \infty$.

Since $Q \subseteq L$, we see that R is normalized by Q. Note also that:

$$|P : QR| \leq |P : L| \ |L : QR| \leq |P : L||L : SR| < \infty.$$

Finally, $|Q \cap R| < \infty$ since S has finite index in Q, therefore $Q \cap R$ is trivial since R is torsion-free. Thus, we have shown that $Q \leq_{vr} P$, as desired. □

[5]A group G is said to be virtually abelian if G contains an abelian subgroup of finite index.

As we will see in the following result, an immediate consequence of the result above and Lemma 3.17 is that subgroups inherit the (LR) property. Moreover, it also shows that any group satisfying the (LR) property is RF and LERF.

Lemma 3.28 ([1], Lemma 5.1, p.15). *Let G be a group.*

(i) Suppose that G satisfies (LR). Then G is RF, and every subgroup $A \leq G$ also satisfies (LR);

(ii) If G satisfies (LR), then G is LERF.

Proof. (i) If $A \leq G$ is a subgroup of G, then since G satisfies (LR), we can take a finitely generated subgroup $H \leq A$ such that H is a virtual retract of G. Thus, by Lemma 3.23 (i), we have that $H \leq_{vr} A$. Therefore, any finitely generated subgroup of A will be a virtual retract of A. That is, A satisfies (LR).

We will show that G is RF. Since G satisfies (LR), every finitely generated subgroup is a virtual retract of G. Let $g \in G \setminus \{1\}$. Then for some finite index subgroup $K \leq G$, containing g, there exists a retraction $\rho : K \longrightarrow \langle g \rangle$. If g has finite order in G, then $\ker(\rho)$ is a subgroup of finite index that does not contain g, since by the Isomorphism Theorem:

$$|K/\ker(\rho)| = |\rho(K)| < \infty \implies |K : \ker(\rho)| < \infty,$$

and by Lagrange's Theorem:

$$|G : \ker(\rho)| = |G : K| \cdot |K : \ker(\rho)| < \infty.$$

On the other hand, suppose that g has infinite order. In this case, consider an epimorphism $\varphi : \langle g \rangle \longrightarrow \mathbb{Z}_2$ given by

$$\varphi(g^n) = n \mod 2, \ \forall n \in \mathbb{Z},$$

where $\mathbb{Z}_2$ is the cyclic group of order 2 generated by $\varphi(g)$. In this case, we have that $g \notin \ker(\varphi \circ \rho)$. Thus, by the Isomorphism Theorem, we have

$$|K/\ker(\varphi \circ \rho)| = |(\varphi \circ \rho)(K)| < \infty \implies |K : \ker(\varphi \circ \rho)| < \infty.$$

By Lagrange's Theorem, we have

$$|G : \ker(\varphi \circ \rho)| = |G : K| \cdot |K : \ker(\varphi \circ \rho)| < \infty.$$

Thus, for each $g \in G \setminus \{1\}$, there exists a finite index subgroup in G that does not contain g. Therefore, G is RF.

(ii) We want to show that any group satisfying (LR) will also be LERF. Indeed, we showed in (i) that G is RF. Since every finitely generated subgroup of G is a virtual retract of G, by Lemma 3.17, we have that all these subgroups will be closed in the profinite topology of G. That is, G is LERF (Lemma 3.10). □

We saw earlier that every free group is RF; we will now show that more than that, every free group satisfies (LR). To demonstrate this important result, we will use M. Hall's Theorem [29] (cf. [[31], Corollary 1]).

Lemma 3.29 ([29], Theorem 5.1, p.429)**.** *Free groups satisfy (LR).*

Proof. Every finitely generated subgroup of a free group is a free factor of a subgroup of finite index by M. Hall's Theorem [29]. That is, any finitely generated subgroup H of a free group F will be a retract of a subgroup of finite index of F containing H. Therefore, free groups of arbitrary rank satisfy (LR). □

Since free groups satisfy the (LR) property, they also satisfy LERF. Furthermore, we know that finite groups satisfy both properties as well. Therefore, Example 2.20, involving the wreath product $S_3 \wr \mathbb{Z}$, is an example of a wreath product of two groups satisfying LERF/(LR) that is not RF.

Moreover, Example 3.9 also illustrates a direct product of two free groups that does not satisfy the (LR) property, and Example 2.3 presents an instance of an HNN extension of a free group that is not RF, and hence does not satisfy LERF/(LR).

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	not necessarily	not necessarily
$A \times B$	yes	not necessarily	not necessarily
$A \rtimes B$	not necessarily	not necessarily	not necessarily
$A * B$	yes	?	?
HNN	not necessarily	not necessarily	not necessarily
$A *_H B$	not necessarily	?	?

Table 3.2: **Table of Residual Properties**

It is important to note that a group G does not inherit the property (LR) from any subgroup $H \leq G$ that satisfies (LR), as presented by A. Minasyan through the following proposition:

Proposition 3.30 ([1], Proposition 1.7, p.4)**.** *The group $G = \mathbb{Z}_2^2 \wr \mathbb{Z}$ satisfies (LR) but is a subgroup of index 2 of a group $\tilde{G}$ that does not satisfy (LR).*

In addition to this proposition, the following example shows, in a clearer way, how we can obtain a group that does not satisfy (LR) and has a subgroup of index 2 with the property (LR).

Example 3.31 ([1], Example 3.6, p. 10). *Let X be any group and $G = X \wr \langle\alpha\rangle_2 \cong X \wr \mathbb{Z}_2$ the wreath product of X with a cyclic group of order 2 generated by α. In other words, $G = (X \times X) \rtimes \langle\alpha\rangle_2$, where α acts on $X \times X$ by alternating the two factors: $\alpha(x,y)\alpha^{-1} = (y,x)$ for all $(x,y) \in X \times X$. Note that the diagonal subgroup $H = \{(x,x) \in X \times X\}$ is centralized by α, and therefore $A = C_G(\alpha) = H\langle\alpha\rangle$ is a subgroup of G such that $|A : H| = 2$. Also note that $H \cong X$.*

Observe that H is a retract of $X \times X$ (the retraction could be defined by $(x,y) \mapsto (x,x)$), so $H \leq_{vr} G$. On the other hand, Proposition 3.32 below shows that A will be a virtual retract only if A is virtually abelian. Thus, if X is, for example, the free group of rank 2, then the subgroup H is a virtual retract of $G = X \wr \langle\alpha\rangle_2$, but H is a subgroup of index 2 of A, which will not be a virtual retract of G.

The next proposition uses the same notation as the previous example.

Proposition 3.32 ([1], Proposition 3.7, p.10). *If $A \leq_{vr} G$, then $X \cong H$ has a finite index abelian subgroup.*

The idea of the proof is the following: If $A \leq_{vr} G$, then there exists N, normalized by A, such that $|G : NA| < \infty$, $N \cap A = 1$ (Lemma 3.22). We define $K = NA$. Since $|G : (X \times X)| = 2$, $X \times X$ is normal in G, and without loss of generality we can replace N by $N \cap (X \times \{1\})$ ensuring $N \subseteq X \times X$. We define $X_1 = X \times \{1\} \leq G$ and $M = K \cap X_1$. Our goal is to show that M is going to be the abelian subgroup of finite index in X.

The key steps are as follows:

- If $(x,1) \in N$, then $(x,x) \in A \cap N$. Since $A \cap N = \{1\}$, this implies $X_1 \cap N = \{(1,1)\}$.
- The commutator $[(x,1),(a,b)]$ is computed, showing that $[x,a] = 1$. Thus, $(x,1) \in M$ commutes with certain elements of X_1, suggesting that M is abelian.
- Each $(a,1) \in M$ is shown to have the form $\alpha^\gamma(c,c)(d,e)$, with $\gamma \in \{0,1\}$, $(c,c) \in H$, and $(d,e) \in N$. This analysis concludes that M is central and thus abelian.

  Since $|X_1 : M| < \infty$ and M is abelian, $X_1 \cong X$ is virtually abelian. Hence, X (or H) has a finite-index abelian subgroup.

Thus, we see that the property (LR) is not invariant under commensurability[6]. An important result to be considered regarding the invariance under commensurability of virtually $\mathcal{X}$ groups, where $\mathcal{X}$ is a class of groups, is the following:

Proposition 3.33 (A. Magidin, [34]). *Let $\mathcal{X}$ be a class of groups that is closed under*

[6] G_1 and G_2 are said to be commensurable if there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ of finite index such that $H_1 \cong H_2$.

subgroups. If G is virtually $\mathcal{X}$[7] *and $H \leq G$, then H is virtually $\mathcal{X}$.*

Its proof follows from the following lemma:

Lemma 3.34 (A. Magidin, [34])**.** *Let G be a group, and let H and K be subgroups. Then*

$$|H : H \cap K| \leq |G : K|.$$

In particular, if $K \leq_{f.i.} G$, then $H \cap K \leq_{f.i.} H$.

Proof. We define a map:

$$\rho : \{h(H \cap K) \mid h \in H\} \longrightarrow \{gK \mid g \in G\}.$$

We will show that ρ is well-defined and injective. Indeed,

$$\begin{aligned} h(H \cap K) = \overline{h}(H \cap K) &\iff (\overline{h})^{-1}h \in H \cap K \\ &\iff (\overline{h})^{-1}h \in K \\ &\iff hK = \overline{h}K. \end{aligned}$$

The last equality follows by contradiction. Suppose $hK \neq \overline{h}K$. Then we would have:

$$(\overline{h})^{-1}hK \neq K \implies (\overline{h})^{-1}h \notin K.$$

This is a contradiction. Therefore,

$$|H : H \cap K| \leq |G : K|.$$

□

Let K be of finite index in G such that $K \in \mathcal{X}$. By Lemma 3.34, $H \cap K$ is of finite index in H. Since $\mathcal{X}$ is closed under subgroups, $H \cap K \in \mathcal{X}$. Hence, H is virtually-$\mathcal{X}$.

Now our objective will be to study the invariance, or lack thereof, of this property under various group products. The following lemma will provide us with a tool to be used when dealing with direct products and subdirect products. First lets recall the definition of subdirect products.

Definition 3.35. *A subgroup $H \leq F_1 \times F_2$ is called a* ***subdirect product of groups****, if for each $i \in \{1, 2\}$ we have that $\rho_i(G) = F_i$.*

Lemma 3.36 ([33], Lemma 2.1.)**.** *Let $G \leq F_1 \times F_2$ be any subdirect product. Then*

[7] A group G is said to be virtually $\mathcal{X}$ if there is a subgroup $H \leq_{f.i.} G$ such that H belongs to the class $\mathcal{X}$.

(i) *For any normal subgroup* $N \lhd G$ *and any* $i \in \{1,2\}$, *the intersection* $N \cap F_i$ *is normal in* $F_1 \times F_2$. *In particular,* $N_i := G \cap F_i$ *is normal in* F_i, *for each* $i = 1, 2$, *and in* G;

(ii) $F_1/N_1 \cong G/(N_1 \times N_2) \cong F_2/N_2$;

(iii) $|(F_1 \times F_2) : G| < \infty$ *if and only if* $|F_1/N_1| < \infty$. *In this case,* $|(F_1 \times F_2) : G| = |F_1/N_1|$.

Proof. (i) Let $(h, 1)$ be an element of $N \cap F_1$ for some $h \in F_1$. Since G is a subdirect product, for every element $f_1 \in F_1$, there exists $f_2 \in F_2$ such that $(f_1, f_2) \in G$. On the one hand, since $N \lhd G$, we have

$$(f_1, 1)(h, 1)(f_1, 1)^{-1} = (f_1, f_2)(h, 1)(f_1, f_2)^{-1} \in N.$$

On the other hand,

$$(f_1, 1)(h, 1)(f_1, 1)^{-1} = (f_1 h f_1^{-1}, 1) \in F_1,$$

for $h \in F_1$. Therefore, $N \cap F_1 \lhd F_1$. Similarly, we can show that $N \cap F_2 \lhd F_2$. Any normal subgroup of F_i is also normal in the direct product $F_1 \times F_2$, for $i = 1, 2$. To prove this, let $H \lhd F_i$ be a normal subgroup of F_i. Then, we need to show that H is normal in $F_1 \times F_2$.

Take any element $(f_1, f_2) \in F_1 \times F_2$ and any $h \in H$. Since $H \lhd F_i$, we have that $f_i h f_i^{-1} \in H$. Suppose that $H \lhd F_1$:

$$(f_1, f_2)(h, 1)(f_1, f_2)^{-1} = (f_1 h f_1^{-1}, 1),$$

where $f_1 h f_1^{-1} \in H$, since $H \lhd F_1$. Thus, $(f_1 h f_1^{-1}, 1) \in H \times \{1\} \subseteq H$. Similarly, if $H \lhd F_2$, then for any $(f_1, f_2) \in F_1 \times F_2$ and $h \in H$, we have:

$$(f_1, f_2)(1, h)(f_1, f_2)^{-1} = (1, f_2 h f_2^{-1}),$$

where $f_2 h f_2^{-1} \in H$. Therefore, $(1, f_2 h f_2^{-1}) \in \{1\} \times H \subseteq H$. Hence, $H \lhd F_1 \times F_2$ whenever $H \lhd F_i$, for $i = 1, 2$.

(ii) Let $\rho_i : F_1 \times F_2 \to F_i$ be the natural projection maps for $i = 1, 2$, where $\rho_1((f_1, f_2)) = f_1$ and $\rho_2((f_1, f_2)) = f_2$. The kernel of ρ_2 is:

$$\ker(\rho_2) = \{(f_1, 1) \mid f_1 \in F_1\}.$$

Clearly, this set corresponds to F_1. By Definition 3.35, $G \leq F_1 \times F_2$, the projection of G onto each factor is surjective. Hence $\rho_2(G) = F_2$. Moreover, we have that:

$$\ker(\rho_2|_G) = G \cap F_1 = N_1.$$

Thus, by the First Isomorphism Theorem:

$$\rho_2(G) \cong G/(G \cap \ker(\rho_2)) = G/N_1.$$

Since $\rho_2(G) = F_2$, we have that $F_2 \cong G/N_1$. By definition, $\rho_2(N_2) = N_2$ since $N_2 \subseteq F_2$. The First Isomorphism Theorem applied to $\rho_2|_G$ gives:

$$\frac{\rho_2(G)}{\rho_2(N_2)} \cong \frac{G}{N_1 \times N_2}.$$

Substituting $\rho_2(G) = F_2$ and $\rho_2(N_2) = N_2$, we get:

$$\frac{F_2}{N_2} \cong \frac{G}{N_1 \times N_2}.$$

Similarly, using the projection ρ_1, we find:

$$\frac{F_1}{N_1} \cong \frac{G}{N_1 \times N_2}.$$

(iii) First, suppose that $|(F_1 \times F_2) : G| < \infty$. Therefore, both $|F_1 : (F_1 \cap G)|$ and $|F_2 : (F_2 \cap G)|$ must be finite, as they are upper bounds for the indices of F_1 and F_2 in $F_1 \times F_2$. This implies that the quotient group F_1/N_1, where $N_1 = F_1 \cap G$, is finite, since $|F_1/N_1| = |F_1 : (F_1 \cap G)|$. Similarly, the same reasoning applies to F_2, and we conclude that $|F_2/N_2|$ is also finite.

Next, assume that F_1/N_1 is finite. By a similar argument, since F_1/N_1 is finite, the quotient F_2/N_2 must also be finite. To see this, observe that the index of F_2/N_2 is determined by the index of G in $F_1 \times F_2$, and since the direct product group structure ensures symmetry between F_1 and F_2, the finiteness of F_1/N_1 implies the finiteness of F_2/N_2.

Now, assume that $|F_1/N_1| < \infty$. The quotient group $(F_1 \times F_2)/(N_1 \times N_2)$ is isomorphic to $F_1/N_1 \times F_2/N_2$, as both N_1 and N_2 are normal in F_1 and F_2, respectively. Since both F_1/N_1 and F_2/N_2 are finite, it follows that the direct product $(F_1 \times F_2)/(N_1 \times N_2)$ is finite. Consequently, we have:

$$|(F_1 \times F_2) : G| = \left| \frac{F_1 \times F_2}{N_1 \times N_2} : \frac{G}{N_1 \times N_2} \right| = \frac{|F_1/N_1| \cdot |F_2/N_2|}{|F_1/N_1|} = |F_1/N_1|.$$

Since $|F_1/N_1| = |F_2/N_2|$ (by (ii)), it follows that $|(F_1 \times F_2) : G| = |F_1/N_1|$, as desired.

□

The next result establishes a condition under which the direct product of groups that satisfy (LR) is also (LR).

Proposition 3.37 ([1], Proposition 5.6, p.17). *Suppose X is finitely generated, virtually abelian, and Y is any group satisfying (LR). Then $X \times Y$ satisfies (LR).*

The proof of the Proposition follows from the next lemma.

Lemma 3.38 ([1], Lemma 5.7, p. 17). *Let X be finitely generated, virtually abelian, and Y be any group. Suppose that $H \leq X \times Y$ is a subgroup such that $\rho_Y(H) \leq_{vr} Y$ where $\rho_Y : X \times Y \longrightarrow Y$ is the natural projection. Then $H \leq_{vr} X \times Y$.*

Proof. Let $\rho_X : X \times Y \longrightarrow X$ denote the natural projection, let $L = \rho_X(H) \leq X$, and $M = \rho_Y(H) \leq Y$. Then $M \leq_{vr} Y$, by hypothesis, and $L \leq_{vr} X$ (by Lemma 3.27). Therefore $L \times M \leq_{vr} X \times Y$ (by Lemma 3.23 (v)). Note that $H \subseteq L \times M$, so (by Lemma 3.23 (iv)), to show that $H \leq_{vr} X \times Y$, it is sufficient to prove that $H \leq_{vr} L \times M$. Note that the subgroup $S = H \cap L$ is normalized by H. Let's understand this last statement. We want to show that

$$\forall h \in H,\ S^h = \{h^{-1}sh \mid s \in S = H \cap L\} = S.$$

We have that $S \subseteq S^h$ because given $s \in S = L \cap H$, we have that $s = h$ for some $h \in H$. In particular $s = s^h \in S^h$. It remains to prove that $S^h \subseteq S$. We have that

$$S^h = h^{-1}Sh \implies h^{-1}Sh \subseteq H \text{ since } S \subseteq H.$$

Since $L \lhd L \times M$, we have that $L^{(l,m)} \subseteq L,\ \forall (l, m) \in L \times M$. But $h \in L \times M$ and $S \subseteq L$, so $h^{-1}Sh \subseteq L$. Finally, we have that

$$h^{-1}Sh \subseteq H \cap L = S.$$

Thus, we have shown that S is indeed normalized by H.

Furthermore, since H projects onto L, we have $S = H \cap L \lhd L$ (by Lemma 3.36 (i)). We will show that the group L is finitely generated and virtually abelian, as it is a subgroup of X. In fact, since the property of being abelian is closed under subgroups, by Proposition 3.33, we have that L is virtually abelian. Let A be the abelian subgroup of finite index in X. As L is a virtual retract of X, which is a finitely generated group, consequently L will be finitely generated (Lemma 3.24).

Thus, since L is a subgroup of finite index in X, and X, which is finitely generated, we have that L is finitely generated (follows from the Reidmeister-Schreier theorem (cf.

[2], Theorem 6.1.8, p. 164)).

As we have seen that L is finitely generated and virtually abelian, we can apply Lemma 3.26 to obtain $R \triangleleft L$ such that $|L : SR| < \infty$ and $S \cap R$ is trivial. Then $R \triangleleft L \times M$, since let $(l, m) \in L \times M$ and $(r, e) \in R \times \{e\}$, we have

$$(l, m)(r, e)(l, m)^{-1} = (lrl^{-1}, mem^{-1}) = (lrl^{-1}, e) \in R \times \{e\}.$$

In particular, R is normalized by H. To understand this, take $H \ni h = (l, m) \in L \times M$ and $r \in R$. We have

$$h^{-1}rh = (l^{-1}, m^{-1})(r, e)(l, m) = (l^{-1}rl, e) \in R \times \{e\}.$$

Furthermore, $R \cap H = R \cap (L \cap H) = R \cap S$ is trivial, so to show that $H \leq_{vr} X \times Y$ (3.22), it remains to show that $|(L \times M) : HR| < \infty$. In fact, note that

$$|L : (L \cap HR)| \leq |L : SR| < \infty$$

since $HR \leq L$. Furthermore, $HR \leq L \times M$ still projects to L and M. Thus, $|(L \times M) : HR| = |L : (L \cap HR)| < \infty$ (by Lemma 3.36, (iii)). Therefore $H \leq_{vr} L \times M$, hence $H \leq_{vr} X \times Y$ as desired. □

Now, to prove Proposition 3.37.

Proof. Let X be a finitely generated virtually abelian group and Y a group satisfying (LR). We want to show that $X \times Y$ also satisfies (LR). Let $H \leq X \times Y$, and let $\rho_Y : X \times Y \longrightarrow Y$ be the natural projection. Note that $\rho_Y(H) \leq_{vr} Y$ is a virtual retract of Y (Lemma 3.28). Thus, we can use Lemma 3.38 to conclude that $H \leq_{vr} X \times Y$. Therefore, we have that $X \times Y$ satisfies (LR). □

In [1], A. Minasyan emphasizes that even though the property (LR) does not pass to subgroups of finite index for the group containing it, it is preserved by some quotients and extensions of finite kernel, as shown by the following result.

Lemma 3.39 ([1], Lemma 5.8, p.18)**.** *Let* $\{1\} \longrightarrow L \hookrightarrow G \overset{\psi}{\twoheadrightarrow} M \longrightarrow \{1\}$ *be a short exact sequence of groups.*

(i) If G satisfies (LR) and L is finitely generated, then M satisfies (LR);

(ii) If M satisfies (LR), L is finite, and G is RF, then G satisfies (LR).

Proof. (i) Consider any finitely generated subgroup $A \leq M$. Since L is finitely generated, $H = \psi^{-1}(A) \leq G$ will also be finitely generated, as any element of H can be expressed as a combination of elements from the finitely generated subgroups $L = \ker(\psi)$ and $\psi^{-1}(\langle A \rangle)$, where $\langle A \rangle$ is the finite set of generators of A.

Thus, we have $H \leq_{vr} G$. Therefore, there exists a subgroup $N \leq G$, such that N is normalized by H, $H \cap N = \{1\}$, and $|G : HN| < \infty$ (by 3.22). Let $B = \psi(N) \leq M$ and note that B is normalized by $A = \psi(H)$; it suffices to apply ψ to N. Moreover, we have $|M : AB| \leq |G : HN| < \infty$ (Correspondence Theorem). Note that

$$\psi^{-1}(A \cap B) = \psi^{-1}(A) \cap \psi^{-1}(B) = H \cap NL = L.$$

The first equality follows from the invariance of the intersection under the homomorphism. The second equality is due to the fact that L is contained in $\psi^{-1}(B)$. The third equality is more laborious. First, note that $L \subseteq H \cap NL$ is trivial. We will show that $H \cap NL \subseteq L$. Indeed, let $g \in H \cap NL$. Then $g = nl$ for some $n \in N,\ l \in L$. Thus,

$$\psi(g) = \psi(nl) = \psi(n)\psi(l) = \psi(n) \cdot 1 = \psi(n).$$

Since $g \in H = \psi^{-1}(A)$, we have $\psi(g) \in A$. But $\psi(n) \in \psi(N) = B$. Moreover, we have $A \cap B = \{1\}$ since $H \cap N = \{1\}$. Therefore,

$$\psi(g) = 1 \implies g \in \ker(\psi) = L.$$

Thus, it has been shown that $A \cap B$ is trivial in $M = G/L$, so $A \leq_{vr} M$. That is, M satisfies (LR).

(ii) Let $H \leq G$ be a finitely generated subgroup. Then $A = \psi(H) \leq_{vr} M$ (since $\psi(H)$ is finitely generated and M satisfies (LR)), hence there exists a subgroup $B \leq M$, normalized by A, with $A \cap B = \{1\}$ and $|M : AB| < \infty$. Note that $N = \psi^{-1}(B) \leq G$ will be normalized by H, because

$$\forall a \in A,\ a^{-1}Ba = B \implies \psi^{-1}(B) = \psi^{-1}(a^{-1}Ba) \implies N = \psi^{-1}(a^{-1})N\psi^{-1}(a) = H^{-1}NH.$$

Additionally, $N \cap H \subseteq \psi^{-1}(B) \cap \psi^{-1}(A) = \psi^{-1}(A \cap B) = \psi^{-1}(1) = \ker(\psi) = L < \infty$. Furthermore, $|G : HN| = |M : AB| < \infty$ because surjective homomorphisms preserve indices.

Since G is RF, there exists a normal subgroup $K \lhd_{f.i.} G$ such that $K \cap (H \cap N) = \{1\}$. By defining $\overline{N} = N \cap K$, we have that $\overline{N}$ is normalized by H, $H \cap \overline{N} = \{1\}$, and $|HN : H\overline{N}| \leq |N : \overline{N}| < \infty$. Thus,

$$|G : H\overline{N}| = |G : HN| \cdot |HN : H\overline{N}| < \infty.$$

Therefore, $H \leq_{vr} G$ and G satisfies (LR). □

Since the amalgamated free product of RF groups is not necessarily RF, it was to be expected that the property (LR) would not be preserved by free products. The following example shows that this property is indeed not preserved.

Example 3.40. *We want to construct a Baumslag-Solitar group $BS(n,m)$ that is not RF but is formed from a free amalgamated product of RF groups. A common choice is the group $BS(2,3)$. Here is why:*

- ***Structure**: We can present $BS(2,3)$ as the amalgamated product $\mathbb{Z} *_{2\mathbb{Z}=3\mathbb{Z}} \mathbb{Z}$. This is a free product with amalgamation over the subgroups $2\mathbb{Z}$ in one copy of $\mathbb{Z}$ and $3\mathbb{Z}$ in the other, which merge according to the relation $a^{-1}b^2a = b^3$.*

- ***Residually Finite Factors**: Each factor $\mathbb{Z}$ is RF, as they are free groups.*

However, the group $BS(2,3)$ is not RF (see 2.3). Consequently, it is neither LERF nor (LR).

The previous example shows that the free product with amalgamation of two groups satisfying the LERF/(LR) properties does not always satisfy the same properties.

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	not necessarily	not necessarily
$A \times B$	yes	not necessarily	not necessarily
$A \rtimes B$	not necessarily	not necessarily	not necessarily
$A * B$	yes	?	?
HNN	not necessarily	not necessarily	not necessarily
$A *_H B$	not necessarily	not necessarily	not necessarily

Table 3.3: **Table of Residual Properties**

As we have seen through Gruenberg's Theorem 2.19, the wreath product of two RF groups is not always RF. Example 2.20 demonstrates that the wreath product of two groups with the (LR)/LERF property can result in a product that does not preserve the (LR)/LERF property, respectively. Now, we will present an important result, originally due to Davis and Olshanskii in [36] that provides an example where the (LR) property is preserved in the wreath product.

Lemma 3.41 ([1], Lemma 9.5, p.30)**.** *Suppose that $G = \mathbb{Z}_p^k \wr \mathbb{Z}$, where $\mathbb{Z}_p$ is the cyclic group of order p and $k \in \mathbb{N}$. Then G satisfies (LR). Furthermore, if $H \leq G$ is a finitely generated subgroup that is not contained in the normal subgroup $W = (\mathbb{Z}_p^k)^{\mathbb{Z}} \lhd G$, then there exists $N \leq W$, such that N is normalized by H, $N \cap H$ is trivial, and $|G : HN| < \infty$.*

An interesting example of an wreath product of two groups that satisfy (LR) but whose wreath product does not satisfy (LR) is given by A. Minasyan in [1]:

Theorem 3.42 ([1], Lemma 9.6, p. 31)**.** *Let $G = A \wr B$, where B is an infinite cyclic group and $A = \mathbb{Z}_{p^m}$ is the cyclic group of order p^m, such that p is prime and $m \geq 2$. Then G does not satisfy (LR).*

Another criterion under which the (LR) property is preserved by wreath products is given by A. Minasyan in [1]. Before stating it, we will define semisimple groups.

Definition 3.43. *A group is said to be* ***semisimple*** *if it is a direct sum of cyclic groups of prime order. For example, $\mathbb{Z}_2^2 \oplus \mathbb{Z}_3$ is semisimple.*

Theorem 3.44 ([1], Theorem 9.7, p. 31)**.** *Suppose that $G = A \wr \mathbb{Z}$, where A is a finitely generated abelian group. Then G satisfies (LR) if and only if A is semisimple.*

Now, with the goal of studying the preservation of the LERF and (LR) properties in free products, we will analyze two fundamental results. The first, due to Romanovskii in [37], addresses the preservation of the LERF property. The second, by Gitik, Margolis, and Steinberg in [7], deals with the preservation of the (LR) property.

3.1 (LR) and LERF for Free Products

At this point, our focus is on studying the preservation of the (LR) and LERF properties in free products. The main result of this section is attributed to Burns and Romanovskii, who state that the free product of LERF groups continues to be a LERF group. Another highly relevant result is the work of Gitik, Margolis, and Steinberg, who demonstrated in [7] that the free product of (LR) groups also retains the (LR) property, showing, in the same article, the result of Burns and Romanovskii regarding LERF groups. Furthermore, another proof of the important Burns-Romanovskii Theorem is presented by Naomi Andrew, who, in [38], uses Bass-Serre Theory to prove the theorem (see Appendix B).

We will begin a series of definitions and auxiliary results that will support the proof that the (LR) and LERF properties are preserved in free products, as presented in [7].

Definition 3.45. *A subgroup $H \leq G$ is said to be* ***malnormal*** *if*

$$g \notin H \implies gHg^{-1} \cap H = 1.$$

We write $H \leq_{mal} G$.

As an example, free factors are malnormal.

Definition 3.46. *We say that a subgroup $H \leq G$ is* ***virtually malnormal*** *in G if there exists $K \leq_{f.i.} G$ such that $H \leq_{mal} K$.*

If all finitely generated subgroups of G are virtually malnormal, we say that G is ***LVM (locally virtually malnormal)****. Free groups are LVM.*

Let P be a property of subgroups, such as: malnormality, being a free factor, being a retract. We will write $H \leq_P G$ if a subgroup H of G has the property P.

Definition 3.47. *A group G is* ***LVP (locally virtually P)*** *if for each $H \leq_{f.g.} G$, there exists $K \leq_{f.i.} G$ such that $H \leq_P K$.*

For example, if P is malnormality, then LVP will be equivalent to LVM.

This class of groups LVP is of great interest to us because if we consider the property P as being a retract, then being LVP will be equivalent to being (LR). In [7], it was shown that if P satisfies the following properties:

- A1. $H \leq_P G$ and $H \leq K \leq G$ imply $H \leq_P K$;
- A2. $H_1 \leq_P G_1$ and $H_2 \leq_P G_2$ imply $H_1 * H_2 \leq_P G_1 * G_2$;
- A3. $1 \leq_P G$, $G \leq_P G$.

Then we could show that the free product of LVP groups, LERF will be a LVP, LERF group.

Note that the property of being a retract satisfies the listed properties:

(A1): To prove A1, it is enough to restrict the homomorphism $\phi : G \to H$ to the subgroup K, obtaining a homomorphism $K \to H$ whose restriction to H remains the identity;

(A2): For A2, since $H_i \leq_r G_i$, there exists a retraction $r_i : G_i \to H_i$ for each $i = 1, 2$, such that $r_i(g) = g$ for all $g \in H_i$.

We will construct a map $r : G_1 * G_2 \to H_1 * H_2$ as follows: for each element $g \in G_1 * G_2$, write g in normal form as a sequence $g = g_1 g_2 \dots g_k$, where the g_i's alternate between G_1 and G_2. We define

$$r(g) = r_{i_1}(g_1) r_{i_2}(g_2) \dots r_{i_k}(g_k).$$

This map r is a retraction, because each r_{i_j} is a retraction from G_{i_j} to H_{i_j}. Therefore, $r(g) = g$ for all $g \in H_1 * H_2$, which implies that $H_1 * H_2$ is a retract of $G_1 * G_2$;

(A3): A3 follows directly.

Therefore, the free product of (LR) groups will be (LR).

The proof of this statement follows from the results we will see next. Note that in the following results, the malnormal property present in [7] has been replaced by the retract property in order to demonstrate that the free product of (LR) groups will be (LR).

Lemma 3.48 ([7], Lemma 2.2, p. 90). *If $H \leq_P K$ and $K \leq_P G$, then $H \leq_P G$. Moreover, if $H \leq_P G$ and $H \leq K \leq G$, then $H \leq_P K$. In particular, if $H \leq_P K$ and $G = K * L$, then $H \leq_P G$.*

Proof. Let $H \leq K$ be a retract of K and $K \leq G$ be a retract of G.

Then, there exist homomorphisms $\phi_K : K \to H$ and $\phi_G : G \to K$ such that $\phi_K(h) = h$ for all $h \in H$ and $\phi_G(k) = k$ for all $k \in K$. Consider the homomorphism $\psi : G \to H$ defined by $\psi = \phi_K \circ \phi_G$. For $h \in H$, we have:

$$\psi(h) = \phi_K(\phi_G(h)) = \phi_K(h) = h.$$

Thus, ψ restricted to H is the identity on H. Therefore, H is a retract of G.

The second statement follows from (A1).

□

Now we will state and prove a version of Lemma 4.3 [7] replacing the malnormal property, originally present in the article, with the retract property.

Lemma 3.49 ([7], Lemma 4.3, p. 95). *Suppose that $H_i \leq_r G_i$ for $i = 1, \dots, n$. Then*

$$H_1 * \cdots * H_n \leq_r G_1 * \cdots * G_n.$$

Proof. This follows directly by induction, using item (A2) right after Definition 3.47. □

The next theorem, together with Lemma 3.48 and Lemma 3.49, is fundamental in the proof of 3.51.

Theorem 3.50 ([7], Theorem 4.1, p.93). *Let G_v, $v \in V$, be a collection of LERF groups, and let G be the free product of the G_v's. Suppose that $H \leq_{f.g.} G$ and $A \subset G$ is finite with $H \cap A = \emptyset$. Then there exist:*

- *$v_1, \dots, v_r \in V$ (not necessarily distinct);*
- *subgroups $H_i \leq_{f.g.} K_i \leq_{f.i.} G_{v_i}$, for $i = 1, \dots, r$;*
- *elements $g_1, \dots, g_r \in G$;*
- *subgroups $F_1, F_2, K_0 \subset G$;*

such that:

$$H = H_1^{g_1} * \cdots * H_r^{g_r} * F_1 \quad \textit{and} \quad K = K_1^{g_1} * \cdots * K_r^{g_r} * F_1 * F_2 * K_0 \leq_{f.i.} G,$$

where F_1 and F_2 are free, K_0 is a finite free product of conjugates of some of the G_v's, and $K \cap A = \emptyset$.

Moreover, if G_{v_i} is LVM, $i \in \{1, \ldots, r\}$, we can take $H_i \leq_{mal} K_i$; if G_{v_i} is finite, we can take $H_i = K_i$.

The proof of this theorem, in [7], uses the concepts of 2-complexes, CW complexes, and fundamental groups of topological spaces. For this reason, its proof will not be addressed in this work.

Note that, again, the previous result holds for LERF groups, and by replacing the LERF property with the (LR) property, it would be possible to adapt the proof, as the property of being a retract satisfies conditions A1, A2, and A3.

Finally, we can state the most important theorem of this section.

Theorem 3.51 ([7], Theorem 1.3, p.88)**.** *The free product of LVP, LERF groups will be an LVP, LERF group.*

Proof. First, we will show that the free product of LERF groups is LERF.

By Theorem 3.50, given a free product $G = *_{v \in V} G_v$ of LERF groups, any finitely generated subgroup $H \leq_{\text{f.g.}} G$ can be described as

$$H = H_1^{g_1} * \cdots * H_r^{g_r} * F,$$

where $H_i \leq_{\text{f.g.}} K_i \leq_{\text{f.i.}} G_{v_i}$ for some elements $g_1, \ldots, g_r \in G$ and subgroups K_i of finite index in G_{v_i}. Here, F is a free subgroup.

Since each G_{v_i} is LERF, any finitely generated subgroup H_i of G_{v_i} is separable. That is, there exists a finite index subgroup $K_i \leq_{\text{f.i.}} G_{v_i}$ such that $H_i \subset K_i$. Thus, each conjugate $H_i^{g_i}$ is contained in a conjugate of K_i in G, and we can construct a finite index subgroup $L \leq_{f.i.} G$ containing H as a free product of the conjugates $K_i^{g_i}$ and the free subgroup F. This subgroup L contains H and excludes any element $g \in G \backslash H$. Therefore, we have shown that H is separable in G, which implies that G is LERF.

Now, we prove that the free product of LVP groups (where P is the retract property) remains LVP. We will show that G is (LR), i.e., that every finitely generated subgroup of G is a virtual retract.

Consider a finitely generated subgroup $H \leq_{\text{f.g.}} G$. By Kurosh's Theorem, H can be decomposed as a free product:

$$H = (*_{j \in J} H_j) * F,$$

where each H_j is isomorphic to an intersection $H \cap G_i^{k_i}$ of H with a conjugate of some G_i, and F is a free subgroup. Note that Kurosh's Theorem does not imply that each H_j is finitely generated, even if H is.

For the finitely generated H_j, there exists a finite index subgroup $K_j \leq_{f.i.} G_{v_i}$ such that $H_j \leq_r K_j$ because each G_{v_i} is (LR). That is, each finitely generated H_j is a virtual retract in G_{v_i}.

By Lemma 3.49, if $H_j \leq_r K_j$ for each finitely generated H_j, then the free product of these H_j's is a retract in $K = *K_j$, which is a finite index subgroup of G. Therefore, the part of H generated by the finitely generated H_j's is a virtual retract in G. Thus, every finitely generated subgroup of G is a virtual retract, which implies that G is (LR). □

3.2 Complete Table of Residual Properties

After compiling several results on RF groups, (LR) groups, and LERF groups, as well as their interactions under various group operations, we have prepared the following table:

A and B	RF	LERF	(LR)
$A \wr B$	not necessarily	not necessarily	not necessarily
$A \times B$	yes	not necessarily	not necessarily
$A \rtimes B$	not necessarily	not necessarily	not necessarily
$A * B$	yes	yes	yes
HNN	not necessarily	not necessarily	not necessarily
$A *_H B$	not necessarily	not necessarily	not necessarily

Table 3.4: **Table of Residual Properties**

Concluding Remarks

The study of residual properties, such as RF, LERF, and (LR), shows that these properties are important in Group Theory (see 2.5). In this chapter, we will list some important problems involving RF, LERF, and (LR) groups, so that interested readers can continue exploring the topic, as the author of this work did.

3.2.1 Questions

(1) In [45] and [46], the authors provided a complete classification of LERF Artin groups.

Question: Is it possible to obtain a similar classification for groups that are (LR)?

(2) In [47], surface braid groups (both virtual and singular, see [47]) and their LERF and (LR) properties were studied.

Question: What other generalizations of braid groups are LERF and/or (LR)?

(3) Let G be linear and finitely generated:

Question:

a) Is $G \otimes G$ linear?

b) Is $G \otimes G$ RF, LERF, (LR)?

Here, $\otimes$ denotes the non-abelian tensor product. Item (a) is Problem 19.9 from "The Kourovka Notebook" [48].

(4) In [1], A. Minasyan raised the following question ([1], Question 11.1., p.38)

Question: Do virtually free groups satisfy (LR)?

(5) **Question: Complete the table 3.4 for other free constructions and other residual properties.**

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Appendix A

BASS-SERRE THEORY

This appendix is based on fundamental concepts presented in the book *Trees* by J-P. Serre [44], which is a classical reference in the theory of groups acting on graphs. Some proofs in this chapter have been omitted since they would go beyond the scope of this work.

A.1 Graphs of Groups

Graphs of groups are a combinatorial tool used to describe group actions on trees. Bass-Serre theory allows certain groups to be represented as fundamental groups of graphs of groups. This is particularly useful for studying groups like amalgamated products and HNN extensions, where the group is constructed from smaller pieces, facilitating the analysis of their properties.

A **graph** is a structure formed by a set of vertices $V(\Gamma)$, edges $E(\Gamma)$, and mappings:

Maps ι and τ:

- ι maps each edge e to a vertex $\iota(e)$, representing the initial vertex of the edge.
- τ maps e to the terminal vertex $\tau(e) = \iota(\overline{e})$, where $\overline{e}$ is the inverse edge, i.e., the one going from the terminal vertex to the initial vertex.

Involution $\overline{e}$: It is an operation that associates each edge e with its inverse $\overline{e}$, satisfying $\overline{\overline{e}} = e$.

Definition A.1. *An **orientation** of a graph is the choice of a direction for each edge, i.e., choosing an edge e in each pair $\{e, \overline{e}\}$.*

Definition A.2. *A **graph of groups** $\mathcal{G}$ extends the idea of a simple graph by associating groups to each vertex and edge:*

- ***Vertices and Vertex Groups** G_v**:** Each vertex v of the graph is associated with* *a group G_v, which can represent the stabilizer of the vertex under a group action.*

- ***Edges and Edge Groups** G_e**:** Each edge e is associated with a group G_e such that G_e is equal to the group of the inverse edge $G_{\overline{e}}$. There exists a monomorphism (injection) α_e from the group G_e to the group of the terminal vertex $G_{\tau(e)}$.*

Definition A.3. *A **path** in a graph of groups $\mathcal{G}$ is a sequence formed by elements of the vertex and edge groups, following the structure of the graph. Specifically, a path of length n is a sequence of the form:*

$$g_0 e_1 g_1 e_2 \dots e_n g_n$$

where:

- *g_i is an element of the group G_{v_i} associated with the vertex v_i.*
- *e_i is an edge connecting vertices v_{i-1} and v_i in the graph.*

Notation A.4. *The group $F(\mathcal{G})$ is the group generated by the vertex and edge groups, with relations of the form $e\alpha_e(g)e^{-1} = \alpha_{\overline{e}}(g)$ for g belonging to the edge group G_e.*

Definition A.5. *A **loop** is a closed path where the initial and terminal vertices coincide, i.e., $v_0 = v_n$.*

Definition A.6. *A **reduced path** is a path that has been simplified by removing any unnecessary redundancy. Specifically:*

- *A path **is not reduced** if it contains a sub-sequence of the form $e\alpha_e(g)e^{-1}$, where:*
  - *e is an edge.*
  - *$\alpha_e(g)$ is the mapping of an element g from the edge group G_e to an element of the group associated with the vertex $\tau(e)$ (where the edge e ends).*
  - *e^{-1} is the inverse edge $\overline{e}$.*

This sub-sequence $e\alpha_e(g)e^{-1}$ represents a movement along the edge e, followed by a return along the inverse edge e^{-1}, resulting in a return to the starting point. If this movement is trivial (does not change anything), it can be removed, simplifying the path.

Definition A.7. *A graph that does not contain non-trivial reduced loops is called **acyclic**, and a connected acyclic graph is called a **tree**.*

In this chapter, some arguments using the category of groupoids will be presented. The main difference between groups and groupoids is that, in groupoids, multiplication is partial, being defined under geometric conditions: two arrows compose if and only if the terminal point of one coincides with the initial point of the other. The arrows a and b in the diagram

$$x \xrightarrow{\text{a}} y \xrightarrow{\text{b}} z$$

compose, resulting in the arrow $ba : x \mapsto z$.

The category of groupoids, in a certain sense, is more general and encompasses the category of groups, where every group is a more trivial case of a groupoid. Strictly speaking, we can define groupoids as follows:

Definition A.8. *A **groupoid** consists of two sets G and M, called respectively the **groupoid** and the **base**, along with two mappings α and β from G to M, called the source projection and the target projection, respectively, a mapping $1 : x \mapsto 1_x, M \to G$ called the object inclusion map, and a partial multiplication $(h, g) \mapsto hg$ in G defined on the set $G * G = \{(h, g) \in G \times G \mid \alpha(h) = \beta(g)\}$, subject to the following conditions:*

(i) $\alpha(hg) = \alpha(g)$ *and* $\beta(hg) = \beta(h)$ *for all* $(h, g) \in G * G$*;*

(ii) $j(hg) = (jh)g$ *for all* $j, h, g \in G$ *such that* $\alpha(j) = \beta(h)$ *and* $\alpha(h) = \beta(g)$*;*

(iii) $\alpha(1_x) = \beta(1_x) = x$ *for all* $x \in M$*;*

(iv) $g1_{\alpha(g)} = g$ *and* $1_{\beta(g)}g = g$ *for all* $g \in G$*;*

(v) Each $g \in G$ *has a "two-sided inverse"* g^{-1} *such that* $\alpha(g^{-1}) = \beta(g)$*,* $\beta(g^{-1}) = \alpha(g)$*, and* $g^{-1}g = 1_{\alpha(g)}$*,* $gg^{-1} = 1_{\beta(g)}$*.*

An element of M may be called an **object** of the groupoid G, and an element of G may be called an **arrow**. The arrow 1_x corresponding to an object $x \in M$ may also be called the **unit** or **identity** corresponding to x.

For the reader interested in delving deeper into the study of groupoids, see [39].

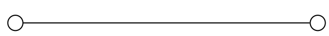
A.1.1 Fundamental Group of a Graph of Groups

Definition A.9. *The **fundamental group** $\pi_1(G, v)$ at a vertex v is the set of all loops in $F(G)$ that start at v.*

It is important to note that the isomorphism class of the fundamental group is independent of the choice of the base vertex.

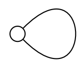
For the computation of the fundamental group at a base vertex, two particular cases related to well-known constructions in combinatorial group theory will be mentioned below.

(i) Let $\mathcal{G}$ be a graph of groups, where Γ is the associated graph of $\mathcal{G}$, with one edge and two vertices:



Let G_u, G_v be the vertex groups and G_e the edge group. Then, $\pi_1(\mathcal{G}, \Gamma) = G_u *_{G_e} G_v$ is a free amalgamated product.

(ii) Now suppose that $\mathcal{G}$ is a graph of groups, where Γ is the associated graph of $\mathcal{G}$ and has only one vertex, that is, Γ is a **petal**:



Let G_v be the vertex group. Then, $\pi_1(\mathcal{G}, \Gamma) = \text{HNN}(H, G_e, t_e, e \in E)$ is an HNN extension with $\{G_e, e \in E\}$ the set of associated subgroups, and $\{t_e \mid e \in E\}$ the set of stable letters.

In fact, the fundamental group $\pi_1(\mathcal{G}, \Gamma)$ is obtained successively by free amalgamated products followed by HNN extensions.

Definition A.10. *The **Bass-Serre tree** T is a structure built from a graph of groups G that serves as a universal cover.*

Vertices in the Tree T: The vertices of the tree T are cosets $G_w p$, where p is a path in the graph of groups connecting the vertices w and v.

Edges in the Tree T: There is an edge connecting two vertices $G_{w_1}p_1$ and $G_{w_2}p_2$ if the paths p_1 and p_2 are related by an edge e of the original graph, following the relation $p_1 = eg_{w_2}p_2$ or $p_2 = eg_{w_1}p_1$, with $g_w \in G_w$.

The fundamental group will act on the Bass-Serre tree, this action will be without inversions and will preserve adjacency.

Preservation of Adjacency: The action must preserve adjacency between the vertices of the tree. This means that if a vertex u is connected to a vertex v by an edge, then the action of the fundamental group must preserve this connection.

Without Inversions: The phrase "without inversions" means that the action does not alter the orientation of the edges of the tree. In other words, if you have an edge going from a vertex u to a vertex v, the group's action should not change this to an edge going from v to u. The edges of the tree maintain their direction and are not "inverted" by the group action.

Definition A.11. *We say that a group G acts on a set X, or that X is a G-set, if there exists a map $G \times X \longrightarrow X$, $(g, x) \mapsto gx$ such that $1x = x$ and $g(g'x) = (gg')(x)$ for all $g, g' \in G$, $x \in X$, and for each $g \in G$, the map $X \longrightarrow X$, $x \mapsto gx$ is a bijection. In other words, there exists a homomorphism from G to the permutation group $Sym(X)$. For each $x \in X$, the **stabilizer** of x is defined as the subgroup $G_x = \{g \in G \mid gx = x\}$. The **orbit** of x is the G-subset $Gx = \{gx \mid g \in G\}$.*

Another way to obtain a graph of groups from a group action on a tree is as follows:

Definition A.12 ([38] Definition 2.7, p.3)**.** *(**Quotient graph of groups**) Suppose that a group G acts on a tree T. Form a graph of groups whose associated graph is the quotient graph of the action, with vertex and edge groups assigned as follows: choose subtrees $T^v \subseteq T^e$ such that T^v contains exactly one representative from each vertex orbit (that is, a lift of a maximal tree in the quotient), and T^e contains exactly one representative from each edge orbit, so that at least one endpoint of each edge lies in T^v. We slightly abuse notation by identifying vertices in T^v and edges in T^e with their orbits (that is, their image in the quotient graph). Define the vertex and edge groups as the stabilizers G_v and G_e. To define the monomorphisms, we choose elements $g_v \in G$ that act to bring each vertex in T^e to T^v: if $v \in T^v$, then define $g_v = 1$, and otherwise, choose any element with this property. Now, we can define the monomorphisms α_e as the composition of inclusion with conjugation by the chosen elements (so $s \mapsto g_{\tau(e)}^{-1} s g_{\tau(e)}$).*

Theorem A.13 ([38], Theorem 2.8, p.4)**.** *Up to isomorphism of the structures involved, the process of constructing the quotient graph of groups and the construction of the fundamental group and Bass-Serre tree are mutually inverse.*

The isomorphisms required in the group actions on trees include:

1. An isomorphism between the original group and the fundamental group.

2. An equivariant isometry[1] between the original tree and the Bass-Serre tree.

From the point of view of graphs of groups, the required isomorphisms are:

1. An isomorphism between the associated graphs.

2. Corresponding isomorphisms between the edge and vertex groups.

These isomorphisms must respect the monomorphisms of the edges. This statement establishes a fundamental link between group actions on trees and group decompositions in terms of graphs of groups. In simple terms, when a group acts on a tree, the resulting structure (the Bass-Serre tree) can be reconstructed from the quotient graph of groups and vice versa. This shows the equivalence of these two constructions, which is essential for understanding the interaction between group actions dynamics and Bass-Serre theory.

A.1.2 Covering Theory

Definition A.14. *Let Γ and Δ be connected graphs. A **covering** of Δ is a local isomorphism $\theta : \Gamma \longrightarrow \Delta$. If Γ is a tree, then such a covering is called a **universal covering**.*

The action of a group G on a Bass-Serre tree is preserved by subgroups, that is, if

[1]A mapping between graphs is said to be **equivariant** if it respects the structure of the graphs relative to the group action.

a group G acts on a tree T, then a subgroup $H \leq G$ will also act on this tree considering the action of G, restricted to H. This action will produce a quotient graph of groups that contains H. The quotient graph will be a covering of the original graph. The definitions to be presented next will focus on the case of free products, that is, in the context of groups acting on trees, the case where all the G_e's are trivial.

Definition A.15 ([38], Definition 2.9, p.3). *Suppose that $\mathcal{H}$ and $\mathcal{G}$ are group graphs with all edge groups trivial. A* ***morphism*** *of group graphs $\Phi : \mathcal{H} \longrightarrow \mathcal{G}$ consists of:*

- *A graph morphism $\varphi : \Gamma_{\mathcal{H}} \longrightarrow \Gamma_{\mathcal{G}}$;*
- *A group homomorphism $\phi : H_v \longrightarrow G_{\varphi(v)}$ for each vertex $v \in \Gamma_{\mathcal{H}}$;*
- *An element λ_v in $\pi_1(\mathcal{G}, \varphi(v))$ for each vertex $v \in \Gamma_{\mathcal{H}}$;*
- *An element $\delta_e \in G_{\varphi(\iota(e))}$ for each edge of $\Gamma_{\mathcal{H}}$.*

Such a morphism induces applications on the structures that are defined from group graphs as shown below:

- A group homomorphism $F(\mathcal{H}) \longrightarrow F(\mathcal{G})$ by $s \mapsto \lambda_v^{-1}\phi_v(s)\lambda_v$ for $s \in H_v$ and $e \mapsto \lambda_{\iota(e)}^{-1}\delta_e^{-1}e\delta_{\overline{e}}\lambda_{\tau(e)}$
- A homomorphism Φ_P of fundamental groupoids, by restricting this map to paths in $F(\mathcal{H})$. Note that, in this case, for each edge e, the extra elements introduced in e, $\iota(e)$ and $\tau(e)$, will cancel out, leaving δ_e and $\delta_{\overline{e}}$, which are elements of the vertex groups at each end.
- A homomorphism $\Phi_v : \pi_1(\mathcal{H}, v) \to \pi_1(\mathcal{G}, \phi(v))$ of fundamental groups, by further restricting the map above to loops at v.
- A map of equivariant graphs $\tilde{\Phi}$ on Bass-Serre trees, defined at the vertices by $H_w p \mapsto G_{\varphi(w)}, \lambda_w \Phi_P(\mathcal{P})$.

Additionally, we can define a local map $\Phi_{v/f}$ for each edge of $\mathcal{G}$. These maps will be useful for understanding locally the image of the Bass-Serre tree under a morphism. To obtain a covering, we will need to ensure that these local maps are injective. In A.18, we will obtain a tool to guarantee this injectivity.

Given a morphism $\Phi : \mathcal{H} \to \mathcal{G}$, let v be a vertex of $\Gamma_{\mathcal{H}}$ and f an edge of $\Gamma_{\mathcal{G}}$ with $\tau(f) = \varphi(v)$.

Define a map

$$\Phi_{v/f} : \coprod_{e \in \varphi^{-1}(f), \tau(e) = v} H_v \to G_{\varphi(v)}$$

by

$$h \mapsto \delta_{\bar{e}}\phi_v(h).$$

Alternatively, we can view $\Phi_{v/f}$ as a map $H_v \times \{e \in E(\mathcal{H}) : \iota(e) = v, \varphi(e) = f\} \to G_{\varphi(v)}$, sending $(H_v, e) \mapsto \delta_e\phi_v(H_v)$.

Given two group actions on trees, and an equivariant map between the trees, we can induce a morphism of group graphs between the quotient graphs of groups. We continue assuming that the actions are free on the edges.

Proposition A.16 ([38], Proposition 2.10, p.5). *Suppose that S is an H-tree, T a G-tree, $\psi : H \longrightarrow G$ is a homomorphism, and $f : S \longrightarrow T$ is a graph map that is ψ-invariant (i.e., f maps vertices to vertices, edges to edges, preserves adjacency, and $vhf = (vf)(h\psi)$). Let $\mathcal{H}$ and $\mathcal{G}$ be the quotient graphs of groups corresponding to the actions of H on S and G on T, respectively. Then ψ and f induce a morphism of group graphs $\mathcal{H} \longrightarrow \mathcal{G}$, which (after the isomorphisms required by Theorem A.13) recovers ψ and f as maps of fundamental groups and Bass-Serre trees.*

In summary, since f is ψ-invariant, it induces a map between the quotient graphs $S/H \longrightarrow T/G$. In cases where ψ is an inclusion and f does not make identifications in the trees, the resulting morphism has advantageous properties, such as being a covering or an immersion.

In the context of graphs without groups, a covering means that locally the map is bijective, while an immersion only requires it to be locally injective. In the case of quotient graphs of groups, the Bass-Serre tree acts as a kind of universal covering, though this covering is not literal, as there may be multiple preimages of an edge at each vertex.

Definition A.17. *A morphism $\Phi : \mathcal{H} \to \mathcal{G}$ is an **immersion** if:*

1. *each $\phi_v : H_v \to G_{\phi(v)}$ is injective, and*
2. *each $\Phi_{v/e}$ is injective.*

*It is a **covering** if the second condition is replaced by:*

2' each $\Phi_{v/e}$ is bijective.

In the above definition, we can think of the morphism $\Phi_{v/e}$ as a mapping from $Star(v)^2$. Bass, in [40], shows that the covering and immersion properties indeed characterize the action of a subgroup on a subtree.

Proposition A.18 ([40], Proposition 2.7). *A morphism Φ is an immersion if, and only if, Φ_{v_0} (in fundamental groups), and $\tilde{\Phi}$ (in Bass-Serre trees) are injective. Additionally, it is a covering if, and only if, Φ_{v_0} (in fundamental groups) is injective and $\tilde{\Phi}$ (in Bass-Serre*

[2] $Star(v)$ is the set of edges incident to v, along with v itself.

trees) is bijective. We will have that $\Phi_{v/f}$ *is injective if, and only if,* ϕ_v *is injective, and* δ_e *represents distinct cosets* $G_{\varphi(v)}/\phi_v(H_v)$.

We can construct immersions by considering subgraphs of subgroups. We will use this tool in the proofs of the theorems addressed in Appendix B.

The following theorem will be fundamental in several occasions, as it characterizes the structure of a subgroup of a free product.

Theorem A.19 (**Kurosh**, [41]). *Suppose* G *is a free product* $*G_{i \in I}$ *and* H *is a subgroup of* G. *Then* $H \cong (*H_j) * F$, *where each* H_j *is isomorphic to an intersection* $H \cap G_i^{k_i}$ *of* H *with a conjugate of some* G_i. *Furthermore, the set* $\{H_j\}$ *is unique up to conjugation and reindexing, and the rank of* F *is uniquely determined.*

The following lemma will be used to calculate the index of a subgroup from the covering graph of groups.

Lemma A.20 ([38], Lemma 3.4, p. 7). *Suppose that a group* G *acts transitively on a set* X, *and* H *is a subgroup of* G. *Let* X_0 *be the set of representatives of the orbits of the action of* H *on* X. *Then*

$$|G : H| = \sum_{x \in X_0} |G_x : H_x|.$$

In particular, $|G : H|$ *is finite if, and only if,* X_0 *is finite, as well as each index* $|G_x : H_x|$.

For a free product, the index of a subgroup can be determined by counting the occurrences of any edge in the same G-orbit. This procedure provides a criterion to identify subgroups of finite index: a subgroup will have finite index if, and only if, the covering graph of groups has finite associated graphs, and each vertex group (or edge group) has finite index in the corresponding vertex group of the original graph of groups.

Appendix B

FREE PRODUCT OF LERF GROUPS

The objective of this appendix is to present the idea of the proof of the Burns and Romanovskii Theorem via Bass-Serre Theory. We will follow the proof and concepts of N. Andrew seen in [38] and introduced in Appendix A on Bass-Serre Theory of this dissertation.

The proof is divided into three steps: first, we complete an immersion of a graph of groups; then, we enlarge the vertex groups to ensure that they have finite index; and, finally, we apply this technique to finitely generated subgroups of free products. The final construction results in a covering graph that contains a subgroup of finite index and excludes elements outside it.

The next theorem provides a way to complete an immersion into a covering when the vertex groups have finite index image in the graph of group.

Theorem B.1 ([38], Theorem 4.1, p.9)**.** *Suppose that G is a free product, expressed as the fundamental group of a graph of groups $\mathcal{G}$, where every edge group is trivial. Suppose that H is a subgroup of G, corresponding to the immersion $\Phi : \mathcal{H} \longrightarrow \mathcal{G}$, where $\Gamma_{\mathcal{H}}$ is finite and each H_v is mapped to a subgroup of finite index $G_{\varphi(v)}$. Then there exists a finite index subgroup M of G containing H as a free factor.*

Proof. **Step 1: Obtain the Degree of the Desired Covering:**

Lemma A.20 provides a formula for calculating the index of a subgroup H in G when G acts transitively on a set X. In the proof, we want to calculate the index of H in G. To do this, we calculate the indices of the stabilizers $|G_u : H_v|$ for each vertex u in $\mathcal{G}$ and their preimages v in $\mathcal{H}$.

For each vertex u of $\mathcal{G}$, we define:

$$d_u = \sum_{v \in \Phi^{-1}(u)} |G_u : H_v|$$

Finiteness of d_u: Since Γ_H is finite and each $|G_u : H_v|$ is finite, d_u is also finite.

Step 2: Equalizing the Values of d_u

We want all preimages of vertices to have the same degree to facilitate the construction of a uniform covering. To do this, we determine the maximum degree d among all d_u:

$$d = \max\{d_u\}$$

Now, for each vertex u in $\mathcal{G}$:

If $d_u < d$, we add $d - d_u$ isolated vertices to $\mathcal{H}$. These isolated vertices are defined in the preimages of u and associated with the subgroup G_u (i.e., $H_v = G_u$).

After adding the isolated vertices, we will have a new value $\tilde{d}$, corresponding to u, note that:

$$\tilde{d_u} = d_u + (d - d_u) \cdot 1 = d$$

Because an isolated vertex adds a factor $|G_u : G_u| = 1$, incrementing d_u until it reaches d. Now, for every vertex u in G, $\tilde{d_u} = d$. As we wanted, since in a covering, the index of a subgroup can be calculated by looking at the preimage of any vertex. That is, we wanted to ensure that all these vertices have the same degree.

Step 3: Extension of the Immersion Φ to New Vertices

After adding isolated vertices, the immersion Φ needs to be extended to include these new vertices. Each new vertex v' added to $\mathcal{H}$ is mapped to the corresponding vertex u in $\mathcal{G}$. The function ϕ_v for these new vertices is defined as the identity function, that is,

$$\phi_v(H_{v'}) = G_u.$$

The immersion Φ now maps all the vertices of H, including the new isolated vertices. However, we still need to handle the edges to ensure that the covering is consistent.

Step 4: Adding Edges to Make the Immersion Bijective

Now, we will ensure that for each edge f in $\mathcal{G}$, there are exactly d preimages of f in $\mathcal{H}$.

Identification of the Preimages of Edges: For each edge f in G, we consider its preimages in the various vertex copies in the covering H.

Partial Bijection Between Cosets: We define a partial bijection between the cosets associated with the preimages of edges. The bijection is given by:

$$\coprod_{v \in \Phi^{-1}(u)} G_{\phi(v)}/\phi_v(H_v) \to \coprod_{x \in \Phi^{-1}(y)} G_{\phi(x)}/\phi_x(H_x),$$

where u and y are the initial and terminal vertices of edge f, respectively.

This bijection is defined by:

$$\phi_{\iota(e)}(H_{\iota(e)})\delta_e \mapsto \phi_{\tau(e)}(H_{\tau(e)})\delta_{\overline{e}}.$$

- $\iota(e)$ and $\tau(e)$ are the initial and terminal vertices of edge e, respectively.
- δ_e and $\delta_{\overline{e}}$ are representatives of the cosets (elements chosen to identify the cosets).

The disjoint unions on both sides of the partial bijection have cardinality d, ensuring that the bijection can be completed. We add new edges to $\mathcal{H}$ corresponding to each edge f in G. For each edge f in G, we add d edges to $\mathcal{H}$, each mapped so that the bijection is satisfied.

We select δ_e and $\delta_{\overline{e}}$ according to the defined bijection, ensuring that the correspondences are consistent.

The immersion Φ now maps all the edges of $\mathcal{H}$ bijectively onto the edges of $\mathcal{G}$. This means that the covering is fully defined, with each edge of G having exactly d preimages in H.

Step 5: Construction of the graph of group $\mathcal{M}$

$\mathcal{M}$: This is the graph of group resulting from the application of the above process to each edge of G.

Extended Immersion: The immersion Φ has been extended to the entire graph of group $\mathcal{M}$, incorporating the new edges and vertices added.

- **Addition of Finite Edges:** The process added only a finite number of edges to H, ensuring that $\mathcal{M}$ remains finite.

- **Connectivity:** Each connected component of $\mathcal{M}$ contains at least one preimage of each vertex of G. Since at least one vertex of G had no preimages added, the underlying graph of $\mathcal{M}$ is connected.

At the end of the process, we can take a base point $b \in \mathcal{M}$ in the preimage of a base point in $\mathcal{G}$ and calculate the fundamental group of $\mathcal{M}$ at this base point. We define:

$$M := \pi_1(\mathcal{M}, b)$$

By construction, we ensure that H has index d, by Lemma A.20. Additionally, H will be a connected subgraph of M, so $H = \pi_1(\mathcal{H})$ will be a partition of the free product of $M = \pi_1(\mathcal{M})$. Thus, H is a free factor of M.

□

In situations where a cover of G has infinite degree (with infinitely many edges in each preimage), controlling the construction of the cover becomes more difficult. To address this challenge, we can modify the vertex groups so that they are mapped to subgroups of finite index. It is important, however, to perform these modifications carefully to ensure that the immersion remains locally injective.

Theorem B.2 ([38], Theorem 4.2, p.10)**.** *Let G be a free product, expressed as the fundamental group of a graph of groups $\mathcal{G}$, where all edge groups are trivial. Let H be a subgroup of G, corresponding to an immersion $\Phi : \mathcal{H} \to \mathcal{G}$ with Γ_H finite. If each $\phi_v(H_v)$ is separable in $G_{\varphi(v)}$, then there exists a finite index subgroup K of G, corresponding to a cover $\mathcal{K}$ that contains $\mathcal{H}$ as a subgraph of subgroups.*

Proof. Our first objective is to alter $\mathcal{H}$ and Φ so that each vertex group is mapped to a finite index subgroup of G_v, while keeping Φ an immersion. To do this, we need to ensure that the elements δ_e continue to represent different cosets of $G_{\varphi(v)}/\phi_v(H_v)$.

For each vertex v of $\mathcal{H}$ and edge f with $\iota(f) = \varphi(v)$, let $X_{v/f}$ be the finite set of elements $\delta_{e_i}^{-1}\delta_{e_j}$, where e_i and e_j are distinct edges with $\iota(e_i) = \iota(e_j) = v$ and $\varphi(e_i) = \varphi(e_j) = f$. Let X_v be the disjoint union of the $X_{v/f}$'s varying over the edges f in $\varphi(v)$.

Since each $\phi_v(H_v)$ is separable in $G_{\varphi(v)}$, there exists a finite index subgroup of $G_{\phi(v)}$ that contains $\phi_v(H_v)$, but does not contain any element of X_v:

First, we want to show that $\delta_{e_i}^{-1}\delta_{e_j} \notin \phi_v(H_v)$. Suppose, for contradiction, that $\delta_{e_i}^{-1}\delta_{e_j} \in \phi_v(H_v)$. Then, we would have $\delta_{e_j} \in \delta_{e_i}\psi_v(H_v)$, and the elements δ_e would represent the same coset, a contradiction via Proposition A.18.

Next, since $\phi_v(H_v)$ is separable in $G_{\varphi(v)}$, for each element in X_v, there will be a finite index subgroup K_v containing $\phi_v(H_v)$ and not containing this element. Thus, we can vary the elements of X_v to cover all of X_v. Finally, we take the intersection of all K_v's to obtain a finite index subgroup K_v, which contains $\phi_v(H_v)$ but does not contain any elements of X_v.

We now extend ϕ_v to K_v so that its image is this subgroup.

This is still an immersion because the vertex mappings remain injective, and we will show that for each edge f in $\varphi(v)$, the elements δ_e, with e in the preimage of f, represent different cosets of $\phi_v(K_v)$. Equivalently,

$$\delta_{e_i}\phi_v(K_v) \neq \delta_{e_j}\phi_v(K_v) \iff \delta_{e_i}^{-1}\delta_{e_j}\phi_v(K_v) \neq \phi_v(K_v) \iff \delta_{e_i}^{-1}\delta_{e_j} \notin \phi_v(K_v), \forall e_i, e_j.$$

But this is exactly what we ensured by requiring that X_v is outside $\phi_v(K_v)$, so this condition is satisfied, and ϕ remains an immersion.

We can now apply Theorem B.1: we have an immersion where the vertex groups correspond to finite index subgroups. This immersion can be completed to a cover corresponding to a finite index subgroup. Since the procedure does not identify any edges or vertices, we can recover $\mathcal{H}$ (and the original immersion) by restricting to a subgraph and subgroups of the vertex groups. □

Now we have the sufficient conditions to present an idea of N. Andrew's proof of the Burns and Romanovskii Theorem.

Theorem B.3. *Let G be a finite free product of LERF groups. Then G is LERF.*

To prove that the free product of LERF groups is LERF, consider a group G which is the free product of groups G_i, where each G_i is LERF. The goal is to show that any finitely generated subgroup $H \subset G$ and any element $g \in G \setminus H$ can be separated by a finite index subgroup of G that contains H but does not contain g.

To achieve this, we begin by fixing a finitely generated subgroup $H \subset G$ and an element $g \in G \setminus H$. Using Proposition A.16, we represent G as the fundamental group of a graph of groups $\mathcal{G}$ with trivial edge groups. The representation of G as the fundamental group of a graph of groups allows us to associate a Bass-Serre tree T to the graph of groups $\mathcal{G}$, on which G acts.

Proposition A.16 guarantees that the subgroup H acts on a finite subtree $T_H \subset T$, thus inducing a graph of groups structure on H, which we denote by $\mathcal{H}$. Therefore, we obtain an immersion of graph of groups $\Phi : \mathcal{H} \longrightarrow \mathcal{G}$ that preserves the structure of the vertex groups and edges of T_H. In other words, $\mathcal{H}$ is realized as a subgraph of $\mathcal{G}$ via the immersion Φ.

We then apply Theorem B.2 to expand each vertex group H_v of $\mathcal{H}$, so that it has finite index in the corresponding vertex group G_v of $\mathcal{G}$. This expansion is possible because we assumed that each G_i is LERF, which implies that the vertex groups of G are separable by subgroups. The expansion is done in such a way as to maintain the immersion Φ without identifying unwanted cosets, ensuring that the immersion structure remains injective and that each H_v now has finite index in the respective G_v.

With these adjustments, we are ready to apply Theorem B.1, which allows us to complete the immersion $\Phi : \mathcal{H} \longrightarrow \mathcal{G}$ into a cover. This theorem states that when the vertex groups of $\mathcal{H}$ have finite index in their counterparts in $\mathcal{G}$, the immersion Φ can be completed into a finite index cover. As a result, we obtain a cover that forms a finite index subgroup $K \subset G$ that contains H and excludes the element g, separating it from H.

Thus, we have shown that for any finitely generated subgroup $H \subset G$ and any element $g \in G \setminus H$, there exists a finite index subgroup $K \subset G$ such that $H \subset K$ and $g \notin K$. This proves that the group G, being the free product of LERF groups, is also LERF.