DESIGN OF ARBITRARY ORDER DYNAMICAL OUTPUT FEEDBACK PROTOCOLS FOR MULTI-AGENT SYSTEMS OVER DIRECTED COMMUNICATION GRAPHS

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TESE DE DOUTORADO EM ENGENHARIA ELÉTRICA

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CESSÃO DE DIREITOS

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To my mother.

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RESUMO

Título: PROJETO DE PROTOCOLOS DINÂMICOS POR REALIMENTAÇÃO DE SAÍDA DE ORDEM ARBITRÁRIA PARA SISTEMAS MULTIAGENTES SOBRE GRAFOS DIRECIONADOS
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Brasília, 17 de julho de 2024

Esta tese propõe novas condições para o controle de sistemas multiagentes homogêneos utilizando protocolos de realimentação dinâmica de saída. Os agentes são descritos por uma dinâmica linear e grafos direcionados modelam a rede de comunicação. Primeiramente, novas condições suficientes para projetar protocolos de realimentação dinâmica de saída de ordem arbitrária - incluindo realimentação de saída estática como um caso particular - são propostas para consenso H_∞ para agentes sujeitos a distúrbios externos. Os agentes podem ou não sofrer perturbações paramétricas e comunicar-se em topologias politópicas incertas. Finalmente, propomos condições suficientes para projetar protocolos para rastreamento de formação de saída variante no tempo. Ao considerar uma representação politópica descrevendo as informações da rede, apresentamos um procedimento que permite o projeto dos ganhos do protocolo para uma família de topologias independente do número de agentes. Todas as condições propostas são baseadas em Desigualdades Matriciais Lineares (LMIs, do inglês *Linear Matrix Inequalities*). Os algoritmos propostos nesta tese podem ser aplicados em diversos problemas do mundo real envolvendo consenso de robôs móveis, drones entre outros sistemas semelhantes. Exemplos numéricos ilustram a eficácia da abordagem proposta.

Palavras-chave: Sistemas Multiagentes, Sistemas Lineares, Controle, Desigualdades Matriciais Lineares.

ABSTRACT

Title: DESIGN OF ARBITRARY ORDER DYNAMICAL OUTPUT FEEDBACK PROTOCOLS FOR MULTI-AGENT SYSTEMS OVER DIRECTED COMMUNICATION GRAPHS
Author: Bruno Martins Calazans Silva
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This thesis proposes new conditions for controlling homogeneous multi-agent systems using dynamic output feedback protocols. The agents are described by linear dynamics, and directed graphs model the communication network. First, new sufficient conditions for designing dynamic output feedback protocols of arbitrary order - including static output feedback as a particular case - are proposed for H_∞ consensus for agents subject to external disturbances. The agents may experience parametric perturbations and communicate in uncertain polytopic topologies. Finally, we propose conditions that design protocols for time-varying output formation tracking. By considering a polytopic representation describing the network information, we present a procedure that allows the design of protocol gains for a family of topologies regardless of the number of agents. All proposed conditions are based on Linear Matrix Inequalities (LMIs). The algorithms proposed in this thesis can be applied to various real-world problems involving the consensus of mobile robots, drones, and other similar systems. Numerical examples illustrate the effectiveness of the proposed approach.

Keywords: Multi-agent Systems, Linear Systems, Control, Linear Matrix Inequalities.

SUMMARY

LIST OF FIGURES

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LIST OF SYMBOLS AND NOTATIONS

SOME NOTATIONS

$He\{X\}$	Notation that stands for $X + X^T$ if X is a real matrix and $X + X^*$ if X is a complex matrix;
I_n	The identity matrix of order n;
X^{-1}	Inverse of a square matrix X;
$\mathcal{L}_2^n[0,\infty)$	Denotes the space of square-integrable vectors of n functions over $[0,\infty)$;
$\mathbb{C}^{m\times n}$	The space of complex matrices with dimension $m \times n$;
$\mathbb{R}^n$	The n-dimensional Euclidean space;
$\mathbb{R}_+^*$	The set of the non-negative real numbers;
$\mathbb{R}^{n\times m}$	The space of real matrices with dimension $n \times m$;
$\lambda_{max}(X)$	The maximum eigenvalue of a matrix X;
$\sigma_{max}(X)$	The maximum singular value of a matrix X;
$\lambda_{min}(X)$	The minimum eigenvalue of a matrix X;
$\sigma_{min}(X)$	The minimum singular value of a matrix X;
$\mathbf{1}_m$	Stands for a vector of ones of dimension m;
$\otimes$	Denotes the Kronecker product;
$X_\perp$	Matrix whose columns form a basis for the right null space of X, i.e. $XX_\perp = 0$;
$X \preceq 0$	Indicates that matrix X is negative semidefinite;
$X \prec 0$	Indicates that matrix X is negative definite;
$X_{\|\|}$	Denotes a $n \times m$ matrix such that $XX_{\|\|} = I_m$;
$X_{\underline{\|\|}}$	Denotes a $n \times m$ matrix such that $X_{\underline{\|\|}}X = I_n$;
$X_{\underline{\perp}}$	Denotes a matrix whose rows that form a basis for the left null space of X, i.e. $X_{\underline{\perp}}X = 0$;
$\star$	Denotes symmetric blocks in partitioned matrices;
$X \succeq 0$	Indicates that matrix X is positive semidefinite;
$X \succ 0$	Indicates that matrix X is positive definite;
X^*	Conjugate transpose of a complex matrix X;
X^T	Transpose of a real matrix X;
$0_{m\times n}$	The null matrix of $m \times n$ order.

LIST OF ACRONYMS AND ABBREVIATIONS

HTVOFT H_∞ Time-Varying Output Formation Tracking. 1, 77, 82

LMI Linear Matrix Inequalities. 1, 11, 18, 22–24, 30, 33, 34, 40, 46, 117

MAS Multi-Agent System. 1, 2, 25

UAV Unmanned Aerial Vehicle. 1

1 INTRODUCTION

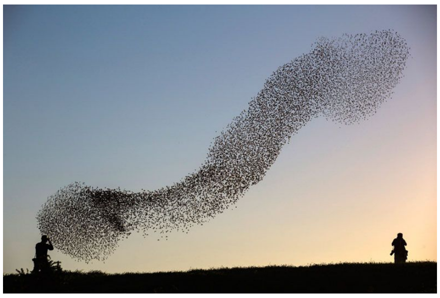
Multi-Agent Systems (MASs) have recently received significant attention due to considerable potential for application in cooperative control of Unmanned Aerial Vehicles (UAVs), multi-satellite control, cloud computing, underwater vehicles, social media, and construction, among others [14], [8], [3]. We can also cite several engineering applications of distributed coordination control for MAS, as indicated in the special issue [59].

The main characteristic of multi-agent systems is the implementation of several nodes or agents distributed around a plant. Agents are entities placed in the environment that sense different variables used to achieve an objective. Each agent has a connection to its neighbor through a communication network. This scheme allows the cooperative control of several subsystems, exchanging information for the agents to achieve the desired purpose [46].

Besides engineering applications, as in comprehensive mathematical models, relations exist between the animals in nature and multi-agent systems concerning the form that some animals move as a group. Various animals have a collective behavior. In some cases, their survival depends on how they move to escape predators, such as a school of fish or migrating birds searching for a more conducive environment. These animals move, creating beautiful and intricate patterns, as shown in Fig. 1.1.

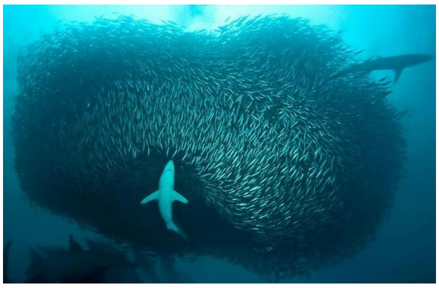
In numerous situations, collective action makes possible the realization of complex tasks that a unique individual can not perform or not achieve easily. As shown in Figure 1.1c, the school of fish can trick sharks, avoiding being preyed upon, without the shoal individuals wasting much energy [42], unlike the sharks. Figure 1.1b shows a flock of migrating birds that, in this formation, can reduce the individual's effort by taking advantage of the wingtip vortex of the ahead bird [26].

A problem studied in obtaining cooperative control is consensus. Consensus relates to the agreement of the agents concerning a variable of the problem. Two known types of consensus are the leaderless consensus, an agent synchronization that does not need a leader, and the leader-follower consensus or consensus tracking when a leader only sends information to the other agents [17]. Comparing the multi-agent systems and the animals' locomotion, observing Figure 1.1a, we see that the locomotion of the birds happens as a cloud, without a specific individual leadership. All birds exchange information and maintain the group unified while moving, representing a leaderless consensus. In Figure 1.1b, it is clear that one individual leads the flying of the migrating birds. All birds create a wingtip vortex sensed and followed by their adjacent birds, which can characterize the leader-follower consensus.



(a) Swarm of birds.

(b) Birds flying in formation.



(c) School of fish trying to escape from sharks.

(d) School of fish in motion.

Figure 1.1 – Locomotion of some animal groups in nature. Source: Google Images

It requires a control law or protocol for existing cooperation between agents, forcing them to reach a consensus with their neighbors and creating a reciprocal agreement regarding a problem variable [46]. The protocol provides the input signal to each agent, usually generated from the information on the interest variables of the neighboring agents. Consensus protocols can assume various forms and use different information about variables depending on the problem. Designing of protocols is widely studied. Depending on the variable used to control the multi-agent system, the design of protocols may be challenging.

1.1 STATE CONSENSUS PROTOCOL DESIGN: AN OVERVIEW

A fundamental task of controlling a network of multi-agent agents is to design a protocol such that all agents work cooperatively and finally reach a consensus. The protocol dictates how agents engage with one another and share information across the network. Many works in the literature deal with the design of protocols for consensus that require the exchange of the agents' states [51], [71], [2], [33]. However, in some situations, the state of the agents is not available, requiring the use of other information about the plant as the agent's outputs.

The problem of designing protocols for consensus of MASs when the states are not available for communication is a recent field of research involving either dynamic output feedback protocols. Some results to design dynamic output feedback protocols have been proposed for undirected communication graphs considering linear homogeneous agents subjected to external disturbances [39, 38, 65, 64], switching networks [39], event or self-triggered mechanisms [38, 70], and adaptive schemes [27]. Most of the previous works consider full-order protocols [38, 70, 65] except for [27], where reduced-order output feedback protocols are designed in several steps subjected to equality constraints, and [39] that designs both reduced- and full-order protocols.

Obtaining design conditions of dynamic output feedback protocols for a directed communication topology is more difficult since the usual transformation of a consensus problem into a stabilizability one leads to nonlinear terms involving the unknown controller parameter matrices. Some solutions reported in the literature assume different hypotheses.

A hypothesis usually considered in implementing dynamic output feedback protocols is the controller interaction, whether the controller transmits state information over the network. Concerning protocols based on controller interaction, [31, 32] deal with directed, [64, 65] with undirected, and [58] with directed and undirected graphs. For instance, in [32], both protocols control the agents with and without controller interaction.

Designing protocols without controller interaction is more involved because there is less information to share [32]. On the other hand, exchanging controller states allows the design of more general protocols that can bring the best consensus performance, with the onus that this type of protocol is less realistic and more challenging to implement [28].

It is possible to design protocols to attenuate the influence of uncertain parameters. The work [69] designs a leader-follower protocol for agents with uncertain parameters with followers connected by undirected networks. In [60], the authors design a new reduced-order protocol for uncertain agents with time-varying interval uncertainties for undirected networks. In [19], an event-triggered protocol is designed based on a state observer for agents with parametric uncertainties and external disturbances distributed in a strongly connected directed network.

Another interesting characteristic found in the literature is the possibility of design protocols that support uncertainty in the topology information. In [34] are designed protocols for continuous and discrete-time agents connected in directed uncertain networks. Some unknown transfer functions or norm-bounded matrices describe the uncertainties. The work [23] studies a class of second-order disturbed multi-agent systems connected by an uncertain network modeled by a polytope.

One of the consensus-based problems receiving ample attention is formation control, which aims for the agents to achieve a specific configuration about a desired variable [45]. In some cases, the prescribed states' formation varies in time, as in [21], which studies the problem of time-varying formation (TVF) control protocols for linear multi-agent systems with actuator failures as in [12], which designs time-varying formation protocols for non-disturbed agents in directed switching networks and [66], that derive LMI conditions for a time-varying formation problem, considering uncertain agents connected in undirected network with communication delays.

In formation control, the problem is called formation tracking or containment, depending on whether one or more leaders guide the agents [45]. In [22] and [24], conditions are proposed to design time-varying formation protocols for disturbed agents guided by a unique leader. The procedure in [22] considers that the agents have uncertain parameters, and in [24], agents have input delays. In [13] and [67], the authors design a time-varying formation protocol that forces the agents to assume a formation following the convex combination of the states of multiple leaders. The formation in [67] is considered in the sampled-data framework, while in [13], in the continuous-time domain. In [20], protocols were designed for the formation of non-disturbed agents for a problem involving followers, formation leaders, and a tracking leader. Finally, [74] studies the design of a static output feedback protocol for the time-varying output containment problem for networks with identical agents.

1.2 MAIN CONTRIBUTIONS

Although the design of protocols for consensus of multi-agent systems has been intensively studied in recent years, to the best of the author's knowledge, no method in the literature can address the problem of designing full, reduced-order, or static output feedback protocols following a single approach.

This work fills a gap in the literature by presenting results on the design of dynamic output-feedback protocols of arbitrary order for multi-agent systems under a directed communication network. Inspired by [1], which addresses the design of output controllers for individual systems, the problem is split into two stages, leading to two LMI design conditions. In the first stage, a stabilizing static state feedback gain is computed. This gain is used in the second stage to obtain the output feedback protocol with the desired order.

The previous literature's works cannot handle the design of dynamic output feedback protocols of arbitrary order for multi-agent systems comprising static, reduced, and full-order controllers. These works of the literature assume more restrictive scenarios, considering full-order controllers (based on full-order observer) [11, 28, 31, 29] reduced controllers with fixed orders, such as the difference between the number of agent's states and inputs [32] or the number of agent's states and outputs[72].

Many works in the literature assume that protocols share controllers' states, as [11, 31, 32] that present multi-step algorithms based on Riccati equations (see Algorithm 1 in [11], Algorithm II in [31], and Algorithm 1 in [32]). For protocols designed by a Riccati equation without relying on the neighbor's controllers' states, one can cite [28, 31, 32]. Our work considers protocols that do not necessarily share controller states (a more difficult scenario [32]) by solving LMI conditions.

We consider the following scenarios:

- State consensus problem: the agents are free of disturbances and uncertainties. The directed network is fixed and known by the designer (Section 3.1).
- H_∞ consensus: the agents are subjected to $\mathcal{L}_2$ disturbances and precisely known. The directed network is fixed and known by the designer (Section 3.2).
- Robust consensus problem: the agents are subjected to $\mathcal{L}_2$ disturbances and parametric uncertainties. The weights of the communication network are uncertain but within known lower and upper bounds (Chapter 4).
- H_∞ time-varying output formation tracking problem: the agents subjected to $\mathcal{L}_2$ disturbances track a leader, and their outputs follow a time-varying formation (Chapter 5). We consider two cases:
  - the weights of the directed communication graph are precisely known by the protocol (Section 5.2);
  - the weights of the directed graph are unknown by the protocols, but the eigenvalues of a certain Laplacian-type matrix are inside a known polytopic region (Section 5.3). The solution, in this case, is appropriate for networks with a large number of agents.

Compared with the existing results in the literature, this work has the following main contributions:

- Conditions of state consensus for designing reduced-order protocols without order constraints. In [32], the proposed algorithm does not design all protocol matrices and presents a solution for a class of reduced-order protocols.
- This work presents the first procedure in the literature to design dynamic H_∞ consensus protocols of any given order for general directed graphs. In particular, this work presents the first technique in the literature to design reduced-order protocols for disturbed agents in directed networks since works in the literature assume only full-order protocols for H_∞ consensus, as in [31] and [28].
- Our work is the first in the literature to propose conditions that design arbitrary-order robust consensus protocols. This novel approach considers time-varying uncertain directed networks, agents subjected to external disturbances, and parametric uncertainty

simultaneously. While [60] considers a time-varying parametric uncertainty, the authors design reduced-order protocols for the agents connected in nominal undirected networks without the influence of external disturbances. In [19], although considering agents with external disturbances and time-varying parametric uncertainty, they design only state observers (full-order protocol) for strongly connected directed graphs (a particular case of directed graphs).

- Regarding the problem of output formation tracking, this work is the first in the literature to provide design conditions for dynamic time-varying output formation tracking protocols with H_∞ performance. Moreover, the design conditions are suitable for any protocol order (including static, reduced, and full-order) chosen by the designer. Among the current works in the literature, only [25] presents conditions for the design of H_∞ static protocols (for the particular case, leader-following). Other related works deal only with systems without disturbance. The works [74] and [25] propose static protocols (for time-varying output containment and leader-following problems, respectively) and [20] present a specific reduced-order leader-following protocol with the order fixed as the difference of the number of the agents' states and inputs.

- We propose conditions for H_∞ time-varying output formation tracking protocols with weaker restrictions in the network communication than the existing similar works in the literature. A particular case of this work is the leader-following problem considered by [25], where the Laplacian-type matrix needs to be normal. In this case, our solution assumes slightly less restrictive conditions for the communication network, where the Laplacian-type matrix needs to be only diagonalizable.

2 PRELIMINARIES

This chapter presents some concepts and results that provide the reader with an introduction to designing output protocols for multi-agent systems.

2.1 GRAPH THEORY

Multi-agent systems are known for the capability of their various agents to perform tasks in cooperation. As the agents exchange information with their neighbors through a communication network, it is necessary to use mathematical techniques to model the agents' connections and how the information travels on the network. Graphs can model the communication network, considering agents as nodes and the connections as edges.

Commonly, literature considers two types of agents' data transmission: agents send and obligatorily receive information from their neighbors or only send or receive information (see, for instance, [65] and [30]). When the network topology has channels that must simultaneously send and acquire data, the network is called undirected; otherwise, the directed ones are less conservative networks that can send or receive information.

(a) Undirected Graph. (b) Directed Graph.

Figure 2.1 – Examples of Graphs.

Figure 2.1 shows in (a) and (b) examples of undirected and directed graphs. The colored circles represent the nodes or agents, and the arrows illustrate the connections or edges between agents, such that the arrows' direction indicates information flow. The following subsection explains the mathematical meaning and modeling of graphs.

2.1.1 Algebraic Graph Theory

A directed graph is denoted by $\mathbb{G}(\mathcal{V}, \mathcal{E}, \mathsf{A})$ where $\mathcal{V} = \{v_1, ..., v_m\}$ is the set of nodes (or vertices), and the set of edges $\mathcal{E} \subseteq \{(v_j, v_i) : i, j = 1, ..., m\}$ describes the connection among nodes. If an ordered pair (v_j, v_i) is an element of $\mathcal{E}$, there is a directed connection from node v_j to node v_i. The adjacency matrix, $\mathsf{A} = [a_{ij}]$, describes the connection weighting with $a_{ij} \geq 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix associated with the graph $\mathbb{G}(\mathcal{V}, \mathcal{E}, \mathsf{A})$ is defined by $L = [l_{ij}]$ with $l_{ii} := \sum_{j=1}^{m} a_{ij}$ and $l_{ij} := -a_{ij}$, for $i \neq j$; and L has complex eigenvalues. A neighbor of node v_i is every node v_j for which $a_{ij} > 0$, and the neighborhood of node v_i is described by the set $\mathcal{N}_i := \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. In an undirected connection it is assumed $a_{ij} = a_{ji} > 0$, and consequently, L is a symmetric positive-semidefinite matrix, with real eigenvalues $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{m-1}$. For any graph, L has the property $L\mathbf{1}_m = 0$ [18].

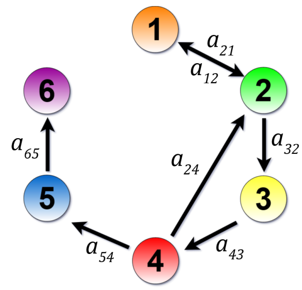


Figure 2.2 – Directed graph and its connection weights.

Figure 2.2 shows that the directed graph's weights definition concerns the network information flow. The node 1 receives and sends information for node 2, then we have $a_{12} = a_{21}$ nonzero weights, and that the node 2, about node 4, only receives information, then we have a_{24} as a nonzero weight. Assuming that all connections' weights assume the value of 1, we can construct the following graph's adjacency and Laplacian matrices

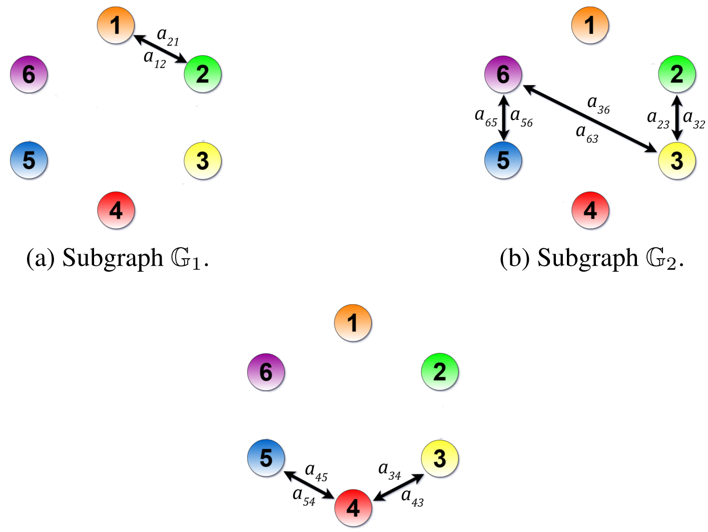
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

2.1.2 Subgraphs

As seen, graph $\mathbb{G}(\mathcal{V}, \mathcal{E}, \mathrm{A})$ is composed of sets of edges and nodes and the adjacency matrix (a set of weights of the graph). Thus, we can consider $\mathbb{G}$ as an union of sets, such that

$$\mathbb{G}(\mathcal{V}, \mathcal{E}, \mathrm{A}) \subseteq \bigcup_{k=1}^{N} \mathbb{G}_k(\mathcal{V}_k, \mathcal{E}_k, \mathrm{A}_k)$$

and consequently, $\mathcal{V} \subseteq \bigcup_{k=1}^{N} \mathcal{V}_k$, $\mathcal{E} \subseteq \bigcup_{k=1}^{N} \mathcal{E}_k$ and $\mathrm{A} = \sum_{k=1}^{N} A_K$. The sets $\mathbb{G}_k$ are called as subgraphs of $\mathbb{G}$. For example, $\mathbb{G}$ is the undirected graph in Figure 2.1a, possibly representing $\mathbb{G}$ as a union of the 3 graphs in Figure 2.3.



(a) Subgraph $\mathbb{G}_1$.

(b) Subgraph $\mathbb{G}_2$.

(c) Subgraph $\mathbb{G}_3$.

Figure 2.3 – Examples of Subgraphs

Based on Figure 2.3 one has that the vertex of each subgraph of $\mathbb{G}$ are $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}_3 \subseteq \{v_1, ..., v_6\}$, their edges are defined as

$$\begin{aligned}
&\mathcal{E}_1 \subseteq \{(v_1, v_2), (v_2, v_1)\}, \\
&\mathcal{E}_2 \subseteq \{(v_2, v_3), (v_3, v_2), (v_3, v_6), (v_6, v_3), (v_6, v_5), (v_5, v_6)\}, \\
&\mathcal{E}_3 \subseteq \{(v_3, v_4), (v_4, v_3), (v_4, v_5), (v_5, v_4)\},
\end{aligned} \tag{2.1}$$

and finally the adjacency matrix A_1 has all inputs zeros except $A_{1,21}$ and $A_{1,12}$; A_2 has the only non-zero inputs $A_{2,32}$, $A_{2,23}$, $A_{2,36}$, $A_{2,63}$, $A_{2,56}$, $A_{2,65}$ and the non-zero inputs of A_3 are $A_{3,54}$, $A_{3,45}$, $A_{3,43}$ and $A_{3,34}$.

Subgraphs are important to define useful concepts of graph theory. In the following subsection, we explore some concepts involving connectivity.

2.1.3 Connectivity

There are some essential concepts regarding the connectivity of the edges and nodes of a graph. In undirected graphs, two vertices are connected when a bidirectional edge exists between them. The trajectory traveled from a node to another connected node is called a **path**. A path that initiates in a vertex and returns for the same vertex, passing through other vertices, is called a **cycle**. Paths and cycles are graphs, where a path or a cycle in a graph $\mathbb{G}$

are its subgraphs [61]. Figure 2.4 illustrates an example of a cycle and path. In green arrows, starting from v_3 is a possible pass for v_4, v_5, and v_6 and returns to v_3, a cycle. A cycle is a path, but otherwise is not valid, i.e., a cycle can include various paths.

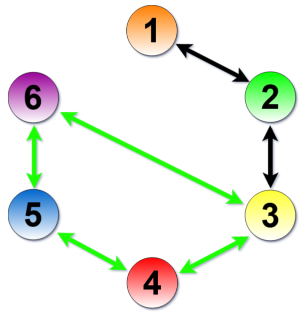


Figure 2.4 – Graph paths

A graph is defined as **connected** when all vertex is connected, forming at least a path. The graph in Figure 2.4 is connected. Even if the edge that links v_3 to v_6 is excluded, the graph in Figure 2.4 is connected. Observe that the subgraphs $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ in Figure 2.3 are not connected graphs. Other interesting definitions are **forest** and **tree**. Using the subgraphs in Fig. 2.3, we can define a graph $\bar{\mathbb{G}} \subseteq \mathbb{G}_1 \cup \mathbb{G}_3$ that has a forest, that is a not connected acyclic graph. A connected forest and every component are called a tree. In the same way, paths are trees [61]. Then $\bar{\mathbb{G}}$ is a forest formed by three trees: the vertex v_6, the connected graph composed by vertices v_3, v_4, v_5 and the connected graph include v_1 and v_2.

A useful definition in consensus problems is the **spanning tree**. In a connected graph, a spanning tree is a subgraph or a tree containing all the graph's vertices [61]; it can also be considered a path connecting all vertices. This definition is important because agents must be connected to achieve consensus in fixed topology cases [48], [32]. Even in switching topology cases [51], the union of switched topologies must form at least a spanning tree. The following lemma is a well-known result in literature that relates spanning trees with a zero eigenvalue in the Laplacian matrix.

LEMMA 2.1 ([48]) A graph has a spanning tree if, and only if, the zero eigenvalue of the associated Laplacian matrix has one as algebraic multiplicity.

Lemma 2.1 does not distinguish the type of graph; directed and undirected graphs satisfy the proposed condition. Moreover, all definitions listed here are easily extended for directed graphs.

3 CONSENSUS FOR NON-DISTURBED AND DISTURBED AGENTS

This chapter presents Linear Matrix Inequalities (LMI) conditions for consensus of homogeneous multi-agent systems subjected to exogenous disturbances in directed communication graphs by dynamic output feedback protocols. The agents under investigation are described as linear dynamics, and the communication network is such that each agent receives only the relative output of neighbor agents as information. We propose new necessary and sufficient conditions for designing dynamic output feedback controllers of arbitrary order – including static output feedback as a particular case – and sufficient Linear Matrix Inequalities (LMIs) for H_∞ consensus.

This chapter presents new conditions for the consensus of agents in directed networks that surpass some gaps in the literature. Some works in the literature design protocols for consensus considering an undirected network. In [38] is presented the design of an observe-based H_∞ protocol for event-triggered consensus. In [64] are presented non-convex conditions that design full-order protocols for agents subjected to uncertainties and external disturbances. The authors in [65] design convex conditions for full-order protocols for disturbed agents. In [27], adaptive fully distributed reduced-order protocols are designed for non-disturbed agents.

Directed networks are more general and can model more complex network connections between the agents. Some work in literature deals with the design of protocols for agents connected in directed networks. In [72], reduced-order protocols are designed for non-disturbed agents. The protocols transmit the agent's inputs through the network communication. In [32], reduced-order protocols are designed for the consensus of non-disturbed agents. Protocols in [32] can deal with controller interaction and not. In [28], full-order protocols are designed for disturbed agents. In [29], full-order protocols are designed for non-disturbed agents. In [31], full-order protocols are designed for uncertain agents with outputs subject to external disturbances.

Table 3.1 summarizes the contribution of this chapter and the previous discussion on the design of protocols concerning the literature. Some works received $\circ$ in the disturbances line in Table 3.1 for not considering disturbances in agents' dynamics and outputs simultaneously. Works that received $\circ$ in the Reduced-Order line have restrictions in protocol order, presenting solutions for a class of reduced-order protocols. Works that received $\circ$ in the Disturbances line do not simultaneously present disturbances in agents' dynamics and outputs. Finally, works that present $\circ$ in the LMI line present a solution where are used a multi-step algorithm based on the LMI and Riccati equation.

	T3.2	T3.4	[32]	[28]	[29]	[31]	[72]	[38]	[65]	[64]	[27]
Static Output Feedback	✓	✓	✓	×	×	×	✓	×	×	×	×
Reduced-Order	✓	✓	○	×	×	×	○	×	×	×	✓
Full-Order	✓	✓	×	✓	✓	✓	×	✓	×	✓	×
Digraph	✓	✓	✓	✓	✓	✓	✓	×	×	×	×
Disturbances	×	✓	×	○	×	○	×	○	✓	✓	×
No Rank Restriction	✓	✓	×	✓	✓	✓	×	×	✓	✓	×
No Controller Interaction	✓	✓	✓	×	×	×	✓	×	×	×	×
LMI	✓	✓	×	✓	✓	○	×	✓	✓	×	×

Table 3.1 – Comparison between Theorems 3.2 (T3.2) and 3.4 (T3.4) concerning literature results. The symbols ✓means "yes", $\times$ means "no" and $\circ$ means "partially".

3.1 STATE CONSENSUS PROTOCOLS

3.1.1 The State Consensus Problem

Consider m agents in a directed network, each with the following dynamic model

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_u u_i(t), \quad i = 1, ..., m \\ y_i(t) &= C_y x_i(t) \end{aligned} \tag{3.1}$$

where $x_i(t) \in \mathbb{R}^n$ are the state variables, $y_i(t) \in \mathbb{R}^q$ the outputs, $u_i(t) \in \mathbb{R}^{n_u}$ the control inputs, $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times n_u}$ and $C_y \in \mathbb{R}^{q\times n}$. We assume that the overall system has a distributed pattern where each agent embeds a local controller according to the following structure

$$\begin{aligned} \dot{x}_{c,i}(t) &= A_c x_{c,i}(t) + B_c \nu_i(t), \quad i = 1, ..., m \\ u_i(t) &= C_c x_{c,i}(t) + D_c \nu_i(t), \end{aligned} \tag{3.2}$$

where $x_{c,i}(t) \in \mathbb{R}^{n_c}$ is the state variables of the distributed dynamical controller, $A_c \in \mathbb{R}^{n_c\times n_c}$, $B_c \in \mathbb{R}^{n_c\times q}$, $C_c \in \mathbb{R}^{s\times n_c}$, $D_c \in \mathbb{R}^{s\times q}$ the dynamic controller parameters to be found, and $\nu_i(t) \in \mathbb{R}^q$ a signal that has access only to local information defined in terms of its own agent output and the output of its neighboring agents is provided by the communication network as

$$\nu_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(y_i(t) - y_j(t)) \quad i = 1, ..., m. \tag{3.3}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ weight the communication from agent j to agent i, and $\mathcal{N}_i$ is the set of neighbor agents of i.

PROBLEM 3.1 (State Consensus Problem) For the multi-agent system (3.1), design protocol (3.2)-(3.3) of given order n_c, $0 \leq n_c \leq n$, such that the resulting closed-loop multi-agent system achieves asymptotic overall state consensus for all initial conditions, defined as

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, \quad \lim_{t\to\infty} ||x_{c,i}(t) - x_{c,j}(t)|| = 0, \; i, j = 1, ..., m.$$

3.1.2 Transformed Closed-loop Multi-Agent System

In this section, the state consensus problem of the multi-agent system (3.1) under the protocol (3.2)-(3.3) is transformed into a stabilization problem. We define the concatenation of indexed vectors $x_1, \ldots, x_N$, as $x \triangleq [x_1^T \; x_2^T \; \cdots \; x_N^T]^T$. The augmented system with the concatenated variables $x(t)$, $y(t)$ and $u(t)$ considering the m agents in (3.1) is given by

$$\begin{aligned} \dot{x}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B_u)u(t), \\ y(t) &= (I_m \otimes C_y)x(t). \end{aligned} \tag{3.4}$$

The augmented dynamical controller is given by

$$\begin{aligned} \dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) + (I_m \otimes B_c)\nu(t), \\ u(t) &= (I_m \otimes C_c)x_c(t) + (I_m \otimes D_c)\nu(t), \end{aligned} \tag{3.5}$$

where the augmented dynamic controller state variable is concatenated and defined as $x_c(t)$, and the concatenated signal $\nu(t)$ can be expressed by

$$\begin{aligned} \nu(t) &= -\begin{bmatrix} \sum_{j\neq 1, j=1}^{m} a_{1j}(y_1 - y_j) \\ \vdots \\ \sum_{j\neq m, j=1}^{m} a_{mj}(y_m - y_j) \end{bmatrix} = -\begin{bmatrix} l_{11}C_y x_1 + \sum_{j=2}^{m} l_{1j}C_y x_j \\ \vdots \\ \sum_{j=1}^{m-1} l_{mj}C_y x_j + l_{mm}C_y x_m \end{bmatrix}, \\ &= -\begin{bmatrix} \sum_{j=1}^{m} l_{1j}C_y x_j \\ \vdots \\ \sum_{j=1}^{m} l_{mj}C_y x_j \end{bmatrix} = -\begin{bmatrix} l_{11}C_y & \cdots & l_{1m}C_y \\ \vdots & \ddots & \vdots \\ l_{m1}C_y & \cdots & l_{mm}C_y \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \\ &= -(L \otimes C_y)x(t). \end{aligned} \tag{3.6}$$

Consequently, by applying the Kronecker mixed product property, one has $(I_m \otimes B_c)(L \otimes$

$C_y) = (L \otimes B_c C_y)$ and the augmented protocol (3.5)-(3.6), can be rewritten as follows

$$\begin{aligned} \dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) - (L \otimes (B_c C_y))x(t), \\ u(t) &= (I_m \otimes C_c)x_c(t) - (L \otimes (D_c C_y))x(t). \end{aligned} \tag{3.7}$$

From (3.7) and (3.4) one has the closed-loop multi-agent system

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix}}_{\dot{\psi}(t)} = \underbrace{\begin{bmatrix} (I_m \otimes A) - (I_m \otimes B_u)(L \otimes (D_c C_y)) & (I_m \otimes B_u)(I_m \otimes C_c) \\ -(L \otimes (B_c C_y)) & (I_m \otimes A_c) \end{bmatrix}}_{\underline{\mathbf{A}}} \underbrace{\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}}_{\psi(t)}. \tag{3.8}$$

From mixed product property of Kronecker product, $(I_m \otimes B_u)(I_m \otimes C_c) = (I_m \otimes B_u C_c)$ and $(I_m \otimes B_u)(L \otimes (D_c C_y)) = (L \otimes (B_u D_c C_y))$, then (3.8) can be rewritten in the following form

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix}}_{\dot{\psi}(t)} = \underbrace{\begin{bmatrix} (I_m \otimes A) - (L \otimes (B_u D_c C_y)) & (I_m \otimes B_u C_c) \\ -(L \otimes (B_c C_y)) & (I_m \otimes A_c) \end{bmatrix}}_{\underline{\mathbf{A}}} \underbrace{\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}}_{\psi(t)} \tag{3.9}$$

Inspired in [55], the use of a tree-type transformation is one form of translating the consensus problem into a stability problem, introducing new variables that represent the disagreement of the agents $\zeta_{1,i} = x_1(t) - x_{i+1}(t)$, as well as the disagreement of the states of dynamic controllers $\zeta_{2,i} = x_{c,1}(t) - x_{c,i+1}(t)$, for $i = 1, ..., m-1$. Observe that the stability of the transformed system guarantees the agents' consensus and dynamic controllers' consensus of their variables.

In order to transform the system in closed-loop (3.9), which is based on $\psi(t)$ variable, into the variable $\zeta(t)$, it is needed to use the following definitions

$$\zeta(t) = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \bar{U}\psi(t), \tag{3.10}$$

where

$$\bar{U} = \begin{bmatrix} U \otimes I_n & 0_{(m-1)n \times mn_c} \\ 0_{(m-1)n_c \times mn} & U \otimes I_{n_c} \end{bmatrix}, U = \begin{bmatrix} \mathbf{1}_{m-1} & -I_{m-1} \end{bmatrix}.$$

The state variable $\psi(t)$ can be recovered by the expression

$$\psi(t) = \begin{bmatrix} \mathbf{1}_m \otimes x_1(t) \\ \mathbf{1}_m \otimes x_{c,1}(t) \end{bmatrix} + \bar{W}\zeta(t), \tag{3.11}$$

where,

$$\bar{W} = \begin{bmatrix} W \otimes I_n & 0_{mn\times(m-1)n_c} \\ 0_{mn_c\times(m-1)n} & W \otimes I_{n_c} \end{bmatrix}, W = \begin{bmatrix} 0_{m-1}^T \\ -I_{m-1} \end{bmatrix}.$$

Considering the time derivative of $\zeta(t) = \bar{U}\psi(t)$, and substituting (3.9) in the result, it can be expressed in the following form

$$\dot{\zeta}(t) \;\; = \;\; \bar{U}\underline{\mathbf{A}}\psi(t). \tag{3.12}$$

Replacing (3.11) in (3.12), it is obtained

$$\dot{\zeta}(t) \;\; = \;\; \bar{U}\left(\underline{\mathbf{A}}\Bigg(\begin{bmatrix} 1_m \otimes x_1(t) \\ 1_m \otimes x_{c,1}(t) \end{bmatrix} + \bar{W}\zeta(t)\Bigg)\right). \tag{3.13}$$

Replacing $\underline{\mathrm{A}}$ in the previous expression, it is obtained

$$\begin{aligned} \dot{\zeta}(t) \;\; = \;\; & \Bigg(\bar{U}\begin{bmatrix} (I_m \otimes A) - (L \otimes (B_u D_c C_y)) & (I_m \otimes B_u C_c) \\ -(L \otimes (B_c C_y)) & (I_m \otimes A_c) \end{bmatrix}\begin{bmatrix} 1_m \otimes x_1(t) \\ 1_m \otimes x_{c,1}(t) \end{bmatrix} \\ & + \bar{U}\begin{bmatrix} (I_m \otimes A) - (L \otimes (B_u D_c C_y)) & (I_m \otimes B_u C_c) \\ -(L \otimes (B_c C_y)) & (I_m \otimes A_c) \end{bmatrix}\bar{W}\zeta(t)\Bigg). \end{aligned} \tag{3.14}$$

Replacing and multiplying the matrices $\bar{U}$ and $\bar{W}$ in (3.14), then applying the Kronecker product property $(X \otimes Y)(N \otimes M) = (XN \otimes YM)$, the expression (3.14) is rewritten as

$$\begin{aligned} \dot{\zeta}(t) \;\; = \;\; & \Bigg(\begin{bmatrix} (U1_m \otimes Ax_1(t)) + (U1_m \otimes B_u C_c x_{c,1}(t)) \\ (U1_m \otimes A_c x_{c,1}(t)) \end{bmatrix} \\ & + \begin{bmatrix} (UW \otimes A) & (UW \otimes B_u C_c) \\ 0_{(m-1)n_c\times(m-1)n} & (UW \otimes A_c) \end{bmatrix}\zeta(t) \\ & - \begin{bmatrix} UL1_m \otimes (B_u D_c C_y)x_1(t) \\ UL1_m \otimes (B_c C_y)x_1(t) \end{bmatrix} \\ & - \begin{bmatrix} ULW \otimes (B_u D_c C_y) & 0_{(m-1)n\times(m-1)n_c} \\ ULW \otimes (B_c C_y) & 0_{(m-1)n_c\times(m-1)n_c} \end{bmatrix}\zeta(t)\Bigg). \end{aligned} \tag{3.15}$$

Using the properties $U1_m = 0$, $L1_m = 0_m$ and $UW = I_{m-1}$, and defining $\bar{L} = ULW$, one has

$$\dot{\zeta}(t) = \tilde{A}\zeta(t), \tag{3.16}$$

where,

$$\tilde{A} = \begin{bmatrix} (I_{m-1} \otimes A) - (\bar{L} \otimes (B_u D_c C_y)) & (I_{m-1} \otimes B_u C_c) \\ -(\bar{L} \otimes (B_c C_y)) & (I_{m-1} \otimes A_c) \end{bmatrix}.$$

Therefore, the state consensus problem (Problem 3.1) is equivalent to the stability of (3.16), i.e., the problem of finding matrices A_c, B_c, C_c and D_c for protocol (3.2)-(3.3) such that multi-agent system (3.1) reach consensus is equivalent to find matrices A_c, B_c, C_c and D_c such that (3.16) is asymptotically stable.

The graph connectivity is essential for agents' protocol design for consensus. For agents 3.1 to achieve consensus, there must exist at least a spanning tree connecting all agents [48], [32]. Therefore, we consider the following assumption.

ASSUMPTION 3.1 The graph $\mathbb{G}$ has a spanning tree.

Even though Assumption 3.1 is not explicitly required in the proof of the proposed conditions, it implies that all agents have at least one communication path between them, which is necessary for reaching a consensus [32], [51]. So, even if Assumption 3.1 is not verified beforehand, the network has a spanning tree if the proposed design conditions are verified.

3.1.3 Design of State Consensus Protocols

In the following, we will derive a design condition for consensus protocol for general directed graphs to design n_c-order protocols ($0 \leq n_c \leq n$) without rank restriction in C_y.

Observe that, using the Kronecker mixed product property, system (3.16) can be rewritten as

$$\dot{\zeta}(t) = \begin{bmatrix} (I_{m-1} \otimes A) - \Xi & (I_{m-1} \otimes B_u)(I_{m-1} \otimes C_c) \\ -(I_{m-1} \otimes B_c)(\bar{L} \otimes C_y) & (I_{m-1} \otimes A_c) \end{bmatrix} \zeta(t), \tag{3.17}$$

with $\Xi = (I_{m-1} \otimes B_u)(I_{m-1} \otimes D_c)(\bar{L} \otimes C_y)$, then the system (3.17) can be rewritten in the following form

$$\dot{\zeta}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C})\zeta(t), \tag{3.18}$$

where,

$$\mathcal{A} = \begin{bmatrix} (I_{m-1} \otimes A) & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 & (I_{m-1} \otimes B_u) \\ I & 0 \end{bmatrix},$$
$$\mathcal{K}_y = \begin{bmatrix} (I_{m-1} \otimes A_c) & (I_{m-1} \otimes B_c) \\ (I_{m-1} \otimes C_c) & (I_{m-1} \otimes D_c) \end{bmatrix}, \mathcal{C} = \begin{bmatrix} 0 & I \\ -(\bar{L} \otimes C_y) & 0 \end{bmatrix}.$$

The following lemma is useful in further developments, with the objective in the search for a $\mathcal{K}_y$ that stabilizes system (3.17).

Lemma 3.1 (Elimination Lemma [7],[54]) Let $\mathcal{Q} \in \mathbb{C}^{n\times n}$, $\mathbf{U} \in \mathbb{C}^{p\times n}$ and $\mathbf{V} \in \mathbb{C}^{q\times n}$ be given matrices with $\mathcal{Q}^T = \mathcal{Q}$. Then, there exists a matrix $\mathcal{X} \in \mathbb{R}^{p\times q}$ such that

$$\mathcal{Q} + \mathbf{U}^T\mathcal{X}\mathbf{V} + \mathbf{V}^T\mathcal{X}^T\mathbf{U} \prec 0, \tag{3.19}$$

if, and only if, the following inequalities are satisfied

$$\begin{matrix} (\mathbf{U}_\perp^T\mathcal{Q}\mathbf{U}_\perp \prec 0 & \text{or} & \mathbf{U}^T\mathbf{U} \succ 0) \\ & \text{and} & \\ (\mathbf{V}_\perp^T\mathcal{Q}\mathbf{V}_\perp \prec 0 & \text{or} & \mathbf{V}^T\mathbf{V} \succ 0). \end{matrix} \tag{3.20}$$

The following theorem presents a new necessary and sufficient condition for solving the output feedback problem to derive design conditions for consensus protocol posteriorly.

Theorem 3.1

There exist a matrix $\mathcal{K}_y$ such that the system (3.18) is asymptotically stable if and only if there exist matrices $\mathcal{K}_x$, $P = P^T \succ 0$, X_1, X_2, $\mathcal{G}$, F_1, F_2, F_3 and $\mathcal{Z}$ such that the following inequality holds

$$\begin{bmatrix} \Psi_1 & \star & \star & \star \\ \Psi_2 & -F_2 - F_2^T & \star & \star \\ \Psi_3 & \mathcal{B}^T F_2^T & -\mathcal{G} - \mathcal{G}^T & \star \\ X_1^T - F_1^T & X_2^T - F_2^T - F_3 & F_3\mathcal{B} & -F_3 - F_3^T \end{bmatrix} \prec 0, \tag{3.21}$$

with

$$\Psi_1 = X_1(\mathcal{A} + \mathcal{B}\mathcal{K}_x) + (\mathcal{A} + \mathcal{B}\mathcal{K}_x)^T X_1^T, \quad \Psi_2 = P - F_1^T + X_2(\mathcal{A} + \mathcal{B}\mathcal{K}_x),$$
$$\Psi_3 = \mathcal{Z}\mathcal{C} - \mathcal{G}\mathcal{K}_x + \mathcal{B}^T F_1^T.$$

Furthermore, if (3.21) holds, $\mathcal{K}_y := \mathcal{G}^{-1}\mathcal{Z}$ is a stabilizing gain for (3.18).

proof.

From Lyapunov stability analysis, the system (3.18) is stabilizable with respect to the parameter matrix $\mathcal{K}_y$ if and only if there exist matrices $\mathcal{K}_y$ and $P = P^T \succ 0$ such that

$$\begin{aligned} 0 &\succ (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C}) \\ &= \underbrace{\begin{bmatrix} I_{n_x} \\ \mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C} \end{bmatrix}^T}_{\mathbf{V}_\perp^{\mathbf{T}}} \underbrace{\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}}_{\mathcal{Q}} \underbrace{\begin{bmatrix} I_{n_x} \\ \mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C} \end{bmatrix}}_{\mathbf{V}_\perp}. \end{aligned} \tag{3.22}$$

Applying the Lemma 3.1 by assuming $\mathbf{U} := I$, the matrices $\mathcal{Q}$ and $\mathbf{V}_\perp$ identified as in (3.22), and using the fact that $\mathbf{V} := \begin{bmatrix} \mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C} & -I_{n_x} \end{bmatrix}$ has $\mathbf{V}_\perp$ as an orthogonal complement, the previous stabilizability characterization is equivalent to the feasibility on matrices $\mathcal{K}_y$, $P = P^T \succ 0$, X_1 and X_2 of the inequality

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + He\left\{ \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C}) & -I \end{bmatrix}}_{\mathbf{V}} \right\} \prec 0, \tag{3.23}$$

or equivalently

$$\begin{bmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C})^T X_1^T + X_1(\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C}) & \star \\ P - X_1^T + X_2(\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C}) & -X_2 - X_2^T \end{bmatrix} \prec 0. \tag{3.24}$$

In turn, one can note that the feasibility of (3.24) implies the feasibility on $\mathcal{K}_x$, $P = P^T \succ 0$, X_1 and X_2 of the inequality

$$\begin{bmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K}_x)^T X_1^T + X_1(\mathcal{A} + \mathcal{B}\mathcal{K}_x) & \star \\ P - X_1^T + X_2(\mathcal{A} + \mathcal{B}\mathcal{K}_x) & -X_2 - X_2^T \end{bmatrix} \prec 0, \tag{3.25}$$

by taken $\mathcal{K}_x := \mathcal{K}_y\mathcal{C}$, for instance. Further, one can note that any $\mathcal{K}_x$ solution of (3.25) is a stabilizing static state-feedback gain (in fact, by using the Lemma 3.1 and the usual LMI Lyapunov stability characterization, (3.25) implies that $\mathcal{K}_x$ is such that $\mathcal{A} + \mathcal{B}\mathcal{K}_x$ is stable).

Conversely, it is clear that the simultaneous feasibility on $\mathcal{K}_y$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1

and X_2 of the inequalities (3.24)-(3.25) is equivalent to the feasibility of

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + He\left\{ \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C}) & -I \end{bmatrix}}_{\mathbf{V}} \right\} \prec 0, \tag{3.26}$$

and

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + He\left\{ \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} (\mathcal{A} + \mathcal{B}\mathcal{K}_x) & -I \end{bmatrix}}_{\mathbf{V}_1} \right\} \prec 0, \tag{3.27}$$

Pre- and post-multiplying inequality (3.26) by $\mathbf{V} := \begin{bmatrix} \mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C} & -I \end{bmatrix}$ and inequality (3.27) by $\mathbf{V}_1 := \begin{bmatrix} \mathcal{A} + \mathcal{B}\mathcal{K}_x & -I \end{bmatrix}$, one has

$$0 \succ (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C})^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C}), \tag{3.28}$$

and

$$0 \succ (\mathcal{A} + \mathcal{B}\mathcal{K}_x)^T P + P(\mathcal{A} + \mathcal{B}\mathcal{K}_x). \tag{3.29}$$

Therefore, one has that stabilizability of (3.18) in terms of $\mathcal{K}_y$ is equivalent to the simultaneous feasibility on $\mathcal{K}_y$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1 and X_2 of the inequalities (3.24) and (3.25).

To couple the design of a stabilizing state feedback gain $\mathcal{K}_x$ to that of a stabilizing output feedback gain $\mathcal{K}_y$, let us rewrite the simultaneous inequalities as just one inequality by using the Lemma 3.1. First, note that the inequalities (3.24)-(3.25) can be rewritten as

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & I \\ \mathcal{K}_y\mathcal{C} - \mathcal{K}_x & 0 \end{bmatrix}^T}_{\mathbf{V}_{1\perp}^{\mathbf{T}}} \mathcal{Q}_1 \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \\ \mathcal{K}_y\mathcal{C} - \mathcal{K}_x & 0 \end{bmatrix}}_{\mathbf{V}_{1\perp}} \prec 0, \tag{3.30}$$

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}^T}_{\mathbf{U}_{1\perp}^{\mathbf{T}}} \mathcal{Q}_1 \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}}_{\mathbf{U}_{1\perp}} \prec 0, \tag{3.31}$$

respectively, where,

$$\mathcal{Q}_1 = \begin{bmatrix} \Psi_1 & \star & \star \\ \Psi_2 - X_1^T + F_1^T & -X_2 - X_2^T & \star \\ \mathcal{B}^T X_1^T & \mathcal{B}^T X_2^T & 0 \end{bmatrix}.$$

Applying the Lemma 3.1 with the identifications $\mathcal{Q}_1$, $\mathbf{U_{1\perp}}$ and $\mathbf{V_{1\perp}}$ in (3.30)-(3.31) and with the choice $\mathbf{U_1} = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$ and $\mathbf{V_1} = \begin{bmatrix} \mathcal{K}_y\mathcal{C} - \mathcal{K}_x & 0 & -I \end{bmatrix}$, one has that simultaneous feasibility of (3.24)-(3.25) is equivalent to feasibility on $\mathcal{K}_y$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1, X_2 and $\mathcal{G}$ of the inequality

$$\underbrace{\begin{bmatrix} \Psi_1 & \star & \star \\ \Psi_2 - X_1^T + F_1^T & -X_2 - X_2^T & \star \\ \mathcal{B}^T X_1^T & \mathcal{B}^T X_2^T & 0 \end{bmatrix}}_{\mathcal{Q}_1} + He\{\underbrace{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}}_{\mathbf{U}_1^T} \mathcal{G} \underbrace{\begin{bmatrix} \mathcal{K}_y\mathcal{C} - \mathcal{K}_x & 0 & -I \end{bmatrix}}_{\mathbf{V}_1}\} \prec 0, \quad (3.32)$$

or,

$$\begin{bmatrix} \Psi_1 & \star & \star \\ \Psi_2 - X_1^T + F_1^T & -X_2 - X_2^T & \star \\ \mathcal{B}^T X_1^T + \mathcal{G}\mathcal{K}_y\mathcal{C} - \mathcal{G}\mathcal{K}_x & \mathcal{B}^T X_2^T & -\mathcal{G} - \mathcal{G}^T \end{bmatrix} \prec 0. \quad (3.33)$$

Defining $\mathcal{Z} := \mathcal{G}\mathcal{K}_y$, it is clear that feasibility of (3.33) implies feasibility on $\mathcal{Z}$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1, X_2 and $\mathcal{G}$ of the inequality

$$\begin{bmatrix} \Psi_1 & \star & \star \\ \Psi_2 - X_1^T + F_1^T & -X_2 - X_2^T & \star \\ \mathcal{B}^T X_1^T + \mathcal{Z}\mathcal{C} - \mathcal{G}\mathcal{K}_x & \mathcal{B}^T X_2^T & -\mathcal{G} - \mathcal{G}^T \end{bmatrix} \prec 0. \quad (3.34)$$

Conversely, if (3.34) is feasible, then from its $(3, 3)$ term, one has that $\mathcal{G}$ is invertible and so, defining $\mathcal{K}_y = \mathcal{G}^{-1}\mathcal{Z}$, the inequality (3.33) is feasible on $\mathcal{K}_y$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1, X_2 and $\mathcal{G}$. Since all steps in the proof up to now are necessary and sufficient, we conclude that stabilizability of (3.18) with respect to $\mathcal{K}_y$ is equivalent to the feasibility on $\mathcal{Z}$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1, X_2 and $\mathcal{G}$ of the inequality (3.34).

Still, for better numerical performance, one can introduce more slack variables by applying the Lemma 3.1 once more. First note that feasibility of (3.34) is equivalent to simultaneous feasibility of (3.34) and of its part

$$0 \succ -\mathcal{G} - \mathcal{G}^T = \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T}_{\mathbf{U_{2\perp}^T}} \underbrace{\begin{bmatrix} \Psi_1 & \star & \star & \star \\ \Psi_2 + F_1^T & 0 & \star & \star \\ \mathcal{G}(\mathcal{K}_y \mathcal{C} - \mathcal{K}_x) & 0 & -\mathcal{G} - \mathcal{G}^T & \star \\ X_1^T & X_2^T & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_2} \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}}_{\mathbf{U_{2\perp}}}. \tag{3.35}$$

Noting also that (3.34) can be rewritten as

$$\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & -I & \mathcal{B} \end{bmatrix}^T}_{\mathbf{V_{2\perp}^T}} \underbrace{\begin{bmatrix} \Psi_1 & \star & \star & \star \\ \Psi_2 + F_1^T & 0 & \star & \star \\ \mathcal{G}(\mathcal{K}_y \mathcal{C} - \mathcal{K}_x) & 0 & -\mathcal{G} - \mathcal{G}^T & \star \\ X_1^T & X_2^T & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_2} \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & -I & \mathcal{B} \end{bmatrix}}_{\mathbf{V_{2\perp}}} \prec 0. \tag{3.36}$$

Consider the identifications $\mathcal{Q}_2$, $\mathbf{U_{2\perp}}$ and $\mathbf{V_{2\perp}}$ in (3.35)-(3.36) and the complements

$$\mathbf{U}_2 := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \ \mathbf{V}_2 := \begin{bmatrix} 0 \\ -I \\ \mathcal{B}^T \\ -I \end{bmatrix}^T, \ \mathcal{X} := \begin{bmatrix} F_1 \\ F_2 \\ H_1 \\ F_3 \end{bmatrix},$$

where F_1, F_2, H_1 and F_3 form a partition of the slack variable $\mathcal{X}$. Then, from the Lemma 3.1, one has that the simultaneous feasibility of (3.35)-(3.36) is equivalent to the feasibility on $\mathcal{Z}$, $\mathcal{K}_x$, $P = P^T \succ 0$, X_1, X_2, $\mathcal{G}$, F_1, F_2 and F_3 of the inequality (3.21). From the previously proved equivalence between the feasibility of (3.34) and stabilizability of (3.18), the statement of the lemma is proved. Furthermore, if (3.21) is feasible, by reverting the sequence of the above arguments, one can see that $\mathcal{K}_x$ is such that $\mathcal{A} + \mathcal{B}\mathcal{K}_x$ is stable, that is, $\mathcal{K}_x$ is a stabilizing static state-feedback gain. ■

Observe that it is not straightforward to extract the controller matrices of (3.2) from $\mathcal{K}_y$ obtained in Theorem 3.1. In order to derive through Theorem 3.1 some tractable LMI conditions for the design of the protocol (3.2), we first rewrite the gains A_c, B_c, C_c and D_c

as

$$A_c = \mathfrak{I}_{11} K_y \mathfrak{I}_{21}^T, \qquad B_c = \mathfrak{I}_{11} K_y \mathfrak{I}_{22}^T,$$
$$C_c = \mathfrak{I}_{12} K_y \mathfrak{I}_{21}^T, \qquad D_c = \mathfrak{I}_{12} K_y \mathfrak{I}_{22}^T,$$

where,

$$\mathfrak{I}_{11} = \begin{bmatrix} I_{n_c} & 0_{n_c\times s} \end{bmatrix}, \mathfrak{I}_{12} = \begin{bmatrix} 0_{s\times n_c} & I_s \end{bmatrix}, K_y = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix},$$
$$\mathfrak{I}_{21} = \begin{bmatrix} I_{n_c} & 0_{n_c\times q} \end{bmatrix}, \mathfrak{I}_{22} = \begin{bmatrix} 0_{q\times n_c} & I_q \end{bmatrix}.$$

Using the mixed product property of the Kronecker product, $\mathcal{K}_y$ can be represented by

$$\begin{aligned} \mathcal{K}_y &= \begin{bmatrix} (I_{m-1} \otimes \mathfrak{I}_{11} K_y \mathfrak{I}_{21}^T) & (I_{m-1} \otimes \mathfrak{I}_{11} K_y \mathfrak{I}_{22}^T) \\ (I_{m-1} \otimes \mathfrak{I}_{12} K_y \mathfrak{I}_{21}^T) & (I_{m-1} \otimes \mathfrak{I}_{12} K_y \mathfrak{I}_{22}^T) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} I_{m-1} \otimes \mathfrak{I}_{11} \\ I_{m-1} \otimes \mathfrak{I}_{12} \end{bmatrix}}_{\mathcal{T}_1} (I_{m-1} \otimes K_y) \underbrace{\begin{bmatrix} I_{m-1} \otimes \mathfrak{I}_{21}^T & I_{m-1} \otimes \mathfrak{I}_{22}^T \end{bmatrix}}_{\mathcal{T}_2}. \end{aligned} \tag{3.37}$$

Therefore, the system (3.18) is equivalent to the system

$$\dot{\zeta}(t) \quad = \quad \Big(\mathcal{A} + \bar{\mathcal{B}}(I_{m-1} \otimes K_y)\bar{\mathcal{C}}\Big)\zeta(t), \tag{3.38}$$

where

$$\bar{\mathcal{B}} = \mathcal{B}\mathcal{T}_1, \quad K_y = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \quad \bar{\mathcal{C}} = \mathcal{T}_2\mathcal{C}. \tag{3.39}$$

In the following, we will see that the transformed system (3.38) enables the design of the $\mathcal{K}_y$ matrix, which has a complicated structure to compute straightforwardly by an LMI condition. System (3.38) and Theorem 3.1 allow us to derive new conditions that design consensus protocols for static, reduced- and full-order control schemes.

Theorem 3.2

If the multi-agent system (3.1) has overall state consensus achievable by the dynamic output feedback protocol (3.2) then there exist matrices K_x, $P = P^T \succ 0$, X_1, X_2, $\mathcal{G}$, F_1, F_2, F_3 and $\mathcal{Z}$ such that (3.21) holds for $\mathcal{K}_x = \mathcal{T}_1 K_x$. Conversely, let K_x be a given matrix such that $\mathcal{A} + \bar{\mathcal{B}} K_x$ is Hurwitz. If there exist matrices $P = P^T \succ 0$, X_1,

X_2, G, F_1, F_2, F_3, Z, such that

$$\begin{bmatrix} \Psi_1 & \star & \star & \star \\ \Psi_2 & -F_2 - F_2^T & \star & \star \\ \Psi_3 & \bar{\mathcal{B}}^T F_2^T & \Psi_4 & \star \\ X_1^T - F_1^T & X_2^T - F_2^T - F_3 & F_3 \bar{\mathcal{B}} & -F_3 - F_3^T \end{bmatrix} \prec 0, \tag{3.40}$$

holds with

$$\Psi_1 = X_1 \left(\mathcal{A} + \bar{\mathcal{B}} K_x\right) + \left(\mathcal{A} + \bar{\mathcal{B}} K_x\right)^T X_1^T, \quad \Psi_2 = P - F_1^T + X_2 \left(\mathcal{A} + \bar{\mathcal{B}} K_x\right),$$
$$\Psi_3 = \left(I_{m-1} \otimes Z\right) \bar{\mathcal{C}} - \left(I_{m-1} \otimes G\right) K_x + \bar{\mathcal{B}}^T F_1^T, \quad \Psi_4 = -I_{m-1} \otimes (G + G^T). \tag{3.41}$$

then the multi-agent system (3.1) under communication (3.3) has overall state consensus achievable by some n_c-order dynamic output feedback controller (3.2). The parameters of one such controller are given by $K_y := G^{-1} Z$.

proof.

[necessity] The proof is immediate. The multi-agent system under dynamic output feedback to be overall state consensus achievable is equivalent to say that (3.38) is stabilizable by static output feedback of the form $I \otimes K_y$. It follows that the inequality (3.21) is feasible by Theorem 3.1.

[sufficiency] The left hand side of matrix (3.40) is a particular case of the matrix in (3.21) with matrices Z and G of the form $\mathcal{Z} = I \otimes Z$ and $\mathcal{G} = I \otimes G$ for system (3.18) with $\mathcal{B}\mathcal{K}_y\mathcal{C} := \mathcal{B}\mathcal{T}_1(I \otimes K_y)\mathcal{T}_2\mathcal{C}$. If inequality (3.40) holds then, from its $(3,3)$ term, G must be nonsingular. Also, from Theorem 3.1, the stability of (3.18) with gain $\mathcal{K}_y$ is equivalent to the stability of (3.38) by the gain K_y with the relation

$$\begin{aligned} \mathcal{K}_y := \mathcal{G}^{-1}\mathcal{Z} &= (I \otimes G)^{-1} (I \otimes Z) \\ &\overset{(a)}{=} \left(I^{-1} \otimes G^{-1}\right) (I \otimes Z) \\ &\overset{(b)}{=} I \otimes \left(G^{-1} Z\right) = I \otimes K_y \end{aligned}$$

where

in (a) we used the property $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

in (b) we used the property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. ■

To obtain LMI conditions for designing an n_c-order controller based on the sufficient part of Theorem 3.2, we need to provide matrix K_x. From (3.25), one has that K_x must be such that $\mathcal{A} + \bar{\mathcal{B}} K_x$ is Hurwitz. In other words, for a gain K_x such that

$$\dot{\zeta}(t) = (\mathcal{A} + \bar{\mathcal{B}} K_x)\zeta(t) \tag{3.42}$$

is Hurwitz stable, one can use gains obtained from any state-feedback design procedure and randomized approaches of the literature as presented in [6].

The following lemma provides a solution that is a slight variation of a result presented in [5]. The solution is based only on the search for a scalar β.

Theorem 3.3

If there exist a scalar $\beta > 0$, a symmetric matrix $W = W^T \in \mathbb{R}^{(m-1)(n+n_c)\times(m-1)(n+n_c)}$, and full matrices $X \in \mathbb{R}^{(m-1)(n+n_c)\times(m-1)(n+n_c)}$, $R_1 \in \mathbb{R}^{n_c\times n}$, $R_2 \in \mathbb{R}^{s\times n}$ and $R_3 \in \mathbb{R}^{n_c\times n_c}$ such that

$$\begin{bmatrix} -X - X^T & X^T\mathcal{A}^T + R^T\bar{\mathcal{B}}^T + W & X^T \\ \mathcal{A}X + \bar{\mathcal{B}}R + W & -W & 0 \\ X & 0 & -W \end{bmatrix} \prec 0 \tag{3.43}$$

where

$$R = \begin{bmatrix} I_{m-1} \otimes \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} & I_{m-1} \otimes \begin{bmatrix} R_3 \\ R_2 Y \end{bmatrix} \end{bmatrix}, \tag{3.44}$$

and $Y = \begin{bmatrix} 0_{n_c\times(n-n_c)} & \beta I_{n_c} \end{bmatrix}^T$, then $\mathcal{A} + \bar{\mathcal{B}} K_x$ is Hurwitz with the state-feedback gain given by $K_x = RX^{-1}$.

The particular choice of the decision variable R in LMIs (3.43) and (3.43) as showed in Theorem 3.3 is addressed to the fact that K_x presents many zeros when R has a free structure. Unnecessary zeros in the gain K_x can imply in a solution K_y in Theorem 3.2 with zero blocks in B_c and C_c, as reported in [57]. When B_c and C_c in (3.2) are null gains, the protocol characterizes only a static output feedback control scheme because u_i does not consider controllers' state information.

REMARK 3.1 When we set the protocols as static, i.e., $n_c = 0$, one has that $R = I_{m-1} \otimes R_2$, and we do not need to tune the β scalar.

REMARK 3.2 Theorem 3.3 solves the state-feedback problem using a unique scalar β to be tuned, which can help to search for a scalar β concerning the feasibility of LMI (3.40). Theorem 3.3 becomes even more important in the next section when we find a H_∞ performance, and searching for a single scalar will be decisive in finding the best possible performance. We present Theorem 3.3, substituting the technique employed in [53], which proposes the solution of the following LMI based on the search of two scalars. If there exist matrices $X \in \mathbb{R}^{(m-1)(n+n_c)\times(m-1)(n+n_c)}$, $J_1 \in \mathbb{R}^{n_c\times n}$, $J_2 \in \mathbb{R}^{s\times n}$, $J_3 \in \mathbb{R}^{n_c\times n_c}$, $J_4 \in \mathbb{R}^{s\times n_c}$, a positive definite matrix $P = P^T \in \mathbb{R}^{(m-1)(n+n_c)\times(m-1)(n+n_c)}$ and given scalars $\xi > 0$ and $\alpha > 0$ such that

$$\begin{bmatrix} \mathcal{A}X + X^T\mathcal{A}^T + \bar{\mathcal{B}}R + R^T\bar{\mathcal{B}}^T & \star \\ P - X + \xi(\mathcal{A}X + \bar{\mathcal{B}}R)^T & -\xi\left(X + X^T\right) \end{bmatrix} \prec 0, \tag{3.45}$$

where

$$R = \begin{bmatrix} I_{m-1} \otimes \begin{bmatrix} J_1 \\ YJ_1 + J_2 \end{bmatrix} & I_{m-1} \otimes \begin{bmatrix} J_3 \\ YJ_3 + J_4 \end{bmatrix} \end{bmatrix},$$

and

$$Y = \begin{cases} \begin{bmatrix} 0_{n_c\times(s-n_c)} & \alpha I_{n_c} \end{bmatrix}^T, \text{ if } \quad n_c < s \\ \begin{bmatrix} 0_{s\times(n_c-s)} & \alpha I_s \end{bmatrix}, \text{ otherwise.} \end{cases}$$

then the state-feedback gain that stabilizes the system (3.42) is given by $K_x = RX^{-1}$.

The following section presents conditions to obtain protocols of arbitrary order for agents with disturbances in dynamics and measurements.

3.2 DESIGN OF H_∞ CONSENSUS PROTOCOLS FOR DISTURBED AGENTS

Some works in the literature of MAS present convex and non-convex solutions for the design of dynamic output feedback protocols with H_∞ consensus for undirected networks (see e.g., [39], [38], [65], [64], [37]). However, the problem becomes complicated for directed networks. For directed networks [28] and [31] design H_∞ full-order protocols. In [28], the H_∞ norm major the plant's transfer function in terms of the agents' disturbances and controlled outputs. For protocols of any order, an H_∞ design scheme brings challenges because the closed-loop system passes by transformations that do not guarantee the same H_∞ performance for the original system. In this section, we consider designing protocols of any order for agents with exogenous disturbances in their dynamics and measurements. We use the H_∞ norm to major the closed-loop system's transfer function to attenuate the agents' disturbance effect.

3.2.1 The H_∞ consensus problem

Consider m agents in a directed network, each with the following dynamic model

$$\dot{x}_i(t) = Ax_i(t) + B_w w_i(t) + B_u u_i(t), \quad i = 1, ..., m, \tag{3.46}$$
$$y_i(t) = C_y x_i(t) + D_y w_i(t) \tag{3.47}$$

where $x_i(t) \in \mathbb{R}^n$ are the state variables, $y_i(t) \in \mathbb{R}^q$ the outputs, $u_i(t) \in \mathbb{R}^s$ the control inputs, $w_i(t) \in \mathbb{R}^{n_w}$ exogenous disturbances that belongs to $\mathcal{L}_2[0, \infty)$. The parameter matrices $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times s}$, $B_w \in \mathbb{R}^{n\times n_w}$, $C_y \in \mathbb{R}^{q\times n}$ and $D_y \in \mathbb{R}^{q\times n_w}$ are supposed real and known. Each agent i, $i = 1, ..., m$ is controlled locally by its corresponding dynamic controller of the form

$$\begin{aligned} \dot{x}_{c,i}(t) &= A_c x_{c,i}(t) + B_c \nu_i(t), \\ u_i(t) &= C_c x_{c,i}(t) + D_c \nu_i(t), \end{aligned} \tag{3.48}$$

where $x_{c,i}(t) \in \mathbb{R}^{n_c}$ is the state of the i-th dynamical controller, $A_c \in \mathbb{R}^{n_c\times n_c}$, $B_c \in \mathbb{R}^{n_c\times q}$, $C_c \in \mathbb{R}^{s\times n_c}$, $D_c \in \mathbb{R}^{s\times q}$ the dynamic controller parameters to find and the agents' relative outputs signal

$$\nu_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(y_i(t) - y_j(t)), i = 1, ..., m, \tag{3.49}$$

where $a_{ij} \geq 0$ are the directed graph weights.

For the H_∞ consensus analysis, following [36], we introduce the balanced consensus error outputs

$$z_i(t) = C_z(x_i(t) - \frac{1}{m}\sum_{j=1}^{m} x_j(t)), \tag{3.50}$$

with $C_z \in \mathbb{R}^{r\times n}$, such that state consensus implies $z_i(t) = 0$ for every i and the matrix C_z balances the relative importance of consensus among particular state components of the agents in the performance analysis. It is considered a system performance evaluation in the H_∞ sense, which relates the overall exogenous disturbance $w(t)$ and the consensus discrepancy $z(t)$ through the inequality

$$\int_0^\infty \|z(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t, \quad \forall w(t) \in \mathcal{L}_2[0, \infty), \tag{3.51}$$

where the scalar $\gamma > 0$ is the H_∞ consensus performance index for the multi-agent system (3.46)-(3.49).

The objective of this section is to address the following problem.

PROBLEM 3.2 For the multi-agent system (3.46)-(3.47), design, if possible, a protocol (3.48)-(3.49) of given order n_c, $0 \leq n_c \leq n$, using only the local information (3.49) such that the resulting closed-loop multi-agent system (3.46)-(3.50):

- without the disturbances $w_i(t)$, for all initial conditions, achieves asymptotic overall state consensus defined as

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, \quad \lim_{t\to\infty} ||x_{c,i}(t) - x_{c,j}(t)|| = 0,\ i, j = 1, ..., m,$$

- in the presence of the disturbance $w_i(t)$ and zero initial conditions, satisfies the H_∞ performance (3.51) for a given gain $\gamma > 0$.

3.2.2 Transformed Closed-loop Multi-Agent System

The augmented system, considering the m agents (3.46)-(3.47) is given by

$$\begin{aligned} \dot{x}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)u(t), \\ z(t) &= (C_g \otimes C_z)x(t), \\ y(t) &= (I_m \otimes C_y)x(t) + (I_m \otimes D_y)w(t), \end{aligned} \tag{3.52}$$

where $C_g = I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^{\mathrm{T}}$.

In the same way, the augmented dynamical controller is given by

$$\begin{aligned} \dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) + (I_m \otimes B_c)\nu(t), \\ u(t) &= (I_m \otimes C_c)x_c(t) + (I_m \otimes D_c)\nu(t). \end{aligned} \tag{3.53}$$

The function $\nu(t)$ is the concatenated form on the left-hand side of (3.49). Similar to the non-disturbed case, one has that

$$
\begin{aligned}
\nu(t) &= -\begin{bmatrix} \sum_{j\neq 1, j=1}^{m} a_{1j}(y_1(t)-y_j(t)) \\ \vdots \\ \sum_{j\neq m, j=1}^{m} a_{mj}(y_m(t)-y_j(t)) \end{bmatrix} = -\begin{bmatrix} l_{11}y_1(t)+\sum_{j=2}^{m} l_{1j}y_j(t) \\ \vdots \\ \sum_{j=1}^{m-1} l_{mj}y_j(t)+l_{mm}y_m(t) \end{bmatrix}, \\
&= -\begin{bmatrix} \sum_{j=1}^{m} l_{1j}(C_y x_j(t)+D_y w_j(t)) \\ \vdots \\ \sum_{j=1}^{m} l_{mj}(C_y x_j(t)+D_y w_j(t)) \end{bmatrix} \\
&= -\begin{bmatrix} l_{11}C_y & \cdots & l_{1m}C_y \\ \vdots & \ddots & \vdots \\ l_{m1}C_y & \cdots & l_{mm}C_y \end{bmatrix}\begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} - \begin{bmatrix} l_{11}D_y & \cdots & l_{1m}D_y \\ \vdots & \ddots & \vdots \\ l_{m1}D_y & \cdots & l_{mm}D_y \end{bmatrix}\begin{bmatrix} w_1(t) \\ \vdots \\ w_m(t) \end{bmatrix}, \\
&= -(L\otimes C_y)x(t)-(L\otimes D_y)w(t).
\end{aligned}
$$

Then, the augmented dynamical controller (3.53) has the following form

$$
\begin{aligned}
\dot{x}_c(t) &= (I_m\otimes A_c)x_c(t)-(I_m\otimes B_c)((L\otimes C_y)x(t)+(L\otimes D_y)w(t)), \\
u(t) &= (I_m\otimes C_c)x_c(t)-(I_m\otimes D_c)((L\otimes C_y)x(t)+(L\otimes D_y)w(t)).
\end{aligned} \tag{3.54}
$$

From Kronecker's mixed product property, the dynamical controller (3.54) can be rewritten as follows

$$
\begin{aligned}
\dot{x}_c(t) &= (I_m\otimes A_c)x_c(t)-L\otimes(B_cC_y)x(t)-L\otimes(B_cD_y)w(t), \\
u(t) &= (I_m\otimes C_c)x_c(t)-L\otimes(D_cC_y)x(t)-L\otimes(D_cD_y)w(t).
\end{aligned} \tag{3.55}
$$

Substituting the dynamical controller (3.55) in (3.52), and using the mixed product property of Kronecker product, $(I_m\otimes B_u)(I_m\otimes C_c)=(I_m\otimes B_uC_c)$ and $(I_m\otimes B_u)(L\otimes(D_cC_y))=(L\otimes(B_2D_cC_y))$, then (3.55) can be rewritten in the following form

$$
\begin{aligned}
\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix}}_{\dot{\psi}(t)} &= \underbrace{\begin{bmatrix} (I_m\otimes A)-(L\otimes(B_uD_cC_y)) & (I_m\otimes B_uC_c) \\ -(L\otimes(B_cC_y)) & (I_m\otimes A_c) \end{bmatrix}}_{\underline{\mathbf{A}}}\underbrace{\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}}_{\psi(t)} \\
&+ \underbrace{\begin{bmatrix} (I_m\otimes B_w)-(L\otimes(B_uD_cD_y)) \\ -L\otimes(B_cD_y) \end{bmatrix}}_{\underline{\mathbf{B}}_w} w(t)
\end{aligned} \tag{3.56}
$$

$$
z(t) = \underbrace{\begin{bmatrix} (C_g\otimes C_z) & 0 \end{bmatrix}}_{\underline{\mathbf{C}}}\underbrace{\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}}_{\psi(t)}. \tag{3.57}
$$

Just as in the non-disturbed case, we will adopt the strategy that transforms the consensus problem into a stability problem, i.e., introducing new variables representing the disagreement of the agents $\zeta_{1,i}(t) = x_1(t) - x_{i+1}(t)$, the disagreement of the state variables of the dynamic controllers $\zeta_{2,i}(t) = x_{c,1}(t) - x_{c,i+1}(t)$, and now, the disagreement of the external disturbances $\eta_i(t) = w_1(t) - w_{i+1}(t)$, such that $i = 1, ..., m-1$.

The augmented transformed disturbance is given by

$$\eta(t) = (U \otimes I_{n_w})w(t). \tag{3.58}$$

Thus, the variable $w(t)$ can be recovered by the expression

$$w(t) = \mathbf{1}_m \otimes w_1(t) + (W \otimes I_{n_w})\eta(t). \tag{3.59}$$

Using the transformed system (3.16), obtained without disturbances, and considering the system (3.57), when deriving $\zeta(t) = \bar{U}\psi(t)$, system (3.56), it can be expressed in the following form

$$\dot{\zeta}(t) \;\; = \;\; \tilde{A}\zeta(t) + \underbrace{\bar{U}\underline{\mathbf{B}}_w w(t)}_{\iota_1(t)}. \tag{3.60}$$

Substituting (3.59) and $\underline{\mathbf{B}}_1$ in $\iota_1(t)$ one has

$$\begin{aligned} \iota_1(t) \;\; &= \;\; \bar{U}\begin{bmatrix} (I_m \otimes B_w) - (L \otimes (B_u D_c D_y)) \\ -L \otimes (B_c D_y) \end{bmatrix} \mathbf{1}_m \otimes w_1(t) \\ &+ \;\; \bar{U}\begin{bmatrix} (I_m \otimes B_w) - (L \otimes (B_u D_c D_y)) \\ -L \otimes (B_c D_y) \end{bmatrix} (W \otimes I_{n_w})\eta(t). \end{aligned} \tag{3.61}$$

Replacing the matrix $\bar{U}$ and operating it is obtained

$$\begin{aligned} \iota_1(t) \;\; = \;\; & \begin{bmatrix} (U\mathbf{1}_m \otimes B_w w_1(t)) - (UL\mathbf{1}_m \otimes (B_u D_c D_y w_1(t))) \\ -UL\mathbf{1}_m \otimes (B_c D_y w_1(t)) \end{bmatrix} \\ & + \begin{bmatrix} (UW \otimes B_w) - (ULW \otimes (B_u D_c D_y)) \\ -ULW \otimes (B_c D_y) \end{bmatrix} \eta(t). \end{aligned}$$

See that, $U1_m = 0$, $L1_m = 0_m$ and $UW = I_{m-1}$, then

$$\iota_1(t) \;\; = \;\; \underbrace{\begin{bmatrix} (I_{m-1} \otimes B_w) - (ULW \otimes (B_u D_c D_y)) \\ -(ULW \otimes (B_c D_y)) \end{bmatrix}}_{\tilde{B}_w} \eta(t). \tag{3.62}$$

Defining $\bar{L} = ULW$, equation (3.60) can be rewritten in the following form

$$\dot{\zeta}(t) = \tilde{A}\zeta(t) + \tilde{B}_w\eta(t), \tag{3.63}$$

where,

$$\begin{aligned}
\tilde{A} &= \begin{bmatrix} (I_{m-1} \otimes A) - (\bar{L} \otimes (B_u D_c C_y)) & (I_{m-1} \otimes B_u C_c) \\ -\bar{L} \otimes (B_c C) & (I_{m-1} \otimes A_c) \end{bmatrix}, \\
\tilde{B}_w &= \begin{bmatrix} (I_{m-1} \otimes B_w) - \bar{L} \otimes (B_u D_c D_y) \\ -\bar{L} \otimes (B_c D_y) \end{bmatrix}.
\end{aligned}$$

The controlled outputs (3.57), substituting (3.11), one has

$$\begin{aligned}
z(t) = {} & \begin{bmatrix} (C_g \otimes C_z) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1}_m \otimes x_1(t) \\ \mathbf{1}_m \otimes x_{c,1}(t) \end{bmatrix} \\
& + \begin{bmatrix} (C_g \otimes C_z) & 0 \end{bmatrix} \begin{bmatrix} W \otimes I_n & 0 \\ 0 & W \otimes I_{n_c} \end{bmatrix} \zeta(t).
\end{aligned} \tag{3.64}$$

Using the Kronecker's mixed product and the property $C_g 1_m = 0$, then

$$z(t) = \begin{bmatrix} (C_g W \otimes C_z) & 0 \end{bmatrix} \zeta(t). \tag{3.65}$$

Observe that (3.63) and (3.65) forms the equivalent system

$$\begin{aligned}
\dot{\zeta}(t) &= (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C})\zeta(t) + (\mathcal{B}_1 + \mathcal{B}\mathcal{K}_y\mathcal{D})\eta(t) \\
z(t) &= \mathcal{C}_z\zeta(t),
\end{aligned} \tag{3.66}$$

with,

$$\mathcal{A} = \begin{bmatrix} (I_{m-1} \otimes A) & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 & (I_{m-1} \otimes B_u) \\ I & 0 \end{bmatrix}, \mathcal{K}_y = \begin{bmatrix} (I_{m-1} \otimes A_c) & (I_{m-1} \otimes B_c) \\ (I_{m-1} \otimes C_c) & (I_{m-1} \otimes D_c) \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 0 & I \\ -(\bar{L} \otimes C_y) & 0 \end{bmatrix}, \mathcal{B}_1 = \begin{bmatrix} (I_{m-1} \otimes B_w) \\ 0 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 0 \\ -(\bar{L} \otimes D_y) \end{bmatrix}, \mathcal{C}_z = \begin{bmatrix} (C_g W \otimes C_z) & 0 \end{bmatrix}.$$

As was seen in the non-disturbed case, the system (3.66) can be rewritten in a form that the gains A_c, B_c, C_c, and D_c can properly be designed by an LMI condition, as follows

$$
\begin{aligned}
\dot{\zeta}(t) &= A_{cl}\zeta(t) + B_{cl}\eta(t) \\
z(t) &= \mathcal{C}_z\zeta(t).
\end{aligned} \tag{3.67}
$$

with

$$
\begin{aligned}
A_{cl} &= \mathcal{A} + \mathcal{B}_2\mathcal{T}_1(I_m \otimes K_y)\mathcal{T}_2\mathcal{C}_y = \mathcal{A} + \bar{\mathcal{B}}_2(I_m \otimes K_y)\bar{\mathcal{C}}, \\
B_{cl} &= \mathcal{B}_1 + \mathcal{B}_2\mathcal{T}_1(I_m \otimes K_y)\mathcal{T}_2\mathcal{D}_y = \mathcal{B}_1 + \bar{\mathcal{B}}_2(I_m \otimes K_y)\bar{\mathcal{D}}.
\end{aligned} \tag{3.68}
$$

A point to note is that the system (3.67) requires a specific H_∞ performance due to the new disturbance variable $\eta(t)$. Then, it is necessary to obtain a relationship between H_∞ performance γ_t of the transformed system (3.67) and H_∞ performance γ of the original system (3.56)-(3.57). The following presents auxiliary lemmas that relate γ with γ_t. First, we present the following lemma that transforms the H_∞ control problem involving the closed-loop system (3.67) in an inequality.

Lemma 3.2 [7](Bounded Real Lemma) A_{cl} is Hurwitz stable and the transfer function of the system (3.67) satisfies $\|T_{z\eta}(s)\|_\infty < \gamma_t$ such that $\gamma_t > 0$, if and only if, there exists a matrix $P^T = P \succ 0$ that satisfies the following matrix inequality

$$
\begin{bmatrix} \mathcal{A}_{cl}^T P + P\mathcal{A}_{cl} + \mathcal{C}_z^T\mathcal{C}_z & PB_{cl} \\ B_{cl}^T P & -\gamma_t^2 I \end{bmatrix} \prec 0, \tag{3.69}
$$

where γ_t the H_∞ performance of the system (3.67).

We enunciate the following auxiliary lemma before relating γ with γ_t.

Lemma 3.3 Let a matrix U, defined as $U = \begin{bmatrix} \mathbf{1}_{m-1} & -I_{m-1} \end{bmatrix}$, where $m > 1$. It follows that $\lambda_{max}(U^TU) = m$.

proof.

To find the eigenvalues of U^TU, we need to find its characteristic polynomial, as follow

$$
\begin{aligned}
\Delta(\lambda) &= det(U^TU - \lambda I_m) \\
&= det \begin{bmatrix} (m-1-\lambda) & -\mathbf{1}_{m-1}^T \\ -\mathbf{1}_{m-1} & I_{m-1} - \lambda I_{m-1} \end{bmatrix}
\end{aligned} \tag{3.70}
$$

Defining $I_\lambda = I_{m-1} - \lambda I_{m-1}$ and applying the determinant rule for block matrices in (3.70), one has

$$\begin{aligned}\Delta(\lambda) &= det(I_\lambda)det((m-1-\lambda) - \mathbf{1}_{m-1}^T(I_\lambda)^{-1}\mathbf{1}_{m-1}) \\ &= det(I_\lambda)det((m-1-\lambda) - \frac{(m-1)}{(1-\lambda)})\end{aligned} \tag{3.71}$$

Observe that,

$$det(I_\lambda) = (1-\lambda)^{m-1}, \tag{3.72}$$

and,

$$det((m-1-\lambda) - \frac{(m-1)}{(1-\lambda)}) = \frac{\lambda(\lambda-m)}{(1-\lambda)} \tag{3.73}$$

From (3.72) and (3.73), equation (3.71) can be rewritten as

$$\Delta(\lambda) = det(U^TU - \lambda I_m) = (1-\lambda)^{m-2}\lambda(\lambda-m). \tag{3.74}$$

Therefore, $\lambda = m$, $\lambda = 0$ and $\lambda = 1$ are the eigenvalues of U^TU. Therefore, since $m > 1$, the maximum eigenvalue of U^TU is m. ■

With Lemmas 3.2 and 3.3 one can enunciate the following lemma.

LEMMA 3.4 If inequality (3.69) assures the H_∞ performance $\gamma_t > 0$ for system (3.67), then $\gamma = \gamma_t\sqrt{m}$ satisfies the condition (3.51) and is the H_∞ performance of the system (3.56)-(3.57).

proof.

Applying Lemma 3.2 in system (3.67), one has that inequality (3.69) holds and $\gamma_t > 0$ is the H_∞ performance of (3.67), if and only if, $\gamma_t > 0$ satisfies the dissipation inequality

$$\int_0^\infty \|z(t)\|^2 \mathrm{d}t < \gamma_t^2 \int_0^\infty \|\eta(t)\|^2 \mathrm{d}t, \quad \forall \eta(t) \in \mathcal{L}_2[0,\infty). \tag{3.75}$$

Observe that

$$\begin{aligned}\int_0^\infty \|z(t)\|^2 dt &< \gamma_t^2 \int_0^\infty \|\eta(t)\|^2, \\ &= \gamma_t^2 \int_0^\infty w^T(t)(U^T \otimes I_{n_w})(U \otimes I_{n_w})w(t), \\ &= \gamma_t^2 \int_0^\infty w^T(t)(U^T U \otimes I_{n_w})w(t), \\ &= \gamma_t^2 \int_0^\infty \|w^T(t)(U^T U \otimes I_{n_w})w(t)\|. \end{aligned} \tag{3.76}$$

Defining $\tilde{U} = (U^T U \otimes I_{n_w})$ and using the Cauchy-Schwarz inequality in (3.76), it is obtained

$$\int_0^\infty \|z(t)\|^2 dt < \gamma_t^2 \int_0^\infty \|w^T(t)\tilde{U}w(t)\| \leq \gamma_t^2 \int_0^\infty \|w^T(t)\|\|\tilde{U}w(t)\|. \tag{3.77}$$

From the matrix norms property, given two matrices X and Y the inequality holds $\|XY\| \leq \|X\|\|Y\|$ [43], considering the inequality (3.77) it is obtained

$$\int_0^\infty \|z(t)\|^2 dt < \int_0^\infty \gamma_t^2 \|w^T(t)\|\|\tilde{U}\|\|w(t)\|. \tag{3.78}$$

Note that $\tilde{U}$ is a positive semi-definite matrix, then $\|\tilde{U}\| = \lambda_{max}(\tilde{U})$. From Kronecker product properties, one has that a given matrix $X \otimes Y$ has eigenvalues $\lambda_x \lambda_y$ where λ_x and λ_y are eigenvalues of X and Y, respectively [49]. Then,

$$\lambda_{max}(\tilde{U}) = \lambda_{max}(U^T U \otimes I_{n_w}) = \lambda_{max}(U^T U). \tag{3.79}$$

Moreover, $\|w^T(t)\| = \|w(t)\|$, then,

$$\int_0^\infty \|z(t)\|^2 dt < \underbrace{\gamma_t^2 \lambda_{max}(U^T U)}_{\gamma^2} \int_0^\infty \|w(t)\|^2, \tag{3.80}$$

where if inequality (3.80) holds then (3.51) is satisfied and from Lemma 3.3 one has $\gamma = \gamma_t \sqrt{m}$ is the H_∞ performance of the system (3.57). ■

Lemma 3.2 provides a necessary and sufficient condition for stability and obtaining the H_∞ performance of (3.67). However, this is not a synthesis condition. With the result provided by Lemma 3.4, we can turn the focus on deriving the LMI condition that stabilizes the closed-loop system (3.67) with an H_∞ performance γ_t. In the following section, we enunciate a theorem that derives a condition for designing protocol gains that stabilize and provide the H_∞ performance of (3.67).

3.2.3 Design of H_∞ Consensus Protocols

Another important lemma applied in later developments is presented below.

LEMMA 3.5 (Schur Complement Lemma [7]) Let $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{Y} \in \mathbb{R}^{n\times m}$ and $\mathbf{Z} \in \mathbb{S}^n$, the following conditions are equivalent

1. $\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^\top & \mathbf{Z} \end{bmatrix} \prec 0;$

2. $\mathbf{X} - \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}^\top \prec 0, \mathbf{Z} \prec 0;$

3. $\mathbf{Z} - \mathbf{Y}^\top\mathbf{X}^{-1}\mathbf{Y} \prec 0, \mathbf{X} \prec 0.$

Theorem 3.4

Let K_x a given matrix such that $\mathcal{A}_{clx} = \mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz stable. If there exist a positive scalar μ and matrices $P = P^T \succ 0$, X_1, X_2, H, G, F_1, F_2, F_4 and Z such that the following LMI holds

$$\begin{bmatrix} \Omega_1 & P + \mathcal{A}_0^T X_2^T - F_1 & F_1\mathcal{B}_1 & X_1 - F_1 & \mathcal{C}_z^T H & \Omega_2 \\ \star & -F_2 - F_2^T & F_2\mathcal{B}_1 & X_2 - F_2 - F_4^T & 0 & F_2\bar{\mathcal{B}} \\ \star & \star & -\mu I & \mathcal{B}_1^T F_4^T & 0 & \Omega_3 \\ \star & \star & \star & -F_4 - F_4^T & 0 & F_4\bar{\mathcal{B}}_2 \\ \star & \star & \star & \star & I - H^T - H & 0 \\ \star & \star & \star & \star & \star & -\Omega_4 \end{bmatrix} \prec 0, \tag{3.81}$$

where,

$$\Omega_1 = X_1\mathcal{A}_{clx} + \mathcal{A}_{clx}^T X_1^T, \Omega_2 = F_1\bar{\mathcal{B}} + \bar{\mathcal{C}}^T(I_{m-1} \otimes Z^T) - \mathcal{K}_x^T(I_{m-1} \otimes G_1^T),$$
$$\Omega_3 = F_3\bar{\mathcal{B}} + \bar{\mathcal{D}}^T(I_{m-1} \otimes Z^T), \Omega_4 = He\{(I_{m-1} \otimes G_1)\},$$

then the multi-agent system (3.46)-(3.47) achieves overall state consensus with H_∞ cost $\gamma > 0$, such that $\gamma = \sqrt{\mu m}$, by the n_c-order dynamic output feedback controller (3.48) with gains recovery by $K_y = G_1^{-1}Z$.

proof.

From Lemma 3.2, the inequality (3.69) can be rewritten as

$$\underbrace{\begin{bmatrix} I & 0 \\ A_{cl} & B_{cl} \\ 0 & I \end{bmatrix}^T}_{\mathbf{V}^{\perp T}} \underbrace{\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -\gamma_t^2 I \end{bmatrix}}_{\mathcal{Q}} \underbrace{\begin{bmatrix} I & 0 \\ A_{cl} & B_{cl} \\ 0 & I \end{bmatrix}}_{\mathbf{V}^{\perp}} \prec 0. \tag{3.82}$$

Besides that, by hypothesis, the following inequality holds

$$0 \succ -\gamma_t^2 I = \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}^T}_{\mathbf{U}^{\perp T}} \underbrace{\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -\gamma_t^2 I \end{bmatrix}}_{\mathcal{Q}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}}_{\mathbf{U}^{\perp}}. \tag{3.83}$$

By the Lemma 3.1 and the identifications in the inequalities (3.82) and (3.83), it is obtained

$$\underbrace{\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -\gamma_t^2 I \end{bmatrix}}_{\mathcal{Q}} + He\left\{ \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{U}^T} \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} A_{cl} & -I & B_{cl} \end{bmatrix}}_{\mathbf{V}} \right\} \prec 0. \tag{3.84}$$

Rewriting the matrix $A_{cl} = \mathcal{A}_{clx} + \bar{\mathcal{B}} S_1$ with $\mathcal{A}_{clx} = \mathcal{A} + \bar{\mathcal{B}} K_x$, $S_1 = (I_m \otimes K_y)\bar{\mathcal{C}} - K_x$ and substituting in (3.84), one has

$$\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -\gamma_t^2 I \end{bmatrix} + He\left\{ \begin{bmatrix} X_1 \\ X_2 \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{clx} + \bar{\mathcal{B}} S_1 & -I & B_{cl} \end{bmatrix} \right\} \prec 0. \tag{3.85}$$

The inequality (3.85) can be rewritten as

$$\begin{bmatrix} \mathcal{A}_{clx}^T X_1^T + X_1 \mathcal{A}_{clx} + \mathcal{C}_z^T\mathcal{C}_z & P + \mathcal{A}_{clx}^T X_2^T & 0 \\ P + X_2 \mathcal{A}_{clx} & 0 & 0 \\ 0 & 0 & -\gamma_t^2 I \end{bmatrix} + He\left\{ \begin{bmatrix} X_1 \\ X_2 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}} S_1 & -I & B_{cl} \end{bmatrix} \right\} \prec 0, \tag{3.86}$$

or, equivalently, inequality (3.86) can be rewritten as

$$\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \bar{\mathcal{B}}S_1 & -I & B_{cl} \end{bmatrix}^T}_{\mathbf{V}_1^{\perp T}} \mathcal{Q}_1 \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \bar{\mathcal{B}}S_1 & -I & B_{cl} \end{bmatrix}}_{\mathbf{V}_1^{\perp}} \prec 0, \tag{3.87}$$

with,

$$\mathcal{Q}_1 = \begin{bmatrix} X_1\mathcal{A}_{clx} + \mathcal{A}_{clx}^T X_1^T + \mathcal{C}_z^T\mathcal{C}_z & P + \mathcal{A}_{clx}^T X_2^T & 0 & X_1 \\ P + X_2\mathcal{A}_{clx} & 0 & 0 & X_2 \\ 0 & 0 & -\gamma_1^2 I & 0 \\ X_1^T & X_2^T & 0 & 0 \end{bmatrix}. \tag{3.88}$$

In the same way, the following inequality holds

$$0 \succ -\gamma_t^2 I = \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T}_{\mathbf{U}_1^{\perp T}} \mathcal{Q}_1 \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}}_{\mathbf{U}_1^{\perp}}. \tag{3.89}$$

According to Lemma 3.1 inequalities (3.82) and (3.83) are equivalent to

$$\underbrace{\begin{bmatrix} X_1\mathcal{A}_{clx} + \mathcal{A}_{clx}^T X_1^T + \mathcal{C}_z^T\mathcal{C}_z & P + \mathcal{A}_{clx}^T X_2^T & 0 & X_1 \\ P + X_2\mathcal{A}_{clx} & 0 & 0 & X_2 \\ 0 & 0 & -\gamma_t^2 I & 0 \\ X_1^T & X_2^T & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_1}$$
$$+He\left\{ \underbrace{\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}}_{\mathbf{U}_1^T} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \underbrace{\begin{bmatrix} \bar{\mathcal{B}}S_1 & -I & B_{cl} & -I \end{bmatrix}}_{\mathbf{V}_1} \right\} \prec 0, \tag{3.90}$$

or, equivalently with introducing of the variable H and substituting $B_{cl} = \mathcal{B}_1 + \bar{\mathcal{B}}_2(I_m \otimes$

$\mathcal{K}_y)\bar{\mathcal{D}}$, one has

$$\underbrace{\begin{bmatrix} X_1\mathcal{A}_{clx} + \mathcal{A}_{clx}^T X_1^T & P + \mathcal{A}_{clx}^T X_2^T - F_1 & F_1\mathcal{B}_1 & X_1 - F_1 \\ \star & -F_2 - F_2^T & F_2\mathcal{B}_1 & X_2 - F_2 - F_4^T \\ \star & \star & -\gamma_t^2 I & \mathcal{B}_1^T F_4^T \\ \star & \star & \star & -F_4 - F_4^T \end{bmatrix}}_{\mathbf{X}_1}$$

$$\underbrace{+He\left\{\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ F_4 \end{bmatrix}\begin{bmatrix} \bar{\mathcal{B}}S_1 & 0 & \bar{\mathcal{B}}S_2 & 0 \end{bmatrix}\right\}}_{\mathbf{X}_2} - \underbrace{\begin{bmatrix} \mathcal{C}_z^T H \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{Y}}(\underbrace{-H^T H}_{\mathbf{Z}})^{-1}\underbrace{\begin{bmatrix} H^T\mathcal{C}_z & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Y}^T} \prec 0. \tag{3.91}$$

Defining $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$ and applying Lemma 3.5 in the inequality (3.91), following its identifications, one has

$$\begin{bmatrix} X_1\mathcal{A}_{clx} + \mathcal{A}_{clx}^T X_1^T & P + \mathcal{A}_{clx}^T X_2^T - F_1 & F_1\mathcal{B}_1 & X_1 - F_1 & \mathcal{C}_z^T H \\ \star & -F_2 - F_2^T & F_2\mathcal{B}_1 & X_2 - F_2 - F_4^T & 0 \\ \star & \star & -\gamma_t^2 I & \mathcal{B}_1^T F_4^T & 0 \\ \star & \star & \star & -F_4 - F_4^T & 0 \\ \star & \star & \star & \star & -H^T H \end{bmatrix}$$
$$+He\left\{\begin{bmatrix} F_1\bar{\mathcal{B}} \\ F_2\bar{\mathcal{B}} \\ 0 \\ F_4\bar{\mathcal{B}} \\ 0 \end{bmatrix}\begin{bmatrix} S_1 & 0 & S_2 & 0 & 0 \end{bmatrix}\right\} \prec 0. \tag{3.92}$$

Note that, the inequality $(I - H)^T(I - H) \geq 0$ holds if, and only if, $-H^T H \leq I - H^T - H$. Substituting $-H^T H$ by $I - H^T - H$, one has the feasibility of the following inequality

$$\begin{bmatrix} X_1\mathcal{A}_{clx}+\mathcal{A}_{clx}^TX_1^T & P+\mathcal{A}_{clx}^TX_2^T-F_1 & F_1\mathcal{B}_1 & X_1-F_1 & \mathcal{C}_z^TH \\ \star & -F_2-F_2^T & F_2\mathcal{B}_1 & X_2-F_2-F_4^T & 0 \\ \star & \star & -\gamma_t^2I & \mathcal{B}_1^TF_4^T & 0 \\ \star & \star & \star & -F_4-F_4^T & 0 \\ \star & \star & \star & \star & I-H^T-H \end{bmatrix}$$
$$+He\left\{\begin{bmatrix} F_1\bar{\mathcal{B}} \\ F_2\bar{\mathcal{B}} \\ 0 \\ F_4\bar{\mathcal{B}} \\ 0 \end{bmatrix}\begin{bmatrix} S_1 & 0 & S_2 & 0 & 0 \end{bmatrix}\right\} \prec 0 \tag{3.93}$$

implies in the feasibility of inequality (3.92). Rewriting inequality (3.93) one has

$$\underbrace{\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ S_1 & 0 & S_2 & 0 & 0 \end{bmatrix}^T}_{\mathbf{V}_2^{\perp T}} \mathcal{Q}_2 \underbrace{\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ S_1 & 0 & S_2 & 0 & 0 \end{bmatrix}}_{\mathbf{V}_2^{\perp}} \prec 0, \tag{3.94}$$

with,

$$\mathcal{Q}_2 = \begin{bmatrix} \Omega_1 & P+\mathcal{A}_{clx}^TX_2^T-F_1 & F_1\mathcal{B}_1 & X_1-F_1 & \mathcal{C}_z^TH & F_1\bar{\mathcal{B}} \\ \star & -F_2-F_2^T & F_2\mathcal{B}_1 & X_2-F_2-F_4^T & 0 & F_2\bar{\mathcal{B}} \\ \star & \star & -\gamma_t^2I & \mathcal{B}_1^TF_4^T & 0 & 0 \\ \star & \star & \star & -F_4-F_4^T & 0 & F_4\bar{\mathcal{B}} \\ \star & \star & \star & \star & I-H^T-H & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix},$$
$$\Omega_1 = X_1\mathcal{A}_{clx}+\mathcal{A}_{clx}^TX_1^T. \tag{3.95}$$

By hypothesis, the following inequality holds

$$0 \succ -\gamma_t^2 I = \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T}_{\mathbf{U}_2^{\perp T}} \mathcal{Q}_2 \underbrace{\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{U}_2^{\perp}} . \tag{3.96}$$

By Lemma 3.1 with the identifications in inequalities (3.94) and (3.96), one has

$$\underbrace{\begin{bmatrix} \Omega_1 & P + \mathcal{A}_0^T X_2^T - F_1 & F_1\mathcal{B}_1 & X_1 - F_1 & \mathcal{C}_z^T H & F_1\bar{\mathcal{B}} \\ \star & -F_2 - F_2^T & F_2\mathcal{B}_1 & X_2 - F_2 - F_4^T & 0 & F_2\bar{\mathcal{B}} \\ \star & \star & -\gamma_t^2 I & \mathcal{B}_1^T F_4^T & 0 & 0 \\ \star & \star & \star & -F_4 - F_4^T & 0 & F_4\bar{\mathcal{B}} \\ \star & \star & \star & \star & I - H^T - H & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}}_{\mathcal{Q}_2}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}}_{\mathbf{U}_2^T} G \underbrace{\begin{bmatrix} S_1 & 0 & S_2 & 0 & 0 & -I \end{bmatrix}}_{\mathbf{V}_2} + \underbrace{\begin{bmatrix} S_1^T \\ 0 \\ S_2^T \\ 0 \\ 0 \\ -I \end{bmatrix}}_{\mathbf{V}_2^T} G^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}}_{\mathbf{U}_2} \prec 0. \tag{3.97}$$

Inequality (3.97) can be rewritten equivalently as

$$\begin{bmatrix} \Omega_1 & P + \mathcal{A}_{clx}^T X_2^T - F_1 & F_1\mathcal{B}_1 & X_1 - F_1 & \mathcal{C}_z^T H & F_1\bar{\mathcal{B}} + S_1^T G^T \\ \star & -F_2 - F_2^T & F_2\mathcal{B}_1 & X_2 - F_2 - F_4^T & 0 & F_2\bar{\mathcal{B}} \\ \star & \star & -\gamma_t^2 I & \mathcal{B}_1^T F_4^T & 0 & F_3\bar{\mathcal{B}} + S_2^T G^T \\ \star & \star & \star & -F_4 - F_4^T & 0 & F_4\bar{\mathcal{B}} \\ \star & \star & \star & \star & I - H^T - H & 0 \\ \star & \star & \star & \star & \star & -(G + G^T) \end{bmatrix} \prec 0. \tag{3.98}$$

Substituting $S_1 = (I_m \otimes K_y)\bar{\mathcal{C}} - K_x$ and $S_2 = (I_m \otimes K_y)\bar{\mathcal{D}}$ in (3.98), it can be rewritten as follow

$$\begin{bmatrix} \Omega_1 & P + \mathcal{A}_{clx}^T X_2^T - F_1 & F_1\mathcal{B}_1 & X_1 - F_1 & \mathcal{C}_z^T H & \Omega_2 \\ \star & -F_2 - F_2^T & F_2\mathcal{B}_1 & X_2 - F_2 - F_4^T & 0 & F_2\bar{\mathcal{B}} \\ \star & \star & -\gamma_t^2 I & \mathcal{B}_1^T F_4^T & 0 & \Omega_3 \\ \star & \star & \star & -F_4 - F_4^T & 0 & F_4\bar{\mathcal{B}} \\ \star & \star & \star & \star & I - H^T - H & 0 \\ \star & \star & \star & \star & \star & -(G + G^T) \end{bmatrix} \prec 0, \tag{3.99}$$

where $\Omega_2 = F_1\bar{\mathcal{B}} + \bar{\mathcal{C}}^T(I_m \otimes K_y^T)G^T - K_x^T G^T$ and $\Omega_3 = F_3\bar{\mathcal{B}} + \bar{\mathcal{D}}^T(I_m \otimes K_y)^T G^T$.

Inequality (3.99) is bilinear in the terms Ω_2, Ω_3, and γ_t^2. Replacing $G := (I_{m-1} \otimes G_1)$ and the scalar variable $\mu := \gamma_t^2$, using the Kronecker mixed product $(I_m \otimes G_1)(I_m \otimes K_y) = (I_m \otimes G_1 K_y)$ and defining $Z := G_1 K_y$, enable us to rewrite the inequality (3.99) as the LMI (3.81). Therefore, if the LMI (3.81) is feasible, then inequality (3.99) is feasible and the protocol gains A_c, B_c, C_c, D_c are recovered by $K_y = G_1^{-1}Z$. Moreover, by Lemma 3.4, the H_∞ performance of the system (3.57) is given by $\gamma = \gamma_t\sqrt{m}$ with $\gamma_t = \sqrt{\mu}$. ■

REMARK 3.3 In Theorems 3.2 and 3.4 is not required full rank for the matrices B_u and C_y. Therefore, these theorems allow the design of protocols for a broader class of systems than that assumed in the literature so far (see, for instance, [32], [35], [72], [38]).

3.3 NUMERICAL EXAMPLES

We present two numeric examples to show the proposed conditions' effectiveness in achieving consensus. Theorems 3.2 and 3.4 are implemented to design reduced and full-order controllers. The simulations consider a plant with eight agents distributed in a communication network modeled by the directed graph shown in Fig. 3.1, with connection weights equal to 1. The algorithms were implemented in the Python 3.11.4 software employing library CVXPY [10] and solver MOSEK [4]. The temporal response of agents is obtained by employing the Euler discretization technique in agents and protocol dynamics.

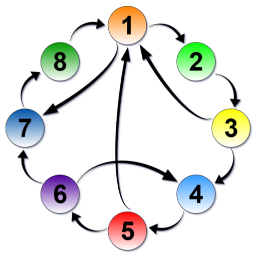


Figure 3.1 – Directed graph that models the communication between the agents.

3.3.1 Example 1

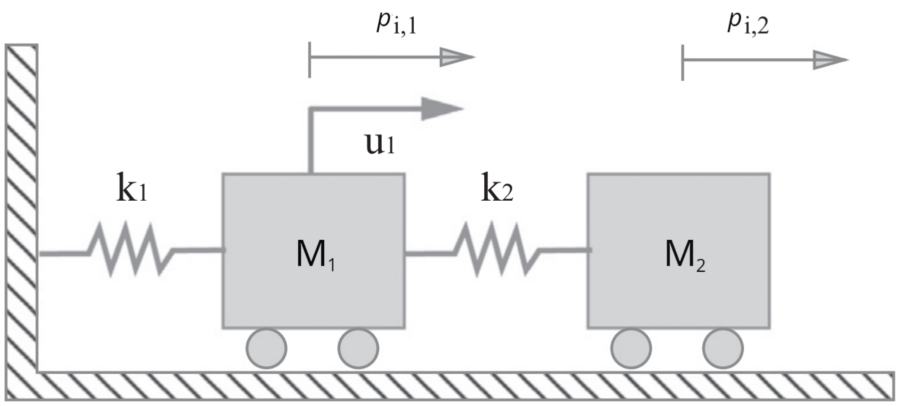


Figure 3.2 – Agent composed by two mass-spring system (Image adapted from [71]).

This example illustrates the design procedure in a mass-spring system that models some industrial applications, for instance, vibration in mechanical systems, etc [71]. Based on [71], which only deals with consensus using neighbors' states, the states of the two-mass–spring multi-agent system are defined as $x_i(t) = \begin{bmatrix} p_{1,i}(t) & \dot{p}_{1,i}(t) & p_{2,i}(t) & \dot{p}_{2,i}(t) \end{bmatrix}^T$, where $p_{1,i}(t)$ and $p_{2,i}(t)$ are the positions and $\dot{p}_{1,i}(t)$ and $\dot{p}_{2,i}(t)$ the velocities of each mass for agents $i \in \{1, \ldots, 8\}$. The dynamics are given by (3.1) with $w_i(t) = 0$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{M_1} & 0 & \frac{k_2}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{M_2} & 0 & \frac{-k_2}{M_2} & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix}, \quad C_y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \tag{3.100}$$

formed by eight two-mass-spring agents (Figure 3.2), with stiffness coefficients $k_1 = 1.5\,\mathrm{N/m}$ and $k_2 = 1\,\mathrm{N/m}$, and mass constants $M_1 = 1.1\,\mathrm{kg}$ and $M_2 = 0.9\,\mathrm{kg}$. The matrix K_x is computed solving Theorem 3.3 with $\beta = 2.5$. Using K_x as input in Theorem 3.2, the protocol (3.2) is obtained with

$$D_c = 0.9802, \tag{3.101}$$

for $n_c = 0$ ($u_i(t) = D_c\nu_i(t)$),

$$\begin{aligned} A_c &= -2.8943, \quad B_c = -0.0192, \\ C_c &= -0.7201, \quad D_c = 1.0003, \end{aligned} \tag{3.102}$$

for $n_c = 1$,

$$\begin{aligned} A_c &= \begin{bmatrix} -2.9376 & -0.0111 \\ -0.0047 & -2.8931 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.0002 \\ -0.0153 \end{bmatrix}, \\ C_c &= \begin{bmatrix} 0.0456 & -0.7191 \end{bmatrix}, \quad D_c = 1.0102, \end{aligned} \tag{3.103}$$

for $n_c = 2$,

$$\begin{aligned} A_c &= \begin{bmatrix} -2.6807 & -0.1299 & 0.2383 \\ -0.0930 & -2.8859 & -0.0966 \\ 0.2436 & -0.1097 & -2.7189 \end{bmatrix}, \quad B_c = \begin{bmatrix} -0.1693 \\ 0.0390 \\ -0.1513 \end{bmatrix}, \\ C_c &= \begin{bmatrix} -0.8057 & 0.2214 & -0.7247 \end{bmatrix}, \quad D_c = 0.6939, \end{aligned} \tag{3.104}$$

for $n_c = 3$,

$$\begin{aligned} A_c &= \begin{bmatrix} -2.8942 & -0.0120 & -0.0318 & -0.0010 \\ 9.2056e-03 & -2.9342 & 2.3564e-02 & -3.0528e-03 \\ -6.8013e-03 & 1.4405e-03 & -2.9103 & 6.0217e-02 \\ 0.0063 & -0.0223 & -0.0407 & -2.9261 \end{bmatrix}, B_c = \begin{bmatrix} 0.1123 \\ -0.085 \\ -0.014 \\ -0.0766 \end{bmatrix}, \\ C_c &= \begin{bmatrix} 0.8276 & -0.7612 & -0.0132 & -0.5628 \end{bmatrix}, \quad D_c = 0.5122, \end{aligned} \tag{3.105}$$

for $n_c = 4$.

This example confirms that we provide the first method to design reduced-order protocols for directed networks with no restriction in the protocols' order. In contrast, other procedures in the literature allow one to design reduced-order protocols only of a particular order for the present two-mass–spring multi-agent system (3.100). In fact, with the works of [32] and [72] it is possible only to design reduced-order protocols of order $n_c := n - s = 3$ and $n_c := n - q = 3$, respectively.

Figure 3.3 shows the agents' temporal response, with random initial conditions for $n_c = 1$. Each depicted agent trajectory has color and number related to the graph in Figure 3.1.

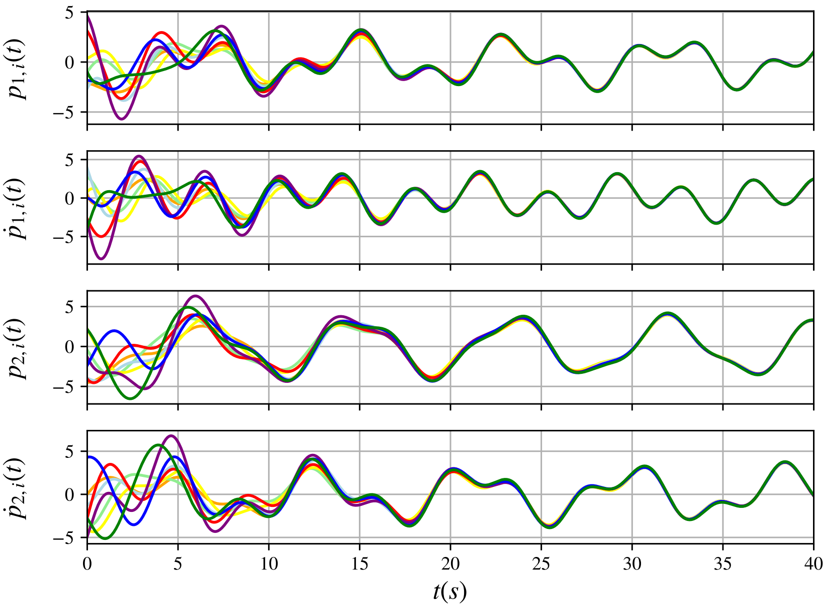


Figure 3.3 – Trajectories of the positions $p_{1,i}(t)$ and $p_{2,i}(t)$, velocities $\dot{p}_{1,i}(t)$ and $\dot{p}_{2,i}(t)$ for agent $i \in \{1, \ldots, 8\}$ considering the protocol (3.2) with gains (3.102) for the topology in Figure 3.1.

3.3.2 Example 2

Consider the model of the translational dynamics of the $i-th$ flying quad-rotor (Figure 3.4) adapted from [52] and [51], as follows

$$\begin{aligned}
\underbrace{\begin{bmatrix}\dot{p}_i(t)\\ \ddot{p}_i(t)\end{bmatrix}}_{\dot{x}_i(t)} &= \underbrace{\begin{bmatrix}0 & 1\\ -c & -b\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}p_i(t)-r_i\\ \dot{p}_i(t)\end{bmatrix}}_{x_i(t)} + \underbrace{\begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}}_{B_u} u_i(t) + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B_w} w_i(t),\\
z_i(t) &= \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{C_z}\left(\underbrace{\begin{bmatrix}p_i(t)-r_i\\ \dot{p}_i(t)\end{bmatrix}}_{x_i(t)} - \frac{1}{8}\sum_{j=1}^{8}\underbrace{\begin{bmatrix}p_j(t)-r_j\\ \dot{p}_j(t)\end{bmatrix}}_{x_j(t)}\right),\\
y_i(t) &= \underbrace{\begin{bmatrix}1 & 2\end{bmatrix}}_{C_y}\underbrace{\begin{bmatrix}p_i(t)-r_i\\ \dot{p}_i(t)\end{bmatrix}}_{x_i(t)} + \underbrace{0.5}_{D_y} w_i(t),\; i = 1, ..., 8,
\end{aligned} \tag{3.106}$$

with $c = 0$ and $b = 1$, and r_i are displacements for agents position considered as $r_1 = 4$ and $r_i = r_{i-1} + 4$ for $i = 2, ..., 8$. Observe that, when agents reach consensus, i.e., $\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0$, each agent's position is equal to its neighbor position incremented by 4. Moreover, we consider $w_i(t) = sin(\frac{3}{i}t)$, if $t \in [8, 16]$, and $w_i(t) = 0$, otherwise.

Figure 3.4 – Example of a quad-rotor. Source: Google Images.

The dynamic protocol (3.48) is designed by Theorem 3.4 for the static ($n_c = 0$), reduced ($n_c = 1$), and full-order ($n_c = 2$) cases with gains K_x designed by Theorem 3.3 with $\beta_{n_c=1} = 0.02$ and $\beta_{n_c=2} = 0.01$ found in a search in the interval $[0.01, 1]$. The obtained H_∞ performance indexes are $\gamma_{n_c=0} = 17.2639$, $\gamma_{n_c=1} = 17.1128$, and $\gamma_{n_c=2} = 17.0811$, with gains

$$D_c = \begin{bmatrix} 1.8174 \\ 0.0367 \end{bmatrix},$$

and

$$\begin{aligned} A_c &= -0.6495, \quad B_c = 0.0033 \\ C_c &= \begin{bmatrix} 80.4055 \\ 2.3616 \end{bmatrix}, \quad D_c = \begin{bmatrix} 1.4501 \\ 0.0231 \end{bmatrix}, \end{aligned} \tag{3.107}$$

and

$$\begin{aligned} A_c &= \begin{bmatrix} -2.6711 & 0.6172 \\ 0.6794 & -0.9803 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.0025 \\ 0.0053 \end{bmatrix} \\ C_c &= \begin{bmatrix} -18.594 & 37.9061 \\ 0.4701 & 1.7088 \end{bmatrix}, \quad D_c = \begin{bmatrix} 1.4924 \\ 0.0083 \end{bmatrix} \end{aligned}$$

for $n_c = 0$, $n_c = 1$ and $n_c = 2$, respectively.

The trajectories of the state variables of the agents depicted in Fig. 3.5, for the case $n_c = 1$, indicate that agents reach consensus. Moreover, the controlled outputs converge to zero.

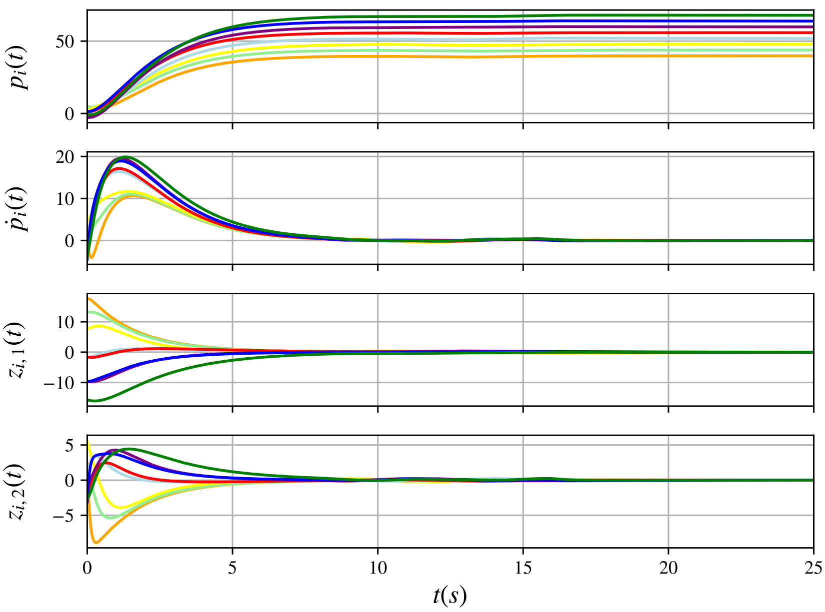


Figure 3.5 – Trajectories of the positions $p_i(t)$, velocities $\dot{p}_i(t)$ and controlled outputs $z_i(t)$ for agent $i \in \{1, \ldots, 8\}$ considering the protocol (3.2) with gains (3.107).

3.4 CHAPTER CONCLUSIONS

This chapter presents conditions for consensus of linear multi-agent systems by dynamic output feedback protocols of arbitrary order under directed communication graphs. The consensus conditions deal with the system under the challenging scenario where the protocol does not share controllers' states; there is no restriction on the agents' dynamical matrices; the communication network is directed and without the requisition of having the Laplacian matrix diagonalizable. Two cases are analyzed: when agents are subject or not to disturbances. This work is the first to propose LMI conditions to design dynamic output feedback controllers in directed networks with protocols that do not assume controllers' states are shared in the network. We also present the design of reduced-order output feedback H_∞ consensus protocols for directed networks as a novelty in the literature. We presented new LMI conditions to design H_∞ static, reduced-, and full-order output protocols.

4 ROBUST PROTOCOLS FOR THE CONSENSUS OF DISTURBED UNCERTAIN AGENTS IN UNCERTAIN NETWORKS

This chapter extends the results in Chapter 3, published in[53], presenting LMI conditions for designing any-order protocols for consensus of disturbed agents with an uncertain parameter and connected in uncertain polytopic networks. Different from Chapter 3, which obtains a bound for the multi-agent system H_∞ performance, in the present chapter, by transforming the consensus problem into a stability problem, we consider a different variable transformation of the applied in Chapter 3, which allows us to compute the original system H_∞ performance directly.

In addition to the above, the results presented in this chapter surpass some gaps in the literature of consensus involving uncertain agents and topologies. Some works in the literature consider that the way agents share information is uncertain, as in [34] studied the design of state protocols for agents connected in uncertain undirected networks. Authors in [34] deal with two cases: continuous-time and discrete-time agents. In [23] is studied the consensuability of second-order disturbed agents connected in a polytopic uncertain network. In [9], a full-order protocol is designed considering that agents are connected in an uncertain fuzzy undirected topology.

The capability of agents to attenuate the influence of parameter uncertainties is another agent feature found in the literature on consensus. The authors in [69] derive conditions that design optimal leader-follower protocols for non-disturbed agents with parameter uncertainty in undirected graphs. The study in [60] deals with designing reduced-order protocols for non-disturbed agents subject to parameter uncertainty connected in an undirected network. The work [19] designs full-order protocols for disturbed agents subject to parametric uncertainties and sharing information in strongly connected directed networks. In [31], two different types of full-order protocols are designed for disturbed agents connected in directed networks. The work [62] designs full-order output regulation protocols for non-disturbed heterogeneous agents subjected to parametric uncertainty connected by a directed graph.

With the above-cited works, designing any order protocols for disturbed agents subjected to a parametric uncertainty connected in uncertain directed networks proves to be a relevant topic for investigation. Table 4.1 summarizes the main characteristics of works in the literature compared to results in this chapter.

	T4.1	C4.1	C4.2	[69]	[60]	[19]	[34]	[23]	[53]	[31]	[62]	[9]
Static Output Feedback	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$
Reduced-Order	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\times$
Full-Order	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$
Digraph	$\checkmark$	$\checkmark$	$\checkmark$	$\circ$	$\times$	$\circ$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$
Parameter Uncertainty	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\times$
Uncertain Topology	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$
Disturbances	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\circ$	$\times$	$\checkmark$	$\checkmark$	$\circ$	$\times$	$\times$
No Rank Restriction	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$
No Controller Interaction	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$
LMI	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\circ$	$\checkmark$	$\checkmark$	$\circ$	$\times$	$\checkmark$

Table 4.1 – Comparison between Theorem 4.1 (T4.1), Corollary 4.1 (C4.1) and 4.2 (C4.2) concerning literature results. The symbols $\checkmark$means "yes", $\times$ means "no" and $\circ$ means partially.

Some works received $\circ$ in the "Digraph," "Disturbances," and "LMI" lines in Table 4.1. Concerning the use of $\circ$ in the "Digraph" line, in [69], the network is directed only in the communication between leader and agents; the rest of the network is undirected. The directed graph in [19] is strongly connected. Concerning disturbances, [19] considers disturbances only in agents' dynamics. In [31], only agents' outputs are subjected to disturbances. Regarding the "Parameter Uncertainty" line, one has that all works consider structured time-varying parametric uncertainties except for [60], which considers time-varying interval uncertainties, and [62], which considers heterogeneous time-invariant uncertainties. Moreover, only [62] considers more than one parameter uncertainty in agents' dynamics. Concerning the use of $\circ$ in the "LMI" line, in [34], only the discrete-time case presents an LMI condition. In [31], the proposed multi-step algorithm has an LMI and a Ricatti equation that must be solved.

4.1 PRELIMINARIES AND PROBLEM STATEMENT

4.1.1 The Robust Consensus Problem

Consider m agents in a time-varying uncertain directed network, each with the following dynamic model

$$\dot{x}_i(t) = (A + \Delta A(t))x_i(t) + B_w w_i(t) + B_u u_i(t), \quad i = 1, ..., m, \tag{4.1}$$
$$y_i(t) = C_y x_i(t) + D_y w_i(t), \tag{4.2}$$

where $x_i(t) \in \mathbb{R}^n$ are the state variables, $y_i(t) \in \mathbb{R}^q$ the outputs, $z_i(t) \in \mathbb{R}^r$ the controlled outputs, $u_i(t) \in \mathbb{R}^s$ the control inputs, $w_i(t) \in \mathbb{R}^{n_w}$ exogenous disturbances that belongs to $\mathcal{L}_2[0,\infty)$, and $A \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times s}$, $B_w \in \mathbb{R}^{n\times n_w}$, $C_z \in \mathbb{R}^{r\times n}$, $C_y \in \mathbb{R}^{q\times n}$ and $D_y \in \mathbb{R}^{q\times n_w}$, are known matrices. The time-varying matrix

$$\Delta A(t) = EF(t)M \tag{4.3}$$

is a parametric uncertainty, where $F(t) \in \mathbb{R}^{f_1\times f_2}$ is a time-varying matrix with Lebesgue measurable elements [63], [19], satisfying $F(t)F^T(t) \le \delta^{-1}I$, $E \in \mathbb{R}^{n\times f_1}$ and $M \in \mathbb{R}^{f_2\times n}$ data matrices. Each agent is controlled locally by each one of the following m dynamic controllers

$$\begin{aligned} \dot{x}_{c,i}(t) &= A_c x_{c,i}(t) + B_c \nu_i(t), \quad i = 1, ..., m, \\ u_i(t) &= C_c x_{c,i}(t) + D_c \nu_i(t), \end{aligned} \tag{4.4}$$

where $x_{c,i}(t) \in \mathbb{R}^{n_c}$ is the state variables of the dynamical controllers, $A_c \in \mathbb{R}^{n_c\times n_c}$, $B_c \in \mathbb{R}^{n_c\times n_\nu}$, $C_c \in \mathbb{R}^{s\times n_c}$, $D_c \in \mathbb{R}^{s\times n_\nu}$ the dynamic controller parameters to find and, the agents' relative output signal

$$\nu_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(t)(y_i(t) - y_j(t)), i = 1, ..., m \tag{4.5}$$

where $a_{ij}(t)$ are time-varying weights whose values are assumed unknown to the controller, but it has information that the weights are within known bounds

$$\underline{a}_{ij} \le a_{ij}(t) \le \overline{a}_{ij}, \ \forall t. \tag{4.6}$$

For consensus analysis, we introduce the balanced consensus outputs

$$z_i(t) = C_z(x_i(t) - \frac{1}{m}\sum_{j=1}^{m} x_j(t)), \tag{4.7}$$

such that state consensus implies $z_i(t) = 0$ for every i, and the matrix C_z balances the relative importance of consensus among particular state components of the agents in the performance analysis. It is considered a system performance evaluation in the H_∞ sense which relates the overall exogenous disturbance $w(t)$ and the consensus discrepancy $z(t)$ through the inequality

$$\int_0^\infty \|z(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t, \quad \forall w(t) \in \mathcal{L}_2[0,\infty), \tag{4.8}$$

where the scalar $\gamma > 0$ is the H_∞ consensus performance index for the multi-agent system

(4.1)-(4.5).

PROBLEM 4.1 (Robust Consensus with Uncertain Disturbed Agents and Time-varying Communication Weights) For the multi-agent system (4.1)-(4.2) with time-varying parameter uncertainty (4.3) and time-varying directed network with weights (4.6), design, if possible, a protocol (4.4)-(4.5) of given order n_c, $0 \leq n_c \leq n$, using only local information (4.5) such that the resulting closed-loop multi-agent system (4.1)-(4.7)

1. satisfies

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, \;\; \lim_{t\to\infty} ||x_{c,i}(t) - x_{c,j}(t)|| = 0, \;\; i, j = 1, ..., m \qquad (4.9)$$

   in the absence of disturbances $w(t)$, for any initial conditions;

2. and satisfies, in presence of disturbances $w(t)$ and zero initial conditions, the H_∞ performance (4.8) for the controlled output (4.7) with a given gain $\gamma > 0$.

4.1.2 Transformed Problem

We first reframe the representation of the weights. It is easy to see that

$$\Omega_{\mathsf{A}} := \{\mathsf{A} = [a_{ij}] \in \mathbb{R}_+^{m\times m} : a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}], \; i, j = 1, ..., m\}$$

is a convex set. Therefore, one can find vertices $\mathsf{A}_k \in \Omega_{\mathsf{A}}, k = 1, \ldots, N$ such that $\Omega_{\mathsf{A}} = Co\{\mathsf{A}_k \in \Omega_{\mathsf{A}}, k = 1, \ldots, N\}$. Further, for any $A(t) = [a_{ij}(t)]$, where $a_{ij}(t) \in [\underline{a}_{ij}, \overline{a}_{ij}]$, there exist $\alpha(t) = (\alpha_1(t), \ldots, \alpha_N(t)) \in \mathbb{R}^N$ satisfying $\alpha_\ell(t) \geq 0$ and $\sum_{\ell=1}^N \alpha_\ell(t) = 1$ such that

$$\mathsf{A}(t) = [a_{ij}(t)] = \sum_{k=1}^N \alpha_k(t)\mathsf{A}_k. \qquad (4.10)$$

With some abuse of notation, for now on, we write $a_{ij}(\alpha(t))$, $\mathsf{A}(\alpha(t))$, in lieu of $a_{ij}(t)$ and $\mathsf{A}(t)$ to evidence that we are considering their representation as a convex combination with parameter $\alpha(t)$.

In this chapter, the following assumption is considered.

ASSUMPTION 4.1 The graphs $\mathbb{G}_k(\mathcal{V}, \mathcal{E}, \mathsf{A}_k)$ of vertices A_k have a spanning tree for $k = 1, ..., N$.

As said in Chapter 3, the assumption that a spanning tree exists in the communication network graph is a prerequisite for consensus. Assumption 4.1 guarantees that all graphs $\mathbb{G}_k$ associated with adjacency matrices A_k have at least one directed path connecting each graph node, since in this chapter we derive conditions based on the vertices of the polytopic Laplacian matrix. In the following, we present two lemmas that relate Assumption 4.1 with the existence of a spanning tree in the convex combination $L(\alpha)$.

LEMMA 4.1 If L_k are Laplacian matrices for $k = 1, ..., N$, the convex combination $L(\alpha(t)) := \sum_{k=1}^{N} \alpha_k(t) L_k$ is a Laplacian matrix for every $\alpha_k(t) \geq 0$ such that $\sum_{k=1}^{N} \alpha_k(t) = 1$.

proof.

Observe that, if $L_k = [l_{ij}^{(k)}]$ are Laplacian matrices, then one has

$$L_k = \begin{cases} l_{ij}^{(k)} = -a_{ij}^{(k)}, \text{for } i \neq j \\ l_{ii}^{(k)} = -\sum_{j=1}^{m} l_{ij}^{(k)}, \end{cases} \tag{4.11}$$

where $a_{ij}^{(k)} \geq 0$. Note that

$$L(\alpha(t)) = \sum_{k=1}^{N} \alpha_k(t) L_k = \begin{cases} \sum_{k=1}^{N} \alpha_k(t) l_{ij}^{(k)} = -\sum_{k=1}^{N} \alpha_k(t) a_{ij}^{(k)}, \text{for } i \neq j, \\ \sum_{k=1}^{N} \alpha_k(t) l_{ii}^{(k)} = -\sum_{k=1}^{N} \alpha_k(t) \sum_{j=1}^{m} l_{ij}^{(k)}, \end{cases} \tag{4.12}$$

Finally, one has

$$L(\alpha(t)) = \begin{cases} l_{ij}(\alpha(t)) = -a_{ij}(\alpha(t)) = -a_{ij}(t), \text{for } i \neq j \\ l_{ii}(\alpha(t)) = -\sum_{j=1}^{m} l_{ij}(\alpha(t)) = -\sum_{j=1}^{m} l_{ij}(t). \end{cases} \tag{4.13}$$

Since $a_{ij}(\alpha(t)) \geq 0$, from (4.13) one can conclude that $L(\alpha(t))$ is a Laplacian matrix for every $\alpha_k(t)$ such that $\sum_{k=1}^{N} \alpha_k(t) = 1$. ■

REMARK 4.1 From Lemma 4.1 one has that $L(\alpha(t))$ is a Laplacian matrix and then $L(\alpha(t))\mathbf{1}_m = 0_m$.

LEMMA 4.2 If graphs $\mathbb{G}_k$ associated with the adjacency matrix A_k has a spanning tree, then graph $\mathbb{G}(\alpha(t))$ associated with $\mathsf{A}(\alpha(t)) = \sum_{k=1}^{N} \alpha_k(t) \mathsf{A}_k$ has a spanning tree for all

$\alpha_k(t) \geq 0$.

proof.

By hypothesis, each graph $\mathbb{G}_k$ has a spanning tree. Then, there exist adjacency matrices A_k, associated with the graphs $\mathbb{G}_k$, with some positive elements (graph weights) that represent at least one directed path through all nodes of each graph $\mathbb{G}_k$. If we take $\alpha_k(t) \geq 0$ and multiply it by A_k, obtaining $\bar{\mathsf{A}}_k = \alpha_k(t)\mathsf{A}_k$, we maintain the connections of the graphs $\mathbb{G}_k$ or cancel A_k. In the same way, the effect of the sum $\sum_{k=1}^{N} \bar{\mathsf{A}}_k$ is to overlap the existing graph connections if more than one $\alpha_k(t)$ are non-zero. Note that $\sum_{k=1}^{N} \bar{\mathsf{A}}_k = \sum_{k=1}^{N} \alpha_k(t)\mathsf{A}_k = \mathsf{A}(\alpha(t))$ represents the convex combination of A_k and is the adjacency matrix associated with the graph $\mathbb{G}(\alpha(t))$. Therefore, $\mathbb{G}(\alpha(t))$ contains at least a spanning tree. ■

The transformed system that will allow us to derive the conditions that solve Problem 4.1 is presented in the next subsection. Unlike Chapter 3, the system transformation proposed here allows us to directly compute the multi-agent system H_∞ performance γ.

4.1.3 Transformed Uncertain Multi-Agent System

The concatenated system, considering the m agents, is given by

$$\begin{aligned} \dot{x}(t) &= (I_m \otimes (A + \Delta A(t)))x(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)u(t), \\ z(t) &= (C_g \otimes C_z)x(t), \\ y(t) &= (I_m \otimes C_y)x(t) + (I_m \otimes D_y)w(t), \end{aligned} \tag{4.14}$$

with C_g defined as in (3.52).

In the same way, the concatenated dynamical controller is given by

$$\begin{aligned} \dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) + (I_m \otimes B_c)\nu(t), \\ u(t) &= (I_m \otimes C_c)x_c(t) + (I_m \otimes D_c)\nu(t). \end{aligned} \tag{4.15}$$

The function $\nu(t)$ is the concatenated form of (3.49). It is easy to see, from the non-disturbed case, that $\nu(t) = -(L(\alpha(t))\otimes C_y)x(t) - (L(\alpha(t))\otimes D_y)w(t)$. Then, the augmented dynamical controller (4.15) has the following form

$$\begin{aligned}\dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) - (I_m \otimes B_c)((L(\alpha(t)) \otimes C_y)x(t) + (L(\alpha(t)) \otimes D_y)w(t)),\\ u(t) &= (I_m \otimes C_c)x_c(t) - (I_m \otimes D_c)((L(\alpha(t)) \otimes C_y)x(t) + (L(\alpha(t)) \otimes D_y)w(t)).\end{aligned} \tag{4.16}$$

From mixed product property of the Kronecker product, $(I_m \otimes B_c)(L(\alpha(t)) \otimes C) = (L(\alpha(t)) \otimes B_cC)$. Then the dynamical controller (4.16) can be rewritten as follows

$$\begin{aligned}\dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) - (L(\alpha(t)) \otimes (B_cC_y))x(t) - (L(\alpha(t)) \otimes (B_cD_y))w(t),\\ u(t) &= (I_m \otimes C_c)x_c(t) - (L(\alpha(t)) \otimes (D_cC_y))x(t) - (L(\alpha(t)) \otimes (D_cD_y))w(t).\end{aligned} \tag{4.17}$$

Substituting the dynamical controller (4.17) in (3.52), and using the mixed product property of Kronecker product, $(I_m \otimes B_u)(I_m \otimes C_c) = (I_m \otimes B_uC_c)$ and $(I_m \otimes B_u)(L(\alpha(t)) \otimes (D_cC_y)) = (L(\alpha(t)) \otimes (B_uD_cC_y))$, then defining $\psi(t) = \begin{bmatrix} x(t)^T & x_c(t)^T \end{bmatrix}^T$ the system (3.55) can be rewritten in the following form

$$\begin{aligned}\dot{\psi}(t) &= \underline{\mathbf{A}}(t)\psi(t) + \underline{\mathbf{B}}_1 w(t),\\ z(t) &= \underline{\mathbf{C}}\psi(t),\end{aligned} \tag{4.18}$$

where,

$$\begin{aligned}\underline{\mathbf{A}}(t) &= \begin{bmatrix} (I_m \otimes (A + \Delta A(t))) - (L(\alpha(t)) \otimes (B_uD_cC_y)) & (I_m \otimes B_uC_c) \\ -(L(\alpha(t)) \otimes (B_cC_y)) & (I_m \otimes A_c) \end{bmatrix},\\ \underline{\mathbf{B}}_1 &= \begin{bmatrix} (I_m \otimes B_w) - (L(\alpha(t)) \otimes (B_uD_cD_y)) \\ -L(\alpha(t)) \otimes (B_cD_y) \end{bmatrix},\\ \underline{\mathbf{C}} &= \begin{bmatrix} (C_g \otimes C_z) & 0 \end{bmatrix}.\end{aligned}$$

Similarly, as in Chapter 3, we need to transform the consensus problem into a stability problem. Inspired by [40], in this chapter, the stability problem is based on a variable defined as the disagreement of the agents and controller states about the average of their neighbors. Therefore, considering a new variable $\varphi(t) = \begin{bmatrix} \varphi_1^T(t) & \varphi_2^T(t) \end{bmatrix}^T$, such that

$$\varphi_1(t) = x(t) - \frac{1}{m}\mathbf{1}_m \otimes \Big(\sum_{j=1}^{m} x_j(t)\Big), \tag{4.19}$$

$$\varphi_2(t) = x_c(t) - \frac{1}{m}\mathbf{1}_m \otimes \Big(\sum_{j=1}^{m} x_{c,j}(t)\Big), \tag{4.20}$$

we can define the following variable transformation

$$\varphi(t) = \underbrace{\begin{bmatrix} (C_g \otimes I_n) & 0 \\ 0 & (C_g \otimes I_{n_c}) \end{bmatrix}}_{\mathcal{C}_g} \psi(t). \tag{4.21}$$

With the derivative of (4.21), system (4.18) can be rewritten as

$$\begin{aligned} \dot{\varphi}(t) &= \mathcal{C}_g \underline{\mathbf{A}}(t)\psi(t) + \mathcal{C}_g \underline{\mathbf{B}}_1 w(t), \\ z(t) &= \underline{\mathbf{C}}\psi(t), \end{aligned}$$

or, equivalently

$$\begin{aligned} \dot{\varphi}(t) &= \begin{bmatrix} (C_g \otimes (A + \Delta A(t))) - (C_g L(\alpha(t)) \otimes (B_u D_c C_y)) & (C_g \otimes B_u C_c) \\ -(C_g L(\alpha(t)) \otimes (B_c C_y)) & (C_g \otimes A_c) \end{bmatrix} \psi(t) \\ &+ \begin{bmatrix} (C_g \otimes B_w) - (C_g L(\alpha(t)) \otimes (B_u D_c D_y)) \\ -C_g L(\alpha(t)) \otimes (B_c D_y) \end{bmatrix} w(t), \\ z(t) &= \begin{bmatrix} (C_g \otimes C_z) & 0 \end{bmatrix} \psi(t). \end{aligned} \tag{4.22}$$

As $C_g = I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^{\mathrm{T}}$, then $\psi(t)$ in (4.22) can be rewritten as

$$\psi(t) = \varphi(t) + \begin{bmatrix} (\frac{1}{m}\mathbf{1}_m\mathbf{1}_m^{\mathrm{T}} \otimes I_n) & 0 \\ 0 & (\frac{1}{m}\mathbf{1}_m\mathbf{1}_m^{\mathrm{T}} \otimes I_{n_c}) \end{bmatrix} \psi(t). \tag{4.23}$$

Substituting (4.23) in (4.22), and remember that $C_g\mathbf{1}_m = 0$ and $L(\alpha(t))\mathbf{1}_m = 0$ (see Remark 4.1), one has

$$\begin{aligned} \dot{\varphi}(t) &= \begin{bmatrix} (C_g \otimes (A + \Delta A(t))) - (C_g L(\alpha(t)) \otimes (B_u D_c C_y)) & (C_g \otimes B_u C_c) \\ -(C_g L(\alpha(t)) \otimes (B_c C_y)) & (C_g \otimes A_c) \end{bmatrix} \varphi(t) \\ &+ \begin{bmatrix} (C_g \otimes B_w) - (C_g L(\alpha(t)) \otimes (B_u D_c D_y)) \\ -(C_g L(\alpha(t)) \otimes (B_c D_y)) \end{bmatrix} w(t), \\ z(t) &= \begin{bmatrix} (C_g \otimes C_z) & 0 \end{bmatrix} \varphi(t). \end{aligned} \tag{4.24}$$

Observe that C_g has a form of a Laplacian matrix with all weights $1/m$; with this information, one has that C_g has a simple zero eigenvalue, and all other eigenvalues are ones. As C_g is a symmetric matrix, there exists a unitary matrix $\mathcal{U} = \begin{bmatrix} \frac{\mathbf{1}_m}{\sqrt{m}} & \tilde{\mathcal{U}} \end{bmatrix}$, such that $\mathcal{U}^T C_g \mathcal{U} = diag(0, 1, ..., 1)$. Defining $\mathfrak{U} = \begin{bmatrix} (\mathcal{U} \otimes I_n) & 0 \\ 0 & (\mathcal{U} \otimes I_{n_c}) \end{bmatrix}$ and the variables $\xi(t) = \mathfrak{U}^T\varphi(t)$, $\bar{w}(t) = (\mathcal{U}^T \otimes I_{n_w})w(t)$ and $\bar{z}(t) = (\mathcal{U}^T \otimes I_r)z(t)$, system (4.24) can be rewritten as

$$
\begin{aligned}
\dot{\xi}(t) &= \begin{bmatrix} ((\mathcal{U}^T C_g \mathcal{U}) \otimes (A + \Delta A(t))) & (\mathcal{U}^T C_g \mathcal{U} \otimes B_u C_c) \\ 0 & (\mathcal{U}^T C_g \mathcal{U} \otimes A_c) \end{bmatrix} \xi(t) \\
&\quad - \begin{bmatrix} (\mathcal{U}^T C_g L(\alpha(t)) \mathcal{U} \otimes (B_u D_c C_y)) & 0 \\ \mathcal{U}^T C_g L(\alpha(t)) \mathcal{U} \otimes (B_c C_y) & 0 \end{bmatrix} \xi(t) \\
&\quad + \begin{bmatrix} (\mathcal{U}^T C_g \mathcal{U} \otimes B_w) - (\mathcal{U}^T C_g L(\alpha(t)) \mathcal{U} \otimes (B_u D_c D_y)) \\ -\mathcal{U}^T C_g L(\alpha(t)) \mathcal{U} \otimes (B_c D_y) \end{bmatrix} \bar{w}(t), \\
\bar{z}(t) &= \begin{bmatrix} (\mathcal{U}^T C_g \mathcal{U} \otimes C_z) & 0 \end{bmatrix} \xi(t). \qquad (4.25)
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
\mathcal{U}^T C_g \mathcal{U} \mathcal{U}^T L(\alpha(t)) \mathcal{U} &= \begin{bmatrix} \frac{1}{\sqrt{m}} \mathbf{1}_m^T \\ \tilde{\mathcal{U}}^T \end{bmatrix} C_g \begin{bmatrix} \frac{1}{\sqrt{m}} \mathbf{1}_m & \tilde{\mathcal{U}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{m}} \mathbf{1}_m^T \\ \tilde{\mathcal{U}}^T \end{bmatrix} L(\alpha(t)) \begin{bmatrix} \frac{1}{\sqrt{m}} \mathbf{1}_m & \tilde{\mathcal{U}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{m} \mathbf{1}_m^T C_g \mathbf{1}_m & \frac{1}{\sqrt{m}} \mathbf{1}_m^T C_g \tilde{\mathcal{U}} \\ \tilde{\mathcal{U}}^T C_g \frac{1}{\sqrt{m}} \mathbf{1}_m & \tilde{\mathcal{U}}^T C_g \tilde{\mathcal{U}} \end{bmatrix} \begin{bmatrix} \frac{1}{m} \mathbf{1}_m^T L(\alpha(t)) \mathbf{1}_m & \frac{1}{\sqrt{m}} \mathbf{1}_m^T L(\alpha(t)) \tilde{\mathcal{U}} \\ \tilde{\mathcal{U}}^T L(\alpha(t)) \frac{1}{\sqrt{m}} \mathbf{1}_m & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix}.
\end{aligned}
$$

Since $L(\alpha(t))$ is a convex combination of Laplacian matrices, then $L(\alpha(t))\mathbf{1}_m = 0$ and as $C_g = I_m - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^{\mathrm{T}}$ then $C_g\mathbf{1}_m = 0_m$ and $\mathbf{1}_m^T C_g = 0_m^T$. Using this fact, one has

$$
\mathcal{U}^T C_g \mathcal{U} \mathcal{U}^T L(\alpha(t)) \mathcal{U} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{U}}^T C_g \tilde{\mathcal{U}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{m}} \mathbf{1}_m^T L(\alpha(t)) \tilde{\mathcal{U}} \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix}. \qquad (4.26)
$$

Since that the congruence transformation $\mathcal{U}^T C_g \mathcal{U}$ diagonalize C_g, one has

$$
\begin{aligned}
\mathcal{U}^T C_g \mathcal{U} \mathcal{U}^T L(\alpha(t)) \mathcal{U} &= \begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{m}} \mathbf{1}_m^T L(\alpha(t)) \tilde{\mathcal{U}} \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix}. \qquad (4.27)
\end{aligned}
$$

Observe that, with (4.27) system (4.25) can be rewritten as

$$
\begin{aligned}
\dot{\xi}(t) = &\begin{bmatrix} (\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix} \otimes (A + \Delta A(t))) & (\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix} \otimes B_u C_c) \\ 0 & (\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix} \otimes A_c) \end{bmatrix} \xi(t) \\
&- \begin{bmatrix} (\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix} \otimes (B_u D_c C_y)) & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix} \otimes (B_c C_y) & 0 \end{bmatrix} \xi(t) \\
&+ \begin{bmatrix} (\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix} \otimes B_w) - (\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix} \otimes (B_u D_c D_y)) \\ - \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} \end{bmatrix} \otimes (B_c D_y) \end{bmatrix} \bar{w}(t), \\
\bar{z}(t) = &\begin{bmatrix} (\begin{bmatrix} 0 & 0 \\ 0 & I_{m-1} \end{bmatrix} \otimes C_z) & 0 \end{bmatrix} \xi(t).
\end{aligned} \tag{4.28}
$$

Note that the state variable of (4.28) is defined as $\xi(t) = \mathfrak{U}^T \varphi(t)$ and can be rewritten as

$$
\xi(t) = \begin{bmatrix} (\begin{bmatrix} \frac{1}{\sqrt{m}} \mathbf{1}_m^T \\ \tilde{\mathcal{U}}^T \end{bmatrix} C_g \otimes I_n) & 0 \\ 0 & (\begin{bmatrix} \frac{1}{\sqrt{m}} \mathbf{1}_m^T \\ \tilde{\mathcal{U}}^T \end{bmatrix} C_g \otimes I_{n_c}) \end{bmatrix} \psi(t). \tag{4.29}
$$

As $\mathbf{1}_m^T C_g = 0_m^T$, one has

$$
\xi(t) = \begin{bmatrix} (\begin{bmatrix} 0_m^T \\ \tilde{\mathcal{U}}^T C_g \end{bmatrix} \otimes I_n) & 0 \\ 0 & (\begin{bmatrix} 0_m^T \\ \tilde{\mathcal{U}}^T C_g \end{bmatrix} \otimes I_{n_c}) \end{bmatrix} \psi(t). \tag{4.30}
$$

From (4.30), one has that can be defined as $\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}$ where $\xi_{1,1}(t) = 0_n$, $\xi_{2,1}(t) = 0_n$. Observe that disturbance $\bar{w}_1(t)$ do not influence system (4.28), then one can rewrite system (4.28) in an equivalent form, discarding $\bar{w}_1(t)$. Defining $\tilde{\xi}(t) = \begin{bmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{bmatrix}$, $\tilde{w}(t) = \begin{bmatrix} \bar{w}_2^T(t) & \cdots & \bar{w}_m^T(t) \end{bmatrix}^T$ and $\tilde{z}(t) = \begin{bmatrix} \bar{z}_2^T(t) & \cdots & \bar{z}_m^T(t) \end{bmatrix}^T$, where $\tilde{\xi}_1(t) = \begin{bmatrix} \xi_{1,2}^T(t) & \cdots & \xi_{1,m}^T(t) \end{bmatrix}^T$ and $\tilde{\xi}_2(t) = \begin{bmatrix} \xi_{2,2}^T(t) & \cdots & \xi_{2,m}^T(t) \end{bmatrix}^T$, we can rewrite the system

(4.28) equivalently as

$$
\begin{aligned}
\dot{\tilde{\xi}}(t) &= \begin{bmatrix} (I_{m-1} \otimes (A + \Delta A(t))) & (I_{m-1} \otimes B_u C_c) \\ 0 & (I_{m-1} \otimes A_c) \end{bmatrix} \tilde{\xi}(t) - \begin{bmatrix} (\tilde{L}(\alpha(t)) \otimes (B_u D_c C_y)) & 0 \\ \tilde{L}(\alpha(t)) \otimes (B_c C_y) & 0 \end{bmatrix} \tilde{\xi}(t) \\
&+ \begin{bmatrix} (I_{m-1} \otimes B_w) - \tilde{L}(\alpha(t)) \otimes (B_u D_c D_y)) \\ -\tilde{L}(\alpha(t)) \otimes (B_c D_y) \end{bmatrix} \tilde{w}(t), \\
\tilde{z}(t) &= \begin{bmatrix} (I_{m-1} \otimes C_z) & 0 \end{bmatrix} \tilde{\xi}(t).
\end{aligned} \tag{4.31}
$$

where, $\tilde{L}(\alpha(t)) = \tilde{\mathcal{U}}^T L(\alpha(t)) \tilde{\mathcal{U}} = \sum_{k=1}^{N} \alpha_k(t) \tilde{\mathcal{U}}^T L_k \tilde{\mathcal{U}}$.

Now, we can rewrite system (4.31) in the equivalent form

$$
\begin{aligned}
\dot{\tilde{\xi}}(t) &= \underbrace{(\mathcal{A} + \Delta\mathcal{A}(t) + \mathcal{B}\mathcal{K}_y \mathcal{C}(\alpha(t)))}_{\mathbb{A}_{cl}(\alpha(t))} \tilde{\xi}(t) + \underbrace{(\mathcal{B}_1 + \mathcal{B}\mathcal{K}_y \mathcal{D}(\alpha(t)))}_{\mathbb{B}_{cl}(\alpha(t))} \tilde{w}(t) \\
\tilde{z}(t) &= \tilde{C} \tilde{\xi}(t),
\end{aligned} \tag{4.32}
$$

where the parameters are defined as

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} I_m \otimes A & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 & I_{m-1} \otimes B_u \\ I_{(m-1)n_c} & 0 \end{bmatrix}, \\
\mathcal{D}(\alpha(t)) &= \begin{bmatrix} 0 \\ -(\tilde{L}(\alpha(t)) \otimes D_y) \end{bmatrix}, \mathcal{K}_y = \begin{bmatrix} I_{m-1} \otimes A_c & I_{m-1} \otimes B_c \\ I_{m-1} \otimes C_c & I_{m-1} \otimes D_c \end{bmatrix}, \\
\mathcal{B}_1 &= \begin{bmatrix} (I_{m-1} \otimes B_1) \\ 0 \end{bmatrix}, \quad \Delta\mathcal{A}(t) = \mathcal{E}\mathcal{F}(t)\mathcal{M}, \\
\mathcal{C}(\alpha(t)) &= \begin{bmatrix} 0 & I_{(m-1)n_c} \\ -\tilde{L}(\alpha(t)) \otimes C_y & 0 \end{bmatrix}, \mathcal{E} = \begin{bmatrix} I_{m-1} \otimes E \\ 0 \end{bmatrix}, \\
\mathcal{F}(t) &= (I_{m-1} \otimes F(t)), \quad \mathcal{M} = \begin{bmatrix} (I_{m-1} \otimes M) & 0 \end{bmatrix}.
\end{aligned} \tag{4.33}
$$

Since matrix $\mathcal{K}_y$ in (4.33) is challenging to design directly with LMI conditions due to its structure. In order to derive tractable LMI conditions for the design of (4.4), one can rewrite

$$
\begin{aligned}
A_c &= \mathfrak{I}_{11} K_y \mathfrak{I}_{21}^T, B_c = \mathfrak{I}_{11} K_y \mathfrak{I}_{22}^T, \\
C_c &= \mathfrak{I}_{12} K_y \mathfrak{I}_{21}^T, D_c = \mathfrak{I}_{12} K_y \mathfrak{I}_{22}^T,
\end{aligned}
$$

where

$$\mathfrak{I}_{11} = \begin{bmatrix} I_{n_c} & 0_{n_c \times s} \end{bmatrix}, \mathfrak{I}_{12} = \begin{bmatrix} 0_{s \times n_c} & I_s \end{bmatrix},$$
$$K_y = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \mathfrak{I}_{21} = \begin{bmatrix} I_{n_c} & 0_{n_c \times q} \end{bmatrix}, \mathfrak{I}_{22} = \begin{bmatrix} 0_{q \times n_c} & I_q \end{bmatrix}.$$

and using some properties of the Kronecker product, $\mathcal{K}_y$ can be represented by

$$\mathcal{K}_y = \mathcal{T}_1(I_{m-1} \otimes K_y)\mathcal{T}_2$$

with

$$\mathcal{T}_1 = \begin{bmatrix} I_{m-1} \otimes \mathfrak{I}_{11} \\ I_{m-1} \otimes \mathfrak{I}_{12} \end{bmatrix}, \mathcal{T}_2 = \begin{bmatrix} I_{m-1} \otimes \mathfrak{I}_{21}^T & I_{m-1} \otimes \mathfrak{I}_{22}^T \end{bmatrix}.$$

Therefore, system (4.32) is rewritten in the following form

$$\dot{\tilde{\xi}}(t) = \underbrace{(\mathcal{A} + \mathcal{E}\mathcal{F}(t)\mathcal{M} + \bar{\mathcal{B}}(I_{m-1} \otimes K_y)\bar{\mathcal{C}}(\alpha(t)))}_{\mathbb{A}_{cl}(\alpha(t))}\tilde{\xi}(t) + \underbrace{(\mathcal{B}_1 + \bar{\mathcal{B}}(I_{m-1} \otimes K_y)\bar{\mathcal{D}}(\alpha(t)))}_{\mathbb{B}_{cl}(\alpha(t))}\tilde{w}(t)$$
$$\tilde{z}(t) = \mathcal{C}_z\tilde{\xi}(t) \tag{4.34}$$

where

$$\bar{\mathcal{B}} = \mathcal{B}\mathcal{T}_1, \quad \bar{\mathcal{C}}(\alpha(t)) = \mathcal{T}_2\mathcal{C}(\alpha(t)), \quad \bar{\mathcal{D}}(\alpha(t)) = \mathcal{T}_2\mathcal{D}(\alpha(t)).$$

REMARK 4.2 Although the use of variable transformations (4.19) and (4.20) represents a more conservative consensus since it forces each agent to synchronize considering the average of all agents, different from Chapter 3 the H_∞ cost it is obtained directly. It is easy to see that finding an H_∞ performance of closed-loop system (4.34) implies finding an H_∞ performance for the closed-loop system (4.18). First, note that $||z(t)||^2 = z(t)^T(\mathcal{U}^T\mathcal{U} \otimes I_r)z(t) = ||\bar{z}(t)||^2$ and $||w(t)||^2 = w(t)^T(\mathcal{U}^T\mathcal{U} \otimes I_r)w(t) = ||\bar{w}(t)||^2$, i.e., the H_∞ performance, is equivalent for (4.25) and (4.18). Finally, it is easy to see that $||\bar{z}(t)||^2 = ||\tilde{z}(t)||^2$ since the first coordinate of $\bar{z}(t)$ is null. Another point to note is that $||\bar{w}(t)||^2 \geq ||\tilde{w}(t)||^2$, then one has

$$\int_0^\infty ||\bar{z}(t)||^2 dt = \int_0^\infty |||\tilde{z}(t)||^2 dt < \gamma \int_0^\infty ||\tilde{w}(t)||^2 dt \leq \gamma \int_0^\infty ||\bar{w}(t)||^2 dt. \tag{4.35}$$

Therefore, since γ is the H_∞ performance of closed-loop system (4.34) then γ is the H_∞ cost of closed-loop system (4.18).

Since the closed-loop system (4.34) was obtained, the following section presents conditions for designing robust consensus protocols.

4.2 DESIGN OF ROBUST CONSENSUS PROTOCOLS FOR UNCERTAIN NETWORKS

This section presents derived conditions that may design the robust consensus protocol (4.4)-(4.5). The following auxiliary lemmas are of great importance in obtaining the proposed conditions.

LEMMA 4.3 ([73]) For any scalar $\beta > 0$ and real matrices $\mathbf{X}$ and $\mathbf{Y}$ of compatible dimensions, one has

$$\mathbf{X}^T\mathbf{Y} + \mathbf{Y}^T\mathbf{X} \preceq \frac{1}{\beta}\mathbf{X}^T\mathbf{X} + \beta\mathbf{Y}^T\mathbf{Y}. \tag{4.36}$$

LEMMA 4.4 The uncertain multi-agent system (4.1)-(4.2) with protocol (4.4)-(4.5) and controlled outputs (4.7) solve Problem 4.1 if there exist $P = P^T \succ 0$ and scalars $\delta > 0$ and $\gamma > 0$ such that the following inequality holds for all $t \geq 0$

$$\begin{bmatrix} \Sigma(\alpha(t)) & P\mathbb{B}_{cl}(\alpha(t)) & P\mathcal{E} \\ \mathbb{B}_{cl}^T(\alpha(t))P & -\gamma^2 I & 0 \\ \mathcal{E}^T P & 0 & -\delta I \end{bmatrix} \prec 0, \tag{4.37}$$

where,

$$\Sigma(\alpha(t)) = \mathcal{A}_{cl}^T(\alpha(t))P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{M}^T\mathcal{M} + \mathcal{C}_z^T\mathcal{C}_z, \tag{4.38}$$

$$\mathcal{A}_{cl}(\alpha(t)) = \mathcal{A} + \bar{\mathcal{B}}(I_{m-1} \otimes K_y)\bar{\mathcal{C}}(\alpha(t)). \tag{4.39}$$

.

proof.

With the identifications in (4.40) we can apply the Schur Complement Lemma (Lemma 3.5) and obtain inequality (4.37) as an equivalent form.

$$\underbrace{\begin{bmatrix} \Sigma(\alpha(t)) & P\mathbb{B}_{cl}(\alpha(t)) \\ \mathbb{B}_{cl}^T(\alpha(t))P & -\gamma^2 I \end{bmatrix}}_{\mathbf{X}} - \underbrace{\begin{bmatrix} P\mathcal{E} \\ 0 \end{bmatrix}}_{\mathbf{Y}} \underbrace{-\delta^{-1} I}_{\mathbf{Z}^{-1}} \underbrace{\begin{bmatrix} \mathcal{E}^T P & 0 \end{bmatrix}}_{\mathbf{Y}^T} \prec 0. \tag{4.40}$$

Inequality (4.40) can be rewritten as

$$\begin{bmatrix} \mathcal{A}_{cl}^T(\alpha(t))P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{M}^T\mathcal{M} + \mathcal{C}_z^T\mathcal{C}_z + \delta^{-1}P\mathcal{E}\mathcal{E}^T P & P\mathbb{B}_{cl}(\alpha(t)) \\ \mathbb{B}_{cl}^T(\alpha(t))P & -\gamma^2 I \end{bmatrix} \prec 0. \tag{4.41}$$

If inequality (4.41) holds, one has that the following inequality holds

$$\mathcal{A}_{cl}^T(\alpha(t))P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{M}^T\mathcal{M} + \mathcal{C}_z^T\mathcal{C}_z + \delta^{-1}P\mathcal{E}\mathcal{E}^T P \prec 0. \tag{4.42}$$

With the identifications $\mathbf{X} := \mathcal{M}$, $\mathbf{Y} := (I_{m-1} \otimes F^T(t))\mathcal{E}^T P$ and $\beta = 1$, and applying the Lemma 4.3 in (4.42) and considering $F(t)F^T(t) \leq \delta^{-1}I$, one has

$$\begin{aligned} \mathcal{A}_{cl}^T(\alpha(t))P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{C}_z^T\mathcal{C}_z + \mathcal{M}^T(I_{m-1} \otimes F^T(t))\mathcal{E}^T P \\ + P\mathcal{E}(I_{m-1} \otimes F(t))\mathcal{M} \preceq \mathcal{A}_{cl}(\alpha(t))^T P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{C}_z^T\mathcal{C}_z \\ + \mathcal{M}^T\mathcal{M} + \delta^{-1}P\mathcal{E}\mathcal{E}^T P \prec 0 \end{aligned} \tag{4.43}$$

Defining $\mathcal{F}(t) = (I_{m-1} \otimes F(t))$, by inequality (4.43) if inequality (4.41) holds then the following inequality holds

$$\Theta(t) = \begin{bmatrix} \mathbb{A}_{cl}^T(\alpha(t))P + P\mathbb{A}_{cl}(\alpha(t)) + \mathcal{C}_z^T\mathcal{C}_z & P\mathbb{B}_{cl}(\alpha(t)) \\ \mathbb{B}_{cl}^T(\alpha(t))P & -\gamma^2 I \end{bmatrix} \prec 0, \tag{4.44}$$

with $\mathbb{A}_{cl}(\alpha(t)) = (\mathcal{A} + \mathcal{E}\mathcal{F}(t)\mathcal{M} + \bar{\mathcal{B}}(I_{m-1} \otimes K_y)\bar{\mathcal{C}}(\alpha(t)))$ and $\mathbb{B}_{cl}(\alpha(t)) = (\mathcal{B}_1 + \bar{\mathcal{B}}(I_{m-1} \otimes K_y)\bar{\mathcal{D}}(\alpha(t)))$ as defined in (4.34).

To deal with the uncertain closed-loop multi-agent system (4.1)-(4.7) consensus with H_∞ performance, we need to show that condition (4.44) implies in the H_∞ definition (4.35). First, introduce for the system (4.34) the Lyapunov function $V(t) = \tilde{\xi}(t)^T P\tilde{\xi}(t)$. Observe that the time derivative of $V(t)$ can be written as

$$\begin{aligned} \dot{V}(t) = \tilde{\xi}^T(t)(P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{A}_{cl}^T(\alpha(t))P + \mathcal{M}^T\mathcal{F}^T(t)\mathcal{E}^T P + P\mathcal{E}\mathcal{F}(t)\mathcal{M})\tilde{\xi}(t) \\ + \tilde{\xi}^T(t)P\mathbb{B}_{cl}(\alpha(t))\tilde{w}(t) + \tilde{w}^T(t)\mathbb{B}_{cl}^T(\alpha(t))P\tilde{\xi}(t). \end{aligned} \tag{4.45}$$

Note that $P \succ 0$ such that $\dot{V}(t) \prec 0$, $t \geq 0$, for $w(t) = 0$ implies $P\mathbb{A}_{cl}(\alpha(t)) + \mathbb{A}_{cl}^T(\alpha(t))P \prec 0$ (quadratic stability [44]). Therefore, $\tilde{\xi}(t) \to 0$ for any initial condition and the requirement 1. in Problem 4.1 is verified.

From linear system theory, we consider now that system (4.34) has influence of $w(t)$, but with zero initial condition $\tilde{\xi}(0) = 0$ and define the following cost functional for any $\tau > 0$,

$$J(\tau) = \int_0^\tau (\|\tilde{z}(t)\|^2 - \gamma^2\|\tilde{w}(t)\|^2)\mathrm{d}t, \tag{4.46}$$

where $\tilde{w}(t)$ and $\tilde{z}(t)$ are the disturbances and outputs of system (4.34).

Since $\int_0^\tau(\dot{V}(t) - \dot{V}(t))\mathrm{d}t = 0$ and $V(0) = 0$ (zero initial condition), the functional (4.46) can be rewritten as

$$J(\tau) = \int_0^\tau (\tilde{\xi}^T(t)(P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{A}_{cl}^T(\alpha(t))P + \mathcal{M}^T\mathcal{F}^T(t)\mathcal{E}^TP + P\mathcal{E}\mathcal{F}(t)\mathcal{M} + \mathcal{C}_z^T\mathcal{C}_z)\tilde{\xi}(t) + \tilde{\xi}^T(t)P\mathbb{B}_{cl}(\alpha(t))\tilde{w}(t) + \tilde{w}^T(t)\mathbb{B}_{cl}^T(\alpha(t))P\tilde{\xi}(t) - \gamma^2\tilde{w}^T(t)\tilde{w}(t))\mathrm{d}t - V(\tau). \tag{4.47}$$

Introducing the variable $\zeta(t) = \begin{bmatrix}\tilde{\xi}^T(t) & \tilde{w}^T(t)\end{bmatrix}^T$ and considering (4.44), one has that

$$J(\tau) = \int_0^\tau (\zeta^T(t)\Theta(t)\zeta(t))\mathrm{d}t - \mathcal{V}(\tau). \tag{4.48}$$

Observe that, $P \succ 0$ and $\Theta(t) \prec 0$ implies that the right-hand side of (4.48) is negative, or equivalently, from (4.46),

$$\int_0^\tau \|\tilde{z}(t)\|^2\mathrm{d}t < \gamma^2 \int_0^\tau \|\tilde{w}(t)\|^2\mathrm{d}t. \tag{4.49}$$

Note that (4.49) is valid for $0 \leq t \leq \tau$ and for all $\tau > 0$. Since $w(t) \in \mathcal{L}_2[0, \infty)$, the integral in right-hand side of (4.49) converges for $\tau \to \infty$. Also, by (4.49), the left-hand side integral is upper bounded and is non-decreasing in τ; therefore, it converges as $\tau \to \infty$. Taking the limit as $\tau \to \infty$ in (4.49) one has that (4.35) holds for any $w(t) \in \mathcal{L}_2[0, \infty)$ and $0 \leq t < \infty$, and therefore, the requirement 2. in Problem 4.1 is also attained. ■

The closed-loop system (4.34) and Lemma 4.4 allow us to derive new conditions that design robust consensus protocols for static, reduced- and full-order control schemes that may solve Problem 4.1. The following theorem presents a sufficient LMI condition for the design of the robust protocol (4.4)-(4.5).

Theorem 4.1

Let K_x a given matrix such that $\mathcal{A}+\bar{\mathcal{B}}K_x$ is Hurwitz. If there exist the scalars $\mu>0$ and $\delta>0$, matrices $P=P^T\succ 0$, $X_{\ell,k}$, for $\ell=1,..,10$ and $k=1,..,N$, $\mathcal{G}$ and Z such that the following LMIs holds,

$$\Upsilon_k=\begin{bmatrix}\Upsilon_{1,k} & \star\\ \Upsilon_{2,k} & \Upsilon_{3,k}\end{bmatrix}\prec 0, \tag{4.50}$$

where,

$$\Upsilon_{1,k}=\begin{bmatrix}\Psi_{1,k} & \star & \star\\ \Psi_{2,k} & -He\{X_{3,k}\} & \star\\ \Psi_{3,k} & -X_{5,k}-X_{4,k}^T & \Psi_{6,k}\end{bmatrix}, \Upsilon_{2,k}=\begin{bmatrix}\Psi_{4,k} & -X_{7,k}+\mathcal{B}_1^TX_{3,k}^T & \Psi_{7,k}\\ \Psi_{5,k} & -X_{9,k}+\mathcal{E}^TX_{3,k}^T & \Psi_{8,k}\\ \bar{\mathcal{B}}^TX_{1,k}^T & \bar{\mathcal{B}}^TX_{3,k}^T & \Psi_{9,k}\end{bmatrix},$$

$$\Upsilon_{3,k}=\begin{bmatrix}\Psi_{10,k} & \star & \star\\ \Psi_{11,k} & \Psi_{13,k} & \star\\ \Psi_{12,k} & \bar{\mathcal{B}}^TX_{9,k}^T & -He\{(I_m\otimes\mathcal{G})\}\end{bmatrix},$$

$$\Psi_{1,k}=X_{1,k}(\mathcal{A}+\bar{\mathcal{B}}K_x)+(\mathcal{A}+\bar{\mathcal{B}}K_x)^TX_{1,k}^T+\mathcal{C}_z^T\mathcal{C}_z+\mathcal{M}^T\mathcal{M}+X_{2,k}+X_{2,k}^T,$$
$$\Psi_{2,k}=P+X_{3,k}(\mathcal{A}+\bar{\mathcal{B}}K_x)+X_{4,k}-X_{1,k}^T, \Psi_{3,k}=X_{5,k}(\mathcal{A}+\bar{\mathcal{B}}K_x)+X_{6,k}-X_{2,k}^T,$$
$$\Psi_{4,k}=X_{7,k}(\mathcal{A}+\bar{\mathcal{B}}K_x)+X_{8,k}+\mathcal{B}_1^TX_{1,k}^T, \Psi_5=X_{9,k}(\mathcal{A}+\bar{\mathcal{B}}K_x)+X_{10,k}+\mathcal{E}^TX_{1,k}^T,$$
$$\Psi_{6,k}=-He\{X_{6,k}\}, \Psi_{7,k}=-X_{8,k}+\mathcal{B}_1^TX_{5,k}^T, \Psi_{8,k}=-X_{10,k}+\mathcal{E}^TX_{5,k}^T,$$
$$\Psi_{9,k}=\bar{\mathcal{B}}^TX_{5,k}^T+(I_m\otimes Z)\mathcal{C}_k-(I_m\otimes\mathcal{G})K_x, \Psi_{10,k}=-\mu I+He\{X_{7,k}\mathcal{B}_1\},$$
$$\Psi_{11,k}=X_{9,k}\mathcal{B}_1+\mathcal{E}^TX_{7,k}^T, \Psi_{12,k}=\bar{\mathcal{B}}^TX_{7,k}^T+(I_m\otimes Z)\bar{\mathcal{D}}_k,$$
$$\Psi_{13,k}=-\delta I+He\{X_{9,k}\mathcal{E}\},$$

then the uncertain multi-agent system (4.1)-(4.2), connected in a polytopic directed network topology, achieves consensus with H_∞ cost $\gamma=\sqrt{\mu}$ and controlled by the n_c-order dynamic output feedback protocol (4.4)-(4.5) with gains $K_y=\mathcal{G}^{-1}Z$.

proof.

If inequality (4.50) holds, then $\Upsilon(\alpha(t))=\sum_{k=1}^N\alpha_k(t)\Upsilon_k\prec 0$ holds for $\sum_{k=1}^N\alpha_k(t)=1$ and $\alpha_k(t)\geq 0$. Defining the following matrix

$$\mathcal{T}_1(\alpha(t))=\begin{bmatrix}I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\\ 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) & 0\end{bmatrix}, \tag{4.51}$$

where, $S_1(\alpha(t)) = (I_{m-1} \otimes K_y)\bar{\mathcal{C}}(\alpha(t)) - K_x$, $S_2(\alpha(t)) = (I_{m-1} \otimes K_y)\bar{\mathcal{D}}(\alpha(t))$ and $G = (I_m \otimes \mathcal{G})$ one has that $\mathcal{T}_1(\alpha(t))^T \Upsilon(\alpha(t))\mathcal{T}_1(\alpha(t)) \prec 0$ can be rewritten as

$$
\begin{bmatrix}
\Psi_1(\alpha(t)) & \star & \star & \star & \star \\
\Psi_2(\alpha(t)) & -He\{X_3(\alpha(t))\} & \star & \star & \star \\
\Psi_3(\alpha(t)) & -X_5(\alpha(t)) - X_4(\alpha(t))^T & \Psi_6(\alpha(t)) & \star & \star \\
\Psi_4(\alpha(t)) & -X_7(\alpha(t)) + \mathcal{B}_1^T X_3^T(\alpha(t)) & \Psi_7(\alpha(t)) & \Psi_{10}(\alpha(t)) & \star \\
\Psi_5(\alpha(t)) & -X_{10}(\alpha(t)) + \mathcal{E}^T X_3^T(\alpha(t)) & \Psi_8(\alpha(t)) & \Psi_{11}(\alpha(t)) & \Psi_{13}(\alpha(t))
\end{bmatrix}
+He\left\{ \begin{bmatrix} X_1(\alpha(t))\bar{\mathcal{B}} \\ X_3(\alpha(t))\bar{\mathcal{B}} \\ X_5(\alpha(t))\bar{\mathcal{B}} + S_1^T(\alpha(t))G^T \\ X_7(\alpha(t))\bar{\mathcal{B}} + S_2^T(\alpha(t))G^T \\ X_9(\alpha(t))\bar{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) & 0 \end{bmatrix} \right\}
- \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) & 0 \end{bmatrix}^T He\{G\} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) & 0 \end{bmatrix} \prec 0. \tag{4.52}
$$

Observe that (4.52) is equivalent to

$$
\begin{bmatrix}
\Psi_1(\alpha(t)) & \star & \star & \star & \star \\
\Psi_2(\alpha(t)) & -He\{X_3(\alpha(t))\} & \star & \star & \star \\
\Psi_3(\alpha(t)) & -X_5(\alpha(t)) - X_4(\alpha(t))^T & \Psi_6(\alpha(t)) & \star & \star \\
\Psi_4(\alpha(t)) & -X_7(\alpha(t)) + \mathcal{B}_1^T X_3^T(\alpha(t)) & \Psi_7(\alpha(t)) & \Psi_{10}(\alpha(t)) & \star \\
\Psi_5(\alpha(t)) & -X_9(\alpha(t)) + \mathcal{E}^T X_3^T(\alpha(t)) & \Psi_8(\alpha(t)) & \Psi_{11}(\alpha(t)) & \Psi_{13}(\alpha(t))
\end{bmatrix}
+He\left\{ \begin{bmatrix} X_1(\alpha(t))\bar{\mathcal{B}} \\ X_3(\alpha(t))\bar{\mathcal{B}} \\ X_5(\alpha(t))\bar{\mathcal{B}} \\ X_7(\alpha(t))\bar{\mathcal{B}} \\ X_9(\alpha(t))\bar{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) & 0 \end{bmatrix} \right\} \prec 0. \tag{4.53}
$$

Defining $\mu = \gamma^2$ in $\Psi_{10}(\alpha(t))$ and rewriting the inequality (4.53) one has

$$\begin{bmatrix} \mathcal{M}^T\mathcal{M} + \mathcal{C}_z^T\mathcal{C}_z & P & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & -\delta I \end{bmatrix}$$
$$+He\left\{ \begin{bmatrix} X_1(\alpha(t)) & X_2(\alpha(t)) \\ X_3(\alpha(t)) & X_4(\alpha(t)) \\ X_5(\alpha(t)) & X_6(\alpha(t)) \\ X_7(\alpha(t)) & X_8(\alpha(t)) \\ X_9(\alpha(t)) & X_{10}(\alpha(t)) \end{bmatrix} \begin{bmatrix} (\mathcal{A} + \bar{\mathcal{B}}K_x)^T & I \\ -I & 0 \\ S_1^T(\alpha(t))\bar{\mathcal{B}}^T & -I \\ \mathbb{B}_{cl}^T(\alpha(t)) & 0 \\ \mathcal{E}^T & 0 \end{bmatrix}^T \right\} \prec 0, \tag{4.54}$$

where, $\mathbb{B}_{cl}(\alpha(t)) = \mathcal{B}_1 + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{D}}(\alpha(t))$.

Defining

$$\mathcal{T}_2(\alpha(t)) = \begin{bmatrix} I & 0 & 0 \\ \mathcal{A}_{cl}(\alpha(t)) & \mathbb{B}_{cl}(\alpha(t)) & \mathcal{E} \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{4.55}$$

where $\mathcal{A}_{cl}(\alpha(t)) = \mathcal{A} + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{C}}(\alpha(t))$ and pre- and post-multiplying (4.54) by $\mathcal{T}_2^T(\alpha(t))$ and $\mathcal{T}_2(\alpha(t))$, respectively, it is obtained the inequality (4.37). ■

REMARK 4.3 The hypothesis that K_x is such that $\mathcal{A}_x := \mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz does not appear explicitly in the proof of Theorem 4.1. However, this hypothesis is a necessary condition to verify (4.50). If (4.50) holds for some $P \succ 0$, then $\Upsilon_{1,k} \prec 0$. Note that

$$\Upsilon_{1,k} = \underbrace{\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z + \mathcal{M}^T\mathcal{M} & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_4} + He\left\{ \underbrace{\begin{bmatrix} X_{1,k} & X_{2,k} \\ X_{3,k} & X_{4,k} \\ X_{5,k} & X_{6,k} \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} \mathcal{A}_x & -I & 0 \\ I & 0 & -I \end{bmatrix}}_{\mathbf{V}} \right\} \prec 0. \tag{4.56}$$

Choosing $\mathbf{V}_\perp = \begin{bmatrix} I & \mathcal{A}_x^T & I \end{bmatrix}^T$ and pre and post-multiplying (4.56) by $\mathbf{V}_\perp^T$ and $\mathbf{V}_\perp$, one has

$$\mathcal{A}_x^T P + P\mathcal{A}_x \prec 0,$$

with $P \succ 0$, that is, $\mathcal{A}_x = \mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz.

In the following, we consider deriving conditions to solve Problem (4.1) when agents are not subjected to external disturbances. Now, we present the following auxiliary lemma that will be of great importance in deriving the proposed condition that designs the protocol (4.4) when $w(t) = 0$.

LEMMA 4.5 The uncertain multi-agent system (4.14) achieves consensus if there exist $P = P^T \succ 0$ and a scalar $\delta > 0$ such that the following inequality holds

$$\begin{bmatrix} \mathcal{A}_{cl}^T(\alpha(t))P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{M}^T\mathcal{M} & P\mathcal{E} \\ \mathcal{E}^T P & -\delta I \end{bmatrix} \prec 0. \tag{4.57}$$

proof.

Consider the following inequality.

$$\underbrace{\mathcal{A}_{cl}^T(\alpha(t))P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{M}^T\mathcal{M}}_{\mathbf{X}} - \underbrace{P\mathcal{E}}_{\mathbf{Y}} \underbrace{(-\delta I)^{-1}}_{\mathbf{Z}^{-1}} \underbrace{\mathcal{E}^T P}_{\mathbf{Y}^T} \prec 0. \tag{4.58}$$

With the identifications in (4.58), by Lemma 3.5, one has that (4.57) is equivalent to (4.58). Observe that (4.58) is the same inequality (4.42), then (4.57) one has that if inequality (4.58) holds, then inequality (4.43) holds. Defining $\mathcal{F}(t) = (I \otimes F(t))$ and pre- and post-multiplying inequality (4.43) by $\tilde{\xi}^T(t)$ and $\tilde{\xi}(t)$, respectively, one has

$$\tilde{\xi}^T(t)(\mathcal{A}_{cl}^T(\alpha(t))P + \mathcal{M}^T\mathcal{F}(t)\mathcal{E}^T P + P\mathcal{A}_{cl}(\alpha(t)) + P\mathcal{E}\mathcal{F}(t)\mathcal{M})\tilde{\xi}(t) \prec 0. \tag{4.59}$$

Defining the Lyapunov function $\mathbb{V}(t) = \tilde{\xi}^T(t)P\tilde{\xi}(t)$, deriving $\mathbb{V}(t)$ and imposing that $\dot{\mathbb{V}}(t) \prec 0$, then one has the stability condition of system (4.34) when $w(t) = 0$, $\forall t$, as follow

$$\dot{\mathbb{V}}(t) = \dot{\tilde{\xi}}^T(t)P\tilde{\xi}(t) + \tilde{\xi}^T(t)P\dot{\tilde{\xi}}(t) \prec 0. \tag{4.60}$$

Observe that (4.60) is equivalent to (4.59), which completes the proof. ■

The closed-loop system (4.34) and Lemma 4.4 allow us to derive new conditions that design robust consensus protocols for static, reduced- and full-order control schemes that may solve Problem 4.1 when agents (4.2) are not subjected to external disturbances. The following theorem presents a sufficient LMI condition for the design of robust protocol (4.4)-(4.5) when $w(t) = 0$.

COROLLARY 4.1 Let K_x a given matrix such that $\mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz. If there exist the scalar $\delta > 0$, matrices $P = P^T \succ 0$, $X_{p,k}$, $\mathcal{G}$ and Z, for $p = 1, ..., 8$ and $k = 1...N$, such that the following LMIs holds,

$$\tilde{\Upsilon}_k = \begin{bmatrix} \tilde{\Upsilon}_{1,k} & \star \\ \tilde{\Upsilon}_{2,k} & \tilde{\Upsilon}_{3,k} \end{bmatrix} \prec 0, \tag{4.61}$$

where,

$$\tilde{\Upsilon}_{1,k} = \begin{bmatrix} \tilde{\Psi}_{1,k} & \star & \star \\ \tilde{\Psi}_{2,k} & -He\{X_{3,k}\} & \star \\ \tilde{\Psi}_{3,k} & -X_{5,k} - X_{4,k}^T & \tilde{\Psi}_{5,k} \end{bmatrix}, \tilde{\Upsilon}_{2,k} = \begin{bmatrix} \tilde{\Psi}_{4,k} & -X_{7,k} + \mathcal{E}^T X_{3,k}^T & \tilde{\Psi}_{6,k} \\ \bar{\mathcal{B}}^T X_{1,k}^T & \bar{\mathcal{B}}^T X_{3,k}^T & \tilde{\Psi}_{7,k} \end{bmatrix},$$
$$\tilde{\Upsilon}_{3,k} = \begin{bmatrix} \tilde{\Psi}_{8,k} & \star \\ \bar{\mathcal{B}}^T X_{7,k}^T & -He\{(I_m \otimes \mathcal{G})\} \end{bmatrix},$$
$$\tilde{\Psi}_{1,k} = X_{1,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + (\mathcal{A} + \bar{\mathcal{B}}K_x)^T X_{1,k}^T + \mathcal{M}^T\mathcal{M} + X_{2,k} + X_{2,k}^T,$$
$$\tilde{\Psi}_{2,k} = P + X_{3,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_{4,k} - X_{1,k}^T, \tilde{\Psi}_{3,k} = X_{5,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_{6,k} - X_{2,k}^T,$$
$$\tilde{\Psi}_{4,k} = X_{7,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_{8,k} + \mathcal{E}^T X_{1,k}^T, \tilde{\Psi}_{5,k} = -He\{X_{6,k}\},$$
$$\tilde{\Psi}_{6,k} = -X_{8,k} + \mathcal{E}^T X_{5,k}^T, \tilde{\Psi}_{7,k} = \bar{\mathcal{B}}^T X_{5,k}^T + (I_m \otimes Z)\mathcal{C}_k - (I_m \otimes \mathcal{G})K_x,$$
$$\tilde{\Psi}_{8,k} = -\delta I + He\{X_{7,k}\mathcal{E}\}.$$

then the uncertain multi-agent system (4.1)-(4.2), connected in a polytopic directed network topology, in the absence of disturbances $w(t)$, $\forall t$, achieves consensus controlled by n_c-order dynamic output feedback protocol (4.4)-(4.5) with gains $K_y = \mathcal{G}^{-1}Z$.

proof.

If inequality (4.50) holds, then $\tilde{\Upsilon}(\alpha(t)) = \sum_{k=1}^{N} \alpha_k(t)\tilde{\Upsilon}_k \prec 0$ holds for $\sum_{k=1}^{N} \alpha_k(t) = 1$ and $\alpha_k(t) \geq 0$. Defining the following matrices

$$\tilde{\mathcal{T}}_3(\alpha(t)) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & S_1(\alpha(t)) & 0 \end{bmatrix}, \tag{4.62}$$

where $S_1(\alpha(t)) = (I_{m-1} \otimes K_y)\bar{\mathcal{C}}(\alpha(t)) - K_x$ and $G = (I_m \otimes \mathcal{G})$. Pré- and post-multiplying (4.61) by $\mathcal{T}_3^T(\alpha(t))$ and $\mathcal{T}_3(\alpha(t))$, respectively, one has

$$\begin{bmatrix} \tilde{\Psi}_1(\alpha(t)) & \star & \star & \star \\ \tilde{\Psi}_2(\alpha(t)) & -He\{X_3(\alpha(t))\} & \star & \star \\ \tilde{\Psi}_3(\alpha(t)) & -X_5(\alpha(t)) - X_4(\alpha(t))^T & \tilde{\Psi}_5(\alpha(t)) & \star \\ \tilde{\Psi}_4(\alpha(t)) & -X_7(\alpha(t)) + \mathcal{E}^T X_3^T(\alpha(t)) & \tilde{\Psi}_6(\alpha(t)) & \tilde{\Psi}_8(\alpha(t)) \end{bmatrix}$$
$$+He\left\{ \begin{bmatrix} X_1(\alpha(t))\bar{\mathcal{B}} \\ X_3(\alpha(t))\bar{\mathcal{B}} \\ X_5(\alpha(t))\bar{\mathcal{B}} + S_1^T(\alpha(t))G^T \\ X_7(\alpha(t))\bar{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & 0 \end{bmatrix} \right\}$$
$$- \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & 0 \end{bmatrix}^T He\{G\} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & 0 \end{bmatrix} \prec 0. \tag{4.63}$$

Observe that, (4.63) is equivalent to

$$\begin{bmatrix} \tilde{\Psi}_1(\alpha(t)) & \star & \star & \star \\ \tilde{\Psi}_2(\alpha(t)) & -He\{X_3(\alpha(t))\} & \star & \star \\ \tilde{\Psi}_3(\alpha(t)) & -X_5(\alpha(t)) - X_4(\alpha(t))^T & \tilde{\Psi}_5(\alpha(t)) & \star \\ \tilde{\Psi}_4(\alpha(t)) & -X_7(\alpha(t)) + \mathcal{E}^T X_3^T(\alpha(t)) & \tilde{\Psi}_6(\alpha(t)) & \tilde{\Psi}_8(\alpha(t)) \end{bmatrix}$$
$$+He\left\{ \begin{bmatrix} X_1(\alpha(t))\bar{\mathcal{B}} \\ X_3(\alpha(t))\bar{\mathcal{B}} \\ X_5(\alpha(t))\bar{\mathcal{B}} \\ X_7(\alpha(t))\bar{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & 0 \end{bmatrix} \right\} \prec 0. \tag{4.64}$$

Substituting $\tilde{\Psi}_p(\alpha(t))$, for $p = 1, ..., 6$ and $\tilde{\Psi}_8(\alpha(t))$, one can rewrite inequality (4.64) as

$$\begin{bmatrix} \mathcal{M}^T\mathcal{M} & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta I \end{bmatrix}$$
$$+He\left\{ \begin{bmatrix} X_1(\alpha(t)) & X_2(\alpha(t)) \\ X_3(\alpha(t)) & X_4(\alpha(t)) \\ X_5(\alpha(t)) & X_6(\alpha(t)) \\ X_7(\alpha(t)) & X_8(\alpha(t)) \end{bmatrix} \begin{bmatrix} (\mathcal{A} + \bar{\mathcal{B}}K_x)^T & I \\ -I & 0 \\ S_1^T(\alpha(t))\bar{\mathcal{B}}^T & -I \\ \mathcal{E}^T & 0 \end{bmatrix}^T \right\} \prec 0. \tag{4.65}$$

Defining

$$\mathcal{T}_4(\alpha(t)) = \begin{bmatrix} I & 0 \\ \mathcal{A}_{cl}(\alpha(t)) & \mathcal{E} \\ I & 0 \\ 0 & I \end{bmatrix}, \tag{4.66}$$

where $\mathcal{A}_{cl}(\alpha(t)) = \mathcal{A} + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{C}}(\alpha(t))$ and pre- and post-multiplying (4.65) by $\mathcal{T}_4^T(\alpha(t))$ and $\mathcal{T}_4(\alpha(t))$, respectively, it is obtained inequality (4.57). ■

For the sake of comparison with the condition presented in Theorem (3.4) in Chapter 3, we will consider here the case where the agents (4.2) are subject to external disturbances but without the influence of uncertain parameter $\Delta A(t)$. The following corollary may design H_∞ protocols that may solve Problem 4.1 when $F(t) = 0$.

COROLLARY 4.2 Let K_x a given matrix such that $\mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz. If there exist the scalar $\gamma > 0$, matrices $P = P^T \succ 0$, $X_{p,k}$, $\mathcal{G}$ and Z, for $p = 1, ..., 8$ and $k = 1, ..., N$, such that the following LMIs holds,

$$\bar{\Upsilon}_k = \begin{bmatrix} \bar{\Upsilon}_{1,k} & \star \\ \bar{\Upsilon}_{2,k} & \bar{\Upsilon}_{3,k} \end{bmatrix} \prec 0, \tag{4.67}$$

where,

$$\bar{\Upsilon}_{1,k} = \begin{bmatrix} \bar{\Psi}_{1,k} & \star & \star \\ \bar{\Psi}_{2,k} & -He\{X_{3,k}\} & \star \\ \bar{\Psi}_{3,k} & -X_{5,k} - X_{4,k}^T & \bar{\Psi}_{5,k} \end{bmatrix}, \bar{\Upsilon}_{2,k} = \begin{bmatrix} \bar{\Psi}_{4,k} & -X_{7,k} + \mathcal{B}_1^T X_{3,k}^T & \bar{\Psi}_{6,k} \\ \bar{\mathcal{B}}^T X_{1,k}^T & \bar{\mathcal{B}}^T X_{3,k}^T & \bar{\Psi}_{7,k} \end{bmatrix},$$

$$\bar{\Upsilon}_{3,k} = \begin{bmatrix} \bar{\Psi}_{8,k} & \star \\ \bar{\mathcal{B}}^T X_{7,k}^T + (I_m \otimes Z)\bar{\mathcal{D}}_k & -He\{(I_m \otimes \mathcal{G})\} \end{bmatrix},$$

$$\bar{\Psi}_{1,k} = X_{1,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + (\mathcal{A} + \bar{\mathcal{B}}K_x)^T X_{1,k}^T + \mathcal{C}_z^T\mathcal{C}_z + X_{2,k} + X_{2,k}^T,$$

$$\bar{\Psi}_{2,k} = P + X_{3,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_{4,k} - X_{1,k}^T, \bar{\Psi}_{3,k} = X_{5,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_{6,k} - X_{2,k}^T,$$

$$\bar{\Psi}_{4,k} = X_{7,k}(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_{8,k} + \mathcal{B}_1^T X_{1,k}^T, \bar{\Psi}_{5,k} = -He\{X_{6,k}\},$$

$$\bar{\Psi}_{6,k} = -X_{8,k} + \mathcal{B}_1^T X_{5,k}^T, \bar{\Psi}_{7,k} = \bar{\mathcal{B}}^T X_{5,k}^T + (I_m \otimes Z)\mathcal{C}_k - (I_m \otimes \mathcal{G})K_x,$$

$$\bar{\Psi}_{8,k} = -\mu I + He\{X_{7,k}\mathcal{B}_1\}.$$

then the uncertain multi-agent system (4.1)-(4.2), connected in a polytopic directed network topology, achieves consensus with H_∞ cost $\gamma = \sqrt{\mu}$ controlled by n_c-order dynamic output feedback protocol (4.4)-(4.5) with gains $K_y = \mathcal{G}^{-1}Z$.

proof.

If inequality (4.50) holds, then $\bar{\Upsilon}(\alpha(t)) = \sum_{k=1}^{N} \alpha_k(t)\bar{\Upsilon}_k \prec 0$ holds for $\sum_{k=1}^{N} \alpha_k(t) = 1$ and $\alpha_k(t) \geq 0$. Defining the following matrices

$$\mathcal{T}_1(\alpha(t)) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) \end{bmatrix}, \tag{4.68}$$

where, $S_1(\alpha(t)) = (I_{m-1} \otimes K_y)\bar{\mathcal{C}}(\alpha(t)) - K_x$, $S_2(\alpha(t)) = (I_{m-1} \otimes K_y)\bar{\mathcal{D}}(\alpha(t))$, $G = (I_m \otimes \mathcal{G})$ one has that $\mathcal{T}_1(\alpha(t))^T \Upsilon(\alpha(t)) \mathcal{T}_1(\alpha(t)) \prec 0$ can be rewritten as

$$\begin{bmatrix} \bar{\Psi}_1(\alpha(t)) & \star & \star & \star \\ \bar{\Psi}_2(\alpha(t)) & -He\{X_3(\alpha(t))\} & \star & \star \\ \bar{\Psi}_3(\alpha(t)) & -X_5(\alpha(t)) - X_4(\alpha(t))^T & \bar{\Psi}_5(\alpha(t)) & \star \\ \bar{\Psi}_4(\alpha(t)) & -X_7(\alpha(t)) + \mathcal{B}_1^T X_3^T(\alpha(t)) & \bar{\Psi}_6(\alpha(t)) & \bar{\Psi}_8(\alpha(t)) \end{bmatrix}$$
$$+He\left\{ \begin{bmatrix} X_1(\alpha(t))\bar{\mathcal{B}} \\ X_3(\alpha(t))\bar{\mathcal{B}} \\ X_5(\alpha(t))\bar{\mathcal{B}} + S_1^T(\alpha(t))G^T \\ X_7(\alpha(t))\bar{\mathcal{B}} + S_2^T(\alpha(t))G^T \end{bmatrix} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) \end{bmatrix} \right\}$$
$$- \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) \end{bmatrix}^T He\{G\} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) \end{bmatrix} \prec 0. \tag{4.69}$$

Observe that (4.69) is equivalent to

$$\begin{bmatrix} \bar{\Psi}_1(\alpha(t)) & \star & \star & \star \\ \bar{\Psi}_2(\alpha(t)) & -He\{X_3(\alpha(t))\} & \star & \star \\ \bar{\Psi}_3(\alpha(t)) & -X_5(\alpha(t)) - X_4(\alpha(t))^T & \bar{\Psi}_5(\alpha(t)) & \star \\ \bar{\Psi}_4(\alpha(t)) & -X_7(\alpha(t)) + \mathcal{B}_1^T X_3^T(\alpha(t)) & \bar{\Psi}_6(\alpha(t)) & \bar{\Psi}_8(\alpha(t)) \end{bmatrix}$$
$$+He\left\{ \begin{bmatrix} X_1(\alpha(t))\bar{\mathcal{B}} \\ X_3(\alpha(t))\bar{\mathcal{B}} \\ X_5(\alpha(t))\bar{\mathcal{B}} \\ X_7(\alpha(t))\bar{\mathcal{B}} \\ X_9(\alpha(t))\bar{\mathcal{B}} \end{bmatrix} \begin{bmatrix} 0 & 0 & S_1(\alpha(t)) & S_2(\alpha(t)) \end{bmatrix} \right\} \prec 0. \tag{4.70}$$

Defining $\mu = \gamma^2$ in $\Psi_8(\alpha(t))$ and rewriting the inequality (4.70) one has

$$\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$+He\left\{ \begin{bmatrix} X_1(\alpha(t)) & X_2(\alpha(t)) \\ X_3(\alpha(t)) & X_4(\alpha(t)) \\ X_5(\alpha(t)) & X_6(\alpha(t)) \\ X_7(\alpha(t)) & X_8(\alpha(t)) \end{bmatrix} \begin{bmatrix} (\mathcal{A} + \bar{\mathcal{B}}K_x)^T & I \\ -I & 0 \\ S_1^T(\alpha(t))\bar{\mathcal{B}}^T & -I \\ \mathbb{B}_{cl}^T(\alpha(t)) & 0 \end{bmatrix}^T \right\} \prec 0, \tag{4.71}$$

where, $\mathbb{B}_{cl}(\alpha(t)) = \mathcal{B}_1 + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{D}}(\alpha(t))$.

Defining

$$\mathcal{T}_2(\alpha(t)) = \begin{bmatrix} I & 0 \\ \mathcal{A}_{cl}(\alpha(t)) & \mathbb{B}_{cl}(\alpha(t)) \\ I & 0 \\ 0 & I \end{bmatrix}, \tag{4.72}$$

where $\mathcal{A}_{cl}(\alpha(t)) = \mathcal{A} + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{C}}(\alpha(t))$ and pre- and post-multiplying (4.71) by $\mathcal{T}_2^T(\alpha(t))$ and $\mathcal{T}_2(\alpha(t))$, respectively, it is obtained

$$\begin{bmatrix} \mathcal{A}_{cl}(\alpha(t))^T P + P\mathcal{A}_{cl}(\alpha(t)) + \mathcal{C}_z^T\mathcal{C}_z & P\mathbb{B}_{cl}(\alpha(t)) \\ \mathbb{B}_{cl}^T(\alpha(t))P & -\gamma^2 I \end{bmatrix} \prec 0. \tag{4.73}$$

Condition (4.73) is equivalent to the inequality (4.44) when $\Delta A(t) = 0$. Following the same steps from (4.44) to (4.49) we proof that if (4.73) holds the closed-loop system (4.34) is asymptotically stable with H_∞ performance $\gamma > 0$ when $\Delta A(t) = 0$. ■

4.3 NUMERICAL EXAMPLE

In this section, the effectiveness of the presented methods, derived in Theorem 4.1, Corollary 4.1 and 4.2 are implemented to design static, reduced- and full-order protocols. The algorithms were implemented in the Python 3.11.4 software employing library CVXPY [10], and MOSEK [4]. The temporal response of agents is obtained by employing the Euler discretization technique in agents and protocol dynamics. Figure 4.1 represents the agent connection used in the simulations in this section.

For the simulations, we consider the quad-rotor model in [53], modified here, now subjected to the parametric uncertainties $\Delta A(t) = EF(t)M$, as follows

$$
\begin{aligned}
\begin{bmatrix} \dot{p}_i(t) \\ \ddot{p}_i(t) \end{bmatrix} &= \Bigg(\underbrace{\begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{E} F(t) \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M} \Bigg) \begin{bmatrix} p_i(t) - r_i \\ \dot{p}_i(t) \end{bmatrix} \\
&\quad + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}}_{B_u} u_i(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_w} w_i(t), \\
z_i(t) &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C_z} \left(\begin{bmatrix} p_i(t) - r_i \\ \dot{p}_i(t) \end{bmatrix} - \frac{1}{8} \sum_{j=1}^{8} \begin{bmatrix} p_j(t) - r_j \\ \dot{p}_j(t) \end{bmatrix} \right), \\
y_i(t) &= \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{C_y} \begin{bmatrix} p_i(t) - r_i \\ \dot{p}_i(t) \end{bmatrix} + \underbrace{0.5}_{D_y} w_i(t), \; i = 1, ..., 8, \qquad (4.74)
\end{aligned}
$$

with $c = 0$, $b = 1$, r_i the position displacements defined later and $F(t) = \begin{bmatrix} 0 & \frac{\sin(t)}{\sqrt{\delta}} \\ 0 & \frac{\cos(t)}{\sqrt{\delta}} \end{bmatrix}$ such that $F(t)^T F(t) \leq \delta^{-1} I_2$. The network connection is represented by Fig. 4.1 and modeled by the uncertain laplacian matrix (4.75), as follows

$$
L(\alpha(t)) = \begin{bmatrix}
3 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 1 + a_{28}(\alpha(t)) & 0 & 0 & 0 & 0 & 0 & -a_{28}(\alpha(t)) \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -a_{63}(\alpha(t)) & 0 & -1 & 1 + a_{63}(\alpha(t)) & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}, \qquad (4.75)
$$

with $0 \leq a_{28}(\alpha(t)) \leq 1$ and $0 \leq a_{63}(\alpha(t)) \leq 1$, defining a polytope of $N = 4$ vertices.

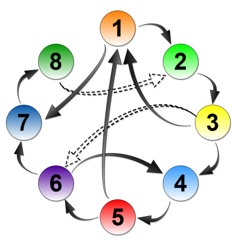


Figure 4.1 – The directed graph that represents the agent's network connection. The dashed arrows mean that the connection is uncertain.

4.3.1 Case 1: Robust consensus with uncertain disturbed agents

In this subsection, we present a numerical example to show the effectiveness of the LMI conditions (4.50) in Theorem 4.1. First, the LMI conditions (3.43) in Theorem 3.3 were solved for $\beta = 0.1$, allowing us to compute K_x. Fixing $\delta_{nc=0} = 27$, $\delta_{nc=1} = 24$ and $\delta_{nc=2} = 25$ and with the computed K_x matrix, the LMI condition (4.50) in Theorem 4.1 was solved and obtained the minimum H_∞ performance indexes $\gamma_{n_c=0} = 12.6452$, $\gamma_{n_c=1} = 11.3275$ and $\gamma_{n_c=2} = 13.4034$, with gains

$$D_c = \begin{bmatrix} 2.5482 \\ 0.0546 \end{bmatrix}, \tag{4.76}$$

for $n_c = 0$,

$$\begin{aligned} &A_c = -0.1528, \quad B_c = 0.0231, \\ &C_c = \begin{bmatrix} 59.1118 \\ 1.2024 \end{bmatrix}, \quad D_c = \begin{bmatrix} 1.9170 \\ 0.0308 \end{bmatrix}, \end{aligned} \tag{4.77}$$

for $n_c = 1$,

$$\begin{aligned} A_c &= \begin{bmatrix} -3.9162 & 1.3726 \\ -3.6490 & 1.1495 \end{bmatrix}, \; B_c = \begin{bmatrix} 0.0075 \\ 0.0240 \end{bmatrix}, \\ C_c &= \begin{bmatrix} -146.6119 & 105.0590 \\ -0.6255 & 1.2959 \end{bmatrix}, \; D_c = \begin{bmatrix} 1.9984 \\ 0.0130 \end{bmatrix}, \end{aligned} \tag{4.78}$$

for $n_c = 2$.

Specifically, communication between the agents 2 and 8 and agents 6 and 3 are uncertain, and modeled by $a_{28}(t) = |sin(t)|$ and $a_{63}(t) = |cos(t)|$. The external disturbances $w_i(t)$ are defined as $w_i(t) = sin(\frac{3}{i}t)$, if $t \in [8, 16]$, and $w_i(t) = 0$, otherwise. We consider the displacements r_i as $r_1 = 5$ and $r_i = r_{i-1} + 5$ for $i = 2, ..., 8$. Observe that, when agents reach consensus, i.e., $\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0$, each agent's position is equal to its neighbor position with a displacement of 5. Figure 4.2 shows the temporal response of the agents (4.74) for the protocol (4.4)-(4.5) with gains (4.77). From $0s$ to $8s$, it is possible to see the influence of parametric uncertainty $A(t)$. Then, when agents almost reach consensus, disturbances act at $8s$ to $16s$, and after $16s$, the agents reach consensus.

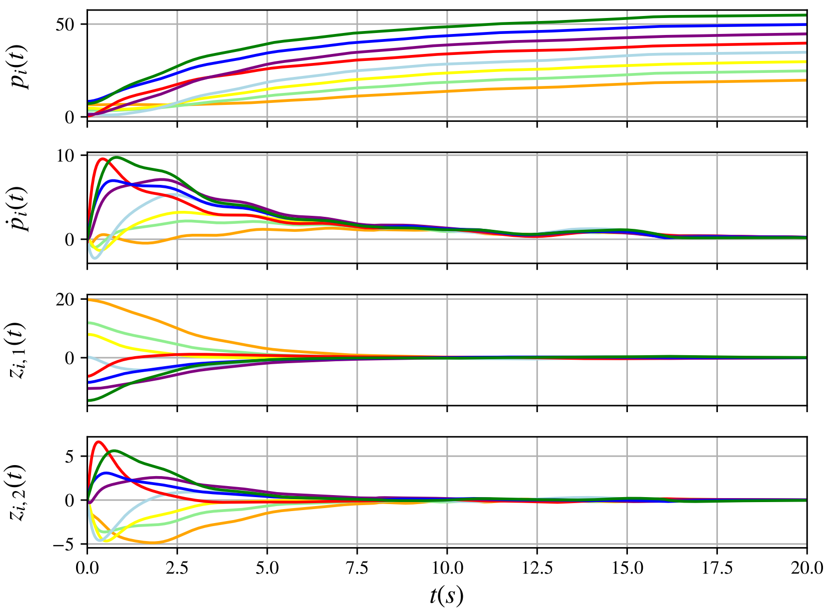


Figure 4.2 – Temporal response of the agents using gains (4.77). The positions $p_i(t)$ and velocities $\dot{p}_i(t)$ of the agents achieve consensus, and the controlled outputs $z_i(t)$ converge to zero. The colors of the agents are identified in Fig. 4.1.

Finally, we can compare the simulation results with some works in the literature. The graph in Figure 4.1 is not strongly connected, a scenario that cannot be handled by [19]. Moreover, [19] does not consider uncertainties in network topology and disturbances in outputs, shares only controller state variables through the communication network, and deals only with full-order protocols. Comparing Theorem 4.1 with Theorem 3.4 (result published in [53, Theorem 4]), we can observe that Theorem 3.4 cannot deal with the scenario proposed by Theorem 4.1 since it considers that agents have an uncertain parameter $\Delta A(t)$ and an uncertain topology. Even considering parametric uncertainty and uncertain topology, Theorem 4.1 may present less conservative results than Theorem 3.4 (concerning feasibility and smaller H_∞ performance γ) that does not consider parametric uncertainty and parametric uncertainty as shown in the LMI solution of Theorem 3.4 presented in Subsection 4.3.3.

4.3.2 Case 2: Robust Consensus for Non-disturbed Agents

In this subsection, we show the effectiveness of the LMI conditions of Corollary 4.1 in designing any-order protocols for multi-agent system (4.74) with displacements as $r_1 = 2$ and $r_i = r_{i-1} + 2$ for $i = 2, ..., 8$ and $w(t) = 0$, $\forall t$. Observe that, when agents reach consensus, i.e., $\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0$, each agent's position is equal to its neighbor position with a displacement of 2. Therefore, this section shows the effectiveness of the conditions stated by Corollary 4.1.

Using same K_x that was used as input in solution of Theorem 4.1 in subsection 4.3.1, we solved Corollary 4.1 and obtained $\delta_{n_c=0} = 12.4167$, $\delta_{n_c=1} = 10.6726$, $\delta_{n_c=2} = 11.5572$, with gains

$$D_c = \begin{bmatrix} 2.7055 \\ 0.0581 \end{bmatrix}, \tag{4.79}$$

for $n_c = 0$,

$$\begin{aligned} A_c &= -0.0978, \quad B_c = 0.0245 \\ C_c &= \begin{bmatrix} 67.1870 \\ 1.3064 \end{bmatrix}, \quad D_c = \begin{bmatrix} 1.9614 \\ 0.0324 \end{bmatrix}, \end{aligned} \tag{4.80}$$

for $n_c = 1$,

$$\begin{aligned} A_c &= \begin{bmatrix} -2.1362 & 0.8815 \\ -0.5935 & 0.1005 \end{bmatrix}, \ B_c = \begin{bmatrix} 0.0059 \\ 0.0257 \end{bmatrix}, \\ C_c &= \begin{bmatrix} -64.7893 & 82.7620 \\ -0.5509 & 1.1875 \end{bmatrix}, \ D_c = \begin{bmatrix} 2.0041 \\ 0.0181 \end{bmatrix}, \end{aligned} \tag{4.81}$$

for $n_c = 2$.

Figure 4.3 shows the agents' temporal response to the gains (4.80). All agents converge to the same position, and the velocity of each agent converges to zero, demonstrating the effectiveness of the proposed technique. Moreover, it is possible to see the influence of parametric uncertainties and their attenuation.

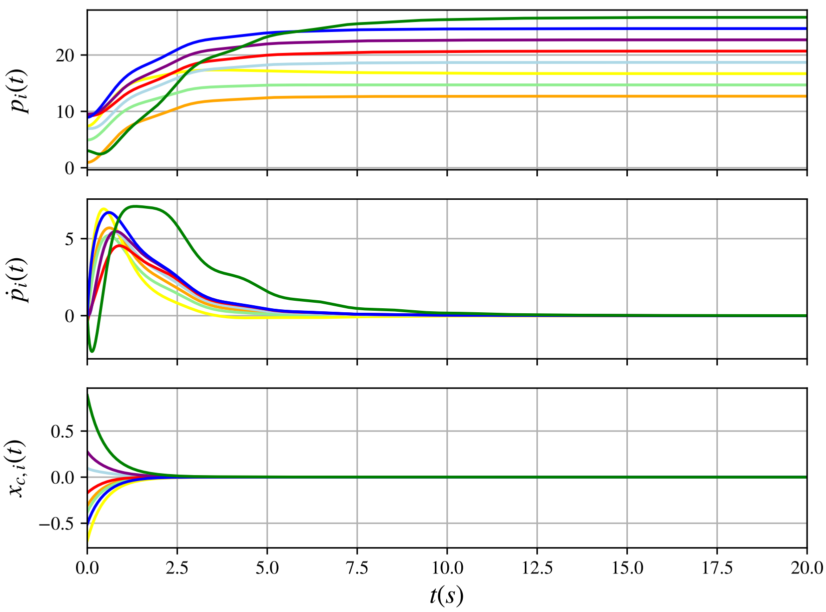


Figure 4.3 – Temporal response of the agents using gains (4.77). The positions $p_i(t)$ and velocities $\dot{p}_i(t)$ of the agents achieve consensus, and the controllers state variables $x_{c,i}(t)$ converge to zero. The colors of the agents are identified in Fig. 4.1.

The scenario presented in this numerical example cannot be handled by [60] since the authors consider only undirected networks in the protocol design. Observe that [60], besides undirected graphs, considers networks without uncertainties and reduced-order protocols a more restricted scenario. Similarly, Theorem 3.2 (result presented in [53, Theorem 2]) cannot deal with the numerical example presented in this subsection since Theorem 3.2 does not consider parametric uncertainties and uncertain topology.

4.3.3 Case 3: Consensus for non-uncertain disturbed agents

For a more suitable comparison with Chapter 3, in this subsection we consider the multi-agent system (4.74) with $F(t) = 0$, $\forall t$, r_i as in Case 2 and external disturbances $w_i(t)$ are defined as $w_i(t) = sin(\frac{3}{i}t)$, if $t \in [7, 11]$, and $w_i(t) = 0$, otherwise. In the following, we show the effectiveness of the conditions stated by Corollary 4.2.

The K_x matrix was obtained with solution of the LMI in Theorem 3.3 for $\beta_{n_c=1} = \beta_{n_c=2} = 0.1$. The gain K_x was used as input in solution of Theorem 4.1 in subsection 4.3.1, we solved Corollary 4.1 and obtained $\gamma_{n_c=0} = 2.4094$, $\gamma_{n_c=1} = 2.4181$, $\gamma_{n_c=2} = 2.4487$, with gains

$$D_c = \begin{bmatrix} 1.5920 \\ 0.0338 \end{bmatrix}, \tag{4.82}$$

for $n_c = 0$,

$$A_c = -1.1120, B_c = 0.0196$$
$$C_c = \begin{bmatrix} 5.4823 \\ 0.2822 \end{bmatrix}, D_c = \begin{bmatrix} 1.5513 \\ 0.0253 \end{bmatrix}, \tag{4.83}$$

for $n_c = 1$,

$$A_c = \begin{bmatrix} -2.3952 & 0.6552 \\ 0.4497 & -1.2134 \end{bmatrix}, B_c = \begin{bmatrix} 0.0056 \\ 0.0242 \end{bmatrix},$$
$$C_c = \begin{bmatrix} -0.3134 & -0.6175 \\ 0.1451 & 0.3875 \end{bmatrix}, D_c = \begin{bmatrix} 1.6215 \\ 0.0146 \end{bmatrix}, \tag{4.84}$$

for $n_c = 2$.

Figure 4.3 shows the agents' temporal response for the gains (4.83). All agents converge to the same position, and the velocity of each agent converges to zero, showing the proposed technique's effectiveness.

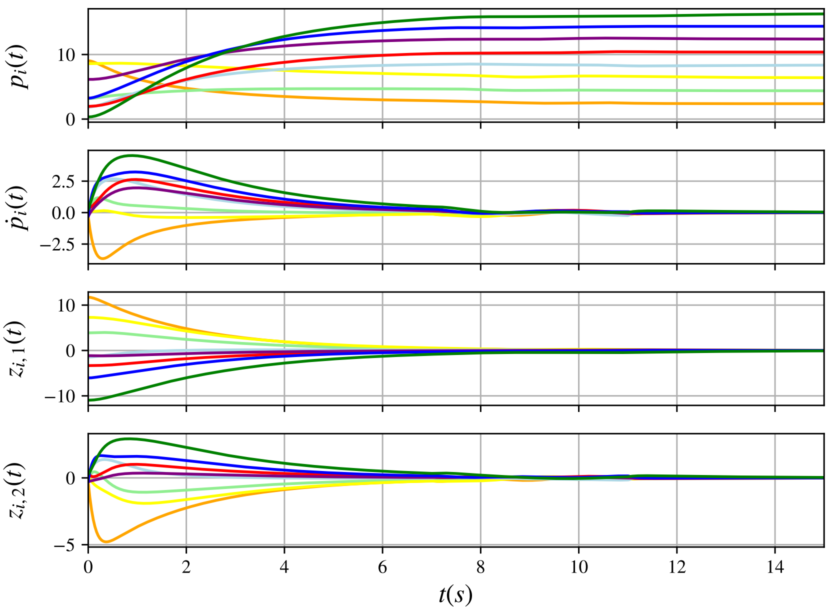


Figure 4.4 – Temporal response of the agents using gains (4.83). The positions $p_i(t)$ and velocities $\dot{p}_i(t)$ of the agents achieve consensus, and the controlled outputs $z_i(t)$ converge to zero. The colors of the agents are identified in the graph in Fig. 4.1.

To compare Corollary 4.2 with Theorem 3.2 (result presented in [53, Theorem 2]), we consider in the simulation that $C_y = I$, $D_y = \begin{bmatrix} 0.5 & 1 \end{bmatrix}^T$ and the uncertain Laplacian matrix (4.13), with $a_{28}(\alpha(t)) = 0$ and $a_{63}(\alpha(t)) = 0$, i.e., the connections from agent 8 to 2 and from agent 3 to 6 in Figure 4.1 do not exist and $L(\alpha(t))$ is a nominal matrix. For $\beta_{n_c=1} = 0.12$ and

$\beta_{n_c=1} = 0.09$ we compute K_x by solving the LMI in Theorem 3.3, and use it for solve [53, Theorem 4] obtaining the H_∞ performance indexes $\gamma_{n_c=0} = 39.4697$, $\gamma_{n_c=1} = 32.3584$ and $\gamma_{n_c=2}$ without solution, and we solve Corollary 4.2 obtaining the H_∞ performance indexes $\gamma_{n_c=0} = 3.312$, $\gamma_{n_c=1} = 3.0425$ and $\gamma_{n_c=2} = 3.0243$.

4.4 CHAPTER CONCLUSIONS

This chapter proposes sufficient conditions for designing output protocols for multi-agent systems subject to external disturbances and parametric uncertainty distributed in a directed network with uncertain communications. The derived LMI conditions may design protocols of arbitrary order for a parameter uncertainty bound and an H_∞ cost concerning the external disturbances. Further, a polytope models the network uncertainties. The conditions proposed in this chapter are the first in the literature to design any-order protocols considering uncertainties in agents' parameters and agents subject to external disturbances connected in an uncertain directed network.

5 TIME-VARYING OUTPUT FORMATION TRACKING OF DISTURBED AGENTS

We design protocols to obtain leaderless consensus in Chapters 3 and 4. In this chapter, we deal with formation tracking, which refers to problems where a group of agents follows a unique leader [45]. More specifically, here we will present the H_∞ Time-Varying Output Formation Tracking (HTVOFT) of linear multi-agent systems by the dynamic output feedback problem. We present two cases for disturbed agents: when the network topology is not restricted, but the condition is not scalable, and when the condition is scalable, but restrictions in network definition exist. Since agents are not subject to external disturbances, we may design protocols that do not restrict network topology through scalable conditions. The presented conditions here can design leader-follower protocols, a particular case of Chapters 3 and 4.

The results of this chapter surpass some gaps in the literature. One can list the main and more recent works in literature that study or are related to time-varying formation problems that correspond with this chapter. For problems where the agents' state variables synchronize, achieving a prescribed formation, one has [41] that studies the design of protocols for time-varying formation containment of linear multi-agent systems for directed switching topologies, [66] that designs time-varying formation protocols for uncertain multi-agent systems with communication delays and nonlinear couplings in undirected graphs. In cases where agent outputs are the variable of interest for time-varying formation, we can cite [74] which studies the design of a static output feedback protocol for time-varying output formation containment problem for networks with identical non-disturbed agents, [16] that studies static protocols for time-varying output formation containment control of homogeneous and heterogeneous descriptor fractional-order multi-agent systems, [68] designs an observer protocol for the adaptive time-varying formation of non-disturbed agents guided by formation leaders and a tracking leader. Since the leader-follower problem is a particular case of time-varying output formation tracking, one can present some works that can design protocols for problems involving a leader and followers as [25] that design static output protocols for disturbed agents that a leader guides, [53] present conditions that may design any order protocols for the consensus of agents subject to external disturbances and [32] that studies reduced-order protocols for non-disturbed agents. Following the above discussion, we present Table 5.1 which summarizes the main characteristics of results in this chapter concerning literature.

In Table 5.1, the work [25] received $\circ$ in the disturbances line for not considering disturbances in agents' dynamics and outputs simultaneously. In "Leaderless" line works [32]

	T5.2	T5.3	T 5.4	T 5.5	[25]	[74]	[32]	[53]	[68]	[41]	[66]	[16]
Static Output Feedback	✓	✓	✓	✓	✓	✓	✓	✓	×	×	×	✓
Reduced-Order	✓	✓	✓	✓	×	×	∘	✓	×	×	×	×
Full-Order	✓	✓	✓	✓	×	×	×	✓	✓	×	×	×
Digraph	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	✓
Disturbances	✓	×	✓	×	∘	×	×	✓	×	×	✓	×
Leaderless	×	×	×	×	×	×	✓	✓	×	×	✓	✓
Formation Problem	✓	✓	✓	✓	×	×	✓	✓	✓	✓	✓	✓
No restrictions in graph weights	✓	✓	×	✓	×	✓	✓	✓	×	✓	×	✓
Large number of agents	×	×	✓	✓	✓	✓	✓	×	✓	✓	×	×
LMI	✓	✓	✓	✓	×	×	×	✓	×	×	✓	×

Table 5.1 – Comparison between Theorems 5.2, 5.3, 5.4 and 5.5 concerning literature results. The symbols ✓means "yes", × means "no" and ∘ means "partially".

and [53] propose conditions for designing consensus protocols that can be used for leader-follower problems, a particular case of Theorems 5.2, 5.3, 5.4, 5.5. The line, "No restriction in graph weights," seeks to highlight works in literature that restrict the graph weights in addition to considering that there is a spanning tree.

5.1 THE TIME-VARYING OUTPUT FORMATION TRACKING PROBLEM

Consider m agents in a directed network, each with the following dynamic model

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_u u_i(t) + B_w w_i(t), \quad i = 1, ..., m \\ y_i(t) &= C_y x_i(t) + D_y w_i(t) \end{aligned} \tag{5.1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^s$, $w_i \in \mathcal{L}_2^{n_w}[0, \infty)$, and $y_i(t) \in \mathbb{R}^q$ are the local state, control input, exogenous disturbance, and measured output of agent i, respectively. The matrices are such that $rank(B_u) = s$, and $rank(C_y) = q$. The agents follow a tracking leader defined as

$$\begin{aligned} \dot{x}_0(t) &= Ax_0(t) + B_u u_0(t), \\ y_0(t) &= C_y x_0(t) \end{aligned} \tag{5.2}$$

where $x_0(t) \in \mathbb{R}^n$, $u_0(t) \in \mathbb{R}^s$, and $y_0(t) \in \mathbb{R}^q$ are the state, the control input, and measured output of the leader, respectively. The leader input $u_0(t)$ is chosen by the designer to define the leader's trajectory, and it is supposed to be known by all agents.

The time-varying output formation is characterized by all agents' outputs positioned about the leader (5.2) according to a prescribed geometry or pattern specified by piecewise continuously differentiable functions $h_i(t) \in \mathbb{R}^q$, $i = 1, ..., m$. To track a formation $\{y_0(t) + h_\ell(t)\}_{\ell=1}^m$, it is assumed that each agent i aggregates the information provided by its neighbors and calculates the information signals

$$\nu_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t) - h_j(t) - (y_i(t) - h_i(t))) - \pi_i e_{y,i}(t), \tag{5.3}$$

$$\upsilon_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_{c,j}(t) - x_{c,i}(t)) - \pi_i x_{c,i}(t), \tag{5.4}$$

to feed the agent's local controller, which is assumed to have the following proposed structure

$$\begin{aligned} \dot{x}_{c,i}(t) &= A_c x_{c,i}(t) + B_c \nu_i(t) + B_{2c} \upsilon_i(t), \\ \tilde{u}_i(t) &= C_c x_{c,i}(t) + D_c \nu_i(t) + D_{2c} \upsilon_i(t), \\ u_i(t) &= \tilde{u}_i(t) + \delta_i(t) + u_0(t), \end{aligned} \tag{5.5}$$

where $e_{y,i}(t) = y_i(t) - h_i(t) - y_0(t)$ is the output formation error, $x_{c,i}(t) \in \mathbb{R}^{n_c}$ is the controller state with $0 \leq n_c \leq n$, controller parameters $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $B_{2c} \in \mathbb{R}^{n_c \times n_c}$, $D_c \in \mathbb{R}^{s \times q}$, $D_{2c} \in \mathbb{R}^{s \times n_c}$, $\delta_i(t)$ is a time-varying formation compensation signal.

The TVOFT problem is characterized by achieving $y_i(t) \to y_0(t) + h_i(t)$. However, the presence of disturbances $w_i(t)$ may prevent the agents from tracking perfectly. To measure the influence of the disturbances in tracking, the following controlled output variable is defined

$$z_i(t) = C_z e_i(t) \tag{5.6}$$

where $e_i(t) = x_i(t) - C_{y||} h_i(t) - x_0(t)$, and $C_{y||}$ such that $C_y C_{y||} = I$. Observe that when the error $e_i(t) \to 0$ implies $z_i(t) = 0$ for every i and the matrix $C_z \in \mathbb{R}^{n_z \times n}$ balances the relative importance of consensus among particular state components of the agents in the performance analysis. It is considered a system performance evaluation in the H_∞ sense which relates the overall exogenous disturbance $w(t)$ and the consensus discrepancy $z(t)$ through the inequality

$$\int_0^\infty \|z(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t, \quad \forall w(t) \in \mathcal{L}_2[0, \infty), \tag{5.7}$$

where the scalar $\gamma > 0$ is the H_∞ consensus performance index for the closed-loop multi-agent system (5.1)-(5.6).

This chapter addresses the following H_∞ time-varying output formation tracking (HTVOFT) problem.

PROBLEM 5.1 For the multi-agent system (5.1) and leader (5.2), design protocol (3.2) with order $0 \leq n_c \leq n$ such that, the resulting closed-loop multi-agent system (5.1)-(5.6):

1. in the absence of disturbance, for any initial conditions, achieves asymptotic overall time-varying output formation tracking defined as

$$\lim_{t\to\infty} (y_i(t) - h_i(t) - y_0(t)) = 0, \quad i = 1, ..., m; \tag{5.8}$$

2. in the presence of the disturbance $w(t)$ and zero initial conditions, satisfies the H_∞ performance (5.7) for some $\gamma > 0$.

REMARK 5.1 Observe that the proposed protocol (5.3)-(5.5) allows the exchange of the controller's states over the network through the signal $v_i(t)$, similar to other works in the literature of MAS (see, for instance, [31], [32]). However, in contrast to other works in the literature, the proposed technique allows the designer to choose whether the controller's states are shared over the network. We show (see Remark 5.11) that the design of protocols without controllers' interaction is a particular case of the approach with controllers' interaction.

5.2 PROBLEM REFRAMING

5.2.1 Notation

Due to the presence of a leader, the communication network is larger by one agent compared to the network of the previous chapters. Then, it is necessary to add some new notational elements. We consider the leader indexed by 0 so that the previous representation of the followers as agents 1 to m remains the same. The graph, adjacency matrix, and Laplacian matrix for the followers remain the same as in the previous chapters.

The connection between all agents and a leader can be represented by an overall graph $\tilde{\mathbb{G}}$ associated with the adjacency matrix $\tilde{\mathsf{A}} = \begin{bmatrix} 0 & 0_{1\times m} \\ \pi & \mathsf{A} \end{bmatrix}$ where $\pi = \begin{bmatrix} \pi_1 & \cdots & \pi_m \end{bmatrix}^T$ is a vector of pinning gains which are such that $\pi_i \geq 0$ if the $i-th$ node is pinned by, or receive information from, an external node (leader), and otherwise $\pi_i = 0$. The Laplacian matrix associated with the overall graph $\tilde{\mathbb{G}}$ is defined as $\tilde{L} = \begin{bmatrix} 0 & 0_{1\times m} \\ -\pi & L+\Pi \end{bmatrix}$, where $\Pi = diag\{\pi_1, ..., \pi_m\}$.

5.2.2 The Transformed Problem

The augmented multi-agent system, considering the m agents in (5.1), with concatenated variables $x(t)$, $y(t)$, $w(t)$, $u(t)$ and $e(t)$, is given by

$$\begin{aligned} \dot{x}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)u(t), \\ z(t) &= (I_m \otimes C_z)e(t), \\ y(t) &= (I_m \otimes C_y)x(t) + (I_m \otimes D_y)w(t) \end{aligned} \tag{5.9}$$

The augmented dynamical controller is given by

$$\begin{aligned} \dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) + (I_m \otimes B_c)\nu(t) + (I_m \otimes B_{2c})\upsilon(t), \\ u(t) &= \tilde{u}(t) + \bar{u}_0(t) + \delta(t), \\ \tilde{u}(t) &= (I_m \otimes C_c)x_c(t) + (I_m \otimes D_c)\nu(t) + (I_m \otimes D_{2c})\upsilon(t), \end{aligned} \tag{5.10}$$

where $x_c(t)$, $e_y(t)$, $\bar{u}_0(t)$, $\delta(t)$, $\nu(t)$ and $\upsilon(t)$ are the concatenated variables, such that

$$\begin{aligned} \nu(t) &= -(\bar{L} \otimes C_y)e(t) - (\bar{L} \otimes D_y)w(t), \\ \upsilon(t) &= -(\bar{L} \otimes I)x_c(t), \\ e(t) &= x(t) - \bar{x}_0(t) - (I_m \otimes C_{y||})h(t), \\ \bar{L} &= L + \Pi, \bar{x}_0(t) = (\mathbf{1}_m \otimes x_0(t)). \end{aligned}$$

From Kronecker mixed product property, one can rewrite (5.10) as

$$\begin{aligned} \dot{x}_c(t) &= (I_m \otimes A_c)x_c(t) - (\bar{L} \otimes B_cC_y)e(t) - (\bar{L} \otimes B_cD_y)w(t) \\ &\quad - (\bar{L} \otimes B_{2c})x_c(t), \\ \tilde{u}(t) &= (I_m \otimes C_c)x_c(t) - (\bar{L} \otimes D_cC_y)e(t) - (\bar{L} \otimes D_cD_y)w(t) \\ &\quad - (\bar{L} \otimes D_{2c})x_c(t). \end{aligned} \tag{5.11}$$

In this chapter, we consider the following assumption.

Assumption 5.1 The graph $\tilde{\mathbb{G}}$ contains a directed spanning tree.

Similar to Chapters 3 and 4, we consider that information flows from the leader for the followers at least for one directed path, i.e., we adopt Assumption 5.1 that is a prerequisite for synchronizing agents and the leader.

In the following, we enunciate a theorem for the HTVOFT analysis of (5.1), that presents a condition to be satisfied by the formation function $h(t)$ in order to solve Problem 5.1.

Theorem 5.1

Consider the MAS (5.1) under protocol (5.3)-(5.5) with the tracking-leader (5.2) and controlled output signal (5.6). Problem 5.1 is solved if the following conditions are satisfied:

1. the formation compensation signal is designed as

$$\delta_i(t) = -B_{u\underline{||}}\left(AC_{y||}h_i(t) - C_{y||}\dot{h}_i(t)\right). \tag{5.12}$$

2. the following output formation tracking feasibility condition is satisfied

$$B_{u\perp}AC_{y||}h_i(t) - B_{u\perp}C_{y||}\dot{h}_i(t) = 0. \tag{5.13}$$

3. the following closed-loop system

$$\begin{aligned}\dot{\phi}(t) &= \underline{\mathbf{A}}\phi(t) + \underline{\mathbf{B}}_w w(t),\\ z(t) &= \mathcal{C}_z\phi(t),\end{aligned} \tag{5.14}$$

with

$$\underline{\mathbf{A}} = \begin{bmatrix} (I_m \otimes A) - (\bar{L} \otimes B_u D_c C) & (I_m \otimes B_u C_c) - (\bar{L} \otimes B_u D_{2c}) \\ -(\bar{L} \otimes B_c C) & (I_m \otimes A_c) - (\bar{L} \otimes B_{2c}) \end{bmatrix},$$
$$\underline{\mathbf{B}}_w = \begin{bmatrix} (I_m \otimes B_w) - (\bar{L} \otimes B_u D_c D_y) \\ -\bar{L} \otimes (B_c D_y) \end{bmatrix}, \mathcal{C}_z = \begin{bmatrix} (I_m \otimes C_z) & 0 \end{bmatrix}, \tag{5.15}$$

is asymptotically stable for $w(t) \equiv 0$, and in the presence of a nonzero disturbance $w(t)$ and zero initial conditions, satisfies (5.7) for some $\gamma > 0$.

proof.

First remember that $u(t) = \tilde{u}(t) + \bar{u}_0(t) + \delta(t)$, then we can rewrite $\dot{x}(t)$ as

$$\begin{aligned}\dot{x}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)\tilde{u}(t)\\ &+ (I_m \otimes B_u)\bar{u}_0(t) + (I_m \otimes B_u)\delta(t),\end{aligned} \tag{5.16}$$

Summing and subtracting the term $(I \otimes C_{y||})\dot{h}(t) - (I \otimes AC_{y||})h(t)$ in the previous expression, it is obtained

$$\begin{aligned}\dot{x}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)\tilde{u}(t) \\ &+ (I_m \otimes B_u)\bar{u}_0(t) + (I_m \otimes B_u)\delta(t) + (I \otimes C_{y||})\dot{h}(t) - (I \otimes AC_{y||})h(t) \\ &- (I \otimes C_{y||})\dot{h}(t) + (I \otimes AC_{y||})h(t),\end{aligned} \tag{5.17}$$

Observe that the derivative of $e(t)$ is

$$\dot{e}(t) = \dot{x}(t) - (I \otimes C_{y||})\dot{h}(t) - \dot{\bar{x}}_0(t). \tag{5.18}$$

Substituting $\dot{x}(t)$ and $\dot{\bar{x}}_0(t)$ in (5.18), it is obtained

$$\begin{aligned}\dot{e}(t) &= (I_m \otimes A)x(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)\tilde{u}(t) + (I_m \otimes B_u)\bar{u}_0(t) \\ &- (I_m \otimes A)\bar{x}_0(t) - (I_m \otimes B_u)\bar{u}_0(t) + (I_m \otimes B_u)\delta(t) - (I \otimes C_{y||})\dot{h}(t), \\ &= (I_m \otimes A)e(t) + (I_m \otimes B_w)w(t) + (I_m \otimes B_u)\tilde{u}(t) \\ &+ \underbrace{(I_m \otimes B_u)\delta(t) - (I \otimes C_{y||})\dot{h}(t) + (I \otimes AC_{y||})h(t)}_{\rho(t)}\end{aligned} \tag{5.19}$$

From (5.11) and (5.19), one can obtain the following closed-loop system concatenating the vector variable $\begin{bmatrix} e(t) \\ x_c(t) \end{bmatrix}$ as follow

$$\begin{aligned}\underbrace{\begin{bmatrix} \dot{e}(t) \\ \dot{x}_c(t) \end{bmatrix}}_{\dot{\phi}(t)} &= \underbrace{\begin{bmatrix} (I_m \otimes A) - (\bar{L} \otimes B_u D_c C) & (I_m \otimes B_u C_c) - (\bar{L} \otimes B_u D_{2c}) \\ -(\bar{L} \otimes B_c C) & (I_m \otimes A_c) - (\bar{L} \otimes B_{2c}) \end{bmatrix}}_{\underline{\mathbf{A}}} \underbrace{\begin{bmatrix} e(t) \\ x_c(t) \end{bmatrix}}_{\phi(t)} \\ &+ \underbrace{\begin{bmatrix} (I_m \otimes B_w) - (\bar{L} \otimes B_u D_c D_y) \\ -\bar{L} \otimes (B_c D_y) \end{bmatrix}}_{\underline{\mathbf{B}}_1} w(t) + \bar{\rho}(t), \\ z(t) &= \underbrace{\begin{bmatrix} (I_m \otimes C_z) & 0 \end{bmatrix}}_{\mathcal{C}_z} \underbrace{\begin{bmatrix} e(t) \\ x_c(t) \end{bmatrix}}_{\phi(t)},\end{aligned} \tag{5.20}$$

with $\bar{\rho}(t) = \begin{bmatrix} \rho(t) \\ 0_{mn_c} \end{bmatrix}$.

Defining $B_{\underline{u||}} \in \mathbb{R}^{s \times n}$ such that $B_{\underline{u||}} B_u = I$ and choosing $\delta_i(t) =$

$-B_{u\underline{||}}\left(AC_{y||}h_i(t) - C_{y||}\dot{h}_i(t)\right)$, one has

$$(I \otimes B_{u\underline{||}})\rho(t) = (I \otimes B_{u\underline{||}})((I \otimes AC_{y||})h(t) - (I \otimes C_{y||})\dot{h}(t) + (I_m \otimes B_u)\delta(t)).$$

As $B_{u\underline{||}}B_u = I$ and using the Kronecker mixed product property, one has

$$(I \otimes B_{u\underline{||}})\rho(t) = 0 \tag{5.21}$$

.

Defining $B_{u\underline{\perp}} \in \mathbb{R}^{(n-s)\times n}$ such that $B_{u\underline{\perp}}B_u = 0$ and supposing that

$$(I_m \otimes B_{u\underline{\perp}}AC_{y||})h(t) - (I_m \otimes B_{u\underline{\perp}}C_{y||})\dot{h}(t) = 0, \tag{5.22}$$

the following equality is equivalent to (5.22)

$$\begin{aligned}(I_m \otimes B_{u\underline{\perp}}AC_{y||})h(t) - (I_m \otimes B_{u\underline{\perp}}C_{y||})\dot{h}(t) + (I_m \otimes B_{u\underline{\perp}}B_u)\delta(t) = 0\\ (I_m \otimes B_{u\underline{\perp}})\rho(t) = 0\end{aligned} \tag{5.23}$$

Then, defining $T = [B_{u\underline{||}}^T \; B_{u\underline{\perp}}^T]^T$, from (5.21) and (5.23) one has

$$(I_m \otimes T)\rho(t) = 0. \tag{5.24}$$

The matrix T is a non-singular matrix, then pre-multiplying (5.24) by $(I_m \otimes T^{-1})$ one has

$$\rho(t) = 0. \tag{5.25}$$

Finally, the stability of (5.14) assures that for $w = 0$, $\phi(t) \to 0$ as $t \to \infty$, then (5.8) is verified. Moreover, we can easily see that if γ satisfies the H_∞ performance (5.7) for the collective dynamics (5.14), then Problem 5.1 (2) is solved. ■

REMARK 5.2 Theorem 5.1 presents conditions for the feasibility of HTVOFT, given an $h_i(t)$ formation function that satisfies (5.13), similarly as in [12], [74], and [68], that present feasibility conditions for its respectively time-varying formation problems. Assumption 2 is fundamentally important in proving Theorem 1 since it helps to obtain

$\rho(t) = 0$.

Verification of the solubility of condition (5.13) and construction of $h_i(t)$ functions satisfying it is a matter of simple algebraic manipulations of the set of differential equations in (5.13). Equation (5.13) is not soluble for any matrices A, B_u, and C_y. However, for suitable matrices A, B_u, and C_y, finding $h_i(t)$ functions that satisfy condition (5.13) is not difficult (see the section Numerical Examples).

REMARK 5.3 Closed-loop system (5.14) looks similar to other closed-loop systems obtained in Chapters 3 and 4 (see (3.63)-(3.65) and (4.32)). Although similar, the problem studied in this chapter forces us to use a different strategy to obtain an analysis condition. Observe that closed-loop system (5.14) was obtained only concatenating variables, as shown in the proof of Theorem 5.1, different of (3.63)-(3.65) and (4.32) that need to pass by a variable transformation, that translates the consensus problem into a stability problem.

Theorem 5.1 present conditions to be satisfied by the closed-loop system 5.14 in order to solve Problem 5.1; from here onwards, we aim to find an equivalent form for 5.14 that allows us to find the protocol gains, since that the protocol gains are dispersed in the closed-loop system (5.14), making the design complex. To facilitate the gain design, system (5.14) is rewritten as

$$\begin{aligned} \dot{\phi}(t) &= (\mathcal{A} + \mathcal{B}\mathcal{K}_y\mathcal{C})\phi(t) + (\mathcal{B}_1 + \mathcal{B}\mathcal{K}_y\mathcal{D})w(t), \\ z(t) &= \mathcal{C}_z\phi(t), \end{aligned} \tag{5.26}$$

with

$$\mathcal{A} = \begin{bmatrix} I_m \otimes A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & I_m \otimes B_u \\ I_{mn_c} & 0 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 0_{(mn_c \times mn_w)} \\ -(\bar{L} \otimes D_y) \\ 0_{(mn_c \times mn_w)} \end{bmatrix}, \tag{5.27}$$

$$\mathcal{K}_y = \begin{bmatrix} I_m \otimes A_c & I_m \otimes B_c & I_m \otimes B_{2c} \\ I_m \otimes C_c & I_m \otimes D_c & I_m \otimes D_{2c} \end{bmatrix}, \mathcal{B}_1 = \begin{bmatrix} (I_m \otimes B_w) \\ 0 \end{bmatrix}, \tag{5.28}$$

$$\mathcal{C} = \begin{bmatrix} 0_{(mn_c \times mn)} & I_{mn_c} \\ -\bar{L} \otimes C_y & 0_{(mq \times mn_c)} \\ 0_{(mn_c \times mn)} & -\bar{L} \otimes I_{n_c} \end{bmatrix}. \tag{5.29}$$

In order to derive tractable LMI conditions, one has

$$A_c = \mathfrak{I}_{11} K_y \mathfrak{I}_{21}^T, B_c = \mathfrak{I}_{11} K_y \mathfrak{I}_{22}^T, B_{2c} = \mathfrak{I}_{11} K_y \mathfrak{I}_{23}^T,$$
$$C_c = \mathfrak{I}_{12} K_y \mathfrak{I}_{21}^T, D_c = \mathfrak{I}_{12} K_y \mathfrak{I}_{22}^T, D_{2c} = \mathfrak{I}_{12} K_y \mathfrak{I}_{23}^T,$$

where

$$\mathfrak{I}_{11} = \begin{bmatrix} I_{n_c} & 0_{n_c \times s} \end{bmatrix}, \mathfrak{I}_{12} = \begin{bmatrix} 0_{s \times n_c} & I_s \end{bmatrix}, \tag{5.30}$$

$$K_y = \begin{bmatrix} A_c & B_c & B_{2c} \\ C_c & D_c & D_{2c} \end{bmatrix}, \mathfrak{I}_{21} = \begin{bmatrix} I_{n_c} & 0_{n_c \times q} & 0_{n_c \times n_c} \end{bmatrix} \tag{5.31}$$

$$\mathfrak{I}_{22} = \begin{bmatrix} 0_{q \times n_c} & I_q & 0_{q \times n_c} \end{bmatrix}, \mathfrak{I}_{23} = \begin{bmatrix} 0_{n_c \times n_c} & 0_{n_c \times q} & I_{n_c} \end{bmatrix}. \tag{5.32}$$

Using some properties of the Kronecker product, $\mathcal{K}_y$ can be represented by

$$\mathcal{K}_y = \mathcal{T}_1 (I_m \otimes K_y) \mathcal{T}_2$$

with

$$\mathcal{T}_1 = \begin{bmatrix} I_m \otimes \mathfrak{I}_{11} \\ I_m \otimes \mathfrak{I}_{12} \end{bmatrix}, \ \mathcal{T}_2 = \begin{bmatrix} I_m \otimes \mathfrak{I}_{21}^T & I_m \otimes \mathfrak{I}_{22}^T & I_m \otimes \mathfrak{I}_{23}^T \end{bmatrix}.$$

Therefore, system (5.26) is rewritten in the following form

$$\begin{aligned} \dot{\phi}(t) &= A_{cl} \phi(t) + B_{cl} w(t), \\ z(t) &= \mathcal{C}_z \phi(t), \end{aligned} \tag{5.33}$$

where

$$A_{cl} = \mathcal{A} + \bar{\mathcal{B}} (I_m \otimes K_y) \bar{\mathcal{C}}, \ \ B_{cl} = \mathcal{B}_1 + \bar{\mathcal{B}} (I_m \otimes K_y) \bar{\mathcal{D}},$$
$$\bar{\mathcal{B}} = \mathcal{B} \mathcal{T}_1, \quad \bar{\mathcal{C}} = \mathcal{T}_2 \mathcal{C}, \quad \bar{\mathcal{D}} = \mathcal{T}_2 \mathcal{D}.$$

We can now state the conditions for the design of K_y in the following section.

5.3 DESIGN OF TIME-VARYING OUTPUT FORMATION TRACKING PROTOCOLS

The following theorem solves the Problem 5.1.

Theorem 5.2

Let $h_i(t)$, for $i = 1, ..., m$, satisfying condition (5.13), the compensation signal $\delta_i(t)$ given by (5.12), and K_x a given matrix such that $\mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz. If there exist a scalar $\mu > 0$ and matrices $P = P^T \succ 0$, $\mathcal{G}$, $\mathcal{Z}$, X_p, for $p = 1, \dots, 8$, such that the following LMI holds

$$\begin{bmatrix} \Phi_1 & \star \\ \Phi_2 & \Phi_3 \end{bmatrix} \prec 0, \tag{5.34}$$

where

$$\Phi_1 = \begin{bmatrix} \Psi_1 + \mathcal{C}_z^T\mathcal{C}_z & \star & \star \\ \Psi_2 & -He\{X_3\} & \star \\ \Psi_3 & -X_5 - X_4^T & -He\{X_6\} \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} \Psi_5 & \Psi_6 & \Psi_7 \\ \bar{\mathcal{B}}^T X_1^T & \bar{\mathcal{B}}^T X_3^T & \Psi_4 \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} \Psi_8 & \star \\ \bar{\mathcal{B}}^T X_7^T + (I_m \otimes \mathcal{Z})\bar{\mathcal{D}} & -He\{(I_m \otimes \mathcal{G})\} \end{bmatrix},$$

$$\begin{aligned}
\Psi_1 &= (\mathcal{A} + \bar{\mathcal{B}}K_x)^T X_1^T + X_1(\mathcal{A} + \bar{\mathcal{B}}K_x) + He\{X_2\}, \\
\Psi_2 &= P + X_3(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_4 - X_1^T, \\
\Psi_3 &= X_5(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_6 - X_2^T, \\
\Psi_4 &= \bar{\mathcal{B}}^T X_5^T + (I_m \otimes \mathcal{Z})\bar{\mathcal{C}} - (I_m \otimes \mathcal{G})K_x, \\
\Psi_5 &= X_7(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_8 + \mathcal{B}_w^T X_1^T, \\
\Psi_6 &= -X_7 + \mathcal{B}_w^T X_3^T, \quad \Psi_7 = -X_8 + \mathcal{B}_w^T X_5^T, \\
\Psi_8 &= -\mu I + He\{\mathcal{B}_w X_7\},
\end{aligned}$$

then the protocol (5.3)-(5.5) with parameter matrices

$$\begin{bmatrix} A_c & B_c & B_{2c} \\ C_c & D_c & D_{2c} \end{bmatrix} = \mathcal{G}^{-1}\mathcal{Z},$$

solves Problem 5.1, with H_∞ cost upper bound $\gamma := \sqrt{\mu}$.

proof.

If (5.34) holds then $-He\{(I_m \otimes \mathcal{G})\} \prec 0$, which implies that $\mathcal{G}$ is a non-singular matrix.

Define $K_y = \mathcal{G}^{-1}\mathcal{Z}$. Replacing $\mathcal{Z}$ by $\mathcal{G}K_y$, inequality (5.35) can be rewritten as

$$\underbrace{\begin{bmatrix} \mathcal{Q}_{31} & \mathcal{X}_b \\ \mathcal{X}_b^* & 0 \end{bmatrix}}_{\mathcal{Q}_3} + He\left\{\mathbf{U}_3^T(I_m \otimes \mathcal{G})\mathbf{V}_3\right\} \prec 0, \tag{5.35}$$

where

$$\mathcal{X}_b = \begin{bmatrix} \bar{\mathcal{B}}^T X_1^T & \bar{\mathcal{B}}^T X_3^T & \bar{\mathcal{B}}^T X_5^T & \bar{\mathcal{B}}^T X_7^T \end{bmatrix}^T,$$
$$\mathcal{Q}_{31} = \begin{bmatrix} \Psi_1 + \mathcal{C}_z^T\mathcal{C}_z & \star & \star & \star \\ \Psi_2 & -He\{X_3\} & \star & \star \\ \Psi_3 & -X_5 - X_4^T & -He\{X_6\} & \star \\ \Psi_5 & \Psi_6 & \Psi_7 & \Psi_8 \end{bmatrix},$$

$\mathbf{U}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & I \end{bmatrix}$, $\mathbf{V}_3 = \begin{bmatrix} 0 & 0 & S & S_2 & -I \end{bmatrix}$, $S = (I_m \otimes K_y)\bar{\mathcal{C}} - K_x$, and $S_2 = (I_m \otimes K_y)\bar{\mathcal{D}}$ and X_p are complex variable matrices for $p = 1, ..., 8$.

Pre- and post-multiplying (5.35) by $\mathbf{V}_{3\perp}^T = \begin{bmatrix} \mathbb{I} & \mathcal{S}^T \end{bmatrix}$ and $\mathbf{V}_{3\perp}$, with $\mathbb{I} = diag\{I, I, I, I\}$ and $\mathcal{S} = \begin{bmatrix} 0 & 0 & S & S_2 \end{bmatrix}$ one has $(\mathbf{V}_{3\perp})^T\mathcal{Q}_3\mathbf{V}_{3\perp} \prec 0$, which can be rewritten as

$$\mathcal{Q}_{31} + He\{\mathcal{X}_b\mathcal{S}\} \prec 0. \tag{5.36}$$

Defining $\mathbf{V}_2 = \begin{bmatrix} \mathcal{A}_x & -I & \bar{\mathcal{B}}S & B_{cl} \\ I & 0 & -I & 0 \end{bmatrix}$ with $B_{cl} = \mathcal{B}_w + \bar{\mathcal{B}}S_2$ and $\mathcal{A}_x = \mathcal{A} + \bar{\mathcal{B}}K_x$, inequality (5.36) can be rewritten as

$$\underbrace{\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu I \end{bmatrix}}_{\mathcal{Q}_2} + He\left\{\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \\ X_5 & X_6 \\ X_7 & X_8 \end{bmatrix}\mathbf{V}_2\right\} \prec 0. \tag{5.37}$$

Pre- and post-multiplying (5.37) by $\mathbf{V}_{2\perp}^T = \begin{bmatrix} I & A_{cl} & I & 0 \\ 0 & B_{cl} & 0 & I \end{bmatrix}^T$ and $\mathbf{V}_{2\perp}$, where $A_{cl} = \mathcal{A} + \bar{\mathcal{B}}K_x + \bar{\mathcal{B}}S$, one has

$$\mathbf{V}_{2\perp}^T\mathcal{Q}_2\mathbf{V}_{2\perp} \prec 0, \tag{5.38}$$

Replacing $\mu = \gamma^2$, inequality (5.38) can be rewritten as

$$\Theta = \begin{bmatrix} A_{cl}^T P + P A_{cl} + \mathcal{C}_z^T \mathcal{C}_z & P B_{cl} \\ B_{cl}^T P & -\gamma^2 I \end{bmatrix} \prec 0. \tag{5.39}$$

In the sequel, to deal with the formation tracking and the H_∞ performance, introduce for the system (5.33) the Lyapunov function $V(t) = \phi^T(t) P \phi(t)$ and the following cost functional for any $\tau > 0$,

$$J(\tau) = \int_0^\tau (\|z(t)\|^2 - \gamma^2 \|w(t)\|^2) \mathrm{d}t. \tag{5.40}$$

Observe that the time derivative of $V(t)$ can be written as

$$\begin{aligned} \dot{V}(t) &= \phi^T(t)(P A_{cl} + (A_{cl})^T P)\phi(t) \\ &\quad + \phi^T(t) P B_{cl} w(t) + w^T(t)(B_{cl})^T P \phi(t). \end{aligned}$$

Since that $\int_0^\tau (\dot{V}(t) - \dot{V}(t)) \mathrm{d}t = 0$ and $V(0) = 0$, the functional (5.40) can be rewritten as

$$\begin{aligned} J(\tau) = \int_0^\tau & (\phi^T(t)(P A_{cl} + (A_{cl})^T P + \mathcal{C}_z^T \mathcal{C}_z)\phi(t), \\ & + \phi^T(t) P B_{cl} w(t) + w^T(t)(B_{cl})^T P \phi(t), \\ & - \gamma^2 w^T(t) w(t)) \mathrm{d}t - V(\tau). \end{aligned} \tag{5.41}$$

Introducing the variable $\zeta(t) = \begin{bmatrix} \phi^T(t) & w^T(t) \end{bmatrix}^T$ and considering (5.39), one has that

$$J(\tau) = \int_0^\tau (\zeta^T(t) \Theta \zeta(t)) \mathrm{d}t - V(\tau). \tag{5.42}$$

Note that $P \succ 0$ such that $\Theta \prec 0$ implies $P A_{cl} + A_{cl}^T P \prec 0$ and $\dot{V}(t) < 0$, $t \geq 0$, for $w(t) = 0$. Therefore, $\phi(t) \to 0$ for any initial condition and the requirement (1) in Problem 5.1 is verified. Also, $P \succ 0$ and $\Theta \prec 0$ implies that the right-hand side of (5.42) is negative, or equivalently, from (5.40),

$$\int_0^\tau \|z(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\tau \|w(t)\|^2 \mathrm{d}t. \tag{5.43}$$

Note that (5.43) is valid for $0 \leq t \leq \tau$ and for all $\tau > 0$. Since $w(t) \in \mathcal{L}_2[0, \infty)$, the integral in right-hand side of (5.43) converges for $\tau \to \infty$. Also, by (5.43), the left-hand side integral is upper bounded and is non-decreasing in τ; therefore, it converges as $\tau \to \infty$. Taking the limit as $\tau \to \infty$ in (5.43) one has that (5.7) holds for any $w \in \mathcal{L}_2[0,\infty)$ and $0 \le t < \infty$, and therefore, the requirement (2) in Problem 5.1 is also attained. ■

REMARK 5.4 Similarly, as in Chapter 4, we consider here the hypothesis that K_x is such that $\mathcal{A}_x := \mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz. In the same way, this hypothesis is necessary to verify (5.34). If (5.34) holds for some $P \succ 0$, then $\Phi_1 \prec 0$. Note that

$$\Phi_1 = \underbrace{\begin{bmatrix} \mathcal{C}_z^T\mathcal{C}_z & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_4} + He\left\{\underbrace{\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \\ X_5 & X_6 \end{bmatrix}}_{\mathcal{X}} \underbrace{\begin{bmatrix} \mathcal{A}_x & -I & 0 \\ I & 0 & -I \end{bmatrix}}_{\mathbf{V}}\right\} \prec 0. \qquad (5.44)$$

Choosing $\mathbf{V}_\perp = \begin{bmatrix} I & \mathcal{A}_x^T & I \end{bmatrix}^T$ and pre- and post-multiplying (5.44) by $\mathbf{V}_\perp^T$ and $\mathbf{V}_\perp$, one has

$$\mathcal{A}_x^T P + P\mathcal{A}_x \prec 0,$$

with $P \succ 0$, that is, $\mathcal{A}_x = \mathcal{A} + \bar{\mathcal{B}}K_x$ is Hurwitz.

The following theorem can be used to design protocols to solve Problem 5.1 (1), using the augmented closed-loop system (5.33), when $w(t) = 0$, $\forall t$.

Theorem 5.3

Let $h_i(t)$, for $i = 1, \ldots, m$, satisfy condition (5.13), the compensation signal $\delta_i(t)$ given by (5.12), and K_x such that $\mathcal{A} + \mathcal{B}K_x$ is Hurwitz stable, then the system (5.33) is asymptotically stable if there exist matrices $P = P^T \succ 0$, $\mathcal{G}$, $\mathcal{Z}$ and X_p, for $p = 1, \ldots, 6$, such that the following inequality holds

$$\begin{bmatrix} \Psi_1 & \star & \star & \star \\ \Psi_2 & -X_3 - X_3^T & \star & \star \\ \Psi_3 & -X_5 - X_4^T & -X_6 - X_6^T & \star \\ \bar{\mathcal{B}}^T X_1^T & \bar{\mathcal{B}}^T X_3^T & \Psi_4 & -He\{(I_m \otimes \mathcal{G})\} \end{bmatrix} \prec 0, \qquad (5.45)$$

with

$$\Psi_1 = (\mathcal{A} + \bar{\mathcal{B}}\mathcal{K}_x)^T X_1^T + X_1(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_2 + X_2^T,$$
$$\Psi_2 = P + X_3(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_4 - X_1^T,$$
$$\Psi_3 = X_5(\mathcal{A} + \bar{\mathcal{B}}K_x) + X_6 - X_2^T,$$
$$\Psi_4 = \bar{\mathcal{B}}^T X_5^T + (I_m \otimes \mathcal{Z})\bar{\mathcal{C}} - (I_m \otimes \mathcal{G})K_x.$$

Furthermore, if (5.45) holds, then the stabilizing gain for (5.33) is given by $K_y := \mathcal{G}^{-1}\mathcal{Z}$.

proof.

As in Theorem 5.2, is easy to see that if (5.34) holds then $-He\{(I_m \otimes \mathcal{G})\} \prec 0$, which implies that $\mathcal{G}$ is a non-singular matrix. Defining $K_y = \mathcal{G}^{-1}\mathcal{Z}$, inequality (5.45) can be rewritten as

$$\underbrace{\begin{bmatrix} \Psi_1 & \star & \star & \star \\ \Psi_2 & -X_3 - X_3^T & \star & \star \\ \Psi_3 & -X_5 - X_4^T & -X_6 - X_6^T & \star \\ \bar{\mathcal{B}}^T X_1^T & \bar{\mathcal{B}}^T X_3^T & \bar{\mathcal{B}}^T X_5^T & 0 \end{bmatrix}}_{\mathcal{Q}_4} + He\left\{\mathbf{U}_4^T (I_m \otimes \mathcal{G})\mathbf{V}_4\right\} \prec 0, \tag{5.46}$$

where $\mathbf{U}_4 = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}$, $\mathbf{V}_4 = \begin{bmatrix} 0 & 0 & S & -I \end{bmatrix}$, $S = (I_m \otimes K_y)\bar{\mathcal{C}} - K_x$ and X_p are complex variable matrices for $p = 1, ..., 6$. Pre- and post-multiplying (5.35) by $\mathbf{V}_{4\perp}^T = \begin{bmatrix} \mathbb{I} & \mathcal{S}^T \end{bmatrix}$ and $\mathbf{V}_{4\perp}$, with $\mathbb{I}_3 = diag\{I, I, I\}$ and $\mathcal{S} = \begin{bmatrix} 0 & 0 & S \end{bmatrix}$ one has $\mathbf{V}_{4\perp}^T \mathcal{Q}_4 \mathbf{V}_{4\perp} \prec 0$, which can be rewritten as

$$\begin{bmatrix} \Psi_1 & \star & \star \\ \Psi_2 & -X_3 - X_3^T & \star \\ \Psi_3 & -X_5 - X_4^T & -X_6 - X_6^T \end{bmatrix} + He\left\{ \begin{bmatrix} X_1\mathcal{B} \\ X_3\mathcal{B} \\ X_5\mathcal{B} \end{bmatrix} \begin{bmatrix} 0 & 0 & S \end{bmatrix} \right\} < 0. \tag{5.47}$$

Note that inequality (5.47) can be rewritten in the following form

$$\underbrace{\begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_5} + He\left\{ \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \\ X_5 & X_6 \end{bmatrix} \underbrace{\begin{bmatrix} \mathcal{A}_x & -I & \bar{\mathcal{B}}S \\ I & 0 & -I \end{bmatrix}}_{\mathbf{V}_5} \right\} \prec 0. \tag{5.48}$$

One can choose a matrix $\mathbf{V}_{5\perp} = \begin{bmatrix} I \\ \mathcal{A}_x + \bar{\mathcal{B}}S \\ I \end{bmatrix}$ which its columns are orthogonal to rows of $\mathbf{V}_5$, then pre- and post-multiplying inequality (5.48) by $\mathbf{V}_{5\perp}^T$ and $\mathbf{V}_{5\perp}$, respectively, one has

$$\begin{aligned} 0 &\succ \underbrace{\begin{bmatrix} I \\ \mathcal{A}_x + \bar{\mathcal{B}}S \\ I \end{bmatrix}^T}_{\mathbf{V}_{5\perp}^T} \underbrace{\begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{Q}_5} \underbrace{\begin{bmatrix} I \\ \mathcal{A}_x + \bar{\mathcal{B}}S \\ I \end{bmatrix}}_{\mathbf{V}_{5\perp}} \\ &= \left(\mathcal{A} + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{C}}\right)^T P + P\left(\mathcal{A} + \bar{\mathcal{B}}(I_m \otimes K_y)\bar{\mathcal{C}}\right), \end{aligned} \tag{5.49}$$

The inequality (5.49) is the condition for Lyapunov stability analysis of the system (5.33) with $w(t) = 0, \forall$. ■

In the next section, we derive conditions that may be used to design protocols for large networks.

5.4 DESIGN OF HTVOFT PROTOCOLS FOR LARGE NETWORKS

For networks with many agents, the results in the previous section and Chapters 3 and 4 yield a high number of variables and LMI rows, making the numerical burden prohibitive. This section presents scalable design conditions for solving Problem 5.1, in which the numerical complexity does not grow with the number of agents, as in Theorems 5.2 and 5.3, but concerning vertices of a polytopic region. When agents are disturbed, this solution is attained by adding mild restrictions (Assumption 5.2) on the class of allowed topologies.

For this new approach, we consider the closed-loop system (5.14), which can be expressed as:

$$\underbrace{\begin{bmatrix}\dot{e}(t)\\ \dot{x}_c(t)\end{bmatrix}}_{\dot{\phi}(t)} = \underbrace{\begin{bmatrix}(I_m\otimes A)-(\bar{L}\otimes BD_cC) & (I_m\otimes BC_c)-(\bar{L}\otimes BD_{2c})\\ -(\bar{L}\otimes B_cC) & (I_m\otimes A_c)-(\bar{L}\otimes B_{2c})\end{bmatrix}}_{\underline{\mathbf{A}}}\underbrace{\begin{bmatrix}e(t)\\ x_c(t)\end{bmatrix}}_{\phi(t)} \tag{5.50}$$

$$+ \underbrace{\begin{bmatrix}(I_m\otimes B_w)-(\bar{L}\otimes BD_cD_y)\\ -\bar{L}\otimes(B_cD_y)\end{bmatrix}}_{\underline{\mathbf{B}}_w} w(t),$$

$$z(t) = \underbrace{\begin{bmatrix}(I_m\otimes C_z) & 0\end{bmatrix}}_{\mathcal{C}_z}\underbrace{\begin{bmatrix}e(t)\\ x_c(t)\end{bmatrix}}_{\phi(t)}. \tag{5.51}$$

Observe that we can define a new variable $\varphi(t) = \begin{bmatrix}I_m\otimes\mathfrak{T}_1 & I_m\otimes\mathfrak{T}_2\end{bmatrix}\phi(t)$, such that $\varphi_i(t) = \begin{bmatrix}e_i^T(t) & x_{c,i}^T(t)\end{bmatrix}^T$, $\phi(t) = \begin{bmatrix}(I_m\otimes\mathfrak{T}_1^T)^T & (I_m\otimes\mathfrak{T}_2^T)^T\end{bmatrix}^T\varphi(t)$, $\mathfrak{T}_1 = \begin{bmatrix}I_n\\ 0_{n_c\times n}\end{bmatrix}$ and $\mathfrak{T}_2 = \begin{bmatrix}0_{n\times n_c}\\ I_{n_c}\end{bmatrix}$. Considering the time derivative of $\varphi(t)$, equation (5.50) can be equivalently rewritten as

$$\begin{aligned}\dot{\varphi}(t) &= \begin{bmatrix}I_m\otimes\mathfrak{T}_1 & I_m\otimes\mathfrak{T}_2\end{bmatrix}(\underline{\mathbf{A}}\phi(t)+\underline{\mathbf{B}}_1w(t))\\ &= \begin{bmatrix}(I_m\otimes\mathfrak{T}_1)\Gamma_1-(I_m\otimes\mathfrak{T}_2)\Gamma_2 & (I_m\otimes\mathfrak{T}_1)\Gamma_3+(I_m\otimes\mathfrak{T}_2)\Gamma_4\end{bmatrix}\phi(t)\\ &+((I_m\otimes\mathfrak{T}_1)((I_m\otimes B_w)-(\bar{L}\otimes BD_cD_y))+(I_m\otimes\mathfrak{T}_2)(-\bar{L}\otimes(B_cD_y)))w(t),\end{aligned} \tag{5.52}$$

where $\Gamma_1 = (I_m\otimes A)-(\bar{L}\otimes BD_cC)$, $\Gamma_2 = (\bar{L}\otimes(B_cC))$, $\Gamma_3 = (I_m\otimes(BC_c))-(\bar{L}\otimes(BD_{2c}))$, $\Gamma_4 = (I_m\otimes A_c)-(\bar{L}\otimes B_{2c})$. Substituting $\phi(t)$ in equations (5.52) and (5.51) one has

$$\begin{aligned}\dot{\varphi}(t) &= ((I_m\otimes\mathfrak{T}_1)\Gamma_1(I_m\otimes\mathfrak{T}_1^T)-(I_m\otimes\mathfrak{T}_2)\Gamma_2(I_m\otimes\mathfrak{T}_1^T)+(I_m\otimes\mathfrak{T}_1)\Gamma_3(I_m\otimes\mathfrak{T}_2^T)\\ &+(I_m\otimes\mathfrak{T}_2)\Gamma_4(I_m\otimes\mathfrak{T}_2^T))\varphi(t)\\ &+((I_m\otimes\mathfrak{T}_1)((I_m\otimes B_w)-(\bar{L}\otimes BD_cD_y))+(I_m\otimes\mathfrak{T}_2)(-\bar{L}\otimes(B_cD_y)))w(t)\end{aligned} \tag{5.53}$$

$$z(t) = (I_m\otimes C_z)(I_m\otimes\mathfrak{T}_1^T)\varphi(t) \tag{5.54}$$

Substituting $\Gamma_1 = (I_m\otimes A)-(\bar{L}\otimes BD_cC)$, $\Gamma_2 = \bar{L}\otimes(B_cC)$, $\Gamma_3 = (I_m\otimes(BC_c))-(\bar{L}\otimes(BD_{2c}))$ and $\Gamma_4 = (I_m\otimes A_c)-(\bar{L}\otimes B_{2c})$ in (5.53) and using the Kronecker mixed product property, system (5.53)-(5.54) has the equivalent form

$$\begin{aligned}
\dot{\varphi}(t) = {} & ((I_m \otimes \mathfrak{T}_1 A \mathfrak{T}_1^T) - (\bar{L} \otimes \mathfrak{T}_1 B D_c C \mathfrak{T}_1^T) \\
& - (\bar{L} \otimes (\mathfrak{T}_2 B_c C \mathfrak{T}_1^T) + (I_m \otimes (\mathfrak{T}_1 B C_c \mathfrak{T}_2^T)) - (\bar{L} \otimes (\mathfrak{T}_1 B D_{2c} \mathfrak{T}_2^T)) \\
& + (I_m \otimes (\mathfrak{T}_2 A_c \mathfrak{T}_2^T)) - (\bar{L} \otimes (\mathfrak{T}_2 B_{2c} \mathfrak{T}_2^T)))\varphi(t) \\
& + (I_m \otimes (\mathfrak{T}_1 B_w) - (\bar{L} \otimes (\mathfrak{T}_1 B D_c D_y)) - (\bar{L} \otimes (\mathfrak{T}_2 B_c D_y)))w(t). && (5.55) \\
z(t) = {} & (I_m \otimes C_z \mathfrak{T}_1^T)\varphi(t) && (5.56)
\end{aligned}$$

Substituting $\mathfrak{T}_1 = \begin{bmatrix} I_n \\ 0_{n_c \times n} \end{bmatrix}$ and $\mathfrak{T}_2 = \begin{bmatrix} 0_{n \times n_c} \\ I_{n_c} \end{bmatrix}$ in equations (5.55) and (5.56) one has

$$\begin{aligned}
\dot{\varphi}(t) = {} & ((I_m \otimes \begin{bmatrix} A & 0_{n \times n_c} \\ 0_{n_c \times n} & 0_{n_c \times n_c} \end{bmatrix}) - (\bar{L} \otimes \begin{bmatrix} BD_cC & 0_{n \times n_c} \\ 0_{n_c \times n} & 0_{n_c \times n_c} \end{bmatrix}) - (\bar{L} \otimes \begin{bmatrix} 0_{n \times n} & 0_{n \times n_c} \\ B_cC & 0_{n_c \times n_c} \end{bmatrix}) \\
& + (I_m \otimes \begin{bmatrix} 0_{n \times n} & BC_c \\ 0_{n_c \times n} & 0_{n_c \times n_c} \end{bmatrix}) - (\bar{L} \otimes \begin{bmatrix} 0_{n \times n} & BD_{2c} \\ 0_{n_c \times n} & 0_{n_c \times n_c} \end{bmatrix}) \\
& + (I_m \otimes \begin{bmatrix} 0_{n \times n} & 0_{n \times n_c} \\ 0_{n_c \times n} & A_c \end{bmatrix}) - (\bar{L} \otimes \begin{bmatrix} 0_{n \times n} & 0_{n \times n_c} \\ 0_{n_c \times n} & B_{2c} \end{bmatrix}))\varphi(t) \\
& + (I_m \otimes \begin{bmatrix} B_w \\ 0_{n_c \times n_w} \end{bmatrix} - \bar{L} \otimes \begin{bmatrix} BD_cD_y \\ 0_{n_c \times n_w} \end{bmatrix} - (\bar{L} \otimes \begin{bmatrix} 0_{n \times n_w} \\ B_cD_y \end{bmatrix}))w(t) && (5.57) \\
z(t) = {} & (I_m \otimes \begin{bmatrix} C_z & 0_{n_z \times n_c} \end{bmatrix})\varphi(t) && (5.58)
\end{aligned}$$

Applying the Kronecker product property $(X_1 \otimes Y_1) + (X_1 \otimes Y_2) = (X_1 \otimes (Y_1 + Y_2))$, system (5.57)-(5.58) can be rewritten as

$$\begin{aligned}
\dot{\phi}(t) &= \left(I_m \otimes \underbrace{\begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}}_{\tilde{A}} + \bar{L} \otimes \underbrace{\begin{bmatrix} -BD_cC & -BD_{2c} \\ -B_cC & -B_{2c} \end{bmatrix}}_{\tilde{B}} \right)\phi(t) \\
&+ \left(I_m \otimes \underbrace{\begin{bmatrix} B_w \\ 0 \end{bmatrix}}_{\bar{B}_w} + \bar{L} \otimes \underbrace{\begin{bmatrix} -BD_cD_y \\ -B_cD_y \end{bmatrix}}_{\tilde{B}_w} \right) w(t). && (5.59) \\
z(t) &= (I_m \otimes \underbrace{\begin{bmatrix} C_z & 0_{n_z \times n_c} \end{bmatrix}}_{\tilde{C}})\varphi(t) && (5.60)
\end{aligned}$$

The aim of rewriting system (5.50)-(5.51) in the form (5.59)-(5.60) lies in decomposing the closed-loop system into m subsystems. For the further developments, the following lemma is enunciated.

Lemma 5.1 Assume that the graph associated with the Laplacian matrix $\tilde{L}$ has a spanning tree. Then all eigenvalues of $\bar{L}$ have positive real part and are the non-zero eigenvalues of $\tilde{L}$.

proof.

By the Lemma 2.1, the Laplacian matrix $\tilde{L}$ has eigenvalues with positive real part and one zero eigenvalue if and only if $\tilde{\mathbb{G}}$ has a spanning tree. Suppose that Assumption 5.1 is valid, i.e., there exists a spanning tree in $\tilde{\mathbb{G}}$.

The characteristic equation of $\tilde{L}$ is given by

$$\begin{aligned} 0 &= det(\tilde{L} - \lambda I_m) \\ 0 &= det \begin{bmatrix} -\lambda & 0_m^T \\ -\pi & \bar{L} - \lambda I_m \end{bmatrix} \end{aligned} \tag{5.61}$$

Applying the determinant rule for block matrices in (5.61), one has

$$\begin{aligned} 0 &= det(-\lambda)det(\bar{L} - \lambda I_m - \pi(\lambda)^{-1}0_m^T) \\ &= (-\lambda)det(\bar{L} - \lambda I_m), \end{aligned} \tag{5.62}$$

Equation (5.62) is satisfied when $\lambda = 0$ or $det(\bar{L} - \lambda I_m) = 0$. Observe that the equation $det(\bar{L} - \lambda I_m) = 0$ is the caracteristic equation of $\bar{L}$. By hypothesis, $\tilde{L}$ has zero as a simple eigenvalue, then the non-zero eigenvalues of $\tilde{L}$ are all eigenvalues of $\bar{L}$ that has positive real part.

■

In addition to Assumption 5.1, we consider the following assumption, which will be of great importance for this section.

Assumption 5.2 The matrix $\bar{L}$ is diagonalizable, that is, there exists a matrix $V \in \mathbb{C}^{m \times m}$ such that $V\bar{L}V^{-1} = \Lambda = diag(\lambda_1, ..., \lambda_m)$.

Now, defining the following variables $\zeta(t) = (V \otimes I)\varphi(t)$, $\bar{w}(t) = (V \otimes I)w(t)$, and

$\bar{z}(t) = (V \otimes I)z(t)$, from (5.59)-(5.60) one has

$$\begin{aligned} \dot{\zeta}(t) &= (I_m \otimes \tilde{A} + \Lambda \otimes \tilde{B})\zeta(t) + (I_m \otimes \tilde{B}_w + \Lambda \otimes \bar{B}_w)\bar{w}(t), \\ \bar{z}(t) &= (I_m \otimes \tilde{C})\zeta(t), \end{aligned} \tag{5.63}$$

which is equivalent to the set of local subsystems

$$\begin{aligned} \dot{\zeta}_i(t) &= (\tilde{\mathcal{A}} + \tilde{\mathcal{B}}K_y\tilde{\mathcal{C}}(\lambda_i))\zeta_i(t) + (\tilde{B}_w + \tilde{\mathcal{B}}K_y\tilde{\mathcal{D}}(\lambda_i))\bar{w}_i(t), \\ \bar{z}_i(t) &= \tilde{C}\zeta_i(t), \quad i = 1, ...m, \end{aligned} \tag{5.64}$$

with

$$\begin{aligned} \tilde{\mathcal{A}} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} A_c & B_c & B_{2c} \\ C_c & D_c & D_{2c} \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} 0 & B_u \\ I_{n_c} & 0 \end{bmatrix}, \\ \tilde{\mathcal{C}}(\lambda_i) &= \begin{bmatrix} 0_{(n_c \times n)} & I_{n_c} \\ -\lambda_i C_y & 0_{(q \times n_c)} \\ 0_{(n_c \times n)} & -\lambda_i I_{n_c} \end{bmatrix}, \quad \tilde{\mathcal{D}}(\lambda_i) = \begin{bmatrix} 0_{(n_c \times n_w)} \\ -\lambda_i D_y \\ 0_{(n_c \times n_w)} \end{bmatrix}. \end{aligned} \tag{5.65}$$

REMARK 5.5 Note that the decomposition applied in system (5.63) cannot be performed in Chapter 3 for system (3.63)-(3.65), since $C_g W$ and $\bar{L}$ would need to be simultaneously diagonalizable, but it is impossible since the multiplication $C_g W$ results in a rectangular matrix. In Chapter 4 for system 4.32, the decomposition cannot be applied directly since all vertices L_k of polytopic matrix $L(\alpha)$ need to be simultaneously diagonalizable.

The decomposed system (5.26) indicates that the HTVOFT problem can, in principle, be solved by a set of m LMIs with fewer scalar variables than (5.34) while keeping the decision variable K_y. Nonetheless, the set of LMIs still depends on the number of agents, which can be a problem for a large number of agents. Inspired by [56], we propose a design strategy that can allow the choice of the number of LMIs smaller than m. We consider the following assumption.

ASSUMPTION 5.3 The set of all eigenvalues λ_i of matrix $\bar{L}$ is inside a polytopic region $\mathcal{P}$ with $N \leq m$ vertices $\hat{\lambda}_k \in \mathbb{C}$ given by:

$$\mathcal{P} := \left\{ \lambda \in \mathbb{C} : \; \lambda = \sum_{k=1}^{N} \alpha_k \hat{\lambda}_k, \; \alpha \in \mathcal{U} \right\} \tag{5.66}$$

with $\mathcal{U} = \{\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N : \; \sum_{k=1}^{N} \alpha_k = 1, \; \alpha_k \geq 0\}$.

Remark 5.6 If the communication network is such that $\bar{L}$ has only real eigenvalues (which is the case of undirected networks), one can always choose $N = 2$ with vertices corresponding to the maximum and minimum eigenvalues of $\bar{L}$. For directed networks, $\bar{L}$ generally has complex eigenvalues, and $\mathcal{P}$ must have at least $N \geq 3$ vertices.

Next, let us relate the H_∞ performance of systems (5.59) and (5.63). Assuming a H_∞ performance γ_d for the transformed system (5.63), the following lemma allows us to find the H_∞ performance γ for the multi-agent system (5.1).

Lemma 5.2 If the auxiliary system (5.63) is asymptotically stable with H_∞ performance γ_d, the H_∞ performance of the multi-agent system (5.1) can be obtained by $\gamma = \gamma_d \kappa(V)$.

proof.

Since auxiliary system (5.63) is asymptotically stable with H_∞ performance γ_d, by definition of H_∞ norm the dissipation inequality holds

$$\int_0^\infty \|\bar{z}(t)\|^2 \mathrm{d}t < \gamma_d^2 \int_0^\infty \|\bar{w}(t)\|^2 \mathrm{d}t, \quad \forall \bar{w}(t) \in \mathcal{L}_2[0, \infty). \tag{5.67}$$

Observe that

$$\int_0^\infty \|\bar{z}(t)\|^2 < \gamma_d^2 \int_0^\infty \|\bar{w}(t)\|^2,$$
$$\int_0^\infty z^T(t)(V^* \otimes I_{n_w})(V \otimes I_{n_w})z(t) < \gamma_d^2 \int_0^\infty w^T(t)(V^* \otimes I)(V \otimes I_{n_w})w(t),$$
$$\int_0^\infty z^T(t)(V^*V \otimes I)z(t) < \gamma_d^2 \int_0^\infty w^T(t)(V^*V \otimes I_{n_w})w(t). \tag{5.68}$$

Observe that, V^*V is a hermitian matrix. Then, using the Rayleigh quotient inequality [43] for any vector $\varrho \neq 0$, one has

$$\lambda_{min}(H \otimes I) \leq \frac{\varrho^*(H \otimes I)\varrho}{\varrho^*\varrho} \leq \lambda_{max}(H \otimes I).$$

and the fact that $\lambda_{max}(X \otimes I) = \lambda_{max}(X)$ and $\lambda_{min}(X \otimes I) = \lambda_{min}(X)$ [49], one has

$$\lambda_{min}(V^*V) \int_0^\infty z^T(t)z(t) \leq \int_0^\infty z^T(t)(V^*V \otimes I)z(t), \tag{5.69}$$
$$\int_0^\infty w(t)^T(V^*V \otimes I)w(t) \leq \lambda_{max}(V^*V) \int_0^\infty w^T(t)w(t). \tag{5.70}$$

Then, by (5.69) and (5.70) one has

$$\lambda_{min}(V^*V)\int_0^\infty z^T(t)z(t) < \gamma_d^2\lambda_{max}(V^*V)\int_0^\infty w^T(t)w(t) \quad (5.71)$$

$$\int_0^\infty \|z(t)\|^2 < \underbrace{\gamma_d^2\frac{\lambda_{max}(V^*V)}{\lambda_{min}(V^*V)}}_{\gamma^2}\int_0^\infty \|w(t)\|^2. \quad (5.72)$$

As system (5.63) is asymptotically stable with H_∞ performance γ_d, then from (5.72) system (5.59)-(5.60) is asymptotically stable with H_∞ performance $\gamma = \gamma_d\kappa(V)$, where $\kappa(V) = \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)}$ is the condition number of V [43]. Therefore, the multi-agent system (5.1) reach HTVOFT with H_∞ performance $\gamma = \gamma_d\kappa(V)$. ■

REMARK 5.7 The choice of matrix V satisfying Assumption 5.2 is not unique, implying different values for γ. A V matrix containing the normalized eigenvectors of $\bar{L}$ may be a good choice. However, following [7], it is possible to compute a matrix V_n with $\kappa(V_n) < \kappa(V)$, solving the problem

$$\begin{aligned} &\text{Minimize}: \kappa_o \\ &\text{Subject to}: P \in \mathbb{R}^{m\times m} \text{ and diagonal, } P \succ 0 \\ &\qquad I_m \preceq V^*PV \preceq \kappa_o I_m \end{aligned} \quad (5.73)$$

If inequality (5.73) is feasible one has $V_n = P^{1/2}V$, where $V_n\bar{L}V_n^{-1} = \Lambda$, and $\kappa(V_n) = \sqrt{\kappa_o}$.

In the following theorem, conditions are derived to find a H_∞ performance for (5.63) and to design K_y such that (5.63) is asymptotically stable.

Theorem 5.4

Let $\mathbb{K}_x$ be a given matrix such that $\tilde{\mathcal{A}}+\tilde{\mathcal{B}}\mathbb{K}_x$ is Hurwitz. With the Assumption 5.1 and considering the vertices $\hat{\lambda}_k$ of $\mathcal{P}$, if there exist real matrices G, Z and complex matrices $P = P^* \succ 0$, $X_{p,k}$, for $p = 1,\dots,8$, such that the LMIs hold for $k = 1,\dots,N$

$$\Xi(\hat{\lambda}_k) := \begin{bmatrix} \tilde{\Phi}_{1,k} & \star \\ \tilde{\Phi}_{2,k}(\hat{\lambda}_k) & \tilde{\Phi}_{3,k}(\hat{\lambda}_k) \end{bmatrix} \prec 0, \quad k = 1,\dots,N, \quad (5.74)$$

where

$$\tilde{\Phi}_{1,k} = \begin{bmatrix} \tilde{\Psi}_{1,k} + \mathcal{C}_z^T \mathcal{C}_z & \star & \star \\ \tilde{\Psi}_{2,k} & -X_{3,k} - X_{3,k}^* & \star \\ \tilde{\Psi}_{3,k} & -X_{5,k} - X_{4,k}^* & -He\{X_{6,k}\} \end{bmatrix},$$

$$\tilde{\Phi}_{2,k}(\hat{\lambda}_k) = \begin{bmatrix} \tilde{\Psi}_{5,k} & \tilde{\Psi}_{6,k} & \tilde{\Psi}_{7,k} \\ \tilde{\mathcal{B}}^T X_{1,k}^* & \tilde{\mathcal{B}}^T X_{3,k}^* & \tilde{\Psi}_{4,k}(\hat{\lambda}_k) \end{bmatrix},$$

$$\tilde{\Phi}_{3,k}(\hat{\lambda}_k) = \begin{bmatrix} \tilde{\Psi}_{8,k} & \star \\ \hat{\mathcal{B}}^T X_{7,k}^* + Z\tilde{\mathcal{D}}(\hat{\lambda}_k) & -He\{G\} \end{bmatrix},$$

$$\begin{aligned}
\tilde{\Psi}_{1,k} &= (\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x)^T X_{1,k}^* + X_{1,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + He\{X_{2,k}\}, \\
\tilde{\Psi}_{2,k} &= P + X_{3,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + X_{4,k} - X_{1,k}^*, \\
\tilde{\Psi}_{3,k} &= X_{5,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + X_{6,k} - X_{2,k}^*, \\
\tilde{\Psi}_{4,k}(\hat{\lambda}_k) &= \tilde{\mathcal{B}}^T X_{5,k}^* + Z\tilde{\mathcal{C}}(\hat{\lambda}_k) - G\mathbb{K}_x, \\
\tilde{\Psi}_{5,k} &= X_{7,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + X_{8,k} + \tilde{B}_w^T X_{1,k}^*, \\
\tilde{\Psi}_{6,k} &= -X_{7,k} + \tilde{B}_w^T X_{3,k}^*, \tilde{\Psi}_{7,k} = -X_{8,k} + \tilde{B}_w^T X_{5,k}^*, \\
\tilde{\Psi}_{8,k} &= -\mu I + He\{\tilde{B}_w X_{7,k}\},
\end{aligned}$$

then the protocol (5.3)-(5.5) with $K_y = G^{-1}Z$ solves Problem 5.1 with H_∞ cost given by $\gamma = \sqrt{\mu}\kappa(V)$.

proof.

First, observe that if $\lambda_i \in \mathcal{P}$, for $i = 1, ..., m$, then $\lambda_i = \sum_{k=1}^N \alpha_{k,i}\hat{\lambda}_k$, with $\sum_{k=1}^N \alpha_{k,i} = 1$, $\alpha_{k,i} \geq 0$. From linearity, inequality (5.74) implies

$$\Xi(\lambda_i) = \sum_{k=1}^N \alpha_{k,i} \begin{bmatrix} \tilde{\Phi}_{1,k} & \star \\ \tilde{\Phi}_{2,k}(\hat{\lambda}_k) & \tilde{\Phi}_{3,k}(\hat{\lambda}_k) \end{bmatrix} \prec 0, \text{ for } i = 1, ..., m. \tag{5.75}$$

Following similar steps of the proof of Theorem 5.2, one shows that $\Xi(\lambda_i) \prec 0$ guarantees

$$\Theta_i = \begin{bmatrix} \bar{A}_{cl,i}^T P + P\bar{A}_{cl,i} + \tilde{C}^T \tilde{C}_z & P\bar{B}_{cl,i} \\ \bar{B}_{cl,i}^T P & -\gamma_d^2 I \end{bmatrix} \prec 0, \tag{5.76}$$

with $\bar{A}_{cl,i} = \tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x + \tilde{\mathcal{B}}S_i$ and $\bar{B}_{cl,i} = \mathcal{B}_1 + \tilde{\mathcal{B}}S_{2,i}$, where $S_i = \mathbb{K}_y\tilde{\mathcal{C}}(\lambda_i) - \mathbb{K}_x$ and $S_{2,i} = \mathbb{K}_y\tilde{\mathcal{D}}(\lambda_i)$, according to the system parameters (5.64). Consequently one has that $P \succ 0$ such that $\Theta_i \prec 0$ implies $PA_{cl,i} + A_{cl,i}^T P \prec 0$ and $\dot{V}_i(t) < 0$ for the Lyapunov function $V_i(t) = \zeta_i^T(t)P\zeta_i(t) > 0$, $t \geq 0$, and $w_i(t) = 0$. Therefore, $\zeta_i(t) \to 0$ for any initial condition and the requirement (1) in Problem 5.1 is verified. Also, $P \succ 0$ and $\Theta_i \prec 0$ implies that

$$\int_0^\tau \|\bar{z}_i(t)\|^2 \mathrm{d}t < \gamma_d^2 \int_0^\tau \|\bar{w}_i(t)\|^2 \mathrm{d}t. \tag{5.77}$$

Since $||\varrho(t)||^2 = \varrho^T(t)\varrho(t) = \sum_{i=1}^m \varrho_i^T(t)\varrho_i(t) = \sum_{i=1}^m ||\varrho_i(t)||^2$, for any vector $\varrho(t)$ with real inputs, one has that if inequality (5.77) holds, then

$$\int_0^\tau \sum_{i=1}^m \|\bar{z}_i(t)\|^2 \mathrm{d}t < \gamma_d^2 \int_0^\tau \sum_{i=1}^m \|\bar{w}_i(t)\|^2 \mathrm{d}t. \tag{5.78}$$

By Lemma (5.2) one has

$$\int_0^\tau \|z(t)\|^2 \mathrm{d}t < \underbrace{\gamma_d^2 \sqrt{\kappa(V)}}_{\gamma^2} \int_0^\tau \|w(t)\|^2 \mathrm{d}t. \tag{5.79}$$

Note that (5.79) is valid for $0 \leq t \leq \tau$ and for all $\tau > 0$. Taking the limit as $\tau \to \infty$ in (5.79) one has that (5.7) holds for any $w \in \mathcal{L}_2[0, \infty)$ and $0 \leq t < \infty$, and therefore, the requirement (2) in Problem 5.1 is also attained. ■

REMARK 5.8 In contrast to Theorem 5.2, Theorem 5.4 does not require the complete information of the network comprised in matrix $\bar{L}$, which may be unavailable to the designer. Instead, only the vertices of $\mathcal{P}$ (usually the extreme values of λ_i) are necessary to design (3.2). However, note that $\gamma = \sqrt{\mu\kappa(V)}$ requires the knowledge of V, which can be a drawback if the computation of the H_∞ cost is necessary. For the particular case where $\bar{L}$ is a normal matrix, the H_∞ cost does not depend on topology because $\kappa(V) = 1$ (V is a unitary matrix and $V^*V = I$).

REMARK 5.9 Table 5.2 shows the number of scalar variables and LMI rows in Theorems 5.2 and 5.4. The table shows that the design conditions in Theorem 5.4 do not depend explicitly on the number m of agents but on the number N of vertices of $\mathcal{P}$. Therefore, an appropriate choice of a polytope with $N << m$ yields less computational burden than considering λ_i, for $i = 1, ..., m$, in the design. This is particularly critical in large networks.

The following subsection shows that the Assumption 5.2 is unnecessary when $w(t) = 0$.

Table 5.2 – Number of scalar variables and LMI rows in Theorems 5.2 (T5.2) and 5.4 (T5.4).

	Scalar variables	LMI rows
T5.2	$m^2\big[7(n+n_c)^2$ $+2(n_w n+n_w n_c)\big]+1$ $+(n_c+s)(3n_c+q+s)$	$m(3n+4n_c+n_w+s)$
T5.4	$N\big[7(n+n_c)^2$ $+2(n_w n+n_w n_c)\big]+1$ $+(n_c+s)(3n_c+q+s)$	$N(3n+4n_c+n_w+s)$

5.4.1 Design of TVOFT protocols for non-disturbed agents

If $w(t)=0, \forall t$, the H_∞ performance is not required, and the design analysis may be extended for a larger class of communication topologies. Assumption 5.2 is no longer required.

From Assumption 5.1 and Lemma 5.1, we can state that the matrix $\bar{L}$ is non-singular and it is possible use the transformation $\mathcal{V}^*\bar{L}\mathcal{V}=\mathcal{J}$, such that $\mathcal{V}$ is a unitary matrix and $\mathcal{J}\in\mathbb{C}^{m\times m}$ a triangular superior matrix with $\mathcal{J}_{ii}=\lambda_i$ the eigenvalues of $\bar{L}$ [50, Theorem 2.3.1], and defining the variable $\varsigma(t)=(\mathcal{V}^*\otimes I_{n+n_c})\varphi(t)$ the following system can be obtained

$$\begin{aligned}(\mathcal{V}^*\otimes I_{n+n_c})\,\dot{\phi}(t)&=(\mathcal{V}^*\otimes I_{n+n_c})\left(\mathcal{V}\otimes\underbrace{\begin{bmatrix}A & BC_c\\ 0 & A_c\end{bmatrix}}_{\bar{A}}\right)(\mathcal{V}^*\otimes I_{n+n_c})\,\phi(t)\\&\quad+(\mathcal{V}^*\otimes I_{n+n_c})\left(\bar{L}\mathcal{V}\otimes\underbrace{\begin{bmatrix}-BD_cC & -BD_{2c}\\ -B_cC & -B_{2c}\end{bmatrix}}_{\bar{B}}\right)(\mathcal{V}^*\otimes I_{n+n_c})\,\phi(t),\end{aligned}\tag{5.80}$$

it can be rewritten as,

$$\dot{\varsigma}(t)=\underbrace{\left(I_m\otimes\underbrace{\begin{bmatrix}A & BC_c\\ 0 & A_c\end{bmatrix}}_{\bar{A}}+\mathcal{J}\otimes\underbrace{\begin{bmatrix}-BD_cC & -BD_{2c}\\ -B_cC & -B_{2c}\end{bmatrix}}_{\bar{B}}\right)}_{\bar{\mathbb{A}}}\varsigma(t).\tag{5.81}$$

To decouple the closed-loop system (5.81) in various closed-loop systems, observe that as $\mathcal{J}$ are triangular superior matrices, only its diagonal elements influence in the stabilizability of the system (5.81). Therefore, defining $\Lambda=diag\{\lambda_i\}$ one has that $\bar{\mathbb{A}}$ is Hurwitz if and only if $\left(I_m\otimes\bar{A}+\Lambda\otimes\bar{B}\right)$ is Hurwitz [32].

Observe that $\left(I_m\otimes\bar{A}+\Lambda\otimes\bar{B}\right)$ is Hurwitz if and only if $\left(\bar{A}+\lambda_i\bar{B}\right)$ for $i=1,...,m$

are Hurwitz. Finally, for $i = 1, ..., m$ one has

$$\bar{A} + \lambda_i \bar{B} = \tilde{\mathcal{A}} + \tilde{\mathcal{B}} \mathbb{K}_y \tilde{\mathcal{C}}_i(\lambda_i), \tag{5.82}$$

with,

$$\tilde{\mathcal{A}} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \mathbb{K}_y = \begin{bmatrix} A_c & B_c & B_{2c} \\ C_c & D_c & D_{2c} \end{bmatrix}$$
$$\tilde{\mathcal{C}}(\lambda_i) = \begin{bmatrix} 0_{(n_c \times n)} & I_{n_c} \\ -\lambda_i C_y & 0_{(q \times n_c)} \\ 0_{(n_c \times n)} & -\lambda_i I_{n_c} \end{bmatrix}, \tilde{\mathcal{B}} = \begin{bmatrix} 0 & B_u \\ I_{n_c} & 0 \end{bmatrix}.$$

With system (5.82), one can derive the condition for TVOFT ($w(t) = 0$, $\forall t$), presented in the following theorem.

Theorem 5.5

Let $\mathbb{K}_x$ a given matrix such that $\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x$ is Hurwitz, $\hat{\lambda}_k$ the vertices of a polytopic region $\mathcal{P}$ containing the eigenvalues of $\bar{L}$, and $h_i(t)$, for $i = 1, ..., m$, satisfying condition (5.13). The TVOFT is reached by the MAS (5.1) with the compensation signal (5.12) and the n_c-order protocol (3.2), if there exist complex matrices, $P = P^* \succ 0$, $X_{\ell,k}$, $\ell = 1, \ldots, 6$, and real matrices G, Z such that the following LMIs hold for $k = 1, \ldots, N$,

$$\tilde{\Xi}(\hat{\lambda}_k) = \begin{bmatrix} \tilde{\Psi}_{1,k} & \star & \star & \star \\ \tilde{\Psi}_{2,k} & -X_{3,k} - X_{3,k}^* & \star & \star \\ \tilde{\Psi}_{3,k} & -X_{5,k} - X_{4,k}^* & -X_{6,k} - X_{6,k}^* & \star \\ \tilde{\mathcal{B}}^T X_{1,k}^* & \tilde{\mathcal{B}}^T X_{3,k}^* & \tilde{\Psi}_{4,k} & -G - G^T \end{bmatrix} \prec 0, \tag{5.83}$$

with

$$\tilde{\Psi}_{1,k} = (\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x)^T X_{1,k}^* + X_{1,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + X_{2,k} + X_{2,k}^*,$$
$$\tilde{\Psi}_{2,k} = P + X_{3,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + X_{4,k} - X_{1,k}^*,$$
$$\tilde{\Psi}_{3,k} = X_{5,k}(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x) + X_{6,k} - X_{2,k}^*,$$
$$\tilde{\Psi}_{4,k} = \tilde{\mathcal{B}}^T X_{5,k}^* + Z\tilde{\mathcal{C}}(\hat{\lambda}_k) - G\mathbb{K}_x.$$

Then, the protocol (5.3)-(5.5) solves the Problem 5.1(1) with $K_y = G^{-1}Z$.

proof.

First, observe that $\lambda_i \in \mathcal{P}$, for $i = 1, ..., m$, then $\lambda_i = \sum_{k=1}^{N} \alpha_{k,i}\hat{\lambda}_k$. From linearity, $\tilde{\Xi}(\hat{\lambda}_k) \prec 0$ implies $\tilde{\Xi}(\hat{\lambda}_i) = \sum_{k=1}^{N} \alpha_{k,i}\tilde{\Xi}(\hat{\lambda}_k) \prec 0$.

Defining $\tilde{\mathcal{A}}_x = \tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_x$ and $S_i = \mathbb{K}_y\tilde{\mathcal{C}}(\lambda_i) - \mathbb{K}_x$, and following similar steps of the proof of Theorem 5.3, one shows that $\tilde{\Xi}(\lambda_i) \prec 0$ guarantees

$$\begin{aligned} 0 \;&\succ\; \begin{bmatrix} I \\ \tilde{\mathcal{A}}_x + \tilde{\mathcal{B}}S_i \\ I \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ \tilde{\mathcal{A}}_x + \tilde{\mathcal{B}}S_i \\ I \end{bmatrix} \\ &= \left(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_y\tilde{\mathcal{C}}_i(\lambda_i)\right)^T P + P\left(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}\mathbb{K}_y\tilde{\mathcal{C}}_i(\lambda_i)\right). \end{aligned} \tag{5.84}$$

Note that inequality (5.84) is the Lyapunov stability condition, i.e., (5.84) holds if, and only if, (5.82) is Hurwitz stable with gain K_y. ■

REMARK 5.10 Theorem 5.5 presents some advantages over the current design conditions from the literature that solve the TVOFT problem for disturbance-free agents. For instance, observe that [74, Theorem 2], which for a single leader becomes a TVOFT problem, designs only static protocols for non-disturbed agents and requires solving a Riccati equation with restrictions, which is harder to solve than LMIs.

REMARK 5.11 It is worth mentioning that Theorems 5.2, 5.3, 5.4, and 5.5 can be adapted to design protocols that do not exchange controller states over the network, avoiding excessive use of the network bandwidth. For this, eliminate the gains B_{2c} and D_{2c} in (3.2) and adopt the following matrices in (5.27), (5.30), and (5.65):

$$\mathcal{C} := \begin{bmatrix} 0 & I_{mn_c} \\ -\bar{L} \otimes C_y & 0 \end{bmatrix}, \mathcal{D} := \begin{bmatrix} 0 \\ -\bar{L} \otimes D_y \end{bmatrix}, K_y := \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix},$$

$$\mathcal{T}_2 = \begin{bmatrix} I_m \otimes \mathfrak{I}_{21} & I_m \otimes \mathfrak{I}_{22} \end{bmatrix},$$

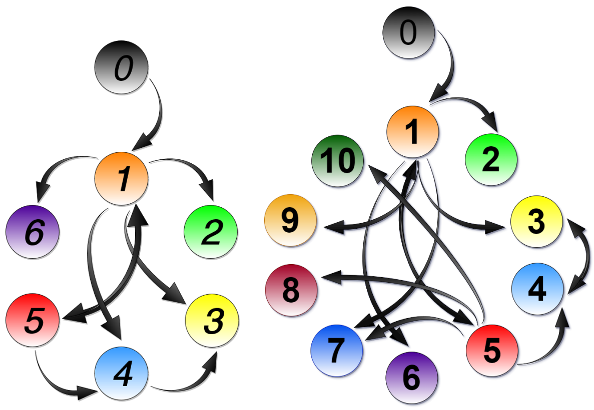
and

$$\tilde{\mathcal{C}}(\lambda_i) := \begin{bmatrix} 0_{(n_c \times n)} & I_{n_c} \\ -\lambda_i C_y & 0_{(q \times n_c)} \end{bmatrix}, \quad \tilde{\mathcal{D}}(\lambda_i) = \begin{bmatrix} 0_{(n_c \times n_w)} \\ -\lambda_i D_y \end{bmatrix}.$$

REMARK 5.12 If $h(t) = 0$, Theorems 5.2, 5.3, 5.4, and 5.5 give new protocol design solutions for the leader-following consensus problem, which is also treated in [25] and Chapter 3. The design conditions of [25] are restricted to only static protocols and are based on a Ricatti equation. Although [53] also considers any order controller using the LMI approach, the conditions here are less conservative and have smaller H_∞ cost (See section 5.5.1). For more detailed comparisons of the proposed solutions with the results of [25] and [53], see Case 1 in Section 5.5.

5.5 NUMERICAL EXAMPLES

We present two numerical examples to show the effectiveness of the proposed conditions. Theorem 5.2 and Theorem 5.4 are implemented to design static, reduced- and full-order protocols. The algorithms were implemented in the Python 3.11.4 software employing library CVXPY [10] and MOSEK [4] packages.



(a) Topology 1. (b) Topology 2.

Figure 5.1 – Directed graphs that model the agents' communication. The leader is identified as the zero agent.

For the first two cases presented below, consider agents (5.1) subjected to external disturbances with matrices adapted from [74]:

$$\begin{gathered} A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_y = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}, \\ B_w = [1\ 1]^T, \quad D_y = [0\ 1]^T, \quad C_z = I. \end{gathered} \tag{5.85}$$

and the leader (5.2) with the same agents' parameters A and B_u, as defined.

5.5.1 Case 1: H_∞ leader-following

Considering the directed connected network in Fig. 5.1a, we design a leader-following protocol ($h(t) = 0$, see Remark 5.12) for the multi-agent system with parameters (5.85) and H_∞ performance obtained by the solution of the condition in Theorem 5.2. For a more suitable comparison with [53], we consider that the protocol does not exchange the controller's states over the network (see Remark 5.11). We assume that the graph in Fig. 5.1a has all weights equal to one and $\pi_1 = 3$, $\pi_i = 0$, $i = 2, \ldots, 6$, forming a non-diagonalizable matrix $\bar{L}$. First, gain K_x is obtained using Theorem 3.3 by making a line search on the values of β in the interval $[0.01, 1]$, yielding $\beta_{n_c=1} = \beta_{n_c=2} = 0.02$. Then, Theorem 5.2 following Remark 5.11 yields protocol gains for $n_c = 0, 1, 2$, with H_∞ performance shown in Table 5.3. For each n_c, with the same corresponding K_x, [53, Theorem 4] provides H_∞ performance indices higher than the ones obtained by Theorem 5.2, as shown in Table 5.3. Since $\bar{L}$ is not a normal matrix, the algorithm proposed by [25, Theorem 2] cannot be applied.

Table 5.3 – Values of the H_∞ performance γ.

	$n_c = 0$	$n_c = 1$	$n_c = 2$
Theorem 5.2	4.7638	4.7211	4.7160
[53, Theorem 4]	8.8614	8.8296	8.8169

Figures 5.2 and 5.3 show the temporal response of the agents for the same initial conditions using reduced-order protocols ($n_c = 1$) designed by [53, Theorem 4] and Theorem 5.2, respectively. The disturbances are chosen as $w_i(t) = \sin\left(\frac{3}{i}t\right)$, for $t \in [5, 10]$, and $w_i(t) = 0$, otherwise. Observe that agents converge in Figure 5.3 faster than Figure 5.2.

5.5.2 Case 2: H_∞ time-varying output formation tracking

We now present the solution for a scenario where the same protocol can be used for two (or more) different topologies with different numbers of agents, as illustrated in Fig. 5.1. For both graphs, the pinning gains are defined as $\pi_1 = 1$ and $\pi_i = 0$, otherwise. The graph in Fig. 5.1a has connection weights $a_{15} = a_{21} = a_{31} = a_{45} = 1$, $a_{34} = 0.5$ and $a_{41} = a_{51} = a_{61} = 1.5$. The graph in Fig. 5.1b has connection weights 0.5 for a_{34} and a_{43}, 1.5 for a_{51}, a_{85}, a_{91} and value 1 for all other depicted connections. Both matrices $\bar{L}$ associated with the graphs in Fig. 5.1 are diagonalizable and not normal with all eigenvalues belonging to a polytope $\mathcal{P}$ formed by vertices $\hat{\lambda}_1 = 0.5 - 0.5i$, $\hat{\lambda}_2 = 3 - 0.5i$, $\hat{\lambda}_3 = 3 + 0.5i$, and $\hat{\lambda}_4 = 0.5 + 0.5i$, i.e., we need to solve only four conditions in Theorem 5.4 independently of the number of agents (see Remark 5.9).

The gains $\mathbb{K}_x$ used as input in Theorem 5.4 are designed by Theorem 3.3, with $\beta_{n_c=1} = 0.22$, and $\beta_{n_c=2} = 0.05$ obtained from a line search in the interval $[0.01, 1]$. Solving Theorem 5.4 it is obtained $\gamma_{d,n_c=0} = 2.9072$, $\gamma_{d,n_c=1} = 2.3784$ and $\gamma_{d,n_c=2} = 2.2736$, providing the

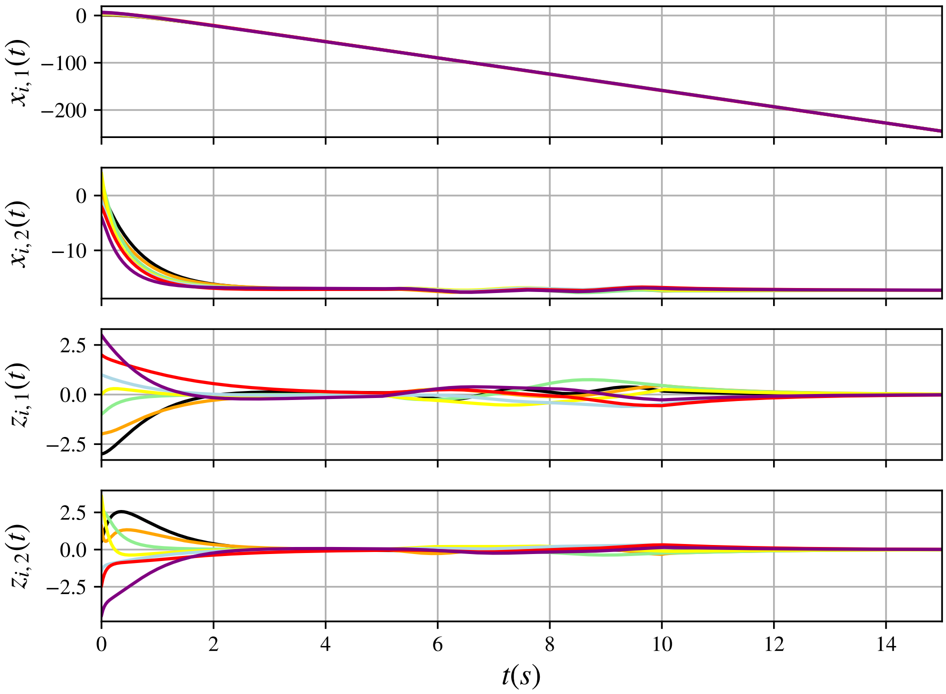


Figure 5.2 – Trajectories of $x_i(t)$ and controlled outputs $z_i(t)$, for the solution of condition in [53, Theorem 4] for a reduced-order protocol ($n_c = 1$).

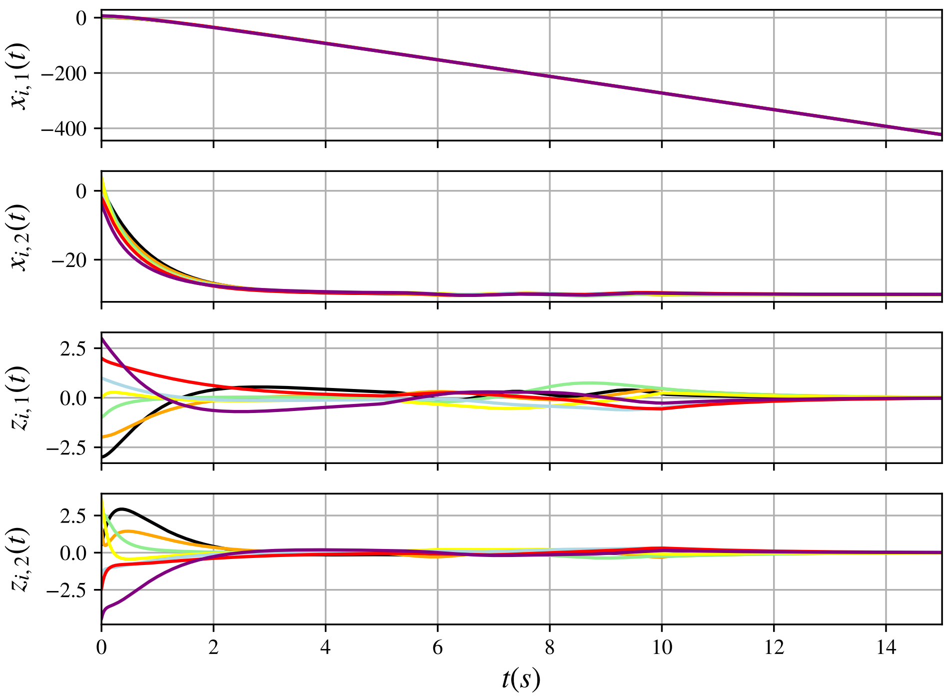


Figure 5.3 – Trajectories of $x_i(t)$ and controlled outputs $z_i(t)$, for the solution of condition in Theorem 5.2 for a reduced-order protocol ($n_c = 1$).

following protocol gains for $n_c = 0$

$$D_c = \begin{bmatrix} 2.4037 & 0.2754 \end{bmatrix}, \tag{5.86}$$

for $n_c = 1$ one has

$$\begin{aligned} A_c &= 0.1226, \quad B_c = \begin{bmatrix} 0.2456 & -0.0039 \end{bmatrix}, \quad B_{2c} = 3.8381, \\ C_c &= 27.8611, \quad D_c = \begin{bmatrix} 2.3581 & 0.037 \end{bmatrix}, \quad D_{2c} = \begin{bmatrix} 32.788 \end{bmatrix}. \end{aligned} \tag{5.87}$$

for $n_c = 2$ one has

$$\begin{aligned} A_c &= \begin{bmatrix} -1.1628 & -0.1763 \\ -15.7391 & -1.0742 \end{bmatrix}, \ B_c = \begin{bmatrix} -0.0071 & 0.0005 \\ 0.0276 & -0.0018] \end{bmatrix}, \\ B_{2c} &= \begin{bmatrix} 3.1467 & -0.3092 \\ -12.6103 & 1.1218 \end{bmatrix}, C_c = \begin{bmatrix} -1977.9633 \\ -66.6008 \end{bmatrix}, \\ D_c &= \begin{bmatrix} 2.3529 & 0.0216 \end{bmatrix}, D_{2c} = \begin{bmatrix} -1723.3713 & -13.5457 \end{bmatrix}. \end{aligned} \tag{5.88}$$

Let $B_{u\underline{||}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $B_{u\underline{\perp}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_{y||} = \begin{bmatrix} 0.5 & -0.25 \\ 0.5 & 0.25 \end{bmatrix}$, and the time-varying formation function $h_i(t) = \begin{bmatrix} 10 \sin\left(t + \frac{(i-1)\pi}{5}\right) \\ 20 \cos\left(t + \frac{(i-1)\pi}{5}\right) \end{bmatrix}$, similarly as in [74], satisfying the condition (5.13) of Theorem 5.1. The signal $\delta_i(t) = 5\cos\left(t + \frac{(i-1)\pi}{5}\right) - 5\sin\left(t + \frac{(i-1)\pi}{5}\right)$, for $i = 1, .., m$, is obtained using (5.12). The disturbances are chosen as $w_i(t) = \sin\left(\frac{3}{i}t\right)$, for $t \in [25, 30]$, and $w_i(t) = 0$, otherwise. The leader input is given by $u_0(t) = 2\sin(t/2)$.

Considering h_i, δ_i, $i = 1, .., 10$, and $\hat{\lambda}_k$, $k = 1, \ldots, 4$, Theorem 5.4 provides robust gains for controller (3.2) for different controller orders $n_c = 0, 1, 2$. The designed gains are suitable for solving Problem 5.1 (1) for both graphs in Fig. 5.1a and 5.1b, but solve Problem 5.1 (2) with different costs. Fig. 5.4 illustrates the time-domain simulation for the topology of Fig. 5.1b ($m = 10$), showing that the output errors, controlled outputs, and controllers' states converge to zero when the disturbance is zeroed. The H_∞ costs associated with the graphs in Fig. 5.1a and 5.1b are obtained following Remark 5.7 for determination of matrices V_a and V_b, and are presented in Table 5.4.

The H_∞ costs associated with the graphs in Fig.5.1a and 5.1b are obtained using numpy.linalg.eig() function in Python, we compute a V matrix such that $V\bar{L}V^{-1} = \Lambda$ for each topology and then obtaining $\kappa(V_a) = 4.8687$ and $\kappa(V_b) = 5.9485$, for $\bar{L}$ associated with the graphs in Figure 5.1a and Figure 5.1b, respectively. Solving condition (5.73) was found $\kappa(V_{n,a}) = 4.3649$ and $\kappa(V_{n,b}) = 5.7192$. Table 5.4 shows the H_∞ performance γ for each topology and protocol order.

Table 5.4 – Values of the H_∞ performance γ for Theorem 5.4.

	$n_c = 0$	$n_c = 1$	$n_c = 2$
Topology 1 (Fig 5.1a)	12.6896	11.5797	11.0695
Topology 2 (Fig 5.1b)	16.6268	13.6025	13.0032

For spatial visualization, consider the simulation depicted in Fig. 5.5, which assumes a system with 10 agents (topology in Fig. 5.1b) designed to reach a decagonal formation. Fig. 5.5 shows the agents reaching the output formation guided by the leader with the protocol (3.2)-(3.49) and the gains (5.86), which evidences the effectiveness of the Theorem 5.4.

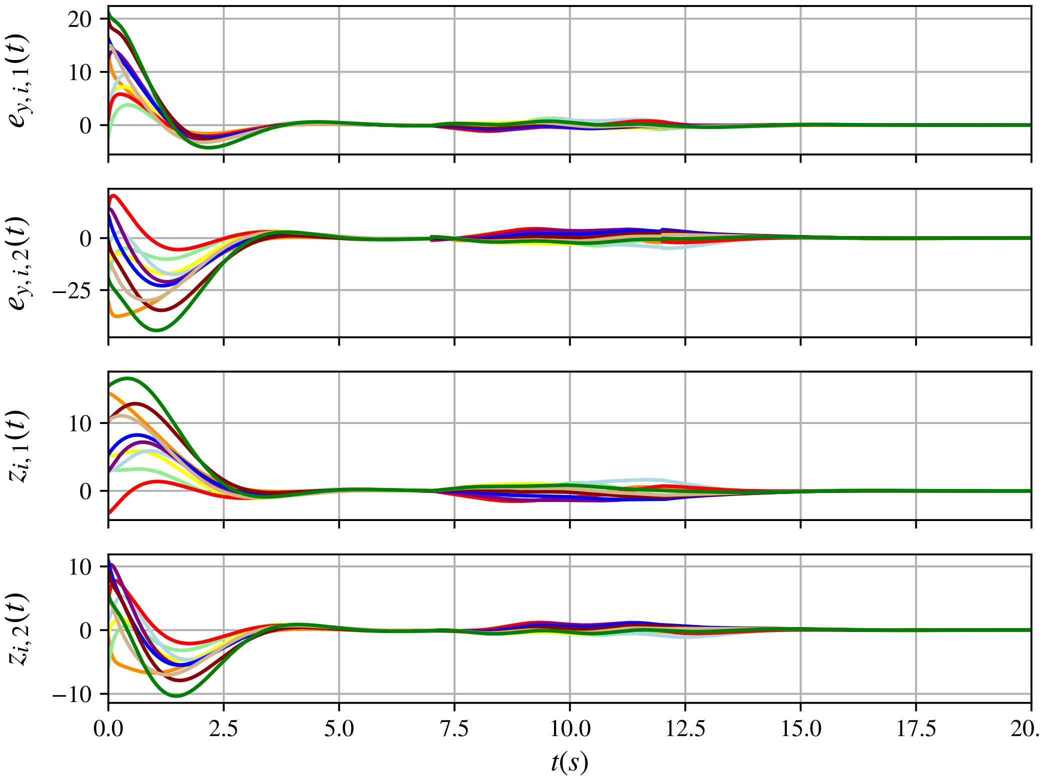


Figure 5.4 – Trajectories of output errors $e_{y,i}(t)$ and controlled outputs $z_i(t)$.

5.5.3 Case 3: time-varying output formation tracking for large agents number

This section presents an example with 50 agents and one leader. Agents (5.1) have same parameters in (5.85), but considering $B_w = 0$, $D_y = 0$ and $C_z = 0$. As defined, Leader (5.2) has the agents' parameter A, B_u. The non-zero graph weights used in this section are presented in Table 5.5, and the pinning gains are defined as $\pi_1 = 0.8799$ and $\pi_i = 0$ otherwise. The matrix $\bar{L}$ obtained has eigenvalues with real part in the interval $[0.2741, 2.6721]$ and

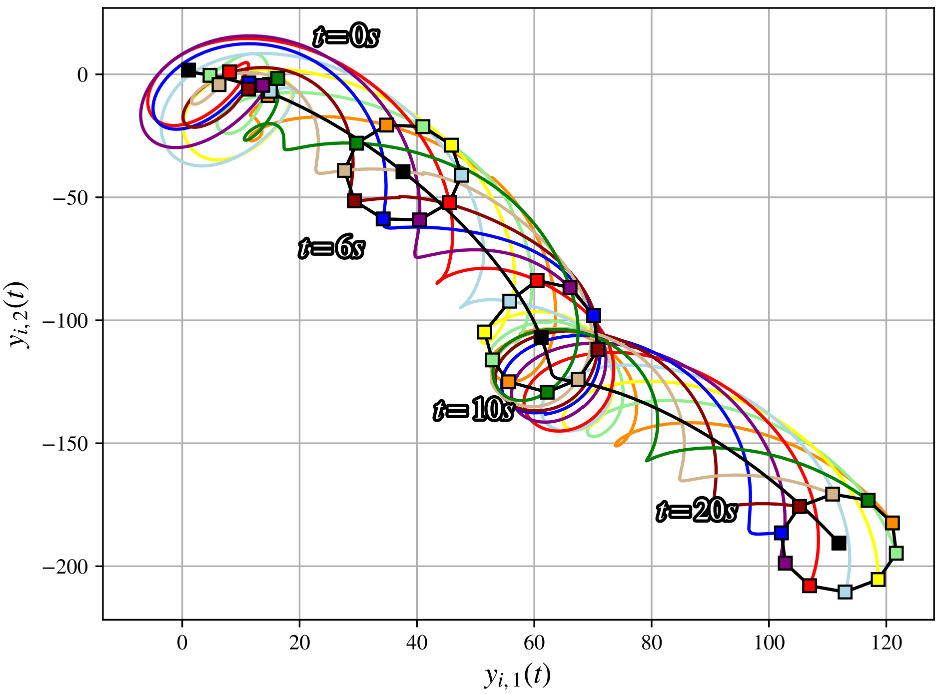


Figure 5.5 – Leader outputs $y_0(t)$, and followers outputs $y_i(t)$ in the plane reaching a decagonal formation. Between 7s and 12s, the disturbances act in agent dynamics ($w(t) \neq 0$), hindering the agents' formation. Finally, at 20s, the agents reach the formation. The color identification of the leader and the followers follows convention in Fig 5.1b.

imaginary part in the interval $[0, 1.1297]$, then we choose a polytope $\mathcal{P}$ formed by vertices $\hat{\lambda}_1 = 0.25 - 1.35i$, $\hat{\lambda}_2 = 0.25 + 1.35i$, $\hat{\lambda}_3 = 2.7 + 1.35i$, and $\hat{\lambda}_4 = 2.7 - 1.35i$.

By solving Theorem 3.3 with $\beta = 0.4$, we compute the $\mathbb{K}_x$ gain. Using $\mathbb{K}_x$ as input in Theorem 5.5, its solution provides the following protocol gains for $n_c = 0$

$$D_c = \begin{bmatrix} 2.2663 & 0.3761 \end{bmatrix}, \tag{5.89}$$

for $n_c = 1$ one has

$$\begin{aligned} A_c &= -3.3507, \quad B_c = \begin{bmatrix} 0.0973 & -0.016 \end{bmatrix}, \quad B_{2c} = -0.0209, \\ C_c &= \begin{bmatrix} 0.0238 \end{bmatrix}, \quad D_c = \begin{bmatrix} 2.1493 & 0.3617 \end{bmatrix}, \quad D_{2c} = \begin{bmatrix} 0.3237 \end{bmatrix}. \end{aligned} \tag{5.90}$$

for $n_c = 2$ one has

$$A_c = \begin{bmatrix} -3.3189 & 0.0231 \\ -0.011 & -3.3505 \end{bmatrix}, \; B_c = \begin{bmatrix} -0.0011 & 0.0028 \\ 0.0977 & -0.0155 \end{bmatrix},$$
$$B_{2c} = \begin{bmatrix} -0.0003 & 0.0037 \\ 0.0016 & -0.0211 \end{bmatrix}, C_c = \begin{bmatrix} -0.0014 & 0.026 \end{bmatrix},$$
$$D_c = \begin{bmatrix} 2.1496 & 0.3607 \end{bmatrix}, D_{2c} = \begin{bmatrix} -0.025 & 0.3248 \end{bmatrix}. \tag{5.91}$$

	Non-zero Graph Weights
$a_{i,j}$	$a_{1,3} = 0.3281$, $a_{1,5} = 0.1939$, $a_{1,8} = 0.1268$, $a_{2,1} = 1.7678$, $a_{3,2} = 1.7906$, $a_{4,3} = 1.4078$, $a_{4,6} = 0.3891$, $a_{5,4} = 1.6481$, $a_{6,5} = 1.7083$, $a_{7,1} = 0.9464$, $a_{7,6} = 1.0705$, $a_{8,4} = 0.9594$, $a_{8,7} = 0.9574$, $a_{9,4} = 0.9772$, $a_{9,8} = 1.0824$, $a_{10,9} = 1.7497$, $a_{11,10} = 1.7521$, $a_{12,5} = 1.0034$, $a_{12,11} = 0.9603$, $a_{13,12} = 1.7855$, $a_{14,13} = 1.7136$, $a_{15,14} = 1.6162$, $a_{16,15} = 1.6278$, $a_{17,16} = 1.7335$, $a_{18,17} = 1.6677$, $a_{19,18} = 1.7663$, $a_{20,19} = 1.6627$, $a_{21,20} = 1.8010$, $a_{22,21} = 1.6553$, $a_{23,22} = 1.7400$, $a_{24,23} = 1.6532$, $a_{25,24} = 1.7199$, $a_{26,25} = 1.6973$, $a_{27,26} = 1.7397$, $a_{28,27} = 1.7440$, $a_{29,28} = 1.7595$, $a_{30,29} = 1.7170$, $a_{31,30} = 1.7408$, $a_{32,31} = 1.7844$, $a_{33,32} = 1.7459$, $a_{34,33} = 1.7212$, $a_{35,34} = 1.7981$, $a_{36,35} = 1.6553$, $a_{37,36} = 1.7145$, $a_{38,37} = 1.6994$, $a_{39,38} = 1.7408$, $a_{40,39} = 1.6666$, $a_{41,40} = 1.6435$, $a_{42,41} = 1.6284$, $a_{43,42} = 1.6910$, $a_{44,43} = 1.6126$, $a_{45,44} = 1.7593$, $a_{46,45} = 1.7001$, $a_{47,46} = 1.8765$, $a_{48,47} = 1.7931$, $a_{49,48} = 1.7962$, $a_{50,49} = 1.7036$.

Table 5.5 – Non-zero graph weights used in the simulation shown in Figure 5.7.

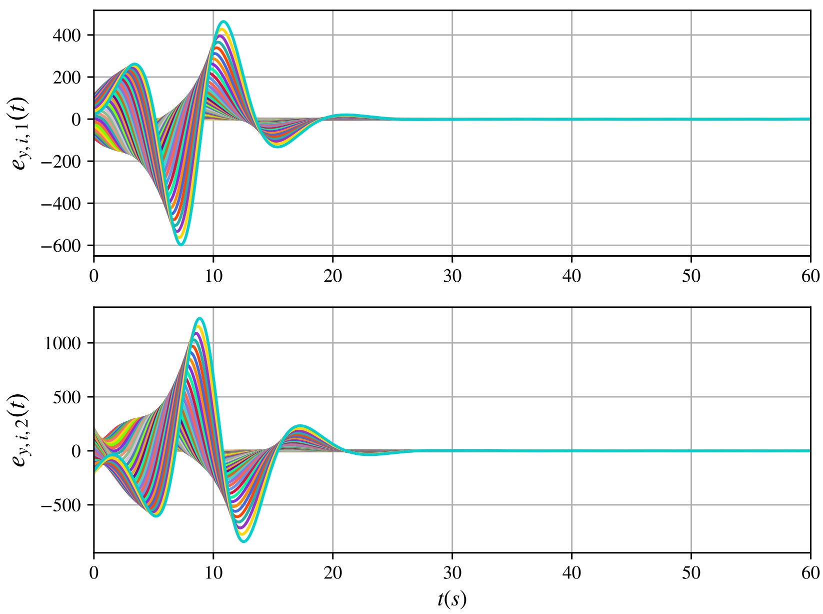


Figure 5.6 – Trajectories of output errors $e_{y,i}(t)$, for $i = 1, ..., 50$.

The simulation considers the same $B_{\underline{u}||}$ and $C_{y||}$ defined in subsection 5.5.2, and the

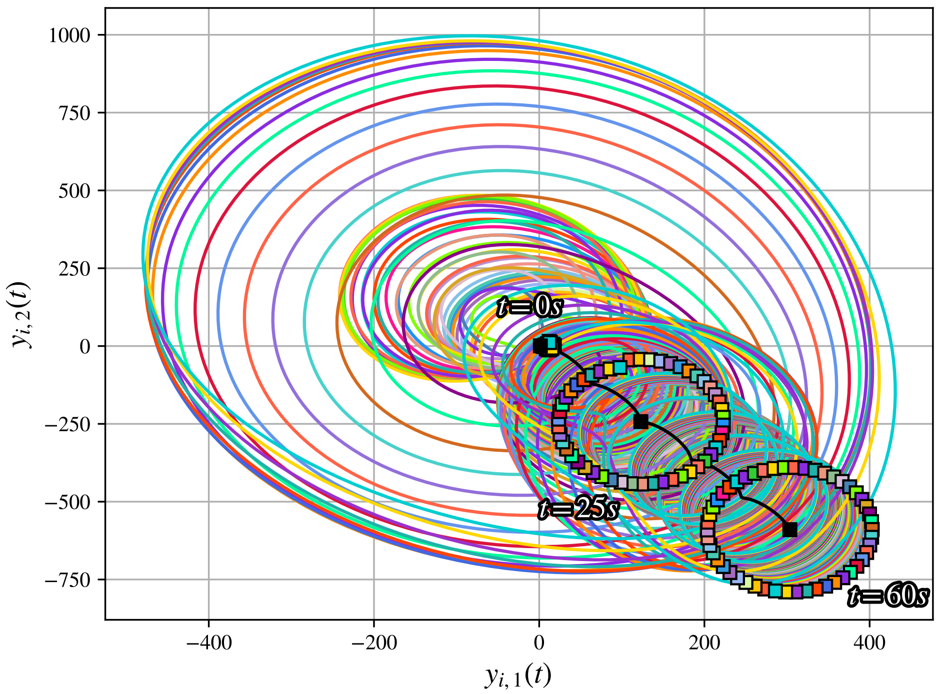


Figure 5.7 – Leader outputs $y_0(t)$ and the 50 followers outputs $y_i(t)$ in the plane reaching the formation.

time-varying formation function is chosen as $h_i(t) = \begin{bmatrix} 100\sin\left(t + \frac{(i-1)\pi}{25}\right) \\ 200\cos\left(t + \frac{(i-1)\pi}{25}\right) \end{bmatrix}$, satisfying the condition (5.13) of Theorem 5.1. The signal $\delta_i(t) = 50\cos\left(t + \frac{(i-1)\pi}{25}\right) - 50\sin\left(t + \frac{(i-1)\pi}{25}\right)$, for $i = 1, .., 50$, is obtained using (5.12). The leader input is given by $u_0(t) = 2\sin(t/2)$.

Figure 5.7 shows 50 agents and a leader in different moments: $t = 0s$, $t = 25s$ and $t = 50s$. The agents (5.1) (with $B_w = 0$ and $D_y = 0$) reach a formation following the leader (5.2) by using protocol (5.3)-(5.5) with gains (5.90), evidencing the effectiveness of Theorem 5.5 in designing TVOFT protocols.

5.5.4 Case 4: time-varying output formation tracking for mobile robots

For this case, we present a simulation in the CoppeliaSim robot simulation platform with the Pioneer mobile robots model, using the framework ROS (Robot Operating System) [47] for the robot's communication. The model used was a linearized model that, according to agents (5.1), is written as

$$\begin{bmatrix} \dot{\mathbf{x}}_i(t) \\ \dot{\mathbf{y}}_i(t) \\ \dot{\mathbf{v}}_{\mathbf{x}i}(t) \\ \dot{\mathbf{v}}_{\mathbf{y}i}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{y}_i(t) \\ \mathbf{v}_{\mathbf{x}i}(t) \\ \mathbf{v}_{\mathbf{y}i}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_u} u_i(t)$$

$$\begin{bmatrix} y_{1_i}(t) \\ y_{2i}(t) \\ y_{3i}(t) \\ y_{4_i}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C_y} \begin{bmatrix} \mathbf{x}_i(t) \\ \mathbf{y}_i(t) \\ \mathbf{v}_{\mathbf{x}i}(t) \\ \mathbf{v}_{\mathbf{y}i}(t) \end{bmatrix}, \quad i = 1, ..., 6, \tag{5.92}$$

where the agents' state variables $\mathbf{x}_i(t)$, $\mathbf{y}_i(t)$, $\mathbf{v}_{\mathbf{x}i}(t)$, $\mathbf{v}_{\mathbf{y}i}(t)$ are the position concerning axis x, the position concerning axis y, the velocity in axis x and the velocity in axis y of each agent i. The leader (5.2) has the same agents' parameters A and B_u.

First, the Theorem 3.3 was solved for $\beta = 3$, allowing us to compute K_x. With the computed K_x matrix, Theorem 5.5 was solved and obtained the following gains for $n_c = 0$ one has

$$D_c = \begin{bmatrix} 1.1343 & 0 & 1.4789 & 0 \\ 0 & 1.1343 & 0 & 1.4789 \end{bmatrix}, \tag{5.93}$$

for $n_c = 1$ one has

$$A_c = -3.4042, \quad B_c = \begin{bmatrix} 0 & 0.0022 & 0 & -0.0793 \end{bmatrix}, \quad B_{2c} = -0.0171$$
$$C_c = \begin{bmatrix} 0 \\ 0.0392 \end{bmatrix}, \quad D_c = \begin{bmatrix} 1.131 & 0 & 1.4614 & 0 \\ 0 & 0.7088 & 0 & 0.9391 \end{bmatrix}, \quad D_{2c} = \begin{bmatrix} 0 \\ 0.0901 \end{bmatrix}, \tag{5.94}$$

for $n_c = 2$ one has

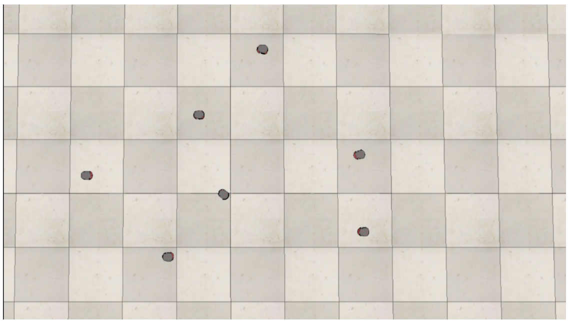
$$A_c = \begin{bmatrix} -3.3907 & 0 \\ 0 & -3.3907 \end{bmatrix}, B_c = \begin{bmatrix} -0.0019 & 0 & -0.0781 & 0 \\ 0 & -0.0019 & 0 & -0.0781 \end{bmatrix},$$
$$B_{2c} = \begin{bmatrix} -0.0169 & 0 \\ 0 & -0.0169 \end{bmatrix}, C_c = \begin{bmatrix} 0.0416 & 0 \\ 0 & 0.0416 \end{bmatrix},$$
$$D_c = \begin{bmatrix} 0.7047 & 0 & 0.9324 & 0 \\ 0 & 0.7047 & 0 & 0.9324 \end{bmatrix}, D_{2c} = \begin{bmatrix} 0.0907 & 0 \\ 0 & 0.0907 \end{bmatrix}, \tag{5.95}$$

for $n_c = 3$ one has

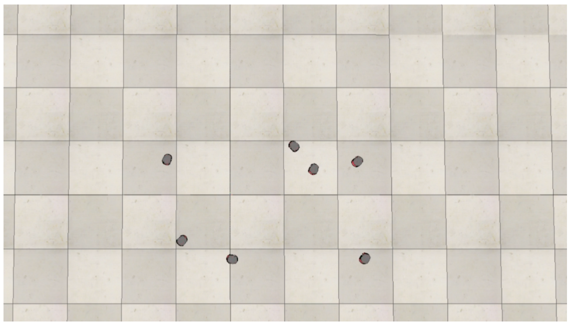
$$A_c = \begin{bmatrix} -3.493 & 0 & -0.6258 \\ 0 & -3.3906 & 0 \\ 0.266 & 0 & -3.0738 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 & -0.1758 & 0 & -0.5109 \\ -0.0017 & 0 & -0.0776 & 0 \\ ]0 & 0.1431 & 0 & 0.1528 \end{bmatrix},$$
$$B_{2c} = \begin{bmatrix} -0.0722 & 0 & -0.0953 \\ 0 & -0.0168 & 0 \\ 0.021 & 0 & 0.0282 \end{bmatrix}, C_c = \begin{bmatrix} 0 & 0.0417 & 0 \\ -0.5443 & 0 & -0.7369 \end{bmatrix},$$
$$D_c = \begin{bmatrix} 0.7038 & 0 & 0.9317 & 0 \\ 0 & 0.2845 & 0 & 0.3821 \end{bmatrix}, D_{2c} = \begin{bmatrix} 0 & 0.0907 & 0 \\ 0.0518 & 0 & 0.0688 \end{bmatrix}. \tag{5.96}$$

Let $B_{u\underline{||}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B_{u\underline{\perp}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $C_{y||} = C_y$, and the time-varying formation function $h_i = \begin{bmatrix} 4cos(0.5t + \frac{i\pi}{3}) \\ 3sin(0.5t + \frac{i\pi}{3}) \\ -2sin(0.5t + \frac{i\pi}{3}) \\ 1.5cos(0.5t + \frac{i\pi}{3}) \end{bmatrix}$, as in [74], satisfying the condition (5.13) of Theorem 5.1. The signal $\delta_i(t) = \begin{bmatrix} -2sin(0.5t + \frac{i\pi}{3}) - cos(0.5t + \frac{i\pi}{3}) \\ 1.5cos(0.5t + \frac{i\pi}{3}) - cos(0.5t + \frac{i\pi}{3}) \end{bmatrix}$, for $i = 1, .., m$, is obtained using (5.12). The leader input is given by $u_0(t) = \begin{bmatrix} -0.9sin(0.3t) \\ 0.9cos(0.3t) \end{bmatrix}$.

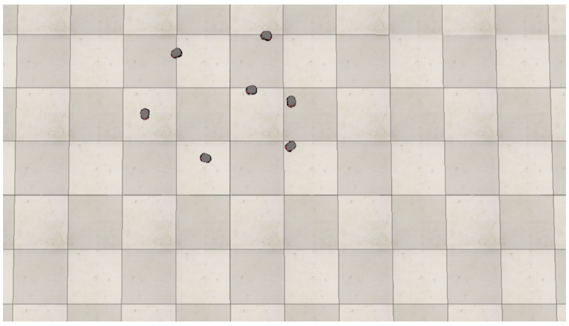
Consider the simulation depicted in Figure 5.8, which assumes a system with 6 followers and a leader (Fig. 5.1b) designed to reach a hexagonal formation. Fig. 5.5 shows the agents reaching the output formation guided by the leader using the protocol (5.3)-(5.5) with the gains obtained by Theorem 5.5. Moreover, Figure 5.9 shows that errors $e_{y,i,1}(t)$ and $e_{y,i,2}(t)$ converges to zero, which evidences the effectiveness of the Theorem 5.5.



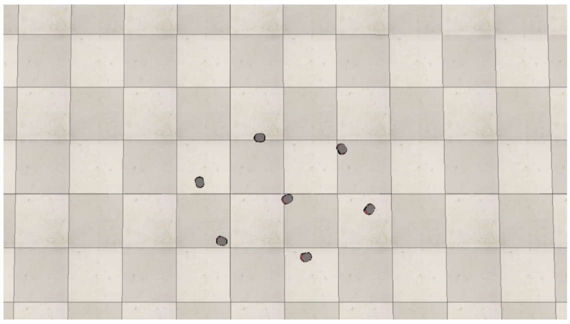
(a) $t = 0s$.



(b) $t = 10s$.



(c) $t = 17s$.



(d) $t = 28s$.

Figure 5.8 – Mobile robots reaching a hexagonal formation with protocol gains (5.95)

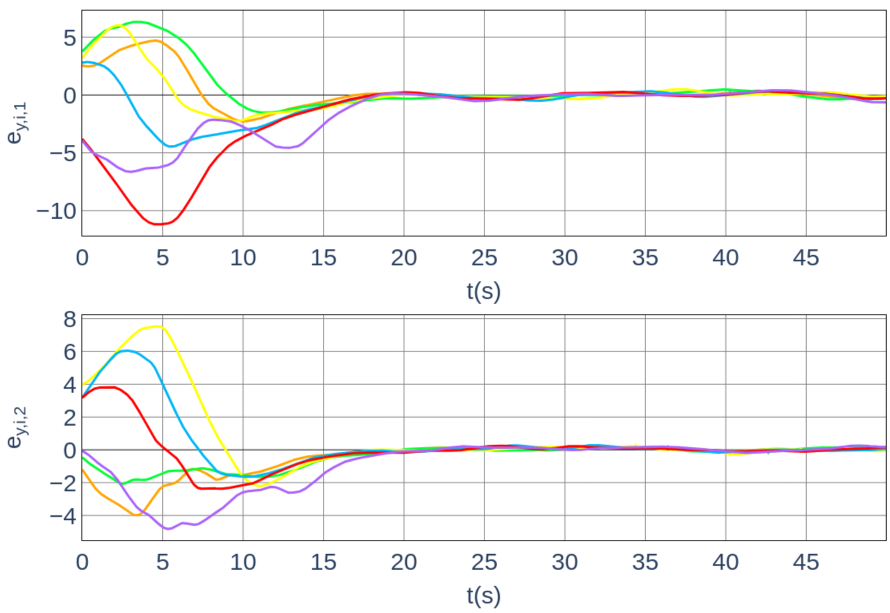


Figure 5.9 – Trajectories of the mobile robots output errors $e_{y,i,1}(t)$ and $e_{y,i,2}(t)$ converging to zero, i.e., the robot position errors converging to zero.

5.6 CHAPTER CONCLUSION

This chapter proposes LMI conditions to solve the time-varying output formation tracking (TVOFT) design problem for arbitrary order dynamic output feedback protocols and with H_∞ performance for agents in directed networks subject to external disturbances. Two solutions are presented: one for relatively small networks and the other for large networks. The advantages of using a polytope to describe a family of topologies are twofold. The designed protocol assures the HTVOFT for a class of topologies, and the computational burden of the design conditions does not increase with the number of agents. In future works, we will consider exploring the leaderless consensus problem for large network communications.

6 CONCLUSION

This work proposed convex conditions for designing dynamic protocols of arbitrary order for multi-agent systems under directed communication graphs using the available output information of neighbor agents. Chapter 3 establishes a new approach to solving the consensus problem for agents with known dynamics. As discussed in Remark 3.3, the method does not restrict the rank of the matrices B_u and C_y. The results presented in Chapter 3 are the first procedure in the literature that designs reduced-order protocols for disturbed agents connected in directed networks.

Chapter 4 presents new conditions for designing protocols for the consensus of homogeneous multi-agent systems subjected to parameter uncertainties and external disturbances. The communication topology is modeled by a polytopic region, which allows us to consider time-varying graph weights in the network connections. Unlike Chapter 3, the transformed system obtained in the analysis is based on a variable transformation that compares each agent with the mean of all agents. This transformation lets us find the H_∞ performance directly. Compared with the literature, the presented conditions are the first to design protocols for uncertain agents with external disturbances in the literature.

Chapter 5 presents new conditions for designing protocols for time-varying output formation tracking of homogeneous disturbed multi-agent systems. We present two cases: conditions that may design protocols that assure topology changes and conditions that may design protocols for many agents but for fixed topologies. The main contribution of this chapter is to present LMI conditions that can design protocols for networks with different numbers of agents since the eigenvalues of $\bar{L}$ lie on a defined polytopic region. Moreover, the presented conditions in Chapter 5 are the first in the literature to allow the design of any-order protocols for disturbed agents without considering the agents' quantity.

The results presented in this work may be extended in further investigations. For example, an interesting improvement for Chapter 4 would be to derive conditions that solve the Robust Consensus problem without depending on the number of agents. It can be challenging since diagonalizing the polytopic laplacian matrix is needed. About Chapter 5, an interesting investigation can be to find precisely a Laplacian-type matrix, such that its eigenvalues are in a given polytopic region. Moreover, in future works, it may be considered an investigation to derive new conditions for designing dynamic output feedback protocols for agents with time-varying delays, event-triggered mechanisms, and heterogeneous dynamics.

6.1 LIST OF PUBLICATIONS

Publication published during the elaboration of this thesis:

- Silva, B. M. C.; Ishihara, J. Y.; Tognetti, E. S. (2019). Consenso de Sistemas Multiagentes por Realimentação Dinâmica de Saída via LMIs. In: Simpósio Brasileiro de Automação Inteligente (SBAI), pages 1457–1462, Ouro Preto - MG, Brazil, 2019.

- Loyola, R. B.; Silva, B.M.C; Araújo, G.F.P.; Ishihara, J. Y.; Borges, G.A. Controle de Formação de Sistemas Multiagentes Não-Holonômicos em Topologia Fixa com Prevenção de Colisão. In: Conferência Brasileira de Dinâmica, Controle e Aplicações (DINCON), pages 1120–1126, São Carlos - SP, Brazil, 2019.

- Silva, B. M. C., Ishihara, J. Y., Tognetti, E. S. LMI-based Consensus of Linear Multi-Agent Systems by Reduced-Order Dynamic Output Feedback. ISA transactions, v. 129, p. 121-129, 2022.

- Borges, N. O.; Silva, B.M.C; Ishihara, Tognetti, E. S. Consenso Robusto de Sistemas Multiagentes Lineares em Redes de Comunicação de Topologia Incerta. In: Congresso Brasileiro de Automatica (CBA), pages 4156–4162, Fortaleza - CE, Brazil, 2022.

- Silva, B. M. C., Ishihara, J. Y., Tognetti, E. S. Time-Varying Output Formation Tracking of Linear Multi-Agent Systems Subject to External Disturbances by Dynamic Protocols. Submitted to Automatica.

[1] C. M. Agulhari, R. C. Oliveira, and P. L. Peres. LMI relaxations for reduced-order robust H_∞ control of continuous-time uncertain linear systems. *IEEE Transactions on Automatic Control*, 57(6):1532–1537, 2011.

[2] A. Amini, A. Asif, and A. Mohammadi. A performance guaranteed sampled-data event-triggered consensus approach for linear multi-agent systems. *Information Sciences*, 484:338–349, 2019.

[3] A. Amirkhani and A. H. Barshooi. Consensus in multi-agent systems: a review. *Artificial Intelligence Review*, 55(5):3897–3935, 2022.

[4] E. D. Andersen and K. D. Andersen. The mosek interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In *High performance optimization*, pages 197–232. Springer, 2000.

[5] P. Apkarian, H. D. Tuan, and J. Bernussou. Continuous-time analysis, eigenstructure assignment, and h/sub 2/synthesis with enhanced linear matrix inequalities (lmi) characterizations. *IEEE Transactions on Automatic Control*, 46(12):1941–1946, 2001.

[6] D. Arzelier, E. N. Gryazina, D. Peaucelle, and B. T. Polyak. Mixed LMI/randomized methods for static output feedback control design. In *Proceedings of the 2010 American control conference*, pages 4683–4688. IEEE, 2010.

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. Siam, 1994.

[8] H. Chehardoli. Decentralized second-order adaptive protocol of multi-agent systems with nonlinear disturbance based on only onboard partial measurement. *ISA transactions*, 104:329–335, 2020.

[9] Y. Chen, H. Xu, L. Dong, and Z. Li. Fully distributed secure observation and consensus for multi-agent systems with uncertain communication topology. *ISA transactions*, 145:176–189, 2024.

[10] S. Diamond and S. Boyd. CVXPY: A python-embedded modeling language for convex optimization. *The Journal of Machine Learning Research*, 17(1):2909–2913, 2016.

[11] X. Dong and G. Hu. Time-varying output formation for linear multiagent systems via dynamic output feedback control. *IEEE Transactions on Control of Network Systems*, 4(2):236–245, 2015.

[12] X. Dong and G. Hu. Time-varying formation control for general linear multi-agent systems with switching directed topologies. *Automatica*, 73:47–55, 2016.

[13] X. Dong and G. Hu. Time-varying formation tracking for linear multiagent systems with multiple leaders. *IEEE Transactions on Automatic Control*, 62(7):3658–3664, 2017.

[14] A. Dorri, S. S. Kanhere, and R. Jurdak. Multi-agent systems: A survey. *IEEE Access*, 6:28573–28593, 2018.

[15] G.-R. Duan and H.-H. Yu. *LMIs in control systems: analysis, design and applications*. CRC press, 2013.

[16] Z. Gao, H. Zhang, Y. Wang, and Y. Mu. Time-varying output formation-containment control for homogeneous/heterogeneous descriptor fractional-order multi-agent systems. *Information Sciences*, 567:146–166, 2021.

[17] X. Ge, Q.-L. Han, D. Ding, X.-M. Zhang, and B. Ning. A survey on recent advances in distributed sampled-data cooperative control of multi-agent systems. *Neurocomputing*, 275:1684–1701, 2018.

[18] C. Godsil and G. F. Royle. *Algebraic graph theory*, volume 207. Springer Science & Business Media, 2001.

[19] W. Hu, Y. Weng, and H. Li. Observer-based output feedback event-triggered robust H_∞ consensus control of uncertain linear multi-agent systems with directed networks. *International Journal of Robust and Nonlinear Control*, 2022.

[20] Y. Hua, X. Dong, L. Han, Q. Li, and Z. Ren. Formation-containment tracking for general linear multi-agent systems with a tracking-leader of unknown control input. *Systems & Control Letters*, 122:67–76, 2018.

[21] Y. Hua, X. Dong, Q. Li, and Z. Ren. Distributed fault-tolerant time-varying formation control for high-order linear multi-agent systems with actuator failures. *ISA transactions*, 71:40–50, 2017.

[22] Y. Hua, X. Dong, Q. Li, and Z. Ren. Distributed time-varying formation robust tracking for general linear multiagent systems with parameter uncertainties and external disturbances. *IEEE transactions on cybernetics*, 47(8):1959–1969, 2017.

[23] W. Huang, Y. Huang, and S. Chen. Robust consensus control for a class of second-order multi-agent systems with uncertain topology and disturbances. *Neurocomputing*, 313:426–435, 2018.

[24] W. Jiang, C. Wang, and Y. Meng. Fully distributed time-varying formation tracking control of linear multi-agent systems with input delay and disturbances. *Systems & Control Letters*, 146:104814, 2020.

[25] Q. Jiao, H. Modares, F. L. Lewis, S. Xu, and L. Xie. Distributed L_2-gain output-feedback control of homogeneous and heterogeneous systems. *Automatica*, 71:361–368, 2016.

[26] F. L. Lewis, H. Zhang, K. Hengster-Movric, and A. Das. *Cooperative control of multi-agent systems: optimal and adaptive design approaches*. Springer Science & Business Media, 2013.

[27] X. Li, F. Liu, M. Buss, and S. Hirche. Fully distributed consensus control for linear multiagent systems: A reduced-order adaptive feedback approach. *IEEE Transactions on Control of Network Systems*, 7(2):967–976, 2019.

[28] X. Li, Y. C. Soh, and L. Xie. Output-feedback H_∞ consensus of linear multi-agent systems over general directed graphs. In *2017 13th IEEE International Conference on Control & Automation (ICCA)*, pages 689–694. IEEE, 2017.

[29] X. Li, Y. C. Soh, and L. Xie. Output-feedback protocols without controller interaction for consensus of homogeneous multi-agent systems: A unified robust control view. *Automatica*, 81:37–45, 2017.

[30] X. Li, Y. C. Soh, and L. Xie. A novel reduced-order protocol for consensus control of linear multi-agent systems. *IEEE Transactions on Automatic Control*, 2018.

[31] X. Li, Y. C. Soh, and L. Xie. Robust consensus of uncertain linear multi-agent systems via dynamic output feedback. *Automatica*, 98:114–123, 2018.

[32] X. Li, Y. C. Soh, and L. Xie. A novel reduced-order protocol for consensus control of linear multi-agent systems. *IEEE Transactions on Automatic Control*, 64(7):3005–3012, July 2019.

[33] X. Li, Y. Tang, and H. R. Karimi. Consensus of multi-agent systems via fully distributed event-triggered control. *Automatica*, 116:108898, 2020.

[34] Z. Li and J. Chen. Robust consensus of linear feedback protocols over uncertain network graphs. *IEEE Transactions on Automatic Control*, 62(8):4251–4258, 2017.

[35] Z. Li, X. Liu, P. Lin, and W. Ren. Consensus of linear multi-agent systems with reduced-order observer-based protocols. *Systems & Control Letters*, 60(7):510–516, 2011.

[36] P. Lin, Y. Jia, and L. Li. Distributed robust H_∞ consensus control in directed networks of agents with time-delay. *Systems & control letters*, 57(8):643–653, 2008.

[37] X. Liu, Y. Xie, F. Li, T. Huang, W. Gui, and W. Li. Admissible H_∞ control of linear descriptor multi-agent systems with external disturbances. *Neurocomputing*, 359:241–248, 2019.

[38] Y. Liu and X. Hou. Event-triggered consensus control of disturbed multi-agent systems using output feedback. *ISA transactions*, 91:166–173, 2019.

[39] Y. Liu and Y. Jia. H_∞ Consensus control of multi-agent systems with switching topology: A dynamic output feedback protocol. *International Journal of Control*, 83(3):527–537, 2010.

[40] Y. Liu and Y. Jia. H_∞ consensus control of multi-agent systems with switching topology: a dynamic output feedback protocol. *International Journal of Control*, 83(3):527–537, 2010.

[41] Y. Lu, X. Dong, Q. Li, J. Lv, and Z. Ren. Time-varying group formation-tracking control for general linear multi-agent systems with switching topologies and unknown input. *International Journal of Robust and Nonlinear Control*, 32(4):1925–1940, 2022.

[42] S. Marras, S. S. Killen, J. Lindström, D. J. McKenzie, J. F. Steffensen, and P. Domenici. Fish swimming in schools save energy regardless of their spatial position. *Behavioral ecology and sociobiology*, 69(2):219–226, 2015.

[43] C. D. Meyer. *Matrix analysis and applied linear algebra*, volume 71. Siam, 2000.

[44] V. Montagner, R. Oliveira, V. J. Leite, and P. L. D. Peres. LMI approach for H_∞ linear parameter-varying state feedback control. *IEE Proceedings-Control Theory and Applications*, 152(2):195–201, 2005.

[45] K.-K. Oh, M.-C. Park, and H.-S. Ahn. A survey of multi-agent formation control. *Automatica*, 53:424–440, 2015.

[46] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[47] M. Quigley, K. Conley, B. Gerkey, J. Faust, T. Foote, J. Leibs, R. Wheeler, A. Y. Ng, et al. ROS: an open-source robot operating system. In *ICRA workshop on open source software*, volume 3, page 5. Kobe, Japan, 2009.

[48] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on automatic control*, 50(5):655–661, 2005.

[49] H. Roger and R. J. Charles. *Topics in matrix analysis*. Cambridge University Press Cambridge, UK, 1994.

[50] C. R. J. Roger A. Horn. *Matrix Analysis*. Cambridge University Press, 2nd edition, 2013.

[51] H. J. Savino, C. R. dos Santos, F. O. Souza, L. C. Pimenta, M. de Oliveira, and R. M. Palhares. Conditions for consensus of multi-agent systems with time-delays and uncertain switching topology. *IEEE Transactions on Industrial Electronics*, 63(2):1258–1267, 2015.

[52] M. Schwager, N. Michael, V. Kumar, and D. Rus. Time scales and stability in networked multi-robot systems. In *2011 IEEE International Conference on Robotics and Automation*, pages 3855–3862. IEEE, 2011.

[53] B. M. C. Silva, J. Y. Ishihara, and E. S. Tognetti. LMI-based consensus of linear multi-agent systems by reduced-order dynamic output feedback. *ISA transactions*, 129:121–129, 2022.

[54] R. E. Skelton, T. Iwasaki, and D. E. Grigoriadis. *A unified algebraic approach to control design*. CRC Press, 1997.

[55] Y. G. Sun and L. Wang. Consensus of multi-agent systems in directed networks with nonuniform time-varying delays. *IEEE Transactions on Automatic Control*, 54(7):1607–1613, 2009.

[56] E. S. Tognetti, T. R. Calliero, I.-C. Morărescu, and J. Daafouz. Synchronization via output feedback for multi-agent singularly perturbed systems with guaranteed cost. *Automatica*, 128:109549, 2021.

[57] E. S. Tognetti, R. C. L. F. Oliveira, and P. L. D. Peres. Reduced-order dynamic output feedback control of continuous-time T–S fuzzy systems. *Fuzzy Sets and Systems*, 207:27–44, November 2012.

[58] H. L. Trentelman, K. Takaba, and N. Monshizadeh. Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, 58(6):1511–1523, 2013.

[59] L. Wang, M. Z. Q. Chen, Z. Gao, C. W. Wu, and Z. Lin. Special issue on “distributed coordination control for multi-agent systems in engineering applications”. *ISA transactions*, 71(Pt 1):1–2, 2017.

[60] X. Wang, X. Wang, H. Su, and J. Lam. Reduced-order interval observer based consensus for mass with time-varying interval uncertainties. *Automatica*, 135:109989, 2022.

[61] D. B. West. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[62] J. Wu, M. Lu, F. Deng, and J. Chen. Cooperative robust output regulation of linear uncertain multi-agent systems: A dynamic event-triggered approach. *International Journal of Robust and Nonlinear Control*, 33(18):11538–11552, 2023.

[63] L. Xie and C. E. de Souza. Robust H/sub infinity/control for linear systems with norm-bounded time-varying uncertainty. In *29th IEEE Conference on Decision and Control*, pages 1034–1035. IEEE, 1990.

[64] X. Xue, F. Wu, and C. Yuan. Robust consensus for linear multi-agent systems with structured uncertainties. *International Journal of Control*, pages 1–12, 2019.

[65] X. Xue, C. Yuan, and F. Wu. Convexified H_∞ output-feedback consensus synthesis for linear multi-agent systems. *IET Control Theory & Applications*, 13(11):1619–1628, 2019.

[66] W. Yang, Z. Shi, and Y. Zhong. Robust time-varying formation control for uncertain multi-agent systems with communication delays and nonlinear couplings. *International Journal of Robust and Nonlinear Control*, 34(1):147–166, 2024.

[67] J. Zhang and H. Su. Time-varying formation for linear multi-agent systems based on sampled data with multiple leaders. *Neurocomputing*, 339:59–65, 2019.

[68] X. Zhang, J. Wu, X. Zhan, T. Han, and H. Yan. Observer-based adaptive time-varying formation-containment tracking for multiagent system with bounded unknown input. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 53(3):1479–1491, 2022.

[69] Z. Zhang, S. Zhang, H. Li, and W. Yan. Cooperative robust optimal control of uncertain multi-agent systems. *Journal of the Franklin Institute*, 357(14):9467–9483, 2020.

[70] J. Zhao, L. Xiang, F. Chen, and W. Lan. Distributed consensus via self-triggered output feedback. *IET Control Theory & Applications*, 10(10):1170–1180, 2016.

[71] Y. Zhao and W. Zhang. Guaranteed cost consensus protocol design for linear multi-agent systems with sampled-data information: An input delay approach. *ISA transactions*, 67:87–97, 2017.

[72] B. Zhou, C. Xu, and G. Duan. Distributed and truncated reduced-order observer based output feedback consensus of multi-agent systems. *IEEE Transactions on Automatic Control*, 59(8):2264–2270, 2014.

[73] K. Zhou and P. P. Khargonekar. Robust stabilization of linear systems with norm-bounded time-varying uncertainty. *Systems & Control Letters*, 10(1):17–20, 1988.

[74] S. Zuo, Y. Song, F. L. Lewis, and A. Davoudi. Time-varying output formation containment of general linear homogeneous and heterogeneous multiagent systems. *IEEE Transactions on Control of Network Systems*, 6(2):537–548, 2018.