Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Approximation Methods for the Dirichlet Problem Involving the Schrödinger Operator and Measure Data

Lucas Menezes de Brito

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Lucas Menezes de Brito

sob orientação do

Prof. Dr. Carlos Alberto Pereira dos Santos

e sob coorientação do

Prof. Dr. Augusto César Ponce

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPG-Mat–UnB, como parte dos requisitos necessários para obtenção do título de Doutor em Matemática.

ATA Nº 16

Aos dezessete dias do mês de dezembro do ano de dois mil e vinte e quatro, instalou-se a banca examinadora de Tese De Doutorado do aluno Lucas Menezes de Brito, matrícula 180145622. A banca examinadora foi composta pelos professores Dr. LUIS HENRIQUE DE MIRANDA (interno - MAT/UnB), Dr. ADILSON EDUARDO PRESOTO (Externo à Instituição - UFSCar), Dr. KAYE OLIVEIRA DA SILVA (Externo à Instituição - UFG), Dr. WILLIAN CINTRA DA SILVA (suplente - MAT/UnB) e Dr. CARLOS ALBERTO PEREIRA DOS SANTOS (orientador/presidente - MAT/UnB). O discente apresentou o trabalho intitulado Approximation Methods for the Dirichlet Problem Involving the Schrödinger Operator and Measure Data. Concluída a exposição, procedeu-se a arguição do(a) candidato(a), e após as considerações dos examinadores o resultado da avaliação do trabalho foi ( ) Pela aprovação do trabalho; ( ) Pela aprovação do trabalho, com revisão de forma, indicando o prazo de até 30 dias para apresentação definitiva do trabalho revisado; ( ) Pela reformulação do trabalho, indicando o prazo de (Nº DE MESES) dias para nova versão; ( ) Pela reprovação do trabalho, conforme as normas vigentes na Universidade de Brasília. Conforme os Artigos 34, 39 e 40 da Resolução 0080/2021 - CEPE, o(a) candidato(a) não terá o título se não cumprir as exigências acima.

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Ata Nº: 16

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(X) Pela aprovação do trabalho;

( ) Pela aprovação do trabalho, com revisão de forma, indicando o prazo de até 30 (trinta) dias para apresentação definitiva do trabalho revisado;

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Resumo

Nesta tese apresentamos uma visão geral e novos resultados relacionados a um problema com operador de Schrödinger elíptico e dados em medida de Borel.

Introduzimos dois métodos de aproximação para esse problema de Schrödinger; no primeiro, apresentamos uma técnica de aproximação no potencial de Schrödinger, que leva à medida reduzida e, consequentemente, a uma subsolução maximal do problema; enquanto, no segundo método, introduzimos uma técnica de aproximação no dado da medida de Borel que possibilita a introdução do conceito de limite reduzido.

Em seguida, provamos propriedades de monotonicidade e semicontinuidade inferior do limite reduzido, em função dos conjuntos de torsão zero e zero universal. Como consequência, mostramos a existência de uma solução (limite reduzido) e a ocorrência do fenômeno de Lavrentiev para um problema de controle optimal. As principais ferramentas usadas são de Teoria Geométrica da Medida e Teoria do Potencial.

Título em português: *Métodos de Aproximação para o Problema de Dirichlet Envolvendo o Operador de Schrödinger e Dado em Medida.*

Palavras-chave: Equações Diferenciais Parciais Elípticas, dados em medida de Borel, conjunto de torsão zero, conjunto zero universal, desigualdade de Kato, capacidades de Sobolev, medida reduzida, limite reduzido, teoria geométrica da medida, teoria do potencial, problema de controle optimal, fenômeno de Lavrentiev.

Abstract

In this thesis we present an overview and new results related to a problem involving an elliptic Schrödinger operator and Borel measure data.

We introduce two approximation methods for this Schrödinger problem; in the first one we present an approximation technique on the Schrödinger potential, that leads to the reduced measure and, consequently, to a maximal subsolution to the problem; while in the second method, we introduce an approximation technique in the Borel measure data, that allows the introduction to the concept of reduced limit.

Next, we prove monotonicity and lower semicontinuity properties of the reduced limit, depending on the torsion and universal zero-sets. As a consequence, we show the existence of a solution (the reduced limit) and the occurrence of the Lavrentiev phenomenon to an optimal control problem. The main tools used are from Geometric Measure Theory and Potential Theory

Keywords: Elliptic Partial Differential Equations, Borel measure data, torsion zero-set, universal zero-set, Kato's inequality, Sobolev capacities, reduced measure, reduced limit, geometric measure theory, potential theory, optimal control problem, Lavrentiev phenomenon.

Notation

- μ, λ denote finite Borel measures;
- $\mathcal{M}(X)$ denote the normed vector space of all finite Borel measures over X;
- $\mu\lfloor A$ is the restriction of the measure μ to the set A;
- Ω denotes an open subset of $\mathbb{R}^N$;
- $\mu_k \overset{*}{\rightharpoonup} \mu$ denotes the weak* convergence;
- $C_c^\infty(\Omega) := \{f : \Omega \to \mathbb{R}; f \in C^\infty(\Omega) \text{ and } \mathrm{supp}(f) \text{ is compact}\}$;
- $C_0^\infty(\Omega) := \{f : \overline{\Omega} \to \mathbb{R}; f|_{\partial\Omega} \equiv 0 \text{ and } f|_\Omega \in C_c^\infty(\Omega)\}$;
- $W^{n,p}(\Omega)$ denotes the Sobolev space of order n, p over Ω;
- $W_0^{n,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{n,p}(\Omega)}}$;
- $\mathrm{cap}_{W^{n,p}}$ denotes the capacity related to the Sobolev space $W^{n,p}$;
- $\widehat{f}$ is the precise representative of the function f;
- $[V; \mu]$ denotes the Schrödinger problem with potential V and density μ:

$$\begin{cases} -\Delta u + Vu = \mu \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega; \end{cases}$$

- ζ_f is the variational solution of the Schrödinger problem with potential V and data f;
- $S \subset \Omega$ denotes the torsion zero-set;
- $Z \subset \Omega$ denotes the universal zero-set;
- $\mu^* \in \mathcal{M}(\Omega)$ denotes the reduced measure related to μ;
- $\mu^\# \in \mathcal{M}(\Omega)$ denotes the reduced limit related to μ.

Contents

1 Introduction

In physics, partial differential equations are ubiquitous, from the study of heat and waves to the study quantum mechanics. We are generally presented with a problem as the following:

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Δ is the Laplace operator operator and it is assumed some regularity on the real function f. Because of analytical reasons, it is extremely hard to derive explicit solutions to these equations, so mathematicians are inclined to weaken the meaning of solution. One of the most useful ways is taking the theory of distributions into account.

Distributions are linear functional on the space of smooth functions with compact support. These functions are very useful because a big range of functions can be treated as such, using the following transformation:

$$\langle f, \phi \rangle := \int_\Omega f\phi.$$

This way of thinking about differential equations and generalized functions was introduced by the mathematicians Sergueï Lvovitch Sobolev (1908-1989) and Laurent Schwartz (1915-2002), when working with Partial Differential Equations.

One of the most important aspects of the space of distributions is that it is big enough to contain a large range of functions, but small enough so that we can have some regularity inside it. For example, we can talk about derivatives of distributions using the duality properties of the space, defining the derivative T' of the distribution T using the integration by parts:

$$\langle T', \phi \rangle := -\langle T, \phi' \rangle, \forall \phi \in C_c^\infty(\Omega).$$

Some phenomenon in physics can not be accurately described by real functions, being set functions the most appropriate object to work with. For these phenomenon we are motivated to use more general objects, like measures. Then, we can start studying equations like the following:

$$\begin{cases} -\Delta u + Vu = \mu \text{ in } \Omega, \\ \qquad\quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

This type of generalized partial differential equation also suffers from the same limitations related to explicit solutions. Therefore we can talk about weak or generalized solutions using the space of distributions. A measure μ can be seen as a distribution according to the following identification:

$$\langle \mu, \phi \rangle := \int_\Omega \phi \, d\mu, \phi \in C_c^\infty(\Omega).$$

This allows us to talk about this type of equation with more ease; and to use several results from the usual box of analytic tools coming from the study of partial differential equations.

Our study will focus on the following partial differential equation:

$$\begin{cases} -\Delta u + Vu = \mu \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, μ is a finite Borel measure on Ω and $V : \Omega \to \mathbb{R}$ is a Lebesgue-measurable function. We call this the **Schrödinger problem with potential V and density μ and denote it by** $[V; \mu]$.

We call the function V a Schrödinger potential, and the operator $\Delta + V$ a Schrödinger operator. This operator appears naturally in physics when studying a force field of the form $-\nabla V$. We can also see this type of equation in quantum mechanics when studying wave functions on a quantum field.

Our measure is not necessarily nonnegative. This is also used in many applications in physics, for example when working with an electric field, where we can have negative values.

Problems with measure data have been studied in Stampacchia's *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus* in 1965 (see [27]).

In this article, Stampacchia studies, in particular, the following problem:

$$\begin{cases} -\Delta u + Vu = \mu \text{ in } \Omega, \\ \qquad\qquad u = c \text{ on } \partial\Omega, \end{cases}$$

for a bounded V and $c \in \mathbb{R}$, proving some existence results and spectral properties on the operator $(-\Delta + V)$.

Malusa and Orsina also study some problems with measure data in *Existence and regularity results for relaxed Dirichlet problems with measure data* ([18]), dealing with a more generalized elliptic operator.

The more modern approach to studying the potential Δu as a measure started with Brezis and Ponce, with results such as the Kato's inequality and the weak maximum principle (see [7], [8] and [9]).

Some of the results we prove in this text are motivated by the ones achieved for the following nonlinear problem

$$\begin{cases} -\Delta u + g(u) = \mu \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

in the papers [6] and [20]. In these works, the authors prove some approximation results, leading to the definitions of reduced measure and reduced limit which inspired us to define their analogous versions in our context of the Schrödinger problem.

Maybe motivated by the techniques that lead to the results about the above problem, Orsina and Ponce studied the Schrödinger problem closely, with the objective of proving maximum principle results, as in [22] and [24], and the Hopf potential [23]. Some of the most important results that we use are the decomposition of the set Ω into sets that obey the maximum principle, the characterization of nonnegative good measures, and the properties of the zero-sets S and Z.

Another source of results and techniques in our approach is the book *Elliptic PDEs, Measures And Capacities* [25] by Ponce, which presents an overview of potential theory and differential equations with measure data. It also proves some comparison methods regarding measures and distributions that are also useful in our text.

With this text, we aim to build on the existent theory, by studying two approximation methods, already used in studying the nonlinear problem. The biggest difficulty is in the treatment of the potential V, that can take the value ∞ in a set of positive measure. Our methods will then heavily use the theory of zero-sets as presented in [24].

Our first objective is to lay down all the known foundations to work with these types of equations with measure data. This is the aim of Chapter 2. This overview of the known literature is necessary for the development of our approximation methods in Chapters 3 and 4.

In Section 2.1 we study finite Borel measures and how they relate to nonnegative Borel semimeasures. The main definition in this section will be the diffuse and concentrated limits of a sequence of measures. These concepts are strongly related to the concepts of absolutely continuous and singular measures from usual measure theory, but here they are generalized and this will allow us to talk more deeply about sequences of measures.

In Section 2.2 we lay the foundations of the area of study known as Geometric Measure Theory (GMT). This is the field that studies fine properties of functions and subsets of $\mathbb{R}^N$, using nonnegative Borel semimeasures that can look deeper into those objects. The most important tools are the Sobolev capacities and Hausdorff measures. We present three different types of capacities and the relationships between them. We also talk about how they relate to the Hausdorff measure.

In Section 2.3 we deal with the concept of precise representatives, which is a tool that allow us to talk about pointwise properties of functions that are defined almost everywhere. This concept derives from the Lebesgue's Differentiation Theorem and the maximal function. We can compute the precise representative $\widehat{f}$ of a Sobolev function f with the integral

$$\widehat{f}(x) = \lim_{r \to 0} \frac{1}{|B(x;r)|} \int_{B(x;r)} f.$$

Then we talk about quasicontinuous functions, and quasi-open and quasi-closed sets. We can then define a class of subsets of $\mathbb{R}^N$ called the Sobolev-open sets, that generate a topology that is going to be very useful to our studies. Similarly, we define the Sobolev-closed sets. We end the section with some useful results describing some properties of measures in Sobolev-open sets.

On Section 2.4 we start our discussion of the Schrödinger problem. We first make clear the differences between distributions, measures and functions and show some relationships between these objects, in order to manipulate them with care.

Then we present the definition of a type of solution to the Schrödinger problem using the theory of distributions. When a function $u \in L^1(\Omega) \cap L^1(\Omega; V)$ satisfies

$$\int_\Omega u(-\Delta\psi + V\psi) = \int_\Omega \psi \, d\mu$$

for every $\psi \in C_0^\infty(\overline{\Omega})$, we call this function a **distributional solution to the problem** $[V;\mu]$. We denote by $\mathcal{G}(V)$ the set of finite Borel measures for which the Schrödinger problem with potential V has a solution. These measures are called good measures.

We make the distinction between a distributional solution with measure data and a solution with L^2 data, which is a function $u \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V)$ that satisfies

$$\int_\Omega u(-\Delta\psi + V\psi) = \int_\Omega \psi \, d\mu$$

for every $\psi \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V)$. We call this function a **variational solution to the problem** $[V;\mu]$. These solutions can also be found by minimizing an energy function, as seen in Definition 2.36. Every variational solution is a distributional solution, and a distributional solution does not always exist.

With the purpose of having a candidate for the distributional solution, we define the **duality solution** that always exists and is unique for every Borel-measurable $V : \Omega \to [0, \infty]$ and $\mu \in \mathcal{M}(\Omega)$. When the distributional solution exists, they are also duality solutions to the same problem.

On the last section (Section 2.5) we move on to zero-sets. The first zero-set, that we denote by S, is related to the set of points of Ω for which the precise representative of the variational solution with $f \equiv 1$, which we call torsion function, is zero. While the set Z is the subset of Ω for which the precise representative of each the distributional solution, for nonnegative $f \in L^\infty(\Omega)$, is zero. We call S the **torsion function zero-set** and Z the **universal zero-set**. In particular, $S \subset Z$.

One of the possible interpretations of these sets, and the one that can be used to obtain S and Z, is related to the strong maximum principle. We know that because of the generality of our potential V, the strong maximum principle does not hold in all cases, i.e., for a variational solution ζ of the Schrödinger problem with potential V and L^∞ data, we do not necessarily have

$$\widehat{\zeta} > 0 \text{ or } \widehat{\zeta} \equiv 0 \text{ in } \Omega,$$

(this is particularly true when dealing with a singular potential V, when we can have, for example, the set $\{x \in \Omega; V(x) = \infty\}$ with positive Lebesgue measure). The set Z is precisely the subset of Ω for which every distributional solution is zero everywhere. In the classical theory of partial differential equations we have the validity of this maximum principle, therefore we do not see the occurrence of these zero-sets.

The set Z gives us a characterization of the existence of solutions to the Schrödinger problem, as seen in Theorem 1.4 from *On the nonexistence of Green's function and failure of the strong maximum principle - Luigi Orsina, Augusto Ponce - 2019* (see [24]). The theorem states that for every $V : \Omega \to [0, +\infty]$, the Schrödinger problem with potential V and nonnegative density μ has a distributional solution if, and only if,

$$\mu(Z) = 0.$$

About S, an important result is the decomposition of Ω in terms of the strong maximum principle using Sobolev-connected sets, as seen in Theorem 1.1 from [24]. We can also use the set S to construct a Schrödinger problem for which a duality solution is a distributional solution. Some remarks about the sizes of these sets can also be made using the capacities and the Hausdorff measure. The set S performs a crucial role in the next chapters.

In the following Chapters 3 and 4 we present our original results and contributions of this thesis to the theory of elliptic differential equation with measure data and some applications.

In the first section of Chapter 3 we will study and connect two types of approximation methods, that we will call reduced measure and reduced limit. The **reduced measure** comes from an approximation on the potential V and gives us a generalized version of the distributional solution. It was first introduced to the theory of elliptic problems with measure by Malusa and Orsina in [18] and, later, a version of the concept of reduced measure was used to study nonlinear equations with measure data.

In order to prove the Theorem 3.8 bellow we were inspired by Malusa and

Orsina's approximation method to work with problems regarding the Schrödinger operator and measure data. We want to find, for every nonnegative $\mu \in \mathcal{M}(\Omega)$, an $L^1(\Omega)$ function that is going to be either the distributional solution of the problem if μ is a good measure, or the maximal subsolution to the problem, if μ is not a good measure. We find that the reduced measure μ^* also has good properties, namely, it is the biggest good measure smaller than μ. Our results can be stated in the following theorem:

Theorem 3.8. *For every nonnegative $\mu \in \mathcal{M}(\Omega)$ and Borel-measurable $V : \Omega \to [0, \infty]$, there exists a measure $\mu^* \in \mathcal{M}(\Omega)$, called the reduced measure of $[V; \mu]$, that satisfies:*

(i) μ^* is a good measure, that is, there exists the distributional solution of the problem $[V; \mu^*]$, say $u^* \in L^1(\Omega)$;

(ii) u^* is a subsolution of $[V; \mu]$ and every subsolution $v \in L^1(\Omega)$ of $[V; \mu]$ satisfies $v \leq u^*$ almost everywhere in Ω;

(iii) $\mu^* \leq \mu$ and for every $\lambda \in \mathcal{G}(V)$ such that $\lambda \leq \mu$, we have $\lambda \leq \mu^*$;

(iv) $\mu^* = \mu\lfloor_{\Omega \setminus Z}$.

In the next section, we introduce the concept of **reduced limit**. We want to talk about sequences of solutions and how they relate to each other. If we take a sequence (μ_k) of measures for which the Schrödinger problem has solutions (u_k), we want to know if $u_k \to u^\#$ in $L^1(\Omega)$ implies that $u^\#$ is a solution to some Schrödinger equation with the same potential V. If the $L^1(\Omega)$ limit $u^\#$ is a solution to a Schrödinger problem with potential V and density $\mu^\#$, we call $\mu^\#$ the reduced limit of the sequence (μ_k).

We answer the question about existence positively:

Theorem 3.25. *Let $(\mu_k)_{k \in \mathbb{N}} \subset \mathcal{G}(V)$, not necessarily nonnegative measures, be a bounded sequence in $\mathcal{M}(\Omega)$. For each $k \in \mathbb{N}$, denote by $u_k \in L^1(\Omega)$ the distributional solution of the Schrödinger problem with Borel-measurable potential $V : \Omega \to [0, \infty]$ and density μ_k. If*

$$u_k \to u^\# \text{ in } L^1(\Omega),$$

then:

(i) the reduced limit of $(\mu_k)_{k \in \mathbb{N}}$ exists and it is unique, say $\mu^\# \in \mathcal{M}(\Omega)$;

(ii) $u^\#$ is the distributional solution of $[V, \mu^\#]$, in particular, $\mu^\#$ is a good measure.

An important property of the reduced limit is the monotonicity over $\Omega \setminus S$ given by the following theorem.

Theorem 3.31. *Let $V : \Omega \to [0, \infty]$ a Borel-measurable function and $(\mu_k)_{k\in\mathbb{N}}$, $(\lambda_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$ be bounded sequences of not necessarily nonnegative measures, with reduced limits $\mu^{\#}, \lambda^{\#} \in \mathcal{M}(\Omega)$, respectively. If, for every $k \in \mathbb{N}$,*

$$\mu_k \geq \lambda_k \text{ in } \Omega \backslash S,$$

then

$$\mu^{\#} \geq \lambda^{\#} \text{ in } \Omega \backslash S.$$

Another important property of the reduced limit concerns the lower semicontinuity with respect to the total variation norm (see Section 2.1).

Theorem 3.32. *Assume $V : \Omega \to [0, \infty]$ a Borel-measurable function and $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$, not necessarily nonnegative measures, be a bounded sequence in $\mathcal{M}(\Omega)$ with reduced limit $\mu^{\#} \in \mathcal{M}(\Omega)$. Then,*

$$\left|\mu^{\#}\right| (\Omega \backslash S) \leq \liminf_{k\to\infty} |\mu_k| (\Omega \backslash S) .$$

Lastly, in Chapter 4 we study two applications of these concepts. We start with an optimization problem known as the Optimal Control problem. This is a type of problem that appears naturally in physics and engineering, and uses tools from the area of Calculus of Variations and Dynamical Systems. We deal with the following Cost Functional:

$$F_{p,u_d}(\mu) = \begin{cases} \|u - u_d\|_{L^p(\Omega)} + \alpha |\mu| (\Omega \backslash S), & \text{if the Schrödinger problem with} \\ & \text{potential V and density } \mu \text{ has a} \\ & \text{distributional solution } u, \\ & \\ \infty, & \text{otherwise,} \end{cases}$$

where the function $u_d : \Omega \to \mathbb{R}$ is a given $L^1(\Omega)$ function that we call "ideal state".

We want to solve the following optimal control problem:

$$\text{find } \mu^{\#} \in \mathcal{M} \text{ such that } F_{p,u_d}\left(\mu^{\#}\right) = \inf_{\mu\in\mathcal{M}} F_{p,u_d}(\mu) . \tag{P}$$

We note that the linear functional F is not lower semicontinuous with respect to the weak* convergence of measures, which makes the problem more difficult.

Our first result gives us a way to deal with the singularity of the zero set S when calculating the total variations of a measure $\mu \in \mathcal{M}(\Omega)$ for which the Schrödinger problem has a solution. In this case we have:

$$\text{If } \mu \in L^1(\Omega) \cap \mathcal{G}(V), \text{ then } \|\mu\|_{L^1(\Omega)} = |\mu| (\Omega \backslash S).$$

Then we prove that our optimal control problem has a solution:

Theorem 4.2. *Assume $V : \Omega \to [0, \infty]$ a Borel-measurable function, $\mu \in \mathcal{M}(\Omega)$ being not necessarily nonnegative, $u_d \in L^1(\Omega)$, $1 \leq p \leq \infty$ and $\alpha > 0$. Then the minimization problem* (P) *has a unique solution $\mu^{\#} \in \mathcal{M}(\Omega)$. Moreover, $\mu^{\#}$ is the reduced limit of any minimizing sequence of the functional F_{p,u_d}, in particular, there exists $u^{\#} \in L^1(\Omega)$, the distributional solution of the problem $[V; \mu^{\#}]$.*

Our second application addresses a type of phenomenon that occurs when we have a solution to an optimization problem that is not summable. This is called the Lavrentiev Phenomenon named after Mikhaïl Lavrentiev (1900-1980). Our result regarding the Lavrentiev phenomenon is the following:

Theorem 4.7. *Let $\mu \in \mathcal{M}(\Omega)$ be not necessarily nonnegative, $N \geq 3$ and $\frac{N}{N-2} \leq p < \infty$. Assume $0 \leq V \in L^{q'}(\Omega)$ for some $1 \leq q < p$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Then there exists a nonnegative and nontrivial $w \in L^q(\Omega) \cap W_0^{1,r}(\Omega)$, for every $1 \leq r < \frac{N}{N-1}$, distributional solution to the problem $[V; \lambda]$, for some nonnegative and nontrivial $\lambda \in \mathcal{M}(\Omega)$, such that the cost functional $F_{p,w}$ satisfies*

$$F_{p,w} \not\equiv \infty \text{ in } \mathcal{M}(\Omega) \text{ and } F_{p,w} \equiv \infty \text{ in } L^1(\Omega).$$

To elaborate this thesis, we took inspiration on the ideas from results of a specific type of problem, in this case, the nonlinear problem, and we aimed to develop some similar and new ones from techniques, with significant changes, to another type of problem, the Schrödinger problem.

We had to take specific care with the singularity of the potential V (since the set $\{x \in \Omega; V(x) = \infty\}$ can have positive Lebesgue measure), and with the help of the zero-sets this study became more viable. The development of a different set of techniques for this specific type of problem was necessary.

2 Preliminaries

In this first chapter we lay the foundations for the study of our main problem.

First we have to study measures and their properties. Our main object of study will be finite Borel measures, and in the first section we define and study these measures.

We first talk about diffuse and concentrated measures, that are the equivalent in potential theory to the usual concepts of absolutely continuous and singular measures with respect to the Newtonian capacity.

Using the weak*-convergence we can also define the diffuse and concentrating limits that will help us look more deeply into finer properties of measures.

The next chapter is dedicated to the Sobolev capacity, a type of measure that allows us to examine closer into sets in the Euclidean space. These "measures" are finer than the usual Lebesgue measure over $\mathbb{R}^N$, and when paired with measure theory can be powerful tool in understanding complicated sets.

Lastly, we introduce the concept of precise representative. Often in analysis we deal with functions defined almost everywhere. When integrating with respect to the Lebesgue measure, this does not present any difficulties. But when working with integrals with respect to arbitrary measures, we have to use functions defined pointwise.

The precise representative is a way to define a function in every point without losing too much information about it and still retaining some important properties. We pair this concept with the ideas of quasicontinuity and Sobolev-open sets, that present a natural way to study some of the functions (notably the solutions to our main problem) and sets (notably the zero-sets) we will see in the next chapters.

Since it is uncommon to have measure as data in a differential equation, in this chapter we also take the time to lay the technical foundation to study the problem we have.

In Section 2.4 we generalize the concept of function using linear functionals defined over continuous functions and Schwartz distributions, and show how these relate to a broader understanding of differential phenomena.

By showing how these ideas are linked to each other we can expand our idea of what a differential equation is, and with this, also expand the tools we can work with to approach, solve and understand partial differential equations.

We then define what we mean by a solution to our PDE, and show how the equation can be seen as a distributional or a functional equation.

In the last section we define other weaker notions of solutions, namely the duality and the variational solutions, that are going to help us study our main problem.

We also see some results concerning existence and uniqueness of solutions and the application of the weak maximum principle. Next we introduce the zero-sets S and Z that are closely reated to the maximum principle results.

We end the chapter with a result about the decomposition of the set Ω into parts that obey the maximum principle, and a characterization of nonnegative good measures, both results from [22].

2.1 Finite Borel measures

Throughout this text we will work with the normed vector space $\mathbb{R}^N$, $N \geq 1$, endowed with the usual topology generated by the Euclidean metric, and pair it with the Borel σ-algebra on $\mathbb{R}^N$. This class of subsets of $\mathbb{R}^N$, that we notate by $\mathcal{B}(\mathbb{R}^N)$, is defined as the σ-algebra generated by the open subsets of $\mathbb{R}^N$ or, equivalently, the smallest σ-algebra that contains all the open subsets of $\mathbb{R}^N$.

We call a set $X \in \mathcal{B}(\mathbb{R}^N)$ a Borel subset of $\mathbb{R}^N$, and define the Borel σ-algebra over X, denoted $\mathcal{B}(X)$, as the natural restriction of $\mathcal{B}(\mathbb{R}^N)$ to the set X:

$$\mathcal{B}(X) := \{A \cap X; A \in \mathcal{B}(\mathbb{R}^N)\}.$$

We begin by defining our main objective of study:

Definition 2.1. *Let $X \in \mathcal{B}(\mathbb{R}^N)$. A finite Borel measure over X is a function $\mu : \mathcal{B}(X) \to \mathbb{R}$ such that*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

for every sequence $(A_k)_{k\in\mathbb{N}} \subset \mathcal{B}(X)$ of disjoint sets. A measure space is a pair $(X;\mu)$, where $X \in \mathcal{B}(\mathbb{R}^N)$ and μ is a finite Borel measure over X.

Remark. *Note that the series $\sum_{k=1}^{\infty} \mu(A_k)$ must be absolutely convergent, so that it does not depend on the order of summation, that is*

$$\sum_{k=1}^{\infty} |\mu(A_k)| < \infty.$$

It follows from the definition that for every measure space $(X;\mu)$, $\mu(\varnothing) = 0$.

Note that finite Borel measures can take negative values. In particular, if the measure μ only takes nonnegative values, we have:

(I) If $A, B \in \mathcal{B}(X)$ and $A \subset B$, then

$$\mu(A) \leq \mu(B).$$

(II) If $(A_k)_{k\in\mathbb{N}} \subset \mathcal{B}(X)$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Given a measure space $(X;\mu)$ and a set $Y \in \mathcal{B}(X)$, one defines the following finite Borel measure:

$$\mu\lfloor_Y(A) := \mu(Y \cap A), \ \forall A \in \mathcal{B}(X).$$

This is called the contraction of μ to Y.

We say that the measure μ is nonnegative, and denote $\mu \geq 0$, if for every $A \in \mathcal{B}(X)$,

$$\mu(A) \geq 0.$$

More generally, given two finite Borel measures over X, μ and λ, we say that the inequality $\mu \leq \lambda$ holds in the sense of measures when

$$\mu(A) \leq \lambda(A), \ \forall A \in \mathcal{B}(X),$$

and we say that the inequality $\mu \leq \lambda$ holds in $Y \in \mathcal{B}(X)$ in the sense of measures, when

$$\mu\lfloor_Y (A) \leq \lambda\lfloor_Y (A), \ \forall A \in \mathcal{B}(X).$$

The next classical result, which is a combination of Theorems 3.3 and 3.4 from [15], gives us a decomposition of the set X in terms of a given finite Borel measure:

Theorem 2.2 (Hahn-Jordan decomposition)**.** *If $(X;\mu)$ is a measure space, then there exist sets $P, N \in \mathcal{B}(X)$ such that*

(i) $P \cup N = X$, $P \cap N = \varnothing$, $\mu(P \cap N) = 0$;

(ii) $\mu\lfloor_P$ *and* $-\mu\lfloor_N$ *are nonnegative finite Borel measures over* X.

In particular, $\mu = \mu\lfloor_P - (-\mu\lfloor_N)$, i.e., we can write μ as the difference of two nonnegative finite Borel measures.

Definition 2.3. *Let $(X;\mu)$ be a measure space, and $P, N \in \mathcal{B}(X)$ be the sets given by the Hahn-Jordan Decomposition. The positive part of μ, notated by μ^+, and the negative part of μ, notated by μ^-, are the nonnegative measures defined as*

$$\mu^+ := \mu\lfloor_P \text{ and } \mu^- := -\mu\lfloor_N.$$

The total variation of μ, notated by $|\mu|$, is defined as

$$|\mu| := \mu^+ + \mu^-.$$

For more about measures, we refer to [11] and [15].

We are also interested in the following alternative way of measuring sets:

Definition 2.4. *Let $X \in \mathcal{B}(\mathbb{R}^N)$. A nonnegative Borel semimeasure over X is a function $T : \mathcal{B}(X) \to [0, \infty]$ such that*

(i) $T(\varnothing) = 0$;

(ii) $T(A) \leq T(B)$ *for every* $A, B \subset \mathcal{B}(X)$ *such that* $A \subset B$;

(iii) there exists a real constant $c > 0$ such that

$$T\left(\bigcup_{k=1}^{\infty} A_k\right) \leq c \sum_{k=1}^{\infty} T\left(A_k\right)$$

for every $(A_k)_{k\in\mathbb{N}} \subset \mathcal{B}(X)$.

Every nonnegative finite Borel measure is a nonnegative Borel semimeasure. The most important example of a nonnegative Borel semimeasure is the Lebesgue measure over $\mathbb{R}^N$. We will in generally denote the Lebesgue measure over $X \in \mathcal{B}(\mathbb{R}^N)$ by

$$|\cdot| : \mathcal{B}\left(X\right) \to \left[0, \infty\right],$$

and we denote the integral with respect to this measure either by omitting the measure in the integral, or by using the symbol dx.

Now we are going to define two possible relationships between a measure and a nonnegative semimeasure:

Definition 2.5. *Let $(X;\mu)$ be a measure space and T be a nonnegative Borel semimeasure over X.*

(I) The measure μ is a diffuse measure with respect to T, or T-diffuse measure, notated by $\mu \ll T$, if

$$\left|\mu\right|\left(A\right) = 0$$

for every $A \in \mathcal{B}(X)$ such that $T(A) = 0$.

(II) The measure μ is a concentrated measure with respect to T, or T-concentrated measure, notated by $\mu \perp T$, if there exists a set $N \in \mathcal{B}(X)$ such that

$$T\left(N\right) = 0 \text{ and } \left|\mu\right|\left(X \backslash N\right) = 0.$$

These concepts are analogous to the usual concepts of absolute continuous and singular measures from classical measure theory.

Next we present an equivalent version of the Lebesgue Decomposition Theorem. Its proof can be found in [25], Theorem 14.12.

Theorem 2.6 (Lebesgue Decomposition Theorem)**.** *Let $(X;\mu)$ be a measure space and T be a nonnegative Borel semimeasure over X. Then, there exist a unique finite Borel measures over X, μ_d and μ_c, such that*

(i) $\mu = \mu_d + \mu_c$;

(ii) $\mu_d \ll T$ and $\mu_c \perp T$.

Now we start working with sequences of measures. The definitions of diffuse and concentrated measures are a particular case of the following definitions for sequences:

Definition 2.7. *Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of finite Borel measures over X, and T be a nonnegative semimeasure over X.*

(I) The sequence $(\mu_k)_{k\in\mathbb{N}}$ is an equidiffuse sequence with respect to T, or T-equidiffuse sequence, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$A \in \mathcal{B}(X) \text{ and } T(A) < \delta \implies |\mu_k|(A) < \varepsilon \; \forall k \in \mathbb{N};$$

(II) The sequence $(\mu_k)_{k\in\mathbb{N}}$ is a concentrating sequence with respect to T, or T-concentrating sequence, if there exists $(A_k)_{k\in\mathbb{N}} \subset \mathcal{B}(X)$, such that

$$\lim_{k\to\infty} T(A_k) = 0 \text{ and } \lim_{k\to\infty} |\mu_k|(X\backslash A_k) = 0.$$

Our objective is to study the convergence of equidiffuse and concentrating sequences. For this we introduce a normed vector space structure over measures. First of all, for every $X \in \mathcal{B}(\mathbb{R}^N)$, we define the space $\mathcal{M}(X)$ as the vector space (over $\mathbb{R}$) of all finite Borel measures over X. In particular, we denote by $\mathcal{M}^+(\Omega)$ the subset of all nonnegative finite Borel measures over X.

We then define the following norm over $\mathcal{M}(X)$, called the total variation norm:

$$\|\mu\|_{\mathcal{M}(X)} := \sup\left\{\mu(A) - \mu(B); A, B \in \mathcal{B}(X)\right\}.$$

Throughout this work we shall use the following characterization of this norm:

$$\|\mu\|_{\mathcal{M}(X)} = \mu^+(X) + \mu^-(X) = |\mu|(X) = \int_X d|\mu|.$$

We can then talk about convergence in norm. Given a sequence $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ we say that it converges to $\mu \in \mathcal{M}(X)$, and denote it by

$$\mu_k \to \mu \text{ in } \mathcal{M}(X),$$

if

$$\lim_{k\to\infty} \|\mu_k - \mu\|_{\mathcal{M}(X)} = 0.$$

This is also called the strong convergence in $\mathcal{M}(X)$. The normed vector space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$ is a Banach space, this means that every Cauchy sequence of elements of $\mathcal{M}(X)$ converges to an measure in $\mathcal{M}(X)$.

From now on we consider the special case of measures over bounded open sets $\Omega \subset \mathbb{R}^N$, and we want to define a weaker type of convergence on $\mathcal{M}(\Omega)$. Let us define the support of a function $f : \Omega \to \mathbb{R}$ as:

$$\text{supp}(f) := \overline{\{x \in \Omega; f(x) \neq 0\}},$$

and the following function space:

$$C_c(\Omega) := \{f : \Omega \to \mathbb{R}; f \text{ is continuous and supp}(f) \text{ is compact}\}.$$

We introduce the concept of weak* convergence on $\mathcal{M}(\Omega)$:

Definition 2.8. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$. We say that the sequence $(\mu_k)_{k\in\mathbb{N}}$ converges weakly* (or vaguely) to μ in $\mathcal{M}(\Omega)$, notated by*

$$\mu_k \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega),$$

if

$$\lim_{k\to\infty} \int_\Omega \phi \, d\mu_k = \int_\Omega \phi \, d\mu$$

for every $\phi \in C_c(\Omega)$. In this case we say that $(\mu_k)_{k\in\mathbb{N}}$ is weakly convergent and that μ is the weak* limit of $(\mu_k)_{k\in\mathbb{N}}$.*

See Section 2.4 for a deeper discussion of the relationship between the spaces $\mathcal{M}(\Omega)$ and $C_c(\Omega)$.

We know that the convergence in norm implies the weak* convergence, and we also have the following compacity and semicontinuity results:

Theorem 2.9. *For every bounded sequence $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$, there exists a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$, and $\mu \in \mathcal{M}(\Omega)$ such that*

$$\mu_{k_j} \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\Omega).$$

Theorem 2.10. *If $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ are such that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, then*

$$\|\mu\|_{\mathcal{M}(\Omega)} \leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}(\Omega)}.$$

The proofs can be found in [25], Propositions 2.6 and 2.8, respectivelly.

The following is an important result regarding weak* limits of T-equidiffuse sequences:

Proposition 2.11. *Let T be a nonnegative Borel semimeasure over Ω, and $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ be a T-equidiffuse sequence. If $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, then, μ is T-diffuse.*

The following result is a generalization for sequences of the analogous version of the Lebesgue Decomposition Theorem and its proof can be found in [10].

Theorem 2.12 (Biting Lemma)**.** *Let T be a nonnegative Borel semimeasure over $\Omega \subset \mathbb{R}^N$, and $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ be bounded. Then, there exist sequences $(\alpha_k)_{k\in\mathbb{N}}, (\sigma_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ such that*

(B_0) $(\alpha_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}}$ are bounded in $\mathcal{M}(X)$;

(B_1) for every $k \in \mathbb{N}$, $\mu_k = \alpha_k + \sigma_k$;

(B_2) $(\alpha_k)_{k\in\mathbb{N}}$ is T-equidiffuse and $(\sigma_k)_{k\in\mathbb{N}}$ is T-concentrating.

When the results of the Biting Lemma hold, we say that $(\alpha_k)_{k\in\mathbb{N}}$ is a T-equidiffuse sequence of $(\mu_k)_{k\in\mathbb{N}}$, and $(\sigma_k)_{k\in\mathbb{N}}$ is a T-concentrating sequence of $(\mu_k)_{k\in\mathbb{N}}$. We can also prove that, for every $k \in \mathbb{N}$, $\alpha_k \perp \sigma_k$.

The next theorem shows that if a T-equidiffuse or T-concentrating sequence of $(\mu_k)_{k\in\mathbb{N}}$ has a weak* limit, then every other subsequence of that type is also weakly* convergent and its limit is the same:

Theorem 2.13. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ be weakly* convergent and bounded, $(\alpha_k)_{k\in\mathbb{N}}$, $(\sigma_k)_{k\in\mathbb{N}}$ and $(\alpha'_k)_{k\in\mathbb{N}}, (\sigma'_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ be two pairs of sequences satisfying the Biting Lemma with respect to $(\mu_k)_{k\in\mathbb{N}}$, and $\alpha, \sigma \in \mathcal{M}(X)$ be measures such that $\alpha_k \overset{*}{\rightharpoonup} \alpha$ and $\sigma_k \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(X)$. Then*

$$\alpha'_k \overset{*}{\rightharpoonup} \alpha \text{ and } \sigma'_k \overset{*}{\rightharpoonup} \sigma \text{ in } \mathcal{M}(X).$$

This proves that the weak* limit of these sequences is independent of the choice of T-equidiffuse and T-concentrating subsequences. We can then define:

Definition 2.14. *Let T be a nonnegative Borel semimeasure over $X \in \mathcal{B}(\mathbb{R}^N)$, $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ be a bounded sequence, and $(\alpha_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}} \subset \mathcal{M}(X)$ be, respectively, T-equidiffuse and T-concentrating sequences of $(\mu_k)_{k\in\mathbb{N}}$.*

(I) The diffuse limit of $(\mu_k)_{k\in\mathbb{N}}$ with respect to T, or the T-diffuse limit of $(\mu_k)_{k\in\mathbb{N}}$, is the weak limit of $(\alpha_k)_{k\in\mathbb{N}}$.*

(II) The concentrated limit of $(\mu_k)_{k\in\mathbb{N}}$ with respect to T, or the T-concentrated limit of $(\mu_k)_{k\in\mathbb{N}}$, is the weak limit of $(\sigma_k)_{k\in\mathbb{N}}$.*

Proposition 2.11 shows us that the $T-$diffuse limit of a sequence is a T-diffuse measure, although it is not true that the T-concentrated limit is a T-concentrated measure.

2.2 The Sobolev capacity

Let us start by taking $X \in \mathcal{B}(\mathbb{R})$ and recalling that a function $f : X \to \mathbb{R}$ is Borel-measurable if, for every $A \in \mathcal{B}(\mathbb{R})$, we have $f^{-1}(A) \in \mathcal{B}(X)$. We then set, for each $1 \leq p < \infty$, the following classes of Borel-measurable functions:

$$\mathcal{L}^p(X) := \left\{ f : X \to \mathbb{R}; \int_X |f|^p < \infty \right\}$$

and

$$\mathcal{L}^\infty(X) := \Big\{ f : X \to \mathbb{R}; \exists M \in \mathbb{R} \text{ with } |\{x \in X; |f(x)| > M\}| = 0 \Big\}.$$

Given Borel-measurable functions $f, g : X \to \mathbb{R}$, we say that $f = g$ almost everywhere (often abbreviated to a.e.) in X if

$$\big|\{x \in X; f(x) \neq g(x)\}\big| = 0.$$

We can then define:

Definition 2.15. *Let $X \in \mathcal{B}(\mathbb{R})$, $1 \leq p \leq \infty$ and $\sim_X$ be the equivalence relation defined by*

$$f \sim_X g \text{ if } f = g \text{ almost everywhere in } X.$$

The space $L^p(X)$ is the quotient space

$$\mathcal{L}^p(X)/{\sim_X}.$$

When $1 \leq p < \infty$ we call $L^p(X)$ the space of Lebesgue p-integrable functions (or p-summable functions) over X. We call $L^\infty(X)$ the space of essentially bounded functions over X.

Given a function $f \in \mathcal{L}^p(X), 1 \leq p \leq \infty$, we denote by $[f] \in L^p(X)$ the equivalence class determined by f. We can define the following norms:

$$\|[f]\|_{L^p(X)} := \left(\int_X |f|^p \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty$$

and

$$\|[f]\|_{L^\infty(X)} := \inf_{M \in \mathbb{R}} \Big\{ |\{x \in X; |f(x)| > M\}| = 0 \Big\}.$$

With this structure, $\big(L^p(X), \|\cdot\|_{L^p(X)}\big), 1 \leq p \leq \infty$, are Banach spaces.

For every $1 \leq p \leq \infty$ we can also define $L^p_{loc}(X)$, called the space of locally p-integrable (if $1 \leq p < \infty$) or locally essentially bounded functions (if $p = \infty$), by:

$$\mathcal{L}^p_{loc}(X) := \{f : X \to \mathbb{R}; [f|_K] \in L^p(K) \text{ for every compact } K \subset X\},$$

and

$$L^p_{loc}(X) := \mathcal{L}^p_{loc}(X)/\sim_X.$$

It is customary to treat the elements of $L^p(X)$ and $L^p_{loc}(X)$ as real functions instead of equivalence classes, often taking a function $f : X \to \mathbb{R}$ and writing expressions like $f \in L^p(X)$ and $\|f\|_{L^p(X)}$ instead of $[f] \in L^p(X)$ and $\|[f]\|_{L^p(X)}$. We will adopt this approach of presentation in order to simplify the notation and match the usual literature.

Now we consider an open set $\Omega \subset \mathbb{R}^N$, and denote the space of infinitely differentiable real functions over Ω, by $C^\infty(\Omega)$. Then we set

$$C^\infty_c(\Omega) := \{f : \Omega \to \mathbb{R}; f \in C^\infty(\Omega) \text{ and } \mathrm{supp}(f) \text{ is compact}\}.$$

This space is called the space of test functions. Since these functions are smooth, we can take a multi-index α as an N-tuple $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$, with

$$|\alpha| := \sum_{i=1}^{N} \alpha_i,$$

and define the differential operator $D^\alpha : C^\infty_c(\Omega) \to C^\infty_c(\Omega)$, that maps $\phi \in C^\infty_c(\Omega)$ into $D^\alpha\phi \in C^\infty_c(\Omega)$ defined by:

$$D^\alpha\phi := \frac{\partial^{|\alpha|}\phi}{\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_N^{\alpha_N}}.$$

We are then ready to define:

Definition 2.16. *Let $\Omega \subset \mathbb{R}^N$ be open, $1 \leq p \leq \infty$ and $n \in \mathbb{N}$. The Sobolev space of order n, p on Ω, denoted by $W^{n,p}(\Omega)$, is the space of functions $\zeta \in L^p(\Omega)$ such that, for each multi-index α with $|\alpha| \leq n$, there exists $f_\alpha \in L^p(\Omega)$ satisfying*

$$\int_\Omega \zeta \cdot D^\alpha\phi = (-1)^{|\alpha|} \int_\Omega f_\alpha \cdot \phi$$

for every $\phi \in C^\infty_c(\Omega)$. In this case we call f_α the weak derivative of order α of ζ and we use the notation $D^\alpha\zeta := f_\alpha$.

The Sobolev space equipped with the norm

$$\|\zeta\|_{W^{n,p}(\Omega)} := \begin{cases} \left(\sum\limits_{0\leq|\alpha|\leq n} \|D^\alpha\zeta\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \sum\limits_{0\leq|\alpha|\leq n} \|D^\alpha\zeta\|_{L^\infty(\Omega)}, & \text{if } p = \infty \end{cases}$$

is a Banach space.

We can also define the spaces $W_0^{n,p}(\Omega)$, for $1 \leq p \leq \infty$ and $n \in N$:

$$W_0^{n,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{n,p}(\Omega)}}.$$

This space contains the functions of $W^{n,p}(\Omega)$ that, in a certain sense, are zero in the border of Ω (see more about trace theory in Chapter 15 of [25]). For more about Sobolev spaces we refer to [5] and [17].

Our study of the Sobolev capacity starts with compact sets. Let $1 \leq p < \infty$ and $n \in \mathbb{N}$. We define the set function $\mathrm{cap}_{W^{n,p}}$ over compact subsets $K \subset \mathbb{R}^N$ as:

$$\mathrm{cap}_{W^{n,p}}(K) := \inf\left\{ \|\phi\|^p_{W^{n,p}(\mathbb{R}^N)} ; \phi \in C_c^\infty(\mathbb{R}^N), \phi \geq 0 \text{ in } \mathbb{R}^N, \phi > 1 \text{ in } K \right\}.$$

This function has several properties:

(I) For every compact $K \subset \mathbb{R}^N$,

$$|K| \leq \mathrm{cap}_{W^{n,p}}(K).$$

(II) If $K, L \subset \mathbb{R}^N$ are compact sets with $K \subset L$, then

$$\mathrm{cap}_{W^{n,p}}(K) \leq \mathrm{cap}_{W^{n,p}}(L).$$

(III) Let $\{K_j\}_{j\in\{1,\dots,m\}}$ be a finite family of compact subsets of $\mathbb{R}^N$. Then, the following finite semi-additivity property holds:

$$\mathrm{cap}_{W^{n,p}}\left(\bigcup_{j=1}^m K_j\right) \leq C \sum_{j=1}^m \mathrm{cap}_{W^{n,p}}(K_j),$$

with $C = C(N, n, p) \geq 1$. In particular, when $p = 1$ we have $C = 1$.

(IV) For every $1 \leq p < \infty$ and every compact $K \subset \mathbb{R}^N$,

$$\mathrm{cap}_{W^{n,p}}(K) \leq \mathrm{cap}_{W^{n+1,p}}(K).$$

The proofs of these facts can be found in Section A.1 from [25].

We can extend this function to open subsets $U \subset \mathbb{R}^N$ by inner regularity:

$$\mathrm{cap}_{W^{n,p}}(U) := \sup\left\{\mathrm{cap}_{W^{n,p}}(K) ; K \subset \mathbb{R}^N \text{ is compact}, K \subset U\right\},$$

and to general subsets of $\mathbb{R}^N$ by outer regularity:

Definition 2.17. *Let $1 \leq p < \infty$ and $n \in \mathbb{N}$. The Sobolev capacity of order n, p, denoted by $cap_{W^{n,p}}$, is the function defined for every $A \subset \mathbb{R}^N$ as*

$$cap_{W^{n,p}}(A) := \inf\left\{cap_{W^{n,p}}(U); U \subset \mathbb{R}^N \text{ is open}, A \subset U\right\}.$$

Using the properties that hold for compact sets, we have the next result (see Proposition A.9 [25]):

Proposition 2.18. *(I) For every $A, B \subset \mathbb{R}^N$ with $A \subset B$ we have*

$$cap_{W^{n,p}}(A) \leq cap_{W^{n,p}}(B).$$

(II) Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}^N$. We have the following semi-additivity property:

$$cap_{W^{n,p}}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq C \sum_{k=1}^{\infty} cap_{W^{n,p}}(A_k),$$

with $C = C(N, k, p) \geq 1$.

When we want to prove relationships between Sobolev capacities and finite Borel measures, it is necessary to restrict $\mathrm{cap}_{W^{n,p}}$ to a good class of subsets of $\mathbb{R}^N$. The next result shows us that the Borel σ-algebra of $\mathbb{R}^N$ serves that purpose well:

Proposition 2.19. *The Sobolev capacity $cap_{W^{n,p}}$ defined over $\mathcal{B}(\mathbb{R}^N)$, the Borel σ-algebra of $\mathbb{R}^N$, is a nonnegative Borel semimeasure.*

Proof: First of all, for every $A \subset \mathbb{R}^N$ (and in particular, for every $A \in \mathcal{B}(\mathbb{R}^N)$), we have

$$\mathrm{cap}_{W^{n,p}}(A) \geq 0.$$

We also have that, since the function $\phi \equiv 0$ is nonnegative and satisfies $\phi > 1$ in $\varnothing$,

$$\mathrm{cap}_{W^{n,p}}(\varnothing) = 0.$$

We conclude by using Proposition 2.18. ∎

With this in hand we can talk about diffuse and concentrated measures with respect to $\mathrm{cap}_{W^{n,p}}$, as well as $\mathrm{cap}_{W^{n,p}}$-equidiffuse and $\mathrm{cap}_{W^{n,p}}$-concentrating sequences of measures. Naturally, everytime we talk about these concepts, we are dealing with $\mathrm{cap}_{W^{n,p}}$ restricted to $\mathcal{B}(\mathbb{R}^N)$ and if there is no risk of confusion we shall make no distinction between $\mathrm{cap}_{W^{n,p}}$ as a set function defined for every subset of $\mathbb{R}^N$ and $\mathrm{cap}_{W^{n,p}}$ as a nonnegative Borel semimeasure.

The next result shows that $\mathrm{cap}_{W^{n,p}}$ is finer than the Lebesgue measure:

Proposition 2.20. *The Lebesgue measure is* $\text{cap}_{W^{n,p}}$*-diffuse.*

Proof: In fact, we have

$$|K| \leq \text{cap}_{W^{n,p}}(K)$$

for every compact $K \subset \mathbb{R}^N$ (note that $\mathcal{B}(\mathbb{R}^N)$ contains every compact subset of $\mathbb{R}^N$). Using the definition of $\text{cap}_{W^{n,p}}$ for open sets, and the outer regularity of the Lebesgue measure, we have

$$|A| \leq \text{cap}_{W^{n,p}}(A)$$

for every $A \in \mathcal{B}(\mathbb{R}^N)$. In particular, if $\text{cap}_{W^{n,p}}(A) = 0$, then $|A| = 0$. ■

The field of Geometric Measure Theory is a vast one. We refer to the references [14] and [21] for more about this topic.

2.3 Precise representatives and quasicontinuity

Thoughout this section, for an integrable function $f : A \to \mathbb{R}$ with $|A| > 0$, we use the notation:

$$⨍_A f := \frac{1}{|A|}\int_A f.$$

The following is a fundamental theorem, whose proof can be found in [25], Proposition 8.1:

Theorem 2.21 (Lebesgue Differentiation Theorem). *Let $f \in L^1(\mathbb{R}^N)$. Then, there exists $A \in \mathcal{B}(\mathbb{R}^N)$ such that*

(i) $|\mathbb{R}^N \backslash A| = 0$*;*

(ii) for every $x \in A$ we have

$$\lim_{r\to 0} ⨍_{B(x;r)} |f - f(x)| = 0.$$

In particular, this implies that, for functions $f \in L^1(\mathbb{R})$, the primitive function $F : \mathbb{R} \to L^1(\mathbb{R})$ defined as

$$F(x) = \int_0^x f,$$

is differentiable, except in a set with Lebesgue measure zero.

We now want to give special importance to the set of points of $\mathbb{R}^N$ for which the Lebesgue's Differentiation Theorem holds.

Definition 2.22. *Let $f \in L^1_{loc}(\mathbb{R}^N)$. A point $x \in \mathbb{R}^N$ is a Lebesgue point of f if there exists $c \in \mathbb{R}$ such that*

$$\lim_{r\to 0} ⨍_{B(x;r)} |f - c| = 0.$$

The set of all Lebesgue points of f is called the Lebesgue set of f and it is denoted by $\mathcal{L}_f$. The set $\mathbb{R}^N \backslash \mathcal{L}_f$ is called the exceptional set of f.

The Lebesgue's Differentiation Theorem shows us that the exceptional set is small with respect to the Lebesgue measure, i.e.,

$$\left|\mathbb{R}^N \backslash \mathcal{L}_f\right| = 0,$$

and that the number $c \in \mathbb{R}$, associated to the point $x \in \mathcal{L}_f$, is $f(x)$ when x is a point for which the limit shown in the theorem holds. Furthermore it is easy to know that we also have:

Proposition 2.23. *Let $f, g \in L^1_{loc}(\mathbb{R}^N)$, $a \in \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous function. Then*

(I) $(\mathcal{L}_f \cap \mathcal{L}_g) \subset \mathcal{L}_{f+g}$*;*

(II) $\mathcal{L}_f \subset \mathcal{L}_{af}$*;*

(III) $\mathcal{L}_f \subset \mathcal{L}_{H(f)}$*.*

Now we define a function associated to f in the set $\mathcal{L}_f$:

Definition 2.24. *Let $f \in L^1_{loc}(\mathbb{R}^N)$ and $\mathcal{L}_f$ the Lebesgue set of f. The precise representative of f, denoted by $\widehat{f} : \mathcal{L}_f \to \mathbb{R}$, is defined as the function that at the point $x \in \mathcal{L}_f$, assumes the value $\widehat{f}(x) \in \mathbb{R}$ that satisfies*

$$\lim_{r \to 0} \fint_{B(x;r)} \left| f - \widehat{f}(x) \right| = 0.$$

Similar to the linearity properties of the Lebesgue sets, we have:

Proposition 2.25. *Let $f, g \in L^1_{loc}(\mathbb{R}^N)$, $a \in \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous function. Then*

(I) $\widehat{f+g} = \widehat{f} + \widehat{g}$ *in* $\mathcal{L}_f \cap \mathcal{L}_g$*;*

(II) $\widehat{af} = a\widehat{f}$ *in* $\mathcal{L}_f$*;*

(III) $\widehat{H(f)} = H(\widehat{f})$ *in* $\mathcal{L}_f$*.*

We know that the precise representative $\widehat{f}$ is defined almost everywhere in $\mathbb{R}^N$. If we denote by A_f the set of points of $\mathbb{R}^N$ for which the limit in the Lebesgue's Differentiation Theorem holds, then

$$A_f \subset \mathcal{L}_f$$

and

$$f = \widehat{f} \text{ in } A_f.$$

And since

$$\left| \fint_{B(x;r)} \big(f - f(x)\big) \right| \leq \fint_{B(x;r)} |f - f(x)|$$

for every $r > 0$, then

$$\widehat{f}(x) = \lim_{r \to 0} \fint_{B(x;r)} f$$

for every $x \in A_f$, which is a nice way to compute the precise representative at almost every point.

We can define all these concepts for functions in $L^1_{loc}(\Omega)$ for $\Omega \subset \mathbb{R}^N$, since the definition is only local, i.e., if $f \in L^1_{loc}(\Omega)$, then the extension $\overline{f} \in L^1_{loc}(\mathbb{R}^N)$ defined by

$$\overline{f} = f \text{ in } \Omega, \text{ and } \overline{f} = 0 \text{ in } \mathbb{R}^N \backslash \Omega,$$

gives a unique $c \in \mathbb{R}$ that satisfies

$$\lim_{r \to 0} ⨍_{B(x;r)} |\overline{f} - c| = \lim_{r \to 0, B(x;r) \subset \Omega} ⨍_{B(x;r)} |f - c| = 0$$

for every $x \in \Omega$.

Now we present the concept of quasicontinuity:

Definition 2.26. *Let $X \in \mathcal{B}(\mathbb{R}^N)$, $f : X \to \mathbb{R}$ be a measurable function and T be a nonnegative Borel semimeasure over X. We say that f is a quasicontinuous function with respect to T, or T-quasicontinuous function, if there exists a sequence $(A_k)_{k \in \mathbb{N}} \subset \mathcal{B}(\mathbb{R}^N)$ such that*

(i) $f|_{A_k}$ *is continuous for every* $k \in \mathbb{N}$*;*

(ii) $\lim_{k \to \infty} T(X \backslash A_k) = 0.$

We also say that a property holds T-quasi-everywhere in X if it holds in the set $X \backslash A$ and $T(A) = 0$.

Definition 2.27. *Let $X \in \mathcal{B}(\mathbb{R}^N)$ and T be a nonnegative Borel semimeasure over X. A set $A \subset X$ is a quasi-open set with respect to T, or a T-quasi-open set, if there exists a sequence $(U_k)_{k \in \mathbb{N}}$, of open subsets of Ω, such that*

(i) $A \cup U_k$ *is an open subset of* X *for every* $k \in \mathbb{N}$*;*

(ii) $\lim_{k \to \infty} T(U_k) = 0.$

Furthermore, $A \subset X$ is a quasi-closed set with respect to T, or a T-quasi-closed set, if $X \backslash A$ is a T-quasi-open set.

We show that the class of T-quasi-open sets is bigger than the class of open subsets of $\mathbb{R}^N$:

Proposition 2.28. *Let T be a nonnegative Borel semimeasure over $\mathbb{R}^N$. Every open subset of $\mathbb{R}^N$ is a T-quasi-open set.*

Proof: It is enough to take a sequence $(U_k)_{k \in \mathbb{N}}$ where

$$U_k := \varnothing, k \in \mathbb{N},$$

and the result follow from the definition. ■

Now, let $\Omega \subset \mathbb{R}^N$ be an open set and let us start to study a particular case of the definitions we layed down. We look more closely at $\text{cap}_{W^{1,2}}$-quasicontinuous functions and $\text{cap}_{W^{1,2}}$-quasi-open and $\text{cap}_{W^{1,2}}$-quasi-closed sets. First we define another class of subsets of Ω:

Definition 2.29. *A set $U \subset \Omega$ is a Sobolev-open set if there exists a nonnegative function $\zeta \in W_0^{1,2}(\Omega)$ such that $\mathcal{L}_\zeta = \Omega$, i.e., every $x \in \Omega$ is a Lebesgue point of ζ, and*

$$U = \left\{ \widehat{\zeta} > 0 \right\},$$

where $\widehat{\zeta}$ is the precise representative of ζ. Moreover, $F \subset \Omega$ is a Sobolev-closed set if $\Omega \backslash F$ is Sobolev-open.

We show a property of functions in $W_0^{1,2}(\Omega)$:

Proposition 2.30. *If $\zeta \in W_0^{1,2}(\Omega)$, then $\widehat{\zeta}$ is $cap_{W^{1,2}}$-quasicontinuous.*

This is Lemma 2.152 from [19]. As an immediate corolary we have:

Corollary 2.31. *Every Sobolev-open set is a $cap_{W^{1,2}}$-quasi-open set.*

2.4 Functions, measures and distributions

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. We want to make sense of the following expression:

$$-\Delta u + Vu = \mu, \tag{1}$$

where u is Lebesgue-integrable over Ω, $V : \Omega \to [0, \infty]$ is a Borel-measurable function such that Vu is Lebesgue-integrable over Ω, and μ is a finite Borel measure on $\mathcal{B}(\Omega)$. Our aim is to look at both sides of this expression as more general elements in a bigger space, in particular where the object Δu is well defined, and then compare these more general objects.

First of all, given a topological real vector space X, we notate the space of all continuous real linear functionals over X, by X', called the continuous dual space of X; and we notate the value of the functional $F \in X'$ acting on $u \in X$, by $\langle F, u\rangle_X$. The bilinearity of the application $\langle \cdot, \cdot \rangle_X$ on $X \times X'$ can be easily seen: if $a, b \in \mathbb{R}$, $u, v \in X$ and $F, G \in X'$, then

$$\langle aF + G, bu + v\rangle_X = ab\langle F, u\rangle_X + a\langle F, v\rangle_X + b\langle G, u\rangle_X + \langle G, v\rangle_X.$$

Let us consider $C_c^\infty(\Omega)$ as a real vector space, and, given a sequence $(\phi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, we say that $\phi_k \to \phi$ in the sense of test functions, if there exists a compact $K \subset \Omega$ such that for every $k \in \mathbb{N}$, $\phi_k|_{\Omega\setminus K} \equiv 0$,

$$\lim_{k\to\infty} \|D^\alpha \phi_k - D^\alpha \phi\|_\infty = 0$$

for every multi-index α, where $\|\cdot\|_\infty$ is called the supremum norm, defined as:

$$\|\phi\|_\infty := \sup\{|\phi(x)|; x \in \Omega\}.$$

(See Section 2.2 for the definition of multi-index and the differential operator D^α).

This notion of convergence is enough to induce a topology τ on $C_c^\infty(\Omega)$ (see the beginning of Chapter 9 from [15]). We denote the topological real vector space $(C_c^\infty(\Omega), \tau)$ simply by $\mathcal{D}(\Omega)$ and when $\phi_k \to \phi$ in the sense of test functions, we write

$$\phi_k \to \phi \text{ in } \mathcal{D}(\Omega).$$

Definition 2.32. *A distribution over Ω is a linear functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$ which is continuous, i.e., for every $(\phi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$, such that $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$, we have*

$$\lim_{k\to\infty} \langle T, \phi_k\rangle_{\mathcal{D}(\Omega)} = \langle T, \phi\rangle_{\mathcal{D}(\Omega)}.$$

We denote the space of distributions over Ω by $\mathcal{D}'(\Omega)$.

It is possible to define certain operators on distributions by knowing how these operator behave on the space of test functions. In particular, the differential operator can be defined on elements of $\mathcal{D}'(\Omega)$. Given $T \in \mathcal{D}'(\Omega)$ and a multi-index α, we define $D^\alpha T$ as the distribution that satisfies

$$\langle D^\alpha T, \phi \rangle_{\mathcal{D}(\Omega)} := (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle_{\mathcal{D}(\Omega)}$$

for every $\phi \in C_c^\infty(\Omega)$. In particular, we define ΔT as the distribution that satisfies

$$\langle \Delta T, \phi \rangle_{\mathcal{D}(\Omega)} := \langle T, \Delta \phi \rangle_{\mathcal{D}(\Omega)}$$

for every $\phi \in C_c^\infty(\Omega)$.

Now, we define an operator $\mathcal{H} : L^1(\Omega) \to \mathcal{M}(\Omega)$ that assigns, for each integrable function f, a finite Borel measure $\mathcal{H}_f$ given by

$$\begin{aligned} \mathcal{H}_f : \mathcal{B}(\Omega) &\to \mathbb{R} \\ X \quad &\mapsto \int_X f. \end{aligned}$$

Another operator, $\mathcal{I} : \mathcal{M}(\Omega) \to \mathcal{D}'(\Omega)$, takes a finite Borel measure $\mu \in \mathcal{M}(\Omega)$ onto a distribution $\mathcal{I}_\mu$ given by

$$\begin{aligned} \mathcal{I}_\mu : C_c^\infty(\Omega) &\to \mathbb{R} \\ \phi \quad &\mapsto \int_\Omega \phi \, d\mu. \end{aligned}$$

Now, denoting by $\mathcal{J}$ the composition $\mathcal{I} \circ \mathcal{H}$, and by $\mathcal{J}_f$ the image of $f \in L^1(\Omega)$, we can write:

$$\begin{aligned} \mathcal{J} : & L^1(\Omega) \to \mathcal{D}'(\Omega) \\ & f \quad \mapsto \mathcal{J}_f : C_c^\infty(\Omega) \to \mathbb{R} \\ & \qquad\qquad\qquad \phi \quad \mapsto \int_\Omega f \cdot \phi. \end{aligned}$$

These operations define bijective isomorphisms over their respective images, with $\mathcal{H}$ being an isometry, and we can draw the following diagram:

$$\begin{array}{ccc} L^1(\Omega) & \xrightarrow{\mathcal{H}} & \mathcal{H}(L^1(\Omega)) \subsetneq \mathcal{M}(\Omega) \\ & \searrow^{\mathcal{J}} & \downarrow \mathcal{I} \\ & & \mathcal{J}(L^1(\Omega)) = \mathcal{I}(\mathcal{M}(\Omega)) \subsetneq \mathcal{D}'(\Omega) \end{array}$$

Now we are ready to make sense of (1). We use the function $\mathcal{J}$ to find a distribution on the right-hand side of the expression, and $\mathcal{I}$ to find a distribution on the left-hand side:

Since $u, Vu \in L^1(\Omega)$, then $\mathcal{J}_u, \mathcal{J}_{Vu} \in \mathcal{D}'(\Omega)$, and using the definition of the differential operator over $\mathcal{D}'(\Omega)$, we can define the distribution $\Delta\mathcal{J}_u$. Then $-\Delta\mathcal{J}_u + \mathcal{J}_{Vu} \in \mathcal{D}'(\Omega)$ is the distribution that satisfies

$$\begin{aligned}
\langle -\Delta\mathcal{J}_u + \mathcal{J}_{Vu}, \phi\rangle_{\mathcal{D}(\Omega)} &= \langle -\Delta\mathcal{J}_u, \phi\rangle_{\mathcal{D}(\Omega)} + \langle \mathcal{J}_{Vu}, \phi\rangle_{\mathcal{D}(\Omega)} \\
&= \langle \mathcal{J}_u, -\Delta\phi\rangle_{\mathcal{D}(\Omega)} + \langle \mathcal{J}_{Vu}, \phi\rangle_{\mathcal{D}(\Omega)} \\
&= \int_\Omega u(-\Delta\phi) + \int_\Omega Vu \cdot \phi \\
&= \int_\Omega u(-\Delta\phi + V\phi)
\end{aligned}$$

for every $\phi \in C_c^\infty(\Omega)$. Analogously, for $\mu \in \mathcal{M}(\Omega)$ we have $\mathcal{I}_\mu \in \mathcal{D}'(\Omega)$. We now compare elements of the same space:

If, given V and μ, we can find $u \in L^1(\Omega)$ such that $Vu \in L^1(\Omega)$, and the distributions $-\Delta\mathcal{J}_u + \mathcal{J}_{Vu}$ and $\mathcal{I}_\mu$ coincide, we will have

$$\begin{aligned}
\int_\Omega u(-\Delta\phi + V\phi) &= \langle -\Delta\mathcal{J}_u + \mathcal{J}_{Vu}, \phi\rangle_{\mathcal{D}(\Omega)} \\
&= \langle \mathcal{I}_\mu, \phi\rangle_{\mathcal{D}(\Omega)} \\
&= \int_\Omega \phi \, d\mu
\end{aligned}$$

for every $\phi \in C_c^\infty(\Omega)$. This is our motivation to define:

Definition 2.33. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty, smooth and bounded open set, $V : \Omega \to [0, \infty]$ be a Borel-measurable function and $\mu \in \mathcal{M}(\Omega)$. We say that $u \in L^1(\Omega)$ solves in the sense of distributions the Schrödinger equation with potential V and density μ, and we denote by*

$$-\Delta u + Vu = \mu \text{ in } \mathcal{D}'(\Omega),$$

if

(i) $Vu \in L^1(\Omega)$;

(ii) for every $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega u(-\Delta\phi + V\phi) = \int_\Omega \phi \, d\mu.$$

Our main objective of study is the following problem, that we call the Schrödinger problem with potential V and density μ:

$$\begin{cases} -\Delta u + Vu = \mu \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega. \end{cases}$$

Since the space of infinitely differentiable functions with compact support is not able to detect boundary values of distributional solutions of the equation with potential V and density μ, we have no information about the value of u on $\partial\Omega$ (formally, we have no information about the trace of the function u). To overcome this obstacle we assign the boundary values of u in the definition:

Definition 2.34. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty, smooth and bounded open set, $V : \Omega \to [0, \infty]$ be a Borel-measurable function and $\mu \in \mathcal{M}(\Omega)$. We say that $u \in L^1(\Omega)$ is a solution in the sense of distributions, or distributional solution, of the Schrödinger problem with potential V and density μ, if*

(i) $u \in W_0^{1,1}(\Omega)$;

(ii) $Vu \in L^1(\Omega)$;

(iii) for every $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega u(-\Delta\phi + V\phi) = \int_\Omega \phi \; d\mu.$$

We can obtain the following characterization of the distributional solution:

Proposition 2.35. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty, smooth and bounded open set, $V : \Omega \to [0, \infty]$ be a Borel-measurable function and $\mu \in \mathcal{M}(\Omega)$. The function $u \in L^1(\Omega)$ is a distributional solution of the Schrödinger problem with potential V and density μ if, and only if*

(i) $Vu \in L^1(\Omega)$;

(ii) for every $\psi \in C_0^\infty(\overline{\Omega})$,

$$\int_\Omega u(-\Delta\psi + V\psi) = \int_\Omega \psi \; d\mu.$$

We can prove this using Proposition 6.3 from [25] for the datum $\mu - Vu$, since $Vu \in L^1(\Omega)$ and therefore $\mu - Vu \in \mathcal{M}(\Omega)$.

Not every Schrödinger problem has a solution in the sense we just defined. We then introduce the concept of duality solution, which can be used as candidates for distributional solutions, since they always exist and they are a generalization of distributional solutions (following Malusa and Orsina [18]. To introduce this concept we first need to talk about the particular case where our measure data is 2-integrable:

Definition 2.36. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty, smooth and bounded open set, $V : \Omega \to [0,\infty]$ be a Borel-measurable function and $f \in L^2(\Omega)$. We say that $u \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V)$, is a variational solution of the problem*

$$\begin{cases} -\Delta u + Vu = f \ \text{in } \Omega, \\ \qquad\qquad u = 0 \ \text{on } \partial\Omega, \end{cases}$$

called the Schrödinger problem with potential V and data f, if it satisfies

$$\int_\Omega (\nabla u \cdot \nabla z + Vuz) = \int_\Omega fz$$

for every $z \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V)$, or, equivalently, if it is the minimizer, in $W_0^{1,2}(\Omega) \cap L^2(\Omega; V)$, of the functional

$$E(z) := \frac{1}{2} \int_\Omega \left(|\nabla z|^2 + Vz^2\right) - \int_\Omega fz.$$

We denote this solution by ζ_f.

From classical minimization techniques (Proposition 22.10 from [25]) we have:

Proposition 2.37. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. For every Borel-measurable $V : \Omega \to [0,\infty]$ and $f \in (W_0^{1,2}(\Omega))'$, the variational solution of the Schrödinger problem with potential V and data f always exists and it is unique.*

For more about the variational treatment of differential equation we refer to [3] and [12].

Let us define a linear functional over $L^\infty(\Omega)$ using the canonical pairing $(u, f) \mapsto \int_\Omega u \cdot f$ for every $u \in L^1(\Omega)$. Using the Hölder's inequality:

$$\begin{aligned} |\langle u, f\rangle_{L^\infty(\Omega)}| &= \left| \int_\Omega u \cdot f \right| \\ &\leq \|u\|_{L^1(\Omega)} \|f\|_{L^\infty(\Omega)}, \end{aligned}$$

i.e., every Lebesgue-integrable function can be seen as a continuous real linear functional over $L^\infty(\Omega)$. Also, if $f \in L^2(\Omega)$, the variational solution ζ_f exists and we can write

$$\int_\Omega (\nabla \zeta_f \cdot \nabla z + V\zeta_f z) = \int_\Omega f \cdot z$$

for every $z \in W_0^{1,2}(\Omega) \cap L^2(\Omega; V)$, and since variational solutions are distributional solutions, we have:

$$\begin{aligned}\int_\Omega \widehat{\zeta_f}\, d\mu &= \int_\Omega (\nabla\zeta_f \cdot \nabla z + V\zeta_f z)\\ &= \int_\Omega f \cdot z,\end{aligned}$$

where $\widehat{\zeta_f}$ is the precise representative of ζ_f (note that $\widehat{\zeta_f}$ is defined pointwise in Ω). If we want a function $u \in L^1(\Omega)$ that can be represented by the integral with respect to μ, we require

$$\int_\Omega u \cdot f = \langle u, f\rangle_{(L^\infty(\Omega))'} = \int_\Omega \widehat{\zeta_f}\, d\mu$$

for every $f \in L^\infty(\Omega)$.

Then we define:

Definition 2.38. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty, smooth and bounded open set, $V : \Omega \to [0, \infty]$ be a Borel-measurable function and $\mu \in \mathcal{M}(\Omega)$. We say that $u \in L^1(\Omega)$ is a duality solution of the Schrödinger problem with potential V and density μ if*

$$\int_\Omega u \cdot f = \int_\Omega \widehat{\zeta_f}\, d\mu$$

for every $f \in L^\infty(\Omega)$, where $\widehat{\zeta_f}$ is the precise representative of the variational solution of the Schrödinger problem with potential V and density f.

For more about the duality solution and its first introduction, see Malusa and Orsina [18], Section 5.

The duality solution always exists and is unique (Proposition 3.3, [24]). To see that we can use duality solutions as candidates for distributional solutions of the problem, we need the following result, whose proof is Proposition 3.3, [22]:

Proposition 2.39. *Every distributional solution to the Schrödinger problem with Borel-measurable potential $V : \Omega \to [0, \infty]$ and density $\mu \in \mathcal{M}(\Omega)$ is a duality solution to the same problem.*

Now, let $u, v \in L^1(\Omega)$ be distributional solutions to the Schrödinger problem with potential V and density μ. Proposition 2.39 tell us that u and v are duality solutions of this problem, and the uniqueness tells us that $u = v$ in $L^1(\Omega)$. This proves the uniqueness of distributional solutions.

While duality solutions always exist they are not very well behaved: two Schrödinger problems with the same potential, but different measures can have the same duality solution. Still, this concept is useful when one wants to use Perron's method of subharmonic functions (see Section 5 of [24]).

So far, we have found a way to rigorously represent the expression $-\Delta u + Vu$ as $-\Delta \mathcal{J}_u + \mathcal{J}_{Vu}$ using the theory of distributions, which allowed us to compare it to the distribution $\mathcal{I}_\mu$, defined by a measure μ. Since $\mathcal{M}(\Omega) \subsetneq \mathcal{D}'(\Omega)$, our objective in the rest of this section is to give a formal meaning to the particular case where

$$-\Delta u + Vu \in \mathcal{M}(\Omega),$$

when $u, Vu \in L^1(\Omega)$. Our first step is finding a way to represent elements of $\mathcal{M}(\Omega)$ as linear functionals. First of all we define the normed vector space $(C_0(\overline{\Omega}), \|\cdot\|_\infty)$, where

$$C_0(\overline{\Omega}) := \left\{ f : \overline{\Omega} \to \mathbb{R}; f \text{ is continuous and } f|_{\partial\Omega} \equiv 0 \right\},$$

and $\|\cdot\|_\infty$, defined for every $\phi \in C_0(\overline{\Omega})$ as

$$\|\phi\|_\infty := \sup \left\{ |\phi(x)|; x \in \overline{\Omega} \right\},$$

is called the supremum norm. We denote the dual of $(C_0(\overline{\Omega}), \|\cdot\|_\infty)$ simply by $(C_0(\overline{\Omega}))'$.

Let $\mathcal{T}$ be the operator that assigns for each $\mu \in \mathcal{M}(\Omega)$ a functional $\mathcal{T}_\mu$, defined over $C_0(\overline{\Omega})$, by

$$\begin{aligned} \mathcal{T}_\mu : C_0(\overline{\Omega}) &\to \mathbb{R} \\ \phi \quad &\mapsto \int_\Omega \phi \, d\mu. \end{aligned}$$

The following is a very important result, which is proved in [15], Proposition 7.17, and is a version of the Riesz Representation Theorem:

Theorem 2.40. *(I) For every $\mu \in \mathcal{M}(\Omega)$,*

$$\mathcal{T}_\mu \in \left(C_0(\overline{\Omega})\right)',$$

i.e., $\mathcal{T}_\mu$ is a continuous linear functional defined over $C_0(\overline{\Omega})$.

(II) The operator $\mathcal{T}$ is a bijection and satisfies

$$\begin{aligned} T_{(\alpha\mu+\lambda)} &= \alpha \mathcal{T}_\mu + \mathcal{T}_\lambda, \text{ and} \\ \|\mu\|_{\mathcal{M}(\Omega)} &= \|\mathcal{T}_\mu\|_{C_0(\overline{\Omega})} \end{aligned}$$

for every $\mu, \lambda \in \mathcal{M}(\Omega)$, and $\alpha \in \mathbb{R}$, i.e., $\mathcal{T}$ is a bijective isometric isomorphism between $\mathcal{M}(\Omega)$ and $(C_0(\overline{\Omega}))'$.

This theorem is the reason why we can identify the spaces $\mathcal{M}(\Omega)$ and $(C_0(\overline{\Omega}))'$ and are able to talk about weak* convergence in the space of finite Borel measures (see Section 2.1, Definition 2.8). It also allows us to define another norm over $\mathcal{M}(\Omega)$ (equivalent to the norms we have defined in Section 2.1) using the usual norm defined for continuous linear functionals:

$$\begin{aligned}\|\mu\|_{\mathcal{M}(\Omega)} &:= \|\mathcal{T}_\mu\|_{(C_0(\overline{\Omega}))'} \\ &= \sup\left\{\int_\Omega \phi \; d\mu; \phi \in C_0\left(\overline{\Omega}\right) \text{ and } \|\phi\|_\infty \leq 1\right\}.\end{aligned}$$

This theorem also gives us a more complete version of the diagram

$$\begin{array}{ccccc} L^1(\Omega) & \xrightarrow{\quad\mathcal{H}\quad} & \mathcal{H}(L^1(\Omega)) \subsetneq \mathcal{M}(\Omega) & \xleftrightarrow{\ \mathcal{T}\ } & \left(C_0(\overline{\Omega})\right)' \\ & \searrow^{\mathcal{J}} & \downarrow \mathcal{I} & & \\ & & \mathcal{J}(L^1(\Omega)) = \mathcal{I}(\mathcal{M}(\Omega)) \subsetneq \mathcal{D}'(\Omega) & & \end{array}$$

Now, let us take $u \in C_0^2(\overline{\Omega})$. First of all, for such functions, the expression $-\Delta u + Vu$ is Lebesgue-integrable, since the Laplacian of u, $\Delta u : \Omega \to \mathbb{R}$, is a real function defined everywhere on Ω and $\Delta u \in C_0(\overline{\Omega})$. That gives us:

$$-\Delta u + Vu \in L^1(\Omega).$$

Now we want to find a way to identify this $L^1(\Omega)$ function with an arbitrary measure $\mu \in \mathcal{M}(\Omega)$.

For every u we define:

$$\begin{aligned}F_u : C_0^2(\overline{\Omega}) &\to \mathbb{R} \\ \phi &\mapsto \int_\Omega u(-\Delta\phi + V\phi).\end{aligned}$$

Then, using the Green's Formula (Gauss-Green Theorem in [13], Appendix C, Theorem 1), for every nonzero $\phi \in C_0^2(\overline{\Omega})$:

$$\begin{aligned}|F_u(\phi)| = \left|\int_\Omega u(-\Delta\phi + V\phi)\right| &= \left|\int_\Omega \phi(-\Delta u + Vu)\right| \\ &\leq \|\phi\|_{L^\infty(\Omega)} \left|\int_\Omega (-\Delta u + Vu)\right|,\end{aligned}$$

which shows that $F_u \in (C_0^2(\overline{\Omega}))'$ (since $\|\phi\|_{L^\infty(\Omega)} \leq \|\phi\|_{C_0^2(\overline{\Omega})}$ for the usual $C_0^2(\overline{\Omega})$ norm).

Now, since $\overline{C_0^2(\overline{\Omega})}^{\|\cdot\|_{C_0(\overline{\Omega})}} = C_0(\overline{\Omega})$, i.e., $C_0^2(\overline{\Omega})$ is dense on $C_0(\overline{\Omega})$, we can find, for every $u \in C_0^2(\overline{\Omega})$, a unique functional $\overline{F_u} \in (C_0(\overline{\Omega}))'$ such that, for every $\phi \in C_0^2(\overline{\Omega})$:

$$\begin{aligned}\overline{F_u}(\phi) &= F_u(\phi)\\ &= \int_\Omega u(-\Delta\phi + V\phi).\end{aligned}$$

Moreover, the right side of expression (1) is a finite Borel measure. From Theorem 2.40, there exists a unique $\mathcal{T}_\mu \in (C_0(\overline{\Omega}))'$ such that

$$\overline{\mathcal{T}}_\mu(\phi) = \int_\Omega \phi \, d\mu$$

for every $\phi \in C_0(\overline{\Omega})$.

Now, we can deal with elements of the same space: $\overline{F_u}, \mathcal{T}_\mu \in (C_0(\overline{\Omega}))'$. If $\overline{F_u} = \mathcal{T}_\mu$, then, in particular

$$\int_\Omega u(-\Delta\phi + V\phi) = F_u(\phi) = \overline{F_u}(\phi) = \mathcal{T}_\mu(\phi) = \int_\Omega \phi \, d\mu \tag{2}$$

for every $\phi \in C_0^2(\overline{\Omega})$.

This allows us to give a meaning to expression (1) by associating the functional $\overline{F_u}$ with $-\Delta u + Vu$ and $\mathcal{T}_\mu$ with μ, by these expressions.

When $\overline{F_u} = \mathcal{T}_\mu$, expression (2) holds, for every $\phi \in C_0^2(\overline{\Omega})$, and we can identify $-\Delta u + Vu \in L^1(\Omega)$ with $\mu \in \mathcal{M}(\Omega)$ using the notation

$$-\Delta u + Vu = \mu \text{ in } (C_0(\overline{\Omega}))'.$$

Then, motivated by this, given a bounded set Ω, $u \in L^1(\Omega)$ and a Borel-measurable function $V : \Omega \to [0, \infty]$ such that $Vu \in L^1(\Omega)$, we write

$$-\Delta u + Vu \in \mathcal{M}(\Omega)$$

if there exists a unique (given by the uniqueness of the solution to the equation with potential V and density μ) $\mu_{u,V} \in \mathcal{M}(\Omega)$ such that

$$\int_\Omega u(-\Delta\phi + V\phi) = \int_\Omega \phi \, d\mu_{u,V} \tag{3}$$

for every $\phi \in C_c^\infty(\Omega)$.

In particular, if $-\Delta u + Vu \in \mathcal{M}(\Omega)$ then, since $Vu \in L^1(\Omega) \subset \mathcal{M}(\Omega)$, we can also write $-\Delta u \in \mathcal{M}(\Omega)$.

It is important to notice that even though (3) is the same expression we look for when searching for a distributional solution to the Schrödinger equation with potential V and density μ, as seen in Definition 2.33, solving the equation requires a fundamentally different approach.

In the first case we are given a measurable potential and a measure for which we want to find an integrable distributional solution u, and in the second case we have functions $u, Vu \in L^1(\Omega)$ and we want to find out whether the distribution $-\Delta u + Vu$ is a measure or not. This difference is made clear when we use the notation $\mu_{u,V}$ to explicit the dependency of this measure.

2.5 Maximum principle and zero-sets

In this section we deal with the maximum principle for the operator $-\Delta + V$. First of all, the weak maximum principle holds for every Borel-measurable potential $V : \Omega \to [0, \infty]$. We want to study the cases where the strong maximum principle fails to hold to describe the subsets of Ω where w_f satisfies

$$\begin{cases} -\Delta w_f + V w_f = f \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases}$$

for some $f \in L^2(\Omega)$ nonnegative, but $w_f = 0$. To make sense of the pointwise values we take the precise representatives of these solutions. We then have the first definition (from [24]):

Definition 2.41. *We denote by S the subset of Ω defined as*

$$S = \left\{ x \in \Omega; \widehat{\zeta_1}(x) = 0 \right\},$$

where $\widehat{\zeta_1}$ is the precise representative of the variational solution of the Schrödinger problem with potential V and data $f \equiv 1$. We call this set the torsion function zero-set.

Note that S is a Sobolev-closed set by definition and that the precise representative of ζ_1 is defined over Ω. Too see why this is true it is enough to note that $\zeta_1 \cdot V \in L^1(\Omega)$ and, using Proposition 8.1 of [23], we obtain

$$-\Delta\zeta_1 + V\zeta_1 \leq 1.$$

This implies that ζ_1 is the difference, almost everywhere, between a continuous and a bounded superharmonic function. From [25], Lemma 8.10, every $x \in \Omega$ is a Lebesgue point of ζ_1.

We can also define more generally:

Definition 2.42. *We define the universal zero-set Z associated to the operator $-\Delta + V$ as the set of points $x \in \Omega$ with the following property: for every nonnegative $f \in L^\infty(\Omega)$ such that*

$$-\Delta w_f + V w_f = f \text{ in } \Omega$$

holds in the distributional sense, for some $w_f \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, such a solution satisfies

$$\widehat{w_f}(x) = 0.$$

It is clear from the definitions that both of these sets are strongly dependent of the potential V. But we have decided to not explicit this relationship in the notation, since our potential is arbitrary most of the time. We will though, clarify the dependency when necessary, to avoid confusion.

We present first results concerning the relationship between S and Z (see Section 4 from [24]).

Lemma 2.43. *The set S can be characterized by:*

$$S = \left\{ x \in \Omega; \widehat{\zeta_f}\,(x) = 0 \text{ for every } f \in L^\infty(\Omega) \right\}.$$

Proof: It is evident that

$$\left\{ x \in \Omega; \widehat{\zeta_f}\,(x) = 0 \text{ for every } f \in L^\infty(\Omega) \right\} \subset S,$$

since $f \equiv 1$ is an $L^\infty(\Omega)$ function. Now let us prove the other inclusion.

Taking a function $f \in L^\infty(\Omega)$, we have the following, in the variational sense:

$$\begin{cases} -\Delta\zeta_f + V\zeta_f = f \text{ in } \Omega, \\ \zeta_f = 0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta\zeta_1 + V\zeta_1 = 1 \text{ in } \Omega, \\ \zeta_1 = 0 \text{ on } \partial\Omega. \end{cases} \tag{5}$$

Using the linearity of the Schrödinger problem, we obtain, from (5):

$$\begin{cases} -\Delta\left(\|f\|_{L^\infty(\Omega)}\,\zeta_1\right) + V\left(\|f\|_{L^\infty(\Omega)}\,\zeta_1\right) = \|f\|_{L^\infty(\Omega)} \text{ in } \Omega, \\ \|f\|_{L^\infty(\Omega)}\,\zeta_1 = 0 \text{ on } \partial\Omega. \end{cases} \tag{6}$$

We now take the sum and the difference between (4) and (6) to have, in the variational sense:

$$\begin{cases} -\Delta\left(\|f\|_{L^\infty(\Omega)}\,\zeta_1 - \zeta_f\right) + V\left(\|f\|_{L^\infty(\Omega)}\,\zeta_1 - \zeta_f\right) = \|f\|_{L^\infty(\Omega)} - f & \text{in } \Omega, \\ \|f\|_{L^\infty(\Omega)}\,\zeta_1 - \zeta_f = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta\left(\|f\|_{L^\infty(\Omega)}\,\zeta_1+\zeta_f\right)+V\left(\|f\|_{L^\infty(\Omega)}\,\zeta_1+\zeta_f\right)=\|f\|_{L^\infty(\Omega)}+f & \text{in } \Omega, \\ \|f\|_{L^\infty(\Omega)}\,\zeta_1+\zeta_f=0 & \text{on } \partial\Omega. \end{cases}$$

Then we have that $\|f\|_{L^\infty(\Omega)}\zeta_1-\zeta_f$ is the variational solution of the Schrödinger problem with potential V and data $\|f\|_{L^\infty(\Omega)}-f$, and $\|f\|_{L^\infty(\Omega)}\zeta_1+\zeta_f$ is the variational solution of the Schrödinger problem with potential V and data $\|f\|_{L^\infty(\Omega)}+f$.

A property following from the definition of the norm in $L^\infty(\Omega)$ gives us

$$\|f\|_{L^\infty(\Omega)}-f\geq 0 \text{ and } \|f\|_{L^\infty(\Omega)}+f\geq 0.$$

Then, since our data is nonnegative, we can use the weak maximum principle to conclude

$$\|f\|_{L^\infty(\Omega)}\,\zeta_1-\zeta_f\geq 0 \text{ and } \|f\|_{L^\infty(\Omega)}\,\zeta_1+\zeta_f\geq 0,$$

which implies

$$|\zeta_f|\leq\|f\|_{L^\infty(\Omega)}\,\zeta_1.$$

The same estimate is satisfied by the precise representatives (because of the monotonicity of limits and integrals), from which we have

$$S\subset\left\{x\in\Omega;\widehat{\zeta_f}\,(x)=0 \text{ for every } f\in L^\infty(\Omega)\right\}.$$

This concludes the proof. ■

Proposition 2.44. *For every $V:\Omega\to[0,+\infty]$, we have $S\subset Z$*

Proof: If $x\in S$ then, from the characterization of S given by the previous proposition

$$\widehat{\zeta_f}(x)=0$$

for every $f\in L^\infty(\Omega)$.

In particular, this holds for every nonnegative $f\in L^\infty(\Omega)$. Since ζ_f is a minimizer of the functional E (see Definition 2.36), then it is also a duality solution of the Schrödinger problem with potential V and data f.

Now, if the Schrödinger problem with potential V and data f (in particular with $f\geq 0$) has a distributional solution w, then this solution is equal to ζ_f (distributional solutions are duality solutions). Therefore we would have

$$\widehat{w}(x)=\widehat{\zeta_f}(x)=0,$$

that is, $x\in Z$. ■

Examples. ***(I)*** *Let $V \in L^1(\Omega)$. Then we have that $cap_{W^{1,2}}(Z) = 0$. Thus, if μ is $cap_{W^{1,2}}$-diffuse, then the Schrödinger problem has a distributional solution, since that implies $\mu(Z) = 0$ (see Theorem 1.1, [24]).*

(II) *Let $V \in L^p(\Omega)$ for $p > 1$. In this case we use the maximum principle from [22] that says that if*

$$-\Delta u + Vu \geq 0,$$

and $\widehat{u} = 0$ in a compact set with positive $cap_{W^{2,p}}$, then $\widehat{u} = 0$ in Ω. This implies that $cap_{W^{2,p}}(Z) = 0$. Then, if μ is $cap_{W^{2,p}}$-diffuse, the Schrödinger problem has a distributional solution.

The introduction of the set S allows us to have the following result, that gives us a distribution solution of the Schrödinger equation:

Proposition 2.45. *Let $u \in L^1(\Omega)$ be a duality solution of the Schrödinger problem with potential V and nonnegative density μ. Then u is the distributional solution of the Schrödinger equation with potential V and density $\mu\lfloor_{\Omega\setminus S} - \lambda$, i.e,*

$$-\Delta u + Vu = \mu\lfloor_{\Omega\setminus S} - \lambda \text{ in } \Omega,$$

where $\lambda \in \mathcal{M}(\Omega)$ is nonnegative, $cap_{W^{1,2}}$-diffuse, and such that $\lambda\lfloor_{\Omega\setminus S} = 0$.

The proof of this proposition can be found in Proposition 4.1 from [24].

The next results are essential to prove the close relationship between the set S and duality solutions.

Proposition 2.46. *Let $\mu \in \mathcal{M}(\Omega)$, and $u \in L^1(\Omega)$ be the duality solution of the Schrödinger problem with potential V and density μ. Then u is also the duality solution of the Schrödinger problem with potential V and density $\mu\lfloor_{\Omega\setminus S}$*

Proof: Using Lemma 2.43 we have

$$\begin{aligned}\int_\Omega uf &= \int_\Omega \widehat{\zeta_f}\, d\mu \\ &= \int_\Omega \widehat{\zeta_f}\, d\mu\lfloor_{\Omega\setminus S}\end{aligned}$$

for every $f \in L^\infty(\Omega)$. This proves the result. ∎

Proposition 2.47. *Let $\mu \in \mathcal{M}(\Omega)$. Then*

$$\mu = 0 \text{ in } \Omega\setminus S$$

if, and only if,

$$\int_{\Omega} \widehat{\zeta_f}\ d\mu = 0$$

holds for every $f \in L^{\infty}(\Omega)$.

Proof: First, let us suppose that $\mu = 0$ in $\Omega \backslash S$. From Lemma 2.43 we know that $\widehat{\zeta_f} = 0$ in S for every $f \in L^{\infty}(\Omega)$. We then have:

$$\begin{aligned} \int_{\Omega} \widehat{\zeta_f}\ d\mu &= \int_{\Omega \backslash S} \widehat{\zeta_f}\ d\mu + \int_{S} \widehat{\zeta_f}\ d\mu \\ &= 0. \end{aligned}$$

This concludes the first part of the proof.

Now let us suppose that $\int_{\Omega} \widehat{\zeta_f}\ d\mu = 0$ for every $f \in L^{\infty}(\Omega)$. ■

Corollary 2.48. *Let $\mu, \lambda \in \mathcal{M}(\Omega)$, and $u, v \in L^1(\Omega)$ be such that u is the duality solution of the Schrödinger problem with potential V and density μ, and v is the duality solution of the Schrödinger problem with potential V and density λ. Then*

$$\mu = \lambda \ \textit{in}\ \Omega \backslash S$$

if, and only if,

$$u = v \ \textit{almost everywhere in}\ \Omega.$$

Proof: Let us suppose that $\mu = \lambda$ in $\Omega \backslash S$. Then $\mu - \lambda = 0$ in $\Omega \backslash S$ and, from Proposition 2.47, this holds if, and only if

$$\int_{\Omega} \widehat{\zeta_f}\ d(\mu - \lambda) = 0$$

for every $f \in L^{\infty}(\Omega)$. This is equivalent to

$$\begin{aligned} \int_{\Omega} uf &= \int_{\Omega} \widehat{\zeta_f}\ d\mu \\ &= \int_{\Omega} \widehat{\zeta_f}\ d\lambda \\ &= \int_{\Omega} vf \end{aligned}$$

for every $f \in L^{\infty}(\Omega)$. And we have that

$$\int_{\Omega} uf = \int_{\Omega} vf$$

holds for every $f \in L^\infty(\Omega)$ if, and only if, $u = v$ almost everywhere in Ω, as we wanted to prove. ■

Now we show some results concerning a decomposition of $\Omega \backslash S$, a maximum principle and a new way to find duality solutions. We present the results without proof, that can be found in the Section 9 of [24], in particular in the proof of Theorem 1.1 from said article.

Definition 2.49. *Let $X \subset \mathbb{R}^N$. We say that a countable family $(A_i)_{i\in I}$, of subsets of X, is a Sobolev-connected-open decomposition of X, if*
(i) the sets $(A_i)_{i\in I}$ are disjoint pairwise,
(ii)

$$X = \bigcup_{i\in I} A_i,$$

(iii) for every $i \in I$, A_i is a Sobolev-connected-open set. In this case we say that X has a Sobolev-connected-open decomposition.

Theorem 2.50. *The set $\Omega \backslash S$ has a Sobolev-connected-open decomposition $(D_i)_{i\in I}$.*

Theorem 2.51. *Let $f \in L^\infty(\Omega)$ be a nonnegative function, $u \in L^1(\Omega)$ be the duality solution of the Schrödinger problem with potential V and density $f\ dx$, and $(D_i)_{i\in I}$ be the Sobolev-connected-open decomposition of $\Omega \backslash S$. Then we have*

$$\text{either } \widehat{u} \equiv 0 \text{ in } D_i \text{ or } \widehat{u} > 0 \text{ in } D_i$$

for every $i \in I$.

Another consequence of the decomposition of the set $\Omega \backslash S$ is the following:

Theorem 2.52. *Let $u \in L^1(\Omega)$ be the duality solution of the Schrödinger problem with potential V and density $\mu \in \mathcal{M}(\Omega)$, and $(D_i)_{i\in I}$ be the Sobolev-connected-open decomposition of $\Omega \backslash S$.*

Then, for every $i \in I$, we have that $u\chi_{D_i} \in L^1(\Omega)$ is the duality solution of the Schrödinger problem with potential V and density $\mu\lfloor_{D_i}$.

In the next result we use the theory of Sobolev-open and Sobolev-closed presented in Section 2.3. and we present the outline of the proof:

Proposition 2.53. *Let $(\mu_k) \subset \mathcal{M}(\Omega)$ be an equidiffuse sequence of measures such that $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$ and $\mu_k \geq 0$ in $\Omega \backslash S$ for every $k \in \mathbb{N}$. Then*

$$\mu \geq 0 \text{ in } \Omega \backslash S.$$

Proof: By definition, we have a nonnegative function

$$\zeta_1 \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$

such that $\widehat{\zeta_1}$ is defined pointwise in Ω and

$$S = \left\{x \in \Omega; \widehat{\zeta_1}(x) = 0\right\}.$$

Thus S is a Sobolev-closed set. We then have that $\Omega \backslash S$ is Sobolev-open set and the desired result. ■

To stablish results concerning the existence of distributional solutions of the Schrödinger problem we use the set Z and the following result that characterizes distributional solution with nonnegative measures:

Theorem 2.54. *For every $V : \Omega \to [0, +\infty]$, the Schrödinger problem with potential V and nonnegative density μ has a distributional solution if, and only if,*

$$\mu(Z) = 0.$$

This is Theorem 1.4 from [24].

3 Methods of approximation

In this chapter we present our contributions to the theory of elliptic differential problems with measure data. We study two different methods of approximation in our differential problem.

The first method is an approximation on the potential V. We define a type of sequence that generalizes the concept of truncation and apply it to the potential V. Then, using some of the methods we gathered in the previous chapters, we prove the convergence of a sequence of distributional solutions, related to a nonnegative measure μ, to a summable function u^* that we use to define the reduced measure.

The reduced measure has some properties related to good measures, namely it it the biggest good measure smaller than μ. Also, the function $u^* \in L^1(\Omega)$, the distributional solution of the problem $[V;\mu^*]$, is the biggest subsolution of $[V;\mu]$.

Using this property and a characterization of nonnegative good measures from [22] we find a representation of the reduced measure that uses the universal zero-set Z.

We then define the reduced measure operator and prove some linear functional properties about it.

In the second part of the chapter we use another type of approximation, this time taken on the measure. From a bounded sequence of measures we can define the reduced limit: a measure $\mu^\#$ that has as a distributional solution a summable function $u^\#$ that is the limit of the approximation taken on the measure μ.

We can prove good properties of this approximation with relation to the total variation norm and the weak* limit of measures.

The properties we prove are the monotonicity of the reduced limit and the lower semicontinuity. Since these results cannot be achieved in the entire domain Ω, we prove them on the set $\Omega\backslash S$, mainly using decomposition and measure methods. These results show the importance of the study of the zero-sets and how they appear naturally when we study properties of the problem. Also, the dependence of the potential V is very explicit in these contexts.

These methods were first introduced by Malusa and Orsina in [18] to a problem involving a general elliptic operator (and later adapted and expanded for the nonlinear problem involving the Laplacian operator, by Brezis, Marcus and Ponce in [6] and Marcus and Ponce in [20]) but without dealing with the case of a singular potential. We adapt this method to our type of problem and obtain similar results.

3.1 Reduced measure

In this section we introduce an adaptation to the theory of linear equations with singular potential of the concept of the reduced measure, closely related to the concept of sub-solution, that we define in the following:

Definition 3.1. *Let $V : \Omega \to [0, \infty]$ be a Borel-measurable function and let $\mu \in \mathcal{M}(\Omega)$. We say that $v \in L^1(\Omega) \cap W_0^{1,1}(\Omega)$ is a sub-solution of the Schrödinger problem with potential V and density μ if*

$$\int_\Omega v(-\Delta\phi + V\phi) \leq \int_\Omega \phi \, d\mu$$

for every $\phi \in C_c^\infty(\Omega)$.

We also define the concept of truncated-type sequence as the following:

Definition 3.2. *Let $V : \Omega \to [0, \infty]$ be a Borel-measurable function. We say that a sequence $(V_k)_{k\in\mathbb{N}}$ is a truncated-type sequence of V if*

(i) for every $k \in \mathbb{N}$, $V_k : \Omega \to [0, \infty]$ is bounded, Borel-measurable, and satisfies $V_k \leq V_{k+1}$;

(ii) $V_k \to V$ pointwise in Ω.

Next we give the example that motivated this definition:

Example. *Given any Borel-measurable $V : \Omega \to [0, \infty]$, we can define a truncated-type sequence of V by setting, for each $k \in \mathbb{N}$, the function defined pointwise by*

$$V_k(x) := \min \{V(x), k\}.$$

This is called the truncation of V at k. It is easy to see that the sequence $(V_k)_{k\in\mathbb{N}}$ thus defined is a truncated-type sequence of V.

Let us present the first technical result (Proposition 14.12 from [25]), a variation of the Lebesgue's decomposition theorem:

Theorem 3.3. *If M is a positive Borel semimeasure and $\mu \in \mathcal{M}(\Omega)$, then we ca uniquely decompose μ as $\mu = \alpha + \sigma$ with $\alpha, \sigma \in \mathcal{M}(\Omega)$, where α is concentrated with respect to M, and σ is diffuse with respect to M. We call α the concentrated part of μ with respect to M, and σ the diffuse part of μ with respect to M.*

In our text we will use a specific positive semimeasure called the Newtonian H^1 capacity (sometimes called the Sobolev $W^{1,2}$ capacity), that we denote simply by *cap*, and is defined on a compact set $K \subset \mathbb{R}^N$ by

$$cap(K) = \inf \left(\|\phi\|^2_{W^{1,2}(\mathbb{R}^N)}; \phi \in C_c^\infty(\mathbb{R}^N), \phi \geq 0 \text{ and } \phi > 1 \text{ in } K \right),$$

and defined on every subset of $\mathbb{R}^N$ by inner regularity. (For more about the Sobolev capacity and related properties, see Section 2.2)

Given $\mu \in \mathcal{M}(\Omega)$ we shall denote by μ_c the concentrated part of μ with respect to *cap*, and μ_d the diffuse part of μ with respect to *cap*.

Let us present the first technical lemmas. The first is the Maximum Principle for weak solutions, and the second is Kato's inequality for potentials:

Lemma 3.4. *Let $u \in L^1(\Omega)$ with $-\Delta u \geq 0$ in the sense of distributions, i. e., for every $\phi \in C_0^\infty(\overline{\Omega})$:*

$$\int_\Omega u(-\Delta\phi) \geq 0.$$

Then $u \geq 0$ almost everywhere in Ω.

This proof can be found in [25], Proposition 6.1.

Lemma 3.5. *Let $u \in L^1(\Omega)$ such that $\Delta u \in \mathcal{M}(\Omega)$. Then we have*

(i) $\Delta u^+ \in \mathcal{M}(\omega)$,

(ii) $(-\Delta u^+)_c = (-\Delta u)_c^+$ *in* ω,

(iii) $(\Delta u^+)_d \geq \chi_{[u\geq 0]}(\Delta u)_d$ *in* ω

for every open set $\omega \subset \Omega$.

This result can be found in Section 6.2 of [25].

Our first result is the following:

Proposition 3.6. *Let $V_1, V_2 : \Omega \to [0,\infty]$ be Borel-measurable functions with $V_1 \leq V_2$ and such that $\mu \in \mathcal{G}(V_1)$ and $\mu \in \mathcal{G}(V_2)$ and let us denote by u_1 and u_2 the distributional solutions of $[V_1;\mu]$ and $[V_2;\mu]$, respectively. Then $u_2 \leq u_1$ almost everywhere in Ω.*

Proof. First of all, since in the sense of distributions

$$-\Delta u_1 + V_1 u_1 = \mu$$

and

$$-\Delta u_2 + V_2 u_2 = \mu,$$

then by subtracting the first equation from the second one, we have

$$-\Delta(u_2 - u_1) = (V_1 u_1 - V_2 u_2). \tag{7}$$

in the sense of distributions.

Since u_1 and u_2 are distributional solutions to Schrödinger problems, from definition we have $u_1, u_2, V_1u_1, V_2u_2 \in L^1(\Omega)$, and from (7) we know that $\Delta(u_2 - u_1)$ is the difference of two $L^1(\Omega)$, and therefore, it is a measure in $\mathcal{M}(\Omega)$.

Then we can apply Kato's inequality to $(u_2 - u_1)$ and obtain:

(i) $(\Delta(u_2 - u_1)^+)_c = (\Delta(u_2 - u_1))_c^+$, and

(ii) $(\Delta(u_2 - u_1)^+)_d \geq \chi_{[u_2 - u_1 \geq 0]}(\Delta(u_2 - u_1))_d$.

But we can rewrite $(\Delta(u_2 - u_1))_c^+$ as $(V_2u_2 - V_1u_1)_c^+$ because of (7), and since $(V_2u_2 - V_1u_1) \in L^1(\Omega)$, then $(\Delta(u_2 - u_1))_c^+ = (V_2u_2 - V_1u_1)_c^+ = 0$. This follow from the fact that functions in $L^1(\Omega)$ are concentrated with respect to the Lebesgue measure and that the Lebesgue measure is concentrated with respect to *cap*.

We also have that

$$\begin{aligned} \chi_{[u_2 - u_1 \geq 0]}(\Delta(u_2 - u_1))_d &= \chi_{[u_2 \geq u_1]}(V_2u_2 - V_1u_1)_d \\ &\geq 0, \end{aligned}$$

since this characteristic function is only nonzero when $u_2 \geq u_1$, and that means, since $V_1 \leq V_2$, that $V_2u_2 - V_1u_1 \geq 0$, and the diffuse part of a nonnegative measure is nonnegative.

We can then write

$$\begin{aligned} \Delta(u_2 - u_1)^+ &= (\Delta(u_2 - u_1)^+)_c + (\Delta(u_2 - u_1)^+)_d \\ &\geq 0 \\ &\implies -\Delta(-(u_2 - u_1)^+) \geq 0. \end{aligned}$$

in the sense of distributions. Then the maximum principle for weak solutions gives us $-(u_2 - u_1)^+ \geq 0$. Then

$$\begin{aligned} -(u_2 - u_1)^+ \geq 0 &\implies (u_2 - u_1)^+ \leq 0 \\ &\implies (u_2 - u_1)^+ = 0 \\ &\implies u_2 \leq u_1 \text{ almost everywhere in } \Omega. \end{aligned}$$

■

In an analogous way we can prove the following result:

Proposition 3.7. *Let $V_1, V_2 : \Omega \to [0, \infty]$ be Borel-measurable functions with $V_1 \leq V_2$ and such that there exist $u_1, u_2 \in L^1(\Omega)$ with $V_1u_1, V_2u_2 \in L^1(\Omega)$ and*

$$-\Delta u_2 + V_2u_2 \leq -\Delta u_1 + V_1u_1$$

in the sense of distributions, then $u_2 \leq u_1$ almost everywhere in Ω.

With this result in hand we can prove that the Schrödinger problem $[V_k; \mu]$, where $(V_k)_{k\in\mathbb{N}}$ is a truncated-type sequence of V, has a distributional solution.

This happens because we can always find real numbers α and β such that β is not in the spectrum of $-\Delta$ and

$$\alpha < V_k < \beta.$$

We can then use a result from Stampacchia (Proposition 9.1 from [27]) to see that the problems $[\alpha; \mu]$ and $[\beta; \mu]$ have distributional solutions, that we can denote by v and w.

From our last result, we have $w \leq v$ almost everywhere in Ω, and w is a subsolution of the problem $[V_k; \mu]$, while v is a supersolution. Then, applying Proposition 22.7 from [25], we have our desired solution to $[V_k; \mu]$. We denote this distributional solution by u_k and our objective is to study the sequence $(u_k)_{k\in\mathbb{N}}$.

First of all we have $0 \leq V_k \leq V_{k+1}$ for every $k \in \mathbb{N}$. Then, from the previous proposition we have

$$u_{k+1} \leq u_k \text{ almost everywhere in } \Omega.$$

In other words, $(u_k)_{k\in\mathbb{N}}$ is monotone.

Let us now prove that the sequence $(u_k)_{k\in\mathbb{N}}$ is convergent. In order to do this we will need our measure μ to be nonnegative since we are using the Monotone Convergence Theorem and Fatou's Lemma. From this point on in this section, let us consider only nonnegative finite Borel measures, unless the contrary is explicitly stated.

Our aim is then to prove the following main theorem:

Theorem 3.8. *For every nonnegative $\mu \in \mathcal{M}(\Omega)$ and Borel-measurable $V : \Omega \to [0, \infty]$, there exists a measure $\mu^* \in \mathcal{M}(\Omega)$, called the reduced measure of $[V; \mu]$, that satisfies:*

(i) μ^* is a good measure, that is, there exists the distributional solution of the problem $[V; \mu^*]$, say $u^* \in L^1(\Omega)$;

(ii) u^* is a subsolution of $[V; \mu]$ and every subsolution $v \in L^1(\Omega)$ of $[V; \mu]$ satisfies $v \leq u^*$ almost everywhere in Ω;

(iii) $\mu^* \leq \mu$ and for every $\lambda \in \mathcal{G}(V)$ such that $\lambda \leq \mu$, we have $\lambda \leq \mu^*$;

(iv) $\mu^* = \mu\lfloor_{\Omega\setminus Z}$.

We begin with a proposition:

Proposition 3.9. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $u_k \in L^1(\Omega)$ be the distribution solution of the Schrödinger problem $[V_k; \mu]$, where (V_k) is a truncated-type sequence of V. Then there exists a function $u^* \in L^1(\Omega)$ such that*

$$u_k \to u^* \text{ in } L^1(\Omega).$$

In particular, we have $-\Delta u^* + Vu^* \in \mathcal{D}'(\Omega)$.

Proof. First of all, expression (2.3) in [24] gives us

$$\|u_k\|_{L^1(\Omega)} \le C\|\mu\|_{\mathcal{M}(\Omega)}$$

for every $k \in \mathbb{N}$, where the constant C does not depend on the potential V_k.

Since $(u_k)_{k\in\mathbb{N}}$ is monotone, from the Monotone Convergence Theorem, we have that there exists an $L^1(\Omega)$ function u^*, such that $u_k \to u^*$ in $L^1(\Omega)$.

Also from [24], expression (2.2), we have that for every $k \in \mathbb{N}$

$$\|V_k u_k\|_{L^1(\Omega)} \le \|\mu\|_{\mathcal{M}(\Omega)},$$

and using Fatou's Lemma we conclude

$$Vu^* \in L^1(\Omega).$$

Now, since $u^*, Vu^* \in L^1(\Omega)$, the distribution $-\Delta u^* + Vu^*$ is well defined, as we wanted to show. ■

Now we want to define a finite measure depending on the functions u^* and V. For that we need to prove the following:

Proposition 3.10. *Let $u^* \in L^1(\Omega)$ be the function given by the previous proposition. Then, $\Delta u^* + Vu^* \in \mathcal{M}(\Omega)$.*

Proof. First of all, for every $k \in \mathbb{N}$ and every $\psi \in C_0^\infty(\overline{\Omega})$:

$$\left| \int_\Omega u_k \Delta\psi + V_k u_k \psi \right| = \left| \int_\Omega \psi \; d\mu \right|.$$

We also have that $u_k \to u^*$ almost everywhere in Ω and since $V_k \to V$ almost everywhere in Ω, then

$$V_k u_k \to Vu^* \text{ almost everywhere in } \Omega.$$

Using Fatou's Lemma:

$$\begin{aligned}\left|\int_\Omega u^*(-\Delta\psi + V\psi)\right| &= \left|\int_\Omega \liminf_{k\to\infty}(u_k\Delta\psi + V_k u_k\psi)\right| \\ &\leq \liminf_{k\to\infty}\left|\int_\Omega u_k\Delta\psi + V_k u_k\psi\right| \\ &= \left|\int_\Omega \psi \; d\mu\right| \\ &\leq \|\psi\|_\infty \|\mu\|_{\mathcal{M}(\Omega)}.\end{aligned}$$

Using the density of the space $C_0^\infty(\overline{\Omega})$ into the space $C_0(\overline{\Omega})$, this expression holds for every $\psi \in C_0(\overline{\Omega})$. This means that the linear functional $\overline{F} : C_0(\overline{\Omega}) \to \mathbb{R}$, defined as the extension of

$$\int_\Omega u^*(-\Delta\psi + V\psi)$$

to $C_0(\overline{\Omega})$, is continuous. This means, from the representation of measures as functionals, Theorem 2.40, there exists a unique measure $\mu_{u^*,V} \in \mathcal{M}(\Omega)$ such that, for every $\psi \in C_0(\overline{\Omega})$,

$$\overline{F}(\psi) = \int_\Omega \psi \; d\mu_{u^*,V}.$$

In particular, if $\psi \in C_c^\infty(\Omega)$:

$$\int_\Omega u^*(-\Delta\psi + V\psi) = \int_\Omega \psi \; d\mu_{u^*,V}.$$

Therefore $-\Delta u^* + Vu^* \in \mathcal{M}(\Omega)$, as we wanted to prove. ∎

We then prove that the limit u^* does not depend on the chosen truncated-type sequence V_k:

Proposition 3.11. *Let $u^* \in L^1(\Omega)$ be the function defined as the L^1 limit of the sequence $(u_k)_{k\in\mathbb{N}}$ of distributional solutions of the Schrödinger problem $[V_k;\mu]$ with $\mu \in \mathcal{M}^+(\Omega)$. Then u^* is a subsolution of $[V;\mu]$ and if v is a subsolution to $[V;\mu]$, then $v \leq u^*$ almost everywhere in Ω, in other words, u^* is the maximal subsolution to the Schrödinger problem with potential V and density μ.*

Proof. Since, for every $\phi \in C_0^\infty(\overline{\Omega})$ we have

$$\int_\Omega u_k(-\Delta\phi + V_k\phi) = \int_\Omega \phi \; d\mu,$$

$u_k \to u^*$ and $V_k u_k \to Vu$ in Ω almost everywhere in Ω, from Fatou's Lemma:

$$\int_\Omega u^*(-\Delta\phi + V\phi) \leq \int_\Omega \phi \, d\mu,$$

for every nonnegative $\phi \in C_0^\infty(\overline{\Omega})$. Therefore, u^* is a subsolution of $[V;\mu]$.

Now, let $v \in L^1(\Omega)$ be a subsolution of $[V;\mu]$. Since $V_k \leq V$ almost everywhere in Ω we have for every nonnegative $\phi \in C_0^\infty(\overline{\Omega})$:

$$\begin{aligned}\int_\Omega v(-\Delta\phi + V_k\phi) &\leq \int_\Omega v(-\Delta\phi + V\phi)\\ &\leq \int_\Omega \phi \, d\mu\\ &= \int_\Omega u_k(-\Delta\phi + V_k\phi).\end{aligned}$$

This is the same as

$$-\Delta v + V_k v \leq -\Delta u_k + V_k u_k \text{ in the sense of distributions in } \Omega.$$

Then we have $v \leq u_k$ almost everywhere in Ω, and taking the limit we have

$$v \leq u^* \text{ almost everywhere in } \Omega,$$

therefore, every subsolution of $[V;\mu]$ is bounded above by u^*, as we wanted to prove. ■

Recalling the measure $\mu_{u^*,V}$ that is given by the Theorem 2.40 (see the proof of Proposition 3.10), we can then define our main objective of study in this text:

Definition 3.12 (Reduced Measure)**.** *Let $V : \Omega \to [0,\infty]$ be a Borel-measurable function and $\mu \in \mathcal{M}^+(\Omega)$. We define the reduced measure of the Schrödinger problem with potential V and density μ as the measure*

$$\mu^* := \mu_{u^*,V}.$$

Now we prove this very important property about the reduced measure:

Proposition 3.13. *Let $V : \Omega \to [0,\infty]$ be a Borel-measurable function, $\mu \in \mathcal{M}^+(\Omega)$ and $\mu^* \in \mathcal{M}(\Omega)$ be the reduced measure of $[V;\mu]$. Then, we have:*

(i) $\mu^* \leq \mu$;

(ii) if there exists $\lambda \in \mathcal{G}(V)$ such that $\lambda \leq \mu$, then

$$\lambda \leq \mu^*.$$

We will use the decomposition of these measures in their diffuse and concentrated parts with respect to *cap*. Kato's inequality is a natural tool to prove this.

We first present a lemma (Corollary 9 from [6]):

Lemma 3.14. *Let $u \in L^1(\Omega)$, $\Delta u \in \mathcal{M}(\Omega)$ and $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation defined by*

$$T_n(s) := n - (n-s)^+, s \in \mathbb{R}.$$

Then,

$$\Delta T_n(u) \leq \chi_{[u \leq n]}(\Delta u)_d + (\Delta u)_c^+.$$

Proof. We can apply Kato's inequality to the function $v := n - u$, obtaining:

$$\begin{aligned}(\Delta T_n(u))_d &= -(\Delta v^+)_d \\ &\leq -\chi_{[v \geq 0]}(\Delta v)_d \\ &= \chi_{[u \leq n]}(\Delta u)_d\end{aligned}$$

and

$$(\Delta T_n(u))_c = (\Delta u)_c^+.$$

Where we used that the Laplacian of a constant and the concentrated part of a constant (with respect to *cap*) are zero. Then,

$$\begin{aligned}\Delta T_n(u) &= (\Delta T_n(u))_d + (\Delta T_n(u))_c \\ &\leq \chi_{[u \leq n]}(\Delta u)_d + (\Delta u)_c^+.\end{aligned}$$

And the result follow from the regularity of the finite Borel measures. ■

Lemma 3.15. *Let $\mu^* \in \mathcal{M}(\Omega)$ be the reduced measure of $[V;\mu]$. Then*

$$\mu_d - \mu_c^- \leq \mu^*.$$

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be the sequence of distributional solutions to the problem $[V_k;\mu]$, where $(V_k)_{k\in\mathbb{N}}$ is a truncated-type sequence of V.

Since we know that, for every $k \in \mathbb{N}$, $u_k \in L^1(\Omega)$ and $\Delta u_k \in \mathcal{M}(\Omega)$, we can apply the previous lemma to each u_k, thus obtaining

$$\Delta T_n(u_k) \leq \chi_{[u_k \leq n]}(\Delta u_k)_d + (\Delta u_k)_c^+.$$

Now we use that the u_k are solutions, and obtain:

$$(\Delta u_k)_d = V_k u_k - \mu_d$$

and

$$(\Delta u_k)_c = -\mu_c.$$

The previous expressions give us

$$-\Delta T_n(u_k) \geq \chi_{[u_k \leq k]}\mu_d - V_k T_n u_k - \mu_c^-.$$

Now, we also have, from the monotonicity of the sequence $(u_k)_{k\in\mathbb{N}}$:

$$[u^* \leq n] \supset [u_k \leq n] \supset [u_1 \leq n]$$

and

$$\chi_{[u_k \leq n]}\mu_d \geq \chi_{[u_1 \leq n]}\mu_d^+ - \chi_{[u^* \leq n]}\mu_d^-.$$

We then have

$$-\Delta T_n(u_k) + V_k T_n u_k \geq \chi_{[u_1 \leq n]}\mu_d^+ - \chi_{[u^* \leq n]}\mu_d^- - \mu_c^-.$$

Now we can use the Dominated Convergence Theorem, again from the monotonicity of $(u_k)_{k\in\mathbb{N}}$ and conclude:

$$-\Delta T_n(u^*) + V T_n u^* \geq \chi_{[u_1 \leq n]}\mu_d^+ - \chi_{[u^* \leq n]}\mu_d^- - \mu_c^-.$$

Finally, we use that the functions u_1 and u^* are quasicontinuous (from Proposition 18.8 in [25] and since both are in $L^1(\Omega)$ and $\Delta u_1, \Delta u^* \in \mathcal{M}(\Omega)$) and that implies that both sets $[u_1 = \infty]$ and $[u^* = \infty]$ have zero capacity.

Then taking the limit $n \to \infty$ in the previous expression we have

$$\begin{aligned}\mu^* &:= -\Delta u^* + V u^* \\ &\geq \mu_d^+ - \mu_d^- - \mu_c^- \\ &= \mu_d - \mu_c,\end{aligned}$$

as we wanted to prove. ∎

We are ready to prove the main result:

Proof of Psoposition 3.13. We know that since u^* is a subsolution of $[V; \mu]$, this means that

$$\begin{aligned}\mu^* &:= -\Delta u^* + V u^* \\ &\leq \mu.\end{aligned}$$

Then, together with the previous Lemma, we can write

$$\mu_d - \mu_c^- \leq \mu^* \leq \mu,$$

and taking the diffuse part we have

$$\begin{aligned} &\mu_d - \mu_c^- \leq \mu^* \leq \mu \\ &\implies \mu_d - (\mu_c^-)_d \leq (\mu^*)_d \leq (\mu)_d \\ &\implies \mu_d \leq (\mu^*)_d \leq (\mu)_d \\ &\implies \mu_d = (\mu^*)_d. \end{aligned}$$

Now, let $\lambda \in \mathcal{G}(V)$ with $\lambda \leq \mu$. Then, the previous expression gives us

$$\lambda_d \leq \mu_d = (\mu^*)_d. \tag{8}$$

Also, since $\lambda \in \mathcal{G}(V)$, let us denote v the distributional solution of the problem $[V; \lambda]$. Then v is a subsolution of $[V; \mu]$, and we have proved that u^* is the largest subsolution to this problem, therefore

$$v \leq u^* \text{ almost everywhere in } \Omega.$$

Then we can apply the inverse maximum principle (Proposition 6.13 from [25]) to the nonnegative function $u^* - v$, obtaining:

$$(-\Delta v)_c \leq (-\Delta u^*)_c.$$

Now we use the fact that $L^1(\Omega)$ functions are diffuse with respect to any Sobolev capacity, and in particular, the one we are dealing with and denoting by *cap*. Then, taking the concentrated part of the equations satisfied by the functions u^* and v we have:

$$\lambda_c = (-\Delta v)_c \leq (-\Delta u^*)_c = (\mu^*)_c. \tag{9}$$

Summing expressions (8) and (9) together, we have

$$\lambda \leq \mu^*,$$

as we wanted to prove. ■

Corollary 3.16. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $V : \Omega \to [0, \infty]$ be Borel-measurable and $\mu^* \in \mathcal{M}(\Omega)$ be the reduced measure of $[V; \mu]$. Then*

$$\mu \in \mathcal{G}(V) \textit{ if, and only if, } \mu = \mu^*.$$

Proof. If $\mu = \mu^*$, then by the very construction of the reduced measure, we have that $\mu = \mu^* \in \mathcal{G}(V)$.

On the other hand, let $\mu \in \mathcal{G}(V)$. The first item of Theorem 3.13 gives us $\mu^* \leq \mu$. But it is evident that $\mu \leq \mu$, which allows us to apply the second item of Theorem 3.13, obtaining $\mu \leq \mu^*$. We conclude $\mu = \mu^*$. ∎

Here we can prove the following result:

Proposition 3.17. *The reduced measure of the (nonnegative) measure μ can be written as:*

$$\mu^* = \mu\lfloor_{\Omega \setminus Z}.$$

This follow from the previous proposition and the characterization of nonnegative reduce measures (Theorem 1.4 from [24]).

Proof of Theorem 3.8 - Conclusion. Using Propositions 3.10, 3.11, 3.13 and 3.17, we conclude the proof of Theorem 3.8. ∎

Now let us define the reduced limit operator and show some properties. For every Borel-measurable $V : \Omega \to [0, \infty]$, we define the function $\mathcal{R}_V$:

$$\begin{aligned} \mathcal{R}_V : \mathcal{M}^+(\Omega) &\to \mathcal{M}(\Omega) \\ \mu \quad &\mapsto \quad \mu^*. \end{aligned}$$

First we want to prove that $\mathcal{R}_V$ is linear. This follows from the linearity of our problem:

Proposition 3.18. *The function $\mathcal{R}_V : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ is a linear operator.*

Proof. Let $\mu, \lambda \in \mathcal{M}(\Omega), \alpha \in \mathbb{R}, V : \Omega \to [0, \infty]$ Borel-measurable, and $(V_k)_{k\in\mathbb{N}}$ a truncated-type sequence of V.

Then, there exist sequences $(u_k)_{k\in\mathbb{N}}, (v_k)_{k\in\mathbb{N}} \subset W_0^{1,1}(\Omega)$ such that in the sense of distributions:

$$-\Delta u_k + V_k u_k = \mu$$

and

$$-\Delta v_k + V_k v_k = \lambda.$$

In particular, using the linearity of the problem, we know that $\alpha u_k + v_k$ is the distributional solution of $[V_k, \alpha\mu + \lambda]$.

We know that $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ converge in $L^1(\Omega)$ to functions that we denote by u^* and v^*, respectively. We also know that $\mathcal{R}_V(\alpha\mu + \lambda)$ is given by the measure representation of the functional represented by $-\Delta w^* + Vw^*$ where w^* is the $L^1(\Omega)$-limit of $\alpha u_k + v_k$. Then

$$\begin{aligned}\mathcal{R}_V(\alpha\mu + \lambda) &:= -\Delta w^* + Vw^* \\ &= -\Delta(\alpha u^* + v^*) + V(\alpha u^* + v^*) \\ &= -\Delta(\alpha u^*) + V\alpha u^* - \Delta v^* + Vv^* \\ &= \alpha\mathcal{R}_V(\mu) + \mathcal{R}_V(\lambda).\end{aligned}$$

Therefore $\mathcal{R}_V$ is a linear operator. ■

We call $\mathcal{R}_V$ the reduced measure operator.

We know from the construction of μ^* that reduced measures are good measures. We can then write

$$\mathcal{R}_V(\mathcal{M}(\Omega)) \subset \mathcal{G}(V),$$

and if $\lambda \in \mathcal{G}(V)$, then $\lambda = \lambda^* \in \mathcal{R}_V(\mathcal{M}(\Omega))$, that is

$$\mathcal{G}(V) \subset \mathcal{R}_V(\mathcal{M}(\Omega)).$$

This proves that

Proposition 3.19. *The image of the reduced measure operator $\mathcal{R}_V : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ is $\mathcal{G}(V)$.*

Using the fact that $\mathcal{R}_V(\mathcal{R}_V(\mu)) = \mathcal{R}_V(\mu)$ (since the image of a good measure is a good measure), we have:

Proposition 3.20. *The reduced measure operator is a projection.*

Proof. It suffices to notice that $\mathcal{R}_V^2 = \mathcal{R}_V$ and $\mathcal{R}_V$ is linear. ■

Since every linear projection creates a direct sum over the vector space it is defined over, this allows us to write the vector space $\mathcal{M}(\Omega)$ as

$$\mathcal{M}(\Omega) = \mathcal{G}(V) \oplus \ker \mathcal{R}_V$$

For every Borel measure $\mu \in \mathcal{M}(\Omega)\backslash\mathcal{G}(V)$ can be written as $\lambda + \nu$, where $\lambda \in \mathcal{G}(V)$ and $\mathcal{R}_V(\nu) = 0$. Then

$$\begin{aligned}\mathcal{R}_V(\mu) &= \mathcal{R}_V(\lambda + \nu) \\ &= \mathcal{R}_V(\lambda) + \mathcal{R}_V(\nu) \\ &= \lambda.\end{aligned}$$

Then, every measure μ can be written as $\mu = \mu^* + \nu$, where $\mathcal{R}_V(\nu) = 0$.

Now we want to prove that $\mathcal{R}_V$ is a continuous operator. We will use the Closed Graph Theorem for this purpose. But first we need to prove some properties about the vector subspaces $\mathcal{G}(V)$ and $\ker \mathcal{R}_V$:

Lemma 3.21. *The vector subspace of all the good measures, $\mathcal{G}(V)$, is a closed subspace of $\mathcal{M}(\Omega)$.*

Proof. Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$ and $\mu_k \to \mu$ strongly in $\mathcal{M}(\Omega)$. Since μ_k is a good measure, there exists, for every $k \in \mathbb{N}$, u_k the distributional solution of the problem $[V; \mu_k]$. Let $k_i, k_j \in \mathbb{N}$. From the linear estimates (identity (2.2), page 80 from [24]) we have:

$$\|Vu_{k_i} - Vu_{k_j}\|_{L^1(\Omega)} \leq \|\mu_{k_i} - \mu_{k_j}\|_{\mathcal{M}(\Omega)}$$

and

$$\|u_{k_i} - u_{k_j}\|_{L^1(\Omega)} \leq C \cdot \|\Delta(u_{k_i} - u_{k_j})\|_{\mathcal{M}(\Omega)} = 2C \cdot \|\mu_{k_i} - \mu_{k_j}\|_{\mathcal{M}(\Omega)}.$$

This implies that the sequences $(u_k)_{k\in\mathbb{N}}$ and $(Vu_k)_{k\in\mathbb{N}}$ are Cauchy sequences in $L^1(\Omega)$, which gives us $u, v \in L^1(\Omega)$ such that in $L^1(\Omega)$:

$$u_k \to u$$

and

$$Vu_k \to v.$$

In particular, $Vu = v$ almost everywhere in Ω.

Now, from the definition of distributional solution, for every $k \in \mathbb{N}$ and every $\phi \in C_0^\infty(\overline{\Omega})$:

$$\int_\Omega -u_k \Delta\phi + \int_\Omega Vu_k\phi = \int_\Omega \phi \, d\mu_k.$$

Since $u_k \to u$ strongly in $L^1(\Omega)$ and $\mu_k \to \mu$ strongly in $\mathcal{M}(\Omega)$, and therefore $\mu_k \overset{*}{\rightharpoonup} \mu$, then

$$\int_\Omega -u\Delta\phi + \int_\Omega Vu\phi = \int_\Omega \phi \, d\mu.$$

We conclude that $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$ and $\mu_k \to \mu$ strongly in $\mathcal{M}(\Omega)$ implies that $\mu \in \mathcal{G}(V)$. That means that $\mathcal{G}(V)$ is a closed subspace of $\mathcal{M}(\Omega)$, as we wanted to prove. ∎

And also:

Lemma 3.22. *The kernel of the linear operator $\mathcal{R}_V$ is a closed subspace of $\mathcal{M}(\Omega)$.*

Proof. Let $(\mu_k)_{k\in\mathbb{N}} \subset \ker \mathcal{R}_V$ and $\mu_k \xrightarrow{\mathcal{M}(\Omega)} \mu$. Then $\mathcal{R}_V(\mu_k - \mu) = \mathcal{R}_V(\mu)$.

Now let $(V_l)_{l\in\mathbb{N}}$ be a truncated-type sequence of V and $u_{l,k} \in W_0^{1,1}$ the distribution solution of the problem $[V_l, \mu_k]$. We know that

$$u_{l,k} \xrightarrow[l\to\infty]{L^1(\Omega)} u_k^*$$

with

$$-\Delta u_k^* + V u_k^* = 0 \ \forall k \in \mathbb{N}.$$

From the usual linear estimates we have, for every $k \in \mathbb{N}$:

$$\|u_k^*\|_{L^1(\Omega)} \leq C \cdot \|\mu\|_{\mathcal{M}(\Omega)} = 0$$
$$\implies u_k^* = 0 \text{ almost everywhere in } \Omega.$$

This implies that $(u_k^*)_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$ to some $v \in L^1(\Omega)$ with $v = 0$ almost everywhere in Ω.

On the other hand, for every $k, l \in \mathbb{N}$ and $\psi \in C_0^\infty(\overline{\Omega})$:

$$\begin{cases} -\Delta u_{l,k} + V_l u_{l,k} = \mu_k \text{ in } \Omega, \\ \qquad\qquad\quad u_{l,k} = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $k_i, k_j \in \mathbb{N}$. Again from the usual linear estimates:

$$\|u_{l,k_i} - u_{l,k_j}\|_{L^1(\Omega)} \leq \|\mu_{k_i} - \mu_{k_j}\|_{\mathcal{M}(\Omega)}.$$

Then $(u_{l,k})_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$:

$$u_{l,k} \xrightarrow[k\to\infty]{L^1(\Omega)} v_l$$

and

$$V_l\ u_{l,k} \xrightarrow[k\to\infty]{L^1(\Omega)} V_l\ v_l.$$

Using the same arguments as the previous lemma we arrive at the following identity in the sense of distributions:

$$-\Delta v_l + V_l v_l = \mu \in \Omega.$$

From the construction of the reduced measure, $(v_l)_{l\in\mathbb{N}}$ converges in $L^1(\Omega)$ (to a function v^*), therefore it converges almost everywhere in Ω. Since for almost every $x \in \Omega$, the following limits exist:

$$\lim_{k\to\infty} u_{l,k}(x) \text{ and } \lim_{l\to\infty} u_{l,k}(x),$$

then, for almost every $x \in \Omega$:

$$\begin{aligned}\lim_{l\to\infty} v_l(x) &= \lim_{l\to\infty}\lim_{k\to\infty} u_{l,k}(x)\\ &= \lim_{k\to\infty}\lim_{l\to\infty} u_{l,k}(x)\\ &= \lim_{k\to\infty} u_k^*(x)\\ &= 0.\end{aligned}$$

Then $\mathcal{R}_V(\mu) = \mu^* := -\Delta v^* + Vv^* = 0$. Therefore, $\ker \mathcal{R}_V$ is a closed subspace of $\mathcal{M}(\Omega)$, as we wanted to prove. ■

We can then prove:

Proposition 3.23. *The reduced measure operator is continuous.*

Proof. First of all let

$$\mu_k \xrightarrow{\mathcal{M}(\Omega)} \mu$$

and

$$\mathcal{R}_V(\mu_k) = \mu_k^* \xrightarrow{\mathcal{M}(\Omega)} \lambda.$$

Since $\mathcal{G}(V)$ is closed, then $\lambda \in \mathcal{G}(V)$, which gives us $\mathcal{R}_V(\lambda) = \lambda$.

But $\mu_k - \mathcal{R}_V(\mu_k) \xrightarrow{\mathcal{M}(\Omega)} \mu - \lambda$. Now, since $\ker \mathcal{R}_V$ is closed and $\mathcal{R}_V(\mu_k - \mathcal{R}_V(\mu_k)) = 0$ then $\mu_k - \mathcal{R}_V(\mu_k) \in \ker \mathcal{R}_V$ and $\mu - \lambda \in \ker \mathcal{R}_V$. This implies

$$\begin{aligned}\mathcal{R}_V(\mu - \lambda) = 0 &\implies \mathcal{R}_V(\mu) - \mathcal{R}_V(\lambda) = 0\\ &\implies \mathcal{R}_V(\mu) = \lambda.\end{aligned}$$

We have that $(\mu_k, \mathcal{R}_V(\mu_k)) \xrightarrow{\mathcal{M}(\Omega)\times\mathcal{M}(\Omega)} (\mu, \lambda)$ implies $\mathcal{R}_V(\mu) = \lambda$. The Closed Graph Theorem allows us to conclude that the operator $\mathcal{R}_V$ is continuous. ■

3.2 Reduced Limit

Now, using the definitions we have laid down, we can talk about a new concept of limit, the reduced limit. This concept was been introduced to nonlinear equations involving measures in 2009 by Marcus and Ponce (see [20]). Here we adapt and expand the concept for linear equations involving a singular Schrödinger potential. This can be done by taking a sequence of measures in $\mathcal{G}(V)$ and whose sequence of distributional solutions converge in $L^1(\Omega)$. We then will prove properties involving this idea.

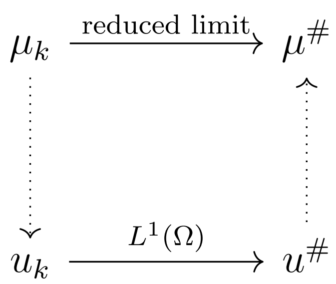
Definition 3.24. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ be a bounded sequence. We say that $\mu^\# \in \mathcal{M}(\Omega)$ is the reduced limit of $(\mu_k)_{k\in\mathbb{N}}$ if*

(i) for every $k \in \mathbb{N}, \mu_k \in \mathcal{G}(V)$, i.e., there exists $(u_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ such that u_k is a distributional solution of the Schrödinger problem with potential V and density μ_k;

(ii) $(u_k)_{k\in\mathbb{N}}$ converges in $L^1(\Omega)$ to a function we denote by $u^\#$;

(iii) $u^\#$ is a distributional solution of the Schrödinger problem with potential V and density $\mu^\#$.

The next diagram shows a graphical way to conceptualize the reduced limit:



The dotted arrows indicate that u_k and $u^\#$ are the distributional solutions of Schrödinger problems associated with the densities μ_k and $\mu^\#$, respectively.

It is clear from the definition, that there is a dependence of the reduced limit on the Borel-measurable and nonnegative potential V. In this text, when we talk about reduced limit, we are always refering to an arbitrary potential, except when explicitly stated.

We now show that, given the two first items of the previous definition, i.e., if we have (i) and (ii) from Definition 3.24, then the reduced limit exists:

Theorem 3.25. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$, not necessarily nonnegative measures, be a bounded sequence in $\mathcal{M}(\Omega)$. For each $k \in \mathbb{N}$, denote by $u_k \in L^1(\Omega)$ the distributional* *solution of the Schrödinger problem with Borel-measurable potential* $V : \Omega \to [0, \infty]$ *and density* μ_k*. If*

$$u_k \to u^{\#} \text{ in } L^1(\Omega),$$

then:

(i) the reduced limit of $(\mu_k)_{k\in\mathbb{N}}$ *exists and it is unique, say* $\mu^{\#} \in \mathcal{M}(\Omega)$*;*

(ii) $u^{\#}$ *is the distributional solution of* $[V, \mu^{\#}]$*, in particular,* $\mu^{\#}$ *is a good measure.*

Proof: First of all, for every $k \in \mathbb{N}$, we have the following estimate that can be found in [24]:

$$\|Vu_k\|_{L^1(\Omega)} \leq \|\mu_k\|_{\mathcal{M}(\Omega)}.$$

From this we conclude that $(Vu_k)_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$.

Now, we note that, since $u_k \to u^{\#}$ in $L^1(\Omega)$, then for a subsequence of $(u_k)_{k\in\mathbb{N}}$ that we can still denote by $(u_k)_{k\in\mathbb{N}}$, we have

$$Vu_k \to Vu^{\#} \text{ a.e. in } \Omega,$$

and using Fatou's Lemma:

$$\begin{aligned} \int_\Omega \left|Vu^{\#}\right| &= \int_\Omega \liminf_{k\to\infty} |Vu_k| \\ &\leq \liminf_{k\to\infty} \int_\Omega |Vu_k| \\ &< \infty, \end{aligned}$$

since $(Vu_k)_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. We can conclude that

$$Vu^{\#} \in L^1(\Omega).$$

Using that $(\mu_k)_{k\in\mathbb{N}}$ and $(Vu_k)_{k\in\mathbb{N}}$ are bounded in $\mathcal{M}(\Omega)$, we use the weak* compacity in this space (Proposition 2.9) to find a subsequences of $(\mu_k)_{k\in\mathbb{N}}$ and $(Vu_k)_{k\in\mathbb{N}}$ (that we can still denote the same way) such that

$$\mu_k \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega)$$

and

$$Vu_k \overset{*}{\rightharpoonup} \lambda \text{ in } \mathcal{M}(\Omega).$$

If we define the following measure on $\mathcal{M}(\Omega)$:

$$\mu^{\#} := \mu + Vu^{\#} - \lambda,$$

we have

$$\begin{cases} -\Delta u^\# + V u^\# = \mu^\# \text{ in } \Omega, \\ u^\# = 0 \text{ on } \partial\Omega, \end{cases}$$

as we wanted to show. ■

We are now interested in proving the lower semicontinuity of the total variation norm with respect to the reduced limit $\mu^\#$ of a sequence $(\mu_k)_{k\in\mathbb{N}}$, i.e., the following identity:

$$\|\mu^\#\|_{\mathcal{M}(\Omega)} \leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}(\Omega)}.$$

Let us first analyze a special case:

Proposition 3.26. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ be a bounded sequence with reduced limit $\mu^\#$. If, for every $k \in \mathbb{N}$,*

$$\mu_k\lfloor_Z = 0,$$

where Z is the universal zero-set, then

$$\|\mu^\#\|_{\mathcal{M}(\Omega)} \leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}(\Omega)}.$$

Proof: Let μ_k^+ and μ_k^- be, respectively, the positive and negative parts of μ_k given by the Jordan Decomposition Theorem. Then by definition, there exist sets $P, N \subset \Omega$ such that

$$\mu_k^+ = \mu_k\lfloor_P \text{ and } \mu_k^- = \mu_k\lfloor_N.$$

Therefore we have $\mu_k^+(Z) = 0$ and $\mu_k^-(Z) = 0$. Using Theorem 2.54 we can conclude that there exist a pair of sequences $(v_k)_{k\in\mathbb{N}}, (w_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ distributional solutions of the Schrödinger problem with density μ_k^+ and μ_k^-.

We then take $\mu_\oplus, \mu_\ominus$ the weak* limits of $(\mu_k^+)_{k\in\mathbb{N}}$ and $(\mu_k^-)_{k\in\mathbb{N}}$, respectively, $v^\#, w^\#$ the $L^1(\Omega)$ limits of $(v_k)_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathbb{N}}$, and $\mu_\oplus^\#$ and $\mu_\ominus^\#$ the reduced limits of $(\mu_k^+)_{k\in\mathbb{N}}$ and $(\mu_k^-)_{k\in\mathbb{N}}$, respectively. Now using the monotonicity of the reduced limit (previous result) we have

$$-\mu_\ominus^\# \leq \mu^\# \leq \mu_\oplus^\#,$$

and using Fatou's Lemma we have

$$-\mu_\ominus \leq \mu_\ominus^\# \text{ and } \mu_\oplus^\# \leq \mu_\oplus.$$

Then by using both inequalities and taking the measure of the set Ω we have:

$$\|\mu^\#\|_{\mathcal{M}(\Omega)} \leq \|\mu_\ominus\|_{\mathcal{M}(\Omega)} + \|\mu_\oplus\|_{\mathcal{M}(\Omega)}.$$

Now, the result follows from the lower semicontinuity of the total variation norm with respect to the weak* convergence:

$$\begin{aligned}\left\|\mu^{\#}\right\|_{\mathcal{M}(\Omega)} &\leq \|\mu_{\ominus}\|_{\mathcal{M}(\Omega)} + \|\mu_{\oplus}\|_{\mathcal{M}(\Omega)} \\ &\leq \liminf_{k\to\infty} \left\|\mu_k^{+}\right\|_{\mathcal{M}(\Omega)} + \liminf_{k\to\infty} \left\|\mu_k^{-}\right\|_{\mathcal{M}(\Omega)} \\ &\leq \liminf_{k\to\infty} \|\mu_k\|_{\mathcal{M}(\Omega)} .\end{aligned}$$

■

Remark. *We know from the theory of Differential Problems with Measure Data, that the set S is very problematic. It is very hard to deal with this set because of its singularity and the fact that our variational solutions are null in S (Lema 2.43). For this reason it is not easy to prove the lower semicontinuity in the general case precisely because we would have to measure S. The solution is then to work only with the set $\Omega\backslash S$. Then, the inequality we aim to prove is as follows:*

$$\left|\mu^{\#}\right|(\Omega\backslash S) \leq \liminf_{k\to\infty} |\mu_k|(\Omega\backslash S),$$

where $\mu^{\#}$ is the reduced limit of $(\mu_k)_{k\in\mathbb{N}}$. This would allow us to solve certain problems without having to deal with the set S and its intricacies. The next results aim to prove the previous inequality.

Proposition 3.27. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$ be such that*

$$\mu_k \geq 0 \ \text{in}\ \Omega\backslash S,$$

$\mu^{\#} \in \mathcal{M}(\Omega)$ be the reduced limit of $(\mu_k)_{k\in\mathbb{N}}$, $u^{\#} \in L^1(\Omega)$ be the distributional solution of the Schrödinger problem with potential V and density $\mu^{\#}$, H be the function given by the Comparison Principle, and $(D_i)_{i\in I}$ be the Sobolev-connected-open decomposition of $\Omega\backslash S$. Then, for every $i \in I$, we have

$$\text{either}\ u^{\#} = 0\ \text{a.e. in}\ D_i,\ \text{or}\ \widehat{\zeta_{H(u^{\#})}} > 0\ \text{in}\ D_i.$$

Proof: First, for every $k \in \mathbb{N}$, let us denote by u_k the distributional solution of the Schrödinger problem with potential V and density μ_k. From Proposition 2.39 we know that the functions $(u_k)_{k\in\mathbb{N}}$ and $u^{\#}$ are also duality solutions with their respective densities. We have, for every $k \in \mathbb{N}$ and every $f \in L^{\infty}(\Omega)$:

$$\int_{\Omega} u_k f = \int_{\Omega} \widehat{\zeta_f}\, d\mu_k \ \text{and}\ \int_{\Omega} u^{\#} f = \int_{\Omega} \widehat{\zeta_f}\, d\mu^{\#}.$$

From Proposition 2.45 we have that, for every $k \in \mathbb{N}$, the function u_k is also the duality solution of the Schrödinger problem with potential V and density $\mu_k \lfloor_{\Omega \setminus S}$. We have also that

$$\mu_k \lfloor_{\Omega \setminus S} \geq 0.$$

Since this measure is nonnegative, we obtain, for every $k \in \mathbb{N}$:

$$u_k \geq \zeta_{H(u_k)} \text{ almost everywhere in } \Omega, \tag{10}$$

where $H : [0, \infty) \to [0, \infty)$ is a bounded ($H \leq M$), continuous and nondecreasing function that satisfies $H(t) > 0$ for $t > 0$, and $\zeta_{H(u_k)}$ is the variational solution of the Schrödinger problem with potential V and data $H(u_k) \in L^\infty(\Omega) \subset L^2(\Omega)$.

We now want to show that the following inequality holds:

$$u^\# \geq \zeta_{H(u^\#)} \text{ almost everywhere in } \Omega. \tag{11}$$

It is enough to note that $\zeta_{H(u_k)} - \zeta_{H(u^\#)}$ is the variational solution of the Schrödinger problem with potential V and data $H(u_k) - H(u^\#)$. Using an estimate of the variational solution:

$$\|\zeta_{H(u_k)} - \zeta_{H(u^\#)}\|_{W^{1,1}(\Omega)} \leq \|H(u_k) - H(u^\#)\|_{L^1(\Omega)}.$$

Since u_k, $u^\# \in L^1(\Omega)$, we have

$$\lim_{k \to \infty} \|\zeta_{H(u_k)} - \zeta_{H(u^\#)}\|_{W^{1,1}(\Omega)} = 0.$$

In particular,

$$\zeta_{H(u_k)} \to \zeta_{H(u^\#)} \text{ almost everywhere in } \Omega..$$

Using this with the identity (10), we conclude (11).

Now we apply the localized strong maximum principle for the Sobolev-connected-open components of $\Omega \setminus S$ (Theorem 2.51) to the function $\zeta_{H(u^\#)}$, which is a duality solution of the Schrödinger problem with potential V and density $H(u^\#)\,dx$, where $H(u^\#) \in L^\infty(\Omega)$. The theorem gives us, for every $i \in I$:

$$\text{either } \widehat{\zeta_{H(u^\#)}} \equiv 0 \text{ in } D_i \text{ or } \widehat{\zeta_{H(u^\#)}} > 0 \text{ in } D_i. \tag{12}$$

If $\widehat{\zeta_{H(u^\#)}} \equiv 0$ in D_i, then, since the precise representative of a function agrees with the function almost everywhere, we have $\zeta_{H(u^\#)} \equiv 0$ almost everywhere in D_i. Using that $\zeta_{H(u^\#)}$ is a duality solution of the Schrödinger problem with potential V

and density $H(u^{\#})\ dx$, we have for every $f \in L^{\infty}(\Omega)$:

$$\int_{\Omega} \zeta_{H(u^{\#})} f = \int_{\Omega} \zeta_f H(u^{\#}).$$

From Theorem 2.52 we know that, for every $f \in L^{\infty}(\Omega)$:

$$\int_{\Omega} \zeta_{H(u^{\#})} \chi_{D_i} f = \int_{\Omega} \zeta_f H(u^{\#}) \chi_{D_i}.$$

Since $\zeta_{H(u^{\#})} \equiv 0$ almost everywhere in D_i, then $\zeta_{H(u^{\#})}\chi_{D_i} = 0$ almost everywhere in Ω and we obtain:

$$\int_{\Omega} \zeta_f H(u^{\#}) \chi_{D_i} = 0 \ \ \forall f \in L^{\infty}(\Omega).$$

Using Proposition 2.47 we conclude that $H(u^{\#})\ dx = 0$ in $\Omega \backslash S$, which implies that $H(u^{\#}) = 0$ almost everywhere in D_i. Since $H(t)$ is nonzero for $t > 0$, we have $u^{\#} = 0$ almost everywhere in D_i. This gives us

$$\widehat{\zeta_{H(u^{\#})}} \equiv 0 \text{ in } D_i \implies u^{\#} = 0 \text{ almost everywhere in } D_i. \tag{13}$$

Then, expressions (12) and (13) conclude the result. ■

For the next proposition we need the following lemma:

Lemma 3.28. *Let $X \subset \mathbb{R}^N$ be a bounded set and $\{A_t\}_{t \in I}$ be a family of disjoint Borel-measurable subsets of X. Then, there exists a countable subset $J \subset I$, such that $|A_t| > 0$ if, and only if, $t \in J$.*

Proof: First, we define the sequence $(B_k)_{k \in \mathbb{N}}$ as:

$$B_1 := \{t \in I; 1 < |A_t|\} \text{ and } B_k := \left\{ t \in I; k > 1, \frac{1}{k} < |A_t| \leq \frac{1}{k-1} \right\}.$$

Our objective is to prove that each set B_k is finite.

Let us fix a $k \in \mathbb{N}$. We know that $t \in B_k$ implies $|A_t| > \frac{1}{k}$; the sets A_t are disjoint; $\bigcup_{t \in B_k} A_t \subset X$; and X is bounded, and therefore, its Lebesgue measure is finite. Then

$$\sum_{t \in B_k} \frac{1}{k} \leq \sum_{t \in B_k} |A_t| = \left| \bigcup_{t \in B_k} A_t \right| \leq |X| < \infty.$$

Since an infinite series with constant terms cannot be finite, we conclude that, for every $k \in \mathbb{N}$, B_k is a finite set. Defining

$$J := \bigcup_{k=1}^{\infty} B_k,$$

we see that $J \subset I$, $t \in J$ if, and only if $|A_t| > 0$, and J is the countable union of finite sets, and therefore it is a countable set. This concludes the proof. ∎

Proposition 3.29. *Let $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$ be such that*

$$\mu_k \geq 0 \ \textit{in} \ \Omega \backslash S,$$

and, for every $k \in \mathbb{N}$, $u_k \in L^1(\Omega)$ be the distributional solution of the Schrödinger problem with potential V and density μ_k, with $u_k \to u^{\#}$ in $L^1(\Omega)$. If $\alpha \in \mathcal{M}(\Omega)$ is the $cap_{W^{1,2}}$-diffuse limit of the sequence $(Vu_k\ dx)_{k\in\mathbb{N}}$, then

$$\alpha = Vu^{\#}\ dx \ \textit{in} \ \left\{\widehat{\zeta_{H(u^{\#})}} > 0\right\}.$$

Proof: First we note that, since, for every $k \in \mathbb{N}$, the measure μ_k is nonnegative in $\Omega \backslash S$, then the function u_k is also nonnegative, and so is $u^{\#}$.

We apply Lemma 3.28 to the family of disjoint sets $\{u^{\#} = t\} \subset \Omega, t \in \mathbb{R}^+$. The lemma tells us that the set $\{t \in \mathbb{R}^+; |\{u^{\#} = t\}| \neq 0\}$ is countable. We can then find a strictly increasing sequence $(C_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $C_j \to \infty$ and

$$\left|\left\{u^{\#} = C_j\right\}\right| = 0.$$

We now define the set

$$A_{k,j,\varepsilon} := \{u_k \leq C_j\} \cap \left\{\varepsilon < \zeta_{H(u^{\#})}\right\}.$$

Now let us work with this set and, for every $\varepsilon > 0$, analyse the double sequence $(Vu_k\chi_{A_{k,j,\varepsilon}})_{j,k\in\mathbb{N}}$.

Our first objective is to find the $L^1(\Omega)$ limit of $(Vu_k\chi_{A_{k,j,\varepsilon}})_{k\in\mathbb{N}}$ using Lebesgue's Dominated Convergence Theorem. From the definition of $A_{k,j,\varepsilon}$ we have:

$$u_k \leq C_j = \frac{C_j}{\varepsilon} \cdot \varepsilon < \frac{C_j}{\varepsilon}\zeta_{H(u^{\#})} \text{ in } A_{k,j,\varepsilon},$$

that implies

$$Vu_k\chi_{A_{k,j,\varepsilon}} \leq \frac{C_j}{\varepsilon} V\zeta_{H(u^{\#})} \in L^1(\Omega), \tag{14}$$

since $V \geq 0$ and $V\zeta_{H(u^{\#})} \in L^1(\Omega)$. We also have that $|\{u^{\#} = C_j\}| = 0$ and $u_k \to u^{\#}$ a.e. in Ω, i.e., there exists a set $N \subset \Omega$ such that $|N| = 0$ and, for every

$x \in \Omega \backslash N, u_k(x) \to u^{\#}(x)$. From this we obtain:

(a) if $x \in \Omega \backslash (N \cup \{u^{\#} = C_j\})$ and $u^{\#}(x) < C_j$, then, for some $k \in \mathbb{N}$ large enough, $u_k(x) \leq C_j$, thus $\chi_{\{u^{\#} \leq C_j\}}(x) = 1$ implies $\chi_{\{u_k \leq C_j\}}(x) \xrightarrow{k \to \infty} 1$;

(b) if $x \in \Omega \backslash (N \cup \{u^{\#} = C_j\})$ and $u^{\#}(x) > C_j$, then, for some $k \in \mathbb{N}$ large enough, $u_k(x) > C_j$, thus $\chi_{\{u^{\#} \leq C_j\}}(x) = 0$ implies $\chi_{\{u_k \leq C_j\}}(x) \xrightarrow{k \to \infty} 0$.

This means that

$$\chi_{\{u_k \leq C_j\}} \xrightarrow{k \to \infty} \chi_{\{u^{\#} \leq C_j\}} \text{ a.e. in } \Omega.$$

Using $\chi_{A \cap B} = \chi_A \chi_B$ and the limit of the product, we have:

$$\begin{aligned} \chi_{A_{k,j,\varepsilon}} &= \chi_{\{u_k \leq C_j\}} \chi_{\{\zeta_{H(u^{\#})} > \varepsilon\}} \\ &\xrightarrow{k \to \infty} \chi_{\{u^{\#} \leq C_j\}} \chi_{\{\zeta_{H(u^{\#})} > \varepsilon\}} \\ &= \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \text{ a.e. in } \Omega. \end{aligned}$$

Then

$$V u_k \chi_{A_{k,j,\varepsilon}} \xrightarrow{k \to \infty} V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \text{ a.e. in } \Omega. \tag{15}$$

From (14), (15) and Lebesgue's Dominated Convergence Theorem we have

$$V u_k \chi_{A_{k,j,\varepsilon}} \xrightarrow{k \to \infty} V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \text{ in } L^1(\Omega). \tag{16}$$

Now we take the sequence $(V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}})_{j \in \mathbb{N}}$ and find its $L^1(\Omega)$ limit using the Monotone Convergence Theorem. First we have that

$$V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \geq 0.$$

and that, for every $j \in \mathbb{N}$:

$$V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \leq V u^{\#} \chi_{\{u^{\#} \leq C_{j+1}\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}},$$

and

$$V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \to V u^{\#} \chi_{\{\zeta_{H(u^{\#})} > \varepsilon\}} \text{ a.e. in } \Omega.$$

since $(C_j)_{j \in \mathbb{N}}$ is increasing and $C_j \to \infty$.

We use the Monotone Convergence Theorem to conclude

$$V u^{\#} \chi_{\{u^{\#} \leq C_j\} \cap \{\zeta_{H(u^{\#})} > \varepsilon\}} \to V u^{\#} \chi_{\{\zeta_{H(u^{\#})} > \varepsilon\}} \text{ in } L^1(\Omega).$$

Now we want to use a diagonal argument to find a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ such that

$$Vu_{k_j}\chi_{A_{k_j,j,\varepsilon}} \to Vu^{\#}\chi_{\{\zeta_{H(u^{\#})}>\varepsilon\}} \text{ in } L^1(\Omega).$$

Let us define $k_1 := 1$. The sequence k_j will then be defined recursively. If k_{j-1} is defined then $k_j > k_{j-1}$ is chosen such that (using (16)):

$$\left\| Vu_{k_j}\chi_{A_{k_j,j,\varepsilon}} - Vu^{\#}\chi_{\{u^{\#}\le C_j\}\cap\{\zeta_{H(u^{\#})}>\varepsilon\}} \right\|_{L^1(\Omega)} \le \frac{1}{2^j}.$$

That implies

$$\begin{aligned}&\left\| Vu_{k_j}\chi_{A_{k_j,j,\varepsilon}} - Vu^{\#}\chi_{\{\zeta_{H(u^{\#})}>\varepsilon\}} \right\|_{L^1(\Omega)}\\ &\le \frac{1}{2^j} + \left\| Vu^{\#}\chi_{\{u^{\#}\le C_j\}\cap\{\zeta_{H(u^{\#})}>\varepsilon\}} - Vu^{\#}\chi_{\{\zeta_{H(u^{\#})}>\varepsilon\}} \right\|_{L^1(\Omega)} \xrightarrow{j\to\infty} 0,\end{aligned}$$

as we wanted to show.

Our objective now is to take the sequence of measures $(Vu_{k_j}\,dx)_{j\in\mathbb{N}}$ and decompose it in $\text{cap}_{W^{1,2}}$-equidiffuse and $\text{cap}_{W^{1,2}}$-concentrating sequences and to analyse these sequences and their weak* limits.

We can write

$$Vu_{k_j} = Vu_{k_j}\chi_{A_{k_j,j,\varepsilon}} + Vu_{k_j}\chi_{\{u_{k_j}>C_j\}} + Vu_{k_j}\chi_{Z_{j,\varepsilon}},$$

where $Z_{j,\varepsilon} = \{u_{k_j} \le C_j\} \cap \{\zeta_{H(u^{\#})} \le \varepsilon\}$. Applying the Biting Lemma to the sequence $(Vu_{k_j}\chi_{Z_{j,\varepsilon}})_{j\in\mathbb{N}}$, we find $\text{cap}_{W^{1,2}}$-equidiffuse and $\text{cap}_{W^{1,2}}$-concentrating subsequences $(\beta_{j,\varepsilon})_{j\in\mathbb{N}}$ and $(\gamma_{j,\varepsilon})_{j\in\mathbb{N}}$ such that, for every $j\in\mathbb{N}$ we have $\beta_{j,\varepsilon}\perp\gamma_{j,\varepsilon}$, i.e., there exist measurable sets $X_{j,\varepsilon}$ and $Y_{j,\varepsilon}$ such that $X_{j,\varepsilon}\cup Y_{j,\varepsilon} = Z_{j,\varepsilon}$, $X_{j,\varepsilon}\cap Y_{j,\varepsilon}=\varnothing$, $\beta_{j,\varepsilon}=0$ in $Y_{j,\varepsilon}$ and $\gamma_{j,\varepsilon}=0$ in $X_{j,\varepsilon}$. This implies: $\beta_{j,\varepsilon} = Vu_{k_j}\chi_{X_{j,\varepsilon}}\,dx$ and $\gamma_{j,\varepsilon} = Vu_{k_j}\chi_{Y_{j,\varepsilon}}\,dx$.

This gives us the following decomposition:

$$Vu_{k_j} = Vu_{k_j}\chi_{A_{k_j,j,\varepsilon}} + Vu_{k_j}\chi_{\{u_{k_j}>C_j\}} + Vu_{k_j}\chi_{X_{j,\varepsilon}} + Vu_{k_j}\chi_{Y_{j,\varepsilon}},$$

where $X_{j,\varepsilon}$ and $Y_{j,\varepsilon}$ are measurable, $X_{j,\varepsilon}\cup Y_{j,\varepsilon} = \{u_{k_j}\le C_j\}\cap\{\zeta_{H(u^{\#})}\le\varepsilon\}$, $X_{j,\varepsilon}\cap Y_{j,\varepsilon}=\varnothing$, and the sequences $(Vu_{k_j}\chi_{X_{j,\varepsilon}})_{j\in\mathbb{N}}$ and $(Vu_{k_j}\chi_{Y_{j,\varepsilon}})_{j\in\mathbb{N}}$ are $\text{cap}_{W^{1,2}}$-equidiffuse and $\text{cap}_{W^{1,2}}$-concentrating, respectively.

Now we note that

$$\text{cap}_{W^{1,2}}\left(\{u_{k_j} > C_j\}\right) \le \frac{c}{C_j} \to 0,$$

from Lemma 3.2 in [20]. Then the sequence of sets $\{u_{k_j} > C_j\}$ satisfy:

$$\mathrm{cap}_{W^{1,2}}\left(\{u_{k_j} > C_j\}\right) \to 0$$

and

$$\int_{\Omega\setminus\{u_{k_j}>C_j\}} V u_{k_j} \chi_{\{u_{k_j}>C_j\}} = 0.$$

This means that the sequence $(V u_{k_j} \chi_{\{u_{k_j}>C_j\}}\ dx)_{j\in\mathbb{N}}$ is $\mathrm{cap}_{W^{1,2}}$-concentrating.

Then

$$\begin{aligned}\alpha_{k_j} &:= V u_{k_j}\chi_{A_{k_j,j,\varepsilon}} + V u_{k_j}\chi_{X_{j,\varepsilon}},\\ \sigma_{k_j} &:= V u_{k_j}\chi_{\{u_{k_j}>C_j\}} + V u_{k_j}\chi_{Y_{j,\varepsilon}}\end{aligned}$$

are, respectively, an $\mathrm{cap}_{W^{1,2}}$-equidiffuse and a $\mathrm{cap}_{W^{1,2}}$-concentrating subsequence of $(V u_{k_j}\ dx)_{j\in\mathbb{N}}$.

Let us calculate α, the weak* limit of $(\alpha_{k_j})_{j\in\mathbb{N}}$, i.e., the $\mathrm{cap}_{W^{1,2}}$-diffuse limit of $(V u_{k_j}\ dx)_{j\in\mathbb{N}}$.

First we know that the weak* limit of $(V u_{k_j}\chi_{A_{k_j,j,\varepsilon}})_{j\in\mathbb{N}}$ is $V u^{\#}\chi_{\{\zeta_{H(u^{\#})}>\varepsilon\}}\ dx$, since this is its $L^1(\Omega)$ limit.

Let us denote the weak* limit of $(V u_{k_j}\chi_{X_{j,\varepsilon}})_{j\in\mathbb{N}}$ by λ_ε. We know that the sequence $(V u_{k_j}\chi_{X_{j,\varepsilon}})_{j\in\mathbb{N}}$ is $\mathrm{cap}_{W^{1,2}}$-equidiffuse and that, since $X_{j,\varepsilon} \subset \{\zeta_{H(u^{\#})} \leq \varepsilon\}$, we have

$$\begin{aligned}\int_{\{\widehat{\zeta_{H(u^{\#})}}>\varepsilon\}} V u_{k_j}\chi_{X_{j,\varepsilon}} &= \int_{\{\zeta_{H(u^{\#})}>\varepsilon\}} V u_{k_j}\chi_{X_{j,\varepsilon}}\\ &= 0.\end{aligned}$$

Since $\widehat{\zeta_{H(u^{\#})}} \in W_0^{1,2}(\Omega)$ and it is defined pointwise in Ω, then, the set $\{\widehat{\zeta_{H(u^{\#})}} > \varepsilon\}$ is a Sobolev-open set, and we know that the $\mathrm{cap}_{W^{1,2}}$-equidiffuse sequence of measures $(V u_{k_j}\chi_{X_{j,\varepsilon}})_{j\in\mathbb{N}}$ equals zero on this set. Therefore, we obtain

$$\lambda_\varepsilon\left(\left\{\widehat{\zeta_{H(u^{\#})}} > \varepsilon\right\}\right) = 0.$$

Then

$$\alpha = V u^{\#}\chi_{\{\widehat{\zeta_{H(u^{\#})}}>\varepsilon\}} + \lambda_\varepsilon,$$

that implies, for every $\varepsilon > 0$:

$$\alpha\llcorner_{\{\widehat{\zeta_{H(u^{\#})}}>\varepsilon\}} = V u^{\#}\chi_{\{\widehat{\zeta_{H(u^{\#})}}>\varepsilon\}}\ dx.$$

Let $k \in \mathbb{N}$ and $A \subset \{\widehat{\zeta_{H(u^\#)}} > 0\}$. Using that

$$A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k} \right\} \subset A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k+1} \right\},$$

$$A = \bigcup_{k \in \mathbb{N}} \left(A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k} \right\} \right),$$

and the continuity from below of finite Borel measures, we have:

$$\begin{aligned}
\alpha(A) &= \alpha \left(\bigcup_{k \in \mathbb{N}} \left(A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k} \right\} \right) \right) \\
&= \lim_{k \to \infty} \alpha \left(A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k} \right\} \right) \\
&= \lim_{k \to \infty} \int_{\left\{ A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k} \right\} \right\}} V u^\# \, dx \\
&= \int_{\left\{ \bigcup_{k \in \mathbb{N}} \left(A \cap \left\{ \widehat{\zeta_{H(u^\#)}} > \frac{1}{k} \right\} \right) \right\}} V u^\# \, dx \\
&= \int_A V u^\# \, dx.
\end{aligned}$$

Then, we conclude $\alpha = V u^\# \, dx$ in $\{\widehat{\zeta_{H(u^\#)}} > 0\}$, as we wanted to prove. ■

The next result is a type of monotonicity result restricted to $\Omega \backslash S$:

Proposition 3.30. *Let $V : \Omega \to [0, \infty]$ a Borel-measurable function and $(\mu_k)_{k \in \mathbb{N}} \subset \mathcal{G}(V)$, not necessarily nonnegative measures, be a bounded sequence in $\mathcal{M}(\Omega)$ with reduced limit $\mu^\# \in \mathcal{M}(\Omega)$. If, for every $k \in \mathbb{N}$,*

$$\mu_k \geq 0 \text{ in } \Omega \backslash S,$$

then

$$\mu^\# \geq 0 \text{ in } \Omega \backslash S.$$

Proof: First, we can use the Sobolev-connected-open decomposition of the set $\Omega \backslash S$ to write

$$\mu^\# \lfloor_{\Omega \backslash S} = \sum_{i \in I} \mu^\# \lfloor_{D_i}.$$

We will prove the result for each measure $\mu^\# \lfloor_{D_i}$ and use this identity to conclude the proof.

From Proposition 3.27 we have that, for each $i \in I$:

$$\text{either } u^{\#} = 0 \text{ a.e. in } D_i, \text{ or } \widehat{\zeta_{H(u^{\#})}} > 0 \text{ in } D_i.$$

We will now consider each one of these cases.

If $u^{\#} = 0$ almost everywhere in D_i, we recall that $u^{\#}$ is the duality solution of the Schrödinger problem with potential V and density $\mu^{\#}$ and we apply Theorem 2.52 to $u^{\#}$, obtaining, for every $f \in L^{\infty}(\Omega)$:

$$\int_{\Omega} \zeta_f \, d\mu^{\#}\lfloor_{D_i} = \int_{\Omega} u^{\#} \chi_{D_i} f = 0.$$

And from Proposition 2.47 we have

$$\mu^{\#}\lfloor_{D_i} = 0 \text{ in } \Omega \backslash S.$$

Since $D_i \subset \Omega \backslash S$, this implies

$$\mu^{\#} = 0 \text{ in } D_i.$$

We now turn our attention to the case $\widehat{\zeta_{H(u^{\#})}} > 0$ in D_i. By noting that $D_i \subset \{\widehat{\zeta_{H(u^{\#})}} > 0\}$, we know, from Theorem 3.29, that the diffuse limit of the sequence $(V u_k \, dx)_{k \in \mathbb{N}}$ is $V u^{\#} \, dx$, in D_i. We denote by ν the weak* limit of $(V u_k \, dx)_{k \in \mathbb{N}}$ and by λ the concentrated limit of this sequence, thus obtaining

$$\nu = V u^{\#} \, dx + \lambda \text{ in } D_i.$$

Now we know that the following equation holds in the sense of distributions:

$$-\Delta u_k + V u_k = \mu_k \text{ in } \Omega.$$

Denoting by μ the weak* limit of $(\mu_k)_{k \in \mathbb{N}}$, recalling that $-\Delta u^{\#}$ is the weak* limit of $(-\Delta u_k)_{k \in \mathbb{N}}$, and taking the weak* limit in the last identity, we have

$$-\Delta u^{\#} + \nu = \mu \text{ in } \Omega.$$

Contracting this identity to D_i:

$$-\Delta u^{\#} + V u^{\#} \, dx + \lambda = \mu \text{ in } D_i.$$

Using this expression and the fact that $u^{\#}$ is the distributional solution of the

Schrödinger equation with potential V and density $\mu^{\#}$:

$$\begin{aligned}\mu^{\#} &= -\Delta u^{\#} + Vu^{\#}\ dx \\ &= \mu - \lambda \text{ in } D_i.\end{aligned} \tag{17}$$

Now, from the Biting Lemma (Theorem 2.12) we can write, for every $k \in \mathbb{N}$:

$$\mu_k = \alpha_k + \sigma_k,$$

where $(\alpha_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ is $\text{cap}_{W^{1,2}}$-equidiffuse, $(\sigma_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega)$ is $\text{cap}_{W^{1,2}}$-concentrating and $\alpha_k \perp \sigma_k$.

Since for every $k \in \mathbb{N}$, $\mu_k \geq 0$ in $\Omega\backslash S$, we also have $(\alpha_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}}$ are nonnegative in D_i. Denoting the diffuse and the concentrated limits of $(\mu_k)_{k\in\mathbb{N}}$, by α and σ respectively, we have $\mu = \alpha + \sigma$. Now we use the fact that, for every $k \in \mathbb{N}$, α_k is nonnegative in $\Omega\backslash S$ and the monotonicity of the weak* limit in Sobolev-open sets, to conclude that $\alpha \geq 0$ in $\Omega\backslash S$, from which follows

$$\begin{aligned}\mu &= \alpha + \sigma \\ &\geq \sigma \text{ in } D_i.\end{aligned}$$

In particular, we have

$$\lambda - \mu \leq \lambda - \sigma \text{ in } D_i. \tag{18}$$

Rewritting Δu_k as

$$\Delta u_k = Vu_k - \mu_k \text{ in } \Omega,$$

and noting that the $\text{cap}_{W^{1,2}}$-concentrated limit of $(Vu_k)_{k\in\mathbb{N}}$ is λ and the $\text{cap}_{W^{1,2}}$-concentrated limit of $(\mu_k)_{k\in\mathbb{N}}$ is σ, we have that $\lambda - \sigma$ is the $\text{cap}_{W^{1,2}}$-concentrated limit of $(\Delta u_k\ dx)_{k\in\mathbb{N}}$. Since $u_k \geq 0$ for every $k \in \mathbb{N}$, we can use the Inverse Maximum Principle for Sequences and conclude that the $\text{cap}_{W^{1,2}}$-concentrated limit of $(\Delta u_k\ dx)_{k\in\mathbb{N}}$ is nonpositive, i.e.

$$\lambda - \sigma \leq 0 \text{ in } \Omega. \tag{19}$$

Using (17), (18) and (19) we obtain

$$\mu^{\#} = \mu - \lambda \geq \sigma - \lambda \geq 0 \text{ in } D_i.$$

This concludes the proof. ■

The next result follows directly from this proposition:

Theorem 3.31. *Let $V : \Omega \to [0, \infty]$ a Borel-measurable function and $(\mu_k)_{k\in\mathbb{N}}$, $(\lambda_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$ be bounded sequences of not necessarily nonnegative measures, with reduced limits $\mu^\#, \lambda^\# \in \mathcal{M}(\Omega)$, respectively. If, for every $k \in \mathbb{N}$,*

$$\mu_k \geq \lambda_k \text{ in } \Omega\backslash S,$$

then

$$\mu^\# \geq \lambda^\# \text{ in } \Omega\backslash S.$$

Proof: First, let us take the sequences $(u_k)_{k\in\mathbb{N}}, (v_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ such that, for every $k \in \mathbb{N}$, u_k is the distributional solution of the Schrödinger problem with potential V and density μ_k, v_k is the distributional solution of the Schrödinger problem with potential V and density λ_k, $u_k \to u^\#$ in $L^1(\Omega)$ and $v_k \to v^\#$ in $L^1(\Omega)$.

Then we define the sequence of measures given by

$$(\mu_k - \lambda_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega).$$

This sequence is nonnegative in $\Omega\backslash S$, and for every $k \in \mathbb{N}$, $u_k - v_k$ is the distributional solution of the Schrödinger problem with potential V and density $\mu_k - \lambda_k$. Moreover, $u_k - v_k \to u^\# - v^\#$ in $L^1(\Omega)$, i.e., $u^\# - v^\#$ is the reduced limit of $(\mu_k - \lambda_k)_{k\in\mathbb{N}}$. Applying Proposition 3.30 we conclude that

$$\mu^\# - \lambda^\# \geq 0 \text{ in } \Omega\backslash S,$$

as we wanted to prove. ■

Now we want to prove the following result concerning the lower semicontinuity of the reduced limit contracted to the set $\Omega\backslash S$:

Theorem 3.32. *Assume $V : \Omega \to [0, \infty]$ a Borel-measurable function and $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{G}(V)$, not necessarily nonnegative measures, be a bounded sequence in $\mathcal{M}(\Omega)$ with reduced limit $\mu^\# \in \mathcal{M}(\Omega)$. Then,*

$$\left|\mu^\#\right| (\Omega\backslash S) \leq \liminf_{k\to\infty} |\mu_k| (\Omega\backslash S).$$

Proof: Let us work with the two following sequences:

$$\left(\mu_k^+\right)_{k\in\mathbb{N}}, \left(\mu_k^-\right)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega).$$

Since $(\mu_k)_{k\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, the sequences $(\mu_k^+)_{k\in\mathbb{N}}$ and $(\mu_k^-)_{k\in\mathbb{N}}$ are also bounded (since $|\mu_k^+|(\Omega) + |\mu_k^-|(\Omega) = |\mu_k|(\Omega)$) and we can define the weak* limits:

$$\mu_k \overset{*}{\rightharpoonup} \mu, \mu_k^+\lfloor_{\Omega\backslash S} \overset{*}{\rightharpoonup} \mu_\oplus, \mu_k^-\lfloor_{\Omega\backslash S} \overset{*}{\rightharpoonup} \mu_\ominus \text{ in } \mathcal{M}(\Omega).$$

And we have that

$$\mu_k = \mu_k^+ - \mu_k^- \implies \mu = \mu_\oplus - \mu_\ominus.$$

We denote the distributional (and duality) solution of the Schrödinger problem with potential V and density μ_k, by $u_k \in L^1(\Omega)$, and the $L^1(\Omega)$ limit of $(u_k)_{k\in\mathbb{N}}$ by $u^\#$. And since the duality solution always exists for any density in $\mathcal{M}(\Omega)$, we denote by v_k and $w_k \in L^1(\Omega)$ the duality solutions of the Schrödinger problem with potential V and density μ_k^+ and μ_k^-, respectively.

Now, we use the fact that the measures μ_k^+ and μ_k^- are nonnegative, and apply Proposition 2.45 to conclude that the following equations hold in the distributional sense:

$$-\Delta v_k + V v_k = \mu_k^+ \lfloor_{\Omega\setminus S} - \lambda_k \text{ in } \Omega,$$

$$-\Delta w_k + V w_k = \mu_k^- \lfloor_{\Omega\setminus S} - \tau_k \text{ in } \Omega,$$

where, for every $k \in \mathbb{N}$, the measures λ_k and τ_k are nonnegative, $\text{cap}_{W^{1,2}}$-diffuse, and $\lambda_k\lfloor_{\Omega\setminus S} = \tau_k\lfloor_{\Omega\setminus S} = 0$. Let us prove that the sequences $(\mu_k^+\lfloor_{\Omega\setminus S} - \lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k^-\lfloor_{\Omega\setminus S} - \tau_k)_{k\in\mathbb{N}}$ are bounded in $\mathcal{M}(\Omega)$.

We know that from identity (2.2) from [24], that

$$\|V v_k\|_{L^1(\Omega)} \leq \|\mu_k^+\|_{\mathcal{M}(\Omega)}.$$

Also from [24], we have, for every $\phi \in C_c^\infty(\Omega)$, that

$$\left| \int_\Omega \phi \, d(\lambda_k - V v_k) \right| \leq C \|\mu_k^+\|_{\mathcal{M}(\Omega)}.$$

By noting that the sequence $(\mu_k^+)_{k\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, we conclude that $(\lambda_k)_{k\in\mathbb{N}}$ is also bounded in $\mathcal{M}(\Omega)$, from which we know that $(\mu_k^+\lfloor_{\Omega\setminus S} - \lambda_k)_{k\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, as we wanted to show. To prove the same result for $(\mu_k^-\lfloor_{\Omega\setminus S} - \tau_k)_{k\in\mathbb{N}}$ we use analogous arguments.

Now, our objective is to prove that the reduced limits of the sequences $(\mu_k^+\lfloor_{\Omega\setminus S} - \lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k^-\lfloor_{\Omega\setminus S} - \tau_k)_{k\in\mathbb{N}}$ exist, i.e., we want to show that $(v_k)_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathbb{N}}$ converge in $L^1(\Omega)$ up to a subsequence.

Since the sequence $(\mu_k^+ - \lambda_k)_{k\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, then we have:

$$\|V v_k\|_{L^1(\Omega)} \leq \|\mu_k^+ - \lambda_k\|_{\mathcal{M}(\Omega)} \leq C,$$

and we conclude that $(\Delta v_k)_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$. This implies that $(v_k)_{k\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ for every $p < \frac{N}{N-1}$. By Rellich-Kondrachov Theorem we have the compact embedded in $L^1(\Omega)$, thus there exists a subsequence of $(v_k)_{k\in\mathbb{N}}$ that converges in $L^1(\Omega)$. We denote this limit in $L^1(\Omega)$ by $v^{\#}$.

We can apply the same reasoning to the sequence $(\mu_k^- - \tau_k)_{k\in\mathbb{N}}$ and find a subsequence of $(w_k)_{k\in\mathbb{N}}$ that converges in $L^1(\Omega)$ to a function we can denote by $w^{\#}$.

Defining the following two measures:

$$\mu_\oplus^{\#} := -\Delta v^{\#} + V v^{\#}, \mu_\ominus^{\#} := -\Delta w^{\#} + V w^{\#} \text{ in } \Omega,$$

in the sense of distributions, we conclude that $\mu_\oplus^{\#}$ and $\mu_\ominus^{\#} \in \mathcal{M}(\Omega)$ are the reduced limits of $(\mu_k^+\llcorner_{\Omega\setminus S} - \lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k^-\llcorner_{\Omega\setminus S} - \tau_k)_{k\in\mathbb{N}}$, respectively.

We can compare the sequences $(\mu_k)_{k\in\mathbb{N}}, (\mu_k^+\llcorner_{\Omega\setminus S} - \lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k^-\llcorner_{\Omega\setminus S} - \tau_k)_{k\in\mathbb{N}}$ in $\Omega\setminus S$, using the definition of positive and negative parts of a measure, and obtain:

$$-\mu_k^- \leq \mu_k \leq \mu_k^+ \text{ in } \Omega \implies -(\mu_k^-\llcorner_{\Omega\setminus S} - \tau_k)\llcorner_{\Omega\setminus S} \leq \mu_k\llcorner_{\Omega\setminus S} \leq (\mu_k^+\llcorner_{\Omega\setminus S} - \lambda_k)\llcorner_{\Omega\setminus S}.$$

Since these sequences have reduced limits, we can apply Theorem 3.31 (and the fact that the reduced limit is linear) to obtain:

$$-\mu_\ominus^{\#} \leq \mu^{\#} \leq \mu_\oplus^{\#} \text{ in } \Omega\setminus S. \tag{20}$$

Let us prove that $\mu_\oplus^{\#} \leq \mu_\oplus$ in Ω.

We want to prove that, for every nonnegative function $\phi \in C_c^\infty(\Omega)$ we have:

$$\int_\Omega -v^{\#}\Delta\phi + V v^{\#}\phi \leq \int_\Omega \phi \, d\mu_\oplus.$$

First of all, we know that since, for every $k \in \mathbb{N}$, $\lambda_k \geq 0$, and we have, in the distributional sense, the equation

$$-\Delta v_k + V v_k = \mu_k^+ - \lambda_k,$$

then

$$-\Delta v_k + V v_k \leq \mu_k^+,$$

that in the distributional sense we can write as

$$\int_\Omega -v_k\Delta\phi + \int_\Omega V v_k\phi \leq \int_\Omega \phi \, d\mu_k^+ \quad \forall\phi \in C_c^\infty(\Omega), \phi \geq 0. \tag{21}$$

Now we can use the fact that $v_k \to v^{\#}$ in $L^1(\Omega)$.

In the first integral of (21), we use the fact that $\Delta\phi$ is bounded, and for this reason, the linear operator $T : L^1(\Omega) \to \mathbb{R}$ defined by

$$T(f) = \int_\Omega f\Delta\phi,$$

is continuous, and therefore, we have

$$\lim_{k\to\infty}\int_\Omega -v_k\Delta\phi = \int_\Omega -v^{\#}\Delta\phi. \tag{22}$$

In the second integral of (21), we use Fatou's Lemma, since $V, v_k, \phi \geq 0$ and $v_k \to v^{\#}$ almost everywhere in Ω, and conclude:

$$\lim_{k\to\infty}\int_\Omega V v_k\phi = \int_\Omega V v^{\#}\phi. \tag{23}$$

Finally, in the last term of (21) we can apply the definition of weak* limit, since $\mu_k^+ \overset{*}{\rightharpoonup} \mu_\oplus$ in $\mathcal{M}(\Omega)$, and obtain:

$$\lim_{k\to\infty}\int_\Omega \phi \; d\mu_k^+. \tag{24}$$

We then have from (21), (22), (23) and (24):

$$\int_\Omega -v^{\#}\Delta\phi + V v^{\#}\phi \leq \int_\Omega \phi \; d\mu_\oplus,$$

as we wanted to conclude.

We prove $-\mu_\ominus \leq -\mu_\ominus^{\#}$ in Ω with an analogous argument. Using these two inequalities and identity (20) we have:

$$-\mu_\ominus \leq \mu^{\#} \leq \mu_\oplus \text{ in } \Omega\backslash S.$$

Taking the positive and negative parts from of the measures in this inequality, we have, respectively:

$$0 \leq (\mu^{\#})^+ \leq \mu_\oplus \text{ in } \Omega\backslash S, \tag{25}$$

and

$$0 \leq (\mu^{\#})^- \leq \mu_\ominus \text{ in } \Omega\backslash S, \tag{26}$$

since we have that $\mu_\oplus, -\mu_\ominus \geq 0$.

Now, we use the weak* limit of $(\mu_k^+\lfloor_{\Omega\backslash S})_{k\in\mathbb{N}}$ and the lower semicontinuity of the total variation norm with respect to the weak* limit, and obtain:

$$\begin{aligned}
\mu_{\oplus}(\Omega\backslash S) &\leq \mu_{\oplus}(\Omega) \\
&= \|\mu_{\oplus}\|_{\mathcal{M}(\Omega)} \\
&\leq \liminf_{k\to\infty} \|\mu_k^{+}\lfloor_{\Omega\backslash S}\|_{\mathcal{M}(\Omega)} \\
&= \liminf_{k\to\infty} \mu_k^{+}(\Omega\backslash S),
\end{aligned}$$

and by the same arguments, we also have

$$\mu_{\ominus}(\Omega\backslash S) \leq \liminf_{k\to\infty} \mu_k^{-}(\Omega\backslash S).$$

Using these two inequalities, together with (25) and (26), we obtain:

$$\begin{aligned}
|\mu^{\#}|(\Omega\backslash S) &\leq \mu_{\oplus}(\Omega\backslash S) + \mu_{\ominus}(\Omega\backslash S) \\
&\leq \liminf_{k\to\infty} \mu_k^{+}(\Omega\backslash S) + \liminf_{k\to\infty} \mu_k^{-}(\Omega\backslash S) \\
&= \liminf_{k\to\infty} |\mu_k|(\Omega\backslash S).
\end{aligned}$$

This concludes the proof. ■

4 Applications

In this chapter we will use the results we achieved to some applications.

We deal with an optimal control problem, a type of optimization problem that involves the minimization of a functional, called the cost functional. Our functional in particular involves a measure.

The first result is an alternative way to measure the L^1-norm of a good measure, which gives us a different way to write the cost functional.

Then we prove the existence of solution to the minimization problem showing that the solution is the reduced limit as defined in the previous chapter. One of the methods we use is proving that the cost functional is lower semicontinuous by using the semicontinuity of the reduced limit, proved on Chapter 3.

Next we investigate a phenomenon called the Lavrentiev phenomenon. This phenomenon in its classical form is about the strict inequality between the infimum of a functional over absolutely continuous trajetories and Lipschitz trajetories.

In our text we prove that for some choice of parameters N and p, our problem satisfies the Lavrentiev phenomenon: the infimum of the cost functional over measures is strictly smaller than the infimum of the cost functional over L^1 functions. This is proved using Sobolev and Newtonian capacities, Hausdorff measures, the properties of the reduced limit and the results proved in the previous chapters.

4.1 Optimal control problem

We want to prove an existence result for a minimization problem related to the Schrödinger problem with potential V and density μ. First let us define a functional $F_p : \mathcal{M}(\Omega) \to [0, \infty]$ in the following way:

$$F_p(\mu) = \begin{cases} \|u - u_d\|_{L^p(\Omega)} + \alpha \|\mu\|_{L^1(\Omega)}, & \text{if the Schrödinger problem with potential V and density } \mu \text{ has a distributional solution } u, \\ \infty, & \text{otherwise,} \end{cases}$$

where $1 \leq p \leq \infty$, $u_d \in L^1(\Omega)$ is a function that we call the ideal state. and $\alpha > 0$.

We want to use a lower semicontinuity type of inequality applied to this functional. Unfortunately, we don't have such result for the $L^1(\Omega)$ norm. The next proposition solves this issue by letting us calculate the total variation of μ in $\Omega \backslash S$ and use the last result from the previous section.

Proposition 4.1. *If $\mu \in L^1(\Omega) \cap \mathcal{G}(V)$, then*

$$\|\mu\|_{L^1(\Omega)} = |\mu| (\Omega \backslash S).$$

Proof: First of all, let's denote by u the distributional solution for the Schrödinger problem with data μ.

Let $\mu = \mu^+ - \mu^-$ be the decomposition of μ in its positive (μ^+) and negative (μ^-) parts given by the Hahn-Jordan Decomposition Theorem. We then have, by Proposition 3.3, [22], that there are $u_1, u_2 \in L^1(\Omega)$ such that u_1 and u_2 are the (unique) duality solutions of the Schrödinger problem with datas μ^+ and μ^-, respectively. By definition, for every $f \in L^\infty(\Omega)$,

$$\int_\Omega u_1 f = \int_\Omega \widehat{\zeta}_f d\mu^+ \text{ and } \int_\Omega u_2 f = \int_\Omega \widehat{\zeta}_f d\mu^-. \tag{27}$$

Now, since μ^+ and μ^- are nonnegative measures, we can use the Remark 4.3 from [24] to conclude that

$$u_1 = u_2 = 0 \text{ a.e. in } S. \tag{28}$$

Taking the difference between the two expressions in (27), we have, for every

$f \in L^\infty(\Omega)$:

$$\int_\Omega (u_1 - u_2) f = \int_\Omega \widehat{\zeta}_f d\left(\mu^+ - \mu^-\right)$$
$$= \int_\Omega \widehat{\zeta}_f d\mu,$$

i.e., $u_1 - u_2$ is a duality solution of the Schrödinger problem with data μ. But we have that u is a distributional solution, then it is also a duality solution of the Schrödinger problem with data μ. Again by Proposition 5, we have that the duality solution is unique, from which we can conclude that $u = u_1 - u_2$, and using (28) we have

$$u = 0 \text{ a.e. in } S.$$

From this, we have that

$$Vu = 0 \text{ a.e. in } S. \tag{29}$$

Next we want to prove estimates about the value of Δu in the set $\{u = 0\}$ (and in particular in S, since $S \subset \{u = 0\}$ as we just concluded). First of all, the absorption estimate from [24] (expression 2.2) states that

$$\|Vu\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}(\Omega)},$$

which means that $Vu \in L^1(\Omega)$. Since we have $-\Delta u + Vu = \mu$, then $\mu \in L^1(\Omega)$ gives us

$$-\Delta u = \mu - Vu \in L^1(\Omega).$$

We can use the Theorem 1.1 from [2]. This theorem gives us (since $u \in L^1_{loc}(\Omega)$ and Δu is locally finite):

$$(\Delta u)_a = 0 \text{ a.e. in } \{u = \alpha\} \cup \{\nabla u = e\}$$

for every $\alpha \in \mathbb{R}$ and every $e \in \mathbb{R}^N$, where $(\Delta u)_a$ is the absolutely continuous part of Δu with respect to the Lebesgue measure. In particular

$$(\Delta u)_a = 0 \text{ a.e. in } \{u = 0\}.$$

It is enough to use the fact that $S \subset \{u = 0\}$ and observe that, since $\Delta u \in L^1(\Omega)$, then $(\Delta u)_a = \Delta u$, to conclude

$$\Delta u = 0 \text{ a.e. in } S. \tag{30}$$

We then have from (29) and (30):

$$\mu = -\Delta u + Vu = 0 \text{ a.e. in } S,$$

from which the result follows directly. ■

Let's now work with the following functional $F : \mathcal{M}(\Omega) \to [0, \infty]$:

$$F(\mu) = \begin{cases} \|u - u_d\|_{L^p(\Omega)} + \alpha\,|\mu|\,(\Omega \backslash S)\,, & \text{if the Schrödinger problem with} \\ & \text{potential V and density } \mu \text{ has a} \\ & \text{distributional solution } u, \\ & \\ \infty, & \text{otherwise,} \end{cases}$$

where $u_d \in L^1(\Omega)$ is a function called the ideal state, $1 \leq p \leq \infty$ and $\alpha > 0$.

Our objective is to prove the following minimization problem:

$$\text{find } \mu^{\#} \in \mathcal{M} \text{ such that } F_{p,u_d}\left(\mu^{\#}\right) = \inf_{\mu \in \mathcal{M}} F_{p,u_d}\left(\mu\right), \tag{P}$$

Our next result states that a solution to this problem exists:

Theorem 4.2. *Assume $V : \Omega \to [0, \infty]$ a Borel-measurable function, $\mu \in \mathcal{M}(\Omega)$ being not necessarily nonnegative, $u_d \in L^1(\Omega)$, $1 \leq p \leq \infty$ and $\alpha > 0$. Then the minimization problem* (P) *has a unique solution $\mu^{\#} \in \mathcal{M}(\Omega)$. Moreover, $\mu^{\#}$ is the reduced limit of any minimizing sequence of the functional F_{p,u_d}, in particular, there exists $u^{\#} \in L^1(\Omega)$, the distributional solution of the problem $[V; \mu^{\#}]$.*

Proof: Our first aim is to prove that F is lower semicontinuous with respect to the reduced limit, i.e., if $(\mu_k) \subset \mathcal{M}$ is a sequence with a reduced limit $\mu^{\#}$, and satisfies, for every $k \in \mathbb{N}, F(\mu_k) < \infty$, then

$$F\left(\mu^{\#}\right) \leq \liminf_{k \to \infty} F\left(\mu_k\right). \tag{31}$$

We start by taking a subsequence of $(\mu_k)_{k \in \mathbb{N}}$ so that the liminf in (31) is a limit. We can keep calling this subsequence $(\mu_k)_{k \in \mathbb{N}}$. Then by the definition of reduced limit we can take, for every $k \in \mathbb{N}$ a function $u_k \in L^1(\Omega)$ so that u_k is a distributional solution of the Schrödinger problem with data μ_k.

Again, using the definition of reduced limit we have that $u_k \to u^{\#}$ in $L^1(\Omega)$ and $u^{\#}$ is the distributional solution of the Schrödinger problem with data $\mu^{\#}$. We can take a subsequence of $(u_k)_{k \in \mathbb{N}}$ to have the almost everywhere convergence, i.e., we have $u_k \to u^{\#}$ a.e. in Ω.

Now if we have $1 \leq p < \infty$ we can use Fatou's Lemma to conclude

$$\left\|u^{\#} - u_d\right\|_{L^p(\Omega)} \leq \liminf_{k \to \infty} \|u_k - u_d\|_{L^p(\Omega)}.$$

If, on the other hand we have $p = \infty$ we have

$$\left|u^{\#} - u_d\right| = \lim_{k\to\infty} |u_k - u_d| \leq \liminf_{k\to\infty} \|u_k - u_d\|_{L^\infty(\Omega)} \text{ a.e. in } \Omega,$$

where the equality comes from the convergence of $(u_k)_{k\in\mathbb{N}}$ and the inequality comes from the definition of the $L^\infty(\Omega)$ norm.

Then, for every $1 \leq p \leq \infty$,

$$\left\|u^{\#} - u_d\right\|_{L^p(\Omega)} \leq \liminf_{k\to\infty} \|u_k - u_d\|_{L^p(\Omega)}. \tag{32}$$

Using the following estimate:

$$\left|\mu^{\#}\right|(\Omega\backslash S) \leq \liminf_{k\to\infty} |\mu_k|(\Omega\backslash S),$$

and (32), we have the semicontinuity of F as we wanted to prove.

The following step is to take a minimizing sequence $(\mu_k) \subset \mathcal{M}(\Omega)$ of F, i.e.,

$$\lim_{k\to\infty} F(\mu_k) = \inf_{\mu\in\mathcal{M}(\Omega)} F(\mu).$$

Since we can always find a measure μ so that the Schrödinger problem has a distributional solution, we have $F \not\equiv \infty$. So we can take our minimizing sequence consisting of measures for which the Schrödinger problem has a distributional solution, i.e., good measures, and conclude, for every $k \in \mathbb{N}, F(\mu_k) < \infty$.

We can also take a subsequence of $(\mu_k)_{k\in\mathbb{N}}$ so that it has a reduced limit $\mu^{\#}$. Using the previously proven lower semicontinuity of F with respect to the reduced limit, we have

$$F\left(\mu^{\#}\right) \leq \liminf_{k\to\infty} F(\mu_k) = \inf_{\mu\in\mathcal{M}(\Omega)} F(\mu).$$

This proves that $\mu^{\#}$ is the minimizer of F in $\mathcal{M}(\Omega)$, i.e. the solution of (P). The theorem is then proved. ∎

4.2 The Lavrentiev phenomenon

Now, we are interested in a version of the Lavrentiev Phenomenon. We wish to prove that the function that minimizes the functional $F_{p,w}$ in $\mathcal{M}(\Omega)$ does not belong to $L^1(\Omega)$. In other words, we want to show that

$$F_{p,w} \not\equiv \infty \text{ in } \mathcal{M}(\Omega) \text{ and } F_{p,w} \equiv \infty \text{ in } L^1(\Omega).$$

We present a lemma that will help us prove a diffusity property. We will not prove it here, but its proof can be found in [26] Proposition 3.1.

Lemma 4.3. *Let $1 \leq p < \infty$, and $K \subset \Omega$ be a compact set such that $cap_{(\Delta,L^p)}(K;\Omega) = 0$. Then there exists a sequence $(\phi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ of nonnegative functions such that*
(i) $\phi_k \to \chi_K$ pointwise, i.e., for every $x \in \Omega$, $\phi_k(x) \to \chi_K(x)$;
(ii) $(\phi_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, i.e., for every $k \in \mathbb{N}$, $\|\phi_k\|_{L^\infty(\Omega)} \leq M$;
(iii) $\Delta\phi_k \to 0$ in $L^p(\Omega)$, i.e., $\|\Delta\phi_k\|_{L^p(\Omega)} \to 0$.

Our first result is the following:

Lemma 4.4. *Let $v \in L^p(\Omega)$, $p > 1$, such that $\Delta v \in \mathcal{M}(\Omega)$. Then Δv is $cap_{(\Delta;L^{p'})}$-diffuse.*

Proof: First of all, we take $\phi \in C_c^\infty(\Omega)$, and since $v \in L^p(\Omega)$ and $\Delta\phi \in L^{p'}(\Omega)$, we have, using the Green's identity and the Hölder inequality:

$$\left|\int_\Omega \phi\Delta v\right| = \left|\int_\Omega v\Delta\phi\right| \leq \|v\|_{L^p(\Omega)}\|\Delta\phi\|_{L^{p'}(\Omega)}. \tag{33}$$

Now let K be a compact subset of Ω such that $\text{cap}_{(\Delta;L^{p'})}(K;\Omega) = 0$. Since $p > 1$ then $1 \leq p' < \infty$ and we can apply Lemma 4.3. Then we know that there exists a sequence $(\phi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ of nonnegative functions such that
(i) $\phi_k \to \chi_K$ pointwise;
(ii) $(\phi_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$;
(iii) $\Delta\phi_k \to 0$ in $L^{p'}(\Omega)$.

Using the expression (33) for the functions ϕ_k, $k \in \mathbb{N}$, we have:

$$\left|\int_\Omega \phi_k\Delta v\right| \leq \|v\|_{L^p(\Omega)}\|\Delta\phi_k\|_{L^{p'}(\Omega)}.$$

Now we use the items (i), (ii) and Lebesgue's Dominated Convergence Theorem to have

$$\left|\int_\Omega \phi_k\Delta v\right| \to \left|\int_K \Delta v\right|.$$

Using (iii) we have:

$$\|v\|_{L^p(\Omega)} \|\Delta \phi_k\|_{Lp'(\Omega)} \to 0.$$

From the last three expressions we conclude

$$\left| \int_K \Delta v \right| = 0.$$

This means that, for every compact $K \subset \Omega$ such that $\mathrm{cap}_{(\Delta;L^{p'})}(K;\Omega) = 0$ we have $(\Delta v \; dx)(K) = 0$.

Let us now take an arbitrary Borel set $A \subset \Omega$ such that $\mathrm{cap}_{(\Delta;L^{p'})}(A;\Omega) = 0$. For every compact K such that $K \subset A$ we also have $\mathrm{cap}_{(\Delta;L^{p'})}(K;\Omega) = 0$ and this implies $(\Delta v \; dx)(K) = 0$. Since $\Delta v \in \mathcal{M}(\Omega)$, we can use the inner regularity of finite Borel measures, that tells us that the measure of any Borel set A can be calculated using only the measure of compact sets contained in A. Then

$$\begin{aligned}(\Delta v \; dx)(A) &= \sup \{(\Delta v \; dx)(K); K \subset A \text{ is compact}\} \\ &= 0\end{aligned}$$

This proves the result. ■

Lemma 4.5. *Let $\frac{N}{N-2} \leq p \leq \infty$, $w \in L^1(\Omega)$ be such that $\Delta w \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ be such that $F_{p,w}(\mu) < \infty$. Then*

$$\mu_c = (-\Delta w)_c,$$

where the subscript c denotes the $\mathrm{cap}_{(\Delta;L^{p'})}$-concentrated part of the measure.

Proof: First of all, since $F_{p,w}(\mu) < \infty$, we have that there exists a distributional solution to the Schrödinger problem with potential V and density μ. Let us denote this solution by u. We also have that $u - w \in L^p(\Omega)$.

Now we know that, since $Vu \in L^1(\Omega)$, then $Vu \in \mathcal{M}(\Omega)$. Also,

$$\Delta u = Vu - \mu \in \mathcal{M}(\Omega),$$

and since $\Delta w \in \mathcal{M}(\Omega)$, we have

$$\Delta(u - w) \in \mathcal{M}(\Omega).$$

By hypothesis, we have that $p \geq \frac{N}{N-2} > 1$. We can then apply Lemma 4.4 to conclude that $\Delta(u - w)$ is $\mathrm{cap}_{(\Delta;L^{p'})}$-diffuse.

We can write:

$$\begin{aligned}\mu &= -\Delta u + Vu \\ &= -\Delta w + \Delta(u-w) + Vu.\end{aligned}$$

We know that $\Delta(u-w)$ and Vu are $\mathrm{cap}_{(\Delta;L^{p'})}$-diffuse, that is, $(\Delta(u-w))_c = (Vu)_c = 0$. Taking the $\mathrm{cap}_{(\Delta;L^{p'})}$-concentrated parts in the last identity gives us

$$\mu_c = (-\Delta w)_c$$

as we wanted to prove. ■

We will also use the following result as a lemma. We will not prove it here, but it is Proposition 2.1 in [24].

Lemma 4.6. *If the Schrödinger problem with potential V and nonnegative density μ has a distributional solution, then, for every $\lambda \in \mathcal{M}(\Omega)$ such that $|\lambda| \leq \mu$, the Schrödinger problem with potential V and density λ also has a distributional solution.*

We wish to prove the following result:

Theorem 4.7. *Let $\mu \in \mathcal{M}(\Omega)$ not necessarily nonnegative, $N \geq 3$ and $\frac{N}{N-2} \leq p < \infty$. Assume $0 \leq V \in L^{q'}(\Omega)$ for some $1 \leq q < p$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Then there exists nonnegative and nontrivial $w \in L^q(\Omega) \cap W_0^{1,r}(\Omega)$, for every $1 \leq r < \frac{N}{N-1}$, distributional solution to the problem $[V;\lambda]$ for some nonnegative and nontrivial $\lambda \in \mathcal{M}(\Omega)$ such that the cost functional $F_{p,w}$ satisfies*

$$F_{p,w} \not\equiv \infty \text{ in } \mathcal{M}(\Omega) \text{ and } F_{p,w} \equiv \infty \text{ in } L^1(\Omega).$$

Proof: Let us split this proof in two cases. The first case is $q \geq \frac{N}{N-2}$.

First of all, we want to find a measure $\mu \in \mathcal{M}(\Omega)$ that is both $\mathrm{cap}_{(\Delta,L^{q'})}$-diffuse and $\mathrm{cap}_{(\Delta,L^{p'})}$-concentrated. For this, we take a Cantor set $K \subset \Omega$ such that

$$0 < \mathcal{H}^{N-2p'}(K) < \infty; \tag{34}$$

for the construction of this set see [4].

We know that K is a compact set and since $\frac{N}{N-2} \leq p < \infty$, then $1 < p' < \frac{N}{2}$. This and (34) gives us (using Theorem 5.1.9 from [1]):

$$\mathrm{cap}_{W^{2,p'}}(K) = 0,$$

and from the Calderón–Zygmund $L^{p'}$ estimates (Corollary 9.10; [16]) we have

$$\mathrm{cap}_{(\Delta,L^{p'})}(K;\Omega) = 0.$$

Since $p > q$, then $p' < q'$ and also

$$\mathrm{cap}_{(\Delta,L^{p'})}(K;\Omega) = 0 \text{ and } \mathrm{cap}_{(\Delta,L^{q'})}(K;\Omega) > 0.$$

Then, using Proposition A.17 from [25] we can conclude the existence of a nonnegative finite Borel measure λ supported in K such that $\lambda(K) = 1$ and, for every nonnegative $\zeta \in C_0^\infty(\overline{\Omega})$,

$$0 \leq \int_\Omega \zeta \, d\lambda \leq C_1 \|\Delta\zeta\|_{L^{q'}(\Omega)}. \tag{35}$$

Also, using the proof of Lemma 4.4, we can conclude that λ is $\mathrm{cap}_{(\Delta,L^{q'})}$-diffuse.

Now we take $u \in L^1(\Omega)$, the distributional solution of the Schrödinger problem with density λ, i.e. u satisfies:

$$\begin{cases} -\Delta u = \lambda \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

and, for every $\zeta \in C_0^\infty(\overline{\Omega})$,

$$\int_\Omega u\Delta\zeta = \int_\Omega \zeta \, d\lambda.$$

Using (35), we have:

$$\left| \int_\Omega u\Delta\zeta \right| \leq C_1 \|\Delta\zeta\|_{L^{q'}(\Omega)}.$$

Using the Riesz Representation Theorem we conclude that $u \in L^q(\Omega)$ and since $V \in L^{q'}(\Omega)$, we have

$$Vu \in L^1(\Omega).$$

Since λ is a nonnegative measure, by the weak maximum principle we have that

$$u \geq 0 \text{ almost everywhere in } \Omega.$$

Now, since $V \geq 0$ (by hypothesis) we have $Vu \geq 0$ and by consequence, for every nonnegative $\zeta \in C_0^\infty(\overline{\Omega})$:

$$\int_\Omega u\Delta\zeta + \int_\Omega Vu\zeta \geq \int_\Omega \zeta \, d\lambda \geq 0.$$

Thus, u is a supersolution for the Schrödinger problem with datum λ. Since the constant function 0 is a subsolution, by [25, Proposition 22.7] there exists a distributional solution of the Schrödinger problem with potential V and density λ. We denote this solution by w.

From Theorem 4.2, the optimal control problem (P) with the desired state w has a solution $\mu^{\#}$ and, in particular, we have

$$F(\mu^{\#}) \leq F(\lambda) = \alpha \|\lambda\|_{\mathcal{M}(\Omega)} < \infty.$$

We now show that for any $\beta \in \mathcal{M}(\Omega)$ with $F(\beta) < \infty$ (which includes $\beta = \mu^{\#}$) we have $\beta \notin L^1(\Omega)$. Indeed, from Lemma (4.5),

$$\alpha_c = (-\Delta w)_c = \lambda_c = \lambda \neq 0. \tag{36}$$

Since being in $L^1(\Omega)$ implies begin diffuse with respect to the capacities, we then have $\alpha \notin L^1(\Omega)$ as claimed.

Now let us prove the remaining case, that is, let us now assume $1 \leq q < \frac{N}{N-2}$. This implies $q' > \frac{N}{2}$, and we have, from the Morrey-Sobolev inequalities that the capacity $\mathrm{cap}_{W^{2,q'}}$ is zero only in the empty set. This means in particular that for every $\mu \in \mathcal{M}(\Omega)$ we have that $\mathrm{cap}_{W^{2,q'}}(A) = 0 \implies A = \varnothing \implies \mu(A) = 0$. This means that

$$\mu \ll \mathrm{cap}_{W^{2,q'}}.$$

Using Proposition 22.8 from [25] we know that, since $V \in L^{q'}(\Omega)$ with $q' > N/2$ there exists a solution for the Schrödinger problem with potential V and every finite measure μ. We take as w the solution associated with $\mu = \mathcal{H}^{N-2p'}\lfloor_K$.

In particular, $F(\mu) < \infty$ and the minimization problem has a finite infimum. We may now proceed as before to deduce (36) for a general measure λ with $F(\lambda) < \infty$.

We then get the desired result when $1 \leq q < \frac{N}{N-2}$. The proof is complete. ■

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