Universidade de Brasília
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by

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À minha mãe.

"Não há ramo da Matemática, por mais abstrato que seja, que não possa um dia vir a ser aplicado aos fenômenos do mundo real." (Lobachevsky)

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Resumo

Neste trabalho, estudamos hipersuperfícies isoparamétricas em variedades produto de dimensão 4. Primeiramente, caracterizamos e classificamos as hipersuperfícies isoparamétricas com curvaturas principais constantes nos espaços produto $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, em que $\mathbb{Q}^2_{c_i}$ é uma forma espacial com curvatura seccional constante c_i, para $c_i \in \{-1, 0, 1\}$ e $c_1 \neq c_2$. Mostramos que tais hipersuperfícies são dadas por conjuntos abertos de uma hipersuperfície produto, em que um dos fatores é uma curva de curvatura constante, ou de uma estrutura diagonal em $\mathbb{H}^2 \times \mathbb{R}^2$, construída a partir de horocírculos em $\mathbb{H}^2$ e retas em $\mathbb{R}^2$.

Em seguida, classificamos as hipersuperfícies em $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ que possuem as três curvaturas principais constantes distintas, em que neste caso $\varepsilon \in \{-1, 1\}$. Mostramos que tais hipersuperfícies são cilindros sobre superfícies isoparamétricas de $\mathbb{Q}^3_\varepsilon$ com duas curvaturas principais distintas e não-nulas. Também provamos que as hipersuperfícies com curvaturas principais constantes em $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ são isoparamétricas. Além disso, fornecemos uma condição necessária e suficiente para uma hipersuperfície isoparamétrica em $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ ter curvaturas principais constantes.

Finalmente, descrevemos a evolução pelo fluxo da curvatura média de hipersuperfícies isoparamétricas em variedades produto de dimensão 4. Mostramos que a evolução de hipersuperfícies isoparamétricas de variedades Riemannianas pelo fluxo da curvatura média é dada por uma reparametrização do fluxo por hipersuperfícies paralelas em um curto espaço de tempo, desde que a unicidade do fluxo de curvatura média seja válida para os dados iniciais e o espaço ambiente correspondente. Através deste resultado, descrevemos a evolução das hipersuperfícies classificadas na primeira e segunda partes do trabalho. Também descrevemos as evoluções de hipersuperfícies isoparamétricas em $\mathbb{S}^2 \times \mathbb{S}^2$ e $\mathbb{H}^2 \times \mathbb{H}^2$, classificadas por Urbano (2019) e Dong Gao, Hui Ma e Zeke Yao (2022), respectivamente, e das hipersuperfícies isoparamétricas em $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ com g curvaturas principais constantes distintas, $g \in \{1, 2\}$, classificadas por Chaves e Santos (2019).

Palavras-chave: hipersuperfícies isoparamétricas, espaços produto, hipersuperfícies paralelas, curvaturas principais constantes, fluxo da curvatura média.

Summary

In this work, we study isoparametric hypersurfaces in product manifolds of dimension 4. First of all, we characterize and classify the isoparametric hypersurfaces with constant principal curvatures in the product spaces $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, where $\mathbb{Q}^2_{c_i}$ is a space form with constant sectional curvature c_i, for $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. We show that such hypersurfaces are given as open subsets of either a product hypersurface, where one factor is a curve of constant curvature, or a diagonal structure in $\mathbb{H}^2 \times \mathbb{R}^2$, constructed from horocycles in $\mathbb{H}^2$ and straight lines in $\mathbb{R}^2$.

Next, we classify the hypersurfaces in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with the three distinct constant principal curvatures, where in this case $\varepsilon \in \{-1, 1\}$. We show that such hypersurfaces are cylinders over isoparametric surfaces of $\mathbb{Q}^3_\varepsilon$ with two non-null distinct principal curvatures. We also prove that the hypersurfaces with constant principal curvatures in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ are isoparametric. Furthermore, we provide a necessary and sufficient condition for an isoparametric hypersurface on $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ to have constant principal curvatures.

Finally, we describe the evolution by the mean curvature flow of isoparametric hypersurfaces in product manifolds of dimension 4. We show that the evolution of isoparametric hypersurfaces of Riemannian manifolds by the mean curvature flow is given by a reparametrization of the flow by parallel hypersurfaces in a short time, as long as the uniqueness of the mean curvature flow holds for the initial data and the corresponding ambient space. Through this result, we describe the evolution of the hypersurfaces classified in the first and second parts of the work. We also describe the evolutions of isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, classified by Urbano (2019) and Dong Gao, Hui Ma and Zeke Yao (2022), respectively, and of isoparametric hypersurfaces in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with g distinct constant principal curvatures, $g \in \{1, 2\}$, classified by Chaves and Santos (2019).

Keywords: isoparametric hypersurfaces, product spaces, parallel hypersurfaces, constant principal curvatures, mean curvature flow.

Contents

Introduction

A hypersurface M^n of a Riemannian manifold $\widetilde{M}^{n+1}$ is said to be isoparametric if it has constant mean curvature as well as its nearby equidistant hypersurfaces (i.e., the correspondent mean curvatures depend only on the distance to M). Equivalently, we say that M is isoparametric if it is the level set of some isoparametric function defined on $\widetilde{M}$. Following Domínguez-Vázquez [15], the first notion of isoparametric surfaces appeared in 1919 in the work of C. Somigliana [39], which deals with the relations between the Huygens principle and geometric optics. This study represented the beginning of an important research line in Differential Geometry, namely the isoparametric hypersurfaces studied by renowned mathematicians such as Beniamino Segre, Élie Cartan, and Tullio Levi-Civita.

When the ambient space is a space form, i.e., a simply connected complete Riemannian manifold with constant sectional curvature, the previous definition of isoparametric hypersurface is equivalent to saying that the hypersurface has constant principal curvatures (see [7] and [15]). However, in other ambient spaces of nonconstant curvature, the equivalence between isoparametric hypersurfaces and hypersurfaces with constant principal curvatures may no longer be true. For instance, Q. M. Wang, in [43], found examples of isoparametric hypersurfaces in complex projective spaces that do not have constant principal curvatures. For more examples, we refer [12], [13] and [21]. Recently, A. Rodríguez-Vázquez, in [33], found an example of a non-isoparametric hypersurface with constant principal curvatures. Another example was given in [22].

In this thesis, we study isoparametric hypersurfaces in product manifolds of dimension 4. More precisely, we consider as ambient spaces the products $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, for $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, and $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$, for $\varepsilon \in \{-1, 1\}$, where $\mathbb{Q}^n_c$ denotes denotes the unit n-sphere $\mathbb{S}^n$, if $c = 1$, the n-dimensional Euclidean space $\mathbb{R}^n$, if $c = 0$, and n-dimensional hyperbolic space $\mathbb{H}^n$, if $c = -1$. Furthermore, among other examples, we study the evolution by the mean curvature flow of isoparametric hypersurfaces that appear in such ambient spaces.

This thesis has four chapters. Chapter 1 is devoted to a brief presentation of some facts already known in the literature about Jacobi field theory and isoparametric hypersurfaces, which will be very useful throughout this work.

In Chapter 2, we consider the Riemannian products of 2-dimensional space forms $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with constant sectional curvatures c_1 and c_2, respectively, with $c_1 \neq c_2$, where $c_i = 1$, 0 or -1, $i = 1, 2$. Such a kind of ambient space was firstly considered in this context by Urbano [41], where it was obtained, among other results, the classification of isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, i.e., when $c_1 = c_2 = 1$.

The case where $c_1 = 1$ and $c_2 = 0$, that is, when the ambient space is $\mathbb{S}^2 \times \mathbb{R}^2$, was considered by Julio-Batalla in [25] where he obtained a complete classification of isoparametric hypersurfaces with constant principal curvatures. Using some ideas developed by Urbano in [41], Julio-Batalla showed that if Σ is an isoparametric hypersurface in $\mathbb{S}^2 \times \mathbb{R}^2$, with constant principal curvatures and unit normal $N = N_1 + N_2$, then $|N_1|$ and $|N_2|$ are constant, where N_1 and N_2 denote the components of N in $\mathbb{S}^2$ and $\mathbb{R}^2$, respectively. The classification continues by showing that $|N_1| = 1$ and $|N_2| = 0$ or $|N_1| = 0$ and $|N_2| = 1$. Thus, the hypersurface families obtained are $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^1(r)$ (for $r \in \mathbb{R}^+$), or $\mathbb{S}^1(t) \times \mathbb{R}^2$ (for $t \in (0, 1]$). Recently, and also following some of Urbano's ideas and techniques, D. Gao, H. Ma and Z. Yao [19] classified, among other results, the isoparametric hypersurfaces of $\mathbb{H}^2 \times \mathbb{H}^2$ and the hypersurfaces with at most two distinct constant principal curvatures. The case of hypersurfaces of $\mathbb{H}^2 \times \mathbb{H}^2$ with three distinct principal curvatures was also considered under some additional conditions.

In this work, we extend and improve the results of [25] in the following sense. Considering the ambient space $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, we prove (see Theorem 2.1)

Theorem 0.1. *Let Σ be an isoparametric hypersurface in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, and unit normal $N = N_1 + N_2$, where N_1 and N_2 denote the components of N in $\mathbb{Q}^2_{c_1}$ and $\mathbb{Q}^2_{c_2}$, respectively. Then the principal curvatures of Σ are constant if and only if $|N_1|$ and $|N_2|$ are constant.*

In addition to the converse of a result obtained by Julio-Batalla, which states that if $|N_1|$ and $|N_2|$ are constant, then Σ has constant principal curvatures, Theorem 0.1 also provides the equivalence for the entire class of ambient spaces $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. To get this Theorem, we use the theory of Jacobi fields, based on the ideas developed by Domínguez-Vázquez and Manzano in [16], to analyze the extrinsic geometry of hypersurfaces parallel to Σ. It is interesting to note that Jacobi field theory allows us to obtain an alternative proof of Julio-Batalla's result. Moreover, we obtain the following general classification of isoparametric hypersurfaces with constant principal curvatures in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, which includes the classification for $\mathbb{S}^2 \times \mathbb{R}^2$ given in [25] (see Theorem 2.2):

Theorem 0.2. *Let Σ be an isoparametric hypersurface with constant principal curvatures in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. Then, up to rigid motions, Σ is an open subset of one of the following hypersurfaces:*

a) $\mathcal{C}^1(\kappa_j) \times \mathbb{Q}^2_{c_2}$ *or* $\mathbb{Q}^2_{c_1} \times \mathcal{C}^1(\kappa_j)$, *where $\mathcal{C}^1(\kappa_j)$ is a complete curve with constant geodesic curvature κ_j in $\mathbb{Q}^2_{c_j}$.*

b) $\Psi(\mathbb{R}^3) \subset \mathbb{H}^2 \times \mathbb{R}^2$, *where $\Psi : \mathbb{R}^3 \to \mathbb{H}^2 \times \mathbb{R}^2$ is an immersion given by*

$$\begin{aligned} \Psi(s, u, v) = e^{-b\,s}(\alpha(u), \vec{0}) + \Big(\cosh(-b\,s), 0, \sinh(-b\,s), V_0 s\Big) \\ + \Big(\vec{0}, p_0 + W_0 v\Big), \end{aligned} \tag{0.1}$$

where $\mathbb{H}^2 \subset \mathbb{L}^3$ is given as the standard model of the hyperbolic space in the Lorentz 3-space $\mathbb{L}^3$, the curve α is given by $\alpha(u) = \left(\frac{u^2}{2}, u, -\frac{u^2}{2}\right)$, $p_0 \in \mathbb{R}^2$, V_0 and W_0 are constant orthogonal vectors in $\mathbb{R}^2$ such that $||W_0|| = 1$ and $b = \sqrt{1 - ||V_0||^2}$, with $b \neq \{1, 0\}$.

Remember that, besides the geodesics, the complete curves $\mathcal{C}^1(\kappa_j) \subset \mathbb{Q}^2_{c_j}$ with constant geodesic curvature are given by: $\mathbb{S}^1(t) \subset \mathbb{S}^2$ for $t \in (0, 1)$; circles, horocycles or hypercycles in $\mathbb{H}^2$ (see for example [36]); and $\mathbb{S}^1(r) \subset \mathbb{R}^2$ for $r \in \mathbb{R}^+$. Regarding the hypersurfaces given in Theorem 2.2.b), geometrically, $\Psi(\mathbb{R}^3)$ provides a hypersurface given as the union of a family of geodesically parallel surfaces given by the products $\mathcal{C}^1(1) \times \mathbb{R}$, where $\mathcal{C}^1(1) \subset \mathbb{H}^2$ is a horocycle (see Remark 3). Furthermore, $\Psi(\mathbb{R}^3)$ is an extrinsically homogeneous hypersurface, i.e., it is a codimension-one orbit of a subgroup of the group of isometries of $\mathbb{H}^2 \times \mathbb{R}^2$ (see Remark 4).

Chapter 3 is devoted to the study of hypersurfaces with constant principal curvatures in the product spaces $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$, where $\mathbb{Q}_\varepsilon^3$ denotes the unit sphere $\mathbb{S}^3$ if $\varepsilon = 1$, and the hyperbolic space $\mathbb{H}^3$ if $\varepsilon = -1$. Our main objective is to classify the hypersurfaces of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ that have the three distinct constant principal curvatures. As a consequence, we will conclude that the hypersurfaces of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ with constant principal curvatures are isoparametric. Furthermore, we will provide a necessary and sufficient condition for the converse to hold.

In recent years, several geometers have dedicated themselves to the study of hypersurfaces in the product spaces $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$. In [40], Tojeiro locally classified the hypersurfaces of $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ that have a special field T as a principal direction, and with that, he also obtained the classification of hypersurfaces with constant angle. Given a hypersurface Σ in $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$, the tangent field T and the angle function θ are defined by

$$\partial_t = T + \cos(\theta)N,$$

where N is the unit normal field to Σ and ∂_t is a unit field tangent to the second factor $\mathbb{R}$.

In [8], Chaves and Santos classified the hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, $n \geq 2$, with g distinct constant principal curvatures, $g \in \{1, 2, 3\}$, where $n \geq 4$ if $g = 3$. Moreover, they proved that such hypersurfaces are isoparametric in those spaces. Motivated by the results of Chaves and Santos, in this work, we obtain the classification of the hypersurfaces with constant principal curvatures when $g = 3$ and $n = 3$, that is, we classify the hypersurfaces in $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ that have the three distinct constant principal curvatures. In order to do that, we show that if Σ is a hypersurface of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ with three distinct constant principal curvatures, then θ is constant. Using this characterization, we obtain the following result (see Theorem 3.6):

Theorem 0.3. *Let Σ^3 be a hypersurface of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ with three distinct constant principal curvatures. Then Σ^3 is an open part of the following hypersurfaces:*

a) $\mathbb{S}^1(c_1) \times \mathbb{S}^1(c_2) \times \mathbb{R}$, *when* $\varepsilon = 1$;

b) $\mathbb{S}^1(c_1) \times \mathbb{H}^1(c_2) \times \mathbb{R}$, *when* $\varepsilon = -1$,

where $c_1 \neq c_2$, $\dfrac{1}{c_1} + \dfrac{1}{c_2} = \varepsilon$ *and the principal curvatures of* Σ^3 *are given by* 0, $\dfrac{c_1}{\sqrt{c_1 + c_2}}$ *and* $\dfrac{-c_2}{\sqrt{c_1 + c_2}}$.

The theorem above complements the classification of the hypersurfaces of $\mathbb{Q}_\varepsilon^n \times \mathbb{R}$ that have g distinct constant principal curvatures, $g \in \{1, 2, 3\}$, stated in [8, Theorem 6.1]. It is worth mentioning that the problem when $g \geq 4$ remains open. Furthermore, as a consequence of Theorem 0.3 and Chaves and Santos classification mentioned above, we show that (see Corollary 3.7)

Corollary 0.4. *Let Σ^3 be a hypersurface of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ with constant principal curvatures. Then Σ^3 is isoparametric.*

In the second Theorem of the Chapter 3 (Theorem 3.8), we obtained a necessary and sufficient condition for an isoparametric hypersurface in $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ to have constant principal curvatures. In [16], Domínguez-Vázquez and Manzano, using Jacobi field theory, showed the equivalence between being isoparametric and having constant principal curvatures is true for hypersurfaces of homogeneous 3-manifolds with 4-dimensional isometry group, which include the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Besides that, a classification for such surfaces is given and, in the case of product spaces, they showed that isoparametric surfaces of $\mathbb{Q}_\varepsilon^2 \times \mathbb{R}$ have constant angle function θ. For dimension $n = 3$, using a similar approach employed by Domínguez-Vázquez and Manzano, we show that

Theorem 0.5. *Let Σ be an isoparametric hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$. Then Σ has constant principal curvatures if and only if θ is constant.*

We point out that Chaves and Santos [8] showed that an isoparametric hypersurface Σ of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ having T as principal direction has constant principal curvatures if and only if, $||T||$ is constant. Therefore, Theorem 0.5 tells us that, at least for $n = 3$, we can improve Chaves and Santos' result, since we do not use the assumption of T being a principal direction. In fact, since ∂_t is an unit vector field, it follows that $||T||^2 + \cos(\theta)^2 = 1$. Thus, Theorem 3.8 says that an isoparametric hypersurface in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ has constant principal curvatures if and only if $||T||$ is constant.

In Chapter 4, our goal is to study the evolution of isoparametric hypersurfaces of a Riemannian manifolds by the mean curvature flow. Given a hypersurface M^n of a Riemannian manifold $\widetilde{M}^{n+1}$, we say that M evolves by the mean curvature flow (MCF) if there is a time-dependent family of smooth hypersurfaces with M as initial data such that the velocity of the evolution at each point of such family is given by the mean curvature vector field of the correspondent hypersurface at that point. There is an extensive literature on the study of MCF, mainly when the ambient space $\widetilde{M}^{n+1}$ is the Euclidean space $\mathbb{R}^{n+1}$. However, cases where the ambient space is a general Riemannian manifold and when the codimension is greater than one have also been considered recently. We suggest the surveys [10, 38] and references therein for a good overview of the mentioned topics.

In [37] the authors showed that a hypersurface M^n of a space form $\mathbb{Q}^{n+1}_\varepsilon$ is the initial data for a solution for the MCF given by a reparametrization of the flow of parallel hypersurfaces if and only if M^n is an isoparametric hypersurface. In the sequence, the authors showed in [37] that the MCF given in this way is reduced to an ordinary differential equation, and provided explicit solutions. From such solutions, the exact collapsing times of the singularities are provided. Following the ideas of [37], a version of their results was provided in [18], for a class of isoparametric hypersurfaces of the product spaces $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ and $\mathbb{Q}^n_\varepsilon \times \mathbb{S}^1$. Recently, the author in [28] also used a reparametrization of the flow by parallel hypersurfaces of isoparametric hypersurfaces to consider the Weingarten flow in Riemannian manifolds, which has as a particular case the MCF. In this case, it is important to point out that, following [5], isoparametric hypersurfaces are defined in [28] as those whose parallels have constant principal curvatures, thus including the case in which the ambient spaces are space forms. For submanifolds with higher codimensions, the MCF with initial data given by an isoparametric submanifold was considered in [30, 31], when the ambient space is a space form. For a class of ambient spaces (which includes the space forms with non-negative curvature), the relation between singular Riemannian foliations in which the leaves are isoparametric submanifolds (in the sense of [23]) with the MCF was investigated in [2, 3, 29].

Here, we characterize reparametrizations of the flows by parallel hypersurfaces as the unique solution for the MCF with isoparametric hypersurfaces as initial data in general ambient spaces. Namely, for an ambient space given by a complete Riemannian manifold such that the curvature and its covariant derivatives up to order 2 are bounded, and with injectivity radius bounded from below by a positive constant, we prove that (see Theorem 4.3)

Theorem 0.6. *Let $\widetilde{M}^{n+1}$ be a complete Riemannian manifold such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant. Let Σ^n be a hypersurface of $\widetilde{M}^{n+1}$ such that the solution $F : \Sigma^n \times [0, T) \to \widetilde{M}^{n+1}$ of the MCF with initial data Σ^n has bounded second fundamental form on $[0, T_-]$ for all $T_- < T$. Then, Σ^n is isoparametric if and only if F is the flow by parallels for some $\delta_0 \leq T$. Moreover, suppose that $[0, \delta)$ is the maximal interval where F is* *a reparametrization of the parallel flow. If $\delta < T$ then $F(.,\delta)$ is a hypersurface that is not isoparametric.*

This result provides an extension to general ambient spaces of [18, 37] and an extension of [28] to general isoparametric hypersurfaces when the MCF is considered. Moreover, we also supply an improvement of their results since we show that isoparametric hypersurfaces, besides providing solutions of the MCF through their parallel hypersurfaces, uniquely determined such evolution as initial data, that is, the flow is well described through the flow by parallel hypersurfaces and the solution of an ordinary differential equation. A crucial element for proving the theorem above is the use of a uniqueness theorem for the solution of MCF for general ambient spaces. The compact case is provided by Lemma 3.2 in [24]. For the complete non-compact hypersurfaces, the uniqueness is obtained under conditions on the curvature of the ambient space, and on the second fundamental form (see [9]).

We end Chapter 4 by describing the evolution by the mean curvature flow of isoparametric hypersurfaces in product manifolds of dimension 4, classified in Chapters 2 and 3. Moreover, we also describe the evolutions of isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, classified by Urbano (2019) and Dong Gao, Hui Ma and Zeke Yao (2022), respectively, and of isoparametric hypersurfaces in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with g distinct constant principal curvatures, $g \in \{1, 2\}$, classified by Chaves and Santos (2019).

Chapter 1

Basic concepts and notations

Throughout this chapter, we will briefly establish the basic notations and concepts that will be common in the remaining chapters. For a better reading of this work, specific concepts will be introduced at the beginning of each chapter.

1.1 The ambient spaces

In this brief section, in order to establish some notations, we will define the ambient spaces in which we will work throughout the thesis.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with the canonical metric

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_m^2,$$

and $\mathbb{L}^m$ the m-dimensional Lorentzian space with the canonical metric

$$ds^2 = -dx_1^2 + dx_2^2 + \ldots + dx_m^2.$$

Let $\mathbb{Q}_c^n$ be n-dimensional space form of constant sectional curvature c. When $c = 0$, we have the n-dimensional Euclidean space $\mathbb{R}^n$. For $c \neq 0$, we have the following cases: if $c > 0$, $\mathbb{Q}_c^n$ will denote the n-dimensional sphere

$$\mathbb{S}^n(c) = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} |\, x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = \frac{1}{c}\right\} \subset \mathbb{R}^{n+1},$$

and, if $c < 0$, $\mathbb{Q}_c^n$ will denote the n-dimensional hyperbolic space

$$\mathbb{H}^n(c) = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{L}^{n+1} | -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = \frac{1}{c}\right\} \subset \mathbb{L}^{n+1}.$$

In particular, we have the unit sphere $\mathbb{S}^n(1) = \mathbb{S}^n$ if $c = 1$, and the hyperbolic space $\mathbb{H}^n(-1) = \mathbb{H}^n$ if $c = -1$. Thus, $\mathbb{S}^n$ and $\mathbb{H}^n$ will be considered as submanifolds of $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+1}$, respectively, with the metric induced by such spaces.

In Chapter 2, we will consider the product space $\mathbb{Q}_{c_1}^2 \times \mathbb{Q}_{c_2}^2$, where $\mathbb{Q}_{c_1}^2$ and $\mathbb{Q}_{c_2}^2$ are two 2-dimensional space forms of constant sectional curvatures c_1 and c_2, respectively, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, and in Chapter 3, we will consider the product space $\mathbb{Q}_c^n \times \mathbb{R}$, $c \neq 0$, with the metric induced by the ambient space, given by $\mathbb{S}^n \times \mathbb{R}$, if $c = 1$, and $\mathbb{H}^n \times \mathbb{R}$, if $c = -1$.

1.2 Jacobi field theory

In this section we shall present the notation, basic concepts and results of the Jacobi field theory, which will be used in this work. This theory is an important tool to analyze the extrinsic geometry of hypersurfaces equidistant to any hypersurface. In what follows, we will give a brief description of this theory, following [4] and [15], where the reader can find more details about it.

Given a hypersurface Σ^n of a Riemannian manifold $\widetilde{M}^{n+1}$ with unit normal vector field N, let ε be a positive real number and, for $r \in (-\varepsilon, \varepsilon)$, consider the application

$$\begin{array}{rcl} \Phi_r : \Sigma^n & \to & \widetilde{M}^{n+1}, \\ p & \mapsto & \exp_p(rN(p)), \end{array} \tag{1.1}$$

where $\exp_p : T_p\widetilde{M} \to \widetilde{M}$ denotes the exponential map of $\widetilde{M}^{n+1}$ at $p \in \Sigma$. For $\varepsilon > 0$ small enough, the map Φ_r is smooth and it parametrizes the parallel displacement of Σ at an oriented distance r in the direction N. The parallel hypersurface $\Phi_r(\Sigma)$ will be denoted by Σ_r.

Let $\gamma : I \to \widetilde{M}$ be a geodesic parametrized by arc length with $0 \in I \subset \mathbb{R}$, $p = \gamma(0) \in \Sigma$ and $\dot{\gamma}(0) = N(p)$. Let $c : I \to \Sigma$ be a smooth curve with $c(0) = p$ and $\dot{c}(0) \in T_p\Sigma$, where $\dot{\alpha}$ denotes the tangent vector field of a smooth curve α. Observe that $V(s,t) = \Phi_t(c(s)) = \gamma_s(t)$ is a smooth geodesic variation of $\gamma = \gamma_0$, where $c(s) = \gamma_s(0) \in \Sigma$ and $N \circ c(s) = \dot{\gamma}_s(0) \in T^{\perp}_{c(s)}\Sigma$ for all s. This variation generates the Jacobi field $\xi(s) = \frac{d}{ds}V(s,0)$ determined by the initial values

$$\xi(0) = \dot{c}(0) \in T_p\Sigma \quad \text{and} \quad \xi'(0) = -A\xi(0),$$

where A is the shape operator of Σ and ξ' denotes the covariant derivative of ξ along γ. A Jacobi field ξ along γ whose initial values satisfy these two conditions is called an Σ-Jacobi field. When Σ_r is a hypersurface, the main properties of Σ-Jacobi field in relation to the extrinsic geometry of Σ_r are given as follows:

i) $T_{\gamma(r)}\Sigma_r = \{\xi(r) : \xi \text{ is a } \Sigma\text{-Jacobi field along } \gamma\}$;

ii) $\dot{\gamma}(r)$ provides a unit normal to Σ_r with corresponding shape operator A^r given by $A^r\xi(r) = -\xi'(r)$.

In what follows, we describe another very interesting way to determine the shape operator A^r.

For each r let us define the endomorphism $D(r) : \dot{\gamma}(r)^{\perp} \longrightarrow \dot{\gamma}(r)^{\perp}$ as follows. If $Z \in T_p\Sigma$ and $\widetilde{P}_\gamma$ is the parallel transport along γ, then D is defined such that $\xi = D \circ \widetilde{P}_\gamma Z$ is the Jacobi field along γ with initial values $\xi(0) = Z$ and $\xi'(0) = -AZ$. It follows that D is a solution of

$$D'' + \widetilde{R}(D, \dot{\gamma})\dot{\gamma} = 0, \quad D(0) = id_{T_p\Sigma}, \quad D'(0) = -A, \tag{1.2}$$

where D' and D'' stand for the first and second covariant derivatives of the tensor field D, respectively, and $id_{T_p\Sigma}$ is the identity operator of $T_p\Sigma$. Since Σ_r is a hypersurface, then $D(r)$ is regular and we have that

$$A^r(D \circ \widetilde{P}_\gamma Z)(r) = A^r(\xi(r)) = -\xi'(r) = -(D \circ \widetilde{P}_\gamma Z)'(r) = -D'(r)\widetilde{P}_\gamma Z.$$

Therefore, the shape operator of the parallel hypersurface M_r associated to the unit normal $\dot{\gamma}(r)$ is given by

$$A^r = -(D' \circ D^{-1})(r). \tag{1.3}$$

Consequently, by the Jacobi formula, the mean curvature of the hypersurface Σ_r is given by

$$h(r) = -\frac{(\det D)'}{n \det D}(r). \tag{1.4}$$

1.3 Isoparametric hypersurfaces

In this section, we introduce some known facts about isoparametric hypersurfaces of Riemannian manifolds in general. For more details on this topic, the reader can look up the reference [15].

We start by presenting the definition of isoparametric function. According to [15], this definition was possibly introduced by T. Levi-Civita [27] in 1937.

Definition 1.1. *Let $(\widetilde{M}, g)$ be a connected Riemannian manifold. A non-constant smooth function $f : \widetilde{M} \longrightarrow \mathbb{R}$ is called isoparametric if there exist smooth functions $a, b : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$(1)\ ||\nabla f||^2 = a(f) \quad \textit{and} \quad (2)\ \Delta f = b(f).$$

The smooth hypersurfaces $\Sigma_r = f^{-1}(r)$ for r regular value of f are called isoparametric hypersurfaces. Note that $||\nabla f||$ and Δf are constant along the level sets of f. As we can see in [15], the condition of the gradient of an isoparametric function f means, roughly speaking, that its level sets are equidistant to each other. On the other hand, the condition on the Laplacian of f has also a geometric meaning: the regular level sets of f have constant mean curvature. These facts are summarized in the following theorem (see [15]).

Theorem 1.1. *Let $\widetilde{M}$ be a Riemannian Manifold. Let $f : \widetilde{M} \longrightarrow \mathbb{R}$ be an isoparametric map $r \in \mathbb{R}$ a regular value for f, and $\Sigma = f^{-1}(r)$ the corresponding level hypersurface. Then Σ is an isoparametric hypersurface.*

Conversely, if Σ is an isoparametric hypersurface in $\widetilde{M}$, then for each $p \in \Sigma$ there is an open neighborhood U such that U is a regular level set of an isoparametric map $f : V \longrightarrow \mathbb{R}$, for some open subset V of $\widetilde{M}$.

Remark 1. *An interesting fact that is used in the proof of the above theorem, and that we will use later, is that the normal vector field $N = \frac{\nabla f}{||\nabla f||}$ of the hypersurface $\Sigma = f^{-1}(r)$ is a geodesic field, that is, $\widetilde{\nabla}_N N = 0$.*

Using the theorem above, we can establish the following definition of isoparametric hypersurface, which is equivalent to the Definition 1.1.

Definition 1.2. *An immersed hypersurface M of a Riemannian manifold $\widetilde{M}$ is called an isoparametric hypersurface if, for each $p \in M$, there exists an open neighborhood U of p in M such that U and the nearby equidistant hypersurfaces to U have constant mean curvature.*

When the ambient space is a space form, that is, a simply connected complete Riemannian manifold with constant sectional curvature, the previous definition of isoparametric hypersurface is equivalent to saying that the hypersurface has constant principal curvatures. This important result was obtained in 1938 by Cartan [7]. More precisely, Cartan showed the following theorem, whose an alternative proof can be found in [15].

Theorem 1.2. *Let Σ be a hypersurface in a space form $\mathbb{Q}^n_c$. Then Σ is isoparametric if and only if Σ has constant principal curvatures.*

In this work, we will use the classification of isoparametric surfaces in the 3-dimensional space forms $\mathbb{S}^3$ and $\mathbb{H}^3$. Such a classification is due to Cartan [7], and in addition to the umbilical surfaces, such surfaces are given by certain products of curves of constant curvature. Specifically, for future reference in this text, following the notation according to [34] and [35], we will enunciate in a theorem the classification of isoparametric surfaces with two distinct principal curvatures in the hyperbolic space $\mathbb{H}^3$ and in the unit sphere $\mathbb{S}^3$ (see Theorem 1 in [35]).

Theorem 1.3. *Let Σ^2 be an isoparametric surface with two distinct principal curvatures in $\mathbb{Q}^3_\varepsilon$, with $\varepsilon^2 = 1$. Then Σ^2 is an open subset of one of the following surfaces:*

a) $\mathbb{S}^1(c_1) \times \mathbb{S}^1(c_2)$, *when* $\varepsilon = 1$,

b) $\mathbb{S}^1(c_1) \times \mathbb{H}^1(c_2)$, *when* $\varepsilon = -1$,

where $c_1 \neq c_2$, $\dfrac{1}{c_1} + \dfrac{1}{c_2} = \varepsilon$ *and the principal curvatures of* Σ^2 *are given by* $\dfrac{c_1}{\sqrt{c_1 + c_2}}$ *and* $\dfrac{-c_2}{\sqrt{c_1 + c_2}}$.

Unlike space forms, in arbitrary ambient spaces, the isoparametricity of a hypersurface and the constancy of the principal curvatures are, a priori, unrelated conditions. For instance, examples of isoparametric hypersurfaces with nonconstant principal curvatures were given in [12], [13], [21] and [43], and examples of non-isoparametric hypersurfaces with constant principal curvatures, can be found in [22] and [33].

Chapter 2

Isoparametric Hypersurfaces in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$

In this chapter, we consider the ambient space $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, where $\mathbb{Q}^2_{c_i}$ is a space form with constant sectional curvature c_i, for $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. We aim to characterize and classify the isoparametric hypersurfaces with constant principal curvatures in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. For this, we combine the techniques of Domínguez-Vázquez and Manzano [16], Urbano [41] and Julio-Batalla [25]. In [16], Domínguez-Vazquéz and Manzano provided the classification of the isoparametric surfaces and surfaces with constant principal curvatures in $\mathbb{E}(\kappa, \tau)$, in [41], Urbano classified the homogeneous and isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, and in [25], Julio-Batalla obtained the classification of the isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{R}^2$ with constant principal curvatures. In addition, like Domínguez-Vázquez and Manzano, we will use Jacobi field theory to describe the geometry of the family of parallels hypersurfaces to a given one. This theory was briefly described in the Section 1.2.

The content of this chapter is a joint work with João Paulo do Santos [17], entitled "Isoparametric hypersurfaces in product spaces", to appear in "Differential Geometry and its Applications".

2.1 Preliminary notions and results

Before stating and proving the main results of this chapter, we will briefly present some background content in the product space $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$.

For $i = 1, 2$, we denote by $\langle , \rangle_i$ and L_i the standard metric and the standard complex structure in $\mathbb{Q}^2_{c_i}$, respectively. If $\mathbb{Q}^2_{c_i}$ is the 2-dimensional $\mathbb{S}^2$ of curvature $c_i = 1$, L_i is given by

$$\begin{aligned} L_i : T\mathbb{S}^2 &\longrightarrow T\mathbb{S}^2 \\ v &\longrightarrow L_i(v) = p \times v, \end{aligned}$$

for $p \in \mathbb{S}^2$, $v \in T_p\mathbb{S}^2$, see [14]. When $\mathbb{Q}^2_{c_i}$ is the hyperbolic space $\mathbb{H}^2$ of curvature $c_i = -1$, as stated in Chapter 1, we will consider its standard Lorentzian model. In this case, the 3-dimensional Minkowski space $\mathbb{L}^3$ is endowed with the Lorentzian cross product $\boxtimes$, defined by

$$(a_1, a_2, a_3) \boxtimes (b_1, b_2, b_3) = (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

In this model, L_i is given by

$$\begin{aligned} L_i : T\mathbb{H}^2 &\longrightarrow T\mathbb{H}^2 \\ v &\longrightarrow L_i(v) = p \boxtimes v, \end{aligned}$$

for $p \in \mathbb{H}^2$, $v \in T_p\mathbb{H}^2$, see [14] and [20]. Finally, if $\mathbb{Q}^2_{c_i}$ is the space form $\mathbb{R}^2$ of curvature $c_i = 0$, L_i is defined by

$$\begin{aligned} L_i : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ v &\longrightarrow L_i(q_1, q_2) = (-q_2, q_1), \end{aligned}$$

see [25].

It is easy to see that L_i satisfies the following properties:

$$L_i^2 = -Id, \quad \text{and} \quad \langle L_i(v), L_i(w)\rangle = \langle v, w\rangle. \tag{2.1}$$

The Kähler 2-form associated to standard complex structure L_i on $\mathbb{Q}^2_{c_i}$ is defined by

$$\omega_i(v, w) = \langle L_i(v), w\rangle_i,$$

for all $v, w \in T\mathbb{Q}^2_{c_i}$. Observe that, since $\mathbb{Q}^2_{c_i}$ has dimension 2, it follows that $d\omega_i = 0$, that is, the Kähler 2-form ω_i is closed, which implies that $\mathbb{Q}^2_{c_i}$ is a Kähler manifold. It is well-known that the Kähler 2-form of a Riemannian manifold is closed if and only if its standard complex structure is parallel with respect to the covariant derivative, see [26]. Thus, we conclude that L_i is parallel on $\mathbb{Q}^2_{c_i}$.

We endow $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ with the standard product metric, denoted by $\langle , \rangle$. Moreover, given $Y \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$, we write $Y = Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}}$, where the components $Y^{\mathbb{Q}^2_{c_1}}$ and $Y^{\mathbb{Q}^2_{c_2}}$ of Y are given as its tangent parts to $\mathbb{Q}^2_{c_1}$ and $\mathbb{Q}^2_{c_2}$, respectively. We define on $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ the complex strutures

$$J_1 = L_1 + L_2, \quad J_2 = L_1 - L_2,$$

and we denote by $\widetilde{\nabla}$ and $\widetilde{R}$ its Levi-Civita connection and curvature tensor, respectively. Note that, using (2.1), the complex structures J_i, $i = 1, 2$, satisfy

$$J_i^2 = -Id, \quad \text{and} \quad \langle J_i(Y), J_i(Z)\rangle = \langle Y, Z\rangle,$$

for all $Y, Z \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$. In addition, since L_i is parallel on $\mathbb{Q}^2_{c_i}$, we have that J_1 and J_2 are parallel on $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ with respect to the Levi-Civita connection $\widetilde{\nabla}$, that is, $\widetilde{\nabla} J_1 = \widetilde{\nabla} J_2 = 0$, and hence, $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ is a kähler manifold.

Now, let us consider the product structure P in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ defined by

$$P\Big(Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}}\Big) = Y^{\mathbb{Q}^2_{c_1}} - Y^{\mathbb{Q}^2_{c_2}},$$

for any vector $Y \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$. Note that P satisfies

$$P = -J_1 J_2 = -J_2 J_1.$$

In fact, given $Y \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$, we have that

$$\begin{aligned} J_1 J_2(Y) &= J_1 J_2(Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}}) = J_1\Big(L_1(Y^{\mathbb{Q}^2_{c_1}}) - L_2(Y^{\mathbb{Q}^2_{c_2}})\Big) = \Big(L_1^2(Y^{\mathbb{Q}^2_{c_1}}) - L_2^2(Y^{\mathbb{Q}^2_{c_2}})\Big) \\ &= -Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}} = -P(Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}}), \end{aligned}$$

that is, $P = -J_1 J_2$. Analogously, we obtain $P = -J_2 J_1$.

Moreover, the product structure P of $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ has the following properties:

$$P^2 = I \ (P \neq I), \quad \langle PY, Z\rangle = \langle Y, PZ\rangle, \quad \text{and} \quad (\widetilde{\nabla}_Y P)(Z) = 0,$$

for any vector fields $Y, Z \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$. Indeed, the first and second properties are immediate. For the third property, given vector fields $Y, Z \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$ and denoting by $\nabla^{\mathbb{Q}^2_{c_1}}$ and $\nabla^{\mathbb{Q}^2_{c_2}}$ the Levi-Civita connection of $\mathbb{Q}^2_{c_1}$ and $\mathbb{Q}^2_{c_2}$, respectively, it follows that

$$\begin{aligned}
(\widetilde{\nabla}_Y P)(Z) &= \widetilde{\nabla}_Y PZ - P\widetilde{\nabla}_Y Z \\
&= \widetilde{\nabla}_{Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}}} P(Z^{\mathbb{Q}^2_{c_1}} + Z^{\mathbb{Q}^2_{c_2}}) - P\Big(\nabla^{\mathbb{Q}^2_{c_1}}_{Y^{\mathbb{Q}^2_{c_1}}} Z^{\mathbb{Q}^2_{c_1}} + \nabla^{\mathbb{Q}^2_{c_2}}_{Y^{\mathbb{Q}^2_{c_2}}} Z^{\mathbb{Q}^2_{c_2}}\Big) \\
&= \widetilde{\nabla}_{Y^{\mathbb{Q}^2_{c_1}} + Y^{\mathbb{Q}^2_{c_2}}} (Z^{\mathbb{Q}^2_{c_1}} - Z^{\mathbb{Q}^2_{c_2}}) - \Big(\nabla^{\mathbb{Q}^2_{c_1}}_{Y^{\mathbb{Q}^2_{c_1}}} Z^{\mathbb{Q}^2_{c_1}} - \nabla^{\mathbb{Q}^2_{c_2}}_{Y^{\mathbb{Q}^2_{c_2}}} Z^{\mathbb{Q}^2_{c_2}}\Big) \\
&= \nabla^{\mathbb{Q}^2_{c_1}}_{Y^{\mathbb{Q}^2_{c_1}}} Z^{\mathbb{Q}^2_{c_1}} - \nabla^{\mathbb{Q}^2_{c_2}}_{Y^{\mathbb{Q}^2_{c_2}}} Z^{\mathbb{Q}^2_{c_2}} - \Big(\nabla^{\mathbb{Q}^2_{c_1}}_{Y^{\mathbb{Q}^2_{c_1}}} Z^{\mathbb{Q}^2_{c_1}} - \nabla^{\mathbb{Q}^2_{c_2}}_{Y^{\mathbb{Q}^2_{c_2}}} Z^{\mathbb{Q}^2_{c_2}}\Big) \\
&= 0.
\end{aligned}$$

Let us write $\widetilde{R}$ in terms of P. Denoting by R^{c_i} the curvature tensor of $\mathbb{Q}^2_{c_i}$, and using the curvature tensor formula of a manifold of constant sectional curvature, it follows that

$$\begin{aligned}
\widetilde{R}(V, W, Z, Y) &= R^{c_1}(V^{\mathbb{Q}^2_{c_1}}, W^{\mathbb{Q}^2_{c_1}}, Z^{\mathbb{Q}^2_{c_1}}, Y^{\mathbb{Q}^2_{c_1}}) + R^{c_2}(V^{\mathbb{Q}^2_{c_2}}, W^{\mathbb{Q}^2_{c_2}}, Z^{\mathbb{Q}^2_{c_2}}, Y^{\mathbb{Q}^2_{c_2}}) \\
&= c_1\Big\{\langle V^{\mathbb{Q}^2_{c_1}}, Y^{\mathbb{Q}^2_{c_1}}\rangle_1 \langle W^{\mathbb{Q}^2_{c_1}}, Z^{\mathbb{Q}^2_{c_1}}\rangle_1 - \langle V^{\mathbb{Q}^2_{c_1}}, Z^{\mathbb{Q}^2_{c_1}}\rangle_1 \langle W^{\mathbb{Q}^2_{c_1}}, Y^{\mathbb{Q}^2_{c_1}}\rangle_1\Big\} \\
&\quad + c_2\Big\{\langle V^{\mathbb{Q}^2_{c_2}}, Y^{\mathbb{Q}^2_{c_2}}\rangle_2 \langle W^{\mathbb{Q}^2_{c_2}}, Z^{\mathbb{Q}^2_{c_2}}\rangle_2 - \langle V^{\mathbb{Q}^2_{c_2}}, Z^{\mathbb{Q}^2_{c_2}}\rangle_2 \langle W^{\mathbb{Q}^2_{c_2}}, Y^{\mathbb{Q}^2_{c_2}}\rangle_2\Big\},
\end{aligned}$$

for any vector fields $V, W, Z, Y \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$. Note that for all $T \in T(\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2})$, we have

$$T^{\mathbb{Q}^2_{c_1}} = \frac{PT + T}{2}, \quad T^{\mathbb{Q}^2_{c_2}} = \frac{T - PT}{2}.$$

Thus, we get

$$\begin{aligned}
\langle V^{\mathbb{Q}^2_{c_1}}, Y^{\mathbb{Q}^2_{c_1}}\rangle_1 &= \frac{1}{2}\langle V, PY + Y\rangle, \quad & \langle W^{\mathbb{Q}^2_{c_1}}, Z^{\mathbb{Q}^2_{c_1}}\rangle_1 &= \frac{1}{2}\langle PW + W, Z\rangle, \\
\langle V^{\mathbb{Q}^2_{c_2}}, Y^{\mathbb{Q}^2_{c_2}}\rangle_2 &= -\frac{1}{2}\langle V, PY - Y\rangle, \quad & \langle W^{\mathbb{Q}^2_{c_2}}, Z^{\mathbb{Q}^2_{c_2}}\rangle_2 &= -\frac{1}{2}\langle PW - W, Z\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{R}(V, W, Z, Y) &= \frac{c_1}{4}\Big\{\langle V, PY + Y\rangle\langle PW + W, Z\rangle - \langle W, PY + Y\rangle\langle PV + V, Z\rangle\Big\} \\
&\quad + \frac{c_2}{4}\Big\{\langle V, PY - Y\rangle\langle PW - W, Z\rangle - \langle W, PY - Y\rangle\langle PV - V, Z\rangle\Big\}.
\end{aligned}$$

Let $\Sigma^3 \subset \mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ be an oriented hypersurface with unit normal vector $N = N_1 + N_2$, where N_1 and N_2 denote the components of N in $\mathbb{Q}^2_{c_1}$ and $\mathbb{Q}^2_{c_2}$, respectively, and Levi-Civita connection ∇. We define in Σ^3 a smooth function C and a tangent vector field X by

$$C = \langle PN, N\rangle \quad \text{and} \quad X = PN - CN. \tag{2.2}$$

Observe that X is the tangential component of PN and $|X|^2 = 1 - C^2$, which implies $-1 \leq C \leq 1$.

Let A be the shape operator of Σ. Using the curvature tensor of $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, the Codazzi equation of Σ is given by

$$\nabla S(V, W, Z) - \nabla S(W, V, Z) = \widetilde{R}(V, W, Z, N), \tag{2.3}$$

where

$$\begin{aligned}
\widetilde{R}(V, W, Z, N) &= \frac{c_1}{4}\Big\{\langle V, PN + N\rangle\langle PW + W, Z\rangle - \langle W, PN + N\rangle\langle PV + V, Z\rangle\Big\} \\
&\quad + \frac{c_2}{4}\Big\{\langle V, PN - N\rangle\langle PW - W, Z\rangle - \langle W, PN - N\rangle\langle PV - V, Z\rangle\Big\} \\
&= \frac{c_1}{4}\Big\{\langle V, X\rangle\langle PW + W, Z\rangle - \langle W, X\rangle\langle PV + V, Z\rangle\Big\} \\
&\quad + \frac{c_2}{4}\Big\{\langle V, X\rangle\langle PW - W, Z\rangle - \langle W, X\rangle\langle PV - V, Z\rangle\Big\},
\end{aligned}$$

with $V, W, Z \in T\Sigma$.

In what follows, we are going to compute the gradient of the function C. Given $Y \in T\Sigma$, since P is parallel, we have

$$\begin{aligned}
\langle \nabla C, Y\rangle &= Y(C) = Y\langle PN, N\rangle \\
&= \langle \nabla_Y PN, N\rangle + \langle PN, \nabla_Y N\rangle \\
&= \langle P\nabla_Y N, N\rangle + \langle PN, \nabla_Y N\rangle \\
&= 2\langle PN, \nabla_Y N\rangle = -2\langle X + CN, AY\rangle \\
&= -2\langle X, AY\rangle = -2\langle AX, Y\rangle,
\end{aligned}$$

which implies that the gradient of C is given by

$$\nabla C = -2AX.$$

2.2 Main results

We are now in a position to prove the main results of this chapter. Our first result characterizes the isoparametric hypersurfaces with constant principal curvatures in the product spaces $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, for $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$.

Theorem 2.1. *Let Σ be an isoparametric hypersurface in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, and unit normal $N = N_1 + N_2$, where N_1 and N_2 denote the components of N in $\mathbb{Q}^2_{c_1}$ and $\mathbb{Q}^2_{c_2}$, respectively. Then the principal curvatures of Σ are constant if and only if $|N_1|$ and $|N_2|$ are constant.*

Proof. Let Σ be an isoparametric hypersurface in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ with $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$, and unit normal $N = N_1 + N_2$. In order to prove Theorem 2.1, it is enough to show that the principal curvatures of Σ are constant if and only if the function C, given in (2.2), is constant. In fact, as $|N_1|^2 = \frac{1+C}{2}$ and $|N_2|^2 = \frac{1-C}{2}$, it follows that $|N_1|$ and $|N_2|$ are constant if and only if C is constant.

In what follows, we will establish some relations between the function C and the shape operator of Σ. Recall that the family of hypersurfaces parallel to Σ in the direction of N is given by (1.1) and the parallel hypersurface at an oriented distance r is denoted by

Σ_r. We first observe that, since Σ is isoparametric and the product structure P is parallel, the function C, defined on the family of parallel hypersurfaces, does not depend on the displacement parameter r, once $N(C) = 0$. In fact, since $C = \langle PN, N\rangle$ and $\nabla_N N = 0$, we have

$$N(C) = \langle \nabla_N N, PN\rangle + \langle N, P\nabla_N N\rangle = 0.$$

Let us recall that $|C| \leq 1$. Consider the open set

$$U = \{p \in \Sigma \mid C^2(p) < 1\}.$$

We can assume that $U \neq \varnothing$, otherwise $C^2 = 1$ on Σ. In this case, let us take in U the following orthonormal frame

$$B = \left\{ B_1 = \frac{X}{\sqrt{1-C^2}}, B_2 = \frac{J_1N + J_2N}{\sqrt{2(1+C)}}, B_3 = \frac{J_1N - J_2N}{\sqrt{2(1-C)}} \right\},$$

where $X = PN - CN$.

Given $p \in \Sigma$, let γ_p be a geodesic of $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ with $\gamma_p(0) = p$ and $\dot\gamma_p(0) = N(p)$. By the definition of Σ_r we have that $\dot\gamma_q(r)$ is a normal vector to Σ_r at $\gamma_q(r)$. Thus, we can extend the unit normal N to $U \times (-\epsilon, \epsilon)$ by $N(\gamma_q(r)) = \dot\gamma_q(r)$, $q \in U$. Consequently, we also can extend the fields B_i.

Recall that a Jacobi field along γ_p is a vector field ξ satisfying the Jacobi equation $\xi'' + R(\xi, \dot\gamma_p)\dot\gamma_p = 0$. For each $j \in \{1, 2, 3\}$, take the Jacobi field ξ_j along γ_p with the initial conditions

$$\xi_j(0) = B_j \quad \text{and} \quad \xi_j'(0) = -AB_j,$$

where A is the shape operator of Σ associated with N.

Since these initial conditions are orthogonal to $\dot\gamma_p(0)$, each Jacobi field ξ_j is also orthogonal to $N(\gamma_p(r)) = \dot\gamma_p(r)$ and, hence, it can be written as

$$\xi_j = b_{1j}B_1 + b_{2j}B_2 + b_{3j}B_3,$$

for certain smooth functions b_{ij} on $(-\epsilon, \epsilon)$.

Let us observe that $\nabla_N B_i = 0$, for all $i = 1, 2, 3$. In fact, since $N(C) = 0$ and P is parallel, we have $\nabla_N X = 0$, which implies $\nabla_N B_1 = 0$. Furthermore, since J_i is also parallel, for $i = 1, 2$, we conclude that $\nabla_N B_j = 0$, $j = 2, 3$. Thus, we have, on the one hand,

$$\xi_j'' = b_{1j}''B_1 + b_{2j}''B_2 + b_{3j}''B_3. \tag{2.4}$$

On the other hand, if we denote by R^{c_i} the curvature tensor of $\mathbb{Q}^2_{c_i}$, we get

$$\begin{aligned}
\widetilde{R}(B_1, N)N &= R^{c_1}(B_1^{\mathbb{Q}^2_{c_1}}, N_1)N_1 + R^{c_2}(B_1^{\mathbb{Q}^2_{c_2}}, N_2)N_2 \\
&= \frac{1}{8\sqrt{1-C^2}}\Big(R^{c_1}(X + PX, N + PN)(N + PN) \\
&\quad + R^{c_2}(X - PX, N - PN)(N - PN)\Big) \\
&= 0,
\end{aligned}$$

since $X + PX = (1 - C)(N + PN)$ and $X - PX = -(1 + C)(N - PN)$. Now, using the curvature tensor formula of a manifold of constant sectional curvature, we get

$$
\begin{aligned}
R(B_2,N)N &= c_1\Big(\langle N_1,N_1\rangle_1 B_2 - \langle B_2,N_1\rangle_1 N_1\Big)\\
&= \frac{c_1\|N+PN\|^2}{4}B_2,\\
R(B_3,N)N &= c_2\Big(\langle N_2,N_2\rangle_2 B_3 - \langle B_3,N_2\rangle_2 N_2\Big)\\
&= \frac{c_2\|N-PN\|^2}{4}B_3.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widetilde{R}(\xi_j,\dot\gamma_p)\dot\gamma_p &= \widetilde{R}(\xi_j,N)N\\
&= b_{1j}R(B_1,N)N + b_{2j}R(B_2,N)N + b_{3j}R(B_3,N)N\\
&= b_{2j}\frac{c_1\|N+PN\|^2}{4}B_2 + b_{3j}\frac{c_2\|N-PN\|^2}{4}B_3\\
&= b_{2j}\frac{c_1(1+C)}{2}B_2 + b_{3j}\frac{c_2(1-C)}{2}B_3.
\end{aligned} \tag{2.5}
$$

Since ξ_j is a Jacobi field, we have from (2.4) and (2.5) the following homogeneous linear system of ordinary differential equations

$$
b''_{1j}=0,\quad b''_{2j}+\delta_1 b_{2j}=0,\quad b''_{3j}+\delta_2 b_{3j}=0, \tag{2.6}
$$

where $\delta_1=\frac{c_1(1+C)}{2}$ and $\delta_2=\frac{c_2(1-C)}{2}$.

In what follows, we describe the initial conditions of the system (2.6). Firstly, as $\xi_j(0)=B_j$, we get

$$
\begin{array}{lll}
b_{11}(0)=1, & b_{12}(0)=0, & b_{13}(0)=0,\\
b_{21}(0)=0, & b_{22}(0)=1, & b_{23}(0)=0,\\
b_{31}(0)=0, & b_{32}(0)=0, & b_{33}(0)=1.
\end{array} \tag{2.7}
$$

Secondly, let the shape operator of Σ be determined by the relations $AB_i=\sigma_{i1}B_1+\sigma_{i2}B_2+\sigma_{i3}B_3$, for certain smooth functions σ_{ij}. Since A is symmetric, we have $\sigma_{12}=\sigma_{21}$, $\sigma_{13}=\sigma_{31}$ and $\sigma_{32}=\sigma_{23}$. Furthermore, taking into account that $\xi'_j=\widetilde{\nabla}_N\xi_j=-A\xi_j$, we obtain

$$
\begin{array}{lll}
b'_{11}(0)=-\sigma_{11}, & b'_{12}(0)=-\sigma_{21}, & b'_{13}(0)=-\sigma_{31},\\
b'_{21}(0)=-\sigma_{12}, & b'_{22}(0)=-\sigma_{22}, & b'_{23}(0)=-\sigma_{23},\\
b'_{31}(0)=-\sigma_{13}, & b'_{32}(0)=-\sigma_{23}, & b'_{33}(0)=-\sigma_{33}.
\end{array} \tag{2.8}
$$

With the initial conditions (2.7) and (2.8), the solution of system (2.6) is given by

$$
\begin{aligned}
b_{11}(r) &= -\sigma_{11}r+1,\\
b_{12}(r) &= -\sigma_{12}r,\\
b_{13}(r) &= -\sigma_{13}r,\\
b_{21}(r) &= -\sigma_{12}S_{\delta_1}(r),\\
b_{22}(r) &= -\sigma_{22}S_{\delta_1}(r)+C_{\delta_1}(r),\\
b_{23}(r) &= -\sigma_{32}S_{\delta_1}(r),\\
b_{31}(r) &= -\sigma_{13}S_{\delta_2}(r),\\
b_{32}(r) &= -\sigma_{32}S_{\delta_2}(r),\\
b_{33}(r) &= -\sigma_{33}S_{\delta_2}(r)+C_{\delta_2}(r),
\end{aligned} \tag{2.9}
$$

where we consider the auxiliary functions

$$S_{\delta_i}(r) = \begin{cases} r & \text{if } \delta_i = 0, \\ \frac{1}{\sqrt{-\delta_i}} \sinh(r\sqrt{-\delta_i}) & \text{if } \delta_i < 0, \\ \frac{1}{\sqrt{\delta_i}} \sin(r\sqrt{\delta_i}) & \text{if } \delta_i > 0, \end{cases} \qquad C_{\delta_i}(r) = \begin{cases} 1 & \text{if } \delta_i = 0, \\ \cosh(r\sqrt{-\delta_i}) & \text{if } \delta_i < 0, \\ \cos(r\sqrt{\delta_i}) & \text{if } \delta_i > 0. \end{cases}$$

for $i \in \{1, 2\}$.

For every r, the shape operator A_r of Σ_r with respect to the normal $\dot{\gamma}_p(r)$ is given by (1.3), where $D(r)$ is a linear endomorphism of $T_{\gamma_p(r)}\Sigma_r$, determined by the relations

$$D(r)B_j(\gamma_p(r)) = \xi_j(r), \quad D'(r)B_j(\gamma_p(r)) = \xi_j'(r).$$

Considering the orthonormal basis $\{B_1(\gamma_p(r)), B_2(\gamma_p(r)), B_3(\gamma_p(r))\}$ of $T_{\gamma_p(r)}\Sigma_r$, the matrix form of the operator $D(r)$ is given by

$$D(r) = \begin{pmatrix} b_{11}(r) & b_{12}(r) & b_{13}(r) \\ b_{21}(r) & b_{22}(r) & b_{23}(r) \\ b_{31}(r) & b_{32}(r) & b_{33}(r) \end{pmatrix}, \tag{2.10}$$

From now on, our strategy is as follows. Firstly, we are going to get explicitly the formulas of $\det D(r)$ and $\frac{d}{dr}(\det D(r))$ in terms of the functions b_{ij} and its derivatives. Secondly, we will apply such formulas to construct

$$f(r) = \frac{d}{dr}(\det D(r)) + 3h(r)\det D(r),$$

which vanishes identically on $(-\epsilon, \epsilon)$, by equation (1.4). Finally, we will use the fact that $f \equiv 0$ as well as its derivatives to obtain some algebraic relations between the components of A on the basis $\{B_i\}_{i=1}^3$ and the function C.

From (2.9), we have that

$$\begin{aligned} b_{11}b_{22}b_{33} &= -\sigma_{11}\sigma_{22}\sigma_{33}rS_{\delta_1}(r)S_{\delta_2}(r) + \sigma_{11}\sigma_{22}rS_{\delta_1}(r)C_{\delta_2}(r) + \sigma_{11}\sigma_{33}rS_{\delta_2}(r)C_{\delta_1}(r) \\ &\quad - \sigma_{11}rC_{\delta_1}(r)C_{\delta_2}(r) + \sigma_{22}\sigma_{33}S_{\delta_1}(r)S_{\delta_2}(r) - \sigma_{22}S_{\delta_1}(r)C_{\delta_2}(r) \\ &\quad - \sigma_{33}S_{\delta_2}(r)C_{\delta_1}(r) + C_{\delta_1}(r)C_{\delta_2}(r), \\ b_{12}b_{23}b_{31} &= -\sigma_{21}\sigma_{32}\sigma_{13}rS_{\delta_1}(r)S_{\delta_2}(r), \\ b_{13}b_{32}b_{21} &= -\sigma_{31}\sigma_{23}\sigma_{12}rS_{\delta_1}(r)S_{\delta_2}(r), \\ b_{31}b_{22}b_{13} &= -\sigma_{13}^2\sigma_{22}rS_{\delta_1}(r)S_{\delta_2}(r) + \sigma_{13}^2 rS_{\delta_2}(r)C_{\delta_1}(r), \\ b_{32}b_{23}b_{11} &= -\sigma_{11}\sigma_{23}^2 rS_{\delta_1}(r)S_{\delta_2}(r) + \sigma_{23}^2 S_{\delta_1}(r)S_{\delta_2}(r), \\ b_{12}b_{21}b_{33} &= -\sigma_{33}\sigma_{12}^2 rS_{\delta_1}(r)S_{\delta_2}(r) + \sigma_{12}^2 rS_{\delta_1}(r)C_{\delta_2}(r). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \det D(r) &= b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{32}b_{21} - b_{31}b_{22}b_{13} - b_{32}b_{23}b_{11} - b_{33}b_{12}b_{21} \\ &= rS_{\delta_1}(r)S_{\delta_2}(r)(-\sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{21}\sigma_{32}\sigma_{13} - \sigma_{31}\sigma_{32}\sigma_{12} \\ &\quad + \sigma_{13}^2\sigma_{22} + \sigma_{23}^2\sigma_{11} + \sigma_{12}^2\sigma_{33}) \\ &\quad + rS_{\delta_1}(r)C_{\delta_2}(r)(\sigma_{11}\sigma_{22} - \sigma_{12}^2) + rS_{\delta_2}(r)C_{\delta_1}(r)(\sigma_{11}\sigma_{33} - \sigma_{13}^2) \\ &\quad - \sigma_{11}rC_{\delta_1}(r)C_{\delta_2}(r) + S_{\delta_1}(r)S_{\delta_2}(r)(\sigma_{22}\sigma_{33} - \sigma_{23}^2) - \sigma_{22}S_{\delta_1}(r)C_{\delta_2}(r) \\ &\quad - \sigma_{33}S_{\delta_2}(r)C_{\delta_1}(r) + C_{\delta_1}(r)C_{\delta_2}(r) \\ &= A_1rS_{\delta_1}(r)S_{\delta_2}(r) + A_2rS_{\delta_1}(r)C_{\delta_2}(r) + A_3rS_{\delta_2}(r)C_{\delta_1}(r) \\ &\quad + A_4S_{\delta_1}(r)S_{\delta_2}(r) - \sigma_{11}rC_{\delta_1}(r)C_{\delta_2}(r) - \sigma_{22}S_{\delta_1}(r)C_{\delta_2}(r) \\ &\quad - \sigma_{33}S_{\delta_2}(r)C_{\delta_1}(r) + C_{\delta_1}(r)C_{\delta_2}(r), \end{aligned}$$

where

$$
\begin{aligned}
A_1 &= -\det A, & A_2 &= \sigma_{11}\sigma_{22} - \sigma_{12}^2,\\
A_3 &= \sigma_{11}\sigma_{33} - \sigma_{13}^2, & A_4 &= \sigma_{22}\sigma_{33} - \sigma_{23}^2.
\end{aligned} \tag{2.11}
$$

Now, taking into account that $S'_{\delta_i}(r) = C_{\delta_i}(r)$ and $C'_{\delta_i}(r) = -\delta_i S_{\delta_i}(r)$, we obtain

$$
\begin{aligned}
\frac{d}{dr}(\det D(r)) = {} & A_1\left(S_{\delta_1}(r)S_{\delta_2}(r) + rC_{\delta_1}(r)S_{\delta_2}(r) + rS_{\delta_1}(r)C_{\delta_2}(r)\right)\\
& + A_2\left(S_{\delta_1}(r)C_{\delta_2}(r) + rC_{\delta_1}(r)C_{\delta_2}(r) - r\delta_2 S_{\delta_1}(r)S_{\delta_2}(r)\right)\\
& + A_3\left(S_{\delta_2}(r)C_{\delta_1}(r) + rC_{\delta_2}(r)C_{\delta_1}(r) - r\delta_1 S_{\delta_2}(r)S_{\delta_1}(r)\right)\\
& + A_4\left(C_{\delta_1}(r)S_{\delta_2}(r) + S_{\delta_1}(r)C_{\delta_2}(r)\right)\\
& - \sigma_{11}\left(C_{\delta_1}(r)C_{\delta_2}(r) - r\delta_1 S_{\delta_1}(r)C_{\delta_2}(r) - r\delta_2 C_{\delta_1}(r)S_{\delta_2}(r)\right)\\
& - \sigma_{22}\left(C_{\delta_1}(r)C_{\delta_2}(r) - \delta_2 S_{\delta_1}(r)S_{\delta_2}(r)\right)\\
& - \sigma_{33}\left(C_{\delta_2}(r)C_{\delta_1}(r) - \delta_1 S_{\delta_2}(r)S_{\delta_1}(r)\right)\\
& - \delta_1 S_{\delta_1}(r)C_{\delta_2}(r) - \delta_2 C_{\delta_1}(r)S_{\delta_2}(r).
\end{aligned}
$$

Thus, the function f is given explicitly as

$$
\begin{aligned}
f(r) = {} & A_1\big(S_{\delta_1}(r)S_{\delta_2}(r) + rC_{\delta_1}(r)S_{\delta_2}(r) + rS_{\delta_1}(r)C_{\delta_2}(r)\\
& + 3rh(r)S_{\delta_1}(r)S_{\delta_2}(r)\big)\\
& + A_2\big(S_{\delta_1}(r)C_{\delta_2}(r) + rC_{\delta_1}(r)C_{\delta_2}(r) - r\delta_2 S_{\delta_1}(r)S_{\delta_2}(r)\\
& + 3rh(r)S_{\delta_1}(r)C_{\delta_2}(r)\big)\\
& + A_3\big(S_{\delta_2}(r)C_{\delta_1}(r) + rC_{\delta_2}(r)C_{\delta_1}(r) - r\delta_1 S_{\delta_2}(r)S_{\delta_1}(r)\\
& + 3rh(r)S_{\delta_2}(r)C_{\delta_1}(r)\big)\\
& + A_4\big(C_{\delta_1}(r)S_{\delta_2}(r) + S_{\delta_1}(r)C_{\delta_2}(r) + 3h(r)S_{\delta_1}(r)S_{\delta_2}(r)\big)\\
& - \sigma_{11}\big(C_{\delta_1}(r)C_{\delta_2}(r) - r\delta_1 S_{\delta_1}(r)C_{\delta_2}(r) - r\delta_2 C_{\delta_1}(r)S_{\delta_2}(r)\\
& + 3rh(r)C_{\delta_1}(r)C_{\delta_2}(r)\big)\\
& - \sigma_{22}\big(C_{\delta_1}(r)C_{\delta_2}(r) - \delta_2 S_{\delta_1}(r)S_{\delta_2}(r) + 3h(r)S_{\delta_1}(r)C_{\delta_2}(r)\big)\\
& - \sigma_{33}\big(C_{\delta_2}(r)C_{\delta_1}(r) - \delta_1 S_{\delta_2}(r)S_{\delta_1}(r) + 3h(r)S_{\delta_2}(r)C_{\delta_1}(r)\big)\\
& - \delta_1 S_{\delta_1}(r)C_{\delta_2}(r) - \delta_2 C_{\delta_1}(r)S_{\delta_2}(r) + 3h(r)C_{\delta_1}(r)C_{\delta_2}(r).
\end{aligned} \tag{2.12}
$$

Taking the derivative in (2.12), we get

$$
\begin{aligned}
f'(r) = {} & A_1\Big(2C_{\delta_1}(r)S_{\delta_2}(r) + 2S_{\delta_1}(r)C_{\delta_2}(r) - rS_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)\\
& + 2rC_{\delta_1}(r)C_{\delta_2}(r) + 3h(r)S_{\delta_1}(r)S_{\delta_2}(r) + 3rh'(r)S_{\delta_1}(r)S_{\delta_2}(r)\\
& + 3rh(r)C_{\delta_1}(r)S_{\delta_2}(r) + 3rh(r)S_{\delta_1}(r)C_{\delta_2}(r)\Big)\\
& + A_2\Big(2C_{\delta_1}(r)C_{\delta_2}(r) - 2\delta_2 S_{\delta_1}(r)S_{\delta_2}(r) - rS_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)\\
& - 2\delta_2 rC_{\delta_1}(r)S_{\delta_2}(r) + 3h(r)S_{\delta_1}(r)C_{\delta_2}(r) + 3rh'(r)S_{\delta_1}(r)C_{\delta_2}(r)\\
& + 3rh(r)C_{\delta_1}(r)C_{\delta_2}(r) - 3\delta_2 rh(r)S_{\delta_1}(r)S_{\delta_2}(r)\Big)\\
& + A_3\Big(2C_{\delta_2}(r)C_{\delta_1}(r) - 2\delta_1 S_{\delta_2}(r)S_{\delta_1}(r) - rS_{\delta_2}(r)C_{\delta_1}(r)(\delta_1+\delta_2)\\
& - 2\delta_1 rC_{\delta_2}(r)S_{\delta_1}(r) + 3h(r)S_{\delta_2}(r)C_{\delta_1}(r) + 3rh'(r)S_{\delta_2}(r)C_{\delta_1}(r)\\
& + 3rh(r)C_{\delta_2}(r)C_{\delta_1}(r) - 3\delta_1 rh(r)S_{\delta_2}(r)S_{\delta_1}(r)\Big)
\end{aligned} \tag{2.13}
$$

$$+ A_4\Big(2C_{\delta_1}(r)C_{\delta_2}(r) - S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2) + 3h'(r)S_{\delta_1}(r)S_{\delta_2}(r)$$
$$+ 3h(r)C_{\delta_1}(r)S_{\delta_2}(r) + 3h(r)S_{\delta_1}(r)C_{\delta_2}(r)\Big)$$
$$- \sigma_{11}\Big(- 2\delta_1 S_{\delta_1}(r)C_{\delta_2}(r) - 2\delta_2 C_{\delta_1}(r)S_{\delta_2}(r) - rC_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)$$
$$+ 2\delta_1\delta_2 rS_{\delta_1}(r)S_{\delta_2}(r) + 3h(r)C_{\delta_1}(r)C_{\delta_2}(r) + 3rh'(r)C_{\delta_1}(r)C_{\delta_2}(r)$$
$$- 3\delta_1 rh(r)S_{\delta_1}(r)C_{\delta_2}(r) - 3\delta_2 rh(r)C_{\delta_1}(r)S_{\delta_2}(r)\Big)$$
$$- \sigma_{22}\Big(- S_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2) - 2\delta_2 C_{\delta_1}(r)S_{\delta_2}(r) + 3h'(r)S_{\delta_1}(r)C_{\delta_2}(r)$$
$$+ 3h(r)C_{\delta_1}(r)C_{\delta_2}(r) - 3\delta_2 h(r)S_{\delta_1}(r)S_{\delta_2}(r)\Big)$$
$$- \sigma_{33}\Big(- S_{\delta_2}(r)C_{\delta_1}(r)(\delta_1+\delta_2) - 2\delta_1 C_{\delta_2}(r)S_{\delta_1}(r) + 3h'(r)S_{\delta_2}(r)C_{\delta_1}(r)$$
$$+ 3h(r)C_{\delta_2}(r)C_{\delta_1}(r) - 3\delta_1 h(r)S_{\delta_2}(r)S_{\delta_1}(r)\Big)$$
$$- C_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2) + 2\delta_1\delta_2 S_{\delta_1}(r)S_{\delta_2}(r) + 3h'(r)C_{\delta_1}(r)C_{\delta_2}(r)$$
$$- 3\delta_1 h(r)S_{\delta_1}(r)C_{\delta_2}(r) - 3\delta_2 h(r)C_{\delta_1}(r)S_{\delta_2}(r).$$

As $f \equiv 0$, so is its derivative. Then, applying $r = 0$ in the derivative above, we obtain the following relation:

$$0 = f'(0) = 2(A_2 + A_3 + A_4) - 9h^2(0) + 3h'(0) - (\delta_1 + \delta_2), \tag{2.14}$$

where $h(0)$ is the mean curvature of Σ.

Note that A_i, δ_i, $h(0)$ and $h'(0)$, depend only, in principle, on the base point $p \in \Sigma$. However, by assumption, Σ is isoparametric and hence, $h(0)$ and $h'(0)$ are constants throughout Σ, that is, they are independent of the chosen base point $p \in \Sigma$ of normal geodesic γ_p.

Furthermore, observe that

$$9h^2(0) = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}),$$

and

$$tr(A^2) = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2).$$

Thus, by the definitions of the functions A_i, $i = 1, \ldots, 4$, in (2.11), we have $2(A_2 + A_3 + A_4) - 9h^2(0) = -tr(A^2)$. Substituting in (2.14), we get

$$tr(A^2) = 3h'(0) - (\delta_1 + \delta_2), \tag{2.15}$$

where $\delta_1 + \delta_2 = \frac{1}{2}(C(c_1 - c_2) + c_1 + c_2)$.

We are in position to prove the equivalence claimed in the statement of the theorem. If Σ has constant principal curvatures μ_1, μ_2, μ_3, then $tr(A^2) = \mu_1^2 + \mu_2^2 + \mu_3^2$ is constant and hence, C is constant, since $c_1 \neq c_2$.

Conversely, suppose C is constant. Since the gradient of the function C is given by $\nabla C = -2A(X)$, then $A(X) = 0$. Therefore, $\sigma_{1j} = \sigma_{j1} = 0$, for all $j = 1, 2, 3$. Thus, we have $A_1 = A_2 = A_3 = 0$ and we can rewrite (2.14) as

$$0 = 2A_4 - 9h^2(0) + 3h'(0) - (\delta_1 + \delta_2),$$

and, as a consequence, we have that A_4 is constant.

Moreover, as $\sigma_{1j} = \sigma_{j1} = 0$, the characteristic polynomial Q_A of A is given by

$$Q_A(\lambda) = -\lambda^3 + 3h(0)\lambda^2 - A_4\lambda.$$

Therefore, since A_4 is constant, it follows that the principal curvatures of Σ are constant. $\square$

Remark 2. *It is worth mentioning that Theorem 2.1 holds in a more general setting, where* $c_1 \neq c_2$ *but not necessarily* $c_i \in \{-1, 0, 1\}$*. Adjusting the ambient spaces and their corresponding complex structures for arbitrary values of* c_i*, the computations and arguments in the proof of Theorem 2.1 remain the same only with the assumption that* $c_1 \neq c_2$*. In fact, this hypothesis is used in equation (2.15) to show that if* Σ *has constant principal curvatures, then* C *is constant. The converse holds, however, even if* $c_1 = c_2$*.*

Observe that Theorem 2.1 tells us that the converse of a result obtained by Julio-Batalla [25] holds, that is, we proved that if $|N_1|$ and $|N_2|$ are constant, then Σ has constant principal curvatures. In addition, the Jacobi field theory, used in the proof of Theorem 2.1, allows us to obtain an alternative proof of Julio-Batalla's result.

Our next result classifies the isoparametric hypersurfaces with constant principal curvatures in the product spaces $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, for $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. As previously mentioned, this classification includes the classification obtained by Julio-Batalla in the product space $\mathbb{S}^2 \times \mathbb{R}^2$.

Theorem 2.2. *Let* Σ *be an isoparametric hypersurface with constant principal curvatures in* $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$*, with* $c_i \in \{-1, 0, 1\}$ *and* $c_1 \neq c_2$*. Then, up to rigid motions,* Σ *is an open subset of one of the following hypersurfaces:*

a) $\mathcal{C}^1(\kappa_j) \times \mathbb{Q}^2_{c_2}$ *or* $\mathbb{Q}^2_{c_1} \times \mathcal{C}^1(\kappa_j)$*, where* $\mathcal{C}^1(\kappa_j)$ *is a complete curve with constant geodesic curvature* κ_j *in* $\mathbb{Q}^2_{c_j}$*.*

b) $\Psi(\mathbb{R}^3) \subset \mathbb{H}^2 \times \mathbb{R}^2$*, where* $\Psi : \mathbb{R}^3 \to \mathbb{H}^2 \times \mathbb{R}^2$ *is an immersion given by*

$$\begin{aligned} \Psi(s, u, v) = e^{-b\, s}(\alpha(u), \vec{0}) + \Big(\cosh(-b\, s), 0, \sinh(-b\, s), V_0 s\Big) \\ + \Big(\vec{0}, p_0 + W_0 v\Big), \end{aligned} \tag{2.16}$$

where $\mathbb{H}^2 \subset \mathbb{L}^3$ *is given as the standard model of the hyperbolic space in the Lorentz 3-space* $\mathbb{L}^3$*, the curve* α *is given by* $\alpha(u) = \left(\frac{u^2}{2}, u, -\frac{u^2}{2}\right)$*,* $p_0 \in \mathbb{R}^2$*,* V_0 *and* W_0 *are constant orthogonal vectors in* $\mathbb{R}^2$ *such that* $||W_0|| = 1$ *and* $b = \sqrt{1 - ||V_0||^2}$*, with* $b \neq \{1, 0\}$*.*

Proof. Let Σ be an isoparametric hypersurface in $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ with constant principal curvatures, where $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$. By Theorem 2.1, we have that C is constant. If $C = 1$ we have $PN = N$, and thus, $N = (N_1, 0)$. If $C = -1$ we have $PN = -N$, and then, $N = (0, N_2)$. In such cases, Σ is an open subset of $\mathcal{C}^1(\kappa_j) \times \mathbb{Q}^2_{c_2}$ or $\mathbb{Q}^2_{c_1} \times \mathcal{C}^1(\kappa_j)$, respectively, where $\mathcal{C}^1(\kappa_j)$ is a curve in $\mathbb{Q}^2_{c_j}$ of constant geodesic curvature κ_j.

In fact, let us suppose that $N = (N_1, 0)$, then Σ is an open subset of $\mathcal{C}^1 \times \mathbb{Q}^2_{c_2}$, where $\mathcal{C}^1$ is a regular curve in $\mathbb{Q}^2_{c_1}$. Let ψ be a parametrization by arc length of $\mathcal{C}^1$, with unit normal vector $n_\psi = \pm N_1$. Let $\{e_1, e_2, e_3\}$ a orthonormal frame in $\mathcal{C}^1 \times \mathbb{Q}^2_{c_2}$, with $e_1 = \psi'$ and $\{e_2, e_3\}$ an orthonormal basis in $\mathbb{Q}^2_{c_2}$. If we denote the shape operator of Σ by A, considering without loss of generality that $N_1 = n_\psi$, we have

$$\begin{aligned} Ae_1 &= -\widetilde{\nabla}_{e_1} N_1 = -\widetilde{\nabla}^{\mathbb{Q}^2_{c_1}}_{\psi'} n_\psi = \kappa_j \psi' = \kappa_j e_1, \\ Ae_2 &= -\widetilde{\nabla}_{e_2} N_1 = 0, \\ Ae_3 &= -\widetilde{\nabla}_{e_3} N_1 = 0. \end{aligned}$$

Therefore, the geodesic curvature κ_j of $\mathcal{C}^1$ is a principal curvature of Σ, which implies that κ_j is constant. The case where $N = (0, N_2)$ is analogous.

From now on, we are going to prove that, if $|C|<1$, the only remaining possibility is the case when one c_i is negative. Therefore, in what follows, let us assume that $C\in(-1,1)$. In this case, as in the proof of Theorem 2.1, let us consider the frame

$$B=\left\{B_1=\frac{X}{\sqrt{1-C^2}},B_2=\frac{J_1N+J_2N}{\sqrt{2(1+C)}},B_3=\frac{J_1N-J_2N}{\sqrt{2(1-C)}}\right\},$$

the function f given in (2.12) and its derivative given in (2.13). Again, taking derivative in (2.13), we get

$$\begin{aligned}
f''(r)=&A_1\Big(6C_{\delta_1}(r)C_{\delta_2}(r)-3S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)-rC_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)\\
&-rS_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)-2\delta_1 rS_{\delta_1}(r)C_{\delta_2}(r)-2\delta_2 rC_{\delta_1}(r)S_{\delta_2}(r)\\
&+6h'(r)S_{\delta_1}(r)S_{\delta_2}(r)+6h(r)C_{\delta_1}(r)S_{\delta_2}(r)+6h(r)S_{\delta_1}(r)C_{\delta_2}(r)\\
&+3rh''(r)S_{\delta_1}(r)S_{\delta_2}(r)+6rh'(r)C_{\delta_1}(r)S_{\delta_2}(r)+6rh'(r)S_{\delta_1}(r)C_{\delta_2}(r)\\
&+6rh(r)C_{\delta_1}(r)C_{\delta_2}(r)-3rh(r)S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)\Big)\\
&+A_2\Big(-3S_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)-6\delta_2C_{\delta_1}(r)S_{\delta_2}(r)-rC_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)\\
&+\delta_2 rS_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)+2\delta_1\delta_2 rS_{\delta_1}(r)S_{\delta_2}(r)-2\delta_2 rC_{\delta_1}(r)C_{\delta_2}(r)\\
&+6h'(r)S_{\delta_1}(r)C_{\delta_2}(r)+6h(r)C_{\delta_1}(r)C_{\delta_2}(r)-6\delta_2h(r)S_{\delta_1}(r)S_{\delta_2}(r)\\
&+3rh''(r)S_{\delta_1}(r)C_{\delta_2}(r)+6rh'(r)C_{\delta_1}(r)C_{\delta_2}(r)-6\delta_2 rh'(r)S_{\delta_1}(r)S_{\delta_2}(r)\\
&-3rh(r)S_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)-6\delta_2 rh(r)C_{\delta_1}(r)S_{\delta_2}(r)\Big)\\
&+A_3\Big(-4\delta_1S_{\delta_1}(r)C_{\delta_2}(r)-3C_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)-rC_{\delta_2}(r)C_{\delta_1}(r)(\delta_1+\delta_2)\\
&-r\delta_1S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)-2\delta_1C_{\delta_2}(r)S_{\delta_1}(r)+6h'(r)S_{\delta_2}(r)C_{\delta_1}(r)\\
&+6h(r)C_{\delta_2}(r)C_{\delta_1}(r)-6\delta_1h(r)S_{\delta_2}(r)S_{\delta_1}(r)+3rh''(r)S_{\delta_2}(r)C_{\delta_1}(r)\\
&+6rh'(r)C_{\delta_2}(r)C_{\delta_1}(r)-6\delta_1 rh'(r)S_{\delta_2}(r)S_{\delta_1}(r)-6\delta_1 rh(r)C_{\delta_2}(r)S_{\delta_1}(r)\\
&-3rh(r)S_{\delta_2}(r)C_{\delta_1}(r)(\delta_1+\delta_2)\Big)\\
&+A_4\Big(-2\delta_1S_{\delta_1}(r)C_{\delta_2}(r)-2\delta_2C_{\delta_1}(r)S_{\delta_2}(r)-C_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)\\
&-S_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)+3h''(r)S_{\delta_1}(r)S_{\delta_2}(r)+6h'(r)C_{\delta_1}(r)S_{\delta_2}(r)\\
&+6h'(r)S_{\delta_1}(r)C_{\delta_2}(r)+6h(r)C_{\delta_1}(r)C_{\delta_2}(r)-3h(r)S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)\Big)\\
&-\sigma_{11}\Big(-3C_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)+6\delta_1\delta_2S_{\delta_1}(r)S_{\delta_2}(r)+\delta_1 rS_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)\\
&+\delta_2 rC_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)+2\delta_1\delta_2 rC_{\delta_1}(r)C_{\delta_2}(r)+2\delta_1\delta_2 rS_{\delta_1}(r)C_{\delta_2}(r)\\
&+6h'(r)C_{\delta_1}(r)C_{\delta_2}(r)-6\delta_1h(r)S_{\delta_1}(r)C_{\delta_2}(r)-6\delta_2h(r)C_{\delta_1}(r)S_{\delta_2}(r)\\
&-3rh''(r)C_{\delta_1}(r)C_{\delta_2}(r)-6\delta_1 rh'(r)S_{\delta_1}(r)C_{\delta_2}(r)-6\delta_2 rh'(r)C_{\delta_1}(r)S_{\delta_2}(r)\\
&+6\delta_1\delta_2 rh(r)S_{\delta_1}(r)S_{\delta_2}(r)-3rh(r)C_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)\Big)\\
&-\sigma_{22}\Big(-C_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)+\delta_2S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)+2\delta_1\delta_2S_{\delta_1}(r)S_{\delta_2}(r)\\
&-2\delta_2C_{\delta_1}(r)C_{\delta_2}(r)+3h''(r)S_{\delta_1}(r)C_{\delta_2}(r)+6h'(r)C_{\delta_1}(r)C_{\delta_2}(r)\\
&-6\delta_2h'(r)S_{\delta_1}(r)S_{\delta_2}(r)-3h(r)S_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2)-6\delta_2h(r)C_{\delta_1}(r)S_{\delta_2}(r)\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \sigma_{33}\Big(- C_{\delta_2}(r)C_{\delta_1}(r)(\delta_1+\delta_2) + \delta_1 S_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2) + 2\delta_1\delta_2 S_{\delta_1}(r)S_{\delta_2}(r)\\
&- 2\delta_1 C_{\delta_2}(r)S_{\delta_1}(r) + 3h''(r)S_{\delta_2}(r)C_{\delta_1}(r) + 6h'(r)C_{\delta_2}(r)C_{\delta_1}(r)\\
&- 6\delta_1 h'(r)S_{\delta_1}(r)S_{\delta_2}(r) - 6\delta_1 h(r)C_{\delta_2}(r)S_{\delta_1}(r) - 3h(r)C_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2)\Big)\\
&+ \delta_1 S_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2) + \delta_2 C_{\delta_1}(r)S_{\delta_2}(r)(\delta_1+\delta_2) + 2\delta_1\delta_2 C_{\delta_1}(r)S_{\delta_2}(r)\\
&+ 2\delta_1\delta_2 S_{\delta_1}(r)C_{\delta_2}(r) + 3h''(r)C_{\delta_1}(r)C_{\delta_2}(r) - 6\delta_1 h'(r)S_{\delta_1}(r)C_{\delta_2}(r)\\
&- 6\delta_2 h'(r)C_{\delta_1}(r)S_{\delta_2}(r) + 6\delta_1\delta_2 h(r)S_{\delta_1}(r)S_{\delta_2}(r) - 3h(r)C_{\delta_1}(r)C_{\delta_2}(r)(\delta_1+\delta_2).
\end{aligned}
$$

Now, applying $r=0$ in (2.13) and in the second derivative above, we obtain the following relations:

$$
0 = f'(0) = 2(A_2+A_3+A_4) - 9h^2(0) + 3h'(0) - (\delta_1+\delta_2), \tag{2.17}
$$

$$
\begin{aligned}
0 = f''(0) = {} & 6A_1 + 6h(0)(A_2+A_3+A_4) - 18h'(0)h(0) + 2\sigma_{11}(\delta_1+\delta_2)\\
& + 2\sigma_{22}\delta_2 + 2\sigma_{33}\delta_1 + 3h''(0),
\end{aligned} \tag{2.18}
$$

where the functions A_i, $i=1,\dots,4$, are given in (2.11).

Let us recall that as C is constant we have $\sigma_{1i} = \sigma_{i1} = 0$ (since $A(X) = -\nabla C/2 = 0$), which implies that $A_1 = A_2 = A_3 = 0$. Moreover, since $h(0)$ is the mean curvature of Σ, we also conclude that

$$
3h(0) = \sigma_{22} + \sigma_{33}. \tag{2.19}
$$

Thus, we can rewrite (2.17) and (2.18) as follows:

$$
0 = 2(\sigma_{22}\sigma_{33} - \sigma_{23}^2) - 9h^2(0) + 3h'(0) - (\delta_1+\delta_2), \tag{2.20}
$$

$$
0 = 6h(0)(\sigma_{22}\sigma_{33} - \sigma_{23}^2) - 18h'(0)h(0) + 2\sigma_{22}\delta_2 + 2\sigma_{33}\delta_1 + 3h''(0). \tag{2.21}
$$

Inserting the expression for $2(\sigma_{22}\sigma_{33} - \sigma_{23}^2)$ obtained from (2.20) into (2.21), and using (2.19), we have that

$$
2\sigma_{33}(\delta_1-\delta_2) + 3h(0)(\delta_1+\delta_2) + 6h(0)\delta_2 + 27h^3(0) - 27h'(0)h(0) + 3h''(0) = 0.
$$

Note that $(\delta_1-\delta_2) = \frac{1}{2}(c_1 - c_2 + C(c_1+c_2)) \neq 0$, since $C \in (-1,1)$ and $c_1 \neq c_2$. Therefore σ_{33} is constant and hence, from (2.19) and (2.20), we have that σ_{22} and σ_{23} are also constant.

On the other hand, we are going to use the Codazzi equation to compute $X(\sigma_{22})$, $X(\sigma_{23})$ and $X(\sigma_{33})$, and for this, we will regard A as a $(0,2)$-tensor, that is, $A(B_i,B_j) = \langle AB_i, B_j\rangle = \langle B_i, AB_j\rangle$. Since each J_i is parallel and $A(X)=0$ (since $\nabla C = -2A(X)$), we have $\nabla_X B_j = 0$ for all $j=1,2,3$. In this way, since

$$
X(\sigma_{ij}) = X(A(B_i,B_j)) = \nabla A(X, B_i, B_j)
$$

it follows from the Codazzi equation (2.3) that

$$
\begin{aligned}
X(\sigma_{22}) &= \nabla A(B_2, X, B_2) + \frac{c_1}{4}\{\langle X,X\rangle\langle PB_2 + B_2, B_2\rangle - \langle B_2, X\rangle\langle PX + X, B_2\rangle\}\\
&\quad + \frac{c_2}{4}\{\langle X,X\rangle\langle PB_2 - B_2, B_2\rangle - \langle B_2, X\rangle\langle PX - X, B_2\rangle\}\\
&= -C\langle AB_2, AB_2\rangle + \langle PAB_2, AB_2\rangle + \frac{c_1\|X\|^2}{2}\\
&= -C(\sigma_{22}^2 + \sigma_{23}^2) + \sigma_{22}^2 - \sigma_{23}^2 + \frac{c_1(1-C^2)}{2}\\
&= \frac{c_1(1-C^2)}{2} + (1-C)\sigma_{22}^2 - (1+C)\sigma_{23}^2,
\end{aligned}
$$

$$\begin{aligned}
X(\sigma_{23}) &= \nabla A(B_2, X, B_3) + \frac{c_1}{4}\{\langle X, X\rangle\langle PB_2 + B_2, B_3\rangle - \langle B_2, X\rangle\langle PX + X, B_3\rangle\} \\
&\quad + \frac{c_2}{4}\{\langle X, X\rangle\langle PB_2 - B_2, B_3\rangle - \langle B_2, X\rangle\langle PX - X, B_3\rangle\} \\
&= -C\langle AB_2, AB_3\rangle + \langle PAB_2, AB_3\rangle \\
&= -C(\sigma_{22}\sigma_{23} + \sigma_{23}\sigma_{33}) + \sigma_{22}\sigma_{23} - \sigma_{23}\sigma_{33} \\
&= (1 - C)\sigma_{22}\sigma_{23} - (1 + C)\sigma_{23}\sigma_{33},
\end{aligned}$$

$$\begin{aligned}
X(\sigma_{33}) &= \nabla A(B_3, X, B_3) + \frac{c_1}{4}\{\langle X, X\rangle\langle PB_3 + B_3, B_3\rangle - \langle B_3, X\rangle\langle PX + X, B_3\rangle\} \\
&\quad + \frac{c_2}{4}\{\langle X, X\rangle\langle PB_3 - B_3, B_3\rangle - \langle B_3, X\rangle\langle PX - X, B_3\rangle\} \\
&= -C\langle AB_3, AB_3\rangle + \langle PAB_3, AB_3\rangle - \frac{c_2||X||^2}{2} \\
&= -C(\sigma_{23}^2 + \sigma_{33}^2) + \sigma_{23}^2 - \sigma_{33}^2 - \frac{c_2(1 - C^2)}{2} \\
&= \frac{c_2(C^2 - 1)}{2} + (1 - C)\sigma_{23}^2 - (1 + C)\sigma_{33}^2.
\end{aligned}$$

Therefore,

$$\frac{c_1(1 - C^2)}{2} + (1 - C)\sigma_{22}^2 - (1 + C)\sigma_{23}^2 = 0, \tag{2.22}$$

$$\frac{c_2(C^2 - 1)}{2} + (1 - C)\sigma_{23}^2 - (1 + C)\sigma_{33}^2 = 0, \tag{2.23}$$

$$(1 - C)\sigma_{22}\sigma_{23} - (1 + C)\sigma_{23}\sigma_{33} = 0. \tag{2.24}$$

Let us show that $\sigma_{23} = 0$. Suppose by contradiction that $\sigma_{23} \neq 0$. From (2.24), we have

$$(1 - C)^2\sigma_{22}^2 - (1 + C)^2\sigma_{33}^2 = 0. \tag{2.25}$$

Now, multiplying (2.22) by $1 - C$ and (2.23) by $1 + C$, we have

$$\frac{c_1(1 - C)(1 - C^2)}{2} + (1 - C)^2\sigma_{22}^2 - (1 - C^2)\sigma_{23}^2 = 0, \tag{2.26}$$

$$\frac{c_2(1 + C)(C^2 - 1)}{2} + (1 - C^2)\sigma_{23}^2 - (1 + C)^2\sigma_{33}^2 = 0. \tag{2.27}$$

Adding (2.26) to (2.27) and using (2.25), we get

$$c_1(1 - C) = c_2(1 + C),$$

Since $C \in (-1, 1)$ and $c_1 \neq c_2$, we have a contradiction. Therefore $\sigma_{23} = 0$.

If $\sigma_{23} = 0$, the system given by equations (2.22), (2.23) and (2.24) is reduced to

$$\sigma_{22}^2 = -\frac{c_1(1 + C)}{2}, \quad \sigma_{33}^2 = -\frac{c_2(1 - C)}{2}. \tag{2.28}$$

Observe that the only possibility of solving (2.28) is to consider that one c_i is negative and the other is zero. Then, without loss of generality, let us assume from now on that $c_1 = -1$ and $c_2 = 0$. Thus, the previous computation shows us that $\sigma_{ij} = 0$, for $i \neq j$ and $\sigma_{11} = \sigma_{33} = 0$. Therefore, we conclude that $\{B_1, B_2, B_3\}$ must be a frame of principal directions of Σ, with principal curvatures

$$\mu_1 = 0, \quad \mu_2 = \pm\sqrt{\frac{1 + C}{2}}, \quad \mu_3 = 0.$$

In what follows, we consider the case when $\mu_2 = \sqrt{\frac{1+C}{2}}$. The shape operator A is given, with respect to the frame B, by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, since

$$\begin{aligned}
PB_1 &= \frac{PX}{\sqrt{1-C^2}} \\
&= \frac{1}{\sqrt{1-C^2}}\Big(P^2N - CPN\Big) \\
&= \frac{1}{\sqrt{1-C^2}}\Big(N - CPN + C^2N - C^2N\Big) \\
&= \frac{1}{\sqrt{1-C^2}}\Big(-C(PN - CN) + (1-C^2)N\Big) \\
&= \frac{1}{\sqrt{1-C^2}}\Big(-CX + (1-C^2)N\Big) \\
&= -CB_1 + \sqrt{1-C^2}N, \\
PB_2 &= \frac{P(J_1N + J_2N)}{\sqrt{2(1+C)}} = \frac{P(2L_1(N_1), 0)}{\sqrt{2(1+C)}} \\
&= \frac{(2L_1(N_1), 0)}{\sqrt{2(1+C)}} = \frac{(J_1N + J_2N)}{\sqrt{2(1+C)}} \\
&= B_2, \\
PB_3 &= \frac{P(J_1N - J_2N)}{\sqrt{2(1-C)}} = \frac{P(0, 2L_2(N_2))}{\sqrt{2(1-C)}} \\
&= \frac{(0, -2L_2(N_2))}{\sqrt{2(1-C)}} = -\frac{(J_1N - J_2N)}{\sqrt{2(1-C)}} \\
&= -B_3,
\end{aligned}$$

the tangential component of the product structure P^T is given, with respect to the frame B, by

$$P^T = \begin{pmatrix} -C & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now, let us determine the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{H}^2 \times \mathbb{R}^2$ in the frame B. Note that

$$\widetilde{\nabla}_{B_i} B_j = \nabla_{B_i} B_j + \sigma_{ij} N.$$

Since P and J_i are parallel and $\sigma_{1i} = \sigma_{i1} = 0$, we get $\widetilde{\nabla}_{B_1} B_j = 0$, for $i = 1, 2, 3$. Moreover, we have that

$$\begin{aligned}
\nabla_{B_2} B_3 &= \frac{1}{\sqrt{2(1-C)}}(\widetilde{\nabla}_{B_2} J_1N - J_2N)^\top \\
&= \frac{1}{\sqrt{2(1-C)}}(-J_1AB_2 + J_2AB_2)^\top \\
&= \frac{1}{\sqrt{2(1-C)}}(-\sigma_{22}J_1B_2 - \sigma_{23}J_1B_3 + \sigma_{22}J_2B_2 + \sigma_{23}J_2B_3)^\top.
\end{aligned}$$

Since $\sigma_{23} = 0$ and

$$J_1 B_2 = \frac{J_1(J_1 N + J_2 N)}{\sqrt{2(1+C)}} = \frac{-N - PN}{\sqrt{2(1+C)}},$$
$$J_2 B_2 = \frac{J_2(J_1 N + J_2 N)}{\sqrt{2(1+C)}} = \frac{-PN - N}{\sqrt{2(1+C)}},$$

it follows that $\nabla_{B_2} B_3 = 0$, and hence, $\widetilde{\nabla}_{B_2} B_3 = 0$. Analogously, we get

$$\begin{array}{lll} \widetilde{\nabla}_{B_1} B_i = 0, & \widetilde{\nabla}_{B_2} B_3 = 0, & \widetilde{\nabla}_{B_3} B_2 = 0, \\ \widetilde{\nabla}_{B_2} B_1 = -\sqrt{\frac{1-C}{2}} B_2, & \widetilde{\nabla}_{B_2} B_2 = \frac{PN+N}{\sqrt{2(1+C)}}, & \widetilde{\nabla}_{B_3} B_1 = 0, \\ \widetilde{\nabla}_{B_3} B_3 = 0. & & \end{array}$$

Note that $[B_1, B_3] = [B_2, B_3] = 0$. Now, let λ a function such that

$$B_1(\lambda) = -\lambda\sqrt{\frac{1-C}{2}}, \quad B_2(\lambda) = 0 \quad \text{and} \quad B_3(\lambda) = 0.$$

In this way, we have

$$[B_1, \lambda B_2] = \left(B_1(\lambda) + \lambda\sqrt{\frac{1-C}{2}} \right) B_2 = 0 \tag{2.29}$$

and $[\lambda B_2, B_3] = 0$. Therefore, by Frobenius Theorem there is a parametrization $\Psi : \Omega \subset \mathbb{R}^3 \longrightarrow \Sigma$, where Ω is an open subset of $\mathbb{R}^3$ with coordinates $(t,\, u,\, v)$, such that

$$\Psi_t = B_1, \quad \Psi_u = \lambda B_2 \quad \text{and} \quad \Psi_v = B_3.$$

Now, we are going to construct the parametrization Ψ. Since $\Psi_v = B_3$, B_3 has no component in $\mathbb{H}^2$, and $\widetilde{\nabla}_{B_3} B_3 = 0$, i.e., B_3 is a geodesic field of $\mathbb{H}^2 \times \mathbb{R}^2$, when we integrate it with respect to v, we have

$$\Psi = \left(\Psi^{\mathbb{H}^2}(t,u), \beta(t,u) + B_3 v \right),$$

where $\Psi^{\mathbb{H}^2}$ is the component of Ψ in $\mathbb{H}^2$.

Before integrating with respect to the variable u, we first observe that B_2 has no component in $\mathbb{R}^2$ and

$$\widetilde{\nabla}_{B_2} B_2 = \nabla^{\mathbb{H}^2}_{B_2^{\mathbb{H}^2}} B_2^{\mathbb{H}^2} = \frac{PN + N}{\sqrt{2(1+C)}}.$$

Therefore,

$$\begin{aligned} ||\nabla^{\mathbb{H}^2}_{B_2^{\mathbb{H}^2}} B_2^{\mathbb{H}^2}||^2 = \langle \nabla^{\mathbb{H}^2}_{B_2^{\mathbb{H}^2}} B_2^{\mathbb{H}^2}, \nabla^{\mathbb{H}^2}_{B_2^{\mathbb{H}^2}} B_2^{\mathbb{H}^2} \rangle &= \frac{1}{2(1+C)} \Big(2\langle PN, N\rangle + \langle PN, PN\rangle + \langle N, N\rangle \Big) \\ &= \frac{1}{2(1+C)}(2C + 2) = 1, \end{aligned}$$

that is, if φ is a curve parametrized by arc length, with $\varphi' = B_2^{\mathbb{H}^2}$, then the geodesic curvature k_g of φ is $k_g = 1$, and hence φ is a horocycle. Up to rigid motions, φ is given by

$$\varphi(u) = \left(1 + \frac{u^2}{2}, u, -\frac{u^2}{2} \right) \in \mathbb{L}^3.$$

Moreover, as $\Psi_u = \left(\Psi_u^{\mathbb{H}^2}, \beta_u\right) = \lambda B_2$, then it follows that β does not depend on u. Thus, $\Psi_u = \lambda B_2 = \lambda(t)\,(u,\,1,\,-u,\,0,\,0)$, once $B_2(\lambda) = B_3(\lambda) = 0$. When we integrate $\Psi_u^{\mathbb{H}^2}$ with respect to u, we have

$$\Psi^{\mathbb{H}^2}(t,u) = \lambda(t)\left(\frac{u^2}{2}, u, -\frac{u^2}{2}\right) + \Lambda(t),$$

where $\Lambda(t)$ is a smooth curve in $\mathbb{H}^2$. Hence,

$$\Psi(t,u,v) = \left(\lambda(t)\alpha(u) + \Lambda(t), \beta(t) + B_3 v\right), \tag{2.30}$$

with $\alpha(u) = \left(\frac{u^2}{2}, u, -\frac{u^2}{2}\right)$.

Finally, we integrate $B_1 = \Psi_t = \left(\lambda'(t)\alpha(u) + \Lambda'(t), \beta'(t)\right)$. Since $\widetilde{\nabla}_{B_1} B_1 = 0$, B_1 is also a geodesic field of $\mathbb{H}^2 \times \mathbb{R}^2$. Therefore, $\beta(t) = p_0 + V_0 t$. Considering $\gamma(t,u) = \lambda(t)\alpha(u) + \Lambda(t)$, we have $\Psi_t = \left(\gamma_t, V_0\right) = B_1$, with $V_0 = B_1^{\mathbb{R}^2}$. It follows by the definition of B_1 that $\|B_1^{\mathbb{R}^2}\| = \sqrt{\frac{1+C}{2}}$. As $\|\gamma_t\|^2 + \|B_1^{\mathbb{R}^2}\|^2 = 1$, we get $\|\gamma_t\| = \|B_1^{\mathbb{H}^2}\| = \sqrt{\frac{1-C}{2}}$.

For any u_0 fixed, we note that

$$\begin{aligned} \frac{D\gamma_t}{dt}(t,u_0) &= \gamma_{tt}(t,u_0) - \|B_1^{\mathbb{H}^2}\|^2\gamma(t,u_0) \\ &= \alpha(u_0)\left(\lambda''(t) - \|B_1^{\mathbb{H}^2}\|^2\lambda(t)\right) + \Lambda''(t) - \|B_1^{\mathbb{H}^2}\|^2\Lambda(t). \end{aligned}$$

Since $\gamma(t,u_0)$ is a geodesic in $\mathbb{H}^2$, u_0 is arbitrary and α does not depend on t, we have that

$$\lambda''(t) - \|B_1^{\mathbb{H}^2}\|^2\lambda(t) = 0 \quad \text{and} \quad \Lambda''(t) - \|B_1^{\mathbb{H}^2}\|^2\Lambda(t) = 0,$$

and hence $\lambda(t)$ and $\Lambda(t)$ are given by

$$\begin{aligned} \lambda(t) &= b_1\cosh(\omega t) + b_2\sinh(\omega t), \\ \Lambda(t) &= V_1\cosh(\omega t) + V_2\sinh(\omega t), \end{aligned} \tag{2.31}$$

where $\omega = \pm\|B_1^{\mathbb{H}^2}\|$, b_i are real constants and V_i orthonormal vectors. If $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$, using $\langle\gamma,\gamma\rangle = -1$, it follows that

$$\begin{aligned} -1 &= \langle\lambda\alpha + \Lambda, \lambda\alpha + \Lambda\rangle \\ &= \lambda^2\langle\alpha,\alpha\rangle + 2\lambda\langle\alpha,\Lambda\rangle + \langle\Lambda,\Lambda\rangle \\ &= \lambda^2 u^2 + 2\lambda(-\frac{u^2}{2}\Lambda_1 + \Lambda_2 u - \frac{u^2}{2}\Lambda_3) - 1, \end{aligned}$$

which implies

$$\lambda^2 u^2 + 2\lambda(-\frac{u^2}{2}\Lambda_1 + \Lambda_2 u - \frac{u^2}{2}\Lambda_3) = 0.$$

Thus, we obtain the following polynomial equation in u:

$$(\lambda - (\Lambda_1 + \Lambda_3))u^2 + 2\Lambda_2 u = 0,$$

that is,

$$\lambda - (\Lambda_1 + \Lambda_3) = 0 \quad \text{and} \quad \Lambda_2 = 0. \tag{2.32}$$

If $V_1 = (v_{11}, v_{12}, v_{13})$ and $V_2 = (v_{21}, v_{22}, v_{23})$, we have

$$\begin{aligned} \Lambda_1 + \Lambda_3 &= (v_{11} + v_{13}) \cosh(\omega t) + (v_{21} + v_{23}) \sinh(\omega t), \\ \Lambda_2 &= v_{12} \cosh(\omega t) + v_{22} \sinh(\omega t), \end{aligned} \tag{2.33}$$

and consequently, combining (2.31), (2.32) and (2.33) it follows that $v_{12} = v_{22} = 0$, $b_1 = v_{11} + v_{13}$ and $b_2 = v_{21} + v_{23}$. Now we know that the second coordinate of V_1 and V_2 is zero. Then, writing $V_1 = (\cosh(a_1), 0, \sinh(a_1))$ and $V_2 = (\sinh(a_1), 0, \cosh(a_1))$, we get $b_1 = b_2 = e^{a_1}$. Thus, we conclude that

$$\begin{aligned} \lambda(t) &= e^{\omega t + a_1}, \\ \Lambda(t) &= \Big(\cosh(\omega t + a_1), 0, \sinh(\omega t + a_1) \Big). \end{aligned}$$

From (2.29), it follows that

$$\omega + \sqrt{\frac{1 - C}{2}} = 0.$$

Thus, we obtain that $\omega = -\|B_1^{\mathbb{H}^2}\|$, and therefore, using the linear change of variable $s = t + \dfrac{a_1}{\omega}$, we write

$$\begin{aligned} \lambda(s) &= e^{-\|B_1^{\mathbb{H}^2}\| s}, \\ \Lambda(s) &= \Big(\cosh(-\|B_1^{\mathbb{H}^2}\| s), 0, \sinh(-\|B_1^{\mathbb{H}^2}\| s) \Big). \end{aligned} \tag{2.34}$$

Writing $b = ||B_1^{\mathbb{H}^2}|| = \sqrt{1 - ||B_1^{\mathbb{R}^2}||^2} = \sqrt{1 - ||V_0||^2}$ and $W_0 = B_3$, when we replace (2.34) in (2.30), we obtain the parametrization (2.16).

For the converse, suppose that Σ is parametrized by (2.16). Note that

$$\begin{aligned} \Psi_s &= -b\Big(e^{-bs}(\alpha(u), \vec{0}) + \Big(\sinh(-bs), 0, \cosh(-bs), -\frac{V_0}{b} \Big) \Big), \\ \Psi_u &= e^{-bs}(\alpha'(u), \vec{0}), \\ \Psi_v &= \Big(\vec{0}, W_0 \Big). \end{aligned}$$

Now, let us to find a unit normal vector field to Σ. Consider the position vector

$$\Psi^{\mathbb{H}^2} = e^{-b\,s}(\alpha(u), \vec{0}) + \Big(\cosh(-b\,s), 0, \sinh(-b\,s), \vec{0} \Big).$$

Taking the cross product of $e^{bs}\Psi_u$ with $\Psi^{\mathbb{H}^2}$, we get

$$e^{bs}\Psi_u \boxtimes \Psi^{\mathbb{H}^2} = -\Big(\sinh(-b\,s) + \frac{e^{-bs}u^2}{2}, e^{-bs}u, \cosh(-b\,s) - \frac{e^{-bs}u^2}{2}, \vec{0} \Big),$$

that is, the normal vector takes the form

$$\widetilde{N} = -\Big(\sinh(-b\,s) + \frac{e^{-bs}u^2}{2}, e^{-bs}u, \cosh(-b\,s) - \frac{e^{-bs}u^2}{2}, \mu V_0 \Big),$$

where μ is a real constant.

It is easy to see that $\langle \widetilde{N}, \Psi_u\rangle = \langle \widetilde{N}, \Psi_v\rangle = 0$. Moreover, since

$$
\begin{aligned}
\langle \widetilde{N}, \Psi_s\rangle &= -b\Big(\sinh(-b\,s) + \frac{e^{-bs}u^2}{2}\Big)\Big(\sinh(-b\,s) + \frac{e^{-bs}u^2}{2}\Big) + be^{-2bs}u^2\\
&\quad + b\Big(\cosh(-b\,s) - \frac{e^{-bs}u^2}{2}\Big)\Big(\cosh(-b\,s) - \frac{e^{-bs}u^2}{2}\Big) - \mu||V_0||^2\\
&= b - be^{-bs}u^2(\cosh(-b\,s) + \sinh(-b\,s)) + be^{-2bs}u^2 - \mu||V_0||^2\\
&= b - \mu||V_0||^2,
\end{aligned}
$$

it follows that $\langle \widetilde{N}, \Psi_s\rangle = 0$ if and only if $\mu = \dfrac{b}{||V_0||^2}$. Thus, we have

$$
\widetilde{N} = -\Big(\sinh(-b\,s) + \frac{e^{-bs}u^2}{2}, e^{-bs}u, \cosh(-b\,s) - \frac{e^{-bs}u^2}{2}, \frac{b}{||V_0||^2}V_0\Big).
$$

Finally, since

$$
\begin{aligned}
\langle \widetilde{N}, \widetilde{N}\rangle =& \Big(\sinh(-b\,s) + \frac{e^{-bs}u^2}{2}\Big)^2 + \Big(\cosh(-b\,s) - \frac{e^{-bs}u^2}{2}\Big)\\
&+ e^{-2bs}u^2 + \frac{b^2}{||V_0||^2}\\
=&\, 1 + \frac{b^2}{||V_0||^2}\\
=& \frac{1}{||V_0||^2},
\end{aligned}
$$

we conclude that an unit normal vector field N to Σ is given by

$$
N = -||V_0||\Big(e^{-bs}(\alpha(u), \vec{0}) + \Big(\sinh(-bs), 0, \cosh(-bs), \frac{b}{||V_0||^2}V_0\Big)\Big).
$$

Denoting by $\tilde{D}$ the covariant derivative in $\mathbb{L}^3$, we obtain

$$
\begin{aligned}
\tilde{D}_{\Psi_s}N &= b||V_0||\Big(e^{-bs}\alpha(u) + \Big(\cosh(-bs), 0, \sinh(-bs)\Big), \vec{0}\Big)\\
&= b||V_0||\Psi^{\mathbb{H}^2},\\
\tilde{D}_{\Psi_u}N &= -||V_0||e^{-bs}(\alpha'(u), \vec{0})\\
&= -||V_0||\Psi_u,\\
\tilde{D}_{\Psi_v}N &= 0.
\end{aligned}
$$

It follows immediately from the derivatives above and the parametrization Ψ that

$$
\langle \tilde{D}_{\Psi_u}N, \Psi^{\mathbb{H}^2}\rangle = \langle \tilde{D}_{\Psi_v}N, \Psi^{\mathbb{H}^2}\rangle = 0 \quad \text{and} \quad \langle \tilde{D}_{\Psi_s}N, \Psi^{\mathbb{H}^2}\rangle = -b||V_0||.
$$

Therefore, since $\widetilde{\nabla}_V W = \tilde{D}_V W + \langle \tilde{D}_V W, \Psi^{\mathbb{H}^2}\rangle \Psi^{\mathbb{H}^2}$, we get

$$
\begin{aligned}
\widetilde{\nabla}_{\Psi_s}N &= 0,\\
\widetilde{\nabla}_{\Psi_u}N &= -||V_0||\Psi_u,\\
\widetilde{\nabla}_{\Psi_v}N &= 0,
\end{aligned}
$$

which implies that Σ has principal curvatures $\mu_1 = 0$, $\mu_2 = ||V_0||$ and $\mu_3 = 0$. Finally, since

$$PN = -||V_0||\Big(e^{-bs}(\alpha(u),\vec{0}) + \Big(\sinh(-bs), 0, \cosh(-bs), -\frac{b}{||V_0||^2}V_0\Big)\Big)$$

and $b = \sqrt{1-||V_0||^2}$, it follows that

$$\begin{aligned} C &= \langle PN, N\rangle \\ &= ||V_0||^2\Big(e^{-2bs}u^2 + 2e^{-bs}\Big(-\frac{u^2}{2}\sinh(-bs) - \frac{u^2}{2}\cosh(-bs)\Big) \\ &\qquad - \sinh^2(-bs) + \cosh^2(-bs) - \frac{b^2}{||V_0||^2}\Big) \\ &= ||V_0||^2\Big(1 - \frac{b^2}{||V_0||^2}\Big) \\ &= 2||V_0||^2 - 1, \end{aligned}$$

which implies that $||V_0|| = \sqrt{\frac{1+C}{2}}$.

To conclude, let us show that Σ is isoparametric. Note that in this case, using (2.9), the matrix D (2.10) is given by

$$D(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sinh(\sqrt{-\delta_1}r) + \cosh(\sqrt{-\delta_1}r) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

since $\sigma_{11} = \sigma_{12} = \sigma_{13} = \sigma_{32} = \sigma_{33} = 0$, $\sigma_{22} = \sqrt{\frac{1+C}{2}}$, $\delta_1 = \frac{-(1+C)}{2}$ and $\delta_2 = 0$. Thus, we get

$$\det D(r) = -\sinh(\sqrt{-\delta_1}r) + \cosh(\sqrt{-\delta_1}r),$$

and consequently

$$(\det D(r))' = -\sqrt{-\delta_1}(\cosh(\sqrt{-\delta_1}r) - \sinh(\sqrt{-\delta_1}r)).$$

Therefore, from (1.4), we obtain

$$\begin{aligned} h(r) &= -\frac{(\det D)'}{3\det D}(r) \\ &= -\frac{-\sqrt{-\delta_1}(\cosh(\sqrt{-\delta_1}r) - \sinh(\sqrt{-\delta_1}r))}{3(-\sinh(\sqrt{-\delta_1}r) + \cosh(\sqrt{-\delta_1}r))} \\ &= \frac{1}{3}\sqrt{\frac{1+C}{2}} \\ &= \frac{1}{3}||V_0||, \end{aligned}$$

that is, the mean curvature of the parallel hypersurfaces to Σ is constant, and hence Σ is isoparametric. □

Remark 3. *Following the notation established in the proof of Theorem 2.2, let us provide a geometric description of the hypersurface given by the parametrization Ψ. Note that a unit normal vector to the horocycle*

$$\varphi(u) = \Big(1 + \frac{u^2}{2}, u, -\frac{u^2}{2}\Big),$$

is given by

$$n(u) = \left(\frac{u^2}{2}, u, 1 - \frac{u^2}{2}\right).$$

Fixing u, $v \in \mathbb{R}$, *let us consider in* $\mathbb{H}^2 \times \mathbb{R}^2$ *the following geodesic parametrized by arc length*

$$\gamma(s) = \Big(\cosh(\omega s)\varphi(u) + \sinh(\omega s)n(u), g(v) + V_0 s\Big),$$

where $g(v) = p_0 + W_0 v$ *is a geodesic in* $\mathbb{R}^2$ *with normal vector* V_0. *Since*

$$\gamma'(s) = \Big(\omega\sinh(\omega s)\varphi(u) + \omega\cosh(\omega s)n(u), V_0\Big),$$

it follows that

$$1 = ||\gamma'(s)||^2 = \omega^2 + ||V_0||^2,$$

which implies $\omega = \pm\sqrt{1 - ||V_0||^2} = \pm b$. *Considering* $\omega = -b$, *we get*

$$\begin{aligned}\gamma(s) = e^{-b\,s}(\alpha(u), \vec{0}) &+ \Big(\cosh(-b\,s), 0, \sinh(-b\,s), V_0 s\Big)\\ &+ \Big(\vec{0}, p_0 + W_0 v\Big).\end{aligned}$$

Varying the parameters $(s, u, v) \in \mathbb{R}^3$, *the construction above provides exactly the parametrization* Ψ. *Therefore, the hypersurface* $\Psi(\mathbb{R}^3)$ *is the union of a family of geodesically parallel hypersurfaces of* $\mathbb{H}^2 \times \mathbb{R}^2$, *given by products of horocycles in* $\mathbb{H}^2$ *and straight lines in* $\mathbb{R}^2$.

Remark 4. *Another interesting property of the hypersurface* $\Psi(\mathbb{R}^3)$ *is its extrinsic homogeneity. A hypersurface of a Riemannian manifold* M *is called extrinsically homogeneous if it is a codimension-one orbit of a subgroup of the group of isometries of* M. *Let us show in this remark that* $\Psi(\mathbb{R}^3) \subset \mathbb{H}^2 \times \mathbb{R}^2$ *is a homogeneous hypersurface.*

Following Domínguez-Vázquez and Manzano [16], we consider the family of complete graphs $\widetilde{\Sigma}_h \subset \mathbb{H}^2 \times \mathbb{R}$ *of constant mean curvature* h, *with* $4h^2 - 1 < 0$, *called* parabolic helicoids. *Considering the half-space model of* $\mathbb{H}^2 = (\mathbb{R}^2_+, g)$ *where*

$$\mathbb{R}^2_+ = \{(x,\, y),\, y > 0\}\,, \quad \text{and} \quad g = \frac{dx^2 + dy^2}{y^2},$$

such surfaces are parametrized in $\mathbb{H}^2 \times \mathbb{R}$ *as*

$$\Phi_h(x, y) = (x,\, y,\, a\log(y)), \ \textit{with } a = \frac{2h}{\sqrt{1 - (2h)^2}}. \tag{2.35}$$

Domínguez-Vázquez and Manzano showed that the parabolic helicoids are homogeneous surfaces. Now, if we consider the product $\widetilde{\Sigma}_h \times \mathbb{R} \subset (\mathbb{H}^2 \times \mathbb{R}) \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}^2$, *we conclude that* $\widetilde{\Sigma}_h \times \mathbb{R}$ *is a homogeneous hypersurface in* $\mathbb{H}^2 \times \mathbb{R}^2$. *Since a homogeneous hypersurface is isoparametric and has constant principal curvatures (see for instance [15, Proposition 2.10]), it follows from Theorem 2.2 that* $\widetilde{\Sigma}_h \times \mathbb{R}$ *is congruent to either a product hypersurface as given in item a), or to* $\Psi(\mathbb{R}^3)$, *as in item b). A straightforward computation shows that a unit normal to* $\widetilde{\Sigma}_h \times \mathbb{R}$ *is given by* $N = N_1 + N_2$, *where* $N_1 = (0,\, ay)/\sqrt{1 + a^2}$ *and* $N_2 = (-1,\, 0)/\sqrt{1 + a^2}$. *Thus, for* $a \neq 0$, *we conclude that* $\widetilde{\Sigma}_h \times \mathbb{R}$ *must be congruent to* $\Psi(\mathbb{R}^3)$, *which shows that* $\Psi(\mathbb{R}^3)$ *is homogeneous.*

We can go further and write the parametrization (2.16) using the half-space model of $\mathbb{H}^2$. *Recall that an isometry between the Lorentzian model and the half-space model is given by*

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_2}{x_1 + x_3}, \frac{1}{x_1 + x_3}\right).$$

Considering the first three coordinate functions of parametrization Ψ

$$\begin{aligned} x_1 &= \cosh(-bs) + \frac{u^2}{2} e^{-b\,s}, \\ x_2 &= u e^{-b\,s}, \\ x_3 &= \sinh(-bs) - \frac{u^2}{2} e^{-b\,s}, \end{aligned}$$

it follows that

$$\left(\frac{x_2}{x_1 + x_3}, \frac{1}{x_1 + x_3} \right) = \left(u, e^{b\,s} \right).$$

Therefore, the parametrization (2.16) *is rewritten as follows*

$$\Psi(s, u, v) = \left(u, e^{b\,s}, p_0 + W_0 v + V_0 s \right).$$

Applying an isometry I_{W_0,V_0} *such that*

$$I_{W_0,V_0} \left(u, e^{b\,s}, p_0 + W_0 v + V_0 s \right) = \left(u, e^{b\,s}, e_2 v + ||V_0|| e_1 s \right),$$

where $\{e_1, e_2\}$ *is the canonical basis of* $\mathbb{R}^2$*, we obtain*

$$I_{W_0,V_0} \circ \Psi(s, u, v) = \left(u, e^{b\,s}, ||V_0|| s, v \right).$$

Finally, making the change of variables $s = \dfrac{1}{b} \log(y)$*, we have*

$$\begin{aligned} I_{W_0,V_0} \circ \Psi(y, u, v) &= \left(u, y, \frac{||V_0||}{b} \log(y), v \right) \\ &= \left(u, y, \frac{||V_0||}{\sqrt{1 - ||V_0||^2}} \log(y), v \right). \end{aligned} \tag{2.36}$$

On the one hand, we know that the mean curvature of $\Psi(\mathbb{R}^3)$ *is* $||V_0||/3$*. On the other hand, since the principal curvatures of* $\widetilde{\Sigma}_h \times \mathbb{R}$ *are* κ_1*,* κ_2 *and* 0*, where* κ_1 *and* κ_2 *are the principal curvatures of* $\widetilde{\Sigma}_h$*, the mean curvature* $\overline{h}$ *of* $\widetilde{\Sigma}_h \times \mathbb{R}$ *is given by* $\overline{h} = 2h/3$*. Therefore,* $2h = ||V_0||$ *and, by Equations* (2.35) *and* (2.36)*, we conclude that*

$$I_{W_0,V_0} \circ \Psi(y, u, v) = (\Phi_h(u, y),\ v).$$

Chapter 3

Hypersurfaces of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with constant principal curvatures

In this chapter, we will consider the ambient space $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$, where $\mathbb{Q}^3_\varepsilon$ denotes the unit sphere $\mathbb{S}^3$ if $\varepsilon = 1$, or the hyperbolic space $\mathbb{H}^3$ if $\varepsilon = -1$. Our main objective is to characterize and classify the hypersurfaces with three distinct constant principal curvatures in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$. We also show that the hypersurfaces with constant principal curvatures in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ are isoparametric. At last, we provide a necessary and sufficient condition for an isoparametric hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ to have constant principal curvatures.

The results presented in this chapter will compose a joint work with Fernando Manfio, João Paulo dos Santos and Joeri Van der Veken.

The chapter is organized as follows. In Section 3.1, we will present some basic content and results in the product space $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ already known in the literature, and Section 3.2 will be devoted to the proof of the main results of this chapter.

3.1 Preliminary concepts and results

Let Σ^n be a hypersurface in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ with unit normal N and let ∂_t be the coordinate vector field of the second factor $\mathbb{R}$. The orthogonal projection of ∂_t onto the tangent space of Σ^n will be denoted by T. Also, let θ be the angle function between N and ∂_t. Then we have the following decomposition

$$\partial_t = T + \cos\theta N.$$

Since ∂_t is a vector field of norm 1, it follows that

$$\cos^2(\theta) + ||T||^2 = 1.$$

In what follows, we will present the main equations that a hypersurface of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ satisfies, as obtained in [11]. Such equations are important since they provide the necessary and sufficient conditions for the existence of a hypersurface on $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, and they will be used throughout the chapter. The description of the equations below follows the structure according to [1].

We will denote by $\langle , \rangle$, $\bar{R}$ and $\overline{\nabla}$ the metric, the curvature tensor and the Riemannian connection of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, respectively, and by ∇, R, A the Riemannian connection, the curvature tensor and the shape operator of a hypersurface Σ^n in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, respectively. We will consider

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

We begin by presenting the Gauss and Codazzi equations.

Proposition 3.1. *Let Σ^n be a hypersurface of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$. The Gauss and Codazzi equations of Σ^n are given, respectively, by*

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \varepsilon\Big(&\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle \\ &+ \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle + \langle Y,T\rangle\langle W,T\rangle\langle X,Z\rangle \\ &- \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle - \langle X,T\rangle\langle W,T\rangle\langle Y,Z\rangle\Big) \\ &+ \langle AX,W\rangle\langle AY,Z\rangle - \langle AX,Z\rangle\langle AY,W\rangle,\end{aligned} \tag{3.1}$$

$$\nabla_X(AY) - \nabla_Y(AX) - A[X,Y] = \varepsilon\cos\theta[\langle Y,T\rangle X - \langle X,T\rangle Y], \tag{3.2}$$

where X, Y, Z, $W \in T\Sigma^n$.

Proof. Let $\Sigma^n \subset \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a hypersurface with unit normal vector denoted by N. In this case, since Σ^n is an isometric immersion in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, its Gauss equation is given by

$$\langle \bar{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle B(Y,W),B(X,Z)\rangle - \langle B(X,W),B(Y,Z)\rangle,$$

where B denotes the second fundamental form of Σ^n and X, Y, Z, $W \in T\Sigma^n$. Note that, since $B(X_1,X_2) = \langle A(X_1),X_2\rangle N$, it follows that

$$\langle \bar{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle + \langle A(Y),W\rangle\langle A(X),Z\rangle - \langle A(X),W\rangle\langle A(Y),Z\rangle.$$

Let $\nabla^{\mathbb{Q}^n_\varepsilon}$ and $\nabla^{\mathbb{R}}$ the Riemannian connections and $R^{\mathbb{Q}^n_\varepsilon}$ and $R^{\mathbb{R}}$ the curvature tensors of $\mathbb{Q}^n_\varepsilon$ and $\mathbb{R}$, respectively. Then, since $\mathbb{Q}^n_\varepsilon$ has constant sectional curvature ε, we obtain that

$$\langle \bar{R}(X,Y)Z,W\rangle = \varepsilon\Big(\langle Y^{\mathbb{Q}^n_\varepsilon},Z^{\mathbb{Q}^n_\varepsilon}\rangle\langle X^{\mathbb{Q}^n_\varepsilon},W^{\mathbb{Q}^n_\varepsilon}\rangle - \langle X^{\mathbb{Q}^n_\varepsilon},Z^{\mathbb{Q}^n_\varepsilon}\rangle\langle Y^{\mathbb{Q}^n_\varepsilon},W^{\mathbb{Q}^n_\varepsilon}\rangle\Big),$$

and hence

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \varepsilon\Big(&\langle Y^{\mathbb{Q}^n_\varepsilon},Z^{\mathbb{Q}^n_\varepsilon}\rangle\langle X^{\mathbb{Q}^n_\varepsilon},W^{\mathbb{Q}^n_\varepsilon}\rangle - \langle X^{\mathbb{Q}^n_\varepsilon},Z^{\mathbb{Q}^n_\varepsilon}\rangle\langle Y^{\mathbb{Q}^n_\varepsilon},W^{\mathbb{Q}^n_\varepsilon}\rangle\Big) \\ &- \langle A(Y),W\rangle\langle A(X),Z\rangle + \langle A(X),W\rangle\langle A(Y),Z\rangle,\end{aligned} \tag{3.3}$$

where the component $Y^{\mathbb{Q}^n_\varepsilon}$ of Y is given as its tangent part to $\mathbb{Q}^n_\varepsilon$.

Now, decomposing $X \in \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ into $X = X^{\mathbb{Q}^n_\varepsilon} + \langle X,T\rangle\partial_t$, we have

$$\begin{aligned}\langle Y^{\mathbb{Q}^n_\varepsilon},Z^{\mathbb{Q}^n_\varepsilon}\rangle&\langle X^{\mathbb{Q}^n_\varepsilon},W^{\mathbb{Q}^n_\varepsilon}\rangle - \langle X^{\mathbb{Q}^n_\varepsilon},Z^{\mathbb{Q}^n_\varepsilon}\rangle\langle Y^{\mathbb{Q}^n_\varepsilon},W^{\mathbb{Q}^n_\varepsilon}\rangle \\ &= \langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle \\ &\quad + \langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle X,W\rangle\langle Y,T\rangle\langle Z,T\rangle \\ &\quad + \langle Y,W\rangle\langle X,T\rangle\langle Z,T\rangle,\end{aligned}$$

and replacing this value in (3.3), we obtain

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \varepsilon\Big(&\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,T\rangle\langle W,T\rangle \\ &+ \langle X,Z\rangle\langle Y,T\rangle\langle W,T\rangle - \langle X,W\rangle\langle Y,T\rangle\langle Z,T\rangle + \langle Y,W\rangle\langle X,T\rangle\langle Z,T\rangle\Big) \\ &- \langle A(Y),W\rangle\langle A(X),Z\rangle + \langle A(X),W\rangle\langle A(Y),Z\rangle.\end{aligned}$$

For the Codazzi equation, remember that for a hypersurface on a Riemannian manifold, the Codazzi equation is given by

$$\bar{R}(X,Y)N = \nabla_Y AX - \nabla_X AY + A[X,Y].$$

On the other hand, since $N^{\mathbb{Q}^n_\varepsilon} = N - \langle N, \partial_t\rangle\partial_t$, we get

$$\begin{aligned}
\bar{R}(X,Y)N &= \varepsilon\Big(\langle Y^{\mathbb{Q}^n_\varepsilon}, N^{\mathbb{Q}^n_\varepsilon}\rangle X^{\mathbb{Q}^n_\varepsilon} - \langle X^{\mathbb{Q}^n_\varepsilon}, N^{\mathbb{Q}^n_\varepsilon}\rangle Y^{\mathbb{Q}^n_\varepsilon}\Big) \\
&= \varepsilon\Big(\langle Y - \langle Y,\partial_t\rangle\partial_t, N - \langle N,\partial_t\rangle\partial_t\rangle(X - \langle X,\partial_t\rangle\partial_t) \\
&\quad - \langle X - \langle X,\partial_t\rangle\partial_t, N - \langle N,\partial_t\rangle\partial_t\rangle(Y - \langle Y,\partial_t\rangle\partial_t)\Big) \\
&= \varepsilon\Big(- \langle Y,\partial_t\rangle\langle N,\partial_t\rangle(X - \langle X,\partial_t\rangle\partial_t) + \langle X,\partial_t\rangle\langle N,\partial_t\rangle(Y - \langle Y,\partial_t\rangle\partial_t)\Big) \\
&= \varepsilon\Big(- \langle Y,\partial_t\rangle\langle N,\partial_t\rangle X + \langle X,\partial_t\rangle\langle N,\partial_t\rangle Y\Big) \\
&= \varepsilon\cos(\theta)\Big(- \langle Y,T\rangle X + \langle X,T\rangle Y\Big).
\end{aligned}$$

Therefore, we conclude that

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = \varepsilon\cos(\theta)\Big(\langle Y,T\rangle X - \langle X,T\rangle Y\Big).$$

□

We will now obtain two interesting properties that T and $\cos(\theta)$ satisfy and which will be of great importance. First, observe that ∂_t is a parallel field in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$. Indeed, if $\nabla^{\mathbb{Q}^n_\varepsilon}$ and $\nabla^{\mathbb{R}}$ denote the Riemannian connections of $\mathbb{Q}^n_\varepsilon$ and $\mathbb{R}$, respectively, it follows that

$$\overline{\nabla}_X\partial_t = \nabla^{\mathbb{Q}^n_\varepsilon}_{X^{\mathbb{Q}^n_\varepsilon}}\partial_t + \nabla^{\mathbb{R}}_{X^{\mathbb{R}}}\partial_t = \nabla^{\mathbb{Q}^n_\varepsilon}_{X^{\mathbb{Q}^n_\varepsilon}} 0 + \nabla^{\mathbb{R}}_{X^{\mathbb{R}}}\partial_t = \nabla^{\mathbb{R}}_{X^{\mathbb{R}}}\partial_t,$$

for $X \in T(\mathbb{Q}^n_\varepsilon \times \mathbb{R})$. Since $X^{\mathbb{R}} = \langle X, \partial_t\rangle\partial_t$, we get

$$\overline{\nabla}_X\partial_t = \nabla^{\mathbb{R}}_{\langle X,\partial_t\rangle\partial_t}\partial_t = \langle X,\partial_t\rangle\nabla^{\mathbb{R}}_{\partial_t}\partial_t = 0.$$

Due to this fact, we have the following result.

Proposition 3.2. *Let Σ^n be a hypersurface of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$. Given X tangent to Σ^n, it holds that*

$$\nabla_X T = \cos(\theta)AX, \tag{3.4}$$
$$X(\cos(\theta)) = -\langle AX, T\rangle. \tag{3.5}$$

Proof. Since ∂_t is parallel in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, we have that

$$0 = \overline{\nabla}_X\partial_t = \overline{\nabla}_X(T + \cos(\theta)N) = \overline{\nabla}_X T + \cos(\theta)\overline{\nabla}_X N + X(\cos(\theta))N, \tag{3.6}$$

for all $X \in T\Sigma^n$. Now, since

$$\overline{\nabla}_X T = \nabla_X T + \langle T, AX\rangle N,$$

replacing this in (3.6), we have

$$\begin{aligned}
0 &= \nabla_X T + \langle T, AX\rangle N - \cos(\theta)AX + X(\cos(\theta))N \\
&= \nabla_X T - \cos(\theta)AX + [\langle T, AX\rangle + X(\cos(\theta))]N.
\end{aligned}$$

Therefore, once $\nabla_X T - \cos(\theta)AX$ is tangent to Σ and $[\langle T, AX\rangle + X(\cos(\theta))]N$ is orthogonal, it follows that

$$\nabla_X T - \cos(\theta)AX = 0, \quad \text{and} \quad \langle T, AX\rangle + X(\cos(\theta)) = 0.$$

□

We finish this section with some interesting results about hypersurfaces of $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, which will be useful in the next section. The first result is due to Fernando Manfio and Ruy Tojeiro [32], which classifies hypersurfaces when T and $\cos(\theta)$ vanish identically.

Proposition 3.3. *Let $f : \Sigma^n \longrightarrow \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a hypersurface.*

i) If T vanishes identically, then $f(\Sigma)$ is an open subset of a slice $\mathbb{Q}^n_\varepsilon \times \{t\}$.

ii) If $\cos(\theta)$ vanishes identically, then $f(\Sigma)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a hypersurface of $\mathbb{Q}^n_\varepsilon$.

Using the Codazzi equation Joeri Van der Veken and Luc Vrancken [42] classify the totally geodesic hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$, and in [6], also using the Codazzi equation, Giovanni Calvaruso, Daniel Kowalczyk and Joeri Van der Veken show that this classification is also obtained in $\mathbb{H}^n \times \mathbb{R}$. In short, they got the following

Theorem 3.4. *Let $f : \Sigma^n \longrightarrow \mathbb{Q}^n_\varepsilon \times \mathbb{R}$ be a totally geodesic hypersurface. Then $f(\Sigma)$ is an open part of a hypersurface $\mathbb{Q}^n_\varepsilon \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, or of a hypersurface $M^{n-1} \times \mathbb{R}$, where M^{n-1} is a totally geodesic hypersurface of $\mathbb{Q}^n_\varepsilon$.*

3.2 Main results

As we pointed out before, in this section, we will present the proof of the main results of this chapter. We start by characterizing, in terms of the angle function, the hypersurfaces of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ that have constant and distinct principal curvatures. Through this characterization, we provide an explicit classification of such hypersurfaces. Furthermore, we prove that the hypersurfaces of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with constant principal curvatures are isoparametric in those spaces. We finish this section by showing that an isoparametric hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ has constant principal curvatures if and only if the angle function is constant.

Lemma 3.5. *Let Σ be a hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with three distinct constant principal curvatures. Then $\cos(\theta)$ is constant on Σ.*

Proof. Let Σ^3 be a hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with distinct constant principal curvatures μ_1, μ_2 and μ_3. Consider $\{e_1, e_2, e_3\}$ a frame of orthonormal principal directions with μ_i being the principal curvature associated to e_i, $i = 1, 2, 3$, that is,

$$Ae_1 = \mu_1 e_1, \quad Ae_2 = \mu_2 e_2, \quad \text{and} \quad Ae_3 = \mu_3 e_3,$$

where A denotes the shape operator of Σ^3. Since T is tangent to Σ^3, we can write

$$T = \sum_{i=1}^{3} b_i e_i, \tag{3.7}$$

where $b_1, b_2, b_3 : \Sigma^3 \longrightarrow \mathbb{R}$ are smooth functions. Furthermore, if ∇ denotes the Levi-Civita connection of Σ^3, let us write

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_j^k(e_i) e_k, \tag{3.8}$$

where ω_j^k are the connection forms of Σ^3. With this notation and the hypothesis that μ_i are constant functions, let us get the consequences of equations (3.1), (3.2), (3.4) and (3.5). In what follows, we will use the indices i, j and k for computations and n, m and l for distinct indices in $\{1, 2, 3\}$ to stablish the consequences.

We begin with the Codazzi equation (3.2) with $X = e_i$ and $Y = e_j$, $i \neq j$. Using equations (3.7) and (3.8) we have

$$\begin{aligned}\varepsilon\cos(\theta)\,(b_j e_i - b_i e_j) &= \mu_j \nabla_{e_i} e_j - \mu_i \nabla_{e_j} e_i - A(\nabla_{e_i} e_j - \nabla_{e_j} e_i)\\ &= \sum_{k=1}^{3}\left[\mu_j \omega_j^k(e_i) - \mu_i \omega_i^k(e_j) - \mu_k \omega_j^k(e_i) + \mu_k \omega_i^k(e_j)\right] e_k.\end{aligned}$$

Thus, we get

$$\sum_{k=1}^{3}\left[(\mu_j - \mu_k)\,\omega_j^k(e_i) - (\mu_i - \mu_k)\,\omega_i^k(e_j)\right] e_k = \varepsilon\cos(\theta)\,(b_j e_i - b_i e_j)\,.$$

Considering the coefficients of each principal direction we conclude that

$$\begin{aligned}(\mu_m - \mu_n)\omega_m^n(e_n) &= \varepsilon\cos(\theta) b_m,\\ (\mu_m - \mu_l)\,\omega_m^l(e_n) &= (\mu_n - \mu_l)\,\omega_n^l(e_m).\end{aligned}$$

Consequently, there is a function Ω such that

$$\omega_m^n(e_n) = \frac{\varepsilon\cos(\theta) b_m}{\mu_m - \mu_n}, \tag{3.9}$$

$$\omega_m^l(e_n) = \frac{\varepsilon\cos(\theta)\Omega}{\mu_m - \mu_l}. \tag{3.10}$$

Now, from equations (3.4) and (3.7), we have

$$\begin{aligned}\cos(\theta)\mu_n e_n &= \sum_{i=1}^{3} e_n(b_i) e_i + \sum_{i=1}^{3} b_i \nabla_{e_n} e_i\\ &= \sum_{i=1}^{3} e_n(b_i) e_i + \sum_{k=1}^{3} b_k \nabla_{e_n} e_k\\ &= \sum_{i=1}^{3} e_n(b_i) e_i + \sum_{k=1}^{3}\sum_{i=1}^{3} b_k \omega_k^i(e_n) e_i\\ &= \sum_{i=1}^{3}\left[e_n(b_i) + \sum_{k=1}^{3} b_k \omega_k^i(e_n)\right] e_i,\end{aligned}$$

that is,

$$e_n(b_n) = \cos(\theta)\mu_n e_n - \sum_{k=1}^{3} b_k \omega_k^n(e_n),$$

$$e_n(b_m) = -\sum_{k=1}^{3} b_k \omega_k^m(e_n) = \sum_{k=1}^{3} b_k \omega_m^k(e_n).$$

Thus, using the equations (3.9) and (3.10), we conclude that

$$e_n(b_n) = \left[\mu_n - \varepsilon\left(\frac{b_l^2}{\mu_l - \mu_n} + \frac{b_m^2}{\mu_m - \mu_n}\right)\right]\cos(\theta), \tag{3.11}$$

$$e_n(b_m) = \varepsilon\left(\frac{\Omega b_l}{\mu_m - \mu_l} + \frac{b_n b_m}{\mu_m - \mu_n}\right)\cos(\theta). \tag{3.12}$$

On the other hand, equation (3.5) supplies

$$e_n(\cos(\theta)) = -\mu_n b_n. \tag{3.13}$$

Now we proceed with Gauss equation. In order to do that we compute the intrinsic curvature of Σ^3. Firstly, the bracket between principal directions is given by

$$\begin{aligned}
[e_m,\, e_n] &= \sum_{k=1}^{3} \Big(\omega_n^k(e_m) - \omega_m^k(e_n)\Big)\, e_k \\
&= \omega_n^m(e_m)e_m - \omega_m^n(e_n)e_n + \big(\omega_n^l(e_m) - \omega_m^l(e_n)\big)\, e_l,
\end{aligned}$$

which enables to conclude that

$$\begin{aligned}
R(e_m, e_n)e_m &= \nabla_{e_m}\nabla_{e_n}e_m - \nabla_{e_n}\nabla_{e_m}e_m - \nabla_{[e_m,e_n]}e_m \\
&= \nabla_{e_m}\left(\sum_{k=1}^{3}\omega_m^k(e_n)e_k\right) - \nabla_{e_n}\left(\sum_{k=1}^{3}\omega_m^k(e_m)e_k\right) \\
&\quad - \omega_n^m(e_m)\nabla_{e_m}e_m + \omega_m^n(e_n)\nabla_{e_n}e_m - \Big(\omega_n^l(e_m) - \omega_m^l(e_n)\Big)\nabla_{e_l}e_m \\
&= \sum_{k=1}^{3}\Big(e_m(\omega_m^k(e_n))e_k + \omega_m^k(e_n)\nabla_{e_m}e_k\Big) - \sum_{k=1}^{3}\Big(e_n(\omega_m^k(e_m))e_k + \omega_m^k(e_m)\nabla_{e_n}e_k\Big) \\
&\quad - \omega_n^m(e_m)\nabla_{e_m}e_m + \omega_m^n(e_n)\nabla_{e_n}e_m - \Big(\omega_n^l(e_m) - \omega_m^l(e_n)\Big)\nabla_{e_l}e_m \\
&= \sum_{k=1}^{3} e_m(\omega_m^k(e_n))e_k + \sum_{k=1}^{3}\sum_{j=1}^{3}\omega_m^k(e_n)\omega_k^j(e_m)e_j - \sum_{k=1}^{3} e_n(\omega_m^k(e_m))e_k \\
&\quad - \sum_{k=1}^{3}\sum_{j=1}^{3}\omega_m^k(e_m)\omega_k^j(e_n)e_j - \sum_{k=1}^{3}\omega_n^m(e_m)\omega_m^k(e_m)e_k \\
&\quad + \sum_{k=1}^{3}\omega_m^n(e_n)\omega_m^k(e_n)e_k - \sum_{k=1}^{3}\Big(\omega_n^l(e_m) - \omega_m^l(e_n)\Big)\omega_m^k(e_l)e_k.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\langle R(e_m, e_n)e_m, e_n\rangle &= e_m(\omega_m^n(e_n)) + \sum_{k=1}^{3}\omega_m^k(e_n)\omega_k^n(e_m) - e_n(\omega_m^n(e_m)) \\
&\quad - \sum_{k=1}^{3}\omega_m^k(e_m)\omega_k^n(e_n) - \omega_n^m(e_m)\omega_m^n(e_m) + (\omega_m^n(e_n))^2 \\
&\quad - \Big(\omega_n^l(e_m) - \omega_m^l(e_n)\Big)\omega_m^n(e_l) \\
&= \omega_m^l(e_n)\omega_l^n(e_m) + e_m(\omega_m^n(e_n)) - \omega_m^l(e_m)\omega_l^n(e_n) - e_n(\omega_m^n(e_m)) \\
&\quad + (\omega_n^m(e_m))^2 + (\omega_m^n(e_n))^2 - \omega_n^l(e_m)\omega_m^n(e_l) + \omega_m^l(e_n)\omega_m^n(e_l) \\
&= \frac{\cos^2(\theta)\Omega^2}{(\mu_m - \mu_l)(\mu_l - \mu_n)} + e_m\left(\frac{\varepsilon\cos(\theta)b_m}{\mu_m - \mu_n}\right) + \frac{\cos^2(\theta)b_l^2}{(\mu_l - \mu_m)(\mu_l - \mu_n)} \\
&\quad + e_n\left(\frac{\varepsilon\cos(\theta)b_n}{\mu_n - \mu_m}\right) + \frac{\cos^2(\theta)b_n^2}{(\mu_n - \mu_m)^2} + \frac{\cos^2(\theta)b_m^2}{(\mu_n - \mu_m)^2} - \frac{\cos^2(\theta)\Omega^2}{(\mu_n - \mu_l)(\mu_m - \mu_n)} \\
&\quad + \frac{\cos^2(\theta)\Omega^2}{(\mu_m - \mu_l)(\mu_m - \mu_n)},
\end{aligned}$$

that is,

$$\begin{aligned}\langle R(e_m,e_n)e_m,e_n\rangle &= \frac{\cos^2(\theta)\Omega^2}{(\mu_m-\mu_l)(\mu_l-\mu_n)} + \frac{\cos^2(\theta)b_l^2}{(\mu_l-\mu_m)(\mu_l-\mu_n)} + \frac{\cos^2(\theta)(b_n^2+b_m^2)}{(\mu_n-\mu_m)^2}\\ &+ \frac{\varepsilon}{\mu_m-\mu_n}\left(e_m(\cos(\theta))b_m+\cos(\theta)e_m(b_m)\right) - \frac{\cos^2(\theta)\Omega^2}{(\mu_n-\mu_l)(\mu_m-\mu_n)}\\ &+ \frac{\varepsilon}{\mu_n-\mu_m}\left(e_n(\cos(\theta))b_n+\cos(\theta)e_n(b_n)\right) + \frac{\cos^2(\theta)\Omega^2}{(\mu_m-\mu_l)(\mu_m-\mu_n)}.\end{aligned}$$

Now, using (3.11) and (3.13), we get

$$\begin{aligned}\langle R(e_m,e_n)e_m,e_n\rangle &= \frac{\cos^2(\theta)\Omega^2}{(\mu_m-\mu_l)(\mu_l-\mu_n)} + \frac{\cos^2(\theta)b_l^2}{(\mu_l-\mu_m)(\mu_l-\mu_n)} + \frac{\cos^2(\theta)(b_n^2+b_m^2)}{(\mu_n-\mu_m)^2}\\ &- \frac{\cos^2(\theta)\Omega^2}{(\mu_m-\mu_n)}\Big(\frac{1}{(\mu_n-\mu_l)}-\frac{1}{(\mu_m-\mu_l)}\Big) - \frac{\varepsilon\mu_m b_m^2}{\mu_m-\mu_n} - \frac{\varepsilon\mu_n b_n^2}{\mu_n-\mu_m}\\ &+ \frac{\varepsilon\cos^2(\theta)}{\mu_m-\mu_n}\left[\mu_m-\varepsilon\Big(\frac{b_l^2}{\mu_l-\mu_m}+\frac{b_n^2}{\mu_n-\mu_m}\Big)\right]\\ &+ \frac{\varepsilon\cos^2(\theta)}{\mu_n-\mu_m}\left[\mu_n-\varepsilon\Big(\frac{b_l^2}{\mu_l-\mu_n}+\frac{b_m^2}{\mu_m-\mu_n}\Big)\right]\\ &= \frac{2\cos^2(\theta)\Omega^2}{(\mu_m-\mu_l)(\mu_l-\mu_n)} + \frac{\cos^2(\theta)b_l^2}{(\mu_l-\mu_m)(\mu_l-\mu_n)} + \frac{2\cos^2(\theta)(b_n^2+b_m^2)}{(\mu_n-\mu_m)^2}\\ &+ \frac{\varepsilon(\mu_n b_n^2-\mu_m b_m^2)}{\mu_m-\mu_n} + \varepsilon\cos^2(\theta) + \frac{\cos^2(\theta)b_l^2}{(\mu_n-\mu_m)}\Big(\frac{1}{(\mu_l-\mu_m)}-\frac{1}{(\mu_l-\mu_n)}\Big)\\ &= \frac{2\cos^2(\theta)\Omega^2}{(\mu_m-\mu_l)(\mu_l-\mu_n)} + \frac{\varepsilon(\mu_n b_n^2-\mu_m b_m^2)}{\mu_m-\mu_n} + \varepsilon\cos^2(\theta) + \frac{2\cos^2(\theta)(b_n^2+b_m^2)}{(\mu_n-\mu_m)^2}.\end{aligned}$$

On the other hand, by Gauss equation (3.1), we have

$$\begin{aligned}\langle R(e_m,e_n)e_m,e_n\rangle &= \varepsilon\Big(\langle e_m,e_n\rangle\langle e_n,e_m\rangle - \langle e_m,e_m\rangle\langle e_n,e_n\rangle\\ &+ \langle e_m,T\rangle\langle e_m,T\rangle\langle e_n,e_n\rangle + \langle e_n,T\rangle\langle e_n,T\rangle\langle e_m,e_m\rangle\\ &- \langle e_n,T\rangle\langle e_m,T\rangle\langle e_m,e_n\rangle - \langle e_m,T\rangle\langle e_n,T\rangle\langle e_n,e_m\rangle\Big)\\ &+ \langle Ae_m,e_n\rangle\langle Ae_n,e_m\rangle - \langle Ae_m,e_m\rangle\langle Ae_n,e_n\rangle\\ &= \varepsilon(b_m^2+b_n^2-1) - \mu_m\mu_n,\end{aligned}$$

and therefore,

$$\frac{2\cos^2(\theta)\Omega^2}{(\mu_m-\mu_l)(\mu_l-\mu_n)} + \frac{\varepsilon(\mu_n b_n^2-\mu_m b_m^2)}{\mu_m-\mu_n} + 2\varepsilon\cos^2(\theta) + \frac{2\cos^2(\theta)(b_n^2+b_m^2)}{(\mu_n-\mu_m)^2} + \varepsilon b_l^2 + \mu_m\mu_n = 0.$$

Consequently, from the above equation, we get:

$$\begin{aligned}\frac{2\cos^2(\theta)\Omega^2}{(\mu_3-\mu_1)(\mu_3-\mu_2)} &= \frac{\varepsilon(\mu_1 b_1^2-\mu_2 b_2^2)}{\mu_2-\mu_1} + \frac{2\cos^2(\theta)(b_1^2+b_2^2)}{(\mu_2-\mu_1)^2} + 2\varepsilon\cos^2(\theta)\\ &+ \mu_1\mu_2 + \varepsilon b_3^2,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\frac{2\cos^2(\theta)\Omega^2}{(\mu_2-\mu_1)(\mu_3-\mu_1)} &= \frac{\varepsilon(\mu_2 b_2^2-\mu_3 b_3^2)}{\mu_3-\mu_2} + \frac{2\cos^2(\theta)(b_2^2+b_3^2)}{(\mu_3-\mu_2)^2} + 2\varepsilon\cos^2(\theta)\\ &+ \mu_2\mu_3 + \varepsilon b_1^2,\end{aligned} \tag{3.15}$$

$$\frac{2\cos^2(\theta)\Omega^2}{(\mu_2-\mu_1)(\mu_3-\mu_2)} = \frac{\varepsilon(\mu_1 b_1^2 - \mu_3 b_3^2)}{\mu_3-\mu_1} + \frac{2\cos^2(\theta)(b_1^2+b_3^2)}{(\mu_3-\mu_1)^2} + 2\varepsilon\cos^2(\theta) + \mu_1\mu_3 + \varepsilon b_2^2. \tag{3.16}$$

In what follows, we will eliminate Ω in the above equations. Replacing (3.14) in (3.15) and (3.16), we get the following equations on the variables b_1^2, b_2^2 and b_3^2:

$$\begin{aligned}
&\left[\frac{\varepsilon\mu_1(\mu_3-\mu_2)}{\mu_2-\mu_1} + \frac{2\cos^2(\theta)(\mu_3-\mu_2)}{(\mu_2-\mu_1)^2} - \varepsilon(\mu_2-\mu_1)\right] b_1^2\\
&+\left[\frac{\varepsilon\mu_2(\mu_2-\mu_3)}{\mu_2-\mu_1} - \frac{\varepsilon\mu_2(\mu_2-\mu_1)}{\mu_3-\mu_2} + \frac{2\cos^2(\theta)(\mu_3-\mu_2)}{(\mu_2-\mu_1)^2} - \frac{2\cos^2(\theta)(\mu_2-\mu_1)}{(\mu_3-\mu_2)^2}\right] b_2^2\\
&+\left[\frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_2} - \frac{2\cos^2(\theta)(\mu_2-\mu_1)}{(\mu_3-\mu_2)^2} + \varepsilon(\mu_3-\mu_2)\right] b_3^2\\
&+2\varepsilon\cos^2(\theta)(\mu_3-\mu_2) - 2\varepsilon\cos^2(\theta)(\mu_2-\mu_1) + \mu_1\mu_2(\mu_3-\mu_2) - \mu_2\mu_3(\mu_2-\mu_1) = 0
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\left[\frac{\varepsilon\mu_1(\mu_3-\mu_1)}{\mu_2-\mu_1} - \frac{\varepsilon\mu_1(\mu_2-\mu_1)}{\mu_3-\mu_1} + \frac{2\cos^2(\theta)(\mu_3-\mu_1)}{(\mu_2-\mu_1)^2} - \frac{2\cos^2(\theta)(\mu_2-\mu_1)}{(\mu_3-\mu_1)^2}\right] b_1^2\\
&+\left[\frac{\varepsilon\mu_2(\mu_1-\mu_3)}{\mu_2-\mu_1} + \frac{2\cos^2(\theta)(\mu_3-\mu_1)}{(\mu_2-\mu_1)^2} - \varepsilon(\mu_2-\mu_1)\right] b_2^2\\
&+\left[\frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_1} - \frac{2\cos^2(\theta)(\mu_2-\mu_1)}{(\mu_3-\mu_1)^2} + \varepsilon(\mu_3-\mu_1)\right] b_3^2\\
&+2\varepsilon\cos^2(\theta)(\mu_3-\mu_1) - 2\varepsilon\cos^2(\theta)(\mu_2-\mu_1) + \mu_1\mu_2(\mu_3-\mu_1) - \mu_1\mu_3(\mu_2-\mu_1) = 0.
\end{aligned} \tag{3.18}$$

Since $||T||^2 + \cos(\theta)^2 = 1$, writing $t = \cos^2(\theta)$, we have $b_3^2 = 1 - t - b_1^2 - b_2^2$. Then, substituting such expression for b_3^2 in (3.17) and (3.18), we obtain

$$\begin{aligned}
&\left[\frac{\varepsilon\mu_1(\mu_3-\mu_2)}{\mu_2-\mu_1} - \frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_2} + \frac{2t(\mu_3-\mu_2)}{(\mu_2-\mu_1)^2} + \frac{2t(\mu_2-\mu_1)}{(\mu_3-\mu_2)^2} + \varepsilon(\mu_1-\mu_3)\right] b_1^2\\
&+\left[\frac{\varepsilon\mu_2(\mu_2-\mu_3)}{\mu_2-\mu_1} - \frac{\varepsilon\mu_2(\mu_2-\mu_1)}{\mu_3-\mu_2} - \frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_2} + \frac{2t(\mu_3-\mu_2)}{(\mu_2-\mu_1)^2} - \varepsilon(\mu_3-\mu_2)\right] b_2^2\\
&+\frac{2t^2(\mu_2-\mu_1)}{(\mu_3-\mu_2)^2} + \left(\frac{2(\mu_1-\mu_2)}{(\mu_3-\mu_2)^2} + \frac{\varepsilon\mu_3(\mu_1-\mu_2)}{\mu_3-\mu_2} + \varepsilon(\mu_3-\mu_2) + 2\varepsilon(\mu_1-\mu_2)\right) t\\
&+\frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_2} + \mu_1\mu_2(\mu_3-\mu_2) + \mu_2\mu_3(\mu_1-\mu_2) + \varepsilon(\mu_3-\mu_2) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\left[\frac{\varepsilon\mu_1(\mu_3-\mu_1)}{\mu_2-\mu_1} - \frac{\varepsilon\mu_1(\mu_2-\mu_1)}{\mu_3-\mu_1} - \frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_1} + \frac{2t(\mu_3-\mu_1)}{(\mu_2-\mu_1)^2} - \varepsilon(\mu_3-\mu_1)\right] b_1^2\\
&+\left[\frac{\varepsilon\mu_2(\mu_1-\mu_3)}{\mu_2-\mu_1} - \frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_1} + \frac{2t(\mu_3-\mu_1)}{(\mu_2-\mu_1)^2} + \frac{2t(\mu_2-\mu_1)}{(\mu_3-\mu_1)^2}\right.\\
&\left. - \varepsilon(\mu_3-\mu_1) - \varepsilon(\mu_2-\mu_1)\right] b_2^2 + \frac{2t^2(\mu_2-\mu_1)}{(\mu_3-\mu_1)^2}\\
&+\left(\frac{2(\mu_1-\mu_2)}{(\mu_3-\mu_1)^2} + \frac{\varepsilon\mu_3(\mu_1-\mu_2)}{\mu_3-\mu_1} + \varepsilon(\mu_3-\mu_1) + 2\varepsilon(\mu_1-\mu_2)\right) t\\
&+\frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_1} + \mu_1\mu_2(\mu_3-\mu_1) + \mu_1\mu_3(\mu_1-\mu_2) + \varepsilon(\mu_3-\mu_1) = 0,
\end{aligned}$$

that is, we obtain the following system of variables b_1^2 and b_2^2:

$$\left[\frac{2t(\mu_2-\mu_3)^3+2t(\mu_2-\mu_1)^3+C_{11}}{(\mu_2-\mu_1)^2(\mu_3-\mu_2)^2}\right]b_1^2+\left[\frac{2t(\mu_3-\mu_2)+C_{12}}{(\mu_2-\mu_1)^2}\right]b_2^2$$
$$+\frac{2t^2(\mu_2-\mu_1)}{(\mu_3-\mu_2)^2}+q_1(t)=0,$$

$$\left[\frac{2t(\mu_3-\mu_1)+C_{21}}{(\mu_2-\mu_1)^2}\right]b_1^2+\left[\frac{2t(\mu_3-\mu_1)^3+2t(\mu_2-\mu_1)^3+C_{22}}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2}\right]b_2^2$$
$$+\frac{2t^2(\mu_2-\mu_1)}{(\mu_3-\mu_1)^2}+q_2(t)=0.$$

Here, C_{11}, C_{12}, C_{21} and C_{22} are real constants that depend on μ_1, μ_2, μ_3, ε and $q_1(t)$ and $q_2(t)$ are linear functions on the variable t, given by

$$q_1(t)=\left(\frac{2(\mu_1-\mu_2)}{(\mu_3-\mu_2)^2}+\frac{\varepsilon\mu_3(\mu_1-\mu_2)}{\mu_3-\mu_2}+\varepsilon(\mu_3-\mu_2)+2\varepsilon(\mu_1-\mu_2)\right)t$$
$$+\frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_2}+\mu_1\mu_2(\mu_3-\mu_2)+\mu_2\mu_3(\mu_1-\mu_2)+\varepsilon(\mu_3-\mu_2)$$

and

$$q_2(t)=\left(\frac{2(\mu_1-\mu_2)}{(\mu_3-\mu_1)^2}+\frac{\varepsilon\mu_3(\mu_1-\mu_2)}{\mu_3-\mu_1}+\varepsilon(\mu_3-\mu_1)+2\varepsilon(\mu_1-\mu_2)\right)t$$
$$+\frac{\varepsilon\mu_3(\mu_2-\mu_1)}{\mu_3-\mu_1}+\mu_1\mu_2(\mu_3-\mu_1)+\mu_1\mu_3(\mu_1-\mu_2)+\varepsilon(\mu_3-\mu_1).$$

Then, b_1^2 and b_2^2 are solutions of the linear system

$$\begin{pmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{pmatrix}\begin{pmatrix} b_1^2\\ b_2^2\end{pmatrix}=\begin{pmatrix}\dfrac{2t^2(\mu_1-\mu_2)}{(\mu_3-\mu_2)^2}-q_1(t)\\ \dfrac{2t^2(\mu_1-\mu_2)}{(\mu_3-\mu_1)^2}-q_2(t)\end{pmatrix},$$

where d_{ij} are the elements of the matrix of coefficients $\tilde{D}=(d_{ij})$, given by

$$\tilde{D}=\begin{pmatrix}\dfrac{2t(\mu_2-\mu_3)^3+2t(\mu_2-\mu_1)^3+C_{11}}{(\mu_2-\mu_1)^2(\mu_3-\mu_2)^2} & \dfrac{2t(\mu_3-\mu_2)+C_{12}}{(\mu_2-\mu_1)^2}\\ \dfrac{2t(\mu_3-\mu_1)+C_{21}}{(\mu_2-\mu_1)^2} & \dfrac{2t(\mu_3-\mu_1)^3+2t(\mu_2-\mu_1)^3+C_{22}}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2}\end{pmatrix}.$$

Thus, we have

$$b_1^2=\frac{1}{\det\tilde{D}}\Bigg[\left(\frac{2t(\mu_3-\mu_1)^3+2t(\mu_2-\mu_1)^3+C_{22}}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2}\right)\left(\frac{2t^2(\mu_1-\mu_2)-(\mu_3-\mu_2)^2q_1(t)}{(\mu_3-\mu_2)^2}\right)$$
$$-\left(\frac{2t(\mu_3-\mu_2)+C_{12}}{(\mu_2-\mu_1)^2}\right)\left(\frac{2t^2(\mu_1-\mu_2)-(\mu_3-\mu_1)^2q_2(t)}{(\mu_3-\mu_1)^2}\right)\Bigg]$$
$$=\frac{1}{\det\tilde{D}}\Bigg[\frac{4t^3(\mu_1-\mu_2)\left((\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3\right)}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2(\mu_3-\mu_2)^2}+\frac{4t^3(\mu_1-\mu_2)(\mu_2-\mu_3)}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2}$$
$$+\frac{2t^2C_{22}(\mu_1-\mu_2)-2t(\mu_3-\mu_2)^2q_1(t)\left((\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3\right)-C_{22}(\mu_3-\mu_2)^2q_1(t)}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2(\mu_3-\mu_2)^2}$$
$$-\frac{2t^2C_{12}(\mu_1-\mu_2)-2t(\mu_3-\mu_1)^2(\mu_3-\mu_2)q_2(t)-C_{12}(\mu_3-\mu_1)^2q_2(t)}{(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2}\Bigg]$$
$$=\frac{4t^3(\mu_1-\mu_2)\left((\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3+(\mu_2-\mu_3)^3\right)+q_3(t)}{\det\tilde{D}(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2(\mu_3-\mu_2)^2},$$

where $q_3(t)$ is a polynomial of degree 2, given by

$$\begin{aligned}q_3(t) &= 2t^2(\mu_1-\mu_2)(C_{22}-C_{12}(\mu_3-\mu_2)^2) - C_{22}(\mu_3-\mu_2)^2 q_1(t)\\ &\quad - 2t(\mu_3-\mu_2)^2 q_1(t)((\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3)\\ &\quad + 2t(\mu_3-\mu_1)^2(\mu_3-\mu_2)^3 q_2(t) + C_{12}(\mu_3-\mu_1)^2(\mu_3-\mu_2)^2 q_2(t).\end{aligned}$$

Now we consider the coefficient of degree 3. Since

$$\begin{aligned}&4\Big((\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3+(\mu_2-\mu_3)^3\Big)\\ &\qquad= 12\mu_3^2(\mu_2-\mu_1) - 12\mu_3(\mu_2^2-\mu_1^2) + 8(\mu_2^3-\mu_1^3) - 12\mu_1\mu_2(\mu_2-\mu_1)\\ &\qquad= (\mu_2-\mu_1)\left(12\mu_3^2 - 12\mu_1\mu_3 - 12\mu_2\mu_3 + 8\mu_2^2 + 8\mu_1^2 - 4\mu_1\mu_2\right)\\ &\qquad= 2(\mu_2-\mu_1)\left(3(\mu_1-\mu_3)^2 + 3(\mu_2-\mu_3)^2 + (\mu_1-\mu_2)^2\right),\end{aligned}$$

it follows that

$$b_1^2 = \frac{-2t^3(\mu_1-\mu_2)^2\Big(3(\mu_1-\mu_3)^2+3(\mu_2-\mu_3)^2+(\mu_1-\mu_2)^2\Big) + q_3(t)}{\det\tilde{D}(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2(\mu_3-\mu_2)^2}.$$

Analogous computations provide

$$b_2^2 = \frac{12t^3(\mu_1-\mu_2)^2(\mu_1-\mu_3)(\mu_2-\mu_3) + q_4(t)}{\det\tilde{D}(\mu_2-\mu_1)^2(\mu_3-\mu_1)^2(\mu_3-\mu_2)^2},$$

where $q_4(t)$ is a polynomial of degree 2, given by

$$\begin{aligned}q_4(t) &= 2t^2(\mu_1-\mu_2)(C_{11}-C_{21}(\mu_3-\mu_1)^2) - C_{11}(\mu_3-\mu_1)^2 q_2(t)\\ &\quad - 2t(\mu_3-\mu_1)^2 q_2(t)((\mu_3-\mu_2)^3+(\mu_2-\mu_1)^3)\\ &\quad + 2t(\mu_3-\mu_2)^2(\mu_3-\mu_1)^3 q_1(t) + C_{21}(\mu_3-\mu_2)^2(\mu_3-\mu_1)^2 q_1(t).\end{aligned}$$

Now we will compute the determinant of $\tilde{D}$. Note that

$$\begin{aligned}\det\tilde{D} &= \frac{4t^2\Big((\mu_3-\mu_2)^3+(\mu_2-\mu_1)^3\Big)\Big((\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3\Big)}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}\\ &\quad+\frac{2t\Big(C_{11}(\mu_3-\mu_1)^3+C_{11}(\mu_2-\mu_1)^3+C_{22}(\mu_3-\mu_2)^3+C_{22}(\mu_2-\mu_1)^3\Big)+C_{11}C_{22}}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}\\ &\quad+\frac{4t^2(\mu_1-\mu_3)(\mu_3-\mu_2)}{(\mu_1-\mu_2)^4} - \frac{2t\Big(C_{12}(\mu_3-\mu_1)+C_{21}(\mu_3-\mu_2)\Big)+C_{12}C_{21}}{(\mu_1-\mu_2)^4}\\ &= \frac{4t^2(\mu_2-\mu_1)^3\Big((\mu_3-\mu_2)^3+(\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3\Big)}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}\\ &\quad+\frac{2t\Big(C_{11}(\mu_3-\mu_1)^3+C_{11}(\mu_2-\mu_1)^3+C_{22}(\mu_3-\mu_2)^3+C_{22}(\mu_2-\mu_1)^3\Big)}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}\\ &\quad-\frac{2t(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2\Big(C_{12}(\mu_3-\mu_1)+C_{21}(\mu_3-\mu_2)\Big)}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}\\ &\quad+\frac{C_{11}C_{22}+C_{12}C_{21}(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2}.\end{aligned}$$

As before we note that

$$\begin{aligned}
&4\left((\mu_3-\mu_2)^3+(\mu_3-\mu_1)^3+(\mu_2-\mu_1)^3\right)\\
&\qquad = 8(\mu_3^3-\mu_1^3)-12\mu_2(\mu_3^2-\mu_1^2)+12\mu_2^2(\mu_3-\mu_1)-12\mu_1\mu_3(\mu_3-\mu_1)\\
&\qquad = (\mu_3-\mu_1)\left(8\mu_1^2-12\mu_1\mu_2-4\mu_1\mu_3+12\mu_2^2-12\mu_2\mu_3+8\mu_3^2\right)\\
&\qquad = 2(\mu_3-\mu_1)\left((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\right).
\end{aligned}$$

Hence, we conclude that

$$\det\tilde{D} = \frac{2t^2(\mu_1-\mu_3)(\mu_1-\mu_2)^3\left((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\right)+q_5(t)}{(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2},$$

where $q_5(t)$ is a linear function given by

$$\begin{aligned}
q_5(t) = {}&2t\left(C_{11}(\mu_3-\mu_1)^3+C_{11}(\mu_2-\mu_1)^3+C_{22}(\mu_3-\mu_2)^3+C_{22}(\mu_2-\mu_1)^3\right)\\
&-2t(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2\left(C_{12}(\mu_3-\mu_1)+C_{21}(\mu_3-\mu_2)\right)\\
&+C_{11}C_{22}-C_{12}C_{21}(\mu_1-\mu_3)^2(\mu_2-\mu_3)^2.
\end{aligned}$$

Therefore,

$$b_1^2 = \frac{-2t^3(\mu_1-\mu_2)^4\left(3(\mu_1-\mu_3)^2+3(\mu_2-\mu_3)^2+(\mu_1-\mu_2)^2\right)+q_3(t)}{2t^2(\mu_1-\mu_3)(\mu_1-\mu_2)^3\left((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\right)+q_5(t)}$$

and

$$b_2^2 = \frac{12t^3(\mu_1-\mu_2)^4(\mu_1-\mu_3)(\mu_2-\mu_3)+q_4(t)}{2t^2(\mu_1-\mu_3)(\mu_1-\mu_2)^3\left((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\right)+q_5(t)}.$$

Consequently, as $b_3^2 = 1-t-b_1^2-b_2^2$, we have

$$b_3^2 = \frac{-2t^3(\mu_2-\mu_3)(\mu_1-\mu_2)^3\left(3(\mu_1-\mu_2)^2+3(\mu_1-\mu_3)^2+(\mu_2-\mu_3)^2\right)+q_6(t)}{2t^2(\mu_1-\mu_3)(\mu_1-\mu_2)^3\left((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\right)+q_5(t)},$$

where $q_6(t)$ is given by

$$\begin{aligned}
q_6(t) = {}&(1-t)q_5(t)+2t^2(\mu_1-\mu_3)(\mu_1-\mu_2)^3\left((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\right)\\
&-q_3(t)-q_4(t).
\end{aligned}$$

In fact, the coefficient of t^3 is given by

$$
\begin{aligned}
&-2t^3(\mu_1-\mu_3)(\mu_1-\mu_2)^3\Big((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\Big)\\
&+2t^3(\mu_1-\mu_2)^4\Big(3(\mu_1-\mu_3)^2+3(\mu_2-\mu_3)^2+(\mu_1-\mu_2)^2\Big)\\
&+12t^3(\mu_1-\mu_2)^4(\mu_1-\mu_3)(\mu_2-\mu_3)\\
&=-2t^3(\mu_1-\mu_2)^3\Big[(\mu_1-\mu_3)^3+3(\mu_1-\mu_3)(\mu_1-\mu_2)^2+3(\mu_1-\mu_3)(\mu_2-\mu_3)^2\\
&\quad-3(\mu_1-\mu_2)(\mu_1-\mu_3)^2-3(\mu_1-\mu_2)(\mu_2-\mu_3)^2-(\mu_1-\mu_2)^3\\
&\quad+6(\mu_1-\mu_2)(\mu_1-\mu_3)(\mu_2-\mu_3)\Big]\\
&=-2t^3(\mu_1-\mu_2)^3\Big[(\mu_1-\mu_3-\mu_1+\mu_2)^3+3(\mu_2-\mu_3)\Big((\mu_1-\mu_3)(\mu_2-\mu_3)\\
&\qquad-(\mu_1-\mu_2)(\mu_2-\mu_3)+2(\mu_1-\mu_2)(\mu_1-\mu_3)\Big)\Big]\\
&=-2t^3(\mu_2-\mu_3)(\mu_1-\mu_2)^3\Big[(\mu_2-\mu_3)^2+3\Big((\mu_1-\mu_3)(\mu_2-\mu_3)\\
&\qquad-(\mu_1-\mu_2)(\mu_2-\mu_3)+2(\mu_1-\mu_2)(\mu_1-\mu_3)\Big)\Big]\\
&=-2t^3(\mu_2-\mu_3)(\mu_1-\mu_2)^3\Big(3(\mu_1-\mu_2)^2+3(\mu_1-\mu_3)^2+(\mu_2-\mu_3)^2\Big).
\end{aligned}
$$

Therefore, we conclude that

$$
b_i^2(t)=\frac{p_i(t)}{q(t)}, \tag{3.19}
$$

where $p_i(t)$ and $q(t)$ are polynomials on the variable t, of degree 3 and 2, respectively.

Now we will calculate $e_1(b_1^2)$, $e_2(b_2^2)$, $e_3(b_3^2)$ and use (3.11) to find polynomial identities on the variable t. Let us start with $e_1(b_1^2)$. First, note that $b_i\neq 0$ for all $i\in\{1,2,3\}$, otherwise as the principal curvatures are distinct, by (3.19), it follows that t is constant, and with that, we would finish the proof of the lemma. Using (3.13) and (3.19), we have

$$
\begin{aligned}
2b_1e_1(b_1)=e_1(b_1^2)=e_1\left(\frac{p_1(t)}{q(t)}\right)
&=\frac{(p_1'(t)q(t)-p_1(t)q'(t))e_1(t)}{q^2(t)}\\
&=\frac{2\cos(\theta)e_1(\cos(\theta))(p_1'(t)q(t)-p_1(t)q'(t))}{q^2(t)}\\
&=-\frac{2\mu_1\cos(\theta)b_1(p_1'(t)q(t)-p_1(t)q'(t))}{q^2(t)},
\end{aligned}
$$

that is,

$$
e_1(b_1)=-\frac{\mu_1\cos(\theta)(p_1'(t)q(t)-p_1(t)q'(t))}{q^2(t)}.
$$

Now, using (3.11), we get

$$
\left[\mu_1-\varepsilon\left(\frac{b_3^2}{\mu_3-\mu_1}+\frac{b_2^2}{\mu_2-\mu_1}\right)\right]=-\frac{\mu_1(p_1'(t)q(t)-p_1(t)q'(t))}{q^2(t)},
$$

which implies

$$\mu_1(\mu_3-\mu_1)(\mu_1-\mu_2)(p_1'(t)q(t)-p_1(t)q'(t)) = \varepsilon(\mu_1-\mu_2)q^2(t)b_3^2+\varepsilon(\mu_1-\mu_3)q^2(t)b_2^2 + \mu_1(\mu_3-\mu_1)(\mu_2-\mu_1)q^2(t).$$

Thus, replacing (3.19) in the above equation, we conclude that

$$\begin{aligned}\mu_1(\mu_1-\mu_2)(\mu_3-\mu_1)(p_1'(t)q(t)-p_1(t)q'(t)) - \mu_1(\mu_1-\mu_2)(\mu_1-\mu_3)q^2(t)\\ = \varepsilon(\mu_1-\mu_3)q(t)p_2(t)+\varepsilon(\mu_1-\mu_2)q(t)p_3(t).\end{aligned} \tag{3.20}$$

Observe that the left side of (3.20) is a polynomial on t of degree at most 4. On the other hand, since

$$\begin{aligned}\varepsilon(\mu_1-\mu_3)q(t)p_2(t)+\varepsilon(\mu_1-\mu_2)q(t)p_3(t) &= \varepsilon q(t)\Big[12t^3(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)\\ &\quad -2t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)\Big(3(\mu_1-\mu_2)^2+3(\mu_1-\mu_3)^2+(\mu_2-\mu_3)^2\Big)\\ &\quad +(\mu_1-\mu_3)q_4(t)+(\mu_1-\mu_2)q_6(t)\Big]\\ &= \varepsilon q(t)\Big[2t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)\Big(3(\mu_1-\mu_3)^2-3(\mu_1-\mu_2)^2-(\mu_2-\mu_3)^2\Big)\\ &\quad +(\mu_1-\mu_3)q_4(t)+(\mu_1-\mu_2)q_6(t)\Big]\\ &= \varepsilon q(t)\Big[4t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)^2(3\mu_1-2\mu_2-\mu_3)\\ &\quad +(\mu_1-\mu_3)q_4(t)+(\mu_1-\mu_2)q_6(t)\Big],\end{aligned}$$

we obtain a polynomial equation such that the coefficient of greatest degree, which is 5, is given by

$$8\varepsilon(3\mu_1-2\mu_2-\mu_3)(\mu_1-\mu_2)^7(\mu_2-\mu_3)^2(\mu_1-\mu_3)\Big((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\Big). \tag{3.21}$$

Now let us compute $e_2(b_2^2)$. Following similarly to the previous case, we obtain

$$\begin{aligned}\mu_2(\mu_1-\mu_2)(\mu_2-\mu_3)(p_2'(t)q(t)-p_2(t)q'(t)) - \mu_2(\mu_2-\mu_1)(\mu_2-\mu_3)q^2(t)\\ = \varepsilon(\mu_2-\mu_3)q(t)p_1(t)+\varepsilon(\mu_2-\mu_1)q(t)p_3(t).\end{aligned} \tag{3.22}$$

Note that the left side of (3.22) is a polynomial on t of degree at most 4. Now, since

$$\begin{aligned}&\varepsilon(\mu_2-\mu_3)q(t)p_1(t)+\varepsilon(\mu_2-\mu_1)q(t)p_3(t)\\ &\quad= \varepsilon q(t)\Big[-2t^3(\mu_2-\mu_3)(\mu_1-\mu_2)^4\Big(3(\mu_1-\mu_3)^2+3(\mu_2-\mu_3)^2+(\mu_1-\mu_2)^2\Big)\\ &\qquad +2t^3(\mu_2-\mu_3)(\mu_2-\mu_1)^4\Big(3(\mu_1-\mu_2)^2+3(\mu_1-\mu_3)^2+(\mu_2-\mu_3)^2\Big)\\ &\qquad +(\mu_2-\mu_3)q_3(t)+(\mu_2-\mu_1)q_6(t)\Big]\\ &\quad= \varepsilon q(t)\Big[4t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)\Big((\mu_1-\mu_2)^2-(\mu_2-\mu_3)^2\Big)\\ &\qquad +(\mu_2-\mu_3)q_3(t)+(\mu_2-\mu_1)q_6(t)\Big],\end{aligned}$$

that is,

$$\begin{aligned}\varepsilon(\mu_2-\mu_3)q(t)p_1(t)+\varepsilon(\mu_2-\mu_1)q(t)p_3(t)&\\ =\varepsilon q(t)\Big[4t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)(\mu_1-\mu_3)\big(\mu_1-2\mu_2+\mu_3\big)&\\ +(\mu_2-\mu_3)q_3(t)+(\mu_2-\mu_1)q_6(t)\Big],&\end{aligned}$$

we get a polynomial equation of degree 5 on variable t, such that the coefficient of t^5 is given by

$$8\varepsilon(\mu_1-2\mu_2+\mu_3)(\mu_1-\mu_2)^7(\mu_1-\mu_3)^2(\mu_2-\mu_3)\Big((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\Big). \quad (3.23)$$

Finally, let us compute $e_3(b_3^2)$. As before, we obtain

$$\begin{aligned}\mu_3(\mu_1-\mu_3)(\mu_3-\mu_2)(p_2'(t)q(t)-p_2(t)q'(t))-\mu_3(\mu_3-\mu_1)(\mu_3-\mu_3)q^2(t)&\\ =\varepsilon(\mu_3-\mu_1)q(t)p_2(t)+\varepsilon(\mu_3-\mu_2)q(t)p_1(t),&\end{aligned} \quad (3.24)$$

where the left side is a polynomial on t of degree at most 4. Now, since

$$\begin{aligned}\varepsilon(\mu_3-\mu_1)q(t)p_2(t)+\varepsilon(\mu_3-\mu_2)q(t)p_1(t)&=\varepsilon q(t)\Big[-12t^3(\mu_1-\mu_2)^4(\mu_1-\mu_3)^2(\mu_2-\mu_3)\\ &+2t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)\Big(3(\mu_1-\mu_3)^2+3(\mu_2-\mu_3)^2+(\mu_1-\mu_2)^2\Big)\\ &+(\mu_3-\mu_1)q_4(t)+(\mu_3-\mu_2)q_3(t)\Big]\\ &=\varepsilon q(t)\Big[2t^3(\mu_1-\mu_2)^4(\mu_2-\mu_3)\Big(3(\mu_2-\mu_3)^2-3(\mu_1-\mu_3)^2+(\mu_1-\mu_2)^2\Big)\\ &+(\mu_3-\mu_1)q_4(t)+(\mu_3-\mu_2)q_3(t)\Big]\\ &=\varepsilon q(t)\Big[4t^3(\mu_1-\mu_2)^5(\mu_3-\mu_2)\big(\mu_1+2\mu_2-3\mu_3\big)\\ &+(\mu_3-\mu_1)q_4(t)+(\mu_3-\mu_2)q_3(t)\Big],\end{aligned}$$

we have a polynomial equation of degree 5 on vaiable t, such that the coefficient of t^5 is given by

$$8\varepsilon(\mu_1+2\mu_2-3\mu_3)(\mu_1-\mu_2)^8(\mu_1-\mu_3)(\mu_3-\mu_2)\Big((\mu_1-\mu_3)^2+3(\mu_1-\mu_2)^2+3(\mu_2-\mu_3)^2\Big). \quad (3.25)$$

Note that, if one of the coefficients (3.21), (3.23) or (3.25) does not vanish, we conclude that t is constant, and hence, $\cos(\theta)$ is constant.

Then, suppose by contradiction that the three coefficients vanish. As the principal curvatures are all distinct, we have the following homogeneous linear system

$$3\mu_1-2\mu_2-\mu_3=0, \quad (3.26)$$
$$\mu_1-2\mu_2+\mu_3=0, \quad (3.27)$$
$$\mu_1+2\mu_2-3\mu_3=0. \quad (3.28)$$

From (3.26) and (3.27), we obtain that $\mu_1=\mu_2=\mu$, and therefore, from (3.28), we conclude that $\mu_3=\mu$, which contradicts the fact that the principal curvatures are distinct. Thus, we conclude the proof of the lemma. □

In [8], and Santos classified the hypersurfaces in the product space $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$, with g distinct constant principal curvatures, $g \in \{1, 2, 3\}$, with $n \geq 4$ when $g = 3$. In the next result, we obtain the classification of the hypersurfaces of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ that have three distinct principal curvatures.

Theorem 3.6. *Let Σ^3 be a hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with three distinct constant principal curvatures. Then Σ^3 is an open part of the following hypersurfaces:*

a) $\mathbb{S}^1(c_1) \times \mathbb{S}^1(c_2) \times \mathbb{R}$, when $\varepsilon = 1$;

b) $\mathbb{S}^1(c_1) \times \mathbb{H}^1(c_2) \times \mathbb{R}$, when $\varepsilon = -1$,

where $c_1 \neq c_2$, $\dfrac{1}{c_1} + \dfrac{1}{c_2} = \varepsilon$ and the principal curvatures of Σ^3 are given by 0, $\dfrac{c_1}{\sqrt{c_1 + c_2}}$ and $\dfrac{-c_2}{\sqrt{c_1 + c_2}}$.

Proof. Let Σ^3 be a hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with three distinct constant principal curvatures. By Lemma 3.5, $\cos(\theta)$ is constant. Since the principal curvatures are distinct, there is no open Ω where $T \equiv 0$, otherwise Ω is part of a slice $\mathbb{Q}^3_\varepsilon \times \{t_0\}$, $t_0 \in \mathbb{R}$, which is totally geodesic (see Proposition 3.3 and Theorem 3.4). Therefore by (3.5), by continuity, T is a principal direction, with principal curvature associated equal 0.

Suppose that $\cos(\theta) \neq 0$ on Σ. Then, from [8, Theorem 4.1], it follows that $\varepsilon = -1$ and Σ is locally parametrized by $f(p, s) = \tilde{h}_s(p) + Bs\partial_t$, for some $B \in \mathbb{R}$, $B > 0$, where $\tilde{h}_s$ is a family of horospheres in $\mathbb{H}^3$. Moreover, the principal curvature associated to the field T is equal to 0, and the other two principal curvatures are both equal, depending on the choice of orientation, to $\dfrac{B}{\sqrt{1 + B^2}}$ or $-\dfrac{B}{\sqrt{1 + B^2}}$. Thus, we have a contradiction with the assumption that the three principal curvatures are distinct.

Therefore $\cos(\theta) = 0$ on Σ. Then Σ^3 is an open part of a vertical cylinder over an isoparametric surface in $\mathbb{Q}^3_\varepsilon$ with two distinct non-zero constant principal curvatures. In this case, the classification follows from the Theorem 1.3. □

Using the classification of the hypersurfaces in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ with constant principal curvatures obtained by Chaves and Santos in [8], we prove Corollary 3.7.

Corollary 3.7. *Let Σ^3 be a hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with constant principal curvatures. Then Σ^3 is isoparametric.*

Proof. In [8, Theorem 6.1], Chaves and Santos classified the hypersurfaces with constant principal curvatures when $g \in \{1, 2\}$ and $n \geq 2$. In both cases, $\cos(\theta)$ is constant. Moreover, if $g = 3$, i.e., all the principal curvatures are distinct, by Lemma 3.5, we obtain that $\cos(\theta)$ is also constant.

Now, if $T \equiv 0$, which is the case when $g = 1$, Σ^3 is an open part of a slice. Once the normal vector field of a slice is the geodesic vector field ∂_t, we conclude that the parallel hypersurfaces to a slice are slices. Therefore Σ^3 is isoparametric, since each slice is totally geodesic. If $T \neq 0$, since $\cos(\theta)$ is constant, it follows from [8, Corollary 3.4], that Σ^3 is also isoparametric. Thus, we conclude the proof of the corollary. □

In the last result of this chapter, we use the Jacobi field theory to obtain a necessary and sufficient condition for an isoparametric hypersurface of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ to have constant principal curvatures. In [8], Chaves and Santos showed that if Σ is an isoparametric hypersurface in $\mathbb{Q}^n_\varepsilon \times \mathbb{R}$ having T as a principal direction, then Σ has constant principal curvatures if and only if $||T||$ is constant. Our result improves, at least for $n = 3$, their result, since we do not use the assumption of T being a principal direction.

Theorem 3.8. *Let Σ be an isoparametric hypersurface of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$. Then Σ has constant principal curvatures if and only if θ is constant.*

Proof. Let Σ be an isoparametric hypersurface in $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ with unit normal N, and let $\{e_1, e_2, e_3\}$ an orthonormal frame of principal directions with corresponding principal curvatures μ_1, μ_2 and μ_3, respectively.

In order to prove Theorem 3.8, it is enough to show that the principal curvatures of Σ are constant if and only if the function $||T||$ is constant. In fact, as $\partial_t = T + \cos(\theta)N$ is a unit vector field, it follows that $1 = ||T||^2 + \cos^2(\theta)$, and hence, θ is constant if only if $||T||$ is constant.

The family of hypersurfaces parallel to Σ in the direction of N is given by (1.1) and these hypersurfaces are denoted by Σ_r. Given $p \in \Sigma$, let γ_p be a geodesic of $\mathbb{Q}_\varepsilon^3 \times \mathbb{R}$ with $\gamma_p(0) = p$ and $\gamma_p'(0) = N(p)$. By definition of Σ_r, the normal to Σ_r at $\gamma_p(r)$ is given by $\dot\gamma_p(r)$.

For each $i \in \{1,2,3\}$, consider the Jacobi field $\xi_i = DP_{e_i}$ along γ_p, where P_{e_i} is the parallel transport of e_i along γ_p with $P_{e_i}(0) = e_i$, given by the following initial conditions

$$\begin{cases} \xi_i(0) = e_i, \\ \xi_i'(0) = -Ae_i = -\mu_i e_i. \end{cases}$$

Denoting $P_{e_i}(r) = e_i(r)$, we can write

$$D(r)(e_i(r)) = \sum_{j=1}^{3} d_{ij}(r) e_j(r),$$

where $d_{ij}(r)$ are the elements of the matrix that represents $D(r)$, and from this, we see that

$$\xi_i(r) = \sum_{j=1}^{3} d_{ij}(r) e_j(r). \tag{3.29}$$

Remember that, since ξ_i is a Jacobi field along γ_p, then ξ_i satisfies the Jacobi equation $\xi_i'' + \widetilde{R}(\xi_i, \dot\gamma_p)\dot\gamma_p = 0$.

As $e_i(r)$ is parallel for all i, taking the derivatives in (3.29), we have

$$\xi_i''(r) = \sum_{j=1}^{3} d_{ij}''(r) e_j(r). \tag{3.30}$$

Therefore, since

$$\widetilde{R}(\xi_i, \dot\gamma_p)\dot\gamma_p = \sum_{l=1}^{3} \langle \widetilde{R}(\xi_i, \dot\gamma_p)\dot\gamma_p, e_l\rangle e_l = \sum_{j=1}^{3}\sum_{l=1}^{3} d_{ij} \langle \widetilde{R}(e_j, \dot\gamma_p)\dot\gamma_p, e_l\rangle e_l,$$

it follows, from (3.30) and the Jacobi equation, that

$$d_{il}'' + \sum_{j=1}^{3} d_{ij} \langle \widetilde{R}(e_j, \dot\gamma_p)\dot\gamma_p, e_l\rangle = 0. \tag{3.31}$$

Suppose, without loss of generality, that γ_p is parametrized by arc length and is given in the form

$$\gamma_p(r) = \Big(\gamma_{\mathbb{Q}_\varepsilon^3}(||T||r), \gamma_{\mathbb{R}}(\cos(\theta)r)\Big),$$

where $\gamma_{\mathbb{Q}^3_\varepsilon}$ and $\gamma_{\mathbb{R}}$ denote the component of γ_p in $\mathbb{Q}^3_\varepsilon$ and $\mathbb{R}$, respectively. Note that

$$\dot{\gamma}_p(r) = ||T||\dot{\gamma}_{\mathbb{Q}^3_\varepsilon} + \cos(\theta)\partial_r.$$

Writing $e_i = e_i^{\mathbb{Q}^3_\varepsilon} + \langle e_i, \partial_t\rangle\partial_t$, where $e_i^{\mathbb{Q}^3_\varepsilon}$ is the component of e_i in $\mathbb{Q}^3_\varepsilon$, we have that

$$\begin{aligned}\langle \widetilde{R}(e_j, \dot{\gamma}_p)\dot{\gamma}_p, e_l\rangle &= \langle R^{\mathbb{Q}^3_\varepsilon}(e_j^{\mathbb{Q}^3_\varepsilon}, ||T||\dot{\gamma}_{\mathbb{Q}^3_\varepsilon})||T||\dot{\gamma}_{\mathbb{Q}^3_\varepsilon}, e_l^{\mathbb{Q}^3_\varepsilon}\rangle \\ &= \varepsilon\Big(||T||^2\langle e_j^{\mathbb{Q}^3_\varepsilon}, e_l^{\mathbb{Q}^3_\varepsilon}\rangle - ||T||^2\langle e_j^{Q}, \dot{\gamma}_{\mathbb{Q}^3_\varepsilon}\rangle\langle e_l^{\mathbb{Q}^3_\varepsilon}, \dot{\gamma}_{\mathbb{Q}^3_\varepsilon}\rangle\Big),\end{aligned}$$

once $\mathbb{Q}^3_\varepsilon$ has constant sectional curvature ε. Furthermore, since

$$0 = \langle e_j, N\rangle = \langle e_j, \dot{\gamma}\rangle = \langle e_j^{\mathbb{Q}^3_\varepsilon}, ||T||\dot{\gamma}_{\mathbb{Q}^3_\varepsilon}\rangle + \langle e_j, \partial_t\rangle\cos(\theta),$$

and

$$\delta_{jl} = \langle e_j^{\mathbb{Q}^3_\varepsilon}, e_l^{\mathbb{Q}^3_\varepsilon}\rangle + \langle e_j, \partial_t\rangle\langle e_l, \partial_t\rangle,$$

we get

$$\langle e_j^{\mathbb{Q}^3_\varepsilon}, \dot{\gamma}_{\mathbb{Q}^3_\varepsilon}\rangle = -\frac{T_j\cos(\theta)}{||T||} \quad \text{and} \quad \langle e_j^{\mathbb{Q}^3_\varepsilon}, e_l^{\mathbb{Q}^3_\varepsilon}\rangle = \delta_{jl} - T_jT_l,$$

with $T_j = \langle e_j, T\rangle$. Thus, we obtain

$$\begin{aligned}\langle \widetilde{R}(e_j, \dot{\gamma}_p)\dot{\gamma}_p, e_l\rangle &= \varepsilon||T||^2\Big(\delta_{jl} - T_jT_l - \frac{T_jT_l\cos^2(\theta)}{||T||^2}\Big) \\ &= \varepsilon||T||^2\Big(\delta_{jl} - \frac{T_jT_l}{||T||^2}(||T||^2 + \cos^2(\theta))\Big) \\ &= \varepsilon(||T||^2\delta_{jl} - T_jT_l),\end{aligned}$$

and replacing this value in (3.31), we conclude that

$$d''_{il} + \varepsilon\sum_{j=1}^{3} d_{ij}\Big[||T||^2\delta_{jl} - T_jT_l\Big] = 0.$$

Hence, the elements d_{ij} are solutions of the following linear system

$$\begin{pmatrix} d''_{11} & d''_{12} & d''_{13} \\ d''_{21} & d''_{22} & d''_{23} \\ d''_{31} & d''_{32} & d''_{33}\end{pmatrix} = -\varepsilon\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33}\end{pmatrix}\begin{pmatrix} ||T||^2 - T_1^2 & T_1T_2 & -T_1T_3 \\ -T_2T1 & ||T||^2 - T_2^2 & -T_2T_3 \\ -T_3T_1 & -T_3T_2 & ||T||^2 - T_3^2\end{pmatrix}.$$

Since $\xi_j(0) = e_j$, we have

$$\begin{aligned} &d_{11}(0) = 1, \quad d_{12}(0) = 0, \quad d_{13}(0) = 0, \\ &d_{21}(0) = 0, \quad d_{22}(0) = 1, \quad d_{23}(0) = 0, \\ &d_{31}(0) = 0, \quad d_{32}(0) = 0, \quad d_{33}(0) = 1.\end{aligned}$$

Thus,

$$\begin{pmatrix} d''_{11}(0) & d''_{12}(0) & d''_{13}(0) \\ d''_{21}(0) & d''_{22}(0) & d''_{23}(0) \\ d''_{31}(0) & d''_{32}(0) & d''_{33}(0)\end{pmatrix} = -\varepsilon\begin{pmatrix} ||T||^2 - T_1^2 & T_1T_2 & -T_1T_3 \\ -T_2T1 & ||T||^2 - T_2^2 & -T_2T_3 \\ -T_3T_1 & -T_3T_2 & ||T||^2 - T_3^2\end{pmatrix}. \tag{3.32}$$

By (1.4), we can consider the function

$$f(r) = \frac{d}{dr}(\det D(r)) + 3h(r)\det D(r),$$

which vanishes identically. Note that

$$f'(r) = \frac{d^2}{dr^2}(\det D(r)) + 3h'(r)\det D(r) + 3h(r)\frac{d}{dr}(\det D(r)).$$

In what follows, we are going to obtain explicitly the formulas of $\frac{d}{dt}(\det D(t)) \mid_{t=0}$ and $\frac{d^2}{dt^2}(\det D(t)) \mid_{t=0}$. First, taking into account that $\xi_i = \widetilde{\nabla}_N \xi_i = -\mu_i e_i$, we get

$$\begin{array}{lll} d'_{11}(0) = -\mu_1, & d'_{12}(0) = 0, & d'_{13}(0) = 0, \\ d'_{21}(0) = 0, & d'_{22}(0) = -\mu_2, & d'_{23}(0) = 0, \\ d'_{31}(0) = 0, & d'_{32}(0) = 0, & d'_{33}(0) = -\mu_3, \end{array}$$

that is, $D'(0) = -A$. By Jacobi formula, we have

$$\frac{d}{dr}(\det D(r)) = \det D(r) tr(D(r)^{-1}D'(r)). \tag{3.33}$$

Hence, at $r = 0$, we get

$$\frac{d}{dr}(\det D(r)) \mid_{r=0} = -trA = -3h(0),$$

where $h(0)$ is the mean curvature of Σ. Now, taking the derivative in (3.33), it follows that

$$\begin{aligned} \frac{d^2}{dr^2}(\det D(r)) = & \frac{d}{dr}(\det D(r)) tr(D(r)^{-1}D'(r)) \\ & + \det D(r) tr\Big((D(r)^{-1})'D'(r) + D(r)^{-1}D''(t)\Big). \end{aligned}$$

Since $D(r)^{-1}D(r) = Id$, we obtain

$$(D(r)^{-1})' = -D(r)^{-1}D'(r)D(r)^{-1}.$$

This means that $(D(0)^{-1})' = A$. Therefore, taking the trace in (3.32), we get

$$\begin{aligned} \frac{d^2}{dt^2}(\det D(t)) \mid_{t=0} &= (tr(A))^2 - tr(A^2) + tr(D''(0)) \\ &= 9h(0)^2 - tr(A^2) - 2\varepsilon||T||^2. \end{aligned}$$

In this way, at $r = 0$, we have that

$$f'(0) = -tr(A^2) + 3h'(0) - 2\varepsilon||T||^2.$$

As $f \equiv 0$, so is its derivative. This shows that

$$tr(A^2) = 3h'(0) - 2\varepsilon||T||^2. \tag{3.34}$$

By assumption, Σ is isoparametric and hence, $h'(0)$ is constant throughout Σ. Therefore, if Σ has constant principal curvatures μ_1, μ_2 and μ_3, then $tr(A^2) = \mu_1^2 + \mu_2^2 + \mu_3^2$ is constant, and hence, $||T||$ is constant.

Conversely, if $||T||$ is constant, from (3.34), we have that $tr(A^2)$ is constant. It easy to see that the characteristic polynomial Q_A of A is given by

$$Q_A(\lambda) = -\lambda^3 + 3h(0)\lambda^2 - (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)\lambda + \det A.$$

Now, observe that

$$9h^2(0) = \mu_1^2 + \mu_2^2 + \mu_3^2 + 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3).$$

Thus, we have

$$9h^2(0) - tr(A^2) = 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3),$$

which implies

$$Q_A(\lambda) = -\lambda^3 + 3h(0)\lambda^2 - \frac{9h^2(0) - tr(A^2)}{2}\lambda + \det A.$$

Notice that if $T \equiv 0$, then Σ^3 is an open part of a slice $\mathbb{Q}_\varepsilon^3 \times \{t_0\}$, $t_0 \in \mathbb{R}$, which is totally geodesic, and hence, has constant principal curvatures. Thus, suppose that $T \neq 0$. Since $||T||$ is constant it follows that $\cos(\theta)$ is constant, which implies that T is a principal direction with correspondent principal curvature 0. This means that $\det A = 0$. Therefore, since Σ is isoparametric, then $h(0)$ is constant, and thus we conclude that the coefficients of characteristic polynomial Q_A are real constants, and hence, Σ has constant principal curvatures. □

Chapter 4

Solutions to the mean curvature flow

In this chapter, we describe the evolution by the mean curvature flow (MCF) of isoparametric hypersurfaces in product manifolds of dimension 4. We show that the evolution of isoparametric hypersurfaces of Riemannian manifolds by the mean curvature flow is given by a reparametrization of the flow by parallel hypersurfaces in a short time, as long as the uniqueness of the mean curvature flow holds for the initial data and the corresponding ambient space. Such a result is given in a general sense, not necessarily restricted to the ambient spaces considered in this work, which has its own interest. Through this result, we describe the evolution of the hypersurfaces classified in Chapters 2 and 3. We also describe the evolutions of isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, classified by Urbano (2019) and Dong Gao, Hui Ma and Zeke Yao (2022), respectively, and of isoparametric hypersurfaces in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with g distinct constant principal curvatures, $g \in \{1, 2\}$, classified by Chaves and Santos (2019).

Part of the results of this chapter composes a joint work with João Paulo do Santos and Felipe Guimarães [22], entitled "Isoparametric hypersurfaces of Riemannian manifolds as initial data for the mean curvature flow".

4.1 An isoparametric hypersurface as initial data for MCF

Before stating and proving the main results and their applications, let us present some short background content on the uniqueness of the mean curvature flow.

Let Σ^n be a 2-sided hypersurface of a Riemannian manifold $\widetilde{M}^{n+1}$. The family of hypersurfaces $F : \Sigma^n \times I \to \widetilde{M}^{n+1}$, $0 \in I \subset \mathbb{R}$, is a solution to the *mean curvature flow* (abbreviated as MCF) with initial data Σ, if

$$\begin{cases} \partial_t F(p,t) = h(p,t)N(p,t), & (p,t) \in \Sigma^n \times I; \\ F(p,0) = p \in \Sigma. \end{cases} \tag{4.1}$$

Here, $h(p,t)$ is the mean curvature and $N(p,t)$ is a unit normal vector field of the hypersurface $\Sigma_t := F(\Sigma, t)$.

Remark 5. *Observe that it is enough to ask for 2-sided hypersurface, since we only need a well-defined normal vector field. In case the manifold $\widetilde{M}^{n+1}$ is orientable, such condition is equivalent to Σ^n being an orientable hypersurface.*

In order to characterize the evolution by the mean curvature flow of isoparametric hypersurfaces, we will need to understand the uniqueness of the MCF. It is well-known that it holds when the initial data is compact (as can be seen, for example, in [24, Lemma 3.2]). As for the noncompact case, we will use the following result, which will be stated here in the hypersurface case:

Theorem 4.1 (Chen-Yin [9]). *Let $(\widetilde{M}^{n+1}, \widetilde{g})$ be a complete Riemannian manifold of dimension $n+1$ such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant, i.e., there are constants $\widetilde{C}$ and $\widetilde{\delta}$ such that*

$$(|\widetilde{R}| + |\widetilde{\nabla}\widetilde{R}| + |\widetilde{\nabla}^2\widetilde{R}|)(p) \leq \widetilde{C}, \;\; inj(\widetilde{M}^{n+1}, p) > \widetilde{\delta},$$

for all $p \in \widetilde{M}^{n+1}$. Let $F_0 : \Sigma^n \to \widetilde{M}^{n+1}$ be an isometrically immersed Riemannian manifold with bounded second fundamental form in $\widetilde{M}^{n+1}$. Suppose F_1 and F_2 are two solutions to the mean curvature flow on $\Sigma^n \times [0,T]$ with the same F_0 as initial data and with bounded second fundamental forms on $[0,T]$. Then $F_1 = F_2$ for all $(p,t) \in \Sigma^n \times [0,T]$.

In what follows, we will study the properties of hypersurfaces that have a particular solution for MCF:

Definition 4.1. *Let $F : \Sigma^n \times [0,T) \to \widetilde{M}^{n+1}$ be a solution to the MCF in $\widetilde{M}^{n+1}$ with initial data Σ^n. We say that this solution is a reparametrization of the parallel flow (abbreviated as RPF) in $[0,\delta)$, $0 < \delta \leq T$, with parameter $\epsilon : [0,\delta) \to \mathbb{R}$, $\epsilon(0) = 0$ if*

$$F(p,t) = \exp_p(\epsilon(t)N(p)), \tag{4.2}$$

for all $t \in [0,\delta)$, where $\exp_p : T_p\widetilde{M} \to \widetilde{M}$ denotes the exponential map of $\widetilde{M}$ at $p \in \Sigma$, and N is a unit normal vector field of the hypersurface Σ.

Our first result is given by Lemma below, which provides a necessary condition for a solution to the MCF to be an reparametrization of the parallel flow. Lemma 4.2 has its own interest since it extends to Riemannian manifolds the results of [37] for space forms. Furthermore, the first part of Lemma 4.2 coincides with Proposition 1 [28] when the MCF is considered, complementing it with the second part, since it provides the corresponding ordinary differential equation concretely in terms of the endomorphism D presented in Section 1.2. For completeness, we will present its entire proof.

Lemma 4.2. *Let Σ^n be a 2−sided hypersurface of $\widetilde{M}^{n+1}$, such that Σ^n is the initial data of a solution $F : \Sigma \times [0,T) \to \widetilde{M}^{n+1}$ for the MCF. If F restricted to $\Sigma \times [0,\delta)$ for some $0 < \delta \leq T$ is a RPF with parameter $\epsilon : [0,\delta) \to \mathbb{R}$, then Σ is an isoparametric hypersurface of $\widetilde{M}^{n+1}$. Moreover, ϵ satisfies the ODE*

$$\epsilon'(t) = -\frac{(\det D)'}{n \det D}\left(\epsilon(t)\right), \tag{4.3}$$

where D is a solution of (1.2), and the right-hand side of (4.3) is independent of $p \in \Sigma$.

Proof. By hypothesis we have that $F(p,t) = \exp_p(\epsilon(t)N(p))$ satisfies

$$\partial_t F(p,t) = h(p,t)\widetilde{N}(p,t),$$

where $\widetilde{N}(\cdot,t)$ and $h(\cdot,t)$ stand for the normal unit vector field and the mean curvature of the hypersurface $F(\cdot,t)$, for $t \in [0,\delta)$, respectively. On the one hand, we have

$$\partial_t F(p,t) = \epsilon'(t)\left(d\exp_p\right)_{\epsilon(t)N(p)} N(p).$$

On the other hand, it follows from Gauss's lemma that

$$\left(d\exp_p\right)_{\epsilon(t)N(p)} N(p) = \widetilde{N}(p,t)$$

for any $p \in \Sigma^n$ and $t \in [0,\delta)$. Thus, $\epsilon'(t) = h(p,t)$ and the hypersurface $\Sigma_t = F(\Sigma, t)$ has constant mean curvature $\epsilon'(t)$. In particular, Σ^n is an isoparametric hypersurface.

Since $\epsilon'(t)$ is the mean curvature of the hypersurface $\Sigma_t = F(\Sigma,t)$ and the MCF is RPF with parameter ϵ, we have from (1.4) that

$$\epsilon'(t) = h(\epsilon(t)) = -\frac{(\det D)'}{n \det D}(\epsilon(t)),$$

where D is the solution of (1.2), which will be independent of the choice of $p \in \Sigma$, once Σ is isoparametric. □

Lemma 4.2 says that if a solution of the MCF is given by an RPF, then the initial hypersurface of this solution must be isoparametric. We will make use of it and Theorem 4.1 to obtain the Theorem 4.3 below, which supplies the characterization of the MCF when the initial data is an isoparametric hypersurface, for a regular enough ambient space (in the sense of Theorem 4.1). For this, we will need the following formula, which is a direct consequence of the Riccati equation (see [33, Section 3]):

$$h'(r) = \widetilde{\mathrm{Ric}}(\dot\gamma(r), \dot\gamma(r)) + |A_{\dot\gamma(r)}|^2, \tag{4.4}$$

where $\widetilde{\mathrm{Ric}}$ is the Ricci tensor of $\widetilde{M}$.

Theorem 4.3. *Let $\widetilde{M}^{n+1}$ be a complete Riemannian manifold such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant. Let Σ^n be a hypersurface of $\widetilde{M}^{n+1}$ such that the solution $F : \Sigma^n \times [0,T) \to \widetilde{M}^{n+1}$ of the MCF with initial data Σ^n has bounded second fundamental form on $[0, T_-]$ for all $T_- < T$. Then, Σ^n is isoparametric if and only if F is the flow by parallels for some $\delta_0 \leq T$. Moreover, suppose that $[0,\delta)$ is the maximal interval where F is a reparametrization of the parallel flow. If $\delta < T$ then $F(.,\delta)$ is a hypersurface that is not isoparametric.*

Proof. Let Σ be an isoparametric hypersurface of $\widetilde{M}^{n+1}$. Then the mean curvatures of its nearby parallel hypersurfaces depend only on the parallel displacement $r \geq 0$. In this case, if D is a solution of (1.2), then the right-hand side of (1.4), which provides the mean curvature of a parallel hypersurface of Σ, depends only on r. Therefore the ODE

$$\epsilon'(t) = -\frac{(\det D)'}{n \det D}\left(\epsilon(t)\right),$$

is well defined in a neighborhood of $r = 0$. Let ϵ be a solution of such ODE, with $\epsilon(0) = 0$, defined in $[0,\delta_0)$ for some $\delta_0 > 0$ such that $|\epsilon(t))| < \widetilde{\delta}$, for all $t \in [0,\delta_0)$ where $\widetilde{\delta}$ is the uniform bound for the injectivity radius of $\widetilde{M}$. Thus, proceeding as in the proof of Lemma 4.2, the family $\overline{F} : \Sigma^n \times [0,\delta) \to \widetilde{M}^{n+1}$ given by $\overline{F}(p,t) = \exp_{f(p)}(\epsilon(t)N(p))$, where N is a unit normal vector field of Σ^n, whose direction is given by the vector mean curvature, is a solution of the MCF, with initial data given by Σ^n. Since the ambient space satisfies the conditions of Theorem 4.1 and Σ^n is isoparametric, it follows by equation (4.4) that the second fundamental form is bounded for all $t \in [0,\delta)$. Consequently, it follows from Theorem 4.1 that $F = \overline{F}$ in $[0,\delta)$. The converse follows from Lemma 4.2.

Let $[0,\delta)$ be the maximal interval where the solution of the MCF $F : \Sigma^n \times [0,T) \to \widetilde{M}^{n+1}$ is RPF. When $\delta < T$, we firstly observe that $F(.,\delta)$ is a regular hypersurface, since the F

is defined at $t = \delta$. Secondly, we claim that $F(\Sigma, \delta)$ is not a isoparametric hypersurface. In fact, suppose by contradiction that $F(\Sigma, \delta)$ is isoparametric. Then we can consider it as an initial data for the mean curvature flow and, by the uniqueness, we will then extend F as RPF beyond δ, which contradicts the maximality of $[0, \delta)$. □

Remark 6. *Theorem 4.3 provides a refinement of Theorem 2.2 of [37] when the ambient space is a space form. It assures that the unique solution for the MCF in a short time, with initial data being an isoparametric hypersurface, is given by the family of parallel hypersurfaces provided by the parameter ϵ arising as the unique solution of the ordinary differential equation (4.3). Similarly, for Riemannian manifolds where being isoparametric is equivalent to having constant principal curvatures as ambient spaces and when the MCF is considered, the unique solution by parallel hypersurfaces given in Corollary 2 of [28], will be, in fact, the unique solution to the MCF, as long as the conditions of Theorem 4.1 are satisfied (recall that the author in [28] defines isoparametric hypersurfaces as those with constant principal curvatures).*

4.2 Applications: evolution of isoparametric hypersurfaces in product spaces

In this section, we will study the evolution of isoparametric hypersurfaces in the product spaces $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$ and $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$. We highlight that these ambient spaces are homogeneous manifolds and products of spaces of constant curvature. Therefore, they satisfy the conditions of Theorem 4.1.

To study the evolution of such hypersurfaces, we will need the parallel surfaces and curves given in the space forms. By (1.1), if Σ is surface in $\mathbb{Q}^3_\varepsilon$ or a curve in $\mathbb{Q}^2_{c_i}$, then the parallels to Σ are given by

$$\Phi_r(p) = C_\varepsilon(r)p + S_\varepsilon(r)N(p), \tag{4.5}$$

where $p \in \Sigma$, $N(p)$ is the unit normal to Σ at p, and the functions $S_\varepsilon(r)$ and $C_\varepsilon(r)$ are given by

$$S_\varepsilon(r) = \begin{cases} r & \text{if } \varepsilon = 0, \\ \sinh(r) & \text{if } \varepsilon = -1, \\ \sin(r) & \text{if } \varepsilon = 1, \end{cases} \qquad C_\varepsilon(r) = \begin{cases} 1 & \text{if } \varepsilon = 0, \\ \cosh(r) & \text{if } \varepsilon = -1, \\ \cos(r) & \text{if } \varepsilon = 1. \end{cases} \tag{4.6}$$

Moreover, if μ denotes either a principal curvature or the curvature of Σ (the last case when Σ is a curve), then a principal curvature or the curvature of the parallels to Σ are given by

$$\mu_r = \frac{\varepsilon S_\varepsilon(r) + \mu C_\varepsilon(r)}{C_\varepsilon(r) - \mu S_\varepsilon(r)}, \tag{4.7}$$

see [15].

4.2.1 On hypersurfaces of $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$

In this subsection, we will study the evolution of isoparametric hypersurfaces by the MCF in the following ambient spaces: $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{R}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$.

We will start by considering the hypersurfaces classified by Theorem 2.2, i.e., when the ambient spaces are given by $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, for $c_i \in \{-1, 0, 1\}$ and $c_1 \neq c_2$.

i) On hypersurfaces of $\mathbb{S}^2 \times \mathbb{R}^2$: The isoparametric hypersurfaces with constant principal curvatures in $\mathbb{S}^2 \times \mathbb{R}^2$ are of the form $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^1(b)$ (for $b \in \mathbb{R}^+$) or $\mathbb{S}^1(a) \times \mathbb{R}^2$ (for $a \in (0, 1]$), where $\mathbb{S}^1(r)$ is the circle with radius $r \in \mathbb{R}$ (see also [25]). As was pointed out previously, these hypersurfaces are characterized by a constant function C defined as $C = \langle PN, N\rangle$, where P is a product structure in $\mathbb{S}^2 \times \mathbb{R}^2$ defined by $P(v_1, v_2) = (v_1, -v_2)$ and N is the unit normal. On these hypersurfaces, the function C assumes the values 1 or -1.

Let us take a look at each case separately.

First, for $\mathbb{S}^2 \times \mathbb{R}$, we have $C = -1$ and the unit normal is of the form $N = (0, N_2)$, where N_2 is the component of N in $\mathbb{R}^2$ with $|N_2|^2 = \frac{1-C}{2}$. Given $v = (v_1, v_2) \in T(\mathbb{S}^2 \times \mathbb{R})$, we have

$$A_N(v) = -\widetilde{\nabla}_v N = -\nabla^{\mathbb{R}^2}_{v_2} N_2 = -dN_2(v_2) = 0,$$

where $\widetilde{\nabla}$ is the Levi Civita connection of $\mathbb{S}^2 \times \mathbb{R}^2$. Then, we have that $\mathbb{S}^2 \times \mathbb{R}$ is totally geodesic and $h = 0$. Since we are in the conditions of Theorem 4.1, the flow is stationary, i.e., $\epsilon(t) = 0$ for all t.

For $\mathbb{S}^2 \times \mathbb{S}^1(b)$ (for $b \in \mathbb{R}^+$), we also have $C = -1$ and $N = (0, N_2)$. For any $w = (w_1, w_2) \in T(\mathbb{S}^2 \times \mathbb{S}^1(b))$, we get

$$A_N(w) = -\widetilde{\nabla}_w N = -\nabla^{\mathbb{R}^2}_{w_2} N_2 = -dN_2(w_2) = -\frac{1}{b} w_2.$$

Then, given an orthonormal basis $\{u_1, u_2, u_3\}$ in $\mathbb{S}^2 \times \mathbb{S}^1(b)$, with $u_1, u_2 \in T\mathbb{S}^2$ and $u_3 \in T\mathbb{S}^1(b)$, we have

$$A_N(u_1) = A_N(u_2) = 0 \quad \text{and} \quad A_N(u_3) = -\frac{1}{b} u_3,$$

that is, the principal curvatures of $\mathbb{S}^2 \times \mathbb{S}^1(b)$ are 0, 0 and $-\frac{1}{b}$. Moreover, from (4.5), the displacement of $\mathbb{S}^2 \times \mathbb{S}^1(b)$ in direction N at distance r is given by

$$\begin{aligned} \Phi_r(p, q) &= \exp_{(p,q)} (rN(p, q)) \\ &= \Big(p, q + rN_2(q)\Big) \\ &= \mathbb{S}^2 \times \mathbb{S}^1(b + r), \end{aligned} \tag{4.8}$$

and from (4.7), the principal curvatures of the parallel hypersurface $\mathbb{S}^2 \times \mathbb{S}^1(b+r)$ are given by 0, 0 and $-\dfrac{1}{r + b}$. Thus, its mean curvature is given by $h(r) = -\dfrac{1}{3(r + b)}$, which implies that the MCF with initial data $\mathbb{S}^2 \times \mathbb{S}^1(b)$ is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = -\frac{1}{3(\epsilon(t) + b)}, \tag{4.9}$$

that is, $3(\epsilon(t) + b)^2 = K_1 - 2t$, where K_1 is a constant. Therefore $\epsilon(t) = \sqrt{b^2 - \dfrac{2t}{3}} - b$.

Finally, for $\mathbb{S}^1(a) \times \mathbb{R}^2$ with $a \in (0, 1)$, we have $C = 1$ and the unit normal is of the form $N = (N_1, 0)$, where N_1 is the component of N in $\mathbb{S}^2$ with $|N_1|^2 = \frac{1+C}{2}$. For any $u = (u_1, u_2) \in T(\mathbb{S}^1(a) \times \mathbb{R}^2)$, we have

$$A_N(u) = -\widetilde{\nabla}_u N = -\nabla^{\mathbb{S}^2}_{u_1} N_1 = -dN_1(u_1) = \cot(\phi_a) u_1,$$

where $0 < \phi_a < \pi$ and $\widetilde{\nabla}$ is the Levi Civita connection of $\mathbb{S}^2 \times \mathbb{R}^2$. Then, given an orthonormal basis $\{v_1, v_2, v_3\}$ in $\mathbb{S}^1(a) \times \mathbb{R}^2$, with $v_1 \in T\mathbb{S}^1(a)$ and $v_2, v_3 \in T\mathbb{R}^2$, we have

$$A_N(v_1) = \cot(\phi_a) v_1 \quad \text{and} \quad A_N(v_2) = A_N(v_3) = 0,$$

that is, the principal curvatures of $\mathbb{S}^1(a)\times\mathbb{R}^2$ are 0, 0 and $\cot(\phi_a)$. By (4.5), the displacement of $\mathbb{S}^1(a)\times\mathbb{R}^2$ in direction N at distance r is given by

$$\Phi_r(p,q) = \Big((\cos r)p + (\sin r)N_1(p), q\Big),$$

and from (4.7), its principal curvatures are 0, 0 and $\cot(\phi_a - r)$, and hence, its mean curvature is given by $h(r) = \dfrac{\cot(\phi_a - r)}{3}$. Therefore, the MCF with initial data $\mathbb{S}^1(a)\times\mathbb{R}^2$ with $a \in (0,1)$, is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = \frac{\cot(\phi_a - \epsilon(t))}{3}, \tag{4.10}$$

that is, $\cos(\phi_a - \epsilon(t)) = \cos(\phi_a)e^{\frac{t}{3}}$. Note that when $a = 1$, we have that $\mathbb{S}^1\times\mathbb{R}^2$ is totally geodesic, and therefore the MCF with initial data $\mathbb{S}^1\times\mathbb{R}^2$ is stationary, i.e., $\epsilon(t) = 0$ for all t.

ii) On hypersurfaces of $\mathbb{S}^2\times\mathbb{H}^2$: Isoparametric hypersurfaces with constant principal curvatures in $\mathbb{S}^2\times\mathbb{H}^2$ are of the form $\mathbb{S}^1(a)\times\mathbb{H}^2$ (for $a\in(0,1]$) or $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$ where $\mathbb{S}^1(a)$ is the circle with radius a and $\mathcal{C}^1(\kappa_j)$ is a complete curve of constant geodesic curvature in $\mathbb{H}^2$, that is, besides the geodesics, the complete curve $\mathcal{C}^1(\kappa_j)\subset\mathbb{H}^2$ is either a circle, a horocycle or a hypercycle. As in the previous case, these hypersurfaces are characterized by a constant function C defined as $C = \langle PN, N\rangle$, where in this case P is a product structure in $\mathbb{S}^2\times\mathbb{H}^2$ also defined by $P(v_1,v_2) = (v_1,-v_2)$ and N is the unit normal. Moreover, on these hypersurfaces, the function C also assumes the values 1 or -1.

The solution of the MCF with initial data $\mathbb{S}^1(a)\times\mathbb{H}^2$ is analogous to the case i) with the same function $\epsilon(t)$, so here we will present the ODE (4.3) of the MCF with initial data $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$.

In this case, we have $C = -1$ and the unit normal is of the form $N = (0, N_2)$, where N_2 is the component of N in $\mathbb{H}^2$ with $|N_2|^2 = \frac{1-C}{2}$. For any $v = (v_1,v_2)\in T(\mathbb{S}^2\times\mathcal{C}^1(\kappa_j))$, we have

$$A_N(v) = -\widetilde{\nabla}_v N = -\nabla^{\mathbb{H}^2}_{v_2} N_2 = -dN_2(v_2) = \kappa_j v_2,$$

where $\widetilde{\nabla}$ is the Levi Civita connection of $\mathbb{S}^2\times\mathbb{H}^2$. Thus, given an orthonormal basis $\{u_1,u_2,u_3\}$ in $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$, with $u_1,u_2\in T\mathbb{S}^2$ and $u_3\in T\mathcal{C}^1(\kappa_j)$, we have

$$A_N(u_1) = A_N(u_2) = 0 \quad \text{and} \quad A_N(u_3) = \kappa_j u_3,$$

that is, the principal curvatures of $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$ are 0, 0 and κ_j, which implies that the mean curvature is $h_{\kappa_j} = \dfrac{\kappa_j}{3}$. Moreover, using (4.5), the displacement of $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$ in direction N at distance r is given by

$$\begin{aligned}\Phi_r(p,q) &= \exp_{(p,q)}(rN(p,q))\\ &= \Big(p, (\cosh r)q + (\sinh r)N_2(q)\Big).\end{aligned}$$

Observe that when $\mathcal{C}^1(\kappa_j)$ is a geodesic, we have $h_{\kappa_j} = 0$, which implies that the flow is stationary, i.e., $\epsilon(t) = 0$ for all t.

If $\mathcal{C}^1(\kappa_j)$ is a horocycle, we have $\kappa_j = 1$, which implies that the principal curvatures of $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$ are 0, 0 and 1. From (4.7), the principal curvatures of the parallel hypersurfaces to $\mathbb{S}^2\times\mathcal{C}^1(\kappa_j)$ at distance r are also given by 0, 0 and 1, and hence, its mean curvature is

$h(r) = \dfrac{1}{3}$. Therefore the MCF with initial data $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = \frac{1}{3}, \tag{4.11}$$

that is, $\epsilon(t) = \dfrac{1}{3}t$.

If $\mathcal{C}^1(\kappa_j)$ is a hypercycle, it follows that the curvature κ_j is given by $\kappa_j = \tanh(\phi_\alpha)$, see [36]. Thus, it follows that the principal curvatures of $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ are 0, 0 and $\tanh(\phi_\alpha)$, and from (4.7), the principal curvatures of the parallel hypersurfaces to $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ at distance r are given by 0, 0 and $\tanh(\phi_\alpha - r)$, and hence, its mean curvature is given by $h(r) = \dfrac{\tanh(\phi_\alpha - r)}{3}$. Therefore, the MCF with initial data $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ with $\mathcal{C}^1(\kappa_j)$ being a hypercycle, is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = \frac{\tanh(\phi_\alpha - \epsilon(t))}{3}, \tag{4.12}$$

that is, $\sinh(\phi_\alpha - \epsilon(t)) = \sinh(\phi_\alpha)e^{-\frac{t}{3}}$.

Finally, if $\mathcal{C}^1(\kappa_j)$ is a circle, we have that the curvature κ_j is given by $\kappa_j = \coth(\phi_\alpha)$, see [36]. Thus, it follows that the principal curvatures of $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ are 0, 0 and $\coth(\phi_\alpha)$. From (4.7), the principal curvatures of the parallel hypersurfaces to $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ at distance r are given by 0, 0 and $\coth(\phi_\alpha - r)$, which implies that its mean curvature is given by $h(r) = \dfrac{\coth(\phi_\alpha - r)}{3}$. Therefore, the MCF with initial data $\mathbb{S}^2 \times \mathcal{C}^1(\kappa_j)$ with $\mathcal{C}^1(\kappa_j)$ being a circle, is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = \frac{\coth(\phi_\alpha - \epsilon(t))}{3}, \tag{4.13}$$

that is, $\cosh(\phi_\alpha - \epsilon(t)) = \cosh(\phi_\alpha)e^{-\frac{t}{3}}$.

iii) On hypersurfaces of $\mathbb{H}^2 \times \mathbb{R}^2$: In this case, if Σ is an isoparametric hypersurface with constant principal curvatures in $\mathbb{H}^2 \times \mathbb{R}^2$, then it follows that Σ is an open part of one of the following hypersurfaces:

a) $\mathcal{C}^1(\kappa_j) \times \mathbb{R}^2$, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{S}^1(b)$ (for $b \in \mathbb{R}^+$), where $\mathbb{S}^1(b)$ is a circle with radius b in $\mathbb{R}^2$ and $\mathcal{C}^1(\kappa_j)$ is a complete curve with constant geodesic curvature κ_j in $\mathbb{H}^2$.

b) $\Psi(\mathbb{R}^3) \subset \mathbb{H}^2 \times \mathbb{R}^2$, where $\Psi : \mathbb{R}^3 \to \mathbb{H}^2 \times \mathbb{R}^2$ is an immersion given by

$$\begin{aligned}\Psi(s,u,v) = e^{-b\,s}(\alpha(u),\vec{0}) + \Big(\cosh(-b\,s), 0, \sinh(-b\,s), V_0 s\Big)\\ + \Big(\vec{0}, p_0 + W_0 v\Big),\end{aligned} \tag{4.14}$$

where $\mathbb{H}^2 \subset \mathbb{L}^3$ is given as the standard model of the hyperbolic space in the Lorentz 3-space $\mathbb{L}^3$, the curve α is given by $\alpha(u) = \left(\dfrac{u^2}{2}, u, -\dfrac{u^2}{2}\right)$, $p_0 \in \mathbb{R}^2$, V_0 and W_0 are constant orthogonal vectors in $\mathbb{R}^2$ such that $||W_0|| = 1$ and $b = \sqrt{1 - ||V_0||^2}$, with $b \neq \{1, 0\}$.

Note that the solution of the MCF with the initial data being a hypersurface of item a) is obtained analogously to cases i) and ii).

Now, let us provide the solution of the MCF with the initial data being the hypersurface parametrized by (4.14). Remember that the unit normal vector field N to Σ is given by

$$N = -||V_0||\left(e^{-bs}(\alpha(u),\vec{0}) + \left(\sinh(-bs), 0, \cosh(-bs), \frac{b}{||V_0||^2}V_0\right)\right),$$

and the principal curvatures of Σ are given by

$$\mu_1 = 0, \quad \mu_2 = \sqrt{\frac{1+C}{2}}, \quad \mu_3 = 0,$$

where the function C is defined by (2.2). Thus, the mean curvature of Σ is given by $h = \frac{1}{3}\sqrt{\frac{1+C}{2}}$. Note that, as N has a component in both factors, the displacement of Σ in direction N at distance r is given by

$$\begin{aligned}
\Phi_r(p,q) &= \exp_{(p,q)}(rN(p,q)) \\
&= \Big(\cosh(||N_1||r)p + ||N_1||^{-1}\sinh(||N_1||r)N_1(p), q + rN_2(q) \Big) \\
&= \Big(\cosh(||N_1||r)\big(e^{-bs}\alpha(u) + (\cosh(-bs), 0, \sinh(-bs))\big) \\
&\quad - ||N_1||^{-1}||V_0||\sinh(||N_1||r)\big(e^{-bs}\alpha(u) + (\sinh(-bs), 0, \cosh(-bs))\big), \\
&\quad p_0 + W_0 v + V_0 s - \frac{brV_0}{||V_0||}\Big) \\
&= \Big(e^{-bs}\alpha(u)(\cosh(||V_0||r) - \sinh(||V_0||r)) \\
&\quad + (\cosh(-||V_0||r - bs), 0, \sinh(-||V_0||r - bs)), p_0 + W_0 v + V_0 s - \frac{brV_0}{||V_0||}\Big) \\
&= e^{-||V_0||r - bs}(\alpha(u), \vec{0}) + \Big(\cosh(-||V_0||r - bs), 0, \sinh(-||V_0||r - bs), V_0 s\Big) \\
&\quad \Big(\vec{0}, p_0 + W_0 v - \frac{brV_0}{||V_0||}\Big).
\end{aligned}$$

Furthermore, as we saw in the proof of Theorem 2.2, the mean curvature of the parallel hypersurfaces to Σ is also given by $h(r) = \frac{1}{3}\sqrt{\frac{1+C}{2}}$. In this way, the MCF with initial data Σ is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = \frac{1}{3}\sqrt{\frac{1+C}{2}},$$

that is, $\epsilon(t) = \frac{1}{3}\sqrt{\frac{1+C}{2}}t$.

iv) On hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$: In this case, the isoparametric hypersurfaces were classified in [41]. They are congruent to $\mathbb{S}^1(a) \times \mathbb{S}^2$, $a \in (0,1]$, or to M_t, $t \in (-1,1)$, which is defined as

$$M_t = \{(p,q) \in \mathbb{S}^2 \times \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^3 : \langle p, q\rangle_{\mathbb{R}^3} = t\}.$$

The solution of the MCF with initial data $\mathbb{S}^1(a) \times \mathbb{S}^2$ is essentially the same as in the case i), so here we will present the ODE 4.3 of the MCF with initial data M_t for $t \in (-1,1)$.

In what follows, the products $\langle , \rangle$ are all in $\mathbb{R}^3$. In [41] it was provided the normal vector field

$$N(p,q) = \frac{1}{\sqrt{1-t^2}}(q - tp, p - tq),$$

and the mean curvature $h_t = \frac{\sqrt{2}t}{3\sqrt{1-t^2}}$ of M_t. A straightforward computation, using the coordinates of the normal given above, shows that the displacement of M_t in direction N

at distance r is given by

$$\begin{aligned}\Phi_r(p,q) &= \exp_{(p,q)}(rN(p,q))\\ &= \left(\left(\cos\frac{r}{\sqrt{2}}\right)p + \left(\sin\frac{r}{\sqrt{2}}\right)\frac{q-tp}{\sqrt{1-t^2}}, \left(\cos\frac{r}{\sqrt{2}}\right)q + \left(\sin\frac{r}{\sqrt{2}}\right)\frac{p-tq}{\sqrt{1-t^2}}\right)\\ &= \left(\mathcal{P}_r(p,q), \mathcal{Q}_r(p,q)\right).\end{aligned}$$

Since $\langle \mathcal{P}_r(p,q), \mathcal{Q}_r(p,q)\rangle = t\cos\left(\sqrt{2}r\right) + \sqrt{1-t^2}\sin\left(\sqrt{2}r\right)$, it follows that $\Phi_r(M_t) = M_{\phi(r,t)}$, where $\phi(r,t) = t\cos\left(\sqrt{2}r\right) + \sqrt{1-t^2}\sin\left(\sqrt{2}r\right)$.

The MCF with initial data M_s is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\begin{aligned}\epsilon'(t) &= h_{\phi(\epsilon(t),s)}\\ &= \frac{\sqrt{2}\phi(\epsilon(t),s)}{3\sqrt{1-\phi(\epsilon(t),s)^2}}\\ &= \frac{\sqrt{2}\left(s\cos\left(\sqrt{2}\epsilon(t)\right) + \sqrt{1-s^2}\sin\left(\sqrt{2}\epsilon(t)\right)\right)}{3\sqrt{1-\left(s\cos\left(\sqrt{2}\epsilon(t)\right) + \sqrt{1-s^2}\sin\left(\sqrt{2}\epsilon(t)\right)\right)^2}}.\end{aligned} \tag{4.15}$$

v) On hypersurfaces of $\mathbb{H}^2\times\mathbb{H}^2$: The isoparametric hypersurfaces of $\mathbb{H}^2\times\mathbb{H}^2$ were classified in [19]. Such hypersurfaces are also characterized by a constant function C defined as $C = \langle PN, N\rangle$, where P is a product structure in $\mathbb{H}^2\times\mathbb{H}^2$ defined by $P(v_1,v_2) = (v_1,-v_2)$ and N is the unit normal.

Let us start with M_Γ, where M_Γ is of the form $M_\Gamma = \left\{(x,y)\in\mathbb{H}^2\times\mathbb{H}^2 \,|\, x\in\Gamma, y\in\mathbb{H}^2\right\}$, whith Γ being a curve of $\mathbb{H}^2$ with constant geodesic curvature. On this hypersurface, the function C assumes the value 1 and the unit normal is of the form $N = (N_1, 0)$, where N_1 is the component of N in $\mathbb{H}^2$ with $|N_1|^2 = \frac{1+C}{2}$. Now, if κ_Γ denotes the curvature of Γ in $\mathbb{H}^2$, then M_Γ has principal curvatures κ_Γ, 0 and 0, see Example 3.1 of [19]. Therefore the mean curvature of M_Γ is given by $h = \dfrac{\kappa_\Gamma}{3}$, and the solution of the MCF with initial data M_Γ is also obtained analogously to case ii).

Now we will present the ODE 4.3 of the MCF with initial data $M^c_{1,-1}$ for $c\in(0,1)$, where the hypersurface $M^c_{1,-1}$ is parametrized by

$$\begin{aligned}\Psi(t,u,v) = \Big(&\cosh(\sqrt{c}t)\gamma(u) + \sinh(\sqrt{c}t)n(u),\\ &\cosh(\sqrt{1-c}t)\tilde{\gamma}(v) + \sinh(\sqrt{1-c}t)\tilde{n}(v)\Big),\end{aligned} \tag{4.16}$$

with unit normal vector field N given by

$$\begin{aligned}N = \Big(&\sqrt{1-c}\sinh(\sqrt{c}t)\gamma(u) + \sqrt{1-c}\cosh(\sqrt{c}t)n(u),\\ &-\sqrt{c}\sinh(\sqrt{1-c}t)\tilde{\gamma}(v) - \sqrt{c}\cosh(\sqrt{1-c}t)\tilde{n}(v)\Big),\end{aligned} \tag{4.17}$$

where $\gamma(u)$ and $\tilde{\gamma}(v)$ are horocycles parametrized by arc length in $\mathbb{H}^2$, i.e.,

$$\gamma(u) = \left(1+\frac{u^2}{2}, u, \frac{u^2}{2}\right)\subset\mathbb{L}^3, \quad \tilde{\gamma}(v) = \left(1+\frac{v^2}{2}, v, \frac{v^2}{2}\right)\subset\mathbb{L}^3,$$

with n, $\tilde{n}$ being its unit normal vector fields given, respectively, by

$$n(u) = \Big(-\frac{u^2}{2}, -u, 1 - \frac{u^2}{2} \Big), \quad \tilde{n}(v) = \Big(\frac{v^2}{2}, v, -1 + \frac{v^2}{2} \Big),$$

see [19].

The hypersurface $M^c_{1,-1}$ has constant $C = 1 - 2c$ and constant principal curvatures 0, $\sqrt{1-c}$ and $\sqrt{c}$ (see example 3.6 of [19]), that is, its mean curvature is given by $h_c = \frac{\sqrt{1-c} + \sqrt{c}}{3}$.

Moreover, if we write $N = (N_1, N_2)$, since $||N_1||^2 = 1 - c$ and $||N_2||^2 = c$, then the displacement of $M^c_{1,-1}$ in direction N at distance r is given by

$$\begin{aligned} \Phi_r(p,q) &= \exp_{(p,q)}(rN(p,q)) \\ &= \Big(\cosh(\sqrt{1-c}r)p + \frac{1}{\sqrt{1-c}} \sinh(\sqrt{1-c}r)N_1(p), \\ &\qquad \cosh(\sqrt{c}r)q + \frac{1}{\sqrt{c}} \sinh(\sqrt{c}r)N_2(p) \Big) \\ &= \Big(\cosh(\sqrt{c}t + \sqrt{1-c}r)\gamma(u) + \sinh(\sqrt{c}t + \sqrt{1-c}r)n(u), \\ &\qquad \cosh(\sqrt{1-c}t - \sqrt{c}r)\tilde{\gamma}(v) + \sinh(\sqrt{1-c}t - \sqrt{c}r)\tilde{n}(v) \Big). \end{aligned} \tag{4.18}$$

Since the principal curvatures of $\Phi_r(M^c_{1,-1})$ are the same as $M^c_{1,-1}$ (see Example 3.6 of [19]), it follows that the mean curvature of the parallel hypersurfaces to $M^c_{1,-1}$ is also given by $h(r) = \frac{\sqrt{1-c} + \sqrt{c}}{3}$. Thus, the MCF with initial data $M^c_{1,-1}$ is given by $\Phi_{\epsilon(s)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(s) = \frac{\sqrt{1-c} + \sqrt{c}}{3}, \tag{4.19}$$

that is, $\epsilon(s) = \frac{(\sqrt{1-c} + \sqrt{c})s}{3}$.

Now, let us analyze the case where $M^c_{1,1}$ for $c \in (0,1)$, is the initial data, where $M^c_{1,1}$ is also parametrized by (4.16) and its unit normal vector field is also given by (4.17), where in this case, $\tilde{n}(v)$ is given by

$$\tilde{n}(v) = \Big(-\frac{v^2}{2}, -v, 1 - \frac{v^2}{2} \Big),$$

see [19].

Moreover, the principal curvatures of $M^c_{1,1}$ are given by 0, $\sqrt{1-c}$ and $-\sqrt{c}$ (see example 3.7 of [19]), and hence, the mean curvature of $M^c_{1,-1}$ is given by $h_c = \frac{\sqrt{1-c} - \sqrt{c}}{3}$. Observe that when $c = \frac{1}{2}$, we have that $h_{\frac{1}{2}} = 0$, and therefore, the flow is stationary. For $c \neq \frac{1}{2}$, the displacement of $M^c_{1,1}$ in direction N at distance r is also given by (4.18).

As in the previous case, the principal curvatures of $\Phi_r(M^c_{1,1})$ are the same as $M^c_{1,1}$ (see Example 3.7 of [19]), which implies that the mean curvature of the parallel hypersurfaces to $M^c_{1,1}$ is also given by $h(r) = \frac{\sqrt{1-c} - \sqrt{c}}{3}$. Therefore, the MCF with initial data $M^c_{1,1}$ is given by $\Phi_{\epsilon(s)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(s) = \frac{\sqrt{1-c} - \sqrt{c}}{3}, \tag{4.20}$$

that is, $\epsilon(s) = \dfrac{(\sqrt{1-c}-\sqrt{c})s}{3}$.

Finally, following the ideas of Urbano [41], we construct a class of isoparametric hypersurfaces with three distinct (constant) principal curvatures, which coincides with the family of hypersurfaces M_τ (for $\tau < -1$), obtained in [19], and we provide the ODE (4.3) of the MCF whose such hypersurface is the initial data.

Let

$$\widetilde{M_t} = \{(p,q) \in \mathbb{H}^2 \times \mathbb{H}^2 \hookrightarrow \mathbb{L}^3 \times \mathbb{L}^3 : \langle p, q\rangle_{\mathbb{L}^3} = -t\},$$

for $t > 1$. In this subsection all products will be taken in $\mathbb{L}^3$. Then it is easy to check that $\widetilde{M_t}$ is a hypersurface of $\mathbb{H}^2 \times \mathbb{H}^2$ with normal vector field

$$N(p,q) = \frac{1}{\sqrt{2(-1+t^2)}}\left(q - tp, p - tq\right).$$

Note that $C = 0$. Let $(v_1, v_2) \in T_{(p,q)}\widetilde{M_t}$ and $\gamma(s) = (p(s), q(s)) : I \to \widetilde{M_t}$ with $\gamma(0) = (p,q)$ and $\gamma'(0) = (v_1, v_2)$, thus ${}^{\mathbb{L}}\nabla_{(v_1,v_2)}N = \frac{d}{ds}N \circ \gamma(s)|_{s=0}$, where ${}^{\mathbb{L}}\nabla$ stands as the connection in the lorentzian space, and

$${}^{\mathbb{L}}\nabla_{(v_1,v_2)}N = \frac{1}{\sqrt{2(-1+t^2)}}\left((v_2, v_1) - t(v_1, v_2)\right),$$

as $\mathbb{H}^2$ is an umbilical hypersurface of $\mathbb{L}^3$ we have the following equation

$${}^{\mathbb{L}}\nabla_{(v_1,v_2)}N = {}^{\mathbb{H}}\nabla_{(v_1,v_2)}N + \alpha((v_1,v_2),N) = {}^{\mathbb{H}}\nabla_{(v_1,v_2)}N + \frac{1}{\sqrt{2(-1+t^2)}}(\langle v_1, q\rangle p, \langle v_2, p\rangle q),$$

where ${}^{\mathbb{H}}\nabla$ stands as the connection in the hyperbolic space, and it follows that

$$A(v_1, v_2) = \frac{1}{\sqrt{2\,(-1+t^2)}}\left(t(v_1, v_2) - (v_2, v_1) + (\langle v_1, q\rangle p, \langle v_2, p\rangle q)\right).$$

We will need an orthonormal basis to calculate the mean curvature. Let $w \in T\mathbb{H}^2$ with $\langle w, w\rangle_{\mathbb{L}^3} = \frac{1}{2}$ such that $\langle w, p\rangle_{\mathbb{L}^3} = \langle w, q\rangle_{\mathbb{L}^3} = 0$, thus $\langle (w, -w), N_{(p,q)}\rangle = \langle w, q\rangle - t\langle w, p\rangle - \langle p, w\rangle + t\langle q, w\rangle = 0$ and we have that $(w, -w) \in T\widetilde{M_t}$. Using the same argument, we have that $(w, w) \in T\widetilde{M_t}$. A straightforward calculation shows that $\{(w, -w), (w, w), (q - tp, -p + tq)\}$ is an orthornormal basis of $T\widetilde{M_t}$. Observe that

$$A(w, -w) = \frac{1}{\sqrt{2\,(-1+t^2)}}\left(t(w, -w) + (w, -w)\right) = \frac{1}{\sqrt{2}}\sqrt{\frac{t+1}{t-1}}(w, -w),$$

$$A(w, w) = \frac{1}{\sqrt{2}}\sqrt{\frac{t-1}{t+1}}(w, w),$$

$$A(q - tp, -p + tq) = \frac{1}{\sqrt{2(-1+t^2)}}\big((p(1-t^2), q(t^2-1)) + ((t^2-1)p, (1-t^2)q)\big) = 0.$$

It follows that $h_t = \frac{\sqrt{2}t}{3\sqrt{-1+t^2}}$. Observe that the displacement of M_t in direction N at distance r is given by

$$\begin{aligned}
\Phi_r(p,q) &= \exp_{(p,q)}\left(rN(p,q)\right) \\
&= \left(\left(\cosh\frac{r}{\sqrt{2}}\right)p + \left(\sinh\frac{r}{\sqrt{2}}\right)\frac{q - tp}{\sqrt{-1+t^2}}, \left(\cosh\frac{r}{\sqrt{2}}\right)q + \left(\sinh\frac{r}{\sqrt{2}}\right)\frac{p - tq}{\sqrt{-1+t^2}}\right) \\
&= \left(\mathcal{P}_r(p,q), \mathcal{Q}_r(p,q)\right).
\end{aligned} \tag{4.21}$$

Since $\langle \mathcal{P}_r(p,q), \mathcal{Q}_r(p,q)\rangle = -t\cosh\left(\sqrt{2}r\right)+\sqrt{-1+t^2}\sinh\left(\sqrt{2}r\right)$, it follows that $\Phi_r(\widetilde{M}_t) = \widetilde{M}_{\phi(r,t)}$, where $\phi(r,t) = t\cosh\left(\sqrt{2}r\right) - \sqrt{-1+t^2}\sinh\left(\sqrt{2}r\right)$.

The MCF with initial data M_s is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\begin{aligned}
\epsilon'(t) &= h_{\phi(\epsilon(t),s)} \\
&= \frac{\sqrt{2}\phi(\epsilon(t),s)}{3\sqrt{-1+\phi(\epsilon(t),s)^2}} \\
&= \frac{\sqrt{2}\left(s\cosh\left(\sqrt{2}\epsilon(t)\right) - \sqrt{-1+s^2}\sinh\left(\sqrt{2}\epsilon(t)\right)\right)}{3\sqrt{-1+\left(s\cosh\left(\sqrt{2}\epsilon(t)\right) - \sqrt{-1+s^2}\sinh\left(\sqrt{2}\epsilon(t)\right)\right)^2}}.
\end{aligned} \tag{4.22}$$

4.2.2 On hypersurfaces of $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$

In this subsection, we will study the evolution of isoparametric hypersurfaces by the MCF in the ambient space $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$, where $\mathbb{Q}^3_\varepsilon$ denotes the unit sphere $\mathbb{S}^3$ if $\varepsilon = 1$, or the hyperbolic space $\mathbb{H}^3$ if $\varepsilon = -1$.

By Corolary 3.7, if Σ is a hypersurface with constant principal curvatures in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$, then Σ is isoparametric. Thus, by Theorem 3.6 and Theorem 6.1 of [8] (items (i) and (ii)), if Σ is an isoparametric hypersurface in $\mathbb{Q}^3_\varepsilon \times \mathbb{R}$ with g constant distinct principal curvatures, we have:

a) If $g = 1$, then Σ is an open part of a slice $\mathbb{Q}^3_\varepsilon \times \{t_0\}$, for any $t_0 \in \mathbb{R}$ or an open subset of a Riemannian product $\Sigma^2 \times \mathbb{R}$. In the latter case, if $\varepsilon = 1$, Σ^2 is a totally geodesic sphere in $\mathbb{S}^3$, and if $\varepsilon = -1$, Σ^2 is a totally geodesic hyperplane in $\mathbb{H}^3$,

b) If $g = 2$, then $\varepsilon = -1$ and Σ is locally parametrized by $f(p,s) = \tilde{h}_s(p) + Bs\partial_t$, for some $B \in \mathbb{R}$, $B > 0$, with $\Sigma^3 = \Sigma^2 \times I$, where $\tilde{h}_s$ is a family of horospheres in $\mathbb{H}^3$, or Σ^3 is an open part of a Riemannian product $\Sigma^2 \times \mathbb{R}$. In the latter case, if $\varepsilon = 1$, Σ^2 is a non-totally geodesic sphere in $\mathbb{S}^3$, and if $\varepsilon = -1$, Σ^2 is an equidistant hypersurface to a totally geodesic $\mathbb{H}^2$, a horosphere, or a hypersphere in $\mathbb{H}^3$,

c) If $g = 3$, then Σ^3 is an open part of the following hypersurfaces:

   i) $\mathbb{S}^1(c_1) \times \mathbb{S}^1(c_2) \times \mathbb{R}$, when $\varepsilon = 1$;

   ii) $\mathbb{S}^1(c_1) \times \mathbb{H}^1(c_2) \times \mathbb{R}$, when $\varepsilon = -1$,

   where $c_1 \neq c_2$, $\dfrac{1}{c_1} + \dfrac{1}{c_2} = \varepsilon$ and the principal curvatures of Σ^3 are given by 0, $\dfrac{c_1}{\sqrt{c_1+c_2}}$ and $\dfrac{-c_2}{\sqrt{c_1+c_2}}$.

Let us analyze each case separately.

In item a), since Σ is totally geodesic, the principal curvatures are all zero, and hence $h = 0$, which implies that the flow is stationary, i.e., $\epsilon(t) = 0$ for all t.

In item b) we have two cases. First, we will deal with the case where Σ is parametrized by $f(p,s) = \tilde{h}_s(p) + Bs\partial_t$, for some $B \in \mathbb{R}$, $B > 0$. It follows from [8, Theorem 4.1] that one principal curvature is 0, and the other two are both equal, depending on the orientation, to $\dfrac{B}{\sqrt{1+B^2}}$ or $-\dfrac{B}{\sqrt{1+B^2}}$. Without loss of generality, assume that the principal curvatures of Σ are 0, $\frac{B}{\sqrt{1+B^2}}$ and $\frac{B}{\sqrt{1+B^2}}$, which implies that the mean curvature of Σ is given by $h_B = \frac{2B}{3\sqrt{1+B^2}}$.

In this case, since $\varepsilon = -1$ and the unit normal is $N = (N_{\mathbb{Q}}, \cos(\theta))$, where $N_{\mathbb{Q}}$ is the component of N in $\mathbb{Q}^3_\varepsilon$, the displacement of Σ in direction N at distance r is given by

$$\begin{aligned}\Phi_r(p,q) &= \exp_{(p,q)}(rN(p,q)) \\ &= \Big(\cosh(||T||r)p + ||T||^{-1}\sinh(||T||r)N_{\mathbb{Q}}, q + r\cos(\theta)(q)\Big),\end{aligned}$$

where $\cos(\theta) = \frac{1}{\sqrt{1+B^2}}$ and $||T|| = ||N_{\mathbb{Q}}|| = \frac{B}{\sqrt{1+B^2}}$, see [32].

Moreover, it follows from [8, Proposition 3.1] that the principal curvatures of the parallel hypersurfaces to Σ at distance r are also given by 0, $\frac{B}{\sqrt{1+B^2}}$ and $\frac{B}{\sqrt{1+B^2}}$, which implies that its mean curvature is also given by $h(r) = \frac{2B}{3\sqrt{1+B^2}}$. Therefore the MCF with initial data Σ is given by $\Phi_{\epsilon(t)}$, where ϵ is the solution of the ODE (4.3):

$$\epsilon'(t) = \frac{2B}{3\sqrt{1+B^2}},$$

that is, $\epsilon(t) = \frac{2Bt}{3\sqrt{1+B^2}}$.

In what follows, we will simultaneously consider the remaining hypersurfaces in item b) and the hypersurfaces in item c). Observe that in both cases, the hypersurface Σ is a cylinder over an isoparametric surface of $\mathbb{Q}^3_\varepsilon$, i.e., Σ is of the form $\Sigma^2 \times \mathbb{R}$, where Σ^2 is an isoparametric surface in $\mathbb{Q}^3_\varepsilon$.

Since $||T|| = 1$, from (4.5), the displacement of $\Sigma^2 \times \mathbb{R}$ in direction $N = (N_{\mathbb{Q}}, 0)$ at distance r is given by

$$\begin{aligned}\Phi_r(p,q) &= \exp_{(p,q)}(rN(p,q)) \\ &= \Big(C_\varepsilon(r)p + S_\varepsilon(r)N_{\mathbb{Q}}(p), q\Big),\end{aligned}$$

where the functions $S_\varepsilon(r)$ and $C_\varepsilon(r)$ are given in (4.6). It follows that the evolution of Σ is reduced to the evolution of Σ^2 in $\mathbb{Q}^3_\varepsilon$, that is, to the evolution of an isoparametric surface in space form. Therefore the MCF with initial data $\Sigma^2 \times \mathbb{R}$ is given by $\Phi_{\epsilon(t)}(p,q) = (\tilde{\Phi}_{\epsilon(t)}(p), q)$, where $\tilde{\Phi}$ is given according to each $\Sigma^2 \subset \mathbb{Q}^3_\varepsilon$, at the following propositions of [37]:

1. Proposition 2.5, when $\Sigma^2 \subset \mathbb{H}^3$ is a horosphere;
2. Proposition 2.6, when $\Sigma^2 \subset \mathbb{H}^3$ is either a hypersphere or an equidistant hypersurface to a totally geodesic $\mathbb{H}^2$;
3. Proposition 2.7, when $\Sigma^2 = \mathbb{S}^1(c_1) \times \mathbb{H}^1(c_2) \subset \mathbb{H}^3$;
4. Proposition 2.8, when $\Sigma^2 \subset \mathbb{S}^3$ is a sphere (not totally geodesic);
5. Proposition 2.9, when $\Sigma^2 = \mathbb{S}^1(c_1) \times \mathbb{S}^1(c_2) \subset \mathbb{S}^3$.

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