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Dynamics and Topology on Maximal Compact Subgroups

This dissertation is submitted as partial requirement
for obtaining the title of *doctor of mathematics*.

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by

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The main idea of the research was to apply the methods and techniques used to study flags to the study of the maximal compact subgroups. So naturally the first objective was to understand the results from flags to then adapt the techniques used to show them. First we did this for general actions on the compact group following [2]. After this we studied the topology of the compact group using the Bruhat cells that are the unstable varieties of the hyperbolic action and provide a natural division of K in cell complexes following [16].

The study of an area of research in contact with diverse areas in mathematics was challenging and interesting. Many opportunities were indispensable in this course: an excellent study group with my advisor, Professor Lucas Ferreira and Laércio dos Santos for studying [18]; many seminars about Dynamical Systems and Lie Groups as well as thoughtful advisor and teachers. All these things motivate me to be a better mathematics researcher and teacher. I thank all that were with me in this journey. I hope to contribute to future research and to the development of the group of Lie Group and Dynamical Systems at UnB.

Thank You.

Abstract

In this work we study the dynamics and topology of the action of a semisimple Lie group G on its maximal subgroup K, first we study hyperbolic actions on K and then general translations. For this we find the minimal Morse components and stable and unstable manifolds and prove that the minimal Morse components are normally hyperbolic. The unstable manifolds correspond to Bruhat cells whose closure are the Schubert cells. This division of K by Schubert cells creates a cell complex that permit the calculation of the homology groups of K. We focus on the case of split real forms. The boundary operator is found in general and the example $\mathrm{SO}(3)$ is calculated geometrically and then algebraically by the formulas we obtain here.

Keywords: Semisimple Lie groups, Morse decomposition, Normal hyperbolicity, Iwasawa decomposition, Cellular homology.

Dinâmica e Topologia em Subgrupos Maximais Compactos

Resumo

Neste trabalho estudamos a dinâmica e topologia da ação de um grupo de Lie semissimples G em seu grupo maximal K, primeiro estudamos as ações hiperbólicas em K e depois estudamos translações gerais. Para isto achamos as componentes de Morse minimais e as variedades estáveis e instáveis e provamos que as componentes de Morse minimais são normalmente hiperbólicas. As variedades instáveis correspondem as células de Bruhar cujos fechos correspondem às células de Schubert. Esta divisão de K por células de Schubert gera um complexo celular que permite o cálculo de grupos de homologia de K. Focamos no caso de formas reais normais. O operador fronteira é descoberto em geral e o exemplo do SO(3) é calculado geometricamente e depois algebricamente pelas fórmulas obtidas aqui.

Contents

Chapter 1

General Introduction

The first main results in this work follow similar results in flags manifolds that can be found in [7], [14], [15] and are also collected in [2]. These results study the dynamics of hyperbolic and general actions on flags, the fixed points, stable, and unstable manifolds for the hyperbolic action are all found and a linearization of the stable sets is possible. For the general action the recurrent set is found and a linearization is also possible. The general methods were maintained with only slight alterations to obtain similar results.

In [5] it is considered a continuous-time flow generated by semisimple element $H \in \mathfrak{g}$ acting on the flag manifolds of $\mathfrak{g}$: they show that it is a Morse-Bott gradient flow, and describe its fixed point set and stable manifolds. In [10] it is analyzed a continuous-time flow generated by an element which is the sum of two commuting elements of $\mathfrak{g}$, one of which induces a gradient vector field and the other generates a one-parameter field of isometries.

Then normal hyperbolicity was first established when the matrix is diagonalizable over the complex numbers [5], [7], [10]. In [15] it is proven that this normal hyperbolicity is true in a far more general context of an arbitrary element of a semisimple Lie group acting on generalized flag manifolds: the so called translations on flag manifolds.

In [16], the Bruhat and Schubert cells are used in flag manifolds to divide the them in cell complexes to then calculate the topology of partial and complete flags. Formulas for the boundary operator are then found. We also followed the results of this work for complete flags to produce similar results for the maximal compact subgroup.

The results of [16] were already partially found for flag manifolds by Kocherlakota [19]. In the realm of Morse homology in Theorem 1.1.4 [19] it is proven that the boundary operator for the Morse-Witten complexes are intimately related since

Bruhat cells are unstable manifolds of the gradient flow of a Morse function (see Duistermatt-Kolk-Varadarajan [5]). Nevertheless the cellular point of view of [16] has the advantage of showing the geometry in a more evident way, in particular, the choice of minimal decompositions for the elements of W fix certain signs ambiguous in the Morse-Witten complex.

The construction of cellular decompositions of group manifolds and homogeneous spaces is an old theme. For the classical compact Lie groups one can use cells using products of reflections via the product of reflections via the method that goes back to Whitehead [25] and was later developed by [23], [24]. More recently it can be found in section 3.D of [8].

The degeneration of Spectral sequences that occurs for unitary and sympletic groups fails for the orthogonal groups, because in the analogue of the iterated fiber decomposition of the orthogonal groups one encounters spheres of adjacent dimensions (see section 3.2 of [4]).

In the second chapter we will define an action of the semisimple group G by left multiplication on the homogeneous manifold G/AN where A and N come from the Iwasawa decomposition. From this we can define what is an hyperbolic action in K and find its fixed points and stable and unstable manifolds. This decomposition is also called a Bruhat decomposition.

In the hyperbolic case a linearization of the manifolds is possible. To obtain the main result in the chapter we begin by constructing a height function and an appropriate metric in K so that we can show that the system is gradient. The first main result for regular flows is Theorem 2.18 and the main result of the chapter is

Theorem 2.20 *Let h^t be a hyperbolic flow in K.*

(*i*) *The set of fixed points is the disjoint union of connected components*

$$\mathrm{fix}(h^t) = \coprod \{\mathrm{fix}(H,u) : u \in U_H \backslash U\}$$

where $\mathrm{fix}(H,u) = K_H^0 ub$. *The attractors are* $\mathrm{fix}(H,c)$ *for* $c \in C_H \backslash C$ *and the repellers are* $\mathrm{fix}(H, cu^-)$ *for* $c \in C_H \backslash C$.

(*ii*) *The group K decomposes as the disjoint union of stable manifolds,*

$$K = \coprod \{\mathrm{st}(H,u) : u \in U_H \backslash U\} \tag{1.1}$$

where each $\mathrm{st}(H,u) = N_H^- \mathrm{fix}(H,u) = N_H^- K_H^0 ub$ *is diffeomorphic to the stable fiber over* $\mathrm{fix}(H,u)$. *Also, for* $c \in C_H \backslash C$ *the stable manifolds are open in K and their union is dense.*

In the third chapter we first show that a general action is decomposable in elliptic, hyperbolic and nilpotent components that commute with each other, the so called Jordan decomposition. Next we study the example of Sl(2) with $K = SO(2)$ and see that the system is not gradient. This motivates the study of chain recurrence and then by working with concrete examples we found Lemma 3.4 where we show that the system is equivariant by right multiplication by M. This provides a symmetry necessary to show that chain recurrent components are the fixed points for the hyperbolic flow $\mathrm{fix}(h^t)$ that is Theorem 3.10. Later we obtain a result for recurrent points:

Theorem 3.11 *Let g^t be the translations flow in G and $g^t = e^t h^t u^t$ is its Jordan decomposition. The recurrent set of translations g^t induced in K is given by*

$$\mathcal{R}(g^t) = \mathrm{fix}(h^t) \cap \mathcal{R}(u^t)$$

and $\mathcal{R}(u^t) = \pi^{-1}(\mathrm{fix}_{\mathbb{F}}(u^t))$, where π is the projection of K in $K/M = \mathbb{F}$.

Later we also obtain a linearization of the general flow in the stable manifolds.

In the fourth chapter we use the Bruhat and Schubert cells to calculate the homology of K for split real forms in particular at the end we calculate the homology of $SO(3)$. In this we follow [16] to first construct the skeleton and then the boundary map to find algebraic expressions for the degrees of the maps. The calculations for the degrees in SO(3) are done in two ways, first geometrically and then algebraically to better illustrate the results obtained. A general result is:

Theorem 4.14 *Let $\sigma(u,v)$ be as in Equation 4.3. Then if $v = v_1 v_2$ then*

$$c(u,v) = \deg\left(\Psi'^{-1}_{v} \circ \Psi_v\right)(-1)^j$$

and if $v = v_1 m_j v_2$ then

$$c(u,v) = \deg\left(\Psi'^{-1}_{v} \circ \Psi_v\right)(-1)^{j+1+\sigma(u,v)}$$

where

$$\sigma(u,v) = \sum_{\beta \in \Pi_{v_2}} \frac{2\langle \alpha_j, \beta \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

as in Proposition 4.13. Whereas Theorem 4.16 is a result most useful when the calculations from Theorem 4.14 turn too long.

Chapter 2

Hyperbolic translations of K

2.1 Action and metric in K

In this chapter we will first define an action of the semisimple group G by left multiplication on the homogeneous manifold G/AN where A and N come from the Iwasawa decomposition. From this we can define an hyperbolic action in K and we find its fixed points and its stable and unstable manifolds, and show that a linearization of the stable manifold is possible. For this we first construct a height function and an appropriate metric in K so that we can show that the system is gradient.

Let G be a real connected semisimple Lie group and let K, A, N be Lie subgroups of G obtained by *a fixed Iwasawa decomposition* $G = KAN$ (see Theorem 13.3.8 from [12]). The natural action of G in G/AN is given by left multiplication (as in section 10.1 of [12]). In this section we will study K as the homogeneous manifold G/AN taking the *base of* G/AN, or the left coset AN, as b. So the isotropy subalgebra of the base will be $\mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is the Lie algebra of A and $\mathfrak{n}$ is the Lie algebra of N. From this point on we will assume the Lie group K is compact. In Theorem 6.31 item (f) of [18] we notice that the Lie group K is compact if and only if the center of G is finite.

Definition 2.1 *Let G be a real connected semisimple Lie group and let K,A, N as in a fixed Iwasawa decomposition. Define the action of G in G/AN by left multiplication. From the Iwasawa decomposition and since G is transitive in G/AN for all $g \in G$ there exists a unique $k \in K$ such that*

$$gb = kb$$

and there are also unique $a \in A$ and $n \in N$ and $g = kan$.

Define the quotient map $\phi : G \to G/AN$ as $\phi(g) = gb = kb$.

The Iwasawa decomposition is in fact a generalization of the Gram-Schmidt orthogonalization process on the column vector of the matrices, where the orthogonal result is in fact the k matrix in the decomposition. One can use this to build some intuition on the action before. Also, in the Sl(2) example it turns out that just dividing by the norm after the linear action in G is enough to understand the action in K. In the examples, we will take a slightly different approach to get k with less calculations.

Now, note that,

$$\phi|_K : K \to G/AN, \quad \phi|_K : k \mapsto kb$$

is a K-equivariant diffeomorphism of a Lie group into a homogeneous manifold, and similarly,

$$\phi|_K^{-1} : G/AN \to K, \quad \phi|_K^{-1} : kb \mapsto k$$

is also K-equivariant.

Informally, we can identify the Lie group K with the homogeneous manifold G/AN. When necessary we will use these diffeomorphisms to relate the group K with the homogeneous manifold.

Example: Let $G = \mathrm{Sl}(2)$ and the Iwasawa subgroups are: $K := \mathrm{SO}(2)$,

$$A := \left\{ \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, h > 0, h \in \mathbb{R} \right\} \quad N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\}$$

Now we will study the hyperbolic action on the compact group SO(2)

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} b = \begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix} b$$

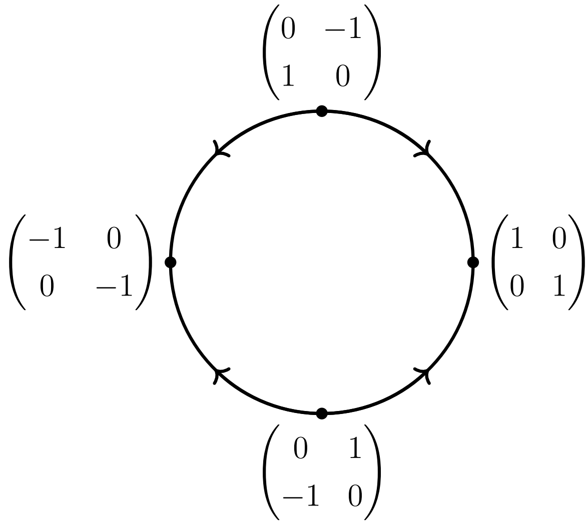
so we need to find the $\alpha(t)$ or the compact component for the matrix

$$\begin{pmatrix} e^t \cos\alpha & -e^t \sin\alpha \\ e^{-t} \sin\alpha & e^{-t} \cos\alpha \end{pmatrix}$$

if we multiply the Iwasawa decomposition taking $h(t)$ to be h and $x(t)$ to be x,

$$\begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

From the elements in the first column $e^t \cos\alpha = h \cos\alpha(t)$ and $e^t \sin\alpha = h \sin\alpha(t)$ dividing both terms when $\sin\alpha \neq 0$ we get $e^{2t} \cot\alpha = \cot\alpha(t)$. So $\alpha(t)$ can be determined. Remember that in the trigonometric circle the cotangent "axis" is tangent to $(0, 1)$. Now we can identify the unstable equilibrium points are when $\cos\alpha = 0$ and the stable equilibrium points are when $\sin\alpha = 0$. The first column of the matrices is used to plot the results in matrix form.



By the action of G in $K \simeq G/AN$ we have that g belongs to the isotropy group of kb if, and only if,

$$gkb = kb \Leftrightarrow gk \in kAN \Leftrightarrow g \in kANk^{-1}$$

Let $\mathfrak{g}_{kb}$ be *the isotropy subalgebra* of kb for $k \in K$ then

$$\mathfrak{g}_{kb} = k(\mathfrak{a} \oplus \mathfrak{n})$$

where the action of the group in the algebra is defined as $gX := \mathrm{Ad}(g)X$ for $X \in \mathfrak{g}$.

Let $M := Z(K, \mathfrak{a})$ be the centralizer of the algebra $\mathfrak{a}$ in K. Let $\mathfrak{k}$ be the subalgebra of K and let θ be the Cartan involution that fixes the elements in $\mathfrak{k}$, *Define the subalgebra* $\mathfrak{s}$ of $\mathfrak{g}$ by the elements $X \in \mathfrak{g}$ such that $\theta(X) = -X$.

Note that $\mathfrak{n}$ can be decomposed by the rootspaces of positive roots, so when fixing a Iwasawa decomposition we have already chosen the set positive roots Σ, the set of $H \in \mathfrak{a}$ such that $\alpha(H) > 0$ for all positive roots is the *positive chamber* $\mathfrak{a}^+$ of $\mathfrak{a}$.

Now we introduce an important immersion of K/M in $\mathfrak{s}$. *Let* $H_r \in \mathfrak{a}^+$ be a *regular* element.

Proposition 2.2 *If* $M := Z(K, \mathfrak{a})$, *or the centralizer of* $\mathfrak{a}$ *in* K, *and* $H_r \in \mathfrak{a}^+$ *be a regular element then the map.*

$$j : K/M \to \mathfrak{s}, \quad kM \mapsto kH_r, \quad k \in K$$

is a K*-equivariant differentiable immersion of* K/M *in* $\mathfrak{s}$.

Proof: The map j is well defined, indeed since if $k \in K$ and $m, m' \in M$ then $kmH_r = km'H_r = kH_r$. To show that the map j is injective let $k' \in K$ and

$kmH_r = k'm'H_r$ then $kH_r = k'H_r$ and $k^{-1}k' \in Z(H_r) \cap K$ the centralizer of H_r in K. By Theorem 4.21 in [2], since H_r is regular $K_{H_r} = Z(H_r) \cap K = M$. Then $k^{-1}k' \in M$ and $k'M = kM$.

The map j is K-equivariant since $j(k.k'M) = k.k'H_r = k.j(k'M)$ and by Corollary C.7 of [2] it is also differentiable. Since K/M is compact and j is differentiable then j is an immersion. □

Now, we will take the application j to be defined with domain K where it fails to be an immersion.

The Cartan involution θ that fixes the subalgebra $\mathfrak{k}$ of K also fixes a Cartan inner product $\langle \cdot, \cdot \rangle$. Now, let H be a *fixed* element of $\mathrm{cl}\mathfrak{a}^+$ and define *the height function* f_H of the application j in the *direction* H as:

$$f_H : K = G/AN \to \mathbb{R}, \qquad x \mapsto \langle j(x), H \rangle$$

and then $f_H(kb) = \langle kH_r, H \rangle$. *Define* G_H as the centralizer of H in G and *define* $K_H := G_H \cap K$ as the centralizer of H in K.

Proposition 2.3 *The function f_H is K_H-invariant and its differential is*

$$f'_H(kb)k(Z \cdot b) = \langle [Z, H_r],\, k^{-1}H \rangle$$

where $k \in K$ and $Z \in \mathfrak{k}$.

Proof: To show K_H-invariance, let $k \in K_H$. By K-equivariance of j

$$f_H(kx) = \langle kj(x), H \rangle = \langle j(x), k^{-1}H \rangle = \langle j(x), H \rangle$$

where k is $\langle \cdot, \cdot \rangle$-orthogonal (see Proposition 2.40 of [2]). Lets evaluate its differential at kb in the direction kZ with $Z \in \mathfrak{k}$,

$$\begin{aligned} f'_H(kb)k(Z \cdot b) &= d/dt|_{t=0}\; f_H(k\exp(tZ)b) \\ &= d/dt|_{t=0}\; \langle k\exp(tZ)H_r,\, H \rangle \\ &= d/dt|_{t=0}\; \langle \mathrm{e}^{t\mathrm{ad}(Z)}H_r,\, k^{-1}H \rangle \\ &= \langle [Z, H_r],\, k^{-1}H \rangle \end{aligned}$$

since k is $\langle \cdot, \cdot \rangle$-orthogonal. □

Now, our next objective is to define a metric in K such that the field induced by H in K is the the gradient of the height function f_H. Using the Cartan inner product, we get a corresponding orthogonal decomposition of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{n}^-$$

by Theorem 2.29 from [2], then

$$(\mathfrak{a} \oplus \mathfrak{n})^{\perp} = \mathfrak{m} \oplus \mathfrak{n}^{-} \tag{2.1}$$

Note that by Theorem 4.13 from [2], $K_{\varnothing} = K(\varnothing)M = M$. Then, by Proposition C.13 (iii) from [2], an inner product in $\mathfrak{m} \oplus \mathfrak{n}^{-}$ is $K_{\varnothing}$-invariant or M-invariant extends to K-invariant metric in B_{kb} for $k \in K$ given by

$$B_{kb}(k(X \cdot b), k(Y \cdot b)) := B(X, Y) \text{ where } X, Y \in \mathfrak{m} \oplus \mathfrak{n}^{-}$$

By decomposing $\mathfrak{n}^{-}$ in M-invariant subspaces we can describe the inner products B that are M-invariant.

Let λ be a real number, and $\mathfrak{b}_{\lambda}$ be the λ-eigenspace of $\mathrm{ad}(H_r)$ in $\mathfrak{g}$ given by

$$\mathfrak{b}_{\lambda} := \{X \in \mathfrak{g} : \mathrm{ad}(H_r)X = \lambda X\}$$

Note that

$$\mathfrak{b}_{\lambda} = \sum_{\alpha(H_r)=\lambda} \mathfrak{g}_{\alpha}$$

and

$$\mathfrak{n}^{-} = \sum_{\lambda<0} \mathfrak{b}_{\lambda}$$

Since M normalizes each $\mathfrak{g}_{\alpha}$ by Proposition 3.25 [2] and M normalizes $\mathfrak{m}$ since $M_0 = \exp(\mathfrak{m})$ is a normal subgroup of M we have that M normalizes the inner product B in $\mathfrak{m} \oplus \mathfrak{n}^{-}$.

For $X \in \mathfrak{g}$ let X_{λ} be the orthogonal projection of X in $\mathfrak{b}_{\lambda}$ and let X_0 be the orthogonal projection of X in $\mathfrak{m}$. Note that for $m \in M$, $(mX)_{\lambda} = mX_{\lambda}$. Let c_{λ} and c_0 be positive real numbers associated to $\mathfrak{b}_{\lambda}$ and $\mathfrak{m}$, respectively. Lets define the inner product in $\mathfrak{m} \oplus \mathfrak{n}^{-}$ given by

$$B(X, Y) := \sum_{\lambda \leq 0} c_{\lambda} \langle X_{\lambda}, Y_{\lambda} \rangle \text{ where } X, Y \in \mathfrak{m} \oplus \mathfrak{n}^{-} \tag{2.2}$$

Since $\langle \cdot, \cdot \rangle$ is M-invariant then B is also M-invariant.

Using the notation $H \cdot x = d/dt|_{t=0}(e^{tH}x)$ and the notation $H\cdot$ for the induced field by H in K, so that $H\cdot$ is in the space of tangent fields $\Gamma(TK)$ of K.

Theorem 2.4 *Taking $c_{\lambda} = -2\lambda$ for $\lambda < 0$ in equation 2.2 then*

$$H\cdot = \mathrm{grad}_B\,(f_H)$$

that is, the induced field by $H \in \mathfrak{s}$ in K is the gradient of the height function f_H with respect to the K-invariant metric B. Also,

$$B(X, Y) = c_0 \langle X_0, Y_0 \rangle + 2\langle [X, H_r], Y \rangle \qquad X, Y \in \mathfrak{m} \oplus \mathfrak{n}^{-}$$

where X_0 and Y_0 are the components of X and Y in $\mathfrak{m}$. And c_0 is arbitrary positive.

Proof: By the definition of grad_B for $k \in K$ and $X \in \mathfrak{m} \oplus \mathfrak{n}^-$,

$$B_{kb}(k(X \cdot b), \mathrm{grad}_B(f_H)(kb)) = f'_H(kb)k(X \cdot b)$$

So to prove the first statement we need to show that

$$B_{kb}\left(k(X \cdot b),\, H \cdot kb\right) = f'_H(kb)k(X \cdot b) \tag{2.3}$$

To evaluate the left side, let Y_- be the orthogonal projection of $k^{-1}H$ in $\mathfrak{m} \oplus \mathfrak{n}^-$, that is parallel to $\mathfrak{a} \oplus \mathfrak{n}$. Note that since $k^{-1}H \in \mathfrak{s}$ then $Y_- \in \mathfrak{n}^-$.

Then $Y_- \cdot b = k^{-1}H \cdot b$, and $k(Y_- \cdot b) = H \cdot kb$. Let $X = X_0 + X_-$ where $X_0 \in \mathfrak{m}$ and $X_- \in \mathfrak{n}^-$. By the K-invariance of the metric the left side is

$$\begin{aligned} B_{kb}\left(k(X \cdot b),\, H \cdot kb\right) &= B_{kb}\left(k(X \cdot b),\, k(Y_- \cdot b)\right) \\ &= B(X_-, Y_-) \\ &= \sum_{\lambda<0} c_\lambda \langle X_\lambda, Y_\lambda \rangle \end{aligned} \tag{2.4}$$

To evaluate the right side, let $Z = X_0 + X_- + \theta X_- \in \mathfrak{k}$. Since $\theta X_- \in \mathfrak{n}$, then $Z \cdot b = (X_0 + X_-) \cdot b$. By Proposition 2.3 then

$$f'_H(kb)k((X_0 + X_-) \cdot b) = \langle [Z, H_r],\, k^{-1}H \rangle \tag{2.5}$$

To evaluate $[Z, H_r]$, first note that

$$[H_r, X_-] = \sum_{\lambda<0} \mathrm{ad}(H_r)X_\lambda = \sum_{\lambda<0} \lambda X_\lambda$$

$$[H_r, \theta X_-] = -[\theta H_r, \theta X_-] = -\theta[H_r, X_-] = -\sum_{\lambda<0} \lambda \theta X_\lambda$$

and that $[H_r, X_0] = 0$. Then

$$[Z, H_r] = \sum_{\lambda<0} \lambda(\theta X_\lambda - X_\lambda)$$

Since $k^{-1}H \in \mathfrak{s}$ (see Proposition 2.40 from [2]) then

$$\langle \theta X_\lambda,\, k^{-1}H \rangle = -\langle \theta X_\lambda,\, \theta k^{-1}H \rangle = -\langle X_\lambda,\, k^{-1}H \rangle$$

and

$$\langle [Z, H_r],\, k^{-1}H \rangle = \sum_{\lambda<0} -2\lambda \langle X_\lambda,\, k^{-1}H \rangle = \sum_{\lambda<0} -2\lambda \langle X_\lambda,\, Y_\lambda \rangle$$

since $Y_- = \sum_{\lambda<0} Y_\lambda$ is the projection of $k^{-1}H$ at $\mathfrak{m} \oplus \mathfrak{n}^-$. From (2.5) and since $X = X_0 + X_-$ then

$$f'_H(kb)k\left(X \cdot b\right) = \sum_{\lambda<0} -2\lambda \langle X_\lambda,\, Y_\lambda \rangle \tag{2.6}$$

Now, to prove equation (2.3) then by (2.4) and (2.6) the metric B has $c_\lambda = -2\lambda$ for $\lambda < 0$.

To prove the last statement, let $X_-, Y_- \in \mathfrak{n}^-$ then

$$\begin{aligned} B(X_-, Y_-) &= -2\sum_{\lambda<0} \lambda\langle X_\lambda, Y_\lambda\rangle \\ &= -2\langle \sum_{\lambda<0} \lambda X_\lambda, Y_-\rangle \\ &= -2\langle [H_r, X], Y_-\rangle \\ &= 2\langle [X, H_r], Y_-\rangle \end{aligned}$$

Since $Y = Y_0 + Y_-$ with $Y_0 \in \mathfrak{m}$ then

$$B(X, Y) = B(X_0, Y_0) + B(X_-, Y_-)$$

□

The Riemannian metric B constructed on the last Theorem is an extension of the *Borel metric* of $\mathbb{F}$.

2.2 Fixed points

Let $M_* := N(K, \mathfrak{a})$ and $M := Z(K, \mathfrak{a})$, respectively, the normalizer and centralizer of $\mathfrak{a}$ in K. Note that in general M is not connected so that $M_0 = \exp(\mathfrak{m})$ is, in general, a proper subgroup of M (see $\mathfrak{sl}(n, \mathbb{C})$). To study the fixed points in K it is convenient to define the *group*

$$U := M_*/M_0$$

This group will play a similar role to the Weyl group in the study of fixed points in flags [2]. Since the Weyl group W is M_*/M (see Corollary 3.24 [2]) then $W = (M_*/M_0)/(M/M_0)$ so that $W = U/C$ where we *define*

$$C := M/M_0$$

Note that this implies that U is then a group extension of W by C. Also note that W is not a subgroup of U and for each element in W there is a corresponding coset of C in U.

Remember from the Iwasawa decomposition that G has subgroups K, A, N now, let $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ be their respective subalgebras.

The definitions of U and C have some useful consequences. For $c \in C$, $cH = H$ for all $H \in \mathfrak{a}$. For $u \in U$, $uA = Au$. Also, for any $\alpha \in \Pi$, $c\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ so $cNc^{-1} = N$. In general, if $u \in wC$, then $u\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$.

In analogy to w^-, the principal involution, we fix some $u^- \in U$ such that $u^- \in w^- C$, so that $u^- \mathfrak{n} = \mathfrak{n}^-$ and $u^- N (u^-)^{-1} = N^-$.

Let us study the flow $h^t = \exp(tH)$, $t \in \mathbb{R}$ and $H \in \mathrm{cl}\mathfrak{a}^+$ defined in K using the action of the beginning of the chapter. In this section, we will describe the fixed points of h^t in K as orbits of G_H^0, *the identity component* of the centralizer of H in G. In this description, the orbit of b by $M_* = N(K, \mathfrak{a})$ has a central role.

From Lemma 3.1 the hyperbolic component H of any action can be assumed to be in the closure of the positive chamber by changing the Iwasawa decomposition. Now note that since Iwasawa decompositions are conjugated to one another the corresponding flows are also conjugated. So we can assume without loss of generality that $H \in \mathrm{cl}\mathfrak{a}^+$.

In the case of complete flags, by Proposition 3.25 of [2], since $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, then the action of M_* in the isotropy subalgebras is the same as the action of W or,

$$M_*(\mathfrak{m} \oplus a \oplus \mathfrak{n}) = W(\mathfrak{m} \oplus a \oplus \mathfrak{n})$$

For the study of actions in the isotropy algebras of K note that,

$$M_*(\mathfrak{a} \oplus \mathfrak{n}) = U(\mathfrak{a} \oplus \mathfrak{n})$$

By the Iwasawa decomposition the action of G in $K \simeq G/AN$ corresponds to the action of Ad in the isotropy algebra of $\mathfrak{g}_b = \mathfrak{a} \oplus \mathfrak{n}$. We will show that the set of fixed points in K of h^t is the union of the orbits $G_H^0 ub$ for $u \in U$. *Define*

$$\mathrm{fix}(H, u) := G_H^0 ub$$

in Theorem 2.8 we prove that the fixed points of h^t are

$$\mathrm{fix}(h^t) = \bigcup_{u \in U} \mathrm{fix}(H, u) = \bigcup_{u \in U} G_H^0 ub$$

Proposition 2.5 *The flow h^t in $K \simeq G/AN$ is the gradient of the height function in relation to the Borel metric and the following sets coincide:*

(*i*) *The zeros of the vector field $H\cdot$ in K.*

(*ii*) *The critical points of f_H.*

(*iii*) *The fixed points of the flow h^t, $t \in \mathbb{R}$.*

(*iv*) *The fixed points of the flow h^t, $t \in \mathbb{Z}$.*

Proof: By Theorem 2.4 the induced vector field by H is the gradient of f_H in the Borel metric. Since a gradient field is zero precisely in the critic points of its height function, then the zeros of the induced vector field by H coincide with the critic points of f_H. Since h^t, $t \in \mathbb{R}$ is the flow induced by the vector field H in K, then its fixed points are given by the points where the vector field is zero. By the previous argumentation, these points are the critic points of f_H. Note that since h^t is a gradient flow, then it does not have periodic orbits. Since f_H is strictly increasing along a non-trivial orbit of h^t this implies that the fixed points of *(iii)* and *(iv)* coincide. □

Let us define then *the fixed points of* h^t *in* K by the set described in the previous Proposition.

Proposition 2.6 *For any* $x \in K$*, the following sets of* K*:*

(i) the omega limit of x *in the flow* h^t*,* $t \in \mathbb{R}$*,*

(ii) the omega limit of x *in the flow* h^t*,* $t \in \mathbb{Z}$*,*

coincide and are the fixed points of h^t *in* K*.*

Proof: Let $f = f_H$, then by the Theorem 2.4, then the real function $t \mapsto f(h^t x)$ is not-decreasing and since K is compact f limited from above. And $t \mapsto f(h^t x)$ is constant if, and only if, x is a critic point of f. By the Proposition 2.5, this occurs if, and only if, x is a fixed point of h^t.

First let us show that if y is an omega limit of x by h^t then y is a fixed point of h^t, $t \in \mathbb{R}$. Let $\omega(x)$ be the omega limit set of x by h^t, $t \in \mathbb{R}$. Since the real function $t \mapsto f(h^t x)$ is non-increasing and limited from above, there exists a limit $a = \lim_{t\to\infty} f(h^t x)$. Then, for any real sequence $t_n \to \infty$, we have $a = \lim_{n\to\infty} f(h^{t_n} x)$, so that from the continuity of f, for all $z \in \omega(x)$, $f(z) = a$. Since $y \in \omega(x)$ and $\omega(x)$ are invariant, then $h^t y \in \omega(x)$ for all t. And then, $f(h^t y) = a$, for all t, and from the argument in the beginning, y is a fixed point of h^t, $t \in \mathbb{R}$.

The omega limit of x by h^t, $t \in \mathbb{Z}$, is contained in the omega limit of x by h^t, $t \in \mathbb{R}$. For the opposite implication let y be the omega limit of x by h^t, $t \in \mathbb{R}$. Then there is a sequence t_i of real numbers where $t_i \to \infty$ and $h^{t_i} x \to y$, when $i \to \infty$. From the argument at the beginning, y is a fixed point of h^t. Let $\varepsilon_i \in [0, 1]$ be a sequence such that $t_i + \varepsilon_i = n_i \in \mathbb{N}$ for all i, since $[0, 1]$ is compact there is a subsequence of ε_i (and then t_i and n_i) and we can assume $\lim_{i\to\infty} \varepsilon_i = \varepsilon$. By continuity and by the fact that y is a fixed point of the flow, we have

$$\lim_{i\to\infty} h^{n_i} x = \lim_{i\to\infty} h^{\varepsilon_i}(h^{t_i} x) = h^{\varepsilon} y = y$$

and y is the omega limit of x by h^t, $t \in \mathbb{Z}$. □

Remember that, $K_H = G_H \cap K$, now *define*

$$\mathfrak{a}(\Theta) := \text{ generated by } \{H_\alpha : \alpha \in \Theta\}$$

and

$$\mathfrak{a}(\Theta)^+ := \{H \in \mathfrak{a}(\Theta) : \alpha(H) > 0, \forall \alpha \in \langle\Theta\rangle^+\}$$

is a Weyl chamber where,

$$\langle\Theta\rangle^+ := \langle\Theta\rangle \cap \Pi^+$$

We need to find a semisimple algebra such that $\langle\Theta\rangle$ can be seen as a set of roots. The subalgebra of *type* Θ is the subalgebra

$$\mathfrak{g}(\Theta) := \mathfrak{a}(\Theta) \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha$$

This subalgebra is in fact a semisimple algebra (see Proposition 4.2 [2]). *Define*

$$\mathfrak{k}(\Theta) := \mathfrak{g}(\Theta) \cap \mathfrak{k}$$

For a given $H \in \mathfrak{a}$ consider the annihilator of H in Σ to be

$$\Sigma(H) := \{\alpha \in \Sigma : \alpha(H) = 0\}$$

and $\mathfrak{n}(H)^\pm$ are *defined* as the nilpotent algebras generated by $\mathfrak{g}(H)$.

The semisimple group $G(\Theta)$ of *type* Θ of G is the connected subgroup generated by $\mathfrak{g}(\Theta)$. Denote by $K(\Theta)$, $A(\Theta)$, $N(\Theta)^\pm$ the connected subgroups generated, respectively, by $\mathfrak{k}(\Theta)$, $\mathfrak{a}(\Theta)$, $\mathfrak{n}(\Theta)^\pm$. *Define*

$$G(H) := G(\Sigma(H))$$

and its components in the Iwasawa decomposition are given by

$$K(H) := K(\Sigma(H)), \qquad N(H) := N(\Sigma(H)).$$

Since uM_0 are lateral classes of M_*/M_0 then $uM_0 = M_0u$ and we have that $K(H)uM_0 = K(H)M_0u$ and $G(H)uM_0 = G(H)M_0u$. Note also that $uG_Hu^{-1} = G_{wH}$ if $u \in wC$.

Proposition 2.7 *If G_H^0 and K_H^0 are the components of identity of G_H and K_H then*

$$K_H^0ub = G_H^0ub = G(H)M_0ub = K(H)M_0ub$$

is a compact connected submanifold of K.

Proof: To show the first equality from Proposition 4.19 from [2] we get

$$G_{w^{-1}H} = K_{w^{-1}H}A(G_{w^{-1}H} \cap N)$$

Let $u \in U$ such that $u \in wC$. By taking the conjugate of the equation above by u then

$$G_H = K_H A(G_H \cap uNu^{-1})$$

Now, $AuNu^{-1}$ fixes ub, in fact, $AuNb = uANb = ub$ and

$$G_H ub = K_H A(G_H \cap uNu^{-1})ub \subset K_H ub$$

Since the action of G in K is continuous then $G_H^0 ub \subset K_H^0 ub$ and since $K_H^0 \subset G_H^0$ then $G_H^0 ub = K_H^0 ub$.

For the second equality, since $G_H = G_{\Sigma(H)}$ (see Proposition 4.22 [2]), $G(H) = G(\Sigma(H))$ and $G_\Theta = MA_\Theta G(\Theta)$ (see Proposition 4.14 of [2]), then

$$G_H = MA_H G(H)$$

Note that $G(H) \subset G_H = Z(\mathfrak{a}_H)$ normalizes A_H and M_0 normalizes $G(H)$ so

$$M_0 A_H G(H) = M_0 G(H) A_H = G(H) M_0 A_H$$

and $G(H)M_0A_H$ is a subgroup of G_H^0. Since the two subgroups are connected and have the same Lie algebra then, in fact, $G_H^0 = G(H)M_0A_H$.

The term $A_H \subset A$ fixes ub so $Aub = uAb = ub$ and

$$G_H^0 ub = G(H)M_0 ub$$

To get the third equality, we show that $K_H^0 = K(H)M_0$. By Theorem 4.21 of [2], we have $K_H = K(H)M$. First let us show that M_0 normalizes $\kappa(\mathfrak{g}_\alpha)$, where $\kappa(X) = (X + \theta X)/2$, for $X \in \mathfrak{g}$. By Proposition 3.25 of [2], $M_0 \subset M$ normalizes $\mathfrak{g}_\alpha$ for all α. If $m_0 \in M_0$ there is $Z \in \mathfrak{m}$ such that $m_0 = \exp(Z)$, let $X \in \mathfrak{g}_\alpha$ then

$$m_0\kappa(X) = \exp(Z)\kappa(X) = e^{\mathrm{ad} Z}\kappa(X) = \frac{1}{2}e^{\mathrm{ad} Z}(X + \theta X)$$

Since $[Z, \theta X] = \theta[Z, X]$ then $e^{\mathrm{ad} Z}\theta X = \theta e^{\mathrm{ad} Z} X$ and

$$m_0\kappa(X) = \frac{1}{2}(e^{\mathrm{ad} Z}X + \theta e^{\mathrm{ad} Z}X) = \frac{1}{2}(m_0X + \theta(m_0X)) = \kappa(m_0X)$$

then $m_0\kappa(\mathfrak{g}_\alpha) \in \kappa(\mathfrak{g}_\alpha)$ and we get that M_0 normalizes $\kappa(\mathfrak{g}_\alpha)$.

Since $K(H)M_0 = M_0K(H)$ we conclude that $K_H^0 = K(H)M_0$, since these Lie groups have the same Lie algebra. Since $M_0 \subset K_H^0$ then $K_H^0 M_0 u = K_H^0 u$. Note that if $uM_0 = u'M_0$ then $K_H^0 u = K_H^0 u'$ so these two subgroups don't depend on the choice of the element in the coset class. In a similar form, since $M_0 \subset G_H^0$ we get $G_H^0 u = G_H^0 u'$.

For the last statement, since K_H^0 is a compact and connected subgroup of K and the action of G in K is differentiable then $\mathcal{M} = K_H^0 ub$ is a connected and compact submanifold of K. □

Note that if $uM_0 = u'M_0$ since $M_0 \subset K_H^0 = K(H)M_0$ then $K_H^0 M_0 u = K_H^0 u$ and $K_H^0 u = K_H^0 u'$ In the proof of the previous Proposition we showed that $M_0 \subset K_H^0 \subset G_H^0$, then we can write $K_H^0 ub$ in the form $K_H^0 u M_0 b$. Similarly, we can write $G_H^0 ub$ as $G_H^0 u M_0 b$.

Theorem 2.8 *The fixed points of h^t in K are given by the union*

$$\bigcup_{u \in U} K_H^0 ub = \bigcup_{u \in U} G_H^0 ub$$

Proof: By Proposition 2.5, the fixed points of h^t in K are the critical points of the height function f_H. By Proposition 2.7, $K_H^0 ub = K(H)M_0 ub$.

To get the critical points of f_H we rewrite the derivative at the point kb, $k \in K$, in the direction $Z \in \mathfrak{k}$, given by Proposition 2.3, as follows

$$\begin{aligned} f_H'(kb)k(Z \cdot b) &= \langle k[Z, H_r],\, H \rangle \\ &= \langle [kZ, kH_r],\, H \rangle \\ &= -\langle \mathrm{ad}(kH_r)kZ,\, H \rangle \\ &= -\langle kZ,\, \mathrm{ad}(kH_r)H \rangle \\ &= -\langle kZ,\, [kH_r, H] \rangle \end{aligned}$$

where we used that k is $\langle \cdot, \cdot \rangle$-orthogonal, $kH_r \in \mathfrak{s}$ (see Proposition 2.40 of [2]) and $\mathrm{ad}(kH_r)$ is $\langle \cdot, \cdot \rangle$-symmetric (see Proposition 2.23 of [2]). Since $k\mathfrak{k} = \mathfrak{k}$, then kb is a critical point of f_H if, and only if,

$$\langle Z,\, [kH_r, H] \rangle = 0, \qquad \text{for all } Z \in \mathfrak{k}$$

Since kH_r and $H \in \mathfrak{s}$ then $[kH_r, H] \in \mathfrak{k}$ and, since $\langle \cdot, \cdot \rangle$ is an inner product in $\mathfrak{k}$, then the previous equation is true if, and only if, $[kH_r, H] = 0$. Then kb is a critical point of f_H if, and only if, kH_r centralizes H.

For $k \in M_*$ we have $kH_r \in \mathfrak{a}$ then $[kH_r, H] = 0$. Then the points kb, $k \in M_*$, are all critical points. Since $M_* = M_0 U$ then $M_0 ub$ for $u \in U$ are critical points. Now since f_H is K_H-invariant to the left then the orbit $K_H Ub$ consists of critical points. In fact, let $\alpha(t)$ be a differentiable curve such that $\alpha(0) = kub$, where $k \in K_H$.

From the K_H invariance of f_H, then $f_H(\alpha(t)) = f_H(k^{-1}\alpha(t))$, where $k^{-1}\alpha(t)$ is a differentiable curve and $k^{-1}\alpha(0) = ub$. Then

$$(f_H \circ \alpha)'(0) = (f_H \circ k^{-1}\alpha)'(0) = 0$$

since ub is a critic point.

In fact, the points $K_H Ub$ are all the critical points of f_H. Indeed, if kb is critical, $k \in K$, from the previous argument then kH_r centralizes $H \in \text{cl}\mathfrak{a}^+$. By Lemma 4.20 of [2], there is $l \in K(H)$ such that $lkH_r \in \mathfrak{a}$. By Proposition 4.28 of [2] and by Corollary 3.24 of [2], there is $m \in M_*$ such that $lkH_r = mH_r$. Then $m^{-1}lkH_r = H_r$, and

$$m^{-1}lk \in K_{H_r} = M$$

and,

$$k = l^{-1}m(m^{-1}lk) \in K(H)M_*$$

Since $M_* = M_0 U$ then $k \in K(H)M_0 U$. Now, since $K_H^0 = K(H)M_0$ (from the beginning of the proof of Proposition 2.7) then $K_H^0 Ub$ are the only critical points of f_H. The equality in the Theorem is a consequence of Proposition 2.7. □

2.3 Linearization

In this section, we will prove in Theorem 2.14, a linearization of the gradient flow h^t around each component of fixed points $\mathcal{M} = \text{fix}(H, u) := K_H^0 ub$.

First let us define a *normal linearization* of a differentiable flow.

Definition 2.9 *Let ϕ^t be a differentiable flow in a Riemannian manifoldX for $t \in \mathbb{R}$. A invariant manifold $\mathcal{M} \subset X$ is normally hyperbolic if the tangent bundle of X over $\mathcal{M}$ has orthogonal invariant sub-bundles V^-, V^+ and there are positive constants c and $\lambda < \mu$ such that*

(i) $TX|_{\mathcal{M}} = T\mathcal{M} \oplus V^- \oplus V^+$.

(ii) $|D\phi^t v| \leq ce^{-\lambda t}|v|$ *for all* $v \in V^-$ *and* $t \geq 0$.

(iii) $|D\phi^{-t} v| \leq ce^{-\lambda t}|v|$ *for all* $v \in V^+$ *and* $t \geq 0$.

(iv) $|D\phi^t v| \leq ce^{\mu|t|}|v|$ *for all* $v \in T\mathcal{M}$ *and* $t \in \mathbb{R}$.

In the previous definition V^- is called the stable fiber of $\mathcal{M}$, V^+ is the unstable fiber of $\mathcal{M}$ and $V^- \oplus V^+$ is the *normal bundle*

$$V := V^- \oplus V^+$$

We identify the zero section of the normal bundle with the *base* $\mathcal{M}$. Let ϕ^t be a differentiable flow in X for $t \in \mathbb{R}$. *Define* $\mathbf{N}\phi^t$ as the restriction to the normal fiber of the differential

$$\mathbf{N}\phi^t := D\phi^t|_V$$

A *normally hyperbolic* linearization around $\mathcal{M}$ is a local diffeomorphism f from a neighbourhood $\mathcal{A}$ of the zero section at the normal bundle to a neighbourhood $\mathcal{B}$ of $\mathcal{M}$ such that f restricted to the normal bundle is the identity in $\mathcal{M}$ and is a local conjugation

$$f(\mathbf{N}\phi^t(v)) = \phi^t f(v)$$

for all $v \in \mathcal{A}$ such that $\mathbf{N}\phi^t v \in \mathcal{A}$.

Now, define an adequate complement of the isotropy subalgebra $\mathfrak{g}_x$ for x in K to use it as a model for the tangent bundle TK_x. Let us fix in $\mathfrak{g}$ a Cartan inner product $\langle \cdot, \cdot \rangle$, that is K-invariant (Proposition 2.40 of [2]). Consider $\mathfrak{g}_x^\perp$ the orthogonal complement of $\mathfrak{g}_x$ in $\mathfrak{g}$ with respect to this inner product. From Proposition C.13 (i) of [2],

$$\mathfrak{g}_x^\perp \to TK_x \qquad X \mapsto X \cdot x \tag{2.7}$$

is a linear isomorphism, and, for $k \in K$,

$$k(\mathfrak{g}_x^\perp) = \mathfrak{g}_{kx}^\perp \tag{2.8}$$

Now, since $\mathfrak{g}_{kb} = k(\mathfrak{a} \oplus \mathfrak{n})$ and $(\mathfrak{a} \oplus \mathfrak{n})^\perp = \mathfrak{m} \oplus \mathfrak{n}^-$ (equation 2.1). Then, in particular,

$$\mathfrak{g}_{um_0b}^\perp = u(\mathfrak{m} \oplus \mathfrak{n}^-) = \mathfrak{m} \oplus u\mathfrak{n}^- \tag{2.9}$$

since $um_0 \in uM_0 \subset M_*$ and $um_0\mathfrak{m} = \mathfrak{m}$. So we can use $\mathfrak{g}_x^\perp$ as a model for TK_x and consider the application

$$TK_x \to K \qquad X \cdot x \mapsto \exp(X)x \qquad X \in \mathfrak{g}_x^\perp \tag{2.10}$$

We will show that, for the following Riemannian metric in K, the restriction of this application to the normal bundle of $\mathcal{M}$ is the differentiable linearization needed.

Proposition 2.10 *Let*

$$\langle X \cdot x, Y \cdot x \rangle_x := \langle X, Y \rangle \qquad \textit{for } X, Y \in \mathfrak{g}_x^\perp$$

be a K-invariant Riemannian metric in K such that the map defined in (2.7) *is an isometry. Then, for $Y \in \mathfrak{g}$,*

$$|Y \cdot x|_x \leq |Y|$$

with equality if, and only if, $Y \in \mathfrak{g}_x^{\perp}$.

Proof: By definition the map in (2.7) is an isometry and the metric is well defined in each tangent space. To show the K-invariance of the metric, let $X \in \mathfrak{g}_x^{\perp}$, so $k(X \cdot x) = kX \cdot kx$, where $kX \in k(\mathfrak{g}_x^{\perp}) = \mathfrak{g}_{kx}^{\perp}$ and similarly, $kY \in \mathfrak{g}_{kx}^{\perp}$. Then by, the K-invariance of the Cartan inner product and the definition of the metric,

$$\begin{aligned}
\langle k(X \cdot x), k(Y \cdot x)\rangle_{kx} &= \langle kX \cdot kx, kY \cdot kx\rangle_{kx} \\
&= \langle kX, kY\rangle \\
&= \langle X, Y\rangle \\
&= \langle X \cdot x, Y \cdot x\rangle_x
\end{aligned}$$

Because of the K-invariance by Proposition C.13 (iii) of [2] this defines a Riemannian metric.

For the last property, let $Y = Y_1 + Y_2$ where $Y_1 \in \mathfrak{g}_x^{\perp}$ and $Y_2 \oplus \mathfrak{g}_x$. Then $Y \cdot x = Y_1 \cdot x$, and

$$|Y \cdot x|_x = |Y_1| \leq |Y|$$

with equality if, and only if, $Y_2 = 0$ or if, and only if, $Y = Y_1 \in \mathfrak{g}_x^{\perp}$. □

From now on, let us fix the previous metric in K. Note that, in general, this metric is different from the Borel metric used previously. First we define candidates for the stable and unstable bundles orthogonal to the tangent bundle of $\mathcal{M}$. By Proposition C.9 of [2] the tangent space of the orbit

$$\mathcal{M} = G_H^0 ub$$

is given by

$$T\mathcal{M} = G_H^0(\mathfrak{g}_H \cdot ub) \subset TK$$

Define

$$N(H) := N(\Sigma(H)).$$

The nilpotent algebras of $\mathfrak{g}(\Theta)$ are $\mathfrak{n}(\Theta)$ and $\mathfrak{n}(\Theta)^-$ (see Proposition 4.5 of [2])

The subalgebra $\mathfrak{n}_H$ is generated by the positive roots not in $\Sigma(H)$ so that $\mathfrak{n} = \mathfrak{n}(H) \oplus \mathfrak{n}_H$. The definition of n_H^- is analogous.

Consider the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_H \oplus \mathfrak{n}_H^- \oplus \mathfrak{n}_H^+ \tag{2.11}$$

where $\mathfrak{g}_H$ and $\mathfrak{n}_H^{\pm}$ are G_H-invariant by Proposition 4.23 from [2]. *Define*

$$V^{\pm} := G_H^0(\mathfrak{n}_H^{\pm} \cdot ub) \subset TK$$

for $x \in \mathcal{M}$, and subspaces

$$\mathfrak{v}_x^{\pm} := \mathfrak{n}_H^{\pm} \cap \mathfrak{g}_x^{\perp}$$

Proposition 2.11 *(i) The tangent space of K in $\mathcal{M}$ can be decomposed as the Whitney orthogonal sum,*

$$TK|_{\mathcal{M}} = T\mathcal{M} \oplus V^{-} \oplus V^{+}$$

where $V^{\pm}$ are G_H-invariant differentiable vector sub-bundles of $\mathcal{M}$. And, in particular, $V^{-} \oplus V^{+}$ is a normal fiber of $T\mathcal{M}$.

(ii) For $x \in \mathcal{M}$, the map

$$\mathfrak{v}_x^{\pm} \to V_x^{\pm} \qquad Y \mapsto Y \cdot x$$

is a linear isomorphism and $k(\mathfrak{v}_x) = \mathfrak{v}_{kx}$ for $k \in K_H$.

Proof: Since $\mathfrak{n}_H^{\pm}$ are G_H-invariant, by Proposition 4.24 of [2], they are G_H^0-invariant and their image in $\mathfrak{g}/\mathfrak{g}_{ub}$ is invariant by the isotropy of ub in G_H^0. From Proposition C.12 of [2] $V^{\pm}$ are sub-bundles G_H^0-invariant over $\mathcal{M} = G_H^0 ub$. From the G_H^0-invariance of $\mathfrak{n}_H^{\pm}$ it follows that

$$V_x^{\pm} = \{(g\mathfrak{n}_H^{\pm} \cdot gx_0) : \, g \in G_H^0, gx_0 = x\} = \mathfrak{n}_H^{\pm} \cdot x \tag{2.12}$$

To prove the Whitney sum, by the orthogonal decomposition

$$\mathfrak{g} = u(\mathfrak{m} \oplus \mathfrak{n}^{-}) \oplus u(\mathfrak{a} \oplus \mathfrak{n})$$

by the decomposition in root spaces of $\mathfrak{n}_H^{\pm}$ and by equation (2.9) then

$$\begin{aligned} \mathfrak{n}_H^{\pm} &= \big(\mathfrak{n}_H^{\pm} \cap u(\mathfrak{m} \oplus \mathfrak{n}^{-})\big) \oplus \big(\mathfrak{n}_H^{\pm} \cap u(\mathfrak{a} \oplus \mathfrak{n})\big) \\ &= (\mathfrak{n}_H^{\pm} \cap \mathfrak{g}_{ub}^{\perp}) \oplus (\mathfrak{n}_H^{\pm} \cap \mathfrak{g}_{ub}) \end{aligned}$$

For $x \in \mathcal{M}$, then $x = kub$ with $k \in K_H^0$. From equation (2.8) then $k(\mathfrak{g}_{ub}^{\perp}) = \mathfrak{g}_x^{\perp}$ and $k\mathfrak{g}_{ub} = \mathfrak{g}_x$. Since $K_H^0 = K \cap G_H^0$ normalizes $\mathfrak{n}_H^{\pm}$, taking k in both sides of the previous decomposition we get

$$\begin{aligned} \mathfrak{n}_H^{\pm} &= (\mathfrak{n}_H^{\pm} \cap \mathfrak{g}_x^{\perp}) \oplus (\mathfrak{n}_H^{\pm} \cap \mathfrak{g}_x) \\ &= \mathfrak{v}_x^{\pm} \oplus (\mathfrak{n}_H^{\pm} \cap \mathfrak{g}_x) \end{aligned} \tag{2.13}$$

From this and from equation (2.12) then

$$\mathfrak{v}_x^{\pm} \to V_x^{\pm} \qquad Y \mapsto Y \cdot x$$

is a linear isomorphism. Since $\mathfrak{g}_H$ is also G_H^0-invariant, the same argument applies to $T\mathcal{M}$ to get $T\mathcal{M}_x = \mathfrak{g}_H \cdot x$ and

$$\mathfrak{g}_H = (\mathfrak{g}_H \cap \mathfrak{g}_x^{\perp}) \oplus (\mathfrak{g}_H \cap \mathfrak{g}_x) \tag{2.14}$$

and,

$$T\mathcal{M}_x = (\mathfrak{g}_H \cap \mathfrak{g}_x^{\perp}) \cdot x \tag{2.15}$$

Combining the decompositions (2.11), (2.13), and (2.14), we get the orthogonal sum

$$\mathfrak{g}_x^{\perp} = (\mathfrak{g}_H \cap \mathfrak{g}_x^{\perp}) \oplus \mathfrak{v}_x^{-} \oplus \mathfrak{v}_x^{+}$$

The image of this decomposition by the isometry (2.7) is the orthogonal sum

$$TK_x = T\mathcal{M}_x \oplus V_x^{-} \oplus V_x^{+}$$

Since $k \in K_H^0 \subset K_H$ normalizes $\mathfrak{n}_H^{\pm}$, from equation (2.8) then $k\mathfrak{v}_x = \mathfrak{v}_{kx}$. □

To study the dynamics in the fibers $V^{\pm}$, first we study the dynamics in the subalgebras $\mathfrak{n}_H^{\pm}$.

Lemma 2.12 *Let $h = \exp H$, where $H \neq 0$. Then*

$$|h^t Y| \leq \mathrm{e}^{-\mu t}|Y| \quad \textit{for } Y \in \mathfrak{n}_H^{-},\ t \geq 0$$

and

$$|h^{-t} Y| \leq \mathrm{e}^{-\mu t}|Y| \quad \textit{for } Y \in \mathfrak{n}_H^{+},\ t \geq 0$$

where

$$\mu = \min\{\alpha(H) : \alpha(H) > 0,\ \alpha \in \Pi\}$$

Proof: Let $Y \in \mathfrak{n}_H^{\pm}$, then $h^t Y = \mathrm{e}^{t\mathrm{ad}(H)}Y$, where $\mathrm{e}^{t\mathrm{ad}(H)}$ is $\langle \cdot, \cdot \rangle$-symmetric with eigenvalues in $\mathfrak{n}_H^{\pm}$ given by

$$\{\mathrm{e}^{\pm\alpha(H)t} :\ \alpha(H) > 0,\ \alpha \in \Pi\}$$

since $\mathrm{ad}(H)$ is $\langle \cdot, \cdot \rangle$-symmetric (see Proposition 2.23 of [2]) with eigenvalues in $\mathfrak{n}_H^{\pm}$ given by

$$\{\pm\alpha(H) :\ \alpha(H) > 0,\ \alpha \in \Pi\}$$

Since Y is an orthogonal sum of eigenvectors $Y = \sum_\alpha Y_\alpha$, with $Y_\alpha \in \mathfrak{g}_\alpha$ then,

$$\left|h^t Y\right|^2 = \sum_\alpha e^{\pm 2\alpha(H)t} |Y_\alpha|^2$$

For $t > 0$ and $Y \in \mathfrak{n}_H^-$, then

$$\left|h^{-tY}\right|^2 = \sum_\alpha e^{-2\alpha(H)t} |Y_\alpha|^2 \le e^{-2\mu t} \sum_\alpha |Y_\alpha|^2 = e^{-2\mu t} |Y|^2$$

since $e^{-\alpha(H)t} < e^{-\mu t}$, for $t > 0$ and all $\alpha \in \Pi$ with $\alpha(H) > 0$, proving the first statement. For $t > 0$ and $Y \in \mathfrak{n}_H^+$, then

$$\left|h^{-tY}\right|^2 = \sum_\alpha e^{-2\alpha(H)t} |Y_\alpha|^2 \le e^{-2\mu t} \sum_\alpha |Y_\alpha|^2 = e^{-2\mu t} |Y|^2$$

since $e^{-\alpha(H)t} < e^{-\mu t}$, for $t > 0$ and all $\alpha \in \Pi$ with $\alpha(H) > 0$, proving the second statement. □

The next result shows that the set of fixed points $\mathcal{M}$ of h^t in K is normally hyperbolic (see Definition 2.9).

Proposition 2.13 *The fibers $V^\pm$ are h^t invariant and*

$$|h^t v| \le e^{-\mu t}|v| \quad \textit{for } v \in V^-,\ t \ge 0$$

$$|h^{-t} v| \le e^{-\mu t}|v| \quad \textit{for } v \in V^+,\ t \ge 0$$

where $\mu > 0$ is obtained by Lemma 2.12. And,

$$|h^t v| = |v| \quad \textit{for } v \in T\mathcal{M},\ t \in \mathbb{R}$$

Proof: Since $\mathfrak{n}_H^\pm$ and $\mathcal{M}$ are h^t-invariant, from the definition of $V^\pm$ it is h^t-invariant. From Proposition 2.11(ii) then $v \in V^\pm$ is given by

$$v = Y \cdot x \qquad \text{with } Y \in \mathfrak{v}_x = \mathfrak{n}_H^\pm \cap \mathfrak{g}_x^\perp$$

where $|v| = |Y|$ and $x \in \mathcal{M}$. Since $h^t v = h^t Y \cdot x$ then

$$|h^t v| \le |h^t Y|$$

by Proposition 2.10. The inequalities from the Proposition follow from Lemma 2.12 and from $|Y| = |v|$. To prove the last statement, by equation (2.15) and the proof of Proposition 2.11, for $v \in T\mathcal{M}$

$$v = Y \cdot x \qquad \text{with } Y \in \mathfrak{g}_H \cap \mathfrak{g}_x^\perp$$

where $|v| = |Y|$ and $x \in \mathcal{M}$. Then $h^t v = h^t(Y \cdot x) = Y \cdot x$, since h^t centralizes $\mathfrak{g}_H$, so

$$|h^t v| = |Y| = |v|$$

□

Take V to be the normal bundle $\mathbf{N}\mathcal{M}$ of $\mathcal{M}$ in K so that

$$V := V^- \oplus V^+$$

and define the *model of normal bundle* V_x of $x \in \mathcal{M}$ by

$$\mathfrak{v}_x := \mathfrak{v}_x^- \oplus \mathfrak{v}_x^+$$

and define the restriction of the map (equation 2.10) to the normal bundle by

$$\Psi : V \to K \qquad X \cdot x \mapsto \exp(X)x \quad \text{where } x \in \mathcal{M},\ X \in \mathfrak{v}_x \tag{2.16}$$

The next result shows that this is the linearization we wanted to prove so that the flow by h^t in K is normally hyperbolic.

Theorem 2.14 *(i) Ψ is a normal linearization of the flow h^t in a neighbourhood of $\mathcal{M}$.*

(ii) The restriction of Ψ to V^- is a h^t-equivariant diffeomorphism between V^- and $\mathrm{st}(\mathcal{M})$*.*

(iii) Ψ is K_H-equivariant.

Proof: Since $\Psi(0 \cdot x) = x$, then Ψ is a bijection from the zero section V_0 to $\mathcal{M}$.

First we prove the equivariances of Ψ. From Proposition 2.11(ii), $k(\mathfrak{v}_x) = \mathfrak{v}_{kx}$, for $k \in K_H$, so that for $X \in \mathfrak{v}_x$, then

$$\Psi(k(X \cdot x)) = \Psi(kX \cdot kx) = \exp(kX)kx = k\exp(X)x = k\Psi(X \cdot x)$$

that proves the K_H-equivariance. Next we prove that, $h^t(\mathfrak{v}_x) = \mathfrak{v}_x$. Since h^t leaves $x \in \mathcal{M}$ fixed then it normalizes $\mathfrak{g}_x$ and it also normalizes $\mathfrak{g}_x^\perp$, since h^t acts in $\mathfrak{g}$ as a self-adjoint transformation with relation to the Cartan inner product. Note that h^t also normalizes $\mathfrak{n}_H^\pm$, since $h^t \in G_H$, so that h^t normalizes $\mathfrak{v}_x$, and

$$\Psi(h^t(X \cdot x)) = \Psi(h^t X \cdot x) = \exp(h^t X)x = h^t \exp(X)x = h^t\Psi(X \cdot x)$$

so that Ψ is h^t-equivariant.

To prove the differentiability let us consider an adjusted map to the sub-bundle V and use the map from Proposition C.13(ii) of [2]. Taking in the Proposition $K = K_H^0$ and $U = \mathcal{M}$ we can consider just the map ψ since it is possible to define $s : \mathcal{M} \to K_H^0$ in the whole set $\mathcal{M}$, if $x = ub$ then

$$\psi : \mathfrak{v}_{ub} \times \mathcal{M} \to V \qquad (Y, y) \mapsto s(y)(Y \cdot ub)$$

Remember that there is a diffeomorphism,

$$\varphi : G/AN \to K, \quad \varphi : kb \mapsto k$$

Define then the section $s(y) = \varphi(y)u^{-1} \in K_H^0$ and the map $p_x(k) = \varphi^{-1}(ku)$, so that,

$$\mathfrak{v}_{ub} \longrightarrow V_{ub} \xrightarrow{s(y)} V_y \qquad Y \mapsto s(y)(Y \cdot ub)$$

is a linear isomorphism, since it is a composition of isomorphisms. Also, since

$$\psi(Y, y) = s(y)Y \cdot s(y)ub = s(y)Y \cdot y$$

where $s(y)Y \in \mathfrak{v}_y$, and

$$\Psi \circ \psi(Y, y) = \exp(s(y)Y)y = s(y)\exp(Y)ub \tag{2.17}$$

is differentiable in (Y, y).

Next, we prove that Ψ is a local diffeomorphism in a neighbourhood of the zero section V_0. From the inverse image Theorem, it is sufficient to prove that the differential $\Psi'(v)$ is an isomorphism for all $v \in V_0$. Let $x \in \mathcal{M}$ be the base point of $v = 0 \cdot x$, so that $\psi(0, x) = v$. So we must prove that

$$(\Psi \circ \psi)'(0, x) : \mathfrak{v}_{ub} \times T\mathcal{M}_x \to TK_x$$

is an isomorphism.

Let us consider the coordinate curve $(0, \alpha(t))$, where $\alpha(0) = x$ and $\alpha'(0) = q \in T\mathcal{M}_x$. And the coordinate curve (tY, x), where $Y \in \mathfrak{v}_{ub}$, so that the tangent vectors in $t = 0$, are respectively $(0, q)$ and $(Y, 0)$. So that,

$$\Psi \circ \psi(0, \alpha(t)) = s(\alpha(t))ub = \alpha(t)$$

since $s(y)u = y$ and

$$\Psi \circ \psi(tY, x) = s(x)\exp(tY)ub$$

Then

$$(\Psi \circ \psi)'(0, x)(Y, q) = q + s(x)(Y \cdot ub)$$

and the image of the differential $(\Psi \circ \psi)'(0, x)$ is $T\mathcal{M}_x \oplus V_x$ that is all TK_x, by Proposition 2.11. This proves that $\Psi'(v)$ is an isomorphism for all $v \in V_0$.

Now we show that Ψ is injective in a neighbourhood $\mathcal{A}$ of V_0. Assume, by contradiction, that there is no neighbourhood of V_0 where Ψ is injective. So that, there are sequences v_k, $v_k' \in V$ such that $v_k \neq v_k'$, $\Psi(v_k) = \Psi(v_k')$ and $v_k, v_k' \to V_0$, when $k \to \infty$. Since V_0 is compact, we can the assume the neighbourhood is compact and by taking sub-sequences, $v_k \to 0 \cdot x$, $v_k' \to 0 \cdot y$, where $x, y \in \text{fix}(H, u)$. Then

$$x = \Psi(0 \cdot x) = \lim_k \Psi(v_k) = \lim_k \Psi(v_k') = \Psi(0 \cdot y) = y$$

and since $v_k \neq v_k'$, the map Ψ is not locally injective in $0 \cdot x = 0 \cdot y$, contradicting the fact that Ψ is a local diffeomorphism in neighbourhood of the zero section V_0.

To finish the proof, *define* $\mathcal{B} := \Psi(\mathcal{A})$. Item (ii) is then a consequence of Lemma A.5 (iii) [2], since by uniqueness, $\Psi|_{V^-}$ is an extension h^t-equivariant of $\Psi_{V^- \cap \mathcal{A}}$ for all V^-. □

2.4 Bruhat Decomposition

The *stable set* of the invariant set $\mathcal{M}$, written st$\mathcal{M}$, is the set of points of K such that the omega limit is in $\mathcal{M}$. Later we will prove that in our case this set is in fact a manifold. Similarly, the *unstable set* un$\mathcal{M}$, is the set of points of K that the alpha limit is in $\mathcal{M}$. In this section we will show that the stable set of each component of fixed points $\mathcal{M} = \text{fix}(H, u)$ is an immersed submanifold of K given by the orbit

$$N_H^- \mathcal{M}$$

This will provide a decomposition of K which we regard as a general Bruhat decomposition.

Since $h^t \in G_H$, by Proposition 4.24 from [2], h^t normalizes the nilpotent subgroups N_H^-, $N^-(H)$, and nilpotent subalgebras $\mathfrak{n}_H^-$, $\mathfrak{n}^-(H)$. In particular this shows that $N_H^- \mathcal{M}$ is h^t-invariant.

Note that since the orbit $N_H^- \mathcal{M}$ in general is not compact, one of the key points will be to prove that it is still embedded. For this, we study some dynamical properties of the subgroup N_H^-.

Lemma 2.15 *(i)* $N_H^- = \{n \in N^- : \lim_{t \to \infty} h^t n h^{-t} = 1\}$ *and*

$$N^- = N_H^- N^-(H)$$

(ii) If $y \in N_H^- x$, for $x \in \mathcal{M}$, then

$$\lim_{t\to\infty} h^t y = x$$

(iii) If $y = \Psi(v)$ and $x \in \omega(y) \cap \mathcal{M}$. Then $v \in V_x^-$ and $y \in N_H^- x$. In particular, $\omega(y) = \{x\}$.

Proof: To prove (i), first we use that $\exp : \mathfrak{n}^- \to N^-$ is a homeomorphism, by (Theorem 2.41 of [2]), and $\mathfrak{n}^- = \mathfrak{n}_H^- \oplus \mathfrak{n}^-(H)$ (Proposition 4.24 of [2]) then for any $n \in N^-$, $n = \exp(Y_1 + Y_2)$ with $Y_1 \in \mathfrak{n}_H^-$ and $Y_2 \in \mathfrak{n}^-(H)$, from Lemma 2.12 then

$$h^t n h^{-t} = \exp(h^t Y_1 + Y_2) \to \exp(Y_2) \in N^-(H) \tag{2.18}$$

when $t \to \infty$, since h^t centralizes $\mathfrak{n}^-(H)$. So that $h^t n h^{-t} \to 1$, if and only if, $Y_2 = 0$, that is equivalent to $Y = Y_1 \in \mathfrak{n}_H^-$ and $n \in N_H^-$.

To prove that $N^- = N_H^- N^-(H)$ let us prove the inclusion "$\subset$"since the other side is immediate. Let $n \in N^-$ and $n = \exp(Y_1 + Y_2)$ as previously. Consider $n\exp(-Y_2) \in N^-$, since h^t centralizes Y_2 then by equation (2.18)

$$h^t(n\exp(-Y_2))h^{-t} = (h^t n h^{-t})\exp(-Y_2) \to \exp(Y_2)\exp(-Y_2) = 1$$

so that from item (i), $n\exp(-Y_2) = n_1 \in N_H^-$, and $n = n_1 n_2$ where $n_2 = \exp(Y_2) \in N^-(H)$.

For (ii), let $y = nx$, with $n \in N_H^-$ and $h^{-t}x = x$, then from (i)

$$h^t y = h^t n h^{-t} x \to x$$

To prove (iii) let Ψ be the linearization from Theorem 2.14 and let $t_n \to \infty$ so that

$$h^{t_n} y \to x$$

From the h^t-equivariance of Ψ then

$$\Psi(h^{t_n} v) = h^{t_n}\Psi(v) = h^{t_n} y \to x = \Psi(0 \cdot x)$$

Since Ψ is a diffeomorphism in a neighbourhood of V_0, then

$$h^{t_n} v \to 0 \cdot x$$

and $|h^{t_n} v| \to 0$. From Proposition 2.13 then $v \in V^-$. Now let, $v \in V_{x'}^-$, where $v = X \cdot x'$, $X \in \mathfrak{v}_{x'}^-$ then

$$y = \Psi(v) = \exp(X)x' \in N_H^- x'$$

since $\mathfrak{v}_{x'}^- \subset \mathfrak{n}_H^-$. From item (ii),

$$h^{t_n} y \to x' = x$$

and $v \in V_x^-$. Also, $\omega(y) = \{x\}$. $\square$

From items (ii) and (iii) of Lemma 2.15 then the map

$$p : N_H^- \mathcal{M} \to \mathcal{M} \qquad y \mapsto \lim_{t\to\infty} h^t y$$

is surjective and the fiber over $x \in \mathcal{M}$ is $N_H^- x$.

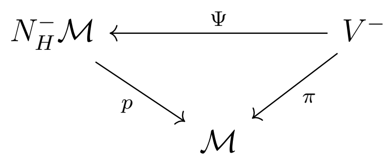
Proposition 2.16 *(i) The stable set of $\mathcal{M}$,*

$$\mathrm{st}\mathcal{M} = N_H^- \mathcal{M}$$

is an immersed submanifold of K.

(ii) The restriction of the linearization Ψ to V^- is an h^t-equivariant diffeomorphism over $N_H^-\mathcal{M}$ that is then, diffeomorphic to the stable bundle V^- over $\mathcal{M}$.

(iii) The following diagram commutes.



In particular, p is a differentiable submersion.

Proof: Let $S = N_H^-\mathcal{M}$. Given $\mathcal{A}$ and $\mathcal{B}$ as in the proof of Proposition 2.14. First we prove that

$$\Psi(V^- \cap \mathcal{A}) = S \cap \mathcal{B} \tag{2.19}$$

By Proposition 2.11, $V_x^- = \mathfrak{v}_x^- \cdot x$, where $\mathfrak{v}_x^- \subset \mathfrak{n}_H^-$ and $x \in \mathcal{M}$, and by the definition of Ψ and $\mathcal{B}$,

$$\Psi(V^- \cap \mathcal{A}) \subset (\exp(\mathfrak{n}_H^-)\mathcal{M}) \cap \mathcal{B} \subset S \cap \mathcal{B}$$

For the converse, let $y \in S \cap \mathcal{B}$. Since $y \in \mathcal{B}$ then $y = \Psi(v)$ for $v \in \mathcal{A}$. And since $y \in S$, by Lemma 2.15 then $h^t y \to x \in \mathcal{M}$ and, by the same Lemma, $v \in V_x^-$. So $y = \Psi(v)$, where $v \in V_x^- \cap \mathcal{A}$, and

$$S \cap \mathcal{B} \subset \Psi(V^- \cap \mathcal{A})$$

Next, we prove that $\Psi(V^-) = S$. Let $y \in S$, since S is h^t-invariant and $\mathcal{B}$ is a neighbourhood of $\mathcal{M}$, by Lemma 2.15 there is $t \geq 0$ such that $h^t y \in S \cap \mathcal{B}$ and $y \in h^{-t}(S \cap \mathcal{B})$. So that

$$S = \bigcup_{t \geq 0} h^{-t}(S \cap \mathcal{B}) \tag{2.20}$$

A similar argument using that h^t invariance of V^-, the neighbourhood $\mathcal{A}$ of V_0 and Proposition 2.13 shows that

$$V^- = \bigcup_{t \geq 0} h^{-t}(V^- \cap \mathcal{A})$$

Then, by equations (2.19) and (2.20) and by the h^t-equivariance of Ψ it follows that

$$\Psi(V^-) = \Psi\left(\bigcup_{t \geq 0} h^{-t}(V^- \cap \mathcal{A})\right) = \bigcup_{t \geq 0} h^{-t}\Psi\left(V^- \cap \mathcal{A}\right) = \bigcup_{t \geq 0} h^{-t}(S \cap \mathcal{B}) = S$$

From Theorem 2.14(ii) it follows that

$$\mathrm{st}\mathcal{M} = S = N_H^- \mathcal{M}$$

and from the same Theorem 2.14(ii), we prove item (ii).

Now we prove that S is an immersed submanifold of K. Since K_H normalizes N_H^-, by Proposition 4.24(ii) of [2], then the product $N_H^- K_H^0$ is a Lie subgroup of G. So $S = N_H^- K_H^0 ub$ is an orbit in K of the Lie subgroup of G, and S is an almost regular, submanifold of K, that is, let L be a locally connected topological space and $\phi : L \to K$ a continuous application. Let ϕ have values in S. Then, $\phi : L \to S$ is continuous with relation to the intrinsic topology (see Theorem C.6 [2], and appendix B of [21]).

Since V^- is an immersed submanifold of V and Ψ is diffeomorphism of $\mathcal{A}$ to $\mathcal{B}$, then from equation (2.19) $S \cap \mathcal{B}$ is an immersed submanifold of K. Let $A \subset S$ be a neighbourhood of $y \in S$ in the intrinsic topology, now we show that A contains a neighbourhood B in the intrinsic topology of S that is induced by an open set U of K, that is, such that, $B = S \cap U$.

By equation (2.20) there is $t > 0$ such that $h^t y \in S \cap \mathcal{B}$. Since h^t is a diffeomorphism of K that leaves invariant the quasi-regular manifold S, then h^t is a diffeomorphism of S (see Proposition C.3 of [2]). Then $h^t A \cap \mathcal{B}$ is an intrinsic neighbourhood of $h^t y$ in $S \cap \mathcal{B}$ and, since $S \cap \mathcal{B}$ is immersed, there is an open set U of K such that

$$h^t A \cap \mathcal{B} = S \cap U$$

So that

$$B = S \cap h^{-t} U = h^{-t}(S \cap U) = h^{-t}(h^t A \cap \mathcal{B}) \subset A$$

is a neighbourhood of y in S in the intrinsic topology that is in A and is induced by the open set $h^{-t}U$ of K, and this proves item (i).

For item (iii), by Proposition 2.11, let $v \in V^-$ such that $v = X \cdot x$ where $X \in \mathfrak{v}_x^- \subset \mathfrak{n}_H^-$ and $x \in \mathcal{M}$. Then $\Psi(v) = \exp(X)x$, where $\exp(X) \in N_H^-$ and $\pi(v) = x$. By Lemma 2.15 then

$$p(\Psi(v)) = x = \pi(v)$$

that is, $p \circ \Psi = \pi$, as needed. Since $\Psi|_{V^-}$ is a diffeomorphism and π is a differentiable submersion, then p also is. □

Now using the various results from the chapter we prove:

Corollary 2.17 *The stable set of the fixed points* $\mathrm{fix}(H,u)$ *is*

$$\mathrm{st}(H,u) = N_H^- \mathrm{fix}(H,u) = N_H^- K_H^0 ub$$

and

$$K = \bigcup_{u \in U} \mathrm{st}(H,u)$$

where each $\mathrm{st}(H,u)$ *is an immersed submanifold that is diffeomorphic to the stable vectorial fiber* V^- *over* $\mathrm{fix}(H,u)$.

Proof: Since $y \in K$ then $\omega(y)$ is a fixed point of h^t by Proposition 2.6. Let $u \in U$ such that $x \in \omega(y) \cap \mathrm{fix}(H,u)$, by Theorem 2.8. By Lemma 2.15, this implies that $y \in N_H^- x$, and then $y \in \mathrm{st}(H,u)$, and this proves the second statement. The first part is Proposition 2.16(i) and the last statement is Proposition 2.16(ii). □

So from now on, in this section the stable set of $\mathcal{M}$ will be taken to mean the stable manifold.

Note that the set of the components of fixed points $\{uM_0 : u \in U\}$ is in a set bijection with $W \times C$ since $W = U/C$. First we study the regular case: one *hyperbolic flow* $h^t = \exp(Ht)$ is defined as *regular* if $H \in \mathfrak{a}^+$.

Theorem 2.18 *Let* h^t *be a regular flow in* K.

(*i*) *The set of fixed points is the* disjoint union

$$\mathrm{fix}(h^t) = \coprod \{\mathrm{fix}(H,u) : u \in U\}$$

where $\mathrm{fix}(H,u) = M_0 ub$. *The attractive components are* $M_0 cb$, *for* $c \in C$ *and the repulsive components are* $M_0 cu^- b$ *where* $c \in C$ *and* $u^- \in w^- C$.

(*ii*) *The manifold* K *decomposes in the disjoint union of stable manifolds*

$$K = \coprod \{\mathrm{st}(H,u) :\ u \in U\} \tag{2.21}$$

where each $\mathrm{st}(H,u) = N^-\mathrm{fix}(H,u) = N^-M_0ub$ *is diffeomorphic to a vector space Cartesian product with* M_0ub. *Also, the union of the attractive stable manifolds* N^-M_0cb, *where* $c \in C$, *is open and dense in* K.

Proof: Since H is regular, then $\mathfrak{n}_H^\pm = \mathfrak{n}^\pm$ and $N_H^\pm = N^\pm$.

For item (i), since H is regular $K_H = M$ (see Theorem 4.21 of [2]) and $K_H^0 = M_0$. By Theorem 2.8 then $\mathrm{fix}(H,u) = M_0ub$ and the union in item (i) is disjoint since $U = M_*/M_0$.

For the last statement of item (i), note that, by Theorem 2.14, M_0ub is an attractor if and only if the zero section of the linearization V around M_0ub is an attractor. By Lemma A.5 of [2] then M_0ub is an attractor if and only if $V^- = V$. And this is true if and only if

$$\mathfrak{n}^- \cap u\mathfrak{n}^- = \mathfrak{n}^- \cap u(\mathfrak{m} \oplus \mathfrak{n}^-) = \mathfrak{v}_{ub}^- = \mathfrak{v}_{ub} = un^-$$

that is equivalent to $u\mathfrak{n}^- \subset \mathfrak{n}^-$. By the definition of $\mathfrak{n}^-$ and since $u\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$, for $u \in wC$ and $w \in W$, this is equivalent to

$$w(\Pi^-) \subset \Pi^-$$

so that, $w = 1$ or $u \in C$ and the attractors of h^t are cM_0b for all $c \in C$.

From a similar argument, uM_0b is a repeller if and only if $u\mathfrak{n}^- \subset \mathfrak{n}^+$, that is equivalent to

$$w(\Pi^-) \subset \Pi^+$$

where $u \in wC$, since $w = ww^-w^-$ (see Proposition 3.10 of [2]), then from item (iv) of Proposition 3.20 of [2] this occurs if and only if $ww^- = 1$ and $w = (w^-)^{-1} = w^-$ that is equivalent to $u \in w^-C$, and from the definition of u^- then $u \in u^-C$. Then the repellers of h^t are u^-cM_0 for any $c \in C$ or cu^-M_0 for any $c \in C$ since C is normal in U.

The first statement of item (ii) follows from Corollary 2.17, noting that the stable manifolds of different components are disjoint.

To show that the union $A := \cup_{c\in C}\mathrm{st}(H,c)$ is open and dense, from Proposition 2.16 $\mathrm{st}(H,u)$ is an immersed submanifold with the same dimension as $\mathfrak{m} \oplus (u\mathfrak{n}^- \cap \mathfrak{n}^-)$. Since uM_0b is an attractor if and only if $u = c \in C$ and the dimension of $\mathrm{st}(H,u)$ is equal to the dimension of $\mathfrak{m} \oplus \mathfrak{n}^-$. Since K has also the same dimension then A is open.

When $u \notin C$ then $\mathrm{st}(H,u)$ has dimension less than the dimension of N^-M_0b. A is a dense manifold then follows from Proposition C.1 of [2], since the complement of A is the union of finitely many manifolds of dimension strictly less than the dimension of K.

The last statement follows from the last statement of Corollary 2.17, since in the regular case each $\mathrm{fix}(H,u)$ is M_0ub, and the stable fiber is a vector space over M_0ub. □

Now we study the general case where $H \in \mathrm{cl}\mathfrak{a}^+$. First note that stable manifolds are disjoint if and only the corresponding fixed points are disjoint so that to study the intersection of stable sets we only need to know the intersection of the corresponding fixed points sets.

To study the general case we need to define *the subgroup* U_H of U,

$$U_H := \frac{K_H^0 \cap M_*}{M_0}$$

Note that U_H is a normal subgroup of W_HC since,

$$W_HC = \frac{K_H \cap M_*}{M} \cdot \frac{M}{M_0} = \frac{K_H \cap M_*}{M_0}$$

and K_H^0 is normal in K_H.

The next result studies the possible intersections of fixed points sets.

Corollary 2.19 *If* $\mathrm{fix}(H,u) \cap \mathrm{fix}(H,v) \neq \varnothing$ *then*

$$v \in U_H u M_0$$

and in this case,

$$\mathrm{fix}(H,u) = \mathrm{fix}(H,v)$$

Proof: If $\mathrm{fix}(H,u) \cap \mathrm{fix}(H,v) \neq \varnothing$ then there are $k_1, k_2 \in K_H^0$ so that $k_1vb = k_2ub$ and $vb = kub$ for $k = (k_1)^{-1}k_2 \in K_H^0$. Then

$$\mathrm{fix}(H,v) = K_H^0 vb = K_H^0 ub = \mathrm{fix}(H,u)$$

Also, $b = v^{-1}kub$ so $v^{-1}ku \in K \cap AN = 1$ and $v = ku$. So that

$$kuM_0 = vM_0 \subset M_*$$

but since $uM_0 \subset M_*$ then $k \in M_*$ and $k \in M_* \cap K_H^0$. Then

$$k \in \frac{M_* \cap K_H^0}{M_0} M_0 = U_H M_0$$

and $v = ku \in U_H M_0 u = U_H u M_0$.

$\square$

Let us now also define *the subgroup* C_H of C.

$$C_H := \frac{K_H^0 \cap M}{M_0}$$

In the general case then the set of fixed points $\{\mathrm{fix}(H,u) : u \in U\}$ are in bijection with the quotient $U_H \backslash U$. Note that also generally, U_H is not normal in U and this quotient is not a group but only right cosets. These cosets can then be used to enumerate the components of fixed points.

Theorem 2.20 *Let h^t be a hyperbolic flow in K.*

(i) *The set of fixed points is the disjoint union of connected components*

$$\mathrm{fix}(h^t) = \coprod \{\mathrm{fix}(H,u) :\ u \in U_H \backslash U\}$$

where $\mathrm{fix}(H,u) = K_H^0 ub$. *The attractors are* $\mathrm{fix}(H,c)$ *for* $c \in C_H \backslash C$ *and the repellers are* $\mathrm{fix}(H, cu^-)$ *for* $c \in C_H \backslash C$.

(ii) *The group K decomposes as the disjoint union of stable manifolds,*

$$K = \coprod \{\mathrm{st}(H,u) :\ u \in U_H \backslash U\} \tag{2.22}$$

where each $\mathrm{st}(H,u) = N_H^- \mathrm{fix}(H,u) = N_H^- K_H^0 ub$ *is diffeomorphic to the stable fiber over* $\mathrm{fix}(H,u)$. *Also, for* $c \in C_H \backslash C$ *the stable manifolds are open in K and their union is dense.*

Proof: For the first statement of item (i), that the set of fixed points is given by such union follows from Theorem 2.8, that this union is disjoint follows from Corollary 2.19.

The first statement of item (ii) follows from Corollary 2.17, noting that stable manifolds of disjoint sets are disjoint. For the second statement of item (ii), from Lemma 2.15 it follows that $N^- = N_H^- N(H)^-$. Since $N(H)^- M_0 \subset G(H) M_0 \subset G_H^0$, then $G_H^0 = G_H^0 N(H)^- M_0$ and

$$\mathrm{fix}(H,c) = G_H^0 cb = G_H^0 N(H)^- M_0 cb$$

Since G_H^0 normalizes N_H^- (see Proposition 4.24 of [2]), then

$$\mathrm{st}(H,c) = N_H^- G_H^0 N(H)^- M_0 cb = G_H^0 N^- M_0 cb$$

Now note that,

$$A := \bigcup_{c\in C} \mathrm{st}(H, c) \supset \bigcup_{c\in C} N^- M_0 cb$$

and the second set, from item (ii) of Theorem 2.18, is open and dense, so that A is also open and dense. This implies that $\cup_{c\in C}\mathrm{fix}(H, c)$ are the only attractors. To show that is enough to select $c \in C_H \backslash C$, note that

$$C \cap U_H = \frac{M}{M_0} \cap \frac{K_H^0 \cap M_*}{M_0} = \frac{K_H^0 \cap M}{M_0} = C_H$$

The argument for the repellers is similar. □

Note that item (ii) from the previous Theorem illustrates that the height function on K with respect to H has $|M/M_0| = |C|$ components that assume the maximum and $|C|$ components that assume the minimum.

Chapter 3

General translations in K

3.1 Preliminaries

In this chapter we study the continuous *flow of translations* in G that is a *linear flow* g^t where $\exp(Xt) = g^t$ in K for $t \in \mathbb{R}$, for any $X \in \mathfrak{g}$.

First we show that every action of G in K is decomposable in elliptic, hyperbolic and nilpotent components that commute with each other. Next we study the example of Sl(2) with $K = SO(2)$ and see that the system is not gradient, this motivates the study of chain recurrence and then in Lemma 3.4 we show that the system is equivariant by right multiplication by M, this provides a symmetry necessary to show that chain recurrent components are given by $\text{fix}(h^t)$.

First, we note that for any $a \in G$, the action $a \exp(Xt)a^{-1} = \exp(aXt)$ is the dynamical conjugate of the action g^t. Lemma 3.1 (i) from [7] helps in decomposing the action in simpler components. Following the proof of this Lemma we can state in full that:

Lemma 3.1 *Let $\mathfrak{g}$ be a semisimple algebra. Then for every $X \in \mathfrak{g}$, there is a Jordan decomposition $X = E + H + N$, these components commute, are in $\mathfrak{g}$, and N is additive nilpotent. There is a Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $E \in \mathfrak{k}_H$ and $H \in \text{cl}\mathfrak{a}^+$.*

This Lemma then helps to conjugate any linear flow to have a more standard hyperbolic component.

The terms $e^t := \exp(Et)$, $h^t = \exp(Ht)$, $u^t = \exp(Nt)$ are called the *Jordan components* of g^t and are called, respectively, the *elliptic*, *hyperbolic*, *unipotent* components of g^t. Let the $H \in \text{cl}\mathfrak{a}^+$ obtained in item (i) of the previous Lemma be the *hyperbolic type* of the translation flow g^t in G.

Note that the term N in this Lemma is a nilpotent component of $X \in \mathfrak{g}$, so that ad$N$ is a nilpotent action in $\mathfrak{g}$. This $N \in \mathfrak{g}$ is not to be confused with the nilpotent group in the Iwasawa decomposition. Also note that, since all Iwasawa decompositions are conjugate changing the Iwasawa decomposition is equivalent to a conjugation by an element in G of X so that also from Lemma 3.1 of [7] we get that:

Proposition 3.2 *There is $a \in G$ so that $aH \in \mathrm{cl}\mathfrak{a}^+$.*

Then, ag^ta^{-1} is a translation flow in G of hyperbolic component

$$ah^ta^{-1} = \exp(aHt)$$

where $aH \in \mathrm{cl}\mathfrak{a}^+$. Note that in K, the homeomorphism induced by a in G gives a conjugation between the flows g^t and ag^ta^{-1}, so that the dynamics of one is topologically equivalent to the other.

From now on, we can then assume that the *hyperbolic type* H of g^t is so that $H \in \mathrm{cl}\mathfrak{a}^+$. Remember now that the gradient dynamics of h^t in K has the following algebraic description given in Theorem 2.20. Let $U = M_*/M_0$. Then the connected components of the fixed points of h^t in K are given by

$$\mathrm{fix}(H, u) = G_H^0 ub = K_H^0 ub, \quad u \in U_H \backslash U$$

where the respective stable manifolds are given by

$$\mathrm{st}(H, u) = N_H^- \mathrm{fix}(H, u) = N_H^- K_H^0 ub, \quad u \in U_H \backslash U$$

so that $\{\mathrm{fix}(H, u) : u \in U_H \backslash U\}$ is the minimal Morse decomposition h^t. The next results prove first that this is also the minimal Morse decomposition of g^t and that each Morse component is normally hyperbolic.

Example: Let $G = \mathrm{Sl}(2)$ where the Iwasawa subgroups are: $K := \mathrm{SO}(2)$,

$$A := \left\{ \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, h > 0, h \in \mathbb{R} \right\} \quad N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\}$$

Now we will study the nilpotent action on the compact group SO(2). First note that

$$\exp\left(t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

so the nilpotent action on K is

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} b = \begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix} b$$

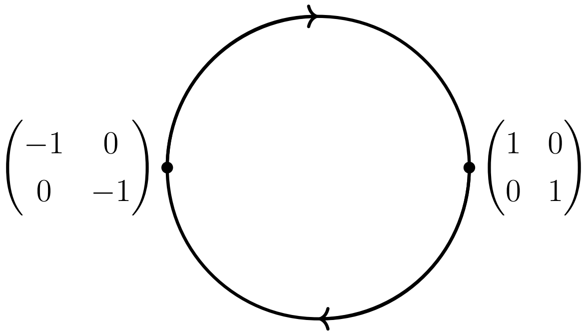
so we need to find the $\alpha(t)$ for the matrix

$$\begin{pmatrix} \cos\alpha + t\sin\alpha & -\sin\alpha + t\cos\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Multiplying the terms of Iwasawa decomposition where $h := h(t)$ and $x := x(t)$

$$\begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and equating both sides, from the elements in the first column $\cos\alpha + t\sin\alpha = h\cos\alpha(t)$ and $\sin\alpha = h\sin\alpha(t)$ dividing both terms when $\sin\alpha \neq 0$ we get $t + \cot\alpha = \cot\alpha(t)$. So $\alpha(t)$ can be determined. Remenber that in the trigonometric circle the cotangent "axis" is tangent to $(0,1)$. Now we can identify that the equilibrium points are when $\sin\alpha = 0$ and that they are both stable in one direction and unstable in the other direction.The first collumn of the matrixes is used to plot the results in matrix form. Note that in this case the system is not gradient.



3.2 Recurrence and chain recurrence

Let X be a metric space with distance d, the *recurrent set* of a flow ϕ^t in a space X is the set of points

$$\mathcal{R}(\phi^t) := \{x \in X : x \in \omega(x)\}$$

An (ϵ, t)-*chain* from x to y is a sequence of points

$$\{x = x_0, \ldots, x_n = y\} \subset X$$

and a sequence of times t_i such that $t_i \geq t$ and $d(\phi^{t_i}(x_i), x_{i+1}) < \epsilon$. The *chain recurrent set* of a flow ϕ^t, $\mathcal{R}_{\mathcal{C}}(\phi^t)$, is the set of points x such that there is (ϵ, t)-chain for every $\epsilon > 0$ and $t > 0$ from x to x.

Proposition 3.3 *Let g^t be a flow of translations in G. The unipotent and hyperbolic components of g^t are given by*

$$u^t := \exp(tN), \qquad h^t := \exp(tH), \qquad t \in \mathbb{R}$$

where $N \in \mathfrak{g}$ is nilpotent and $H \in \mathfrak{g}$ is additive hyperbolic. The flow g^t and all its Jordan components are flows in the centralizer G_H^0.

Proof: The first statement follows from B.24 and the fact that the Jordan components of g^t are in G follows from Theorem B.19 of [2]. The fact that these components are in G_H comes from the fact that the terms E, H, N from Lemma 2.1 3.1 commute. For the last statement, note that these components are in G_H^0 since the action is continuous. □

The hyperbolic element $H \in \mathfrak{g}$ given by the previous Proposition is the *hyperbolic type* of the flow g^t. In continuous time $g^t = \exp(tX)$, the terms N and H are, respectively, the *nilpotent* and *hyperbolic additive parts* of X.

The translation flows g^t and its Jordan components induce flows in K. In this section we will study the recurrence, the chain recurrence and the minimal Morse decomposition of g^t in K.

The κ used next will be defined in the group G and is not related to the kappa, κ, that shows up in Chapter 2 that is based only on the Cartan involution.

Indeed, the following kappa when transposed to the algebra will have kernel $\mathfrak{a} \oplus \mathfrak{n}$ and depends on the Iwasawa decomposition. Whereas the first kappa has kernel $\mathfrak{s}$, the symmetric space.

Lemma 3.4 *Given an Iwasawa decomposition $G = KAN$, define $\kappa : G \to K$ so that $g \in \kappa(g)AN$, then*

$$\kappa(gm) = \kappa(g)m$$

for any $m \in M$.

Proof: If $m \in M$ then $mAm^{-1} = A$ and $mNm^{-1} = N$. So

$$\begin{aligned} g \in \kappa(g)AN \quad &\Leftrightarrow \quad gm \in \kappa(g)ANm \\ &\Leftrightarrow \quad gm \in \kappa(g)mm^{-1}ANm \\ &\Leftrightarrow \quad gm \in \kappa(g)mAN \end{aligned}$$

and since $m \in K$ then $\kappa(gm) = \kappa(g)m$.

□

Assume the metric in K is such that multiplication by left or right are isometries.

Proposition 3.5 *Let G be a semisimple Lie group with Iwasawa decomposition $G = KAN$ and K compact and define $\kappa : G \to K$ so that $g \in \kappa(g)AN$. Let u^t be a unipotent flow.*

(*i*) *If $k \in K$ is a fixed point in K of the flow u^t then km is also a fixed point for any $m \in M$.*

(*ii*) *If $k \in K$ is a fixed point in K of the flow u^t then kb_0 is a fixed point of u^t in the maximal flag $\mathbb{F}$. Equivalently $u^t k \in kAN$ for all $t \in \mathbb{R}$ then $u^t k \in kMAN$ for all $t \in \mathbb{R}$.*

Proof: For item (i), from hypothesis $\kappa(u^t k) = k$ for all $t \in \mathbb{R}$, then by Lemma 3.4

$$\kappa(u^t km) = \kappa(u^t k)m = km \text{ for all } t \in \mathbb{R}$$

For item (ii), if $\kappa(u^t k) = k$ for all $t \in \mathbb{R}$ then, by definition, $u^t k \in kAN \subset kMAN$ for all $t \in \mathbb{R}$ so that $u^t kMAN = kMAN$. □

Lemma 3.6 *Let G be a semisimple Lie group with Iwasawa decomposition $G = KAN$ with K compact and $\kappa : G \to K$ so that $g \in \kappa(g)AN$. Let u^t be a unipotent flow that commutes with the elliptic flow $e^t \in K$.*

(*i*) *There is $s_n \to \infty$ so that $e^{s_n} \to 1$.*

(*ii*) *For all $k \in K$, there is $a_n \in \mathbb{N}$ so that $k^{a_n} \to 1$.*

Proof: For item (i), since K is compact there is $t_n \to \infty$ and $k \in K$ so that $e^{t_n} \to k$. Since $t_n \to \infty$ it is possible to assume taking a subsequence of t_n that $s_n = t_{n+1} - t_n \to \infty$. By the isometry of left multiplication then

$$d(e^{t_{n+1}} e^{-t_n}, 1) = d(e^{t_{n+1}}, e^{t_n}) \le d(e^{t_{n+1}}, k) + d(k, e^{t_n}) \to 0$$

when $n \to \infty$ and then $e^{s_n} \to 1$ when $n \to \infty$.

For item (ii), in a similar form to the previous item, taking the sequence $k^n \in K$ where $n \in \mathbb{N}$, there is a sequence $a_n \in \mathbb{N}$ so that $k^{a_n} \to 1$. □

Lemma 3.7 *For all $\epsilon > 0$, $k \in K$ and $T > 0$ exists $s > T$, $k_0 \in K$ and $m \in M$, so that, $e^s \in B(1, \epsilon)$, $\kappa(u^{-s} k) \in B(k_0, \epsilon)$ and $\kappa(u^s k) \in B(k_0 m, \epsilon)$.*

Proof: From item (i) of Lemma 3.6 there is $n_0 \in \mathbb{N}$ so that $e^{s_n} \in B(1,\epsilon)$ for $n > n_0$. Now, by Lemma 1.8 from [2] applied to the maximal flag for all $k \in K$ there is $k_0' \in K$ so that $u^t kMAN \to k_0'MAN$ when $t \to \pm\infty$. Then

$$\kappa(u^t k)M = \kappa(u^t kM) = \kappa(u^t kMAN) \to \kappa(k_0'MAN) = k_0'M$$

when $t \to \pm\infty$. Since $s_n \to \infty$ then $\omega(\kappa(u^{-s_n}k)) \in k_0'M$, and since M is compact taking a subsequence of s_n we can assume there is $m' \in M$ so that $\kappa(u^{-s_n}k) \to k_0'm'$. For the last limit we can again taking a subsequence of s_n assume that there also is $m'' \in M$ so that $\kappa(u^{s_n}k) \to k_0'm''$.

We conclude then that there is p integer so that $e^{s_p} \in B(1,\epsilon)$, $\kappa(u^{-s_p}k) \in B(k_0'm',\epsilon)$ and $\kappa(u^{s_p}k) \in B(k_0'm'',\epsilon)$, now take $s = s_p$, $k_0 = k_0'm'$ and $m = (m')^{-1}m''$ then $k_0'm'' = k_0'm'(m')^{-1}m'' = k_0m$. $\square$

Proposition 3.8 *Let G be semisimple Lie group with Iwasawa decomposition $G = KAN$ with K compact. Let $\kappa : G \to K$ such that $g \in \kappa(g)AN$. Let e^t, u^t be commuting elliptic and unipotent flows, then $\mathcal{R}_C(e^t u^t) = K$.*

Proof: Let $x \in K$ and $\epsilon > 0$, to prove the result we will show a (ϵ, T)-chain from x to x. First, let us form a $(\epsilon/2, T)$-chain from x to xm^p.

Take ϵ in Lemma 3.7 to be $\epsilon/8$. Then there are $s > T$, $k_0 \in K$ and $m \in M$, so that, $e^s \in B(1,\epsilon/8)$,

$$\kappa(u^{-s}k) \in B(k_0,\epsilon/8) \quad \text{and} \quad \kappa(u^s k) \in B(k_0m,\epsilon/8)$$

By isometry $d(1,e^{-s}) = d(e^s, e^s e^{-s}) = d(e^s, 1)$, so $e^{-s} \in B(1,\epsilon/8)$. From the triangle inequality,

$$\kappa(e^{-s}u^{-s}k) \in B(k_0,\epsilon/4) \quad \text{and} \quad \kappa(e^s u^s k) \in B(k_0m,\epsilon/4)$$

From Lemma 3.4 for $i = 1,\ldots,p-1$.

$$\kappa(e^{-s}u^{-s}km^i) \in B(k_0m^i,\epsilon/4) \quad \text{and} \quad \kappa(e^s u^s km^i) \in B(k_0m^{i+1},\epsilon/4)$$

Now we prove that the sequence

$$\{k, \kappa(e^{-s}u^{-s}km), km, \kappa(e^{-s}u^{-s}km^2), km^2, \ldots, km^p\}$$

is a $(\epsilon/2, T)$-chain from k to km^p with "times" equal to s. First note that,

$$\begin{aligned} d(\kappa(e^s u^s k), \kappa(e^{-s}u^{-s}km)) &\leq d(\kappa(e^s u^s k), k_0m) + d(k_0m, \kappa(e^{-s}u^{-s}km)) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \end{aligned}$$

And $\kappa(e^s u^s e^{-s} u^{-s} km^i) = km^i$. Then from Lemma 3.4 applied to the previous equation

$$d(\kappa(e^s u^s km^i), \kappa(e^{-s} u^{-s} km^{i+1})) < \frac{\epsilon}{2}$$

Proving that the previous sequence is indeed a $(\epsilon/2, T)$-chain from k to km^p.

Now take p to be an positive integer so that $m^p \in B(1, \epsilon/2)$. Since left multiplication by k is an isometry then $km^p \in B(k, \epsilon/2)$. Exchanging the last term in the previous chain by k there is then a (ϵ, T)-chain from k to k. □

3.3 Minimal Morse decomposition

Definition 3.9 *A* Morse decomposition *of a flow ϕ^t in a space X, is a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of subsets of X such that*

(*i*) *Each $\mathcal{M}_i$ is compact and ϕ^t-invariant.*

(*ii*) *For all $x \in X$, $\omega(x)$, $\alpha(x) \subset \cup_i \mathcal{M}_i$.*

(*iii*) *If $\omega(x)$, $\alpha(x) \subset \mathcal{M}_j$ then $x \in \mathcal{M}_j$.*

A *minimal* Morse decomposition is a decomposition that is contained in any other Morse decomposition. Each element of the decomposition is also called a *component* of the decomposition. An important result from dynamical systems (see [6]) is that if the Morse components are connected and

$$\cup_i \mathcal{M}_i = \mathcal{R}_{\mathcal{C}}(\phi^t)$$

then this decomposition is minimal.

Theorem 3.10 *Let g^t be a translation flow in G and consider the induced flow of g^t in $K \simeq G/AN$. Then*

(*i*) *The minimal Morse components are the connected components of* $\mathrm{fix}(h^t)$ *and then*

$$\mathcal{R}_C(g^t) = \mathrm{fix}(h^t) = \coprod \left\{ \mathrm{fix}(H, u) = K_H^0 ub : \ u \in U_H \backslash U \right\}$$

where the only attractors are $\mathrm{fix}(H, c) = K_H^0 cb$ *where $c \in C_H \backslash C$ and the only repellers are* $\mathrm{fix}(H, cu^-) = K_H^0 cu^- b$ *for $c \in C_H \backslash C$.*

(*ii*) *The stable manifolds of* g^t *are* $\mathrm{st}(H,u) = N_H^- K_H^0 ub$ *and the unstable manifolds of* g^t *are* $\mathrm{un}(H,u) = N_H^+ K_H^0 ub$. *Also,* K *can be decomposed as the disjoint union of stable manifolds*

$$K = \coprod \left\{ N_H^- K_H^0 u : \, u \in U_H \backslash U \right\} \tag{3.1}$$

or similarly, the disjoint union of unstable manifolds

$$K = \coprod \left\{ N_H^+ K_H^0 u : \, u \in U_H \backslash U \right\} \tag{3.2}$$

And the union of the stable manifolds of the attractors $\mathrm{fix}(H,c) = K_H^0 cb$ *for all* $c \in C_H \backslash C$ *is open and dense.*

Proof: Since $\mathrm{fix}(H,u) = G_H^0 ub$ and $g^t \in G_H^0$, then $\mathcal{M} = \mathrm{fix}(H,u)$ is g^t-invariant. First we will prove that $S = \mathrm{st}(H,u)$ is a stable set from the g^t-invariant $\mathcal{M}$. By the Bruhat decomposition of K (Theorem 2.20 (ii)), it is enough to show that S is in the stable manifold of $\mathcal{M}$, that is, it is enough to prove that for any $y \in S$ then $\omega(y) \subset \mathcal{M}$.

Let $y \in S$, so $y = \exp(Y) lub$, where $Y \in \mathfrak{n}_H^-$ and $l \in G_H^0$. Then

$$g^t y = g^t \exp(Y) g^{-t} g^t lub = \exp(g^t Y) g^t lub$$

where $g^t lub \in \mathrm{fix}(H,u)$, since $g^t l \in G_H^0$. First, we prove that $g^t Y \to 0$. This follows from Lemma 1.5 of [2], since the spectral radius of the restriction of g to $\mathfrak{n}_H^-$ is less then 1. In fact, from the Jordan decomposition, $r(g)$ is the greatest eigenvalue from its hyperbolic component, is given by the restriction of h to $\mathfrak{n}_H^-$. These eigenvalues are $e^{-\alpha(H)}$, where $\alpha \in \Pi^+$ and $\alpha(H) > 0$, since $r(g) < 1$. Taking then $x \in \omega(y)$ then let $t_j \to \infty$ such that $g^{t_j} y \to x$. Since $\exp(g^{t_j} Y) \to 1$, then the limits of the sequences

$$g^{t_j} y = \exp(g^{t_j} Y) g^{t_j} lub \qquad \text{and} \qquad g^{t_j} lub$$

are the same, so

$$g^{t_j} lub \to x$$

and x is in the closed set $\mathrm{fix}(H,u)$, since each $g^{t_j} lub$ is in $\mathrm{fix}(H,u)$. And $\omega(y) \subset \mathrm{fix}(H,u)$. Similarly, we can prove that $N_H^+ \mathcal{M}$ is the unstable set of the g^t-invariant set $\mathcal{M}$.

Since the stable and unstable of each $\mathrm{fix}(H,u)$ with respect to g^t or to h^t coincide, then it follows that $\{\mathrm{fix}(H,u) : u \in U_H \backslash U\}$ is a Morse decomposition for g^t, since it is a Morse decomposition for h^t. So that

$$\mathcal{R}_C(g^t) \subset \mathrm{fix}(h^t) = \bigcup_{u \in U} \mathrm{fix}(H,u)$$

To show that $\mathrm{fix}(h^t) \subset \mathcal{R}_C(g^t)$ note that the restriction of g^t to $\mathrm{fix}(h^t)$ is given by $e^t u^t$. So by Proposition 3.8, $\mathrm{fix}(h^t) \subset \mathcal{R}_C(e^t u^t h^t) = \mathcal{R}_C(g^t)$ and $\mathcal{R}_C(g^t) = \mathrm{fix}\,(h^t)$. The other statements of items (i) and (ii) follow from Theorem 2.20 noting that stable manifolds and unstable manifolds from g^t are the same as for the flow h^t that is a gradient flow. □

Next we obtain a characterization of the recurrent set.

Theorem 3.11 *Let g^t be the translations flow in G and $g^t = e^t h^t u^t$ is its Jordan decomposition. The recurrent set of translations g^t induced in K is given by*

$$\mathcal{R}(g^t) = \mathrm{fix}(h^t) \cap \mathcal{R}(u^t)$$

and $\mathcal{R}(u^t) = \pi^{-1}(\mathrm{fix}_{\mathbb{F}}(u^t))$, where π is the projection of K in $K/M = \mathbb{F}$.

Proof: First we prove that $\mathrm{fix}(h^t) \cap \mathcal{R}(u^t) \subset \mathcal{R}(g^t)$. If $k \in \mathcal{R}(u^t)$ then kM is a fixed point of the maximal flag $\mathbb{F} = K/M$, so $\kappa(u^t k) = k m_t$ with $m_t \in M$ for all $t \in \mathbb{R}$ (see Lemma 1.8 of [2] applied to complete flags) . Now we prove that $m_{t+s} = m_t.m_s$. Indeed, since $\kappa(k^{-1}u^t k) = m_t$ then $k^{-1}u^t k \in m_t AN$ and

$$k^{-1}u^{t+s}k = k^{-1}u^t k.k^{-1}u^s k \in m_t AN.m_s AN$$

and since M normalizes AN, then

$$m_t AN.m_s AN = m_t m_s m_s^{-1} AN m_s AN = m_t.m_s AN$$

but since, $k^{-1}u^{t+s}k \in m_{t+s}AN$ then $m_{t+s}AN = m_t.m_s AN$ and $m_{t+s} = m_t.m_s$ for all $t,s \in \mathbb{R}$. Since $m_0 = 1$ then $m_{-t} = m_t^{-1}$. And since M is compact, there is t_i and $\widetilde{m} \in M$ so that $u^{t_i}k \to k\widetilde{m}$, taking a subsequence of t_i we can assume that $e^{t_i} \to \widetilde{e}$ and $s_i = t_{i+1} - t_i \to \infty$. Then

$$m_{s_i} = m_{t_{i+1}}.m_{-t_i} \to \widetilde{m}\widetilde{m}^{-1} = 1$$

and $\kappa(u^{s_i}k) = k m_{s_i} \to k$. Since $e^{s_i} \to \widetilde{e}.\widetilde{e}^{-1} = 1$ then

$$\kappa(g^{s_i}k) = \kappa(e^{s_i}u^{s_i}k) \to k$$

and $k \in \mathcal{R}(g^t)$.

Now let us prove that $\mathcal{R}(g^t) \subset \mathrm{fix}(h^t) \cap \mathcal{R}(u^t)$. From item (i) of Theorem 3.10, $k \in \mathcal{R}(g^t) \subset \mathcal{R}_C(g^t) = \mathrm{fix}\,(h^t)$, so next we prove that $k \in \mathcal{R}(u^t)$. Let t_i such that $\kappa(g^{t_i}k) \to k$ and taking a subsequence we can assume that $e^{t_i} \to \widetilde{e}$ and that

$s_i = t_{i+1} - t_i \to \infty$. Since $\kappa(g^{s_i}k) = \kappa(e^{s_i}u^{s_i}k) = e^{s_i}\kappa(u^{s_i}k)$ and since $e^{s_i} \to 1$ then the sequence $\kappa(u^{s_i}k)$ converges to k and $k \in \mathcal{R}(u^t)$.

For the last statement, we need to show that $k \in \mathcal{R}(u^t)$ if and only if kM is a fixed point of the maximal flag K/M. So there is $t_n \to \infty$ such that $\kappa(u^{t_n}k) \to k$ if and only if $\kappa(u^t k) = km_t$ with $m_t \in M$ for $t \in \mathbb{R}$. In fact, if $\kappa(u^{t_n}k) \to k$ then $u^{t_n}kMAN \to kMAN$ and $u^{t_n}kb_0 \to kb_0 \in \mathbb{F}$ so that kb_0 is a fixed point in $\mathbb{F}$.

Now, if b_0 is a fixed point in $\mathbb{F}$ then $\kappa(u^t k) = km_t$ with $m_t \in M$ and since M is compact there is a subsequence t_n such that $m_{t_n} \to \widetilde{m}$ and then $\kappa(k^{-1}u^{t_n}k) \to \widetilde{m}$. Taking a subsequence we can assume that $s_n = t_{n+1} - t_n \to \infty$ so $\kappa(k^{-1}u^{s_n}k) \to \widetilde{m}\widetilde{m}^{-1} = 1$ and $\kappa(u^{s_n}k) \to k$, so that k is a recurrent point of the flow u^t. If π is the canonic projection of K to $K/M = \mathbb{F}$ then

$$\mathcal{R}(u^t) = \pi^{-1}(\mathrm{fix}_{\mathbb{F}}(u^t))$$

□

3.4 Linearization

The next result shows that each minimal Morse component $\mathcal{M} = \mathrm{fix}(H, u)$ of g^t in K is normally hyperbolic (see Definition 2.9). Consider the vector bundles $V = V^- \oplus V^+$ over $\mathcal{M}$ as in Section 2.3.

Proposition 3.12 *The vector bundles $V^\pm$ are g^t-invariant and there are positive numbers c and $\lambda < \mu$ such that*

(i) $|g^t v| \leq c\,\mathrm{e}^{-\lambda t}|v|$ *for* $v \in V^-$ *and* $t \geq 0$.

(ii) $|g^{-t} v| \leq c\,\mathrm{e}^{-\lambda t}|v|$ *for* $v \in V^+$ *and* $t \geq 0$.

(iii) $|g^t v| \leq c\,\mathrm{e}^{\mu|t|}|v|$ *for* $v \in T\mathcal{M}$ *and* $t \in \mathbb{R}$.

Proof: First note that

$$g^t = e^t h^t u^t$$

where $g^t, e^t, h^t, u^t \in G_H^0$. We have $V^\pm = \mathfrak{n}_H^\pm \cdot \mathcal{M}$ and G_H^0 normalizes $\mathfrak{n}_H^\pm$, from Proposition 4.23 of [2], and G_H^0 also leaves $\mathcal{M}$ invariant, so that $V^\pm$ is g^t-invariant. Let $v \in V^\pm$, then $v = Y \cdot x$ with $Y \in \mathfrak{v}_x^\pm = n_H^\pm \cap \mathfrak{g}_x^\perp$ and $x \in \mathcal{M}$, from Proposition 2.10, $|v| = |Y|$ and

$$|g^t v| = |g^t Y \cdot g^t x| \leq |g^t Y|$$

where $g^tY \in \mathfrak{n}_H^{\pm}$. It is enough to show then the inequalities for g^t restricted to $\mathfrak{n}_H^{\pm}$. First we consider the case where $Y \in \mathfrak{n}_H^-$, the next case is proven similarly. From Lemma 2.12, there is $\mu > 0$ such that $|h^tZ| \leq \mathrm{e}^{-\mu t}|Z|$, for $t \geq 0$ and $Z \in \mathfrak{n}_H^-$. Since $e^t \in K_H^0$ and the inner product is K-invariant, then

$$|g^tY| = |h^tu^tY| \leq \mathrm{e}^{-\mu t}|u^tY|$$

where we used that $u^t \in G_H^0$, so $u^tY \in \mathfrak{n}_H^-$. From Theorem B. 24 of [2], $u^t = \exp(tN)$, for $N \in \mathfrak{g}$ nilpotent and $u^tY = \mathrm{e}^{t\mathrm{ad}(N)}Y$. From the triangle inequality then

$$|u^tY| = |\mathrm{e}^{t\mathrm{ad}(N)}Y| \leq \sum_{k\geq 0} \frac{|t^k|}{k!}\|\mathrm{ad}(N)^k\||Y| = p(t)|Y|$$

where $\|\cdot\|$ is the operator norm associated to the norm $|\cdot|$ in $\mathfrak{n}_H^-$ and $p(t)$ is a polynomial, since $\mathrm{ad}(N)$ is nilpotent. Then

$$|g^tY| \leq \mathrm{e}^{-\mu t}p(t)|Y|$$

Since $|g^tv| \leq |g^tY|$ and $|v| = |Y|$ then for $v \in V^-$

$$|g^tv| \leq \mathrm{e}^{-\mu t}p(t)|v|, \quad t \geq 0$$

The case for V^+ is similar so that for $v \in V^+$

$$|g^{-t}v| \leq \mathrm{e}^{-\mu t}p(t)|v|, \quad t \geq 0$$

For $T\mathcal{M}$, note that $x \in \mathcal{M}$ and $g^tx = e^tu^tx$, and g^t acts as e^tu^t in $T\mathcal{M}$. From equation 2.15 from Proposition 2.11 a tangent vector $v \in T\mathcal{M}_x$ is $v = Y \cdot x$, for $Y \in \mathfrak{g}_H \cap \mathfrak{g}_x^{\perp}$. From Proposition 2.10 $|v| = |Y|$ and

$$|g^tv| = |e^tu^tY \cdot e^tu^tx| \leq |e^tu^tY| = |u^tY| \leq p(t)|Y| = p(t)|v|$$

where we used that $e^t \in K_H^0$ and the inequality for $|u^tY|$ previously obtained.

Since $\mathrm{e}^{-\frac{\mu}{2}t}p(t) \to 0$ when $t \to +\infty$, then $\mathrm{e}^{-\frac{\mu}{2}t}p(t)$ is less then c_1 for $t \geq 0$ and

$$\mathrm{e}^{-\mu t}p(t) = \mathrm{e}^{-\frac{\mu}{2}t}\left(\mathrm{e}^{-\frac{\mu}{2}t}p(t)\right) \leq c_1\mathrm{e}^{-\frac{\mu}{2}t}, \quad t \geq 0$$

For the last case, since $\mathrm{e}^{-\mu|t|}p(t) \to 0$ when $t \to \pm\infty$, then $\mathrm{e}^{-\mu|t|}p(t)$ is less then c_2 for $t \in \mathbb{R}$ and

$$p(t) = \mathrm{e}^{\mu|t|}\left(\mathrm{e}^{-\mu|t|}p(t)\right) \leq c_3\mathrm{e}^{\mu|t|}, \quad t \in \mathbb{R}$$

The items (i), (ii) and (iii) of the Proposition follow taking $\lambda = \frac{\mu}{2}$ and taking c to be the maximum of c_1 and c_2. □

Theorem 3.13 *Let $\mathcal{M} \subset X$ compact and normally hyperbolic.*

(i) There is a normal linearization in a neighbourhood of $\mathcal{M}$.

(ii) The restriction of this linearization to a neighbourhood of $\mathbf{N}\mathcal{M}_0$ in V^- extends uniquely to a diffeomorphism from V^- to $\mathrm{st}(\mathcal{M})$ that is is a global conjugation. In particular, the stable set $\mathrm{st}(\mathcal{M})$ is diffeomorphic to the stable fiber V^-.

Proof: The first statement is Theorem 1 from [20] for discrete time and Theorem 2 from [20] for continuous time. The second statement is item (iii) from Lemma A.5 from [2]). □

The following result linearizes the flow of translations g^t around each Morse component $\mathcal{M} = \mathrm{fix}(H, u)$, generalizing for g^t the result of linearization of the hyperbolic flow (Theorem 2.14).

Theorem 3.14 *(i) There is a normal linearization of the flow g^t around $\mathrm{fix}(H, u)$.*

(ii) This linearization extends in a unique fashion to g^t-equivariant diffeomorphism from V^- to $\mathrm{st}(H, u)$.

Proof: Item (i) follows from Theorem 3.13 item (i), after noting that the action of g^t in the normal bundle V is given by the restriction of the action of Dg^t in TK to V, and that the equivariance propriety is equivalent to the property of conjugation of Theorem 3.13. Since $\mathrm{st}(\mathcal{M}) = N_H^-\mathcal{M}$ (Theorem 3.10 (ii)) then item (ii) follows from Theorem 3.13 (ii). □

Chapter 4

Topology of K

4.1 Bruhat and Schubert cells in K

In this chapter we use the Bruhat and Schubert cells to calculate the homology of K for split real forms, in particular, at the end we calculate the homology of $SO(3)$. In this we follow [16] to first construct the skeleton, then the boundary map to find algebraic expressions for the degrees of the maps. The calculations for the degrees in SO(3) are done in two ways, first geometrically and then algebraically to better illustrate the results obtained.

First we will study an example:

Example: Let $G = \mathrm{Sl}(2)$ and the Iwasawa subgroups are: $K := \mathrm{SO}(2)$,

$$A := \left\{ \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, h > 0, h \in \mathbb{R} \right\} \quad N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\}$$

The Bruhat cells in this case are Nub, where $u \in U = M^*$, since $M_0 = 1$. In this case

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

Let $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $s^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $s^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = s^{-1}$. So that $U = \{1, s, s^2, s^3\}$.

The 4 Bruhat cells are then Nb, Nsb, Ns^2b, Ns^3b. Hence $Nb = b$ is a cell of one point and

$$Nsb = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} b = \begin{pmatrix} -x & 1 \\ -1 & 0 \end{pmatrix} b$$

By the Iwasawa decomposition there exists α, $h > 0$ and $y \in \mathbb{R}$ depending on x such that:

$$\begin{pmatrix} -x & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

Calculating the product on the right side, from the first column of the matrix we get: $-x = h\cos\alpha$ and $-1 = -h\sin\alpha$, since $h > 0$ then $\sin\alpha > 0$ and we can consider α in the interval $(0, \pi)$. Dividing both equations we get that $\cot\alpha = -x$ so α can be found. Then for each $x \in \mathbb{R}$ there are unique corresponding $\alpha \in (0, \pi)$, $h > 0$, $y \in \mathbb{R}$.

Taking $F = E_{1,2} - E_{2,1}$ then $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \exp(F\alpha)$ and $s = \exp(F\frac{\pi}{2})$. The 4 Bruhat cells are 2 manifolds of dimension 1:

$$Nsb = \exp(F\alpha)b \quad \text{for } 0 < \alpha < \pi$$

$$Ns^3b = Nss^2b = \exp(F\alpha)s^2b \quad \text{for } 0 < \alpha < \pi$$

And 2 trivial manifolds of one point: $Nb = b$ and $Ns^2b = s^2b$, since $s^2 \in M$.

4.2 Preliminaries

In this section we will be primarily interested in the case that all roots are simple so that for any root α the eigenspace $\mathfrak{g}_\alpha$ is one dimensional. We will study normal real forms and will also assume that $M_0 = 1$ or that the algebra $\mathfrak{m}$ is trivial. Finally we will also assume that the real Lie group G can be complexified, in that case M can be explicitly calculated in function of the roots.

Considering each element $w = \pi(u) \in W$ as a product of simple reflections r_α, the *length* of $\ell(w)$ of $w \in W$ is the number of simple reflections in any reduced expression of w. Another useful result Theorem 4.15.10 of [22] is that $l(w)$ is equal to the cardinality of $\Pi_w = \Pi^+ \cap w\Pi^-$, or the set of positive roots sent to the negative roots by w^{-1}. Let $w = r_1 \cdots r_d$ be a fixed reduced expression of w and $\alpha_i = \alpha_{r_i}$ be the simple positive roots for each r_i, then each root of Π_w can be written explicitly as in Theorem 4.15.10 of [22] as

$$\Pi_w = \{\alpha_1, r_1\alpha_2, \ldots, r_1 \cdots r_{d-1}\alpha_d\} \tag{4.1}$$

Since M is normal in M_* then $C = M/M_0$ is normal in $U = M_*/M_0$. Now note that since $\pi(s_\alpha) = r_\alpha$ then $\pi(s_1 \cdots s_d) = r_1 \cdots r_d = w$. So for any element $u \in U$ there is $m \in M/M_0$ such that $u = s_1 \cdots s_d m$ and $\pi(u) = w$.

Following the Theorems 7.53 and 7.55 from [18], we get that $F = M$ where F is the cartesian product of cyclic groups of order 2 and is generated by

$$\gamma_\alpha = \exp 2\pi i |\alpha|^{-2} H_\alpha$$

In fact, there is a bijective homomorphism from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}(\alpha) := \mathfrak{g}_\alpha \oplus \mathbb{R}H_\alpha \oplus \mathfrak{g}_{-\alpha}$ that can be complexified taking $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{g}(\alpha)^{\mathbb{C}}$. In the corresponding homomorphism between Lie groups then γ_α is the image of $-Id$. Since $\mathfrak{g}_\alpha$ are one dimensional *define* a normalized E_α such that $\mathfrak{g}_\alpha = E_\alpha \mathbb{R}$. Note that γ_α does not depend on the choice of E_α from the original homomorphism.

In a similar manner to [16] we define.

Definition 4.1

$$\psi_j(t_j) := \exp(F_j t_j) \textit{ for } t_j \in [0,\pi]$$

where $F_j = F_{\alpha_j}$ *and* $F_\alpha := E_\alpha + \theta E_\alpha \in \mathfrak{k}$ *for a fixed chosen* $E_\alpha \in \mathfrak{g}_\alpha$.

Let

$$s_\alpha := \exp(F_\alpha(\pi/2)), \quad m_\alpha := \exp(F_\alpha \pi)$$

And similarly, $s_j = s_{\alpha_j}$, $m_j = m_{\alpha_j}$, *so that* $m_j = s_j^2 = \exp(F_j\pi)$.

Now, define the map $\pi : K \to \mathbb{F}$ by $kb \mapsto kb_0$, note that this map is a differentiable finite covering of $\mathbb{F}$ with $|M|$ sheets, since we can take $\mathbb{F}$ the maximal flag to be K/M so that it is a finite quotient. Then for a sufficiently small open set $V \subset \mathbb{F}$ there is a diffeomorphism $\pi^{-1} : V \to B$, where B is an open set of K, such that $\pi\pi^{-1} = \mathrm{id}|_V$ and $\pi^{-1}\pi|_B = \mathrm{id}|_B$. Note also that, for any $m \in M$ and $x \in V$, $\pi(\pi^{-1}(x).m) = x$, so that, for a given V, π^{-1} is not uniquely defined and in fact has $|M|$ possible choices.

Note that $\pi(M_*) = M_*/M = W$, but since we took M_0 to be 1 then

$$U = M_*/M_0 = M_*$$

and

$$\pi(U) = W$$

Definition 4.2 *For ease of notation we define a Bruhat cell in* K *as*

$$\mathcal{B}(u) := Nub$$

for $u \in U = M_*$.

By the dynamical decomposition of the unstable manifolds when H is regular we have $N_H = N$ and $G_H = M_0 = 1$ so from (ii) of 3.10 that

$$K = \bigcup_{\text{disj.}} Nub = \bigcup_{\text{disj.}} \mathcal{B}(u)$$

where the unions are taken over all $u \in U$. Let $\pi(u) = w \in W$ and $w = r_1 \dots r_d$ be a fixed reduced decomposition.

Note that if one prefers to use instead the stable manifolds decomposition its possible to obtain similar results by using that $N^- = u^- N(u^-)^{-1}$ where $u^- \in w^- M$ and w^- is the principal involution.

In an analogous fashion to flags we define in K, the Schubert cells to be the closure of the Bruhat cells.

Definition 4.3 *A Schubert cell in K of $u \in U$ is defined as the closure of the Bruhat cell $\mathcal{B}(u) = Nub$,*

$$\mathcal{S}(u) := \mathrm{cl}(\mathcal{B}(u)) = \mathrm{cl}(Nub)$$

Let $\mathrm{cl}A$ be the closure of the set A. In the present work we take $\partial(A)$ to mean the *frontier* of the set A, that is $\mathrm{cl}A \backslash A$. Note that in this case the frontier of the Bruhat cell is then $\partial\mathcal{B}(u) = \mathcal{S}(u)\backslash\mathcal{B}(u)$. Later, to avoid confusion of notation, we use δ to be the boundary operator.

Lemma 4.4 *The frontier of a Bruhat cell, $\partial\mathcal{B}(u) := \mathcal{S}(u)\backslash\mathcal{B}(u)$ in K is the union of Bruhat cells $\mathcal{B}(v) = Nvb$ for $v \in V(u)$, where $V(u)$ is a subset of U. The cells in $\partial\mathcal{B}(u)$ have smaller dimension then $\mathcal{B}(u)$.*

Proof: Let nvb be in the frontier of $\mathcal{B}(u) = Nub$ for some $n \in N$ then $\mathcal{B}(v) = Nvb$ is also in the frontier of Nub. In fact, since nvb is in the frontier of Nub then there is a sequence n_k in N such that $n_k ub \to nvb$. Let y be any point in Nvb then $y = n'vb$ for some $n' \in N$ and the sequence in Nub, $n'n^{-1}n_k ub$, converges to $n'n^{-1}nvb = n'vb = y$ and y is also in the frontier of $\mathcal{B}(u)$.

Since π is a finite cover it preserves dimension of cells. Let $\pi(u) = w$ then $\pi(\partial\mathcal{B}(u)) \subset \partial\mathcal{B}(w)$, now from Proposition 1.9 (2) of [16] the dimension of $\partial\mathcal{B}(w)$ is less then the dimension of $\mathcal{B}(w)$. □

So for any $u \in U$ there is *a subset* $V(u)$ of U so that the frontier of $\mathcal{S}(u)$ is

$$\bigcup_{v \in V(u)} Nvb$$

so that,

$$\mathcal{S}(u) = \mathrm{cl}(Nub) = Nub \cup \bigcup_{v \in V(u)} Nvb$$

and since multiplication by right by an element $m \in M$ takes Bruhat cells to Bruhat cells

$$\mathcal{S}(um) = \mathrm{cl}(Numb) = Numb \cup \bigcup_{v \in V(u)} Nvmb$$

for any $m \in M$, so that $V(um) = V(u)m$.

Now we construct a map Ψ_u from a closed cube to the Schubert cell $\mathcal{S}(u)$. In this we mainly use Proposition 1.9 from [16]. Let $\pi(u) = w \in W$ and $w = r_1 \dots r_n$ be a fixed reduced decomposition. First we prove to Lemmas about the action of M.

Lemma 4.5 *The root spaces $\mathfrak{g}_\beta$ are invariant by the action of m_α and*

$$(m_\alpha)|_{\mathfrak{g}_\beta} = (-1)^{\epsilon(\alpha,\beta)}\mathrm{id}$$

Proof: If $X \in \mathfrak{g}_\beta$ and i the imaginary unit then,

$$m_\alpha X = \exp(\pi i H_\alpha^\vee)X = e^{\mathrm{ad}(\pi i H_\alpha^\vee)}X$$

where $H_\alpha^\vee = \frac{2H_\alpha}{\langle\alpha,\alpha\rangle}$. Then since $\mathrm{ad}(H_\alpha)X = \langle\alpha,\beta\rangle X$,

$$e^{\mathrm{ad}(\pi i H_\alpha^\vee)}X = e^{\pi i \epsilon(\alpha,\beta)}X$$

where $\epsilon(\alpha,\beta) = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ is the Killing number and so it is an integer from Proposition 2.74 of [18]. □

Lemma 4.6 *Let $t_i \in (0,\pi)$ then $m_i\psi_j(t_j) = \psi_j(t'_j)m'$ for some $t'_j \in (0,\pi)$ and $m' \in M$.*

Proof: Let $t_j \in (0,\pi)$ and $i \neq j$, remember that $\psi_j(t_j) = \exp(t_jF_j)$, where $F_j := E_j + \theta E_j$, where θ is the Cartan involution. First, we calculate $m_i\psi_j(t_j)m_i$. For this, let us calculate $\mathrm{Ad}(m_i)E_j$, since $m_i\mathfrak{g}_{\alpha_j} = \mathfrak{g}_{\alpha_j}$ and $m_i^2 = 1$ then $\mathrm{Ad}(m_i)E_j = \pm E_j$. Since $\mathrm{Ad}(m_i)\theta = \theta\mathrm{Ad}(m_i)$ then if $\mathrm{Ad}(m_i)E_j = E_j$,

$$\mathrm{Ad}(m_i)F_j = \mathrm{Ad}(m_i)E_j + \theta\mathrm{Ad}(m_i)E_j = E_j + \theta E_j = F_j$$

and if $\mathrm{Ad}(m_j)E_i = -E_i$ then

$$\mathrm{Ad}(m_i)F_j = \mathrm{Ad}(m_i)E_j + \theta\mathrm{Ad}(m_j)E_j = -E_j + \theta(-E_j) = -F_j$$

So

$$m_i \exp(t_jF_j)m_j^{-1} = \exp(\mathrm{Ad}(m_i)(t_jF_j)) = \exp(\pm t_jF_j)$$

If $\mathrm{Ad}(m_i)F_j = F_j$ then $m_i\psi_j(t_j) = \psi_j(t_j)m_i$ and if $\mathrm{Ad}(m_i)F_j = -F_j$ then

$$m_i\psi_j(t_j) = \psi_j(-t_j)m_i = \psi_j(\pi - t_j)m_jm_i$$

and $\pi - t_i \in (0,\pi)$. The case $i = j$ is trivial. □

Definition 4.7 *Let $\pi(u) = w = r_1 \cdots r_d$ be a reduced expression for w and let $u = s_1 \cdots s_d m$, for some $m \in M$, then define $\Psi_u : J^d \to K$,*

$$\Psi_u(t_1, \ldots, t_d) := \psi_1(t_1) \cdots \psi_d(t_d) mb$$

for $t_j \in J = [0, \pi]$.

Note that if $I = (0, \pi)$, then with the definition above,

$$ub = s_1 \cdots s_d mb = \psi_1(\pi/2) \cdots \psi_d(\pi/2) mb = \Psi_u(\pi/2, \ldots, \pi/2) \in \Psi_u(I^d)$$

Proposition 4.8 *Let $w = \pi(u) = r_1 \cdots r_d$ be a reduced decomposition and let $u = s_1 \cdots s_d m$, for some $m \in M$. Define the intervals $I = (0, \pi)$ and $J = [0, \pi]$, so that $\partial I^d = J^d \backslash I^d$, then*

$$\mathcal{S}(u) = \mathrm{cl}(\mathcal{B}(u)) = \Psi_u(J^d)$$

The frontier of $\mathcal{B}(u)$ is

$$\Psi_u\left(\partial I^d\right) = \partial \mathcal{B}(u) = \mathcal{S}(u) \backslash \mathcal{B}(u)$$

and $\Psi_u|_{I^d}$ is a diffeomorphism from I^d to $\mathcal{B}(u) = Nub$.

Proof: First we prove that $\pi : K \to \mathbb{F}$ is injective in each Bruhat cell in K. Let n, $n' \in N$ and assume that $nub \neq n'ub$ and $\pi(nub) = \pi(n'ub)$. Then there is $m \in M$ such that $nub = n'umb$, but since the Bruhat cells are disjoint then $m = 1$, that is a contradiction and π is injective in each Bruhat cell.

Since π is G-equivariant then $\pi(Nub) = Nwb_0$, where $\pi(u) = w \in W$, so that π is also surjective in Bruhat cells, and then bijective in Bruhat cells.

Now, from Proposition 1.9 items (1),(2),(3) of [16] and using that $\pi\Psi_u = \Psi_w$ then

1. $\pi(\Psi_u(J^d)) \subset \pi(\mathcal{S}(u))$.

2. $\pi(\Psi_u(\mathbf{t})) \in \pi(\partial \mathcal{B}(u))$ if and only if $\mathbf{t} \in \partial I^d = J^d \backslash I^d \cong S^{d-1}$.

3. $\pi\Psi_u|_{I^d} : I^d \to Nwb_0 = \pi(Nub) = \pi(\mathcal{B}(u))$ is a diffeomorphism.

To show the corresponding statements for K we need first to show that there is a open set containing the Bruhat cell. Take $N' = u^* N (u^*)^{-1}$ where $u^* = u^- u^{-1}$. In Proposition 2.7 of [13]

$$N' = (N' \cap N^-)(N' \cap N)$$

Where in the paper the notation $N_u(B)$ is used for N'. Now note that

$$u^- u^{-1} Nub = u^- u^{-1} Nu (u^-)^{-1} u^- b = N' u^- b = (N' \cap N)(N' \cap N^-) u^- b$$

and

$$(N' \cap N)(N' \cap N^-)u^-b = (N' \cap N)u^-(u^-)^{-1}(N' \cap N^-)u^-b = N' \cap Nu^-b \subset Nu^-b$$

then $Nub \subset u(u^-)^{-1}Nu^-b$ but $u(u^-)^{-1}Nu^-b$ is a translation of an open set so it is open in K. Taking the translates by right multiplication by M of this open set we obtain disconnected open sets with the same image in $\mathbb{F}$ by π. But since $\Psi_u(I^d)$ is connected and contains ub then necessarily $\Psi_u(I^d) = Nub$.

So $\Psi_u(J^d) \subset \mathcal{S}(u)$ also $\psi_u(\mathbf{t}) \in \partial\mathcal{S}(u)$ if and only if $\mathbf{t} \in \partial I^d = J^d \backslash I^d \cong S^{d-1}$ and $\Psi_u|_{I^d} : I^d \to Nub = \mathcal{B}(u)$ is a diffeomorphism. □

The closed cube J^d can be identified by a homeomorphism preserving orientation to a closed ball B_d of dimension d so that the frontier of the cube, $\partial I^d = J^d \backslash I^d$, is identified with a sphere S^{d-1}.

4.3 The skeleton X^d

To construct the skeleton for the CW or cellular decomposition of K obtained here, we will follow page 5 of [8] and construct inductively the d-skeleton X^d, or the skeleton of dimension d, from X^{d-1} starting by the discrete set X^0 and attaching the open d-cells, e_u^d, via *attaching* maps φ_u from $S^{d-1} \to X^{d-1}$.

Remember that the dimension of $\mathcal{B}(u)$ is the length of $w = \pi(u)$. So the dimension is 0 if and only if $l(w) = 0$ which happens if and only if $w = 1$ and $u = m \in M$. By definition $\mathcal{B}(m) = Nmb = mb$, so that $\mathcal{B}(m) = \mathcal{S}(m) = \{mb\}$ for $m \in M$ are the only discrete cells.

To construct the next levels of the skeleton X^d, we must identify the maps φ_u from $S^{d-1} \to X^{d-1}$. For this first we fix, for every dimension d, a homeomorphism from S^d to the frontier of the closed cube $\partial I^d = J^d \backslash I^d$ so that for any $x \in S^{d-1}$ there is $\mathbf{t}(x) \in \partial I^d$. From Proposition 4.8 we define $\varphi_u(x) = \Psi_u(\mathbf{t}(x))$.

By starting with $u \in U$, such that $l(\pi(u)) = 1$, to construct X^1. By induction on the length, we can construct all the skeleton of K. Note that $X^{d'}$ when d' has maximum dimension equals K. Note that there are $|M|$ cells of highest dimension since these correspond to the $u \in U$, such that $\pi(u) = w^-$, the principal involution, and the dimension of these cells is the number of elements in $\Pi_{w^-} = \Pi^-$.

As a consequence of the second equation in the Proposition 4.8, we have the following construction. Let $d = \dim\mathcal{S}(u) = \dim\mathcal{B}(u)$. The sphere S^d is the quotient $J^d/\partial I^d$, where the boundary $\partial I^d = J^d \backslash I^d$ is collapsed to a point. We can do the same to the Schubert cell $\mathcal{S}(u)$. *Define*

$$\sigma_u := \mathcal{S}(u)/\partial\mathcal{B}(u)$$

that is, the space obtained by identifying the complement of the Bruhat cell $\mathcal{B}(u)$ in $\mathcal{S}(u)$ to a point. As $\Psi_u(\partial I^d) \subset \partial\mathcal{B}(u)$, it follows that Ψ_u induces a map $S^d \to \sigma_u$ which is a homeomorphism. The inverse of this homeomorphism will be denoted by

$$\Psi_u^{-1} : \sigma_u \to S^d \tag{4.2}$$

note that this is not the same as the inverse of Ψ_u because of the collapse in the corresponding borders.

The following is Proposition 1.10 of [16] for Schubert cells in $\mathbb{F}$ and it will be useful later on. Remember that all roots are one dimensional.

Proposition 4.9 *Let w, $w' \in W$. The following statements are equivalent:*

1. $\mathcal{S}_w \subset \mathcal{S}_{w'}$ *and* $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$.
2. *If $w = r_1 \cdots r_d$ is a reduced expression of $w \in W$ as a product of simple reflections and for some i, $w' = r_1 \cdots \widehat{r_i} \cdots r_d$ is also a reduced expression.*

Note that in the previous Proposition $\pi(u) = w$.

Let ∂J^d be the frontier of J^d, that is, $\partial J^d = J^d \backslash I^d$. And let $\partial\mathcal{S}(u)$ be the frontier of $\mathcal{S}(u)$, that is, $\partial\mathcal{S}(u) = \mathcal{S}(u)\backslash\mathcal{B}(u) = \cup_{v\in V(u)}\mathcal{B}(v)$.

Let $\pi(u) = w = r_1 \cdots r_d$ and let $v \in U$ such that $\pi(v) < \pi(u)$ in the Bruhat order and the length of $\pi(v)$ is equal to $d-1$ then by Theorem 5.10 of [11], $w' = \pi(v) = r_1 \cdots \widehat{r_i} \cdots r_d$ for a unique integer i as the next Lemma shows. First, note that this expression for $\pi(v)$ is necessarily reduced, since the length of $\pi(v)$ is $d-1$.

Lemma 4.10 *If $w = r_1 \cdots r_d$ is a reduced expression, $w' < w$, in the Bruhat order and also, the length of w' is $d-1$, then there is a unique integer i, $1 \le i \le d$, such that $w' = r_1 \cdots \widehat{r_i} \cdots r_d$.*

Proof: Since $w' < w$ then w' equals a subexpression of $r_1 \cdots r_d$ Theorem 5.10 of [11]. Since the length of w' is $d-1$ then $w' = r_1 \cdots \widehat{r_i} \cdots r_d$ for some i. To show that the integer i is unique, let $i < j$ such that

$$r_1 \cdots \widehat{r_i} \cdots r_d = r_1 \cdots \widehat{r_j} \cdots r_d$$

then by cancellation, $r_{i+1} \cdots r_j = r_i \cdots r_{j-1}$ and by substitution of this on the original expression for w,

$$\begin{aligned} w &= r_1 \cdots (r_i \cdots r_{j-1}) r_j \cdots r_d = r_1 \cdots (r_{i+1} \cdots r_j) r_j \cdots r_d \\ &= r_1 \cdots \widehat{r_i} \cdots \widehat{r_j} \cdots r_d \end{aligned}$$

that is a contradiction with original expression for w being reduced. □

4.4 The boundary map

Let $\mathcal{C}$ be the $\mathbb{Z}$-module freely generated by $\mathcal{B}(u)$, $u \in U$. The *boundary maps* $\delta : \mathcal{C} \to \mathcal{C}$ are defined by

$$\delta\mathcal{B}(u) := \sum_{v \in V(u)} c(u,v)\mathcal{B}(v)$$

where $c(u,v) := 0$, if $\dim\mathcal{B}(u) - \dim\mathcal{B}(v) \neq 1$ and

$$c(u,v) := \deg\left(\phi_{u,v} : S_u^{d-1} \to S_v^{d-1}\right)$$

if $\dim\mathcal{B}(u) - \dim\mathcal{B}(v) = 1$.

Where the map $\phi_{u,v}$ is the *composition of the following maps*:

(a) the attaching map $\Psi_u|_{\partial I^d} : S_u^{d-1} \cong \partial I^d \to \partial\mathcal{B}(u) = \mathcal{S}(u)\backslash\mathcal{B}(u)$.

(b) the quotient map $\partial\mathcal{B}(u) \to \partial\mathcal{B}(u)/(\partial\mathcal{B}(u)\backslash\mathcal{B}(v))$, where we take the cell $\mathcal{B}(v)$ and identify its complement in $\partial\mathcal{B}(u) = \mathcal{S}(u)\backslash\mathcal{B}(u)$ to a point.

(c) the identification $\partial\mathcal{B}(u)/(\partial\mathcal{B}(u)\backslash\mathcal{B}(v)) \cong \mathcal{S}(v)/\partial\mathcal{B}(v) = \sigma_v$, where the last equality comes from the definition of σ_v.

(d) the map defined by equation (4.2), $\Psi_v^{-1} : \sigma_v \to S_v^{d-1}$.

Remark

To compute the degree

$$c(u,v) = \deg\left(\phi_{u,v} : S_u^{d-1} \to S_v^{d-1}\right)$$

when $u = s_1 \cdots s_d m$ and $\pi(v) = r_1 \cdots \widehat{r_j} \cdots r_d$ is a reduced expression for some j then v is one of two options:

$$v = v_1 v_2 \quad or \quad v = v_1 m_j v_2$$

where $v_1 = s_1 \cdots s_{j-1}$ and $v_2 = s_{j+1} \cdots s_d m$. These two options will correspond to 0 and π in the next Lemma.

We will determine the degree of the map $\phi_{u,v}$ in three steps.

Step 1: Domain and co-domain spheres.

First we identify the spheres S_u^{d-1} in the domain and S_v^{d-1} in the co-domain. Remember that the "closed ball", $B_u^d = J^d$, where $J = [0, \pi]$, as in Theorem 4.8 and the domain of $\phi_{u,v}$ is the frontier of J^d:

$$S_u^{d-1} = \{(t_1, \dots, t_d) : \exists j, t_j = 0 \text{ or } \pi\}$$

or the union of the "faces" of the closed cube J^d.

On the other hand, let $B_v^{d-1} = J \times \cdots \times \widehat{J} \times \cdots \times J$, without the interval in the jth position. The co-domain is the sphere S_v^{d-1} obtained by collapsing to a point the boundary of $B_v^{d-1} = J^{d-1}$. This is seen in the items (c) and (d) in the definition of $\phi_{u,v}$.

Step 2: σ_v in the image $\Psi_u(S_u^{d-1})$.

The second step is to see how σ_v sits inside the image $\Psi_u(S_u^{d-1})$. The following Lemma says what is the pre-image of Nvb under Ψ_u.

Lemma 4.11 *If $u = s_1 \cdots s_d m$ and $v = s_1 \cdots m' \cdots s_d m$ with m' replacing s_j where $m' = 1$ or $m' = m_j$. Then $\Psi_u(t_1, \dots, t_d) \in Nvb = \mathcal{B}(v)$ if and only if $t_k \in I = (0, \pi)$, for all $k \neq j$ and $t_j = 0$ for $m' = 1$, and $t_j = \pi$ for $m' = m_j$.*

Proof: If $t_j = 0$ then $\psi_j(0) = 1$ and

$$\Psi_u(t_1, \dots, 0, \dots, t_d) = \psi_1(t_1) \cdots 1 \cdots \psi_d(t_d) mb$$

Now, since $\pi(v)$ is reduced then

$$\Psi_u(t_1, \dots, 0, \dots, t_d) = \Psi_{v_1 v_2}(t_1, \dots, \widehat{t_j}, \dots, t_d)$$

and $\Psi_u(t_1, \dots, 0, \dots, t_d)$ is in $\mathcal{S}(v_1 v_2)$, that is the case $m' = 1$.

If $t_j = \pi$ then $\psi_j(\pi) = m_j$ and

$$\Psi_u(t_1, \dots, \pi, \dots, t_d) = \psi_1(t_1) \cdots m_j \cdots \psi_d(t_d) mb$$

First note that taking $t_k = \pi/2$ for $k \neq j$ and $t_j = \pi$ we obtain

$$s_1 \cdots m_j \cdots s_d m = v_1 m_j v_2$$

Now remember that by Lemma 4.6 for any k, j : $m_j \psi_k(I) = \psi_k(I) m''$, and $m'' \in M$. So that, by successive applications of Lemma 4.6,

$$m_j . \psi_{j+1}(I) \cdots \psi_d(I) = \psi_{j+1}(I) \cdots \psi_d(I) m^*$$

for some $m^* \in M$. So,

$$\Psi_u(t_1, \dots, \pi, \dots, t_d) = \psi_1(t_1) \cdots 1 \cdots \psi_d(t_d') m^* mb$$

and that is in $\mathcal{S}(s_1 \cdots 1 \cdots s_d m^* m)$. By Proposition 4.8, we see that

$$\Psi_u(t_1, \dots, \pi, \dots, t_d) \in Nvb = \mathcal{B}(v)$$

□

In other words the pre-image of $\mathcal{B}(v)$ is contained in B_u^d and is one of the interior of the two faces corresponding to the jth coordinate, that is, the faces where $t_j = 0$ and $t_j = \pi$ in the case $v = v_1 v_2$ and $v = v_1 m_j v_2$, respectively. Note that the second case can be rewritten by $v_1 m_j v_2 = v_1 v_2 (v_2^{-1} m_j v_2)$ so the previous m^* can be determined .

In the quotient $\sigma_v = \mathcal{S}(v)/\partial\mathcal{B}(v)$ the faces of ∂B_u^d corresponding to the kth coordinates $k \neq j$, are all collapsed to a point.

Step 3: Degrees.

The degree of $\phi_{u,v}$ is then the degree of one of the two maps, namely the maps restricted to the two faces

$$\mathcal{F}_j^0 = \{(t_1, \dots, 0, \dots, t_d)\} \text{ or } \mathcal{F}_j^\pi = \{(t_1, \dots, \pi, \dots, t_d)\}$$

The values of $\phi_{u,v}$ in these faces are given by

$$\begin{aligned} f_j^0(\mathbf{t}) &= (\Psi'_v)^{-1} (\psi_1(t_1) \cdots \psi_j(0) \cdots \psi_d(t_d) mb) \\ &= (\Psi'_v)^{-1} (\psi_1(t_1) \cdots 1 \cdots \psi_d(t_d) mb) \end{aligned}$$

$$\begin{aligned} f_j^\pi(\mathbf{t}) &= (\Psi'_v)^{-1} (\psi_1(t_1) \cdots \psi_j(\pi) \cdots \psi_d(t_d) mb) \\ &= (\Psi'_v)^{-1} (\psi_1(t_1) \cdots m_j \cdots \psi_d(t_d) mb) \end{aligned}$$

where $\mathbf{t} = (t_1, \dots, \widehat{t_j}, \dots, t_d)$ and Ψ'_v is given by a choice of reduced expression $v = s'_1 \cdots s'_{d-1} m'$ which can, in principle, be *different* from $s_1 \cdots \widehat{s_j} \cdots s_d m$.

The degree of $\phi_{u,v}$ then is the degree of f_j^0 or f_j^π which may be considered as maps $S^{d-1} \to S^{d-1}$ by collapsing the boundary of the faces to points. Now the degree of a map ϕ can be computed as the local degree in the inverse image of $f^{-1}(\xi)$ of a regular value which has a finite number of points (see [8], Proposition 2.30). In the case of our map $\phi_{u,v}$, the maps f_j^0 and f_j^π are homeomorphisms so that the pre-image $\phi_{u,v}^{-1}(\xi)$ of a point is one point in one of the faces, namely a point x_1 in face $\mathcal{F}_j^0$ if $v = v_1 v_2$ or a point x_2 in $\mathcal{F}_j^\pi$ if $v = v_1 m_j v_2$. The local degree at x_1 is the degree of f_j^0 and the local degree at x_2 is the degree of f_j^π. Finally, the degrees of f_j^0 and f_j^π are ± 1 since they are homeomorphisms.

Summarizing: To get the degree of $\phi_{u,v}$, in each case, we must restrict $(\Psi'_v)^{-1} \circ \Psi_v$ to the faces $\mathcal{F}_j^0$ and $\mathcal{F}_j^\pi$ and view these faces as spheres (with boundaries collapsed to points). The degrees of each one of the restrictions is the degree of $\phi_{u,v}$. The restrictions of $(\Psi'_v)^{-1} \circ \Psi_v$ to the faces $\mathcal{F}_j^0$ and $\mathcal{F}_j^\pi$ are homeomorphisms and hence have degree ± 1.

The term $(\Psi'_v)^{-1} \circ \Psi_v$ can disappear if $\Psi'_v = \Psi_v$ this happens when in the graph of the Bruhat order it is possible to make so that every word immediately bellow is a subword of the word above. When the Bruhat order graph grows more complicated this is not possible. So we need more to use of more then one expression for a given term depending on the relation, see [17].

4.5 Algebraic expressions for the degrees

Here we compute the coefficients $c(u,v)$ in terms of the roots by finding the degrees of the maps involved.

For a diffeomorphism φ of the sphere its degree is the local degree at a point x which in turn is the sign of the determinant $\det d\varphi_x$ with respect to a volume form of S^d. Let us apply this in our context.

We let $u = s_1 \cdots s_d m$ and $v = s_1 \cdots \widehat{s_j} \cdots s_d m$, with $s_j = s_{\alpha_j}$ and $m_j = m_{\alpha_j}$. We must find the degrees of f_j^0 and f_j^π defined respectively by

$$f_j^0(t_1, \dots, 0, \dots, t_d) := {\Psi'_v}^{-1} \left(\psi_1(t_1) \cdots 1 \cdots \psi_d(t_d) m b\right)$$

$$f_j^\pi(t_1, \dots, \pi, \dots, t_d) := {\Psi'}_v^{-1} \left(\psi_1(t_1) \cdots m_j \cdots \psi_d(t_d) m b\right)$$

in these expressions ${\Psi'}_v^{-1}$ is defined by a previously chosen expression $v = s'_1 \cdots s'_{d-1} m'$ which can, in principle, be different from $s_1 \cdots \widehat{s_j} \cdots s_d m$. On the other hand $s_1 \cdots \widehat{s_j} \cdots s_d m$ can be used to define another characteristic map, which will be denoted by Ψ_v. This new characteristic map can then be used to define new functions

$$p_j^0(t_1, \dots, 0, \dots, t_d) := \Psi_v^{-1} \left(\psi_1(t_1) \cdots 1 \cdots \psi_d(t_d) m b\right)$$

$$p_j^\pi(t_1, \dots, \pi, \dots, t_d) := \Psi_v^{-1} \left(\psi_1(t_1) \cdots m_j \cdots \psi_d(t_d) m b\right)$$

The two pairs of functions are then related by

$$f_j^\epsilon = \left({\Psi'}_v^{-1} \circ \Psi_v\right) \circ p_j^\epsilon, \quad \epsilon = 0, \pi$$

The composition ${\Psi'_v}^{-1} \circ \Psi_v$ (also understood as a map between spheres in which the boundary is collapsed to points) is an homeomorphism of spheres and, hence, has degree ± 1.

Before getting these degrees we make the following discussion on the orientation of the faces of the cube $[-1,1]^d$, centered on the origin of $\mathbb{R}^d$, which is given with the basis $\{e_1,\ldots,e_d\}$. Afterwards we will use the cube $[0,\pi]^d$, since there is a translation and magnification taking one into the other, we have corresponding orientations.

Starting with the $(d-1)$-dimensional sphere S^{d-1} we orient the tangent space at $x \in S^{d-1}$ by a basis $\{f_2,\ldots,f_d\}$ such that $\{x, f_2,\ldots,f_d\}$ is positively oriented. The faces of $[-1,1]^d$ are oriented accordingly: given a base vector e_j, we let F_j^- be the face perpendicular to e_j that contains $-e_j$ and F_j^+ be the one that contains e_j. Then F_j^- has the same orientation as the basis $e_1,\ldots,\widehat{e_j},\ldots,e_d$ if j is even (since $-e_j, e_1,\ldots,\widehat{e_j},\ldots,e_d$ is positively oriented if j is even). So that the orientation of F_j^- is $(-1)^j$ the orientation of $e_1,\ldots,\widehat{e_j},\ldots,e_d$. And similarly the orientation of F_j^+ is $(-1)^{j+1}$ the orientation of $e_1,\ldots,\widehat{e_j},\ldots,e_d$.

Following this system, in the case of one dimension the point 1 is positive oriented and -1 is negative oriented, in two dimensions the sides of the squares are oriented counter-clockwise and in the cube the faces are oriented following the right hand rule, we will call these orientations the *standard* orientation.

The following facts about the action of an element $m \in M$ will be used below in the computation of the degrees.

Define Π_u as Π_w where $\pi(u) = w$. The following is Lemma 2.4 from [16] with some changes.

Lemma 4.12 *For a root α consider the action on K of $m = m_\alpha = \exp(\pi F_\alpha)$. Then*

1. *$mub = uu^{-1}mub = um'b$, for $m' = u^{-1}mu \in M$, and $mNm^{-1} = N$*
2. *the restriction of m to $Nub = \mathcal{B}(u)$ is a diffeomorphism from $\mathcal{B}(u)$ to $\mathcal{B}(um')$.*
3. *the differential dm_{ub} identifies to the action of m restricted to the subspace*

$$\sum_{\beta \in \Pi_u} \mathfrak{g}_\beta$$

where $u \in wC$.

Proof:

From Lemma 4.5 for $\beta \in \Pi$ then $m_\alpha \mathfrak{g}_\beta = \mathfrak{g}_\beta$. Since

$$mNub = mNm^{-1}mub = Nmub = Nuu^{-1}mub = Num'b$$

where $m' = u^{-1}mu \in M$.

For the third statement we use the notation $X \cdot k = d/dt(\exp(tX))|_{t=0}$, for $k \in K$ and $X \in \mathfrak{g}$. Also, for $A \subset \mathfrak{g}$ let $A \cdot k = \{X \cdot k : X \in A\}$.

Note that $Nub = uu^{-1}Nub$ and the tangent space to $u^{-1}Nub$ at b is spanned by $\mathfrak{g}_\gamma \cdot b$ with $\gamma < 0$ such that $\gamma = u^{-1}\beta$ and $\beta > 0$, that is, $u\gamma > 0$. Since $(du)(\mathfrak{g}_\gamma \cdot b) = \mathfrak{g}_{u\gamma} \cdot ub$, it follows that the tangent space $T_{ub}(Nub)$ is spanned by $\mathfrak{g}_\beta \cdot b$ where $\beta = u\gamma > 0$ such that $u^{-1}\beta = \gamma < 0$. Hence the result. □

The following is Proposition 2.5 from [16] only separating the degrees, changing notation, and since in our case the multiplicities of roots are always 1 then $I = j$ and $\dim\mathfrak{g}_\beta = 1$ for all β in the Proposition.

Proposition 4.13 *The degrees are:*

1. $\deg(p_j^0) = (-1)^j$.

2. $\deg(p_j^\pi) = (-1)^{j+1+\sigma}$, *where*

$$\sigma = \sigma(u,v) = \sum_{\beta \in \Pi_{v_2}} \frac{2\langle \alpha_j, \beta\rangle}{\langle \alpha_j, \alpha_j\rangle} \tag{4.3}$$

$\Pi_{v_2} = \Pi^+ \cap v_2\Pi^-$ *and* $v_2 = s_{j+1}\cdots s_d m$.

Proof: The map p_j^0 is the projection of the face of a d-dimensional cube onto the face of a $d-1$-dimensional cube, i.e. in coordinates

$$(t_1, \ldots, 0, \ldots, t_d) \mapsto (t_1, \ldots, \widehat{t_j}, \ldots t_d)$$

Note that with respect to the basis $e_1, \ldots, e_d$ the t_j-coordinate appears in the jth position. Hence, by the orientation of the cube, if j is even or odd, the orientation is respectively, positive or negative. Therefore, $\deg(p_j^0) = (-1)^j$.

To get the $\deg(p_j^\pi)$ let m_j be the element of M appearing in the expression of p_j^π. Its left action on K takes any Bruhat cell $\mathcal{B}(u') = Nu'b$ to the cell $Nm_ju'b = Nu'(u')^{-1}m_ju'b = \mathcal{B}(u'm')$, where $m' = (u')^{-1}m_ju'$. And hence it takes the Schubert cell $\mathcal{S}(u')$ to $\mathcal{S}(u'm')$. Moreover the restriction of m_j to $Nu'b$ is a diffeomorphism.

In particular, we restrict the action of m_j to the cell $\mathcal{S}(v_2)$, $v_2 = s_{j+1}\cdots s_d m$. Its action on K takes the Bruhat cell $\mathcal{B}(v_2) = Nv_2b$ to the cell $\mathcal{B}(v_2m'')$ where $m'' = v_2^{-1}m_jv_2$. Using the parametrization of this cell by the cube $B_{v_2} = J^{d-j}$ we get

$$m_j\psi_{j+1}(t_{j+1})\cdots\psi_d(t_d)mb = \psi_{j+1}(t'_{j+1})\cdots\psi_d(t'_d)mb$$

with $(t'_{j+1}, \ldots, t'_d) = \overline{m}_j(t_{j+1}, \ldots, t_d)$ with $\overline{m}_j : B_{v_2} \to B_{v_2}$ continuous and a diffeomorphism of the interior of B_{v_2}.

Hence, $p_j^\pi(t_1, \ldots, \pi, \ldots, t_d)$ becomes the projection of the $(j-1)$ first coordinates and the composition of $\overline{m}_j$ with the projection of the last kth coordinates, $k =$

$j+1,\dots,d$. From the choice of the orientation of $B_u = J^d$, the face $(t_1,\dots,\pi,\dots,t_d)$ of B_u has orientation $(-1)^{j+1}$ with respect to the orientation of the coordinates $(t_1,\dots,\widehat{t_j},\dots,t_d)$. Hence after collapsing the boundary to a point, we get the degree

$$\deg p_j^\pi = (-1)^{j+1}\deg \overline{m}_j$$

The degree of $\overline{m}_j$ equals its degree at one point which in turn is the sign of the determinant of the differential $d(m_j)_{v_2b}$ restricted to the tangent space of the Bruhat cell $B(v_2) = Nv_2b$ at v_2b:

$$\deg p_j^\pi = (-1)^{j+1}\mathrm{sgn}\left[\det\left(d(m_j)_{v_2b}|_{T_{v_2b}(Nv_2b)}\right)\right]$$

By the third statement in Lemma 4.12 $T_{v_2b}(Nv_2b)$ identifies to $\sum_{\beta\in\Pi_{v_2}}\mathfrak{g}_\beta$.

Once we have the generators $\mathfrak{g}_\beta\cdot v_2b$, $\beta\in\Pi_{v_2}$ for $T_{v_2b}(Nv_2b)$ together with the action of m_j over $\mathfrak{g}_\beta$ given by Lemma 4.5, $m_\alpha|_{\mathfrak{g}_\beta} = (-1)^{\epsilon(\alpha,\beta)}\mathrm{id}$, we conclude that the signal of $\det\left(d(m_j)_{v_2b}|_{T_{v_2b}(Nv_2b)}\right) = (-1)^\sigma$ where

$$\sigma = \sum_{\beta\in\Pi_{v_2}}\frac{2\langle\alpha_j,\beta\rangle}{\langle\alpha_j,\alpha_j\rangle}$$

□

Summarizing, we have the following algebraic expression for the coefficient $c(u,v)$.

Theorem 4.14 *Let $\sigma(u,v)$ be as in Equation 4.3. Then if $v = v_1v_2$ then*

$$c(u,v) = \deg\left({\Psi'}_v^{-1}\circ\Psi_v\right)(-1)^j$$

and if $v = v_1m_jv_2$ then

$$c(u,v) = \deg\left({\Psi'}_v^{-1}\circ\Psi_v\right)(-1)^{j+1+\sigma(u,v)}$$

We will now cite another formula for $\sigma(u,v)$,from [16] Proposition 2.7, that does not depend on the reduced expressions of u and v. This formula is similar to the one given by Theorem A of [19] when the dimensions of $\mathfrak{g}_\beta$ are all 1.

For $w\in W$, let

$$\phi(w) := \sum_{\beta\in\Pi_w}\beta$$

be the sum of roots in $\Pi_w = \Pi^+\cap w\Pi^-$. As before let $w = r_1\cdots r_d$, and $w' = r_1\cdots\widehat{r_j}\cdots r_d$ be reduced expressions. And correspondingly $u = s_1\cdots s_dm$ and $v = s_1\cdots\widehat{s_j}\cdots s_dm = v_1v_2$.

Proposition 4.15 *Let β' be the unique root (not necessarily simple) such that $w = r_{\beta'} w'$ that is, $\beta' = r_1 \cdots r_{j-1}\alpha_j$. Then*

$$\phi(w) - \phi(w') = (1-\sigma)\beta'$$

where $\sigma = \sigma(w, w') = \sigma(u, v)$ is the sum in Equation 4.3.

Theorem 4.16 *Let $\sigma(u, v)$ be as in Equation 4.3. Then if $v = v_1 v_2$ then*

$$c(u, v) = \deg\left(\Psi'^{-1}_v \circ \Psi_v\right)(-1)^j$$

and if $v = v_1 m_j v_2$ then

$$c(u, v) = \deg\left(\Psi'^{-1}_v \circ \Psi_v\right)(-1)^{j+\kappa(u,v)}$$

where $\kappa(u, v)$ is the integer defined by $\phi(w) - \phi(w') = \kappa(u, v)\beta'$ and β' is the unique root such that $w = r_{\beta'} w'$. Note that by Proposition 4.15, $\kappa(u, v) = 1 - \sigma(u, v)$.

4.6 Example $G = \mathrm{Sl}(3), K = \mathrm{SO}(3)$

Since right multiplication by $m \in M$ is a diffeomorphism then $\mathcal{B}(um) = \mathcal{B}(u)m$ for $u \in M_*$. So $\delta(\mathcal{B}(um)) = \delta(\mathcal{B}(u))m$. So to obtain $\delta(\mathcal{B}(u))$ for all $u \in M_*$ we need only obtain $\delta(\mathcal{B}(u))$ for the 5 elements s_1, s_2, $s_1 s_2$, $s_2 s_1$, $s_1 s_2 s_1$.

Let $E_{i,j}$ be the matrix with 1 in the position (i, j) and zero elsewhere. Take $A = E_{1,2} - E_{2,1}$ and $B = E_{2,3} - E_{3,2}$ note that $\psi_1(t) = e^{tA}$ and $\psi_2(s) = e^{sB}$ then

$\Psi_1(0) = b$

$\Psi_{s_1}(t) = e^{tA}b,\ t \in [0, \pi]$

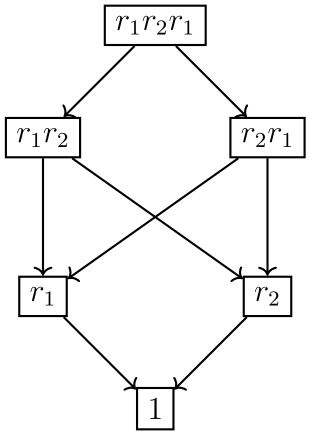
$\Psi_{s_2}(s) = e^{sB}b,\ s \in [0, \pi]$

$\Psi_{s_1 s_2}(t, s) = e^{tA}e^{sB}b,\ (t, s) \in [0, \pi]^2$

$\Psi_{s_2 s_1}(t, s) = e^{tB}e^{sA}b,\ (t, s) \in [0, \pi]^2$

$\Psi_{s_1 s_2 s_1}(t, s, z) = e^{tA}e^{sB}e^{zA}b,\ (t, s, z) \in [0, \pi]^3$

The multiples by m to the right are similar. Then we obtain expressions for $c(u, v)$. The following calculations can be done more geometrically by comparing orientations of the maps with the standard orientation or more algebraically by calculating $\sigma(u, v)$ as in Proposition 4.14. The Bruhat diagram for the Weyl group can be represented as:



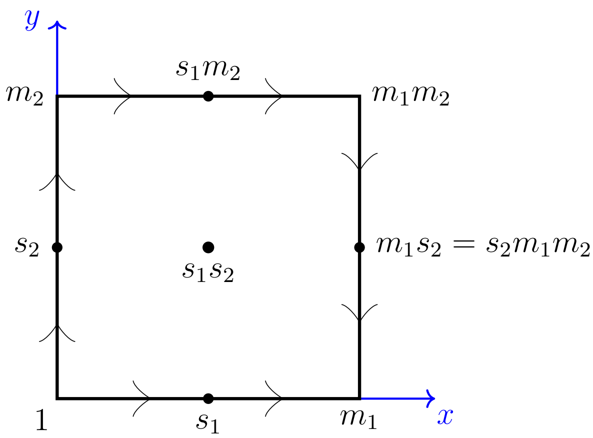
Note that in this case there is no need to account for the factor ${\Psi'}_v^{-1} \circ \Psi_v$ since there is only one expression needed for each of the five elements. More complex cases need more care, see [17].

First calculating geometrically:

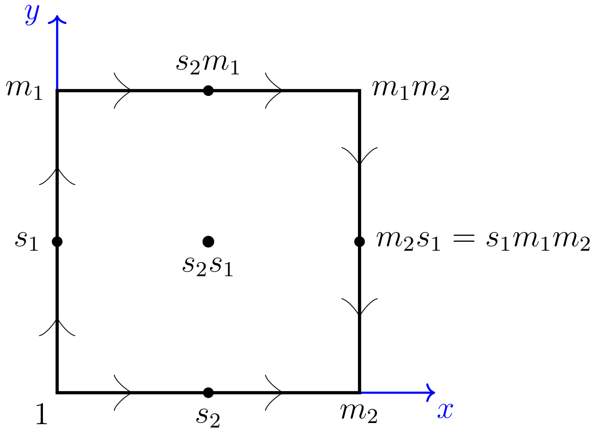
1. $c(s_1, m_1) = 1$ and $c(s_1, 1) = -1$ since $f_1^\pi(\pi) = e^{\pi A} = m_1$ and $f_1^0(0) = 1$. Remember the convention we adopted for orientation of the cube $[-1, 1]^d$: at the final point in the direction of the axis it is $+1$ and in the initial point opposite the axis it is -1.

2. $c(s_2, m_2) = 1$ and $c(s_2, 1) = -1$ since $f_1^\pi(\pi) = e^{\pi B} = m_2$ and $f_1^0(0) = 1$.

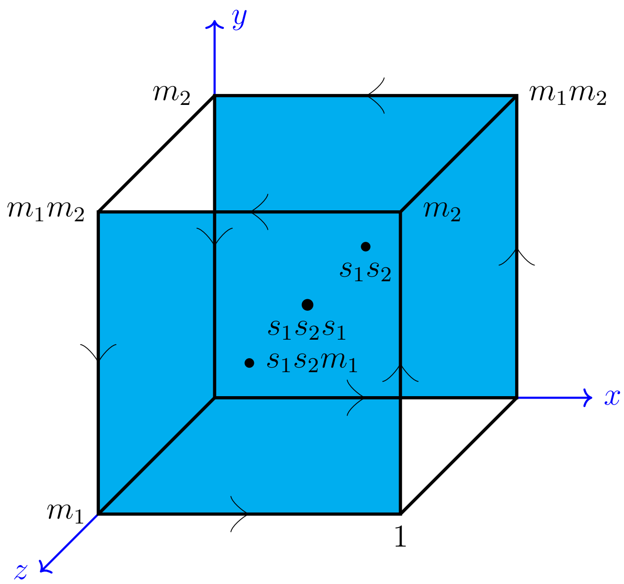
3. $c(s_1s_2, s_1m_2) = -1$, $c(s_1s_2, s_1) = 1$, $c(s_1s_2, m_1s_2) = -1$,$c(s_1s_2, s_2) = -1$. We need to consider the degree of 4 maps: $f_2^\pi(t, \pi) = e^{tA}e^{\pi B} = e^{tA}m_2$, $f_2^0(t, 0) = e^{tA}.1 = e^{tA}$, $f_1^\pi(\pi, s) = e^{\pi A}e^{sB} = m_1e^{sB} = e^{(\pi-s)B}m_2m_1$ since $m_1e^{sB}m_1 = e^{-sB}$, $f_1^0(0, s) = 1.e^{sB} = e^{sB}$. These maps are illustrated with orientations in the next picture. Note that following our convention the positive orientation in this case is the counter-clockwise orientation.

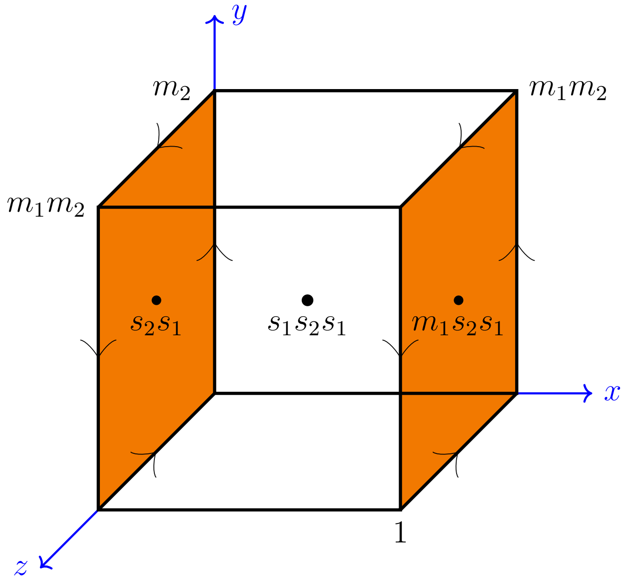


4. $c(s_2s_1, s_2m_1) = -1$, $c(s_2s_1, s_2) = 1$, $c(s_2s_1, m_2s_1) = -1$,$c(s_2s_1, s_1) = -1$. We need to consider the degree of 4 maps: $f_2^\pi(t,\pi) = e^{tB}e^{\pi A} = e^{tB}m_1$, $f_2^0(t,0) = e^{tB}.1 = e^{tB}$, $f_1^\pi(\pi, s) = e^{\pi B}e^{sA} = m_2e^{sA} = e^{(\pi-s)A}m_2m_1$ since $m_2e^{sA}m_2 = e^{-sA}$, $f_1^0(0,s) = 1.e^{sB} = e^{sB}$. These maps are illustrated with orientations in the next picture.



5. $c(s_1s_2s_1, s_1s_2m_1) = 1$, $c(s_1s_2s_1, s_1s_2) = -1$, $c(s_1s_2s_1, m_1s_2s_1) = 1$, $c(s_1s_2s_1, s_2s_1) = -1$. We need to consider 4 maps: $f_2^\pi(t,s,\pi) = e^{tA}e^{sB}m_1$, $f_2^0(t,s,0) = e^{tA}.e^{sB}$, $f_1^\pi(\pi,s,z) = m_1e^{sB}e^{zA} = e^{(\pi-s)B}m_1m_2e^{zA} = e^{(\pi-s)B}e^{(\pi-z)A}m_2$, since $m_1e^{sB}m_1 = e^{-sB}$ and $m_2e^{zA}m_2 = e^{-zA}$, $f_1^0(0,s,z) = e^{sB}e^{zA}$. These 4 maps are illustrated in the next two pictures with orientations of the faces on the edges.





Now we obtain the same results using the more general algebraic method:

Remember we need to find $\Pi_{v_2} = \Pi^+ \cap v_2\Pi^-$ for each case to find the respective $\sigma(u, v)$. For this we use equation 4.1 and that $\Pi_{v_2} = \Pi_{\pi(v_2)}$ and that $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$. Here we will use $*$ to represent m_j or 1.

1. To calculate $c(s_1, *)$ we have $j = 1$ and $v_2 = 1$ so $\Pi_{v_2} = \varnothing$. Then $c(s_1, 1) = (-1)^1$ and $c(s_1, m_1) = (-1)^{1+1}$

2. For $c(s_2, *)$ we have $j = 1$ and $v_2 = 1$ so $\Pi_{v_2} = \varnothing$. Then $c(s_2, 1) = (-1)^1$ and $c(s_2, m_1) = (-1)^{1+1}$

3. For $c(s_1s_2, s_1*)$ we have $j = 2$ and $v_2 = 1$ so $\Pi_{v_2} = \varnothing$. Then $c(s_1s_2, s_1) = (-1)^1$ and $c(s_1s_2, s_1m_2) = (-1)^{1+1}$. Now for $c(s_1s_2, *s_2)$ we have $j = 1$ and $v_2 = s_2$ so $\Pi_{v_2} = \{\alpha_2\}$ by Equation 4.1 and $\sigma = (2.(-1))/2 = -1$. Then $c(s_1s_2, s_2) = (-1)^1$ and $c(s_1s_2, m_1s_2) = (-1)^1$.

4. For $c(s_2s_1, s_2*)$ we have $j = 2$ and $v_2 = 1$ so $\Pi_{v_2} = \varnothing$. Then $c(s_2s_1, s_2) = (-1)^1$ and $c(s_2s_1, s_2m_1) = (-1)^{1+1}$. Now for $c(s_2s_1, *s_1)$ we have $j = 1$ and $v_2 = s_1$ so $\Pi_{v_2} = \{\alpha_1\}$ by Equation 4.1 and $\sigma = (2.(-1))/2 = -1$. Then $c(s_2s_1, s_2) = (-1)^1$ and $c(s_2s_1, m_1s_2) = (-1)^1$.

5. For $c(s_1s_2s_1, s_1s_2*)$ we have $j = 3$ and $v_2 = 1$ so $\Pi_{v_2} = \varnothing$. Then $c(s_1s_2s_1, s_1s_2) = (-1)^1$ and $c(s_1s_2s_1, s_1s_2m_1) = (-1)^{1+1}$. Now for $c(s_1s_2s_1, *s_2s_1)$ we have $j = 1$ and $v_2 = s_2s_1$ so $\Pi_{v_2} = \{\alpha_2, r_2\alpha_1\} = \{\alpha_2, \alpha_1 + \alpha_2\}$ by Equation 4.1 and
$$\sigma = \frac{2\langle\alpha_1, \alpha_2\rangle}{\langle\alpha_1, \alpha_1\rangle} + \frac{2\langle\alpha_1, \alpha_1 + \alpha_2\rangle}{\langle\alpha_1, \alpha_1\rangle} = \frac{2.(-1)}{2} + \frac{2.(1)}{2} = 0$$
so $c(s_1s_2s_1, s_2s_1) = (-1)^1$ and $c(s_1s_2s_1, m_1s_2s_1) = (-1)^{1+1} = 1$

Writing $\mathcal{B}(m)$ as m, for any $m \in M$ we get,

$$\delta_1\mathcal{B}(s_1) = m_1 - 1$$
$$\delta_1\mathcal{B}(s_2) = m_2 - 1$$

$$\delta_2\mathcal{B}(s_1s_2) = \mathcal{B}(s_1)(1 - m_2) - \mathcal{B}(s_2)(1 + m_1m_2)$$
$$\delta_2\mathcal{B}(s_2s_1) = \mathcal{B}(s_2)(1 - m_1) - \mathcal{B}(s_1)(1 + m_1m_2)$$

$$\delta_3\mathcal{B}(s_1s_2s_1) = \mathcal{B}(s_1s_2)(m_1 - 1) + \mathcal{B}(s_2s_1)(m_2 - 1)$$

Calculating the kernel and image of boundary maps δ_k we can then calculate the homology of the compact group K.

$$H_k = \frac{\ker\delta_k}{\operatorname{Im}\delta_{k+1}}$$

Since the operators δ_k are equivariant with relation to right multiplication with members of M then the kernels and images in the previous expression also are invariant by right multiplication by M.

Note also that δ_k are all linear operators so in the matrix format is always possible to find the kernels and images by linear algebra calculations. The calculations for the kernels are all done in the appendix. In the following calculations we will write $\mathcal{B}(s_js_k)$ and $\mathcal{B}(s_j)$ as s_js_k and s_j, respectively.

$$\operatorname{Im}\delta_1 = \langle m_1 - 1, m_2 - 1, m_1m_2 - m_2, m_1m_2 - m_1\rangle$$
$$\operatorname{Im}\delta_1 = \langle m_1 - 1, m_2 - 1, m_1m_2 - 1\rangle$$

Now, we can calculate the homology group,

$$H_0 = \frac{\ker\delta_0}{\operatorname{Im}\delta_1} = \frac{\langle 1, m_1, m_2, m_1m_2\rangle}{\operatorname{Im}\delta_1} = \langle 1\rangle$$

$$\ker\delta_1 = \langle s_1(m_1+1), s_1m_2(m_1+1), s_2(m_2+1), s_2m_1(m_2+1),$$
$$s_1(m_2-1) - s_2(m_1-1)\rangle$$
$$\ker\delta_1 = \langle s_1(m_1+1), s_1m_2(m_1+1), s_2(m_2+1), s_2m_1(m_2+1),$$
$$s_1(m_2-1) - s_2(m_1-1) + s_2m_1(m_2+1)\rangle$$
$$\ker\delta_1 = \langle s_1(m_1+1), s_1m_2(m_1+1), s_2(m_2+1), s_2m_1(m_2+1),$$
$$s_1(m_2-1) + s_2(m_1m_2+1)\rangle$$
$$\ker\delta_1 = \langle s_1(m_2+1)(m_1+1), s_1m_2(m_1+1), s_2(m_1+1)(m_2+1), s_2m_1(m_2+1),$$
$$s_1(m_2-1) + s_2(m_1m_2+1)\rangle$$
$$\ker\delta_1 = \langle s_1(m_2+1)(m_1+1), s_1m_2(m_1+1), s_2(m_1+1)(m_2+1),$$
$$s_1m_2(m_1+1) + s_2m_1(m_2+1), s_1(m_2-1) + s_2(m_1m_2+1)\rangle$$

Let $a = \delta_2(s_1s_2) = s_1(1-m_2) - s_2(1+m_1m_2)$

Let $b = \delta_2(s_2s_1) = s_2(1-m_1) - s_1(1+m_1m_2)$

Since $a.(1+m_2)(1-m_1m_2) = 0$, $b.(1+m_1)(1-m_1m_2) = 0$ and $am_1 + bm_2 = a + b = -(s_1m_2(m_1+1) + s_2m_1(m_2+1))$ then,

$$\operatorname{Im}\delta_2 = \langle a, am_1, am_2, am_1m_2, b, bm_1, bm_2, bm_1m_2\rangle$$
$$\operatorname{Im}\delta_2 = \langle a, am_1, am_2, b, bm_1, bm_2\rangle$$
$$\operatorname{Im}\delta_2 = \langle a, am_1, am_2, b, bm_1\rangle$$
$$\operatorname{Im}\delta_2 = \langle a, am_1, a(1+m_2), b, b(1+m_1)\rangle$$
$$\operatorname{Im}\delta_2 = \langle a, am_1, a(1+m_2), a+b, b(1+m_1)\rangle$$
$$\operatorname{Im}\delta_2 = \langle a, am_1, s_2(1+m_1m_2)(1+m_2), a+b, s_1(1+m_1m_2)(1+m_1)\rangle$$
$$\operatorname{Im}\delta_2 = \langle a, am_1, s_2(1+m_1)(1+m_2), a+b, s_1(1+m_2)(1+m_1)\rangle$$

To the 2nd element add the 1st, 3rd, -5th so that

$$\operatorname{Im}\delta_2 = \langle a, -2s_1m_2(1+m_1), s_2(1+m_1)(1+m_2), a+b, s_1(1+m_2)(1+m_1)\rangle$$

Comparing $\operatorname{Im}\delta_2$ and $\ker\delta_1$, note that they have 4 equal terms and that the 5th term is double the other, since its easy to check that they are linearly independent.

Then,

$$H_1 = \frac{\ker\delta_1}{\operatorname{Im}\delta_2} = \mathbb{Z}/2\mathbb{Z}$$

$$\ker\delta_2 = \langle s_1s_2(m_1-1)(m_2+1), s_1s_2(m_1-1)+s_2s_1(m_2-1),$$
$$s_1s_2(m_1-1)+s_2s_1m_1(1-m_2)\rangle$$
$$\ker\delta_2 = \langle s_1s_2(m_1-1)(m_2+1), s_1s_2(m_1-1)+s_2s_1(m_2-1),$$
$$s_1s_2(m_1-1)+s_2s_1m_1(1-m_2)-(s_1s_2(m_1-1)+s_2s_1(m_2-1))\rangle$$
$$\ker\delta_2 = \langle s_1s_2(m_1-1)(m_2+1), s_1s_2(m_1-1)+s_2s_1(m_2-1),$$
$$s_2s_1(m_1+1)(1-m_2)\rangle$$

Let $c = \delta_2(s_1s_2s_1) = s_1s_2(m_1-1)+s_2s_1(m_2-1)$, Since $c.(m_1+1)(m_2+1)=0$ then,

$$\operatorname{Im}\delta_3 = \langle c, cm_1, cm_2, cm_1m_2\rangle$$
$$\operatorname{Im}\delta_3 = \langle c, cm_1, cm_2\rangle$$
$$\operatorname{Im}\delta_3 = \langle c, c(m_1+1), c(m_2+1)\rangle$$
$$\operatorname{Im}\delta_3 = \langle s_1s_2(m_1-1)+s_2s_1(m_2-1), s_2s_1(m_2-1)(m_1+1), s_1s_2(m_1-1)(m_2+1)\rangle$$

Note then that $\operatorname{Im}\delta_3 = \ker\delta_2$ so that

$$H_2 = \frac{\ker\delta_2}{\operatorname{Im}\delta_3} = \{0\}$$

$$\ker\delta_3 = \langle s_1s_2s_3(m_1+1)(m_2+1)\rangle$$

$$H_3 = \frac{\ker\delta_3}{\operatorname{Im}\delta_4} = \ker\delta_3 \simeq \mathbb{Z}$$

These results agree with SO(3) being homeomorphic to the projective three dimensional space.

4.7 Appendix

In the following calculations we will write $\mathcal{B}(s_js_k)$ and $\mathcal{B}(s_j)$ as s_js_k and s_j, respectively. Also, let $a_i \in \mathbb{Z}$.

Calculating $\ker\delta_1$:

$$\delta_1(s_1(a_1+a_2m_1+a_3m_2+a_4m_1m_2)+s_2(a_5+a_6m_1+a_7m_2+a_8m_1m_2))$$

$$=(m_1-1)(a_1+a_2m_1+a_3m_2+a_4m_1m_2)+(m_2-1)(a_5+a_6m_1+a_7m_2+a_8m_1m_2)$$

$= (-a_1+a_2-a_5+a_7)+m_1(a_1-a_2-a_6+a_8)+m_2(-a_3+a_4+a_5-a_7)+m_1m_2(-a_4+a_3+a_6-a_8)$

So to get $\ker\delta_1$ we need

$$\begin{cases} a_1 + a_5 = a_2 + a_7 \\ a_1 + a_8 = a_2 + a_6 \\ a_3 + a_7 = a_4 + a_5 \\ a_3 + a_6 = a_4 + a_8 \end{cases}$$

Subtracting the second line by the first and the fourth line by the third,

$$\begin{cases} a_1 + a_5 = a_2 + a_7 \\ a_5 + a_6 = a_7 + a_8 \\ a_3 + a_7 = a_4 + a_5 \\ a_5 + a_6 = a_7 + a_8 \end{cases}$$

So

$$\begin{cases} a_1 = a_2 - a_5 + a_7 = a_2 + (a_6 - a_7 - a_8) + a_7 = a_2 + a_6 - a_8 \\ a_5 = -a_6 + a_7 + a_8 \\ a_3 = a_4 + a_5 - a_7 = a_4 + (-a_6 + a_7 + a_8) - a_7 = a_4 - a_6 + a_8 \end{cases}$$

where the a_5 was substituted in the first and third line, so that

$$s_1(a_1 + a_2m_1 + a_3m_2 + a_4m_1m_2) + s_2(a_5 + a_6m_1 + a_7m_2 + a_8m_1m_2)$$

$$= s_1((a_2 + a_6 - a_8) + a_2m_1 + (a_4 - a_6 + a_8)m_2 + a_4m_1m_2)$$

$$+s_2((-a_6 + a_7 + a_8) + a_6m_1 + a_7m_2 + a_8m_1m_2)$$

reorganizing the terms

$$a_2s_1(1 + m_1) + a_4s_1(m_2 + m_1m_2) + a_6(s_1(1 - m_2) + s_2(-1 + m_1))$$

$$+a_7(s_2(1 + m_2)) + a_8(s_1(-1 + m_2) + s_2(1 + m_1m_2))$$

In the previous expression the terms multiplying a_2, a_4, a_6, a_7, a_8 are then generators of $\ker\delta_1$. To simplify adding the term from a_6 to a_8 we get that:

$$\ker\delta_1 = \langle s_1(1+m_1), s_1m_2(1+m_1), s_2(1+m_2), s_2m_1(1+m_2), s_1(1-m_2)-s_2(1-m_1)\rangle$$

Calculating $\ker\delta_2$:

$$\delta_2(s_1s_2(a_1 + a_2m_1 + a_3m_2 + a_4m_1m_2) + s_2s_1(a_5 + a_6m_1 + a_7m_2 + a_8m_1m_2))$$

$$= \begin{matrix} (s_1(1 - m_2) - s_2(1 + m_1m_2))(a_1 + a_2m_1 + a_3m_2 + a_4m_1m_2) \\ +(s_2(1 - m_1) - s_1(1 + m_1m_2))(a_5 + a_6m_1 + a_7m_2 + a_8m_1m_2) \end{matrix}$$

$$
\begin{aligned}
& s_1((a_1 - a_3 - a_5 - a_8) + m_1(a_2 - a_4 - a_6 - a_7) + m_2(a_3 - a_1 - a_6 - a_7) \\
= \quad & +m_1m_2(a_4 - a_2 - a_5 - a_8)) + s_2((a_5 - a_1 - a_4 - a_6) + m_1(a_6 - a_2 - a_3 - a_5) \\
& +m_2(a_7 - a_2 - a_3 - a_8) + m_1m_2(a_8 - a_1 - a_4 - a_7))
\end{aligned}
$$

So to get $\ker \delta_2$ we need

$$
\begin{cases}
a_1 = a_3 + a_5 + a_8 \\
a_2 = a_4 + a_6 + a_7 \\
a_3 = a_1 + a_6 + a_7 = (a_3 + a_5 + a_8) + a_6 + a_7 \\
a_4 = a_2 + a_5 + a_8 = (a_4 + a_6 + a_7) + a_5 + a_8 \\
a_5 = a_1 + a_4 + a_6 = (a_3 + a_5 + a_8) + a_4 + a_6 \\
a_6 = a_2 + a_3 + a_5 = (a_4 + a_6 + a_7) + a_3 + a_5 \\
a_7 = a_2 + a_3 + a_8 = (a_4 + a_6 + a_7) + a_3 + a_8 \\
a_8 = a_1 + a_4 + a_7 = (a_3 + a_5 + a_8) + a_4 + a_7
\end{cases}
$$

where we substituted the the first two lines in the other equations. With some cancellations we note that the fourth, the seventh and eighth are redundant, then

$$
\begin{cases}
a_1 = a_3 + a_5 + a_8 \\
a_2 = a_4 + a_6 + a_7 \\
a_5 + a_8 + a_6 + a_7 = 0 \\
a_3 + a_8 + a_4 + a_6 = 0 \\
a_4 + a_7 + a_3 + a_5 = 0
\end{cases}
$$

Summing the last three equations we get $a_3 + a_4 + a_6 + a_8 + a_5 + a_7 = 0$ and substituting the third, fourth and fifth equation we get:

$$
\begin{cases}
a_1 = a_3 + a_5 + a_8 \\
a_2 = a_4 + a_6 + a_7 \\
a_3 + a_4 = 0 \\
a_5 + a_7 = 0 \\
a_6 + a_8 = 0
\end{cases}
$$

With this we can put every term as function of a_4,a_7,a_8 as

$$
\begin{cases}
a_1 = -a_4 - a_7 + a_8 \\
a_2 = a_4 - a_8 + a_7 \\
a_3 = -a_4 \\
a_5 = -a_7 \\
a_6 = -a_8
\end{cases}
$$

So that

$$
s_1s_2(a_1 + a_2m_1 + a_3m_2 + a_4m_1m_2) + s_2s_1(a_5 + a_6m_1 + a_7m_2 + a_8m_1m_2)
$$

$$= \begin{array}{l} s_1s_2((-a_4 - a_7 + a_8) + (a_4 - a_8 + a_7)m_1 - a_4m_2 + a_4m_1m_2) \\ +s_2s_1(-a_7 - a_8m_1 + a_7m_2 + a_8m_1m_2) \end{array}$$

reorganizing terms,

$$= \begin{array}{l} a_4s_1s_2(-1 + m_1 - m_2 + m_1m_2) + a_7(s_1s_2(-1 + m_1) + s_2s_1(-1 + m_2)) \\ +a_8(s_1s_2(1 - m_1) + s_2s_1(-m_1 + m_1m_2)) \end{array}$$

In the previous expression the three terms multiplying a_4,a_7,a_8 are then a generating set for $\ker \delta_2$ or

$$\ker \delta_2 = \begin{array}{l} \langle s_1s_2(m_1 - 1)(m_2 + 1), s_1s_2(m_1 - 1) + s_2s_1(m_2 - 1), \\ s_1s_2(m_1 - 1) + s_2s_1m_1(1 - m_2)\rangle \end{array}$$

Calculating $\ker \delta_3$:

$$\delta_3(s_1s_2s_1(a_1 + a_2m_1 + a_3m_2 + a_4m_1m_2))$$
$$= (s_1s_2(m_1 - 1) + s_2s_1(m_2 - 1))(a_1 + a_2m_1 + a_3m_2 + a_4m_1m_2)$$

$$= \begin{array}{l} s_1s_2((a_2 - a_1) + (a_1 - a_2)m_1 + (a_4 - a_3)m_2 + (a_3 - a_4)m_1m_2) \\ +s_2s_1((a_3 - a_1) + (a_4 - a_2)m_1 + (a_1 - a_3)m_2 + (a_2 - a_4)m_1m_2) \end{array}$$

So to get $\ker \delta_3$ we need $a_1 = a_2$, $a_3 = a_4$ and $a_1 = a_3$, $a_2 = a_4$ or $a_1 = a_2 = a_3 = a_4$ so

$$\ker \delta_3 = \langle s_1s_2s_1a_1(1 + m_1 + m_2 + m_1m_2)\rangle$$
$$\ker \delta_3 = \langle s_1s_2s_1a_1(m_1 + 1)(m_2 + 1)\rangle$$

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