PhD Thesis

On the Temperature of a Causal Diamond in Algebraic Quantum Field Theory

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Resumo

Nesta tese, estudamos a temperatura de uma região abstrata do espaço-tempo chamada cone duplo, que também é chamado de diamante. A estrutura sob consideração é a teoria de campo escalar sem massa livre tratada dentro da abordagem da teoria quântica de campo algébrico. Antes de discutir o resultado principal nos introduzimos álgebras de von Neumann, sua classificação de tipo, a teoria modular de Tomita-Takesaki (TT) e a condição KMS, que são partes indispensáveis do modelo usado em nossa análise. Também fornecemos uma extensa discussão sobre alguns resultados conhecidos, em particular, transformações geométricas e os operadores modulares correspondentes para regiões interconectadas do espaço-tempo: uma cunha direita, um cone de luz frontal e um diamante. Como resultado principal, apresentamos uma definição intrínseca de temperatura em termos de um campo vetorial inverso de temperatura que pode ser calculado sem se referir a uma trajetória particular de TT. Este campo vetorial reproduz a temperatura Unruh para uma cunha direita. Mais tarde é aplicado para calcular a temperatura de um diamante. Também consideramos alguns limites contra-intuitivos nos quais o fluxo modular de um diamante se assemelha ao do espaço-tempo de Minkowski ou de uma cunha. Enquanto no primeiro caso o comportamento estipulado se encontra no centro de um diamante, afastado das bordas, no segundo caso está próximo das bordas, mais especificamente, próximo aos cantos esquerdo e direito.

Abstract

In this thesis, we study the temperature of an abstract spacetime region called a double cone, which is also referred to as a diamond. The framework under consideration is free massless scalar field theory treated within the algebraic quantum field theory approach. Before discussing the main result we introduce von Neumann algebras, their type classification, the Tomita-Takesaki (TT) modular theory and the KMS condition, which are indispensable parts of the model used in our analysis. We also provide an extensive discussion on some known results, in particular, geometrical transformations and the corresponding modular operators for interconnected spacetime regions: a right wedge, a forward lightcone and a diamond. As a main result, we present an intrinsic definition of temperature in terms of an inverse temperature vector field that can be computed without referring to a particular TT trajectory. This vector field reproduces the Unruh temperature for a right wedge. Later it is applied to compute the temperature of a diamond. We also consider some counterintuitive limits in which the modular flow of a diamond resembles that of Minkowski spacetime or a wedge. While in the former case the stipulated behavior is found in the center of a diamond, away from the boundaries, in the latter case it is close to the boundaries, more specifically, close to left and right corners.

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Chapter 1

Introduction

In 1900, during the conference held in Paris, David Hilbert [1] presented twenty three problems in mathematical sciences that would later revolutionize the way theoretical problems were solved. In particular, the sixth problem was related to physics. Motivated by the foundation of geometry Hilbert's main concern was to treat physical problems with some sort of axiomatization. In fact, in Hilbert's own words, see e.g. [2]:

> The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.

For the time being, the quantum field theory (QFT) was not known yet, development of which would begin later in 20's. Unfortunately, soon after starting off the construction of the theory it was plagued by unprecedented mathematical difficulties due to its lack of rigorous mathematical treatment. In spite of several solutions, such as renormalization techniques, the theory was still incomplete, scarce of a solid foundation. There had been many attempts to fix the conundrum in that regard, some of which are quite effective hitherto. One of such approaches, commenced in late fifties, is Algebraic QFT (AQFT).

In 1957, during the conference held in Lille, one of the pioneers of AQFT, Rudolf Haag, presented an idea of using an axiomatic strategy to tackle problems in QFT [3,4]. Haag's approach, being algebraic, was different from the Wightman's one, presented in the same conference [5]. While the latter was mostly concerned with analytic properties of quantum fields, see e.g. [6], the former exploited local algebras associated with a given spacetime region. On the algebra side, within a decade of Haag's proposal in Lille, Haag and Kastler [7], came up with axioms written in terms of local nets of algebras, $O \mapsto \mathcal{A}(O)$, associated with a family of spacetime regions. That also covered the fundamental physical concepts like Einstein's locality principle as well as Poincaré covariance. This was probably the first time ever the QFT was equipped with rigorous mathematical methods based on axioms fulfilling Hilbert's dream. The algebraic approach then was strengthened further through the work of Haag and Araki [8, 9]. Despite the Wightman approach being successful, an extensive review in relation to that is given in [6, 10], the algebraic approaches, in the spirit of Araki-Haag-Kastler axioms, led to many important works, just to name a few: a criteria for the spectral condition [11], independence of locality and spectral condition [12], an algebraic version of the Reeh–Schlieder theorem [13] (see also references therein), TCP symmetry [14, 15] and so on. As far as progress on the Hilbert's sixth problem is concerned, a detailed account given by Wightman in mid 70's can be found in [2].

A phenomenal work by Tomita [16, 17] in late sixties can be considered as a page turner, which put the von Neumann algebras into action. It was later enhanced by Takesaki [18]. The theory under consideration is known as Tomita-Takesaki modular theory. Use of this theory in AQFT started an era of rigorous results, vid. [19], including a first use of the theory to derive Haag duality [20], as well as Connes' classification of type III factors [21, 22], the type of local algebras [23, 24] and the nuclearity condition [25, 26]. In fact, the significance of this theory, especially in physics, can be seen from some recent works, see e.g. [27] and references therein. A close tie between the modular theory and KMS condition [28] is also one of many important physical aspects of this theory that appeared earlier in Tomita's work.

Very soon after the work of Eckmann and Osterwalder [20], Bisoganano and Wichmann [29, 30] undertook the task of studying wedge dualities using the modular theory in the Wightman's axiomatic settings. One of the main outcomes of their studies was a modular operator for a right (or Rindler) wedge, that acts locally as a Lorentz boost. Later, as Sewell [31] observed, an observer whose time evolution is the boost under consideration is in fact the uniformly accelerated one, which led to the Unruh or the Hawking effect. The upshot of this study is a geometrical interpretation of a modular operator, a feature that exists in a very limited number of cases [19]. Similar results were then found for a forward light cone [32] and a double cone [33]. While the results for a wedge are valid regardless of whether the field is massive or massless, interacting or free, that for a forward light cone and a double cone, hold only for the free massless scalar field theory. As such, in particular, [33] exploits conformal transformations to derive the modular operator for a double cone using that of a forward lightcone or a wedge [29, 32].

Recently, in [34, 35], using this modular operator, the authors attempted studying the temperature of a double cone. Although their effort is commendable, the approach considered is too restrictive as it relies on specific trajectories induced by the corresponding modular operators. In this thesis, we introduce an intrinsic definition of temperature in terms of an inverse temperature vector field that can be computed without referring to particular trajectories. As far as the geometrical TT action for any spacetime region is given, one can compute the corresponding temperature. Our definition coincides with the one given in [36], where the approach under consideration is based on the relative entropy between the vacuum and its perturbation. We also provide some limiting cases in which the behaviour of the modular flow of a double cone looks like that of a wedge or Minkowski spacetime. Further we discuss a ratio of the temperature obtained with respect to the modular flow of a double cone to that of a wedge. This is where one needs to use a particular TT trajectory. At the end we discuss a connection between our results and the conformal factor associated with the geometrical transformation under consideration. Before we embark on discussing the main results of our studies we provide an extensive review on mathematical tools and some known results. The thesis is organized as described in the following section.

1.1 Outline of the Thesis

In chapter 2 we give an introduction to von Neumann algebras, including some definitions of abstract notions used in proving theorems such as a double commutant theorem, as well as the type classification of von Neumann algebras. We also provide one of the main results concerning type III_1 factor, Thm. 2.3.1.

Chapter 3 focuses on fundamental aspects of AQFT, namely axioms. In particular, we review the Haag-Kaster and Haag-Araki axioms in sect. 3.1, and apply them to prove the Reeh-Schielder theorem in sect. 3.2.

Chapter 4 contains a detailed review on the Tomita-Takesaki modular theory that is based on [37]. It starts with the fundamental properties of the Tomita's operator S. After reviewing some intermediate results we prove the main theorem 4.1.1. In subsect. 4.1.1 we give necessary details on a connection between the modular automorphism group induced by the modular operator and the KMS state. Later in subsection 4.1.2 we also discuss a couple of applications.

Chapter 5 , a backbone of our analysis, presents the Hislop-Longo theorem [33] that guarantees the conformal mapping of a modular operator corresponding to a right wedge or a forward lightcone to that of a double cone without tweaking the fundamental axioms of AQFT. In addition, a systematic derivation of geometrical transformations and associated modular operators for the spacetime regions just mentioned is given. A conformal factor corresponding to the transformations used in our analysis is also introduced.

Chapter 6 gives main results of our analysis [38]. In particular, we provide the intrinsic definition of temperature by introducing an inverse temperature vector field. The concerned vector field is used first to obtain the well known Unruh temperature for a right wedge. Later it is applied to study the temperature of a diamond. We also discuss counterintuitve limits in which the modular flow of a diamond resembles that of Minkowski spacetime or a wedge. Analysis on a ratio of temperatures obtained with respect to the TT trajectories of a wedge and a diamond is given. A comparison between these results and the conformal factor introduced in chapter 5 is considered briefly.

Finally, in chapter 7 we conclude the thesis and briefly touch upon further possible studies.

Appendix A focuses on basic notions and mathematical concepts used in the main text, such as operator topologies and states.

Chapter 2

A Mathematical Exordium

Here we shall discuss useful concepts which are imperative in our analysis. For some basics related to the topics considered here we refer to appendix A.

2.1 Von Neumann Algebras

Definition 2.1.1. A *Banach algebra* $\mathcal{A}$ over $\mathbb{C}$ is a normed algebra that is complete and satisfies the following property:

$$\|AB\| \leq \|A\|\|B\| \quad \forall A, B \in \mathcal{A}. \tag{2.1.1}$$

If $\mathcal{A}$ is equipped with a $*$-*operation*/involution $* : A \mapsto A^* \in \mathcal{A}$, such that for all $A, B \in \mathcal{A}$ and for all $\lambda_1, \lambda_2 \in \mathbb{C}$ one has

1. $(A^*)^* = A$,
2. $(\lambda_1 A + \lambda_2 B)^* = \bar{\lambda}_1 A^* + \bar{\lambda}_2 B^*$,
3. $(AB)^* = B^* A^*$,
4. $\|A^*\| = \|A\|$,

then $\mathcal{A}$ is called a *Banach $*$-algebra* or an *involutive Banach algebra.*

Definition 2.1.2. A *C^*-algebra* is an involutive Banach algebra that in addition satisfies

$$\|A^* A\| = \|A^*\|\|A\| \quad \forall\, A \in \mathcal{A}. \tag{2.1.2}$$

Now let us consider bounded operators that play an important role in our studies.

Definition 2.1.3. A *bounded operator* acting on $\mathcal{H}$ is a linear mapping $A : \mathcal{H} \to \mathcal{H}$ defined with norm

$$\|A\| \doteq \sup \left\{ (A\xi, A\xi)^{1/2} \,\middle|\, \xi \in \mathcal{H}, (\xi, \xi) = 1 \right\} < \infty. \tag{2.1.3}$$

Let $\mathfrak{B}(\mathcal{H})$ denote a set of bounded linear operators acting on a Hilbert space $\mathcal{H}$, which is an involutive Banach algebra. The $*$-operation is given by the property $(\eta, A\xi) = (A^*\eta, \xi)$, $\xi, \eta \in \mathcal{H}$. Here A^* denotes the adjoint of A. Now we can define concretely realized algebras.

Definition 2.1.4. Let $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ be a subalgebra. If it is invariant under the $*$-operation, then $\mathcal{A}$ is called a $*$-*subalgebra*. If $\mathbb{1} \in \mathcal{A}$, then $\mathcal{A}$ is called *unital*.

Definition 2.1.5. A *concrete C^*-algebra* $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ is an unital $*$-algebra that is norm closed (see appendix A.1). If $\mathcal{A}$ is also weakly or strongly or σ-weakly closed then it is called *a concrete von Neumann algebra*.

Since the norm topology is the finest among all topologies on $\mathfrak{B}(\mathcal{H})$, a von Neumann algebra is a C^*-algebra algebra but the converse is not always true. Though we will need some abstract definitions of these algebras, it will be worthwhile to keep in mind these concrete versions.

Definition 2.1.6. Let $\mathfrak{B}(\mathcal{H})$ be as above, then a *projection* $P \in \mathfrak{B}(\mathcal{H})$ is a linear map $P : \mathcal{H} \to \mathcal{H}$, which is an idempotent, i.e., $P^2 = P$.

Note that an orthogonal projection satisfies $P^2 = P = P^*$. Here we shall always deal with such projections, unless specified otherwise.

Definition 2.1.7. Let $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$. Then its commutant in $\mathfrak{B}(\mathcal{H})$ is defined as the set of all bounded linear operators from $\mathfrak{B}(\mathcal{H})$ that commute with all operators from $\mathcal{A}$, i.e.,

$$\mathcal{A}' \doteq \left\{ B \in \mathfrak{B}(\mathcal{H}) \,\middle|\, AB = BA \; \forall \, A \in \mathcal{A} \right\}. \tag{2.1.4}$$

The double commutant is defined by $\mathcal{A}'' \doteq (\mathcal{A}')'$. As such, for the multiple commutant the following equalities hold:

$$\mathcal{A} \subseteq \mathcal{A}'' = \mathcal{A}^{4\prime} = \dots \quad \text{and} \quad \mathcal{A}' = \mathcal{A}''' = \mathcal{A}^{5\prime} = \dots \tag{2.1.5}$$

Next, we introduce abstractly realized von Neumann algebras.

Definition 2.1.8. A *von Neumann algebra* $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ is a unital $*$-subalgebra that satisfies $\mathcal{M} = \mathcal{M}''$ (see Thm. 2.1.2). Equivalently, a von Neumann algebra is a C^*-algebra that has a predual (see, e.g., [39]).

We will also need some important subsets of $\mathcal{M}$ and its dual $\mathcal{M}^*$ that we now consider.

Definition 2.1.9. The *predual* $\mathcal{M}_* \subset \mathcal{M}^*$ for a von Neumann algebra $\mathcal{M}$ is the Banach space of all σ-weakly (vid. appendix A.1) continuous linear functionals acting on $\mathcal{M}$.

As we shall see, this set, being the set of normal states (vid. Thm. A.2.1), turns out to be of paramount importance in the approach under consideration.

Henceforth, $\mathcal{M}_1^{*+}$ denotes the set of all positive normalized linear functionals (see appendix A.2) called states and $\mathcal{M}_{1*}^{+}$ the space of all normal states.

Let us now define algebras generated by restricting the one just defined to act on a particular domain that we shall need later on to discuss spectral analysis.

Definition 2.1.10. Let $P \in \mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ denote a projection with range $\mathfrak{R}_P \equiv P\mathcal{H} \subset \mathcal{H}$, where $\mathcal{M}$ is a von Neumann algebra as in def. 2.1.8. Let us consider a set

$$\mathcal{M}^P \doteq \{PAP \mid A \in \mathcal{M}\} \subset \mathfrak{B}(\mathcal{H}). \tag{2.1.6}$$

Then the mapping $\mathcal{M}^P \ni A \mapsto A_P \doteq A|_{\mathfrak{R}_P}$ onto $\mathfrak{R}_P$ gives rise to the set of restricted operators that forms an algebra called the *reduced* von Neumann algebra defined by

$$\mathcal{M}_P \doteq \{A_P \mid A \in \mathcal{M}^P\} \subset \mathfrak{B}(\mathfrak{R}_P). \tag{2.1.7}$$

Remark 2.1.1. It satisfies the equality $[\mathcal{M}_P\mathfrak{K}] = \mathfrak{R}_P$, $\mathfrak{K} \subset \mathfrak{R}_P$ or equivalently, $\mathcal{M}_P$ is non-degenerate (see def. 2.1.13) on $\mathfrak{R}_P$. Furthermore, $[\mathfrak{R}_P]$ turns out to be invariant with respect to $\mathcal{M}^P$, so the kernel of P as well. In the above definition, if $P \in \mathcal{M}' \subset \mathfrak{B}(\mathcal{H})$ instead, then $\mathcal{M}_P$ is called *induction* of $\mathcal{M}$.

Definition 2.1.11. Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ denote a von Neumann algebra. Let $\mathcal{K} \subseteq \mathcal{H}$, then $\mathcal{K}$ is *separating* for $\mathcal{M}$ if for any $A \in \mathcal{M}$ and every $\xi \in \mathcal{K}$, $A\xi = 0$, implies $A = 0$. In other words, $\mathcal{K}$ cannot be annihilated by any nonzero operator. $\mathcal{K}$ is *cyclic* for $\mathcal{M}$ if $[\mathcal{M}\mathcal{K}] = \mathcal{H}$, i.e., $\mathcal{M}\mathcal{K}$ is dense in $\mathcal{H}$.

Here $[\cdot]$ denotes the closure with respect to a given topology. Note that alternatively we also use bar, $\bar{\cdot}$.

Proposition 2.1.1. *Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be a von Neumann algebra and $\mathcal{K} \subseteq \mathcal{H}$. Then the following statements are equivalent.*

1. *$\mathcal{K}$ is cyclic for $\mathcal{M}$,*

2. *$\mathcal{K}$ is separating for $\mathcal{M}'$.*

Proof. 1. $\Longrightarrow$ 2. Suppose $\mathcal{K}$ is cyclic for $\mathcal{M}$, then by definition $[\mathcal{M}\mathcal{K}] = \mathcal{H}$. Let $A' \in \mathcal{M}'$ such that $A'\mathcal{K} = \{0\}$. Then taking arbitrary $B \in \mathcal{M}$, one has

$$A'B\mathcal{K} = BA'\mathcal{K} = 0. \tag{2.1.8}$$

Since it holds for every $B \in \mathcal{M}$, in general, $A'[\mathcal{M}\mathcal{K}] = 0$. Then necessarily $A' = 0$.

2. $\Longrightarrow$ 1. Suppose $\mathcal{K}$ is separating for $\mathcal{M}'$. Let $P \in \mathcal{M}'$ be a projection onto the close subspace $[\mathcal{M}\mathcal{K}]$. Since $P\mathcal{K} = \mathcal{K}$, we have $(\mathbb{1} - P)\mathcal{K} = 0$. But $(\mathbb{1} - P) \in \mathcal{M}'$, so $\mathbb{1} - P = 0$ and (2) implies that $P = \mathbb{1}$, i.e., $[\mathcal{M}\mathcal{K}] = \mathcal{H}$. □

Definition 2.1.12. Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathcal{M}$ is called σ-finite if it contains at most countable families of orthogonal projections.

Theorem 2.1.1. *Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ denote a von Neumann algebra. Then the following conditions are equivalent.*

1. *$\mathcal{M}$ is σ-finite.*

2. *$\exists$ a countable subset $\mathcal{K} \subset \mathcal{H}$ such that $\mathcal{K}$ is separating for $\mathcal{M}$.*

3. *$\exists\ \omega \in \mathcal{M}^+_{*1}$ that is also faithful.*

4. *$\mathcal{M} \cong \pi(\mathcal{M})$— representation of $\mathcal{M}$ (see def. 2.3.2), with $\pi(\mathcal{M})$ carrying a cyclic and separating vector.*

Proof. See prop. 2.5.6 in [40], also prop. II.3.19 in [41]. □

Definition 2.1.13. Let $0 \neq \mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ be a $*$-subalgebra. Let $\mathcal{K} \subset \mathcal{H}$. Define the close subspace

$$\overline{\mathcal{AK}} = \{A\xi \mid A \in \mathcal{A}, \xi \in \mathcal{K}\} \subset \mathcal{H}. \tag{2.1.9}$$

Then an action of $\mathcal{A}$ is called *nondegenerate* on $\mathcal{H}$ if $\overline{\mathcal{AK}} = \mathcal{H}$. $\mathcal{A}$ is called *irreducible* if the only invariant subspace is $\mathcal{H}$. Otherwise $\mathcal{A}$ is *reducible* and $\mathcal{K}$ is *reducing*. Note that irreducibility implies nondegeneracy.

Proposition 2.1.2. *Let $\mathcal{A}$ be a $*$-subalgebra as above and P be a projection onto $\overline{\mathcal{A}\xi}$, $0 \neq \xi \in \mathcal{H}$. Then $P \in \mathcal{A}'$.*

Proof. First of all note that for $A \in \mathcal{A}$, we clearly have $AP\mathcal{H} \subseteq P\mathcal{H}$. Let $P^{\perp} = \mathbb{1} - P$, then

$$P^{\perp}AP = 0 \implies AP = PAP. \tag{2.1.10}$$

Suppose $A = A^*$, then

$$(AP)^* = (PAP)^* \implies PA = PAP \implies PA = AP, \tag{2.1.11}$$

where we used the fact that P is an orthogonal projection (recall def. 2.1.6). Since every $A \in \mathcal{A}$ is a linear combination of two self-adjoint elements, the last equality holds for all $A \in \mathcal{A}$. So it means P commutes with every $A \in \mathcal{M}$. Thus necessarily $P \in \mathcal{A}'$. □

Now we prove the theorem that characterizes von Neumann algebras and it is one of the fundamental results in the subject under consideration. The theorem is known as the double commutant/bicommutant theorem. In fact, the statement of the concerned theorem just a definition 2.1.8 with if and only if condition, proof of which is a trivial consequence of the following one.

Theorem 2.1.2. *[42, lemma. 4.1.4] Let $\mathcal{M} \subseteq \mathfrak{B}(\mathcal{H})$ be a unital $*$-subalgebra, then $\overline{\mathcal{M}}^{\text{SOT}}$ coincides with $\mathcal{M}''$.*

Proof. Let $A \in \mathcal{M}''$ and for $\xi \in \mathcal{H}$ define the close vector subspace

$$\mathcal{K} \doteq \overline{\{B\xi \mid B \in \mathcal{M}\}} \subset \mathcal{H}. \tag{2.1.12}$$

Since $\mathcal{K}$ is invariant, i.e., $\mathcal{K} \subseteq \mathcal{H}$, it is reducible as explained above. Let P be a projection onto $\mathcal{K}$, then prop. 2.1.2 implies $P \in \mathcal{M}'$, i.e., $[P, A] = 0$ for every $A \in \mathcal{M}''$.

Note that for any $A \in \mathcal{M}$, we have $AP\mathcal{H} \subseteq P\mathcal{H} = \mathcal{K}$. Since $\mathcal{M}$ is unital, $\mathbb{1} \in \mathcal{M}$, such that $\mathbb{1}\xi = \xi \in \mathcal{K}$. Then clearly

$$PA\xi = AP\xi = A\xi \in \mathcal{K}. \tag{2.1.13}$$

Then there exists a sequence $\{B_n\} \in \mathcal{A}$ such that $A\xi = \lim_{n\to\infty} B_n\xi$.

Let us now consider a diagonal mapping, in particular, a unital $*$-homomorphism

$$\iota : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}^n),\ \forall\, n \in \mathbb{Z}_+. \tag{2.1.14}$$

Clearly $\iota(\mathcal{M}) \subset \mathfrak{B}(\mathcal{H}^n)$ is a unital $*$-subalgebra. In particular,

$$\mathfrak{B}(\mathcal{H}) \ni A \mapsto A_{ij} = A\delta_{ij} = \operatorname{diag}(A, A, \dots, A) \in \mathfrak{B}(\mathcal{H}^n),\ i, j = \overline{1, n}. \tag{2.1.15}$$

In fact, for any $A \in \mathcal{M}''$, from the construction follows $\iota(A) \in (\iota(\mathcal{M}))''$. As such, if $C \in (\iota(\mathcal{M}))'$ and $B \in \mathcal{M}$, then trivially

$$\iota(B)C = C\iota(B) \qquad \text{equivalently} \qquad BC_{ij} = C_{ij}B, \tag{2.1.16}$$

which is true for all the elements under consideration. Meaning $C_{ij} \in \mathcal{M}'$. As a result,

$$AC_{ij} = C_{ij}A \implies \iota(A)C = C\iota(A). \tag{2.1.17}$$

Let us now take $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathcal{H}^n$, then as above, there exists a sequence $\{B_m\} \in \mathcal{M}$ such that $\iota(A)\xi = \lim_{m\to\infty} \iota(B_m)\xi$. Therefore, in terms of components, one has

$$A\xi_i = \lim_{m\to\infty} B_m\xi_i,\ \forall\ i = \overline{1, n}. \tag{2.1.18}$$

Therefore, for $\epsilon > 0$ there exists some N such that for every $m > N$, one finds

$$\|\iota(A)\xi - \iota(B_m)\xi\| < \epsilon \implies \sum_{i=1}^{n} \|A\xi_i - B_m\xi_i\|^2 < \epsilon^2. \tag{2.1.19}$$

As a result, we have

$$\left\{ \sum_{i=1}^{n} \|A\xi_i - B_m\xi_i\|^2 < \epsilon^2 \right\} \subseteq U_\epsilon(A; \xi_1, \ldots, \xi_n), \tag{2.1.20}$$

where U_ϵ denoting the strong neighbourhood of A. Since $A \in \mathcal{M}''$ is arbitrary, the claim follows. □

As a consequence, we have the following result, which also justifies def. 2.1.5, and establishes equivalence between def. 2.1.5 and def. 2.1.8.

Corollary 2.1.2.1. *Let $\mathcal{M}$ be as described in the theorem 2.1.2, then the following statements are equivalent:*

1. $\mathcal{M}'' = \mathcal{M}$,
2. $\mathcal{M}$ *is weakly closed,*
3. $\mathcal{M}$ *is strongly closed.*

Proof. 1. $\implies$ *2.* Let $\mathcal{N} = \mathcal{M}'$. Take any $B \in \mathcal{N}$ and weakly convergent sequence $\{A_n\} \to A$, with $A_n \in \mathcal{N}'$, $\forall\, n \in I$, then for $\xi, \eta \in \mathcal{H}$

$$\begin{aligned} \lim_{n\to\infty} (\xi, (A_n B - B A_n)\eta) &= \lim_{n\to\infty} [(\xi, A_n B\eta) - (B^*\xi, A_n\eta)] = \\ &= (\xi, AB\eta) - (B^*\xi, A\eta) = (\xi, (AB - BA)\eta) = 0. \end{aligned} \tag{2.1.21}$$

Clearly $\mathcal{N}'$ is weakly closed. Then the claim follows due to

$$\mathcal{M} \subset \mathcal{N}' \subset \mathcal{M}'' = \mathcal{M} \implies \mathcal{M} = \mathcal{N}'. \tag{2.1.22}$$

2. $\implies$ *3.* Weak operator topology being weaker than the strong operator topology (see appendix A.1), any close set in the former one is also closed in the latter so the claim follows.

3. $\implies$ *1.* Direct consequence of the Thm. 2.1.2. □

2.2 Classification of von Neumann Algebras

We collect a few definitions, before we discuss the subject matter. For the historical account we refer to [21, 22] and references therein.

Definition 2.2.1. A von Neumann algebra $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ is called a *factor* if its *center*

$$\mathcal{Z}(\mathcal{M}) \doteq \mathcal{M} \cap \mathcal{M}' \tag{2.2.1}$$

satisfies $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbb{1}$.

This is equivalent to the fact that $\mathfrak{B}(\mathcal{H})$ factors into $\mathcal{M}$ and its commutant $\mathcal{M}'$, i.e.,

$$\mathcal{M} \vee \mathcal{M}' = \left\{ AB \,\middle|\, A \in \mathcal{M}, B \in \mathcal{M}' \right\}'' = \mathfrak{B}(\mathcal{H}) \tag{2.2.2}$$

Definition 2.2.2. Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ denote a von Neumann algebra. Let $P, Q \in \mathcal{M}$ be projections, then if there exists $U \in \mathcal{M}$ such that $P = U^*U$ and $Q = UU^*$, we say that P and Q are *von Neumann-Murray equivalent*, and write it as $P \sim Q$.

The closed subspaces of $\mathcal{H}$ are in one to one correspondence with projections, where the former can be partially ordered by inclusion. Using the correspondence a partial order $\leq$ in the set of projections is defined as $P \leq Q \iff \mathfrak{R}_P \subseteq \mathfrak{R}_Q$.

A projection P is called *abelian* if $P\mathcal{M}P$ is abelian. P is called *minimal* if P is non-zero and $P\mathcal{M}P = \mathbb{C}P$. Note that a minimal projection is abelian.

If $P \sim Q \leq P$, for some non-zero $Q \in \mathcal{M}$, necessarily means $P = Q$, then P is said to be *finite* with respect to $\mathcal{M}$. Otherwise, it is *infinite*. If there is no non-zero $Q \in \mathcal{M}$ such that $Q \leq P$, then P is called *purely infinite*. For every central projection $F \in \mathcal{Z}(\mathcal{M})$ with non-zero FP, if FP is infinite, then P is said to be *properly infinite*. Accordingly, a von Neumann algebra $\mathcal{M}$ is called *finite, infinite, purely infinite, or properly infinite* depending on the property of an identity projection $\mathbb{1}$. $\mathcal{M}$ is *semi-finite* if there is a faithful, normal semi-finite trace (def. A.2.1) acting on $\mathcal{M}$, see Thm. 2.7.17 in [40], also cf. def. 2.3.5.

Remark 2.2.1. If $P \in \mathcal{M}$ is minimal and there exists $Q \in \mathcal{M}$ with $Q \leq P$, then due to $Q = PQP \in P\mathcal{M}P = \mathbb{C}P$, necessarily $Q \in \{0, P\}$.

Let $\mathcal{M}$ denote a factor. It is principally divided into the following three principal types: type I, II, III. Each of them are also divided into subtypes. We shall systematically define here the principal types and their subtypes.

Definition 2.2.3. A factor $\mathcal{M}$ is called *type* I if it contains a minimal projection.

Any type I is isomorphic to $\mathfrak{B}(\mathcal{H})$. As such, a type I factor is *type* I_n if n is the dimension of $\mathcal{H}$.

The type I is crucial when one is dealing with a finite dimensional Hilbert space, such as, the one in quantum mechanics, however, its usage in QFT could result into misleading interpretations, see e.g. [43]. Note that any type I factor is isomorphic to $\mathfrak{B}(\mathcal{H})$.

Definition 2.2.4. A factor $\mathcal{M}$ is *type* II if it is semi-finite and it has no non-zero abelian projection.

If a factor $\mathcal{M}$ is type II and finite in the sense of projection (def. 2.2.2), then it is called *type* II_1. If $\mathcal{M}$ is of type II but there is no non-zero $F \in \mathcal{M}$ that is finite then it is called *type* II_∞. For the latter case $\mathcal{M}$ is properly infinite.

The type II is mathematically rich and there is comparatively more work on type II. For recent works, see for example [44], where they have used such a factor to demonstrate finiteness of entropy. In fact, efforts in that direction can be traced back to the work of Segal in early 60's, [45].

Definition 2.2.5. A factor $\mathcal{M}$ is called *type* III if there is no non-zero projection $P \in \mathcal{M}$ that is finite. In other words, $\mathcal{M}$ is purely infinite (def. 2.2.2).

The type III is the most important one from QFT standpoint, vid. [43]. Its subtypes involve a complicated mathematical structure, as such, we shall consider them with minimal details in the next subsection.

2.2.1 Subtypes of Type III

To start off, we need some basics of modular theory, which play a very important role in this classification. We shall merely recall some definitions here. The details will be given in the next chapter.

Definition 2.2.6. Let $\mathcal{A}$ be a C^*-algebra as defined in sect. 2.1, then $\mathcal{A}$ is called a W^*-algebra if there exists a Banach space $\mathcal{A}_*$ called a predual, such that the dual of $\mathcal{A}_*$ is $\mathcal{A}$, i.e. $(\mathcal{A}_*)^* = \mathcal{A}$. This is basically just an abstract von Neumann algebra in the sense defined in def. 2.1.8.

Definition 2.2.7. Let $\mathcal{A}$ denote a Banach $*$-algebra, then a $*$-*automorphism*, (which is also a $*$-*isomorphism*), or simply *automorphism* of $\mathcal{A}$ is a mapping $\alpha : \mathcal{A} \to \mathcal{A}$ defined by[1]

$$\alpha(A) \doteq uAu^*, \ A \in \mathcal{A}, \tag{2.2.3}$$

with u being an unitary operator that induces the automorphism is not necessarily from $\mathcal{A}$. The group of all such automorphisms will be denoted by $\text{Aut}\mathcal{A}$.

Definition 2.2.8. In the definition 2.2.7 if $u \in \mathcal{A}$, then such automorphisms will be called *inner automorphisms*. A set of these elements is defined as

$$\text{Inn}\mathcal{A} \doteq \Big\{\alpha \in \text{Aut}\mathcal{A} \,\Big|\, \alpha(A) = uAu^*, \ \forall\, A \in \mathcal{A}, u \in \mathcal{A}\Big\}. \tag{2.2.4}$$

An *outer automorphism* is a quotient space $\text{Out}\mathcal{A} \doteq \text{Aut}\mathcal{A}/\text{Inn}\mathcal{A}$.

Remark 2.2.2. Above definitions are quite general. When we consider one parameter group of automorphisms, in particular, $\mathbb{R} \ni t \mapsto \alpha_t$, we shall accordingly use $\alpha_t(\cdot) = u(t) \cdot u(t)^* \equiv u_t \cdot u_t^*$.

Definition 2.2.9. Let $\mathcal{U}$ be either W^* or C^*-algebra, then the corresponding dynamical system (W^* or C^*-dynamical system) is a triple $(\mathcal{U}, G, \alpha)$, where G is a locally compact group and α is a homomorphism from G into $\text{Aut}(\mathcal{U})$.

For the one parameter automorphisms, in accordance to the above discussion, $\alpha_t(A)$ is continuous from $G = \mathbb{R}$ to $\mathcal{U}$ for every $A \in \mathcal{U}$.

Let $L^1(\mathbb{R})$ be the usual L^1-space on $\mathbb{R}$ with the L^1-norm, $\|\cdot\|_1$. In addition, let it be equipped with an operation $*$ called *convolution* such that for every $f, g \in L^1(\mathbb{R})$, one has

$$f * g(t) = \int_{\mathbb{R}} ds f(t-s)g(s). \tag{2.2.5}$$

Clearly $L^1(\mathbb{R})$ is a *commutative Banach algebra*. For every $f \in L^1(\mathbb{R})$, its Fourier transform $\hat{f}$ defined by

$$\hat{f}(t) \doteq \int_{\mathbb{R}} d\lambda f(\lambda) e^{i\lambda t}, \tag{2.2.6}$$

is a smooth function over $\mathbb{R}$, in particular, $\hat{f} \in C_0^\infty(\mathbb{R})$.

[1] In principle, one can also consider abstract definition without referring to unitary elements whatsoever.

Let $M(G)$ denote a collection of all bounded Radon measures over G, such as, the Haar measure when G is locally compact, or the Lebesgue one, when G is simply $\mathbb{R}$, where the latter one is the case that we are interested in. In particular, we have $M(\mathbb{R}) = C_0^\infty(\mathbb{R})^*$. As such, for every $\mu \in M(\mathbb{R})$ there exists $\alpha(\mu) \in B_\alpha(\mathcal{M})$, where $B_\alpha(\mathcal{M})$ stands for the set of all $\sigma(\mathcal{M}, \mathcal{M}_*)$-$\sigma(\mathcal{M}, \mathcal{M}_*)$ continuous (see appendix A.1) linear operators on $\mathcal{M}$. In that case for every $A \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$, one has

$$\omega\left(\alpha(\mu)\right) = \int_{\mathbb{R}} d\mu(t)\, \omega\left(\alpha_t(A)\right). \tag{2.2.7}$$

Clearly for every $f \in L^1(\mathbb{R})$(and $A \in \mathcal{M}, \omega \in \mathcal{M}_*$), such that $||\alpha(f)|| \leq ||f||_1$, rewriting the above formula in terms of individual functions, it reduces to

$$\omega\left(\alpha(f)\right) = \int_{\mathbb{R}} dt\, f(t) \omega\left(\alpha_t(A)\right). \tag{2.2.8}$$

For a closed ideal $J^\alpha = \left\{ f \in L^1(\mathbb{R}) : \alpha(f) = 0 \right\}$ of $L^1(\mathbb{R})$, let

$$J^\perp = \left\{ t \in \mathbb{R} \mid \hat{f}(t) = 0 \; \forall f \in J^\alpha \right\} = \bigcap \left\{ N(\hat{f}) \mid f \in J^\alpha \right\}, \tag{2.2.9}$$

where $N(\hat{f})$ is the zero point set of $\hat{f}$. Now we define one of the important elements of spectral analysis under consideration.

Definition 2.2.10. Let $(\mathcal{M}, G = \mathbb{R}, \alpha)$ be a W^*-system, then the *Arveson spectrum* $\operatorname{Sp}\alpha$ of α is given by

$$\operatorname{Sp}\alpha = J^\perp. \tag{2.2.10}$$

Note that in order to define the *point spectrum* $\operatorname{Sp}_\alpha(A)$ for some $A \in \mathcal{M}$ one just replaces $\alpha(f) \to \alpha(f)(A)$ in (2.2.8) and follows the same construction.

For a given W^*-system $(\mathcal{M}, \mathbb{R}, \alpha)$ we denote by $\mathcal{M}^\alpha$ a *fixed point algebra* defined by

$$\mathcal{M}^\alpha \doteq \left\{ A \in \mathcal{M} \mid \alpha_t(A) = A \; \forall t \in \mathbb{R} \right\}. \tag{2.2.11}$$

Corresponding collection of projections will be denoted by $\operatorname{Proj}(\mathcal{M}^\alpha)$.

Definition 2.2.11. Let $(\mathcal{M}_P = P\mathcal{M}P, G = \mathbb{R}, \alpha^P = \alpha \upharpoonright \mathcal{M}_P)$ be a reduced W^*-system (see def. 2.1.10), whose Arveson spectrum is denoted by $\operatorname{Sp}\alpha^P$, then the *Connes spectrum* is defined by

$$\Gamma(\alpha) \doteq \bigcap \Big\{ \operatorname{Sp}\alpha^P \Big| 0 \neq P \in \operatorname{Proj}(\mathcal{M}^\alpha) \Big\}. \tag{2.2.12}$$

Let $\alpha_t = \sigma^\omega_\lambda$, in particular, $u = \Delta^{i\lambda}$, with $\Delta^{i\lambda}$ being a non-singular operator defined by a product of closure of closable operator, $S_0 : A\Omega \mapsto A^*\Omega$, $A \in \mathcal{M}$, denoted by S and its adjoint $F = S^*$. Here $\Omega \in \mathcal{M}$ is cyclic and separating in the sense defined in def. 2.1.11. More details on this operator will be given in chapter 4.

Definition 2.2.12. σ^ω_λ as described above will be called the *modular automporphism* associated with the pair $(\mathcal{M}, \omega)$.

Here the fixed point algebra defined in (2.2.11), when the underlying automorphism is induced by the modular operator, is defined as follows. For any state $\omega \in \mathcal{M}^{*+}_1$ the corresponding *centralizer* is given by

$$\begin{aligned} \mathcal{M}_\omega &\doteq \Big\{ A \in \mathcal{M} \Big| \omega(AB) = \omega(BA)\ \forall B \in \mathcal{M} \Big\} = \\ &= \Big\{ A \in \mathcal{M} \Big| \sigma^\omega_\lambda(A) = A\ \forall \lambda \in \mathbb{R} \Big\}. \end{aligned} \tag{2.2.13}$$

Here the equality is due to the following result.

Lemma 2.2.1. *Let $A \in \mathcal{M}$ and $\omega \in \mathcal{M}^{*+}_1$, which is also faithful, then the following statements are equivalent.*

1. $\omega(AB) = \omega(BA)\ \forall B \in \mathcal{M}$,

2. $\sigma^\omega_\lambda(A) = A\ \forall \lambda \in \mathbb{R}$.

Proof. *1.* $\Longrightarrow$ *2.* Let $B \doteq \sigma^\omega_\lambda(C)$, $C \in \mathcal{M}$, then by hypothesis we have

$$\omega\left(A\sigma^\omega_\lambda(C)\right) = \omega\left(\sigma^\omega_\lambda(C)A\right). \tag{2.2.14}$$

Then as per the KMS condition (4.1.66) there exists a complex valued bounded function $\mathcal{F}$, which is continuous on and analytic in the closed strip $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Im} z \leq 1\}$, such that, in the present case, one has

$$\mathcal{F}(\lambda) = \mathcal{F}(\lambda + i). \tag{2.2.15}$$

Since $\mathcal{F}$ is analytic and bounded in the domain of our interest, it extends to the entire function. Then using Liouville's theorem, we have that $\mathcal{F}$ is constant. As a consequence,

$$\mathcal{F}(\lambda) = \omega\left(A\sigma_\lambda^\omega(C)\right) = \omega\left(\sigma_\lambda^\omega(A)C\right) = \omega(AC) = \mathcal{F}(0),\ \forall\, C \in \mathcal{M}, \tag{2.2.16}$$

which is independent of λ. Also ω is faithful (def. A.2.1) by hypothesis, so we have the proof.

2. $\Longrightarrow$ *1.* Let $\sigma_\lambda^\omega(A) = A$ for every $A \in \mathcal{A}$, then as above, using (4.1.66) we can write

$$\mathcal{F}(\lambda) = \omega(AB) \qquad \mathcal{F}(\lambda + i) = \omega(AB) \tag{2.2.17}$$

for any $B \in \mathcal{M}$. Since these are the constant functions, claim follows. (See also [18]). □

Now we are in a position to start discussing spectral analysis of von Neumann algebras and corresponding types.

Definition 2.2.13. If $(\mathcal{M}, G = \mathbb{R}, \alpha = \sigma^\omega)$ is a W^*-system with σ^ω being the modular automorphism group of $\mathcal{M}$ associated with ω in the sense of definitions 2.2.9 and 2.2.12, then the *Connes' modular spectrum* is defined by

$$S(\mathcal{M}) = \bigcap \left\{ \operatorname{Sp} \Delta_\omega \;\middle|\; \text{faithful } \omega \in \mathcal{M}_{*1}^+ \right\}. \tag{2.2.18}$$

Let $\mathcal{M}$ be a σ-finite von Neumann algebra, so that due to Thm. 2.1.1 there exists a faithful $\omega \in \mathcal{M}_{*1}^+$ on $\mathcal{M}$. Moreover, define $\Gamma(\mathcal{M}) = \Gamma(\sigma^\omega)$ (independent of a choice of ω, cf. Connes' unitary cocycle thm. A.2.3), then the modular spectrum satisfies

$$\exp\left(\Gamma(\mathcal{M})\right) = S(\mathcal{M}) \bigcap\, (0, \infty) = \bigcap \left\{ \exp\left(\operatorname{Sp} \sigma^\omega\right) \;\middle|\; \text{faithful } \omega \in \mathcal{M}_{*1}^+ \right\}, \tag{2.2.19}$$

where σ_λ^ω is restricted to $\mathcal{M}_P = P\mathcal{M}P$ with $P = \operatorname{supp} \omega$. Note that $\lambda \in \operatorname{Sp} \sigma^\omega \iff e^\lambda \in \operatorname{Sp} \Delta_\omega$.

The subtypes of type III are now defined in terms of the modular spectrum as follows.

Definition 2.2.14. Let $\mathcal{M}$ be a type III factor as defined in def. 2.2.5, then it has the following three subtypes:

- Type III_0 if $S(\mathcal{M}) = \{0, 1\}$,
- Type III_ν with $0 < \nu < 1$ if $S(\mathcal{M}) = \{\nu^n \mid n \in \mathbb{Z}\} \cup \{0\}$,
- Type III_1 if $S(\mathcal{M}) = [0, \infty)$.

Proof of this fact can be found in Thm. 12.1.6, [46]. Naturally one may ask if there is any way to distinguish the algebra of type III_μ from that of III_ν for $\mu \neq \nu$? This question is answered by Connes-Stormer theorem [47]. We shall briefly recall a few definitions and state the main result, which will also help to characterize some types of von Neumann algebras discussed in def. 2.2.14, though we don't use these results in our analysis.

Definition 2.2.15. Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}^+_{1*}$ the corresponding space of normal states (def. 2.1.9), then the *diameter of the state orbit space* $d(\mathcal{M})$ of $\mathcal{M}$ is defined by

$$d(\mathcal{M}) \doteq \sup\Big\{\inf\left\{\|\omega_1\left(\alpha_t\right) - \omega_2\| \mid \alpha_t \in \mathrm{Inn}\mathcal{M}\right\} \Big| \, \omega_1, \omega_2 \in \mathcal{M}^+_{1*}\Big\}, \tag{2.2.20}$$

where $\omega_1(\alpha_t(A))$ for every $A \in \mathcal{M}$ is called the *orbit of the state* ω_1. In (2.2.20) the internal bracketed term is seen as a *distance function* on the norm closure of the state space orbit under inner automorphism, i.e.,

$$\overline{\omega_1} = \overline{\Big\{\omega_1\left(\alpha_t\right) \Big| \, \alpha_t \in \mathrm{Inn}\mathcal{M}\Big\}}^{\|\cdot\|} \tag{2.2.21}$$

so that the concerned function takes the form

$$d\left(\overline{\omega_1}, \overline{\omega_2}\right) = \inf\Big\{\|\omega_1\left(\alpha_t\right) - \omega_2\| \Big| \, \alpha_t \in \mathrm{Inn}\mathcal{M}\Big\} \tag{2.2.22}$$

Now we can state the theorem due to Connes and Stormer [47].

Theorem 2.2.2. *Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be a von Neumann algebra acting on a separable $\mathcal{H}$. Suppose $\mathcal{M}$ is a factor (def. 2.2.1), with $\nu \in [0, 1]$, then*

$$d(\mathcal{M}) = 2\frac{1 - \nu^{\frac{1}{2}}}{1 + \nu^{\frac{1}{2}}}. \tag{2.2.23}$$

While for the type III_0, $\mathcal{M}$, one has $d(\mathcal{M}) = 2$, III_1 gives $d(\mathcal{M}) = 0$, which means, it is not possible to distinguish between two different normal states of such *factor*, since the concerned orbit is norm dense in $\mathcal{M}_1^{*+}$. Type III_1 is the most important factor for the algebraic quantum field theory, which we shall now briefly discuss and highlight its physical relevance.

2.3 Relevance of Type III_1 Factor

Type III factor plays a very important role in algebraic formulation of the standard QFT. Concerning this, a moderately extensive review by Yngvason [43] provides some details with physical examples. Here we shall present some useful results corresponding to general spacetime region that also covers the case of our interest. This will be useful, in particular in sect. 5.1, to conclude on the type of local algebras associated with a given region.

In [24], Fredenhagen showed that in the framework of asymptotically scale invariant theory the spectra of all modular operators associated with local algebras coincides with a real line $\mathbb{R}_+ = [0, \infty)$, which as we saw above corresponds to the type III_1 factor. We now present somewhat generalized case corresponding to what is called lightlike monotone regions, which also covers wedges. Let us start with a definition of this abstract spacetime region. More details on physical aspects will be given in later chapters. Here we provide minimal details and discuss the main result of our interest.

Definition 2.3.1. Let $\mathbb{R}^{1+3}$ denote the usual four dimensional Minkowski spacetime and $O \subset \mathbb{R}^{1+3}$ be any open region, then O is called *lightlike monotone* if $O' \neq \emptyset$ (causal complement of O, see 3.1.1) and any of the following conditions is satisfied:

1. $(\mathbf{L}^+)$: $O + y \subset O + x$, $\mu_x > \mu_y$,

2. $(\mathbf{L}^-)$: $O + x \subset O + y$, $\mu_x > \mu_y$,

where $x, y \in \mathbb{R}^{1+3}$, and their explicit forms are given by $x^\mu = \mu_x \eta^\mu$ and $y^\mu = \mu_y \eta^\mu$ with $\mu_{x,y} \in \mathbb{R}$ and η^μ being lightlike, i.e., $\eta^\mu \eta_\mu = 0$.

The regions satisfying the above conditions are necessarily unbounded, since the translation vector is given purely in terms of light rays. Example of such regions are naturally right and left wedges (5.1.21).

Definition 2.3.2. Let $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ be a C^*-algebra as defined in sect. 2.1. Then a $*$-homomorphism $\pi : \mathcal{A} \to \mathfrak{B}(\mathcal{H})$ is called a *representation* of $\mathcal{A}$ in $\mathfrak{B}(\mathcal{H})$, i.e., π respects the following properties:

1. $\pi(\lambda A + \mu B) = \lambda\pi(A) + \mu\pi(B)$,
2. $\pi(AB) = \pi(A)\pi(B)$,
3. $\pi(A^*) = \pi(A)^*$,

for every $A, B \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. In addition if it is a $*$-isomorphism, such that, $\pi(A) = 0$ necessarily means $A = 0$, then π is called a *faithful representation.* Note that for a C^*-algebra there always exists a faithful representation.

Definition 2.3.3. Let X denote a Borel space and μ a positive measure on X. For a von Neumann algebra $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$, let $X \ni \xi \mapsto \mathcal{M}_\xi$ be the μ-measurable field of $\mathcal{M}$ defined on the corresponding field $\mathcal{H}_\xi$ of the Hilbert space $\mathcal{H}$. Then the algebra $\mathcal{M}$ is *decomposable* and its *decomposition* is given by

$$\mathcal{M} = \int^{\oplus} d\mu(\xi)\ \mathcal{M}_\xi. \tag{2.3.1}$$

Here the field $\mathcal{M}_\xi$ is defined such that for any other field $\mathcal{N}_\xi$ defining the same von Neumann algebra $\mathcal{M}$, one has $\mathcal{M}_\xi = \mathcal{N}_\xi$, μ-almost everywhere, i.e., there exists $Y \in X$ with $\mu(Y) = 0$ such that $\xi \in X \setminus Y$.

Definition 2.3.4. If $\mathcal{M}$ is a von Neumann algebra, then its decomposition with respect to $\mathcal{Z}(\mathcal{M})$ is called a *central decomposition* of $\mathcal{M}$ into factors.

In particular, for a von Neumann algebra $\pi(\mathcal{A})''$ (cf. Sherman-Takeda thm.) its commutant is $\pi(\mathcal{A})'$. As such, for an abelian subalgebra $\mathcal{N} \subset \pi(\mathcal{A})'$, the central decomposition[2] is given by

$$\mathcal{N} = \int^{\oplus} d\mu(\xi)\ \mathcal{N}_\xi, \qquad \text{with} \quad \mathcal{N}_\xi \in \pi(\mathcal{A})'' \cap \pi(\mathcal{A})'. \tag{2.3.2}$$

Now let us quickly recall some useful facts that we shall need below to prove an important result concerning the types of local algebras in QFT.

Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be a von Neumann algebra. Then Connes invariant $T(\mathcal{M})$ is defined by a set of all real numbers such that the corresponding modular automorphism (def. 2.2.12) associated

[2] Study of central decomposition of states was initiated by Sakai. For the details in this direction we refer to [39].

with a state ω acts as an identity, i.e.,

$$T(\mathcal{M}) = \Big\{t_0 \in \mathbb{R} \Big| \exists\, \omega \text{ s.t. } \sigma^{\omega}_{t_0} = \mathbb{1}\Big\}. \tag{2.3.3}$$

Note that states with such properties are called periodic (def. A.2.2). Now due to Thm. 2.3 in [21](see below), alternatively we also have (recall def. 2.2.8)

$$t_0 \in T(\mathcal{M}) \iff \sigma^{\omega}_{t_0} \in \text{Inn}\mathcal{M}, \tag{2.3.4}$$

Therefore, one can redefine the invariant as

$$T(\mathcal{M}) = \Big\{t_0 \in \mathbb{R} \Big| \sigma^{\omega}_{t_0} \in \text{Inn}\mathcal{M}\Big\}. \tag{2.3.5}$$

In what follows next, we set $u_t \equiv u$, $t \in \mathbb{R}$, for the sake of convenience.

Proposition 2.3.1. *Let $T(\mathcal{M})$ be as above and $\omega \in \mathcal{M}^+_{*1}$ be faithful then the following statements are equivalent:*

1. *$t_0 \in T(\mathcal{M})$,*
2. *$\sigma^{\omega}_{t_0} \in \text{Inn}\mathcal{M}$,*
3. *There exists $u \in \mathcal{U}(\mathcal{Z}(\mathcal{M}_\omega))$ such that $\sigma^{\omega}_{t_0} = u \cdot u^*$,*
4. *There exists bounded invertible element $A \in \mathcal{Z}(\mathcal{M}_\omega)^+$ such that for $\omega_A = \omega(A\cdot)$, we have $\sigma^{\omega_A}_{t_0} = \mathbb{1}$.*

Proof. 1. $\Longrightarrow$ 2. When $t_0 \in T(\mathcal{M})$, (2.3.5) shows that the canonical projection of the corresponding automorphism is a trivial element of Out$\mathcal{M}$, hence $\sigma^{\omega}_{t_0} \in \text{Inn}\mathcal{M}$.

2. $\Longrightarrow$ 3. Let $\sigma^{\omega}_{t_0} \in \text{Inn}\mathcal{M}$, then using (2.2.4) clearly we have $u \in \mathcal{M}$ such that $\sigma^{\omega}_{t_0} = u \cdot u^*$. Since ω is a faithful normal state by hypothesis, it satisfies the KMS condition (sect. 4.1.1). As such, since ω is invariant under the action of modular automorphism, i.e.,

$$\omega(\sigma^{\omega}_{t_0}(A)) = \omega(uAu^*) = \omega(A), \tag{2.3.6}$$

taking $A = Bu$ one has $\omega(uB) = \omega(Bu)$. But then using (2.2.13) one concludes that $u \in \mathcal{M}_\omega$.

Now for $u \in \mathcal{M}_\omega$ in accordance with lemma 2.2.1 we have

$$\sigma^\omega_{t_0}(A) = uAu^* = A, \ \forall \ A \in \mathcal{M}. \tag{2.3.7}$$

Meaning $u \cdot u^*$ acts as an identity, which then naturally commutes with every element from $\mathcal{M}_\omega$. So we must have $u \in \mathcal{U}(\mathcal{Z}(\mathcal{M}_\omega))$—unitaries inside of the center of a centralizer.

3. $\Longrightarrow$ *4.* Set $A^{it_0} = u^*$, then using so-called the Connes cocycle theorem A.2.3 one has $\sigma^{\omega_A}_t(\cdot) = A^{it}\sigma^\omega_t(\cdot)A^{-it}$ so that for every $B \in \mathcal{M}$ with $t = t_0$

$$\sigma^{\omega_A}_{t_0}(B) = A^{it_0}\sigma^\omega_{t_0}(B)A^{-it_0} = u^*(uBu^*)u = B \implies \sigma^{\omega_A}_{t_0} = \mathbb{1}. \tag{2.3.8}$$

4. $\Longrightarrow$ *1* Since the corresponding modular automorpshims are in the same equivalence class, the claim follows as a trivial consequence of (2.3.3). □

Let $\mathcal{M}_i$ with $i = \text{I}, \text{II}, \text{III}$ be the principal types of von Neumann algebras as defined in the beginning of sect. 2.2. Then any von Neumann algebra $\mathcal{M}$ can be uniquely decomposed such that $\mathcal{M} = \mathcal{M}_\text{I} \oplus \mathcal{M}_\text{II} \oplus \mathcal{M}_\text{III}$ (see Thm. 5.1.19 in [41] or Thm. 6.5.2 in [48]).

Definition 2.3.5. In the above decomposition if the central projection $z_\text{III} \in \mathcal{Z}(\mathcal{M}_\text{III})$ is zero then $\mathcal{M}$ is called *semi-finite.*

Proposition 2.3.2. *Let $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be a von Neumann algebra that has a separable predual, then the following statements are equivalent.*

1. *$\mathcal{M}$ is semi-finite,*

2. *$T(\mathcal{M}) = \mathbb{R}$.*

Proof. See prop. 2.6 in [21]. □

Now we state and prove the main result of this section, the following theorem that is due to Driessler [23], see also [49].

Theorem 2.3.1. *Let $O \subset \mathbb{R}^{1+3}$ be a lightlike monotone (def. 2.3.1). Suppose axioms, namely*[3] *isotonoy (a), weak-additivity (b), locality (c), Poincaré covariance (d) and spectral condition (e)*

[3] We refer to the next chapter for the details in this regard.

hold. In addition, suppose there is unique (up to a phase) $\Omega \in \mathcal{H}$ *such that* $U(x)\Omega = \Omega$*, where* $U(x) \equiv U(x, \mathbb{1})$ *is a unitary representation in* $\mathcal{H}$ *corresponding to spacetime translations.*

Then the corresponding local (von Neumann) algebras, $\mathcal{M}(O)$*, are of type* III_1

Proof. Since the local algebra under consideration satisfies Tomita-Takesaki theorem 4.1.1, there exists $\Omega \in \mathcal{H}$ with a property given in the hypothesis such that it is cyclic and separating (def. 2.1.11) for $\mathcal{M}(O)$.

Let $\mathcal{M}(O)_\omega$ be the corresponding fixed point algebras as defined in (2.2.13). Then for an arbitrary element $A \in \mathcal{M}(O)_\omega$ one has

$$(\Omega, AB\Omega) = (\Omega, BA\Omega)\,,\ \forall\, B \in \mathcal{M}(O), \tag{2.3.9}$$

where we used the vector state representation of ω (def. A.2.4). Using lemma 2.2.1 and the corresponding facts presented above one clearly sees that for some $\sigma^\omega_\lambda(A) \in \mathcal{M}(O+x)_\omega = U(x)\mathcal{M}(O)_\omega U(-x)$ (Poincaré covariance (d)) and for every $B \in \mathcal{M}(O+x)$ we have

$$\begin{aligned}
(\Omega, \sigma^\omega_\lambda(A)B\Omega) &= (\Omega, B\sigma^\omega_\lambda(A)\Omega) \iff \\
\iff (\Omega, U(x)AU(-x)B\Omega) &= (\Omega, BU(x)AU(-x)\Omega) \iff \\
\iff (\Omega, AU(-x)B\Omega) &= (\Omega, BU(x)A\Omega)\,,
\end{aligned} \tag{2.3.10}$$

where we simply exploited the translation invarince of Ω. By assumption O is lightlike monotone, def. 2.3.1, so every translation vector has a form $x = \mu_x \eta$. Since the corresponding unitary operator, $U(\mu_x\eta)$, is positive by assumption, namely spectral condition (e), left and right hand sides of (2.3.10) can be analytically continued in μ_x, into the lower and upper half plane, respectively. Since boundary values of both functions coincide on real axis, we have bounded entire analytic function, hence the Liouville's theorem implies, it must be constant. As a result, $(\Omega, BU(x)A\Omega)$ is independent of x. Then equating this function for values $x = 0$ and $x = \mu_x\eta$, with $|\mu_x| \to \infty$, the weakly clustering property of states (see e.g. [50]) yields

$$\mathrm{W}-\lim_{|\mu_x| \longrightarrow \infty} (\Omega, BU(\mu_x\eta)A\Omega) = (\Omega, A\Omega)\,(\Omega, B\Omega)\,, \tag{2.3.11}$$

where W-lim stands for the limit with respect to WOT (see appendix A.1). Note Ω is cyclic and separating for $\mathcal{M}(O)$, so for $\mathcal{M}(O+x)$ as well. As a consequence one has

$$A^*\Omega = (\Omega, A\Omega)\,\Omega \iff A^* = (\Omega, A\Omega)\,\mathbb{1}. \tag{2.3.12}$$

Since A is arbitrary, we must have $\mathcal{M}(O)_\omega = \mathbb{C}\mathbb{1}$.

If $\mathcal{M}(O)$ is semifinite then prop. 2.3.2 implies $T(\mathcal{M}(O)) = \mathbb{R}$, but then using (2.3.5), one immediately concludes that $\sigma_t^\omega \in \mathrm{Inn}\mathcal{M}(O)$, for every $t \in \mathbb{R}$.

Now using Thm. 2.3.1, we have $\sigma_t^\omega = u \cdot u^*$ with $u \in \mathcal{U}(\mathcal{Z}(\mathcal{M}_\omega(O)))$. As we proved above, elements of $\mathcal{M}(O)_\omega$ are scalar multiples of identity, then clearly $\mathcal{M}_\omega(O)' = \mathcal{M}(O)$, therefore, $\mathcal{Z}(\mathcal{M}_\omega(O)) = \mathbb{C}\mathbb{1}$. As such, $u \in \mathcal{U}(\mathbb{C}\mathbb{1})$, meaning all such operators are trivial.

As a result all σ_t^ω also acts trivially, but then automatically we have $\mathcal{M}(O) = \mathcal{M}(O)_\omega = \mathbb{C}\mathbb{1}$, which contradicts our assumption that $\mathcal{M}(O)$ is semi-finite (cf. Thm. VIII.3.14, [46]). Therefore, $\mathcal{M}(O)$ must be type III factor.

Since $\mathcal{M}_\omega(O)' = \mathcal{M}(O)$, $\mathcal{M}_\omega(O)' \cap \mathcal{M}(O) \neq \mathbb{C}\mathbb{1}$. Thus using Thm. 4.1 given in [21] one can see that $\mathcal{M}(O)$ cannot be type III$_\nu$, $0 < \nu < 1$. Moreover, Thm. 5.2.1 [22] shows that it cannot be type III$_0$ either. As such, we conclude $\mathcal{M}(O)$ is type III$_1$. □

Though we don't need for our analysis, the results for the central decomposition follow. Since the algebras under consideration are dilation invariant, Thm. 4.2 and prop. 4.3 given in [24] are applicable here as well. We shall return to this theorem in sect. 5.1 when we consider the local algebras for other spacetime regions.

Chapter 3

Elements of Algebraic Quantum Field Theory

Seeds of AQFT were sown in the mid-fifties [3, 4]. Today, AQFT is a framework that renders a rigorous mathematical formulation of standard QFT. Its base relies on a set of axioms [8], involving operator algebras associated with spacetime regions.

Below we outline the axioms, which differ from the Wightman's axioms [6, 10] in the sense that the latter explicitly make use of fields. For our purpose, AQFT seems to be a suitable framework.

3.1 Axioms: A Set of Rules

For convenience, let us gather some definitions and fix the notations. Let $\mathcal{A} \subset \mathfrak{B}(\mathcal{H})$ denote in general an involutive Banach algebra acting on a Hilbert space $\mathcal{H}$, then $\mathcal{A}$ is called a *C^*-algebra* if $\|A^*A\| = \|A\|^2$. $\mathcal{A}$ is called *unital*, if $\mathcal{A}$ contains an identity operator $\mathbb{1}$. A state on a unital C^*-algebra is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ that is positive and normalized, meaning $\omega(A) \geq 0$ for every $A \in \mathcal{A}^+$ and $\omega(\mathbb{1}) = 1$. Here $\mathcal{A}^+$ stands for the positive cone of $\mathcal{A}$. A space of all such states is the *state space*, denoted by $\mathcal{A}_1^{*+}$. The commutant $\mathcal{A}'$ is the set of all elements from $\mathfrak{B}(\mathcal{H})$ that commute with all elements of $\mathcal{A}$. The double commutant is $\mathcal{A}'' = (\mathcal{A}')'$.

Let $\mathbb{R}^{1+d}$ be the $d+1$ dimensional Minkowski spacetime. The metric signature will be mostly minus. Let $\mathcal{P}_+^\uparrow = \mathbb{R}^{1+d} \rtimes \mathcal{L}_+^\uparrow$ be the connected component of the identity of the Poincaré group, with $\mathcal{L}_+^\uparrow$ being the proper orthochronous Lorentz group. Then its unitary representation in a

Hilbert space $\mathcal{H}$ will be denoted by $U(\Lambda, a)$ with $a \in \mathbb{R}^{1+d}$ and $\Lambda \in \mathcal{L}_+^\uparrow$. We shall frequently use the following identifications $U(a) \equiv U(\mathbb{1}, a)$ and $U(\Lambda) \equiv U(\Lambda, \mathbb{1})$.

Fundamental objects in AQFT are observables and states. These are the mathematical objects that in many ways deal with operations in a given region. While the former describes objects that can be measured, the latter are connected to a measurement outcome. Without going into any details of measurements, we simply set the stage by focusing on spacetime regions.

Let $O \subset \mathbb{R}^{1+d}$ denote any open[1] bounded region and $\mathfrak{O}(\mathbb{R}^{1+d})$ be the set of all such regions in $\mathbb{R}^{1+d}$. As such the mapping, $O \mapsto \mathcal{U}(O)$, for every $O \in \mathfrak{O}(\mathbb{R}^{1+d})$ establishes the fundamental object in AQFT called an algebra of local observables. The causal complement for every $O \in \mathfrak{O}(\mathbb{R}^{1+d})$ is defined by

$$O^c \equiv O' \doteq \mathrm{Int}\Big\{x \in \mathbb{R}^{1+d} \,\Big|\, (x-y)^2 < 0 \;\forall\, y \in O\Big\}. \tag{3.1.1}$$

The Haag-Araki(HA) [8,9] formulation of AQFT deals with von Neumann algebras $\mathcal{U}(O) = \mathcal{M}(O)$. On the other hand, the Haag-Kastler(HK) approach [7] is based on C^*-algebras[2] $\mathcal{U}(O) = \mathcal{A}(O)$. Any of these two formulations satisfy the following set of axioms, which can be found in any standard book on AQFT [8, 51]. The main reason to repeat them here is two fold. One, it will help the reader to get acquainted with our notations and conventions, some of which will be used in later chapters. Second, we shall point-out some relevant works or add additional comments as we go through the axioms.

(a) **Isotony**: If $O_1 \subset O_2$, then $\mathcal{M}(O_1) \subset \mathcal{M}(O_2)$ and $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

Let $\{\mathcal{M}(O)\}_{O\in\mathfrak{O}(\mathbb{R}^{1+d})}$ and $\{\mathcal{A}(O)\}_{O\in\mathfrak{O}(\mathbb{R}^{1+d})}$ be the nets of local algebras consisting of local observables. Since for the former W^*-inductive limit and for the latter C^*-inductive limit is defined, see e.g. [39, 52], one considers a sequence $(O_i)_{i\in I}$ of monotonically increasing spacetime regions in $\mathbb{R}^{1+d}$ to approximate an arbitrary unbounded region $\tilde{O}$ that through canonical mapping leads to the corresponding local algebra $\mathcal{U}(\tilde{O})$.

In fact, the C^*-inductive limit of the C^*-algebra $\mathcal{A}(O)$, $O \in \mathfrak{O}(\mathbb{R}^{1+d})$ with $O \subset \tilde{O}$, is the C^*-algebra $\mathcal{A}(\tilde{O})$. As for a von Neumann algebra, being a special case of a C^*-algebra as considered in sect. 2.1, both W^*-inductive limit and C^*-inductive limit exist. As such, for

[1] In principle, it can be closed, the closure of a given region, as well.
[2] Indeed, a von Neumann algebra is the C^* one which has a predual, see def. 2.1.8.

the von Neumann algebra, $\mathcal{M}(O)$, $O \in \mathfrak{O}(\mathbb{R}^{1+d})$ with $O \subset \tilde{O}$, $\mathcal{A}(\tilde{O})'' = \mathcal{M}(O)$ holds.

If $\tilde{O}$ happens to be the whole $\mathbb{R}^{1+d}$, then the corresponding algebras $\mathcal{M}(\mathbb{R}^{1+d}) \equiv \mathcal{M}$ and $\mathcal{A}(\mathbb{R}^{1+d}) \equiv \mathcal{A}$ are known as a *global algebra* and a *quasilocal algebra*, respectively.

(b) **Weak additivity**: Let $O + a$ denote translation of any given region $O \in \mathfrak{O}(\mathbb{R}^{1+d})$ by the four vector $a \in \mathbb{R}^{1+d}$, then for every such O one has

$$\mathcal{M} = \left\{ \bigcup_{a \in \mathbb{R}^{1+d}} \mathcal{M}(O + a) \right\}'' \quad \text{and} \quad \mathcal{A} = \overline{\bigcup_{a \in \mathbb{R}^{1+d}} \mathcal{A}(O + a)}^{C^*}, \tag{3.1.2}$$

where overline with C^* stands for the closure taken with respect to the C^*-norm. Clearly, the global structure can be obtained by gradually translating the region O such that $\mathcal{A}(O + a) \subset \mathcal{A}$ always. This axiom is crucial for the Reeh-Schlieder(R-S) theorem, see sect. 3.2.

A stronger version of this requirement is known as *strong additivity*, which says that if $\{O_i\}_{i \in I}$, $O_i \in \mathfrak{O}(O)$ is a given family of regions such that $O = \bigcup_{i=1}^{n} O_i$ then the corresponding algebra is collectively generated in the following sense

$$\mathcal{M}(O) = \left\{ \bigcup_{i=1}^{n} \mathcal{A}(O_i) \right\}'' \quad \text{and} \quad \mathcal{A}(O) = \overline{\bigcup_{i \in I} \mathcal{A}(O_i)}^{C^*}. \tag{3.1.3}$$

Strong additivity implies weak additivity and even isotony.

(c) **Locality**: For every $O_i \in \mathfrak{O}(\mathbb{R}^{1+d})$, $i = 1, 2$; such that, $O_1 \subset O_2'$, one has

$$\mathcal{M}(O_1) \subset \mathcal{M}(O_2)'. \tag{3.1.4}$$

On the other hand, in the HK approach the corresponding expression is given by

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0. \tag{3.1.5}$$

Note the key difference between algebras, where for the former and latter, global and quasi-local algebras are used respectively. A stronger version of this axiom, known as

Haag duality, is the statement that

$$\mathcal{M}(O_1) = \mathcal{M}(O_2)'. \tag{3.1.6}$$

Duality for free scalar field theory is due to Araki [53] (see also [54] for a somewhat related commutation theorem). Those were Eckmann and Osterwalder [20] who first used Tomita-Takesaki theory (see sect. 4.1) to prove the duality for free Bose fields.

(d) **Poincaré covariance**: In the HA approach there exists a strongly continuous unitary representation $U(\Lambda, a)$ of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$ such that for every $O \in \mathfrak{O}(\mathbb{R}^{1+d})$ we have

$$U(\Lambda, a)\mathcal{M}(O)U(\Lambda, a)^{-1} = \mathcal{M}(\Lambda(O + a)). \tag{3.1.7}$$

For the HK formulation there exists a strongly continuous automorphism $\alpha_{(\Lambda,a)}$ such that

$$\alpha_{(\Lambda,a)}\left(\mathcal{A}(O)\right) = \mathcal{A}(\Lambda(O + a)) \tag{3.1.8}$$

Note that in HK approach there exists a (vacuum) vector, $\Omega \in \mathcal{H}$, that is cyclic (def. 2.1.11) for $\mathcal{A}$ (from GNS construction), and translation invariant, i.e., $U(a)\Omega = \Omega$. On the other hand, in HA approach there is no such notion, see below.

(e) **Spectral condition**: The support of a spectral measure corresponding to the spectrum of $U(a)$ is contained in the closed forward lightcone $\bar{V}_+$. The integral form of $U(a)$ is given by (cf. (3.2.1))

$$U(a) = \int d\mu(p)\, e^{ip\cdot a}, \tag{3.1.9}$$

where $\mu(p)$ is the spectral measure for which

$$\operatorname{supp}\mu(p) \subset \overline{V}_+ \doteq \left\{p \in \mathbb{R}^{1+d}\,\middle|\, p^2 \geq 0,\, p_0 \geq 0\right\}. \tag{3.1.10}$$

Note that in the HK approach, the spectral condition does not exist, as there is no notion of unitary group associated with translations. Whereas, there are some ways to deal with this problem, as we comment below. In any case, a central principle is the same, look for the positive energy representation of the Poincaré group. It is worth noting that this axiom is fundamental like all the others above. In particular, for the quasilocal algebras assuming rest of the axioms independence of this axiom has been verified, see [12].

Remark 3.1.1. In the above formulation of axiom (e), existence of translation covariant representation is crucial, which is guaranteed by the existence of translation invariant state on $\mathcal{A}$, which follows from the Markov-Kakutani fixed point theorem [55, Thm. 5.20]. Here the fixed point theorem is applicable, since the automorphism induced by the unitary representation of the translation subgroup can be considered as a family of continuous affine maps acting the set of states $\mathcal{A}_1^{*+}$, which is in fact a convex compact set, then the concerned theorem implies there exists translation invariant state, i.e., the fixed point of the automorphism.

First condition of such type was worked out by Doplicher in [11], which gives the criterion for the positive representation of $\mathcal{A}$ to exist:

Let $\mathcal{L}^1(\mathbb{R}^{1+3})$ be the set of summable functions such that their Fourier transforms satisfy

$$\hat{f}(p) = 0 \quad \forall f \in \mathcal{L}^1(\mathbb{R}^{1+3}) \quad \text{and} \quad \forall p \in V_+. \tag{3.1.11}$$

For such functions we define

$$\mathcal{B} \doteq \left\{ A_f \in \mathcal{A} \,\middle|\, A \in \mathcal{U},\ (3.1.11)\,\text{holds} \right\} \tag{3.1.12}$$

with $A_f \doteq \int d^4x\, f(x) U(x) A U(x)^{-1}$. Then the criterion is that $\mathcal{A}$ has at least one positive representation satisfying the spectral condition if and only if the minimal translation invariant left ideal, $\mathcal{J} \doteq \text{span}\{\mathcal{A}\mathcal{B}\}$, satisfies $\mathcal{J} \subset \mathcal{A}$. See also [56] for further studies.

3.2 The Reeh-Schlieder Theorem and its Converse

In this section, we consider the Reeh-Schlieder (R-S) theorem [57] in the algebraic setting, as well as, its converse, which is due to Borchers [13]. In particular, our plan here is to tackle the following points, which will allow us to see the axioms coming into action, as such, serve as a nice example of an abstract mathematical setup that emerges from mere application of the axioms without additional requirements.

→ The algebraic version of the R-S theorem.

→ Bounded spectrum of $U(x)$, in particular, the spectrum of the energy-momentum operator, and its consequence on the R-S theorem.

→ The converse of the generalization of the R-S theorem.

→ Independence of the group representation for the converse.

As we shall see later, the R-S theorem plays a crucial role in our analysis. In spite of its counter intuitive features, it holds against alternative ideas. One of such efforts, its physical consequences and its comparison with the R-S theorem are discussed with adequate mathematical rigour in [58].

Let $\mathbb{R}^{1+d}$ be the $d+1$ dimensional Minkowski spacetime and $T_x = \{\mathbb{1}, x\}$ denote the translation subgroup of the Poincaré group. Let $U(x)$ be the unitary representation of this group on a Hilbert space $\mathcal{H}$. Note that the corresponding generator is not necessarily bounded in general. Henceforth the spectrum of $U(x)$ means the spectrum of the corresponding generator, the energy-momentum operator.

We will need the axioms (a), (b), (d)—with $U(\Lambda, a)$ replaced by $U(x)$, i.e., only translations, and (e). Note that we neither require the full Poincaré covariance (d) nor locality condition (c) nor the existence of a vacuum, which is generally added for a matter of convenience (see [59] and references therein for the exposure of some work in this direction). In the current setting, one can say that the proof of the R-S theorem depends on weak-additivity (b).

Before we state and prove the theorem we need a few facts.

Definition 3.2.1. If $\mathcal{V}(0)$ is a neighbourhood filter of the origin in $\mathbb{C}^{1+d}$, then a vector $\xi \in \mathcal{H}$ is said to be *analytic* for $U(x)$ if $U(x)\xi$ can be analytically continued to the entire system $\mathcal{V}(0)$.

Let G denote any locally compact abelian group and $\hat{G}$ its character group, and $U(g)$, $g \in G$ be a strongly continuous unitary representation of G in $\mathcal{H}$, then as per the SNAG(Stone-Naimark-Ambrose-Godement) theorem (vid. e.g. Thm. 4.45 in [60]), which is basically a generalization of the Stone's theorem, there exists a unique regular projection valued measure μ on the character group $\hat{G}$ such that

$$U(g) = \int_{\hat{G}} d\mu(\hat{g}) \, \langle \hat{g}, g \rangle \,, \tag{3.2.1}$$

for all $g \in G$, where $\hat{g} \in \hat{G}$ is some fixed element. Then it is easily shown to hold

$$U(g_1 - g_2) = \int_{\hat{G}} d\mu(\hat{g})\ \langle \hat{g}, g_1 - g_2 \rangle = U(-g_2)U(g_1). \tag{3.2.2}$$

Taking $G = T_x \cong \mathbb{R}^{1+d}$ to be the additive group, for which $\hat{G}$ is the same as G (realized as the momentum space), (3.2.1) and (3.2.2) reduce to

$$U(x) \quad = \quad \int d\mu(p)\, e^{ip\cdot x} \quad \forall\, x \in G,\, p \in \hat{G}, \tag{3.2.3}$$

with $\text{supp}\mu(p) \subset \bar{V}_+$, and

$$U(x_1 - x_2) \quad = \quad U(-x_2)U(x_1) \quad \forall\, x_1, x_2 \in G. \tag{3.2.4}$$

In particular, we have a homomorphism $U(x) \colon G \to \mathcal{U}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$ with $\mathcal{U}(\mathcal{H})$ denoting unitaries acting on $\mathcal{H}$, that satisfies the following properties:

- $U(x + y) = U(x)U(y)$,
- $U(x)^{-1} = U(-x) = U(x)^*$,
- $U(x)$ is strongly continuous in $\mathcal{H}$, i.e., $G \ni x \mapsto U(x)\xi$ is continuous for all $\xi \in \mathcal{H}$.

Let $\mathcal{N} \subset \mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ be von Neumann algebras as defined in the previous chapter. Then with $\mathcal{N}(x) \doteq U(x)\mathcal{N}U(-x)$ we define the algebra $\mathcal{N}_{V_0}$, associated with any open neighbourhood V_0 of the origin in $\mathbb{R}^{1+d}$, by

$$\mathcal{N}_{V_0} \doteq \Big\{ \bigcup_x \mathcal{N}(x) \,\Big|\, x \in V_0 \Big\}''. \tag{3.2.5}$$

Now assuming weak additivity (b) we prove a generalized version of the R-S property. The R-S theorem in its original form states the following: $\Omega \in \mathcal{H}$ with the property[3] $U(\Lambda, a)\Omega = \Omega$ (note that the underlying representation refers to the Poincaré group) is cyclic and separating (def. 2.1.11) for $\mathcal{M}(O)$, $O \subset \mathbb{R}^{1+d}$, such that O and O' are non-empty, more precisely, O contains a finite set other than a point. It is this property of the vector Ω that will be imperative in the next chapter.

[3] Which is what one means by the vacuum in AQFT for it being uniquely defined by the Poincaré invariance.

Here we consider a larger class of vectors from $\mathcal{H}$ that are analytic in the sense of def. 3.2.1. Since cyclic vectors, whose power series have positive radius of convergence, are special instances of analytic vectors, the following results also holds for the case of our interest. We shall refer to this generalization as simply the R-S property.

Theorem 3.2.1. *Let $\mathcal{N} \subset \mathcal{M}$ such that*

$$\mathcal{N}_G = \left\{ \bigcup_{x \in G} \mathcal{N}(x) \right\}'' = \mathcal{M}, \tag{3.2.6}$$

then for any vector $\eta \in \mathcal{H}$ that is analytic for $U(x)$, and for any V_0 in G, one has the following equality

$$\mathcal{N}_{V_0}\eta = \mathcal{M}\eta. \tag{3.2.7}$$

Proof. Let $\eta \in \mathcal{H}$ be analytic for $U(x)$, then using def. 3.2.1, $U(x)\eta$ is analytic in some complex neighbourhood $N \subset \mathcal{V}(0)$. Since $G = T_x$ is abelian, $U(x+y)\eta = U(y)(U(x)\eta)$ is also analytic in N for any $y \in G$. In that case $U(x)$, given in (3.2.3), can be extended to the whole system of complex neighbourhoods $\mathcal{V}(0)$ by analytic continuation such that it yields

$$U(z) = \int d\mu(p)\, e^{ip\cdot z}, \tag{3.2.8}$$

for some complex vector z. $U(z)$ is a strongly continuous function of z in the closed forward tube $\overline{\mathcal{I}} \doteq \{z \mid \operatorname{Im} z \in \overline{V}_+\}$ and holomorphic in the open forward tube $\mathcal{I} \doteq \{z \mid \operatorname{Im} z \in V_+\}$. Altogether, we have $U(z)\xi$ as an operator valued function with values in $\mathcal{H}$, which is holomorphic in $\mathcal{I}$ for any $\xi \in \mathcal{H}$. But the support of μ lies in $\overline{V}_+$, which allows us to conclude that $U(z)$ is the boundary value of $U(z)\xi$. As such, for some vector $\xi \in \mathcal{H}$ we want to show that ξ is orthogonal to the closed subspaces $[\mathcal{N}_G\eta]$ and $[\mathcal{M}\eta]$.

Let us commence by observing that $A_i(x_i) \doteq U(x_i)A_iU(-x_i) \in \mathcal{N}_G$ for every $x_i \in V_0$ and $A_i \in \mathcal{N}$, where $i = 1, 2, \ldots, n$. Assume that ξ is orthogonal to $\mathcal{N}_{V_0}\eta$, then

$$(\xi, A_1(x_1)A_2(x_2)\ldots A_n(x_n)\eta) = 0. \tag{3.2.9}$$

Using (3.2.8) one can define a holomorphic function for every $z_i \in \mathcal{I}$ by

$$\mathcal{F} \equiv (\xi, U(z_1)A_1\,U(z_2)A_2 \ldots U(z_n)A_nU(z_{n+1})\eta). \tag{3.2.10}$$

Then setting $z_i = x_i - x_{i-1}$ with $x_0 = x_{n+1} = 0$ in (3.2.10) and using (3.2.4) one derives the following chain of equalities

$$\begin{aligned} \mathcal{F} &= (\xi, U(x_1)A_1\,U(x_2 - x_1)A_2 \ldots U(x_n - x_{n-1})A_nU(-x_n)\eta) \\ &= (\xi, U(x_1)A_1\,U(-x_1)\,U(x_2)A_2U(-x_2) \ldots U(x_n)A_nU(-x_n)\eta) \\ &= (\xi, A_1(x_1)A_2(x_2) \ldots A_n(x_n)\eta), \end{aligned} \tag{3.2.11}$$

where we used analyticity of η. Comparing (3.2.9) and (3.2.11) one concludes that $\mathcal{F} = 0$ identically. Then by analytic continuation one finds that (3.2.9) is true for all $x_i \in G$. So it follows that ξ is orthogonal to $[\mathcal{N}_G\eta]$, hence due to (3.2.6), to $[\mathcal{M}\eta]$ as well. But from the construction $\mathcal{N}_{V_0}$ is a dense $*$-subalgebra of $\mathcal{N}_G$, so (3.2.7) holds. □

Note that adding (c) to the list of axioms assumed above, provides one with extra structural properties of the associated algebras, see e.g., Thm. 1.3.1 in [51].

Now we shall prove a trivial consequence of the R-S theorem. For that purpose, we need the following setup. Let $\sigma_x \colon A \in \mathcal{M} \mapsto \sigma_x(A)$ for every $x \in G$ be an automorphism (def. 2.2.7) of $\mathcal{M}$ defined by[4] $\sigma_x^U(A) \doteq U(x)AU(-x) = A(x) \in \mathcal{M}$ for every $A \in \mathcal{M}$. Clearly for $U \in \mathcal{M}$, σ_x is just an inner automorphism (def. 2.2.8). We use σ_0, with $U(0) \equiv \mathbb{1} \in \mathcal{U}(\mathcal{H})$, to denote the identity automorphism. Then the spectrum of $U(x)$ is bounded if $\|\sigma_x - \sigma_0\| \to 0$ as $x \to 0$ [61]. The automorphism σ satisfying this requirement will be referred to as norm continuous. Now as a consequence of the Thm. 3.2.1, we have the following result.

Corollary 3.2.1.1. *Suppose the assumptions of Thm. (3.2.1) hold. In addition, if the spectrum of $U(x)$ is bounded then*

$$\mathcal{N}_{V_0} = \mathcal{M}. \tag{3.2.12}$$

Proof. Since the spectrum of $U(x)$ is bounded, equivalently σ is norm continuous, previous

[4] We shall drop the index U, when it is clear from the context.

theorem gives $\mathcal{N}_{V_0}\xi = \mathcal{M}\xi$, $\forall\, \xi \in \mathcal{H}$. Then the claim follows. □

Note that the covariance system, i.e., a pairing $\{\mathcal{M}, \sigma\}$ with a bounded spectrum satisfies $\sigma_x(\mathcal{M}(O)) = \mathcal{M}(O)$ for every $O \subset \mathfrak{O}(\mathbb{R}^{1+d})$ [61]. Since quasilocal algebras are non-commutative, the spectrum of $U(x)$ must be unbounded for a quantum system.

We will need the following important result to prove the converse of the Thm. 3.2.1.

Proposition 3.2.1. *Suppose $U(x)$ and $V(x)$ are two continuous unitary representations of G that satisfy the spectrum condition (e), and for the corresponding induced automorphisms, $A(x) = \sigma^U_x(A) = \sigma^V_x(A)$ holds for every $A \in \mathcal{M}$, then*

$$[\mathcal{M}\eta] = [\mathcal{N}\eta] \tag{3.2.13}$$

for every analytic vectors of $U(x)$ if and only if the equality (3.2.13) holds also for every analytic vectors of $V(x)$.

Proof. Let $W(x)$, $x \in G$, be another representation such that $[W, V] = 0$. Then we define $W(x) \doteq U(x)V(-x)$ that by the construction is a continuous representation of G.

Suppose $[\mathcal{M}\eta] = [\mathcal{N}\eta]$ for every analytic vector $\eta \in \mathcal{H}$ with respect to U. Let some $\xi \in \mathcal{H}$ such that it is analytic for V. Note that $W(x) \in \mathcal{M}'$, as such, we consider a spectral projection E corresponding to a bounded subsets from the spectrum of $W(x)$. Though the spectrum is unbounded as discussed above, one still has $[E, V] = 0$, such that

$$U(x)E\xi = W(x)V(x)E\xi = W(x)E(V(x)\xi). \tag{3.2.14}$$

Since ξ is analytic for V and $W(x)E$ is just an operator valued analytic function of x, we conclude that $E\xi$ is analytic for $U(x)$. Then due to $\mathcal{N} \subset \mathcal{M}$ and $W(x) \in \mathcal{M}'$ one finds

$$[\mathcal{N}E\xi] = [\mathcal{M}E\xi] \implies E([\mathcal{N}\xi] - [\mathcal{M}\xi]) = 0, \tag{3.2.15}$$

which is true for all such projectors E, in particular, from every strong neighbourhood of $\mathbb{1} \subset \mathfrak{B}(\mathcal{H})$. Then we must have $[\mathcal{N}\xi] = [\mathcal{M}\xi]$, which holds for any arbitrary, hence every $\xi \in \mathcal{H}$ that is analytic for V. The converse is proved in a verbatim manner.

To prove the trickier part, namely when $[W, V] \neq 0$ for which W is not a representation, one can take an advantage of splittings of $U(g)$ and $V(g)$ given by $U(x) = U_1(x)U_2(x)$ and $V(x) = V_1(x)V_2(x)$ with $U_1, V_1 \in \mathcal{M}$ and $V_2, U_2 \in \mathcal{M}'$ [62, lemma 3]. This information basically provides us with

$$UU_1 = U_1U_2U_1 = U_1(U_1U_2) = U_1U \quad \text{and} \quad VV_1 = V_1V_2V_1 = V_1(V_1V_2) = V_1V. \tag{3.2.16}$$

Since V_1 and U_1 are from the same algebras, the above equalities extend to the original ones (in terms of W, V, U), where now $W_1 = U_1V_1$, so $[V_1, U_1] = 0$. Similarly, $[V, V_2] = [U, U_2] = [V_2, U_2] = 0$. Ultimately, one has $[V, U] = 0$. Thus,

$$WV = (UV)V = V(UV) = VW. \tag{3.2.17}$$

Then clearly, one can repeat the same procedure as above and establish the desired equalities. □

Now we have the converse of Thm. 3.2.1 as follows.

Theorem 3.2.2. *Let $\mathcal{N} \subset \mathcal{M} \subset \mathfrak{B}(\mathcal{H})$. Let $U(x)$ denote the strongly continuous unitary representation of G, which is arbitrary due to prop. 3.2.1.*

If any vector $\eta \in \mathcal{H}$ that is analytic for $U(x)$ satisfies $[\mathcal{M}\eta] = [\mathcal{N}\eta]$, then

$$\mathcal{M} = \mathcal{N}_G. \tag{3.2.18}$$

Proof. By the spectrum condition (e), the spectrum of the representation $U(x)$ is contained in $\overline{V}_+$, and the algebra $\mathcal{M}$, hence $\mathcal{N}$, is stable under the conjugation by $U(x)$, i.e., $\sigma_x(\mathcal{M}) = \mathcal{M}$, $\forall\, x \in G$. Then by [62, main Thm.,pg. 2] (see also [53, prop. 1] for the exposure of another approach, which antedates [62]) we have, $\sigma_x(A) = A$, $\forall\, A \in \mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$. In other words, the center of $\mathcal{M}$ is point-wise invariant under the conjugation by $U(x)$, which is in fact due to $[A, U(x)] = 0$ for every $A \in \mathcal{Z}(\mathcal{M})$ and for every $x \in G$. Moreover, there exists $V(x) \in \mathcal{M}$ such that $\sigma_x^U(A) = \sigma_x^V(A)$ for all $A \in \mathcal{M}$. So, we also have $U(x) \in \mathcal{M}$ as a bonus.

This allows us to split $U(x)$ in terms of some $U_1(x) \in \mathcal{N}_G$ and $U_2(x) \in \mathcal{N}_G' \cap \mathcal{M}$, which takes the form as $U(x) = U_1(x)U_2(x)$ [62]. Let E denote the spectral projection corresponding to $U_2(x)$. Since $U_2(x)$ is from the joint set of commutant of $\mathcal{N}_G$ and $\mathcal{M}$, E commutes with $U(x)$ and $U_1(x)$. As a result, one finds (3.2.15) to be true here as well. But this time for the set $\{E\xi\}$ with ξ being an analytic vector, we have $[E\xi] = E\mathcal{H}$, meaning that $E\xi$ is dense in the subspace

$E\mathcal{H}$. It basically provides us with the following equality for all $\xi \in \mathcal{H}$

$$\sigma_x^U(E)\xi = E\xi, \tag{3.2.19}$$

hence, $\sigma_x^U(E) = E$. As a result, $E \in \mathcal{M}'$, but then $U_2(x) \in \mathcal{Z}_{\mathcal{M}}$. Due to which one obtains

$$\sigma_x^U(A) = \sigma_x^{U_1}\left(\sigma_x^{U_2}(A)\right) = \sigma_x^{U_1}\left(U_2(x)AU_2(-x)\right) = \sigma_x^{U_1}(A). \tag{3.2.20}$$

But $U_1(x) \in \mathcal{N}_G$ and the last equality holds for every $A \in \mathcal{M}$, so we conclude $U(x) \in \mathcal{N}_G$. Still it remains for us to see the case when $E \in \mathcal{N}_G'$. Clearly, for analytic vectors $\xi \in \mathcal{H}$, $E\xi$ is also analytic. Since the image $E\xi$ is invariant under $\mathcal{M}$ (analogous to (3.2.19)), we have $E \in \mathcal{M}'$. Altogether, it implies $\mathcal{N}_G' \subset \mathcal{M}' \implies \mathcal{M} \subset \mathcal{N}_G$. But $\mathcal{N}_G \subset \mathcal{M}$ from the construction itself. Therefore, $\mathcal{M} = \mathcal{N}_G$. □

Naively one could think of one of the consequences of Thms. 3.2.1 and 3.2.2 that the R-S property is equivalent to the weak-additivity. However, one should be careful before arriving at this conclusion is clear from the following remark.

Remark 3.2.1. It is crucial to note that Thms. 3.2.1 and 3.2.2 are not strictly the converse of one and another. It might be explained better with the following counterexample. Let $\mathcal{H} = l^2$ be the space of square summable sequences $(a_n)_{n\in\mathbb{N}}$ and define the action of one parameter unitary group $U(t)$, $t \in \mathbb{R}$, by $U(t)(a_n)_{n\in\mathbb{N}} \doteq e^{itn}(a_n)_{n\in\mathbb{N}}$. Let P_v, with $v = e^{-n}$ being a vector that is analytic for $U(t)$, be the projection onto the one dimensional subspace $\{zv\}$, $z \in \mathbb{C}$. Let the algebra $\mathcal{N}$ generated by $\mathbb{1}$ and P_v be given by $\mathcal{N} = \{z_1\mathbb{1} + z_2P_\lambda\}$, $z_1, z_2 \in \mathbb{C}$, then one can straightforwardly see that

$$\mathcal{N}_G = \left\{\bigcup_{t\in\mathbb{R}} U(t)\{z_1\mathbb{1} + z_2P_v\}U(-t)\right\}'' = \mathfrak{B}(\mathcal{H}) = \mathcal{M}'' = \mathcal{M}, \tag{3.2.21}$$

which is in fact the claim of the converse given in (3.2.18). Whereas, due to $[\mathcal{N}v] = P_v\mathcal{H}$ and $[\mathcal{M}v] = \mathcal{H}$, i.e., the latter is dense in $\mathcal{H}$ but not the former one, we clearly have $[\mathcal{M}v] \neq [\mathcal{N}v]$. But then a-priori condition of Thm. 3.2.2, namely the R-S property $[\mathcal{M}\xi] = [\mathcal{N}\xi]$ for any analytic vector $\xi \in \mathcal{H}$, fails.

Chapter 4

The Tomita-Takesaki Modular Theory

Modular theory, also known as Tomita-Takesaki (TT) theory [18], is an indispensable part of AQFT. It was introduced by Tomita [16, 17], and enhanced by Takesaki [18], hence the name. The theory itself associates von Neumann algebras with intrinsic dynamics rendered by automorphism group, called modular flows.

4.1 Modular Theory

We consider a special case of TT modular theory, based on [37] (see also [40, 48]), with a cyclic and separating vector Ω. For the generalization of the present case we refer to [18, 46]. Let $\mathcal{H}$ be a complex Hilbert space and $\overline{\mathcal{H}}$ the corresponding conjugate space. We consider $\mathcal{H}$ and $\overline{\mathcal{H}}$ as sets, such that, $\overline{\mathcal{H}}$ is just a set $\mathcal{H}$ with the scalar multiplication given by $z, \xi \mapsto \overline{z}\xi$ for $z \in \mathbb{C}$ and $\xi \in \mathcal{H}$, and the scalar product $\xi, \eta \mapsto \overline{(\xi, \eta)} = (\eta, \xi)$ for $\xi, \eta \in \mathcal{H}$. Here the scalar product $\xi, \eta \mapsto (\xi, \eta)$ corresponds to $\mathcal{H}$, which will be taken antilinear in the first argument and linear in the second argument.

Let $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \overline{\mathcal{H}}$ be a linear operator, where $\mathcal{D}(A)$ is the domain of A, then naturally its adjoint is $A^*\colon \mathcal{D}(A^*) \subset \overline{\mathcal{H}} \to \mathcal{H}$. As such, A and A^* satisfy

$$(\eta, A\xi) = (A^*\eta, \xi) = \overline{(\xi, A^*\eta)} \quad \forall\, \xi \in \mathcal{D}(A), \forall\, \eta \in \mathcal{D}(A^*). \tag{4.1.1}$$

On the other hand if A is antilinear or equivalently conjugate-linear then

$$(\eta, A\xi) = \overline{(A^*\eta, \xi)} = (\xi, A^*\eta) \quad \forall\, \xi \in \mathcal{D}(A), \forall\, \eta \in \mathcal{D}(A^*). \tag{4.1.2}$$

Now before we consider the central object of this chapter we recall a few definitions.

Definition 4.1.1. Let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator. Let the corresponding graph be $\Gamma(A) \doteq \{(\xi, A\xi) \mid \xi \in \mathcal{D}(A)\} \subseteq \mathcal{H} \times \mathcal{H}$. Then A is *closed* if $\Gamma(A)$ is closed. Let $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ denote some other linear operator such that $\Gamma(A) \subset \Gamma(B)$, then B is called an *extension* of A. A smallest such extension is known as the closure of A, which will be denoted by $\overline{A}$. An operator A is *preclosed/closable* if it has a closed extension. Alternatively, densely defined A is closable if A^* is densely defined. In that case, the closure of A is just $\overline{A} = (A^*)^* = A^{**}$.

A starting point of the TT modular theory is to consider a von Neumann algebra $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$ with $\Omega \in \mathcal{H}$ cyclic and separating (vid. R-S theorem 3.2, see also def. 2.1.11) with respect to $\mathcal{M}$. Then one considers a mapping $\mathcal{M}\Omega \ni A\Omega \mapsto A^*\Omega$ that is closable as an anti-linear operator from $S_0 \colon \mathcal{H} \to \overline{\mathcal{H}}$, see below. Let the closure of S_0, i.e., $\overline{S}_0$, be denoted by S, known as Tomita's operator [16], and set $F = S^*$. In what follows next, we shall be using the following actions of S_0 and F_0:

$$\begin{aligned} S_0 A\Omega &= A^*\Omega, \quad A \in \mathcal{M}; \\ F_0 A'\Omega &= A'^*\Omega, \quad A' \in \mathcal{M}'. \end{aligned} \tag{4.1.3}$$

These actions are well defined on the corresponding dense domains $\mathcal{D}(S_0) = \mathcal{M}\Omega$ and $\mathcal{D}(F_0) = \mathcal{M}'\Omega$. (See [63] for the bounded operator approach to the concerned theory.) With this preparation we are now ready to have the very first preliminary result.

Proposition 4.1.1. *Let S_0, S and F be as above. Then S_0 is closable antilinear, and one has $\mathcal{M}'\Omega \subset \mathcal{D}(F)$ and $FA'\Omega = A'^*\Omega$, $A' \in \mathcal{M}'$. Furthermore, S and F are involutions in the following sense: for $\xi \in \mathcal{D}(F) \implies F\xi \in \mathcal{D}(F)$ and $FF\xi = \xi$. Similarly, the equalities hold for S as well.*

Proof. First we want to show that S_0 is closable. For every $A \in \mathcal{M}$ and for every $A' \in \mathcal{M}'$ one finds

$$\big(A'\Omega, S_0 A\Omega\big) \;\; = (A'\Omega, A^*\Omega) = (AA'\Omega, \Omega) = (A'A\Omega, \Omega) = (A\Omega, A'^*\Omega)\,. \tag{4.1.4}$$

Whereupon one immediately infers $S_0^* A'\Omega = A'^*\Omega$, $A'\Omega \in \mathcal{D}(S_0^*)$, but $\mathcal{D}(S_0) = \mathcal{M}\Omega$. Then using (4.1.3), we have $F_0 \subseteq S_0^*$, thus S_0^* is densely defined. Then by definition S_0 is closable. Since $F = S^*$, we have also have the second claim.

Now let us take $\xi \in \mathcal{H}$ and $A \in \mathcal{M}$, then

$$\begin{aligned}(A\Omega, FF\xi) &= (F\xi, F^*A\Omega) = (F\xi, SA\Omega) = \\ &= (F\xi, A^*\Omega) = (SA^*\Omega, \xi) = (A\Omega, \xi)\,.\end{aligned} \tag{4.1.5}$$

Similarly, it follows for S that $(A'\Omega, SS\xi) = (A'\Omega, \xi)$, $A' \in \mathcal{M}'$. □

Now we prove some equalities that will eventually turn out to be useful for the proof of the main theorem below as well as for some intermediate results.

The polar decomposition of the operator S can be written as

$$S = J\Delta^{\frac{1}{2}}, \tag{4.1.6}$$

where $J : \mathcal{H} \to \overline{\mathcal{H}}$ is an antiunitary operator and $\Delta \doteq FS$ is a positive self-adjoint operator (unbounded in general). In particular, J is a partial isometry and $\Delta^{\frac{1}{2}}$ is the square root of the positive operator Δ. Since $F = S^*$, we also have

$$F = \Delta^{\frac{1}{2}} J. \tag{4.1.7}$$

Since $(S_0)^2 = 1$ on its domain, follows $S_0 = (S_0)^{-1}$, then its closure also satisfies $S = S^{-1}$, but then $F = F^{-1}$ as well. In that case, using the fact that J is antiunitary, so that $J^*J = JJ^* = 1 \implies J^{-1} = J^*$ one finds

$$J\Delta^{\frac{1}{2}} = S = S^{-1} = \Delta^{-\frac{1}{2}} J^* \implies J^2\Delta^{\frac{1}{2}} = J\Delta^{-\frac{1}{2}} J^*. \tag{4.1.8}$$

Here $J\Delta^{-\frac{1}{2}}J^*$ is a positive selfadjoint operator, so by the uniqueness of the polar decomposition of S, we have, $J^2 = \mathbb{1}$, therefore, altogether, it yields

$$J = J^{-1} = J^*. \tag{4.1.9}$$

Meaning J is an antiunitary involution, henceforth will be called a *modular conjugation.* As such, one finds the following trivial consequences:

$$\begin{aligned} J\Delta^{\frac{1}{2}} &= S = S^{-1} = \Delta^{-\frac{1}{2}}J, \\ \Delta^{\frac{1}{2}}J &= F = F^{-1} = J\Delta^{-\frac{1}{2}}. \end{aligned} \tag{4.1.10}$$

Also (4.1.8) reduces to

$$\Delta^{\frac{1}{2}} = J\Delta^{-\frac{1}{2}}J, \tag{4.1.11}$$

whereupon taking square on both sides one obtains

$$\Delta = J\Delta^{-1}J \quad \text{equivalently} \quad \Delta^{-1} = J\Delta J. \tag{4.1.12}$$

Then straightforwardly, because for any bounded Borel function f on $\mathbb{C}$, holds $f(\Delta^{-1}) = J\overline{f(\Delta)}J$, as such, with $f(x) = x^{-i\lambda}$, $\lambda \in \mathbb{R}$, one has

$$J\Delta^{i\lambda} = \Delta^{i\lambda}J \quad \forall\, \lambda \in \mathbb{R}. \tag{4.1.13}$$

Note that Δ, which from now on will be referred to as the *modular operator*, is a non-singular operator, i.e., its inverse exists. In particular, we have

$$\Delta^{-1} = S^{-1}F^{-1} = SF. \tag{4.1.14}$$

Finally, since $S\Omega = \Omega$ and $F\Omega = \Omega$, one also finds

$$\Delta\Omega = \Omega \quad \text{and} \quad J\Omega = \Omega. \tag{4.1.15}$$

Now we are ready to prove some other preliminary results.

Proposition 4.1.2. *For every $z \in \mathbb{C}$ with $|z| = 1$, but $z \neq 1$, we have*

$$(\Delta - z)^{-1} \mathcal{M}'\Omega \subseteq \mathcal{M}\Omega. \tag{4.1.16}$$

Proof. Let $k = (1-z)^{-1}$, such that

$$\bar{k} = (1-\bar{z})^{-1} = z(z-1)^{-1}. \tag{4.1.17}$$

Then

$$k + \bar{k} = (1-z)^{-1} + z(z-1)^{-1} = \tfrac{1-z}{1-z} = 1. \tag{4.1.18}$$

Let $A' \in \mathcal{M}'$ with $0 \leq A' \leq 1$. Define for every $A \in \mathfrak{M}$— selfadjoint elements of $\mathcal{M}$, and for every $B \in \mathcal{M}$, the following normal functionals on $\mathcal{M}$:

$$\begin{aligned} f(B) &\doteq (\Omega, B\Omega), \\ g(B) &\doteq \left(A'\Omega, B\Omega\right), \\ h_A(B) &\doteq f\left(kAB + \bar{k}BA\right). \end{aligned} \tag{4.1.19}$$

Clearly, for $A \in \mathfrak{M}$,

$$\overline{h_A(B)} = f\left(\bar{k}B^*A^* + kA^*B^*\right) = f\left(kAB^* + \bar{k}B^*A\right) = h_A(B^*). \tag{4.1.20}$$

Here g is bounded by $0 \leq g \leq f$. Now we want to show that there exists $A \in \mathfrak{M}$ such that $h_A = g$. Let us consider

$$H \doteq \{h_A \,|\, A \in \mathcal{M}, A^* = A, ||A|| \leq 1\} \subset \mathfrak{M}_*, \tag{4.1.21}$$

which, by Banach-Alaoglu theorem A.1.1, is weak*-compact. Suppose $g \notin H$, then as per the geometrical Hahn-Banach theorem there exists $\tilde{A} \in \mathfrak{M}$ and $\lambda \in \mathbb{R}$, such that, for all $A \in \mathfrak{M}$ with $||A|| \leq 1$ the following inequalities are satisfied

$$g\left(\tilde{A}\right) > \lambda \quad \text{and} \quad h_A\left(\tilde{A}\right) \leq \lambda. \tag{4.1.22}$$

Let $\tilde{A} = \tilde{A}_+ - \tilde{A}_-$ denote a canonical decomposition of $\tilde{A}$ into positive and negative parts and $e \equiv \operatorname{supp} \tilde{A}_+$, also let $A_0 = e - (1-e)$, then

$$\tilde{A}A_0 \quad = A_0\tilde{A} = e\tilde{A} - (1-e)\,\tilde{A} = e\tilde{A}_+ - \left(\tilde{A}_+ - \tilde{A}_- - e\tilde{A}_+\right) = \tilde{A}_+ + \tilde{A}_-. \tag{4.1.23}$$

Therefore, combining the above results the following chain of inequalities follows:

$$\begin{aligned}\lambda < g\left(\tilde{A}\right) = g\left(\tilde{A}_+ - \tilde{A}_-\right) \leq g\left(\tilde{A}_+ + \tilde{A}_-\right) \leq f\left(\tilde{A}_+ + \tilde{A}_-\right) = \\ = f\left(A_0\tilde{A}\right) = f\left(\left(k+\bar{k}\right)A_0\tilde{A}\right) = f\left(kA_0\tilde{A} + \bar{k}\tilde{A}A_0\right) = h_{A_0}\left(\tilde{A}\right) \leq \lambda,\end{aligned} \tag{4.1.24}$$

which clearly contradicts (4.1.22). Therefore, we must have $g \in H$.

Let $A \in \mathfrak{M}$ such that $h_A = g$, which is true for every $B \in \mathcal{M}$, so $h_A(B) = g(B)$, then using (4.1.19) we have

$$\begin{aligned}&\left(A'\Omega, B\Omega\right) = f\left(kAB + \bar{k}BA\right) = \left(\Omega, \left(kAB + \bar{k}BA\right)\Omega\right) \\ &= k\left(\Omega, AB\Omega\right) + \bar{k}\left(\Omega, BA\Omega\right) = \left(\bar{k}A\Omega, B\Omega\right) + \left(B^*\Omega, \bar{k}A\Omega\right).\end{aligned} \tag{4.1.25}$$

Whereupon one obtains

$$\begin{aligned}&\left(B^*\Omega, \bar{k}A\Omega\right) = \left(A'\Omega, B\Omega\right) - \left(\bar{k}A\Omega, B\Omega\right) = \\ &= \overline{\left(B\Omega, A'\Omega\right)} - \overline{\left(B\Omega, \bar{k}A\Omega\right)} = \overline{\left(B\Omega, \left(A' - \bar{k}A\right)\Omega\right)}.\end{aligned} \tag{4.1.26}$$

Now using $SB\Omega = B^*\Omega$ (cf. (4.1.3)) and $F = S^*$ yields

$$\left(\left(A' - \bar{k}A\right)\Omega, B\Omega\right) \quad = \left(SB\Omega, \bar{k}A\Omega\right) = \left(F\bar{k}A\Omega, B\Omega\right). \tag{4.1.27}$$

Therefore, we have the following equalities (recall $A \in \mathfrak{M}$)

$$\left(A' - \bar{k}A\right)\Omega \quad = F\bar{k}A\Omega = kFA\Omega = kF(SA^*\Omega) = kF(SA\Omega) = k\Delta A\Omega. \tag{4.1.28}$$

Then clearly

$$A'\Omega = \left(k\Delta + \bar{k}\right)A\Omega. \tag{4.1.29}$$

As such, using (4.1.17) and the definition of k we finally arrive at

$$A\Omega = (1-z)\,(\Delta - z)^{-1}\,A'\Omega. \tag{4.1.30}$$

Since the algebra $\mathcal{M}$ is spanned by the positive elements A' with $0 \leq A' \leq 1$, we have the proof. □

Proposition 4.1.3. *Let $A' \in \mathcal{M}'$ and $z \in \mathbb{C}$ with $|z| = 1$, but $z \neq 1$. Moreover, let $A \in \mathcal{M}$ such that*

$$(\Delta - z)^{-1}\,A'\Omega = A\Omega. \tag{4.1.31}$$

Then for every $\xi, \eta \in \mathcal{D}\left(\Delta^{\frac{1}{2}}\right) \cap \mathcal{D}\left(\Delta^{-\frac{1}{2}}\right)$, we have

$$\big(\xi, A'\eta\big) = \left(\Delta^{\frac{1}{2}}\xi, JA^*J\Delta^{-\frac{1}{2}}\eta\right) - z\left(\Delta^{-\frac{1}{2}}\xi, JA^*J\Delta^{\frac{1}{2}}\eta\right) \tag{4.1.32}$$

Proof. Let $X, Y \in \mathcal{M}$, which approximate the concerned vectors, i.e., $X\Omega = \xi$ and $Y\Omega = \eta$, so that

$$\big(X\Omega, A'Y\Omega\big) = \big(Y^*X\Omega, A'\Omega\big) = \left(Y^*X\Omega, (\Delta - z)A\Omega\right), \tag{4.1.33}$$

where we used (4.1.31). Now using linearity of the scalar product and $\Delta = FS$ along with the fact that S and $F = S^*$ are antilinear, see (4.1.2), we find

$$\begin{aligned} & (Y^*X\Omega, \Delta A\Omega) - (Y^*X\Omega, zA\Omega) = (SA\Omega, SY^*X\Omega) - z\,(Y^*X\Omega, A\Omega) = \\ = \; & (A^*\Omega, X^*Y\Omega) - z\,(X\Omega, YA\Omega) = (XA^*\Omega, Y\Omega) - z\,(X\Omega, YA\Omega) = \\ = \; & (SAX^*\Omega, Y\Omega) - z\,(X\Omega, SA^*Y^*\Omega) = \\ = \; & (SASX\Omega, Y\Omega) - z\,(X\Omega, SA^*SY\Omega). \end{aligned} \tag{4.1.34}$$

Then using one of the consequences of the polar decomposition of S, i.e. (4.1.10), one obtains

$$\begin{aligned} & \left(\Delta^{-\frac{1}{2}}JAJ\Delta^{\frac{1}{2}}X\Omega, Y\Omega\right) - z\left(X\Omega, \Delta^{-\frac{1}{2}}JA^*J\Delta^{\frac{1}{2}}Y\Omega\right) = \\ = \; & \left(\Delta^{\frac{1}{2}}X\Omega, JA^*J\Delta^{-\frac{1}{2}}Y\Omega\right) - z\left(\Delta^{-\frac{1}{2}}X\Omega, JA^*J\Delta^{\frac{1}{2}}Y\Omega\right), \end{aligned} \tag{4.1.35}$$

which is the desired result. □

Proposition 4.1.4. *Let* $z \in \mathbb{C} \setminus [0, \infty)$*. If* $A, B \in \mathfrak{B}(\mathcal{H})$ *satisfy*

$$(\xi, A\eta) = \left(\Delta^{\frac{1}{2}}\xi, B\Delta^{-\frac{1}{2}}\eta\right) - z\left(\Delta^{-\frac{1}{2}}\xi, B\Delta^{\frac{1}{2}}\eta\right) \tag{4.1.36}$$

for every $\xi, \eta \in \mathcal{D}\left(\Delta^{\frac{1}{2}}\right) \cap \mathcal{D}\left(\Delta^{-\frac{1}{2}}\right)$ *then* $B = T_z(A)$ *with*

$$T_z(A) \doteq \int_{-\infty}^{+\infty} d\lambda \frac{(-z)^{i\lambda-\frac{1}{2}}}{e^{\pi\lambda} + e^{-\pi\lambda}} \Delta^{-i\lambda} A \Delta^{i\lambda}. \tag{4.1.37}$$

Proof. Exploiting linearity of the scalar product given in the hypothesis, we have

$$\begin{aligned}(\xi, A\eta) &= \left(\Delta^{\frac{1}{2}}\xi, B\Delta^{-\frac{1}{2}}\eta\right) - z\left(\Delta^{-\frac{1}{2}}\xi, B\Delta^{\frac{1}{2}}\eta\right) \\ &= \left(\xi, \left(\Delta^{\frac{1}{2}}B\Delta^{-\frac{1}{2}} - z\Delta^{-\frac{1}{2}}B\Delta^{\frac{1}{2}}\right)\eta\right)\end{aligned} \tag{4.1.38}$$

Therefore

$$A = \Delta^{\frac{1}{2}}B\Delta^{-\frac{1}{2}} - z\Delta^{-\frac{1}{2}}B\Delta^{\frac{1}{2}}. \tag{4.1.39}$$

As such, we want to show the following:

$$A = \Delta^{\frac{1}{2}}T_z(A)\Delta^{-\frac{1}{2}} - z\Delta^{-\frac{1}{2}}T_z(A)\Delta^{\frac{1}{2}}. \tag{4.1.40}$$

Substituting $T_z(A)$ (4.1.37) in (4.1.40) one finds

$$\begin{aligned}\Delta^{\frac{1}{2}}T_z(A)\Delta^{-\frac{1}{2}} - z\Delta^{-\frac{1}{2}}T_z(A)\Delta^{\frac{1}{2}} &= \int_{-\infty}^{+\infty} d\lambda \tfrac{(-z)^{i\lambda}}{e^{\pi\lambda}+e^{-\pi\lambda}} \left((-z)^{-\frac{1}{2}}\Delta^{-i\left(\lambda+\frac{i}{2}\right)}A\Delta^{i\left(\lambda+\frac{i}{2}\right)} + \right. \\ &\quad \left. + (-z)^{\frac{1}{2}}\Delta^{-i\left(\lambda-\frac{i}{2}\right)}A\Delta^{i\left(\lambda-\frac{i}{2}\right)}\right).\end{aligned} \tag{4.1.41}$$

The spectral decomposition of the positive linear operator $\Delta \equiv \Delta^{\pm i\left(\lambda\pm\frac{i}{2}\right)}$ takes the form

$$\Delta^{\pm i\left(\lambda\pm\frac{i}{2}\right)} = \int d\mu_\Delta \; E^{\pm i\left(\lambda\pm\frac{i}{2}\right)} \tag{4.1.42}$$

Assign E, E' to the positive and negative parts of spectral values, respectively. Then plugging the above relation into (4.1.41), the right hand side yields

$$\int d^2\mu_\Delta \left(\left(-\frac{E'}{zE}\right)^{\frac{1}{2}} + \left(-\frac{zE}{E'}\right)^{\frac{1}{2}}\right) A \int_{-\infty}^{+\infty} d\lambda \frac{\exp\left(i\lambda \ln\left(-\frac{zE}{E'}\right)\right)}{e^{\pi\lambda}+e^{-\pi\lambda}}, \tag{4.1.43}$$

where we used $\left(-\frac{zE}{E'}\right)^{i\lambda} = \exp\left(i\lambda\ln\left(-\frac{\mathrm{zE}}{\mathrm{E}'}\right)\right)$. Now we need the following integral

$$\int_{-\infty}^{+\infty} d\lambda \frac{e^{ia\lambda}}{e^{\pi\lambda}+e^{-\pi\lambda}}, \tag{4.1.44}$$

which has simple poles at $\lambda = i\left(n+\frac{1}{2}\right)$. As such, the residue takes the form

$$\text{Res} = \lim_{\lambda\to i\left(n+\frac{1}{2}\right)} \frac{\left(\lambda - i\left(n+\frac{1}{2}\right)\right)e^{ia\lambda}}{e^{\pi\lambda}+e^{-\pi\lambda}} = \frac{(-1)^n}{2\pi i} e^{-\frac{a}{2}} e^{-na}, \tag{4.1.45}$$

where one makes use of the L'Hôpital's rule and $\cosh(in\pi) = \cos(n\pi) = (-1)^n$. Then the residue theorem immediately gives

$$\int_{-\infty}^{+\infty} d\lambda \frac{e^{ia\lambda}}{e^{\pi\lambda}+e^{-\pi\lambda}} \quad = \tfrac{2\pi i}{2\pi i} e^{-\frac{a}{2}} \sum_{n=0}^{\infty} (\text{-}1)^n\, e^{-na} = \tfrac{1}{e^{\frac{a}{2}}+e^{-\frac{a}{2}}}, \tag{4.1.46}$$

where we used $\sum_{n=0}^{\infty}(-1)^n x^n = (1+x)^{-1}$, $|x| < 1$. Finally, substituting this result with $a = \ln\left(-\frac{zE}{E'}\right)$ into (4.1.43) one arrives at

$$\int d^2\mu_\Delta \left(\left(-\frac{E'}{zE}\right)^{\frac{1}{2}} + \left(-\frac{zE}{E'}\right)^{\frac{1}{2}}\right) \frac{A}{\left(-\frac{zE}{E'}\right)^{\frac{1}{2}} + \left(-\frac{E'}{zE}\right)^{\frac{1}{2}}} = \int d^2\mu_\Delta A = A. \tag{4.1.47}$$

Hence proved. □

Suppose there is another $\tilde{B} = \tilde{T}_z(A) \in \mathfrak{B}(\mathcal{H})$ defined by

$$\tilde{T}_z(A) \doteq \int_{-\infty}^{+\infty} d\lambda \frac{(-z)^{i\lambda-\frac{1}{2}}}{e^{\pi\lambda}+e^{-\pi\lambda}} \Delta^{-i\lambda} A, \; A \in \mathfrak{B}(\mathcal{H}), \tag{4.1.48}$$

then from the proof of prop. 4.1.4 trivially follows

$$\Delta^{\frac{1}{2}}\tilde{T}_z(A) - z\Delta^{-\frac{1}{2}}\tilde{T}_z(A) = \int d^2\mu_{\tilde{\Delta}} A = A \implies \tilde{T}_z(A) = \Delta^{\frac{1}{2}}(\Delta - z)^{-1} A. \tag{4.1.49}$$

We now state and prove the main theorem.

Theorem 4.1.1. *Let J be the modular conjugation and $\Delta^{i\lambda}$, $\lambda \in \mathbb{R}$ be the modular operator. Then J and $\Delta^{i\lambda}$ satisfy the following equalities:*

$$J\mathcal{M}J = \mathcal{M}', \tag{4.1.50}$$

and

$$\Delta^{i\lambda}\mathcal{M}\Delta^{-i\lambda} = \mathcal{M}. \tag{4.1.51}$$

Proof. Let $A', B' \in \mathcal{M}'$ and $1 \neq z \in \mathbb{C}$ such that $|z| = 1$. Then from prop. 4.1.2 we know that there exists $A \in \mathcal{M}$ such that

$$(\Delta - z)^{-1} A'\Omega = A\Omega. \tag{4.1.52}$$

Moreover, from prop. 4.1.3, in particular, using the scalar product given in (4.1.32) one can easily come up with

$$\left(\xi, A'\eta\right) = \left(\xi, \left(\Delta^{\frac{1}{2}} JA^* J\Delta^{-\frac{1}{2}} - z\Delta^{-\frac{1}{2}} JA^* J\Delta^{\frac{1}{2}}\right) \eta\right) \tag{4.1.53}$$

for every $\xi, \eta \in \mathcal{D}\left(\Delta^{\frac{1}{2}}\right) \cap \mathcal{D}\left(\Delta^{-\frac{1}{2}}\right)$. Then clearly

$$A' = \Delta^{\frac{1}{2}} JA^* J\Delta^{-\frac{1}{2}} - z\Delta^{-\frac{1}{2}} JA^* J\Delta^{\frac{1}{2}}. \tag{4.1.54}$$

As such, using prop. 4.1.4 one infers

$$JA^*J = T_z\left(A'\right) \implies A^* = JT_z\left(A'\right)J. \tag{4.1.55}$$

Now applying both sides of the last relation to $B'\Omega$, $B' \in \mathcal{M}'$, we can write

$$A^*B'\Omega = JT_z(A')JB'\Omega. \tag{4.1.56}$$

Furthermore, applying $B'J\Delta^{\frac{1}{2}}$ on both sides of (4.1.52), and after using (4.1.6) and $SA\Omega =$

$A^*\Omega, A \in \mathcal{M}$, to solve the right hand side one finds

$$\begin{aligned} &B'J\Delta^{\frac{1}{2}}\left(\Delta - z\right)^{-1}A'\Omega = B'J\Delta^{\frac{1}{2}}A\Omega = B'SA\Omega \\ &= B'A^*\Omega = A^*B'\Omega = JT_z\left(A'\right)JB'\Omega, \end{aligned} \tag{4.1.57}$$

where in the last equality we used (4.1.56). Then using (4.1.49) and (4.1.37) together with (4.1.48) one rewrites the above expression as follows.

$$B'J\int_{-\infty}^{+\infty}d\lambda\tfrac{(-z)^{i\lambda-\frac{1}{2}}}{e^{\pi\lambda}+e^{-\pi\lambda}}\Delta^{-i\lambda}A'\Delta^{i\lambda}J\Omega = J\int_{-\infty}^{+\infty}d\lambda\tfrac{(-z)^{i\lambda-\frac{1}{2}}}{e^{\pi\lambda}+e^{-\pi\lambda}}\Delta^{-i\lambda}A'\Delta^{i\lambda}JB'\Omega, \tag{4.1.58}$$

where we used (4.1.15) and (4.1.13). As a result, since

$$(-z)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}d\lambda\frac{(-z)^{i\lambda}}{e^{\pi\lambda}+e^{-\pi\lambda}}\left(B'J\Delta^{-i\lambda}A'\Delta^{i\lambda}J\Omega - J\Delta^{-i\lambda}A'\Delta^{i\lambda}JB'\Omega\right) = 0, \tag{4.1.59}$$

we must have

$$B'J\Delta^{-i\lambda}A'\Delta^{i\lambda}J\Omega = J\Delta^{-i\lambda}A'\Delta^{i\lambda}JB'\Omega. \tag{4.1.60}$$

Let $\tilde{B}' \in \mathcal{M}'$, then the last equality trivially gives

$$\tilde{B}'B'J\Delta^{-i\lambda}A'\Delta^{i\lambda}J\Omega \quad = \tilde{B}'J\Delta^{-i\lambda}A'\Delta^{i\lambda}JB'\Omega. \tag{4.1.61}$$

Then using (4.1.60) one more time we arrive at

$$\left[\tilde{B}'B', J\Delta^{-i\lambda}A'\Delta^{i\lambda}J\right]\Omega = 0. \tag{4.1.62}$$

But by construction $\tilde{B}'$, $B' \in \mathcal{M}'$. Therefore,

$$J\Delta^{-i\lambda}A'\Delta^{i\lambda}J \in \mathcal{M} \implies JA'J \in \mathcal{M} \implies J\mathcal{M}'J \subseteq \mathcal{M}. \tag{4.1.63}$$

Similarly, using $\tilde{B}, B, A \in \mathcal{M}$, one can arrive at (4.1.62) with $\tilde{B}' \to \tilde{B}$, $B' \to B$ and $A' \to A$. Thus, as above

$$J\mathcal{M}J \subseteq \mathcal{M}', \tag{4.1.64}$$

which proves (4.1.50). As a consequence, using (4.1.13) one also finds

$$\Delta^{-i\lambda}\mathcal{M}'\Delta^{i\lambda} \subseteq \mathcal{M}' \qquad \text{and} \qquad \Delta^{-i\lambda}\mathcal{M}\Delta^{i\lambda} \subseteq \mathcal{M}. \tag{4.1.65}$$

It completes the proof. □

Having discussed the Tomita-Takeskai theorem, we now briefly consider its connection with the KMS condition.

4.1.1 KMS Condition

Let $(\mathcal{M}, \alpha_t)$ be a W^*-dynamical system (def. 2.2.9) and $\omega \in \mathcal{M}_1^{*+}$, which is also faithful (def. A.2.1). Then ω is called the (α, β)-KMS (Kubo-Martin-Schwinger) state if for every $A, B \in \mathcal{M}$ there exists a complex valued bounded function $\mathcal{F}$, which is continuous on and analytic in the closed strip $S_\beta \doteq \{z \in \mathbb{C} \mid 0 \leq \mathrm{Im} z \leq \beta\}$, such that

$$\mathcal{F}(t) = \omega(\alpha_t(A)B) \qquad \text{and} \qquad \mathcal{F}(t + i\beta) = \omega(B\alpha_t(A)), \tag{4.1.66}$$

where $\beta > 0$. The expressions given in (4.1.66) are generally referred to as the *KMS boundary condition*[1]. Note that for $\beta = 0$, from (4.1.66), one has $\omega(\alpha_t(A)B) = \omega(B\alpha_t(A))$ for every $A, B \in \mathcal{M}$, so in that case $\omega(\cdot)$ is just a tracial state (def. A.2.1). For this α_t must be a trivial automorphism [64].

Proposition 4.1.5. *Let $\mathcal{M}$, α_t and ω be as in the KMS definition such that (4.1.66) is satisfied. Then ω is α_t-invariant. In particular,*

$$\omega(\alpha_t(A)) = A \quad \forall A \in \mathcal{M}, \forall\, t \in \mathbb{R}. \tag{4.1.67}$$

Proof. Let $C \in \mathcal{M}_\omega$, then from (2.2.13), one has $\omega(CA) = \omega(AC)$, for every $A \in \mathcal{M}$. In that case, since ω satisfies, $\mathcal{F}(t) = \omega(\alpha_t(A)C)$ and $\mathcal{F}(t + i) = \omega(C\alpha_t(A))$, with $\mathcal{F}$ being continuous and analytic in the strip S_1, we must have

$$\mathcal{F}(t) = \mathcal{F}(t + i) \quad \forall\, t \in \mathbb{R}. \tag{4.1.68}$$

[1] In the framework of AQFT with regards to the modular theory it was discussed in [28], see also [18].

Such a function extends to a bounded and continuous function on the whole complex plane $\mathbb{C}$, which is also analytic on each of the open strips $\{z \in \mathbb{C} : n < \mathrm{Im} z < n+1\}$, $n \in \mathbb{Z}$. As such, follows that this extended function is analytic on $\mathbb{C}$, cf. the Schwarz reflection principle, then by Liouville's theorem it is constant. So we have

$$\omega(\alpha_t(A)C) = \mathcal{F}(t) = \mathcal{F}(0) = \omega(AC). \tag{4.1.69}$$

Taking $C = \mathbb{1}$, the claim follows (cf. lemma 2.2.1). □

The following results establishes a connection between the KMS condition and the modular automorphism group, of which the latter will play an active role in our studies. Note that the following is a slightly weakened form of the KMS condition, as it is restricted to the self-adjoint elements, but still it is enough to determine the modular automorphism group.

Theorem 4.1.2. *Let $\mathcal{M}$, ω and α_t be as in the KMS condition. Also let $\mathcal{M} = \overline{\mathfrak{M}}^{\mathrm{SOT}}$ (equivalently $\overline{\mathfrak{M}}^{\mathrm{WOT}}$, Cor. 2.1.2.1), with $\mathfrak{M} \subset \mathcal{M}$ denoting subalgebra consisting of self-adjoint elements, such that for any $A, B \in \mathfrak{M}$, we have*

$$\mathcal{F}(t) = \omega(\alpha_t(A)B) \quad \text{and} \quad \mathcal{F}(t+i) = \omega(B\alpha_t(A)), \;\; t \in \mathbb{R}, \tag{4.1.70}$$

with bounded $\mathcal{F}$ which is continuous on and analytic in S_1.

Then α_t satisfies the KMS condition with respect to ω.

Proof. Let $A \in \mathfrak{M}$ and $B \in \mathcal{M}$ with $B = B^*$. Let $\{B_n\}_{n\in I} \in \mathfrak{M}$ be a bounded net such that $\lim_n B_n \to B$ strongly. Then for $A, B_n \in \mathfrak{M}$, we can have $\mathcal{F}_n$ for every $n \in I$ satisfying conditions given in the hypothesis, such that

$$|\mathcal{F}_n(t) - \mathcal{F}_m(t)| = |\omega(\alpha_t(A)(B_n - B_m)|. \tag{4.1.71}$$

Then using linearity of ω along with the fact that square of a self-adjoint operator is positive as well as prop. 4.1.5, one finds

$$|\mathcal{F}_n(t) - \mathcal{F}_m(t)| \leq (\omega(\alpha_t(AA^*)))^{\frac{1}{2}} \left(\omega((B_n - B_m)^2)\right)^{\frac{1}{2}} \leq \|A\| \left(\omega((B_n - B_m)^2)\right)^{\frac{1}{2}}. \tag{4.1.72}$$

Similarly,

$$|\mathcal{F}_n(t+i) - \mathcal{F}_m(t+i)| \leq \|A\| \left(\omega((B_n - B_m)^2)\right)^{\frac{1}{2}} . \tag{4.1.73}$$

Note that $\mathcal{F}_n - \mathcal{F}_m$ has similar properties as of $\mathcal{F}$, as such, for $z \in S_1$ one also has (cf. maximum modulus principle, see e.g. [65])

$$|\mathcal{F}_n(z) - \mathcal{F}_m(z)| \leq \|A\| \left(\omega((B_n - B_m)^2)\right)^{\frac{1}{2}} . \tag{4.1.74}$$

Clearly, since B_n and B_m converges strongly to B by assumption, from the strong operator continuity of ω, follows $\lim\limits_{n,m} \omega((B_n - B_m)^2) \to 0$. Thus the family $\{\mathcal{F}_n\}$ converges uniformly to $\mathcal{F}$ on S_1, which being bounded and continuous on S_1 and analytic in it, one obtains (4.1.70). By linearity it holds for every $A \in \mathfrak{M}$ and $B \in \mathcal{M}$, where the latter is not necessarily self-adjoint.

Now let $B \in \mathcal{M}^{\text{sa}}$ and $A \in \mathcal{M}$ such that $A = A^*$. Also $\{A_n\}_{n \in I} \in \mathfrak{M}$. Then in the above arguments superseding B with A one can easily come with analogous results for A. Note that to approximate the boundary values of $\mathcal{F}_n - \mathcal{F}_m$, as above, in the present case, one makes use of prop. 4.1.5, in particular, $\omega(\alpha_t(A_n - A_m)^2) = \omega((A_n - A_m)^2)$. As a result, α_t satisfies the concerned condition here as well. □

Corollary 4.1.2.1. *Let $\mathcal{M}$ and ω be as in the KMS condition. Then there exists a modular automorphism group $\alpha_t = \sigma^\omega_\lambda$ (def. 2.2.12) of $\mathcal{M}$ that satisfies the KMS condition for $\beta = 1$.*

Proof. An immediate consequence of the theorem 4.1.2. □

4.1.2 Applications

Apart from its major role in type classification of von Neumann algebras (sect. 2.2), the TT modular theory also has various applications in mathematical physics. Here we outline them with a few specific examples. For an extensive review on the subject matter we recommend [8,19,27].

Modular theory was used to prove the Haag duality (3.1.6) [20] for the free field theory. Later, Bisognano and Wichmann (B-W) [29] came up with famous results for the modular operator $\Delta^{i\lambda}$ and the modular conjugation J, for the right wedge W_r, see below. The result is valid for massive, massless, and even interacting theories. Though the framework used in their studies was Wightman axioms [6, 10], the geometrical interpretation of the modular operator led to many other important results, see e.g. [19], one of which [33] will be considered in sect. 5.1. Sewell [31], managed to derive the Unruh/Hawking effect using the B-W results. We comment on this below.

We now state the Bisognano-Wichmann theorem:

Theorem 4.1.3. *Let $\mathcal{M}(W_r)$ denote a local algebra associated with the right wedge W_r (5.1.21) equipped with a cyclic and separating vector $\Omega \in \mathcal{H}$. Let $\Delta_{W_r}^{i\lambda}$, $\lambda \in \mathbb{R}$ and J_{W_r} be the corresponding modular operator and modular conjugation respectively. Then one has*

$$\Delta_{W_r}^{2\pi i\lambda} = U(\Lambda_1(2\pi\lambda), \mathbb{1}) \quad \text{and} \quad J_{W_r} = \Theta U(R_1(\pi), \mathbb{1}), \tag{4.1.75}$$

where $\Lambda_1(\lambda)$ stands for the Lorentz boost in x_1-direction, $R_1(\pi)$ is the rotation about x_1-axis by an angle π and Θ is the PCT operator [6].

In addition, the wedge duality is satisfied, namely

$$\mathcal{M}(W_r) = \mathcal{M}(W_l)'. \tag{4.1.76}$$

Since the analytic condition satisfied by the modular group is equivalent to the KMS condition, Thm. 4.1.2 (see also [28]), Thm. 4.1.3 now implies that the vacuum $\Omega \in \mathcal{H}$ is a thermal equilibrium state for the time evolution given by $\Lambda(\lambda)$. In that case, Sewell [31] noticed that the corresponding observer, the one whose time evolution is the Lorentz boost, is nothing but an uniformly accelerated observer, or equivalently a freely falling observer in a gravitational field. Hence it led to the classical Unruh/Hawking result.

The upshot of B-W studies is the geometrical action induced by the modular operator, which exists only for very limited spacetime regions [19]. Some interconnected cases will be considered in the next chapter with the corresponding geometrical transformations and modular operators. Here we quote one of the fundamental results in this regard which will be a bridge between the modular operators and its geometrical interpretation.

The following result, due to Brunetti-Guido-Longo [66], holds for conformal QFTs.

Theorem 4.1.4. *Let $\mathbb{R}^{1+3} \ni O \mapsto \mathcal{M}(O)$ be a conformally covariant net of von Neumann algebras. Then the modular operators $\Delta^{i\lambda}$ and the modular conjugations J, associated with wedges, lightcones, and double cones, have geometrical actions.*

Chapter 5

The Modular Operators and Their Geometrical Aspects

In this chapter, we derive modular operators for spacetime regions of interest, that include, a right wedge, a forward lightcone and a double cone. In accomplishing this, the Hislop-Longo theorem 5.1.1 plays a crucial role that basically guarantees unitary representation of inversion ρ on the underlying Hilbert space. This eventually allows one to implement geometrical transformations on the Hilbert space to obtain the modular operators. We also discuss the geometrical transformations with necessary details and extract the underlying conformal factor that is related to a concept of temperature discussed in the later chapter. A brief discussion on substantial challenges one might face in the massive case is also provided.

5.1 The Hislop-Longo Approach

In [33], Hislop and Longo managed to take advantage of geometrical action induced by the modular operator of a double cone region deduced in the framework of a massless scalar AQFT. Here we provide necessary ingredients and reproduce the main result.

5.1.1 Rudiments

Let $\mathcal{H} = L^2[H_0, d\mu]$ denote a one-particle Hilbert space, where the upper massshell, which in the present is just the forward lightcone defined on the momentum space, cf. (5.1.19), that takes the form

$$H_0 = \left\{ p \in \mathbb{R}^{1+d} \,\middle|\, p_0 \geq 0, p^2 = 0 \right\} \tag{5.1.1}$$

and the Lorentz invariant integral measure $d\mu$ is given by

$$\int d^{1+d}p\, \theta(p_0)\delta(p^2) \equiv \int \frac{d^d\boldsymbol{p}}{2\omega_{\boldsymbol{p}}}, \tag{5.1.2}$$

with $\omega_{\boldsymbol{p}} = |\boldsymbol{p}|$. Note that up to a multiple constant $d\mu$ is the only Lorentz invariant measure, see e.g. Thm. IX.33 and Thm. IX.37 in [67].

Then the symmetrized or Bosonic Fock space is defined to be

$$\Gamma_s(\mathcal{H}) \doteq \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^{(n)}, \tag{5.1.3}$$

where $\mathcal{H}^{(n)} = \mathcal{H} \otimes \overset{n\,\text{times}}{\cdots} \otimes \mathcal{H}$. Henceforth the n-particle Hilbert space $S_n\mathcal{H}^{(n)}$ will be denoted by $\mathcal{H}_s^{(n)}$. There exists a set $\tilde{\Gamma} \subset \Gamma_s(\mathcal{H})$ of finite particle vectors[1]. Among these vectors the vacuum state vector

$$\Omega = 1 \oplus 0 \oplus 0 \oplus \ldots \in \tilde{\Gamma} \tag{5.1.4}$$

has a special property that we shall see now. Let U denote any arbitrary unitary operator defined on $\mathcal{H}$. Then the *second quantization* $\Gamma(U)$ of U is the unitary operator on $\Gamma_s(\mathcal{H})$ whose action is understood as

$$\Gamma(U) = \bigoplus_{n \geq 0} U_n \qquad \text{with} \qquad U_0 = \mathbb{1}. \tag{5.1.5}$$

Then it is clear that the vacuum vector satisfies $\Gamma(U)\Omega = \Omega$.

The restriction of $\Gamma(U)$ on the n-particle space takes the form

$$\Gamma(U)\big|_{\mathcal{H}_s^{(n)}} = U \otimes \overset{n\,\text{times}}{\cdots} \otimes U, \qquad n > 0. \tag{5.1.6}$$

[1] These are the vectors $\xi^{(n)} \in \mathcal{H}_s^{(n)}$ such that $\xi^{(n)} = 0$ for all but finitely many n, i.e., there exists a set say G with cardinality $|G| \in \mathbb{N}$ such that when $n \notin G$ then $\xi^{(n)} = 0$.

Then its action on any vector $\xi^{(n)} \in \mathcal{H}_s^{(n)}$ is given by[2]

$$\Gamma(U)\xi^{(n)} = U\xi_1 \otimes U\xi_2 \otimes \ldots \otimes U\xi_n. \tag{5.1.7}$$

Suppose A is a self-adjoint operator acting on $\mathcal{H}$ and $U = e^{itA}$ to be one-parameter unitary group on $\mathcal{H}$, then $\Gamma(U) = e^{itd\Gamma(A)}$ is defined on $\{\xi \in \tilde{\Gamma} \mid \otimes_{k=1}^{n} D \;\forall\, n\} \cap \mathcal{H}_s^{(n)}$ where $D \subset \mathcal{H}$ with respect which A is essentially self-adjoint, i.e., symmetric on D and its closure is self-adjoint. In particular, the action of the second quantization $d\Gamma(A)$ of A is given by

$$d\Gamma(A) \doteq \overline{\bigoplus_{n\geq 0} A_n} = 0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus \ldots = 0 \oplus A \oplus (\mathbb{1} \otimes A) \oplus (\mathbb{1} \otimes \mathbb{1} \otimes A) \oplus \ldots, \tag{5.1.8}$$

with $A_0 = 0$. Here A_n on any $\xi^{(n)}$ acts as follows:

$$A_n\xi^{(n)} = A_n\left(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n\right) = \sum_{i=1}^{n} \left(\xi_1 \otimes \xi_2 \otimes \ldots \otimes A\xi_i \otimes \ldots \otimes \xi_n\right), \tag{5.1.9}$$

where the summation in the last equality is due to symmetrization of the tensor product.

It is this action that should be used to derive the action of the n-particle modular operator using that of the one-particle one. Though we shall only work out the expression for the one-particle space, the expressions for the n-particle space or for the Fock space follows from the above construction.

The sets of timelike and spacelike vectors are defined by

$$T \doteq \left\{x \in \mathbb{R}^{1+d} \,\middle|\, x^2 > 0\right\}, \tag{5.1.10}$$

$$S \doteq \left\{x \in \mathbb{R}^{1+d} \,\middle|\, x^2 < 0\right\}. \tag{5.1.11}$$

Then the timelike complement O^t and the spacelike complement O^s for any open region $O \subset T \cup S$ in $\mathbb{R}^{1+3}$ are defined as (cf. (3.1.1))

$$O^t \doteq \mathrm{Int}\left\{x \in \mathbb{R}^{1+d} \,\middle|\, (x-y)^2 > 0 \,\forall\, y \in O\right\}, \tag{5.1.12}$$

$$O^s \doteq \mathrm{Int}\left\{x \in \mathbb{R}^{1+d} \,\middle|\, (x-y)^2 < 0 \,\forall\, y \in O\right\}. \tag{5.1.13}$$

[2] This in principle can be used for the field transformations when the field is written in terms of annihilation and creation operators.

Consider the ray inversion $\rho : x \in \mathbb{R}^{1+d} \to \rho(x)$, which is an involutive diffeomorphism of $T \cup S$, whose action on $\mathbb{R}^{1+d}$ is defined by

$$\rho(x) \doteq \Big(-\frac{x_0}{x^2}, -\frac{\boldsymbol{x}}{x^2}\Big) = -\frac{x}{x^2}. \tag{5.1.14}$$

As such, for some $O \subset T$ we set

$$\rho(O) \doteq \{\rho(x) \mid x \in O\}, \tag{5.1.15}$$

for which the timelike complement transpires to be of the form

$$\rho(O)^t = \mathrm{Int}\Big\{\rho(x) \,\Big|\, \big(\rho(x) - \rho(y)\big)^2 > 0 \quad \forall\, \rho(y) \in \rho(O)\Big\}. \tag{5.1.16}$$

Now let $U \equiv \{x \in \mathbb{R}^{1+d} \mid (x-y)^2 > 0 \; \forall\, y \in O\}$, and $U_i \subset T$, $i \in I$, be all open sets such that $\cup_i U_i \subset U$ and $\cup_i \rho(U_i) \subset \rho(U)$, then it is straightforward to show that

$$\rho(O^t) = \rho(\mathrm{Int}(U)) = \rho(\cup_i U_i) = \cup_i \rho(U_i) = \mathrm{Int}(\rho(U)) = \rho(O)^t, \tag{5.1.17}$$

which clearly demonstrates that taking either the timelike complement first for a given $O \subset T$ and then applying ρ on it or doing it other way around, commute. A similar statement holds for the spacelike complements, i.e.,

$$\rho(O^s) = \rho(O)^s, \quad O \subset S. \tag{5.1.18}$$

Now let us consider two of the fundamental spacetime regions, called a *forward lightcone* and a *backward lightcone*, defined by

$$V_+ \doteq \Big\{x \in \mathbb{R}^{1+d} \,\Big|\, x_0 \geq 0,\, x^2 \geq 0\Big\} \quad \text{and} \quad V_- \doteq \Big\{x \in \mathbb{R}^{1+d} \,\Big|\, x_0 \leq 0,\, x^2 \geq 0\Big\}. \tag{5.1.19}$$

Whereupon we have a definition of a double cone. We shall use simpler formulation, see below, which is enough for all the analysis that we are interested in.

Definition 5.1.1. A *double cone* generated by the forward lightcone V_+ and the backward lightcone V_- is defined by

$$D = D_{x,y} = V_+^y \cap V_-^z = \Big\{ z \in \mathbb{R}^{1+d} \,\Big|\, x - z \in V_+, z - y \in V_+ \Big\}. \tag{5.1.20}$$

A *diamond* is defined by the property $O = O''$. It is an open region (bounded or unbounded) that coincides with its own double causal complement. On the other hand

1. A double cone is a diamond that is bounded, but the converse does not always hold [51].

2. A wedge region, being a special case of lightlike monotone regions (def. 2.3.1), e.g. either left or right (5.1.21), is a diamond which is unbounded.

For a diamond $D_{x_1,y_1} \cup D_{x_2,y_2}$ to be a double cone we must have $D_{x_1,y_1} \cap D_{x_2,y_2} \neq \emptyset$.

The right wedge W_r and the left wedge W_l are defined as follows:

$$W_r \doteq \Big\{ x \in \mathbb{R}^{1+3} \,\Big|\, x^1 > |x^0| \Big\} \quad \text{and} \quad W_l \doteq \Big\{ x \in \mathbb{R}^{1+3} \,\Big|\, -x^1 > |x^0| \Big\}. \tag{5.1.21}$$

Now we give a definition of a double cone that we shall be concerned with for our studies. A double cone $O \subset \mathbb{R}^{1+d}$ is a spacetime region with the following restriction:

$$O \doteq \Big\{ x \in \mathbb{R}^{1+d} \,\Big|\, |x_0| + \|\boldsymbol{x}\| < L \Big\}, \tag{5.1.22}$$

with $L > 0$ being some length scale, in principle, length of O_0— a double cone located at the origin, see below. Henceforth we shall use a double cone and a diamond interchangeably.

Field Transformations

Let $\mathbb{R}^{1+d}$ be the $(1+d)$-dimensional Minkowski spacetime with a metric signature mostly minuses, and call $\mathcal{S}(\mathbb{R}^{1+d})$ the corresponding Schwartz space. We consider a free massless scalar field $\phi(x)$ regarded as a distribution that upon smearing out over spacetime with an averaging test function f gives an unbounded essentially self-adjoint operator[3]

$$\phi[f] = (\phi, f) = \int d^{1+d}x \, \phi(x) f(x) \quad \forall\, f \in \mathcal{S}(\mathbb{R}^{1+d}) \tag{5.1.23}$$

[3] Later we shall consider the set $D(T \cup S)$ of all smooth compactly supported functions defined on the timelike and spacelike vectors to write the action of unitary operators on the vectors generated by applying the field operator to the vacuum.

defined on the domain $D_0 \subset \Gamma_s(\mathcal{H})$, where the domain is obtained by applying polynomial algebra generated by the field operators on the vacuum $\Omega \in \Gamma_s(\mathcal{H})$.

Let $V(\lambda)$, $\lambda \in \mathbb{R}$, be a one parameter unitary group on $\mathcal{H}$ that in general corresponds to the conformal group, the Poincaré group being a subgroup. Then its action on the field operator $\phi[f]$ is given by[4]

$$\Gamma(V(\lambda))\phi[f]\Omega = \phi[f_\lambda]\Omega, \tag{5.1.24}$$

where

$$f_\lambda = \gamma(x,\lambda) f(v(-\lambda)x), \tag{5.1.25}$$

with $v(\lambda)$ denoting the group action on $\mathbb{R}^{1+3}$.

Let $\mathfrak{C}(x)$ denote the scale factor and D the scaling dimension associated with the conformal transformations which has two parts:

- **Coordinate transformation**: A generalized form $x^\mu \to x'^\mu$ of change in coordinate yields

$$\begin{aligned} \phi(x) \to \phi'(x') &= \phi(x), \\ g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= g_{\rho\sigma}\partial_\mu(x^\rho)'\partial_\nu(x^\sigma)' = \mathfrak{C}^{-2}(x) g_{\mu\nu}, \end{aligned} \tag{5.1.26}$$

  where transformation of the metric generates a familiar induced metric.

- **Weyl transformation**:

$$\begin{aligned} \phi(x) \to \tilde{\phi}(x) &= \mathfrak{C}^{-D}(x)\phi(x), \\ g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu} &= \mathfrak{C}^{2}(x) g_{\mu\nu}, \end{aligned} \tag{5.1.27}$$

  where the metric transforms in opposite way to compensate coordinate transformation.

Combination of (5.1.26) and (5.1.27) produce the conformal transformations with true con-

[4] Note that in principle one should first construct the canonical field $\phi[f]$, $f \in \mathcal{S}(\mathbb{R}^{1+3})$, using the Segal field quantization $\phi_S[f]$, $f \in \mathcal{H}$, where the latter is defined in terms of the creation and annihilation operators as $\phi_S[f] \doteq \frac{a(f)+a^*(f)}{\sqrt{2}}$. Then the field transformation follows from that of operators $a(f)$ and $a^*(f)$, see [67] for the details.

formal symmetry, namely the metric remains invariant.

$$\begin{aligned} \tilde{\phi}'(x') &= \mathfrak{C}^{-D}(x)\phi(x), \\ \tilde{g}'_{\mu\nu} &= g_{\mu\nu}. \end{aligned} \tag{5.1.28}$$

We fix the scaling dimension as follows. Consider a massless scalar field action.

$$S = \int d^{1+d}x\sqrt{|g(x)|}\; g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x), \tag{5.1.29}$$

where $g \equiv \det g_{\mu\nu}$. Let the scaling transformation be given by[5], $x \to x' = \lambda x$, so that the scale factor becomes $\mathfrak{C} = \lambda$, then applying first the coordinate transformations (5.1.26), we get

$$\begin{aligned} S' &= \int d^{1+d}x'\sqrt{|g'(x')|}\, g'^{\mu\nu}(x')\partial'_\mu\phi'(x')\partial'_\nu\phi'(x') \\ &= \int \lambda^{d+1}d^{1+d}x\lambda^{-d-1}\sqrt{|g(x)|}\,\lambda^2 g^{\mu\nu}(x)\lambda^{-2}\partial_\mu\phi(x)\partial_\nu\phi(x) \\ &= \int d^{1+d}x\sqrt{|g(x)|}\, g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x), \end{aligned} \tag{5.1.30}$$

where, as it can be easily seen, a coordinate change in the integral measure nullifies due to negative contribution from the metric determinant g, and the change in derivatives cancel out to that of the inverse metric $g^{\mu\nu}$. Above action cannot be considered conformally invariant yet, rather it is just the usual diffeomorphism(reparametrization invariance). But applying the conformal transformation, i.e., simultaneously the coordinate and Weyl ones, we see that the action,

$$\tilde{S}' = \int \lambda^{d+1}d^{1+d}x\sqrt{|g(x)|}\; g^{\mu\nu}(x)\lambda^{-2D}\lambda^{-2}\partial_\mu\phi(x)\partial_\nu\phi(x), \tag{5.1.31}$$

remains scale-invariant if we set $D = \frac{d-1}{2}$, which for $d = 3$ (the case of our interest) simply gives $D = 1$.

This preparation allows us to derive the transformation of a test function. First of all, note

[5] One can follow the same procedure with more complicated forms of conformal transformation, but here our only aim is to fix the scaling dimension.

that with $x \in \mathbb{R}^{1+d}$, the transformation of a scalar field takes the form[6]

$$\Gamma(V(\lambda))\phi(x)\Gamma(V(\lambda))^{-1} = \mathfrak{C}^{-1}(x)\phi(v(\lambda)x). \tag{5.1.32}$$

Then clearly we have

$$\phi[f] \to \phi_\lambda[f] = \Gamma(V(\lambda))\phi[f]\Gamma(V(\lambda))^{-1} = \int d^4x\, \mathfrak{C}^{-1}(x)\phi(v(\lambda)x)f(x), \tag{5.1.33}$$

where we used (5.1.32). Now making the change of variable $x \to v(-\lambda)x$, one finds

$$\begin{aligned}\phi_\lambda[f] &= \int d^4(v(-\lambda)x)\, \mathfrak{C}^{-1}(v(-\lambda)x)\phi(v(\lambda)v(-\lambda)x)f(v(-\lambda)x) \\ &= \int d^4x \mathcal{J}\, \mathfrak{C}^{-1}(x)\phi(x)f(v(-\lambda)x) = \phi[f_{-\lambda}]\end{aligned} \tag{5.1.34}$$

which is what we call the tranformation of a test function f. Here $\mathcal{J}$ stands for the Jacobian associated with the transformation under consideration, combination of which, with a scalar factor, namely, $\gamma(x;\lambda) = \mathcal{J}\mathfrak{C}^{-1}(x)$ is non other than a conformal factor generated out of the conformal transformations. Note that a conformal factor of the field transformation is just $\gamma(x;\lambda) = \mathfrak{C}^{-1}(x)$, i.e., without Jacobian. Note that $\mathcal{J}$ could have an additional multiplicative term, depending on a given coordinate system.

Example 5.1.1. Here we consider inversion and dilation, and find the corresponding conformal factors. As for the inversion $\rho\colon x \to \rho(x) = -\frac{x}{x^2}$ one finds $\mathcal{J} = -(x^2)^{-4}$, then using (5.1.26) one fixes the scale factor to be $\mathfrak{C}(x)^{-1} = x^2$, which together with $\mathcal{J}$, yields $\gamma_\rho = (x^2)^{-3}$. Similarly, for dilation $\varsigma\colon x \to x' = e^\lambda x$, we have $\mathcal{J} = e^{-4\lambda}$ and $\mathfrak{C}(x)^{-1} = e^\lambda$, therefore, $\gamma_\varsigma = e^{-3\lambda}$.

5.1.2 Massless Case

Since the theory under consideration is free, everything is controlled by the Wightman function (or Wightman distribution or vacuum expectation value or n-point function), which in the current setting, in particular, for the massless case[7], takes the form

$$\mathcal{W}_2(x-y) \doteq \frac{1}{(2\pi)^d}\int \frac{d^d\boldsymbol{p}}{2|\boldsymbol{p}|}e^{-ip(x-y)} = (\phi(x)\Omega, \phi(y)\Omega), \tag{5.1.35}$$

[6] The expression given below is legitimate only when both sides act on the vector $\Omega \in \mathcal{H}$, which satisfies $V(\lambda)\Omega = \Omega$.

[7] Note that for the massive case, one has $p_0 = \omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$, which for $m = 0$ simplifies to $p_0 = \omega_{\boldsymbol{p}} = |\boldsymbol{p}|$.

When the last equation is written more precisely in terms of Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^{1+d})$, it looks as follows

$$\begin{aligned}(\phi[f]\Omega, \phi[g]\Omega) &= \int\int d^{1+d}x\, d^{1+d}y\, \mathcal{W}_2(x-y)\overline{f(x)}g(y) \\ &= \frac{1}{(2\pi)^n}\int \frac{d^d\boldsymbol{p}}{2|\boldsymbol{p}|}\,\overline{\tilde{f}(-|\boldsymbol{p}|, -\boldsymbol{p})}\tilde{g}(|\boldsymbol{p}|, \boldsymbol{p}),\end{aligned} \tag{5.1.36}$$

where $\tilde{f}(p)$ stands for the Fourier transform defined by

$$\tilde{f}(p) \doteq \int d^{1+d}x\, e^{ipx} f(x). \tag{5.1.37}$$

Using (5.1.36) one straightforwardly gets the following expressions:

$$(\phi[f]\Omega, \pi[g]\Omega) = (\phi[f]\Omega, i|\boldsymbol{p}|\phi[g]\Omega), \tag{5.1.38}$$

and

$$(\pi[f]\Omega, \pi[g]\Omega) = (\phi[f]\Omega, |\boldsymbol{p}|^2\phi[g]\Omega). \tag{5.1.39}$$

Let us now define ϕ_0 and π_0 to be the field and the momentum restricted to the d-dimensional time-zero submanifold of $\mathbb{R}^{1+d}$. In that case, $f \in \mathcal{S}(\mathbb{R}^{1+d})$ has a natural restriction to $\mathbb{R}^d$, see [67]. Then for every $f \in \mathcal{S}(\mathbb{R}^d)$ there exists a distribution $\hat{f}(x) \doteq \delta(t)f(\boldsymbol{x})$ such that[8] $\phi_0[f] \doteq (\phi_0, \hat{f})$ and $d+1$-dimensional Fourier transform given in (5.1.37) reduces to d-dimensional one:

$$\tilde{f}(p) = \int d^{1+d}x\, e^{ipx}\delta(t)f(\boldsymbol{x}) = \int d^d\boldsymbol{x}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\bar{f}(\boldsymbol{x}), \tag{5.1.40}$$

where we used $\int dt\, e^{i|\boldsymbol{p}|t}\delta(t) = 1$. The expression given in (5.1.40) will be reflected in the two point functions written in terms of time zero field ϕ_0 and momentum π_0. Rewriting (5.1.36),

[8] The restriction of a distribution to a submanifold may not exist.

(5.1.38) and (5.1.39) for the case in our hand we obtain[9]

$$
\begin{aligned}
(\phi_0[f]\Omega, \phi_0[g]\Omega) &= \frac{1}{(2\pi)^d} \int \frac{d^d\boldsymbol{p}}{2|\boldsymbol{p}|}\, \mathcal{F}, \\
(\pi_0[f]\Omega, \pi_0[g]\Omega) &= \frac{1}{2(2\pi)^d} \int d^d\boldsymbol{p}\, |\boldsymbol{p}|\, \mathcal{F}, \\
(\phi_0[f]\Omega, \pi_0[g]\Omega) &= \frac{i}{2(2\pi)^d} \int d^d\boldsymbol{p}\, \mathcal{F},
\end{aligned} \tag{5.1.41}
$$

where $\mathcal{F}$ is given by

$$
\mathcal{F} \equiv \int\!\!\int d^d\boldsymbol{x}\, d^d\boldsymbol{y}\, e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}\overline{f(\boldsymbol{x})}g(\boldsymbol{y}). \tag{5.1.42}
$$

Let us now consider the very first equation from (5.1.41) and rewrite it in polar coordinates for $n = 3$ so that we have

$$
(\phi_0[f]\Omega, \phi_0[g]\Omega) = \int_0^\infty\!\!\int_0^\pi \frac{d|\boldsymbol{p}|d\theta}{2(2\pi)^2}\, |\boldsymbol{p}|\, \sin\theta \iint d^3\boldsymbol{x}d^3\boldsymbol{y}\, e^{i|\boldsymbol{p}|\,|\boldsymbol{r}|\cos\theta}\, \overline{f(\boldsymbol{x})}g(\boldsymbol{y}), \tag{5.1.43}
$$

where $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{y}$. Note that if the field under consideration was massive, meaning $\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$, then a cancellation of $\omega_{\boldsymbol{p}}$ given in the integral measure with $|\boldsymbol{p}| \neq \omega_{\boldsymbol{p}}$ (from the surface element) for such a case would not have been possible. As we shall see below, this situation certainly affects the results quite significantly. Introducing a factor $e^{-\epsilon|\boldsymbol{p}|}$ in (5.1.43) that basically allows to change the order of integration, and setting $\vartheta = \cos\theta$ one easily finds

$$
(\phi_0[f]\Omega, \phi_0[g]\Omega) = \lim_{\epsilon\to 0} \iint d^3\boldsymbol{x}d^3\boldsymbol{y} \int_0^\infty \frac{d|\boldsymbol{p}|}{(2\pi)^2}\, \frac{\sin(|\boldsymbol{r}|\,|\boldsymbol{p}|)}{|\boldsymbol{x}-\boldsymbol{y}|} e^{-\epsilon|\boldsymbol{p}|}\, \overline{f(\boldsymbol{x})}g(\boldsymbol{y}). \tag{5.1.44}
$$

One can show that

$$
\lim_{\epsilon\to 0} \int_0^\infty d|\boldsymbol{p}| \sin(r|\boldsymbol{p}|) e^{-\epsilon|\boldsymbol{p}|} = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \tag{5.1.45}
$$

which upon plugging into (5.1.44) yields

$$
(\phi_0[f]\Omega, \phi_0[g]\Omega) = \frac{1}{4\pi^2} \int\!\!\int d^3\boldsymbol{x}d^3\boldsymbol{y}\, \frac{\overline{f(\boldsymbol{x})}g(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}. \tag{5.1.46}
$$

[9] In order to write these two point functions in terms of Fourier transforms corresponding to momentum (see (5.1.36)) one needs to use the d-dimensional form given in (5.1.40).

Similarly, expanding the second equation from (5.1.41) for $d = 3$ one can straightforwardly come up with

$$\begin{aligned}(\pi_0[f]\Omega, \pi_0[g]\Omega) &= \frac{1}{4\pi^2}\int \frac{d^3\boldsymbol{p}}{2|\boldsymbol{p}|}\int\int d^3\boldsymbol{x}d^3\boldsymbol{y}\,\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}}e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}\overline{f(\boldsymbol{x})}g(\boldsymbol{y}) = \\ &= \frac{1}{4\pi^2}\int \frac{d^3\boldsymbol{p}}{2|\boldsymbol{p}|}\int\int d^3\boldsymbol{x}d^3\boldsymbol{y}\,e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}\overline{\nabla_{\boldsymbol{x}}f(\boldsymbol{x})}\cdot\nabla_{\boldsymbol{y}}g(\boldsymbol{y}), \qquad (5.1.47)\end{aligned}$$

where in the last equality we integrated twice by parts. Note that the last line given in (5.1.47) is exactly the same as of the previous case[10] (first equation in (5.1.41)) with $\overline{f(\boldsymbol{x})}g(\boldsymbol{y}) \to \overline{\nabla_{\boldsymbol{x}}f(\boldsymbol{x})}\cdot\nabla_{\boldsymbol{y}}g(\boldsymbol{y})$. Therefore,

$$(\pi_0[f]\Omega, \pi_0[g]\Omega) = \frac{1}{4\pi^2}\int\int d^3\boldsymbol{x}d^3\boldsymbol{y}\,\frac{\overline{\nabla_{\boldsymbol{x}}f(\boldsymbol{x})}\cdot\nabla_{\boldsymbol{y}}g(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}. \qquad (5.1.48)$$

Expanding the last equation given in (5.1.41) and utilizing $\delta(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{p}\,e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}$ one arrives at

$$(\phi_0[f]\Omega, \pi_0[g]\Omega) = \frac{i}{2}\int d^3\boldsymbol{x}\,\overline{f(\boldsymbol{x})}g(\boldsymbol{x}). \qquad (5.1.49)$$

For $h \in D(\mathbb{R}^{1+3})$ and $k \in D(\mathbb{R}^3)$, we find

$$(\phi[h]\Omega, \phi_0[k]\Omega) = -\frac{1}{4\pi^2}\int\int d^4x\,d^4y\,\frac{\overline{h(x)}k(y)}{(x-y)^2}, \qquad (5.1.50)$$

and

$$(\phi[h]\Omega, \pi_0[k]\Omega) = \frac{1}{2\pi^2}\int\int d^4x\,d^4y\,x_0\,\frac{\overline{h(x)}k(y)}{(x-y)^2}, \qquad (5.1.51)$$

where $y_0 = 0$ and $\mathrm{supp}(h) - \mathrm{supp}(k) \subset T \cup S$ with T and S being the sets of strictly timelike and spacelike vectors, see (5.1.10) and (5.1.11).

As we shall see in a moment, the above results play a crucial role in proving some relations for the operator U_ρ induced by the inversion ρ (basically involutive diffeomorphism of $T \cup S$), where ρ is defined by $\rho(x) \doteq -\frac{x}{x^2}$ with $x \in \mathbb{R}^{1+3}$ being an arbitrary four vector. We shall come back to this point in the next section with some pragmatic calculations. Here we are interested in motivating the idea behind U_ρ being a self-adjoint unitary operator by proving a useful theorem.

[10] Thus the equality $\omega_{\boldsymbol{p}} = |\boldsymbol{p}|$ plays an imperative role here as well.

For this purpose, let us define $\mathbb{R}^3_* \doteq \mathbb{R}^3 \setminus \{0\}$ to be the three dimensional punctured Euclidean space and denote by $\rho_0 : \mathbb{R}^3_* \to \mathbb{R}^3_*$ the involutive diffeomorphism that is defined by $\rho_0(\boldsymbol{x}) \doteq \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}$. As such, there exists an operator U_{ρ_0}, to be precise, $\Gamma(U_{\rho_0})$,whose action on the domain (see below for the details on $D(R^3_*)$ functions)

$$D(\Gamma(U_{\rho_0})) \doteq \mathrm{Span}_{\mathbb{C}}\Big\{\phi_0[f]\Omega \,\Big|\, f \in D(\mathbb{R}^3_*)\Big\} \subset \Gamma_s(\mathcal{H}) \tag{5.1.52}$$

is given by (see also example 5.1.1)

$$\Gamma(U_{\rho_0})\phi_0[f]\Omega = \phi_0[f_{\rho_0}]\Omega, \tag{5.1.53}$$

where $f_{\rho_0} \equiv f_{\rho_0}(\boldsymbol{x})$ has the following form

$$f_{\rho_0}(\boldsymbol{x}) = \mathcal{J}_0(\boldsymbol{x})|\boldsymbol{x}|^2 f(\rho_0(\boldsymbol{x})) = |\boldsymbol{x}|^{-4} f(\rho_0(\boldsymbol{x})) \tag{5.1.54}$$

with $\mathcal{J}_0(\boldsymbol{x}) = |\boldsymbol{x}|^{-6}$ being the Jacobian corresponding to the coordinate transformations $\boldsymbol{x} \to \boldsymbol{x}/|\boldsymbol{x}|^2$. Clearly the conformal factor is $\gamma_{\rho_0} = |\boldsymbol{x}|^{-4}$. Analogously, for the operator $\Gamma(U_\rho)$ on the domain

$$D(\Gamma(U_\rho)) \doteq \mathrm{Span}_{\mathbb{C}}\Big\{\phi[f]\Omega \,\Big|\, f \in D(T \cup S)\Big\} \tag{5.1.55}$$

we have

$$\Gamma(U_\rho)\phi[f]\Omega = \phi[f_\rho]\Omega, \tag{5.1.56}$$

where $f_\rho \equiv f_\rho(x)$ turns out to be

$$f_\rho(x) = \mathcal{J} x^2 f(\rho(x)) = -(x^2)^{-3} f(\rho(x)), \tag{5.1.57}$$

where, as above, $\mathcal{J} = -(x^2)^{-4}$ and $\gamma_\rho = -(x^2)^{-3}$, exactly as it appears in the example 5.1.1. For some four vectors $x, y \in (T \cup S)$ one finds

$$[\rho(x) - \rho(y)]^2 = x^{-2} y^{-2} (x - y)^2. \tag{5.1.58}$$

In what follows, we will restrict our attention to the one-particle Hilbert space $\mathcal{H}$. In particular, the vectors generated out of $\phi[f]\Omega$ and $\phi_0[f]\Omega$ will be identified as vectors from the domains $D(U_\rho) \subset \mathcal{H}$ and $D(U_{\rho_0}) \subset L^2[\mathbb{R}^3, d\mu_0]$, respectively. Note that $L^2[\mathbb{R}^3, d\mu_0]$ is the one-particle Hilbert space associated with the Cauchy data that is obtained by restricting[11] $\mathcal{H}$ to the time-zero submanifold, where $d\mu_0$ is the integral measure over $\mathbb{R}^3$, cf. (5.1.2).

In addition, we assume U_{ρ_0} to be densely defined symmetric operator on its domain $\mathcal{D}(U_{\rho_0}) \subset \mathcal{H}$, i.e., (see [68])

$$(\xi, U_{\rho_0}\eta) = (U_{\rho_0}\xi, \eta) \qquad \forall\, \xi, \eta \in \mathcal{D}(U_{\rho_0}). \tag{5.1.59}$$

Here U_{ρ_0} is densely defined follows from the fact that the Fourier transform of $D(\mathbb{R}^3_*)$ is dense in $L^2[\mathbb{R}^3, d\mu_0]$ as explained below. Note that U_{ρ_0} is an involution, i.e., $U_{\rho_0}^2 = \mathbb{1}$, on its domain.

Before we prove the main theorem we look at a couple of intermediate results.

Proposition 5.1.1. *Let (X, Σ, μ) be a measure space with X denoting locally compact T_2 space, Σ a σ-algebra containing all compact sets $K \subset X$ and μ finite regular measure. Let $C_c(X)$ denote the set all of all compactly supported continuous functions on X. Then for every $p \in [1, \infty)$, $C_c(X)$ is dense in $L^p(X)$.*

Proof. Note that $C_c(X) \subset L^p(X)$. Let $A \in \Sigma$ with finite measure, i.e, $\mu(A) < \infty$. Let χ_A be the characteristic function defined by

$$\chi_A(x) \doteq \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in X \setminus A. \end{cases} \tag{5.1.60}$$

Then we want to show that χ_A can be approximated in the L^p-norm by $C_c(X)$ functions.

Since μ is regular by hypothesis there exists an open set $U \subset X$ and a compact set $K \subset X$ such that $K \subset A \subset U$ and for $\epsilon > 0$ one has

$$\mu(U \setminus K) = \mu(U) - \mu(K) < \epsilon, \tag{5.1.61}$$

where one uses σ-additivity, finiteness and monotonicity of μ.

[11] This should be done carefully, since due to Sard's theorem the integral measure on the submanifold turns out to be zero. Thus, it requires to construct intermediate Banach space to find vectors that can be restricted to the Hilbert space of Cauchy data, see e.g. [67] for the details.

Using Uryshon's lemma for locally compact T_2 spaces, there exists $f \in C_c(X)$ such that $f \in [0,1]$ and $suppf \subset U$. Then

$$(\|\chi_A - f\|_p)^p = \int_X d\mu \, |\chi_A - f|^p = \int_{U\setminus K} d\mu \, |\chi_A - f|^p < \epsilon. \tag{5.1.62}$$

Let $S(X)$ be the set of simple functions, i.e., all functions of the form $g \doteq \sum_{i=1}^n c_i \chi_{A_i}$, where every $A_i \subset X$ is a measurable set. Then as in (5.1.62) simple functions can also be approximated by $C_c(X)$ functions. Clearly $S(X) \subset L^p(X)$.

Let $f \in L^p(X)$ then there exists (see Thm. 2.10 in [69]) a sequence $\{g_n\} \in S(X)$ such that $g_n \to f$ pointwise and $|g_n| \leq |f|$. Then $g_n \in L^p(X)$ and

$$|f - g_n|^p \leq |f|^p \in L^1(X). \tag{5.1.63}$$

Then the dominated convergence theorem implies

$$\int d\mu \ |f - g_n|^p \to 0 \qquad \text{as} \quad n \to \infty. \tag{5.1.64}$$

Meaning the simple functions are dense in $L^p(X)$, as a result, $C_c(X)$ is also dense in $L^p(X)$. □

Now applying prop. (5.1.1) to $D(\mathbb{R}^3_*)$, in particular, with $X = \mathbb{R}^3$, one has the following chain of arguments. Since $D(\mathbb{R}^3_*) \subset L^2(\mathbb{R}^3)$ is a subset of compactly supported smooth functions (cf. D(TUS)), the Fourier transform of $D(\mathbb{R}^3_*)$ is again in $L^2(\mathbb{R}^3)$ due to Hausdorff-Young inequality theorem (see e.g. [67]). In particular,

$$\forall \ f \in D(\mathbb{R}^3_*) \subset L^2(\mathbb{R}^3) \quad \text{the image} \quad \hat{f} \in L^2(\mathbb{R}^3). \tag{5.1.65}$$

Moreover, smooth functions belong to the class C^∞ that are continuous, thus prop. 5.1.1 implies any function $f \in L^2(\mathbb{R}^3)$ can be approximated by the smooth functions. Altogether Fourier transform of $D(\mathbb{R}^3_*)$ is dense in $L^2(\mathbb{R}^3)$.

Proposition 5.1.2. *For $f \in D(\mathbb{R}^3_*)$ the following equality holds*

$$U_{\rho_0} \pi_0[f]\Omega = \pi_0[f_-]\Omega, \tag{5.1.66}$$

where $f_-(\boldsymbol{x}) = |\boldsymbol{x}|^2 f_{\rho_0}(\boldsymbol{x})$.

Proof. Let $f, g \in D(\mathbb{R}^3_*)$, then

$$\begin{aligned}
&(\phi_0[f]\Omega, U_{\rho_0}\pi_0[g]\Omega) = (U_{\rho_0}\phi_0[f]\Omega, \pi_0[g]\Omega) = (\phi_0[f_{\rho_0}]\Omega, \pi_0[g]\Omega) \\
&= \frac{i}{2}\int d^3\boldsymbol{x}\, \overline{f_{\rho_0}(\boldsymbol{x})} g(\boldsymbol{x}) = \frac{i}{2}\int d^3\boldsymbol{x}\, |\boldsymbol{x}|^{-4}\overline{f(\rho_0(\boldsymbol{x}))} g(\boldsymbol{x}) \\
&= \frac{i}{2}\int d^3\boldsymbol{x}\, |\boldsymbol{x}|^{-6}|\boldsymbol{x}|^4\overline{f(\boldsymbol{x})} g(\rho_0(\boldsymbol{x})) = \frac{i}{2}\int d^3\boldsymbol{x}\overline{f(\boldsymbol{x})}|\boldsymbol{x}|^2\left[|\boldsymbol{x}|^{-4} g(\rho_0(\boldsymbol{x}))\right] \\
&= \frac{i}{2}\int d^3\boldsymbol{x}\, \overline{f(\boldsymbol{x})} g_-(\boldsymbol{x}) = (\phi_0[f]\Omega, \pi_0[g_-]\Omega), \qquad (5.1.67)
\end{aligned}$$

where we used (5.1.49), (5.1.54), f_- defined in (5.1.66), $U^*_{\rho_0} = U_{\rho_0}$ (see below), and the scaling coordinate transformations $\boldsymbol{x} \to \boldsymbol{x}/|\boldsymbol{x}|^{-2} \equiv \rho_0(\boldsymbol{x})$ in the third line. As well as the fact that ρ_0 is an involution. □

This preparation makes it feasible for us to state and prove the following theorem.

Theorem 5.1.1. *The symmetric densely defined operator U_{ρ_0} has a unique self-adjoint, unitary extension, denoted again by U_{ρ_0}, on $\mathcal{H}$. In addition, the closure of the operator U_ρ, i.e., U^{**}_ρ, is U_{ρ_0} and U_ρ is self-adjoint unitary operator on $\mathcal{H}$.*

Proof. First we show that U_{ρ_0} is an isometry. Let $f \in D(\mathbb{R}^3_*)$, then

$$\begin{aligned}
&||U_{\rho_0}\phi_0[f]\Omega|| = ||\phi_0[f_{\rho_0}]\Omega|| = [(\phi_0[f_{\rho_0}]\Omega, \phi_0[f_{\rho_0}]\Omega)]^{\frac{1}{2}} \\
&= \left[\frac{1}{4\pi^2}\int\int d^3\boldsymbol{x} d^3\boldsymbol{y}\, \frac{\overline{f_{\rho_0}(\boldsymbol{x})} f_{\rho_0}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{4\pi^2}\int\int d^3\boldsymbol{x} d^3\boldsymbol{y}\, \frac{\overline{f(\rho_0(\boldsymbol{x}))} f(\rho_0(\boldsymbol{y}))}{|\boldsymbol{x}-\boldsymbol{y}|^2|\boldsymbol{x}|^4|\boldsymbol{y}|^4}\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{4\pi^2}\int\int d^3\boldsymbol{x} d^3\boldsymbol{y}\, \frac{|\boldsymbol{x}|^4\overline{f(\boldsymbol{x})}|\boldsymbol{y}|^4 f(\boldsymbol{y})}{|\boldsymbol{x}|^6|\boldsymbol{y}|^6|\rho_0(\boldsymbol{x})-\rho_0(\boldsymbol{y})|^2}\right]^{\frac{1}{2}} \\
&= \left[\frac{1}{4\pi^2}\int\int d^3\boldsymbol{x} d^3\boldsymbol{y}\, \left(\frac{|\boldsymbol{x}|^2|\boldsymbol{y}|^2}{|\boldsymbol{x}|^2|\boldsymbol{y}|^2|\boldsymbol{x}-\boldsymbol{y}|^2}\right)\overline{f(\boldsymbol{x})} f(\boldsymbol{y})\right]^{\frac{1}{2}} \\
&= [(\phi_0[f]\Omega, \phi_0[f]\Omega)]^{\frac{1}{2}} = ||\phi_0[f]\Omega||, \qquad (5.1.68)
\end{aligned}$$

where we used (5.1.46) in the second line[12], (5.1.54) in the third line, coordinate transformations

[12] We would like to highlight here once again the importance of a relation $\omega_{\boldsymbol{p}} = |\boldsymbol{p}|$, which holds only for the massless case. Moreover, the expressions that enter in the proof of this theorem, as we saw above, explicitly depend on this relation. Hence, it should be clear by now that this theorem may not be proven for the massive

$\boldsymbol{x} \to \boldsymbol{x}/|\boldsymbol{x}|^2 \equiv \rho_0(\boldsymbol{x})$ and $\boldsymbol{y} \to \boldsymbol{y}/|\boldsymbol{y}|^2 \equiv \rho_0(\boldsymbol{y})$ in the fourth line, hence, (5.1.58) for spatial three dimensional vectors in the third last equality.

As a result, $U_{\rho 0}$ is bounded on $\mathcal{D}(U_{\rho 0})$ with $\|U_{\rho 0}\| = 1$. Then using the continuous linear extension theorem, also known as BLT theorem (see Thm. I.7 in [55]), follows that an extension $U_{\rho 0}$ on $\mathcal{H}$ is bounded. But symmetric bounded (linear) operator is self-adjoint. Hence unitary, since $U_{\rho 0}$ is an isometry (or alternatively from $U_{\rho 0}^2 = \mathbb{1}$).

Now we prove the second claim. Let us consider $h_1 \in D(T \cup S)$ and the corresponding open and bounded open set $\mathcal{O}_{h_1} \subset T \cup S$ such that $\operatorname{supp} h_1 \subset \mathcal{O}_{h_1}$. We define

$$\mathcal{O}_{h_1} - \mathcal{V} \doteq \Big\{ x - y \,\Big|\, x \in \mathcal{O}_{h_1},\, y \in \mathcal{V} \text{ with } y_0 = 0 \Big\}, \tag{5.1.69}$$

for an arbitrary open set $\mathcal{V} \subset \mathbb{R}^3_*$. Let $\mathcal{H}' \subset \mathcal{H}$ be a dense subspace generated by the set $\{\phi_0[h_1]\Omega, \pi_0[h_2]\Omega \mid h_1, h_2 \in D(\mathcal{V})\}$.

Since $U_{\rho 0}$ is self-adjoint, it is closed. But for the closed symmetric self-adjoint operator, $U_{\rho 0}$, one has $U_{\rho 0} = U_{\rho 0}^{**} = U_{\rho 0}^*$. Then due to $(U_{\rho 0} \restriction \mathcal{H}') \subset U_\rho^*$ and $(U_{\rho 0} \restriction \mathcal{H}')^{**} = U_{\rho 0}$, follows $U_\rho^{**} = U_{\rho 0}$.

Now using (5.1.50), (5.1.51) and prop. 5.1.2 one finds

$$(U_\rho \phi[h]\Omega, U_{\rho 0}\phi_0[k]\Omega) = (\phi[h_\rho]\Omega, \phi_0[k_{\rho 0}]\Omega) = (\phi[h]\Omega, \phi_0[k]\Omega), \tag{5.1.70}$$

and

$$(U_\rho \phi[h]\Omega, U_{\rho 0}\pi_0[k]\Omega) = (\phi[h_\rho]\Omega, \pi_0[k_{\rho 0}]\Omega) = (\phi[h]\Omega, \pi_0[k]\Omega). \tag{5.1.71}$$

Computation for the last two equations is quite similar to the case considered in (5.1.68). The only change is to take two different types of transformations using ρ and ρ_0 for four and three dimensional vectors, respectively, and use suitable analogue of the expression given in (5.1.58).

As it is shown in (5.1.70) and (5.1.71), U_ρ along with $U_{\rho 0}$ preserve the scalar products. So one has $U_\rho^* U_{\rho 0} = \mathbb{1}$, but $U_{\rho 0} = U_\rho^{**}$, thus $U_\rho^* U_\rho^{**} = \mathbb{1}$. Since U_ρ is an isometry on the domain $D(U_\rho)$ (cf. (5.1.68)), i.e., $U_\rho^* U_\rho = \mathbb{1}$, using the above arguments it also extends to the bounded self-adjoint operator on $\mathcal{H}$. Moreover, since U_ρ is an involution, altogether U_ρ is

theories. So, the whole construction considered in our studies is valid just for the massless or conformal scalar theories.

self-adjoint unitary. □

As a consequence of the theorem 5.1.1, $\mathcal{M}(V_+)$, $\mathcal{M}(O)$ and $\mathcal{M}(W_r)$ are spatially isomorphic[13]. We shall use this fact to derive relevant geometrical transformations in the next chapter. Most importantly, combined this theorem with Thm. 2.3.1 one can now see that the local algebras for the double cones are of type III_1.

Using the above theorem, basically the fact that U_ρ is a self-adjoint unitary operator, we shall now derive a couple of useful relations and conclude this subsection. Let P_μ, D, $M_{\mu\nu}$ and K_μ denote the generators corresponding to translations, dilation/dilatation, Lorentz transformation and special conformal transformations (SCT), respectively. It is quite well known that these generators satisfy the following algebra:

$$\begin{aligned}&[P_\mu, D] = -iP_\mu; \quad [K_\mu, D] = iK_\mu; \quad [P_\mu, K_\nu] = -2i\left(\eta_{\mu\nu}D + M_{\mu\nu}\right);\\&[P_\mu, M_{\nu\rho}] = i\left(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\right); \quad [K_\mu, M_{\nu\rho}] = i\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right),\end{aligned} \tag{5.1.72}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. All other commutators vanish. Let us now consider $\varsigma(\cdot)$, $t(\cdot)$ and $k(\cdot)$ representing dilation, translations and SCT on $\mathbb{R}^{1+3}$. Let α be some real parameter and $x \in \mathbb{R}^{1+3}$ so that

$$\rho(\varsigma(\alpha)\rho(x)) = \rho\left(-e^\alpha \frac{x}{x^2}\right) = e^{-\alpha}x, \tag{5.1.73}$$

meaning that dilation conjugated by inversion yields inverse dilation in the function space. The corresponding generator is essentially the same as a dilation generator with negative sign, i.e., $-D$. Thus, on a Hilbert space we deduce the expression

$$U_\rho e^{i\lambda D} U_\rho = e^{-i\lambda D}, \qquad \lambda \in \mathbb{R}. \tag{5.1.74}$$

Similarly,

$$\rho(t(\alpha)\rho(x)) = \rho\left(t(\alpha)\left(-\frac{x^\mu}{x^2}\right)\right) = \rho\left(\frac{x^2\alpha^\mu - x^\mu}{x^2}\right) = \frac{x^\mu - \alpha^\mu x^2}{1 - 2\alpha x + \alpha^2 x^2}. \tag{5.1.75}$$

[13] Meaning there exists the unitary operator $\Gamma(U_\rho) : \Gamma(\mathcal{H}_1) \to \Gamma(\mathcal{H}_2)$ induced by $U_\rho : \mathcal{H}_1 \to \mathcal{H}_2$, where the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are the ones on which the underlying local algebras live. For example, $\Gamma(T(1)U_\rho T(\frac{1}{2}))\mathcal{M}(V_+)\Gamma(T(-\frac{1}{2})U_\rho T(-1)) = \mathcal{M}(O)$ (see (5.3.3)) with $\mathcal{M}(V_+) \subset \mathfrak{B}(\Gamma(\mathcal{H}_1))$ and $\mathcal{M}(O) \subset \mathfrak{B}(\Gamma(\mathcal{H}_2))$.

It implies that on a Hilbert space we have

$$U_\rho e^{i\lambda P_\mu} U_\rho = e^{i\lambda K_\mu}, \qquad \lambda \in \mathbb{R}. \tag{5.1.76}$$

5.1.3 Massive Case

Let $\mathcal{F}(\chi(\boldsymbol{p}))$ denote the Fourier transform of $\chi(\boldsymbol{p})$ in the momentum space, i.e.,

$$\hat{\chi}(\boldsymbol{r}) = \frac{1}{(2\pi)^d} \int d^d\boldsymbol{p}\, \chi(\boldsymbol{p})\, e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}, \tag{5.1.77}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. In (5.1.35), proceeding with $\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$, one finds the following formulas for the time zero field ϕ_0 and the canonical momentum π_0 (we restrict ourselves to $d = 3$ spatial dimension):

$$(\phi_0[f]\Omega, \phi_0[g]\Omega) = \frac{1}{2} \int d^3\boldsymbol{x} d^3\boldsymbol{y}\; \overline{f(\boldsymbol{x})}\; \widehat{\omega_{\boldsymbol{p}}^{-1}}(\boldsymbol{r})\, g(\boldsymbol{y}). \tag{5.1.78}$$

$$(\pi_0[f]\Omega, \pi_0[g]\Omega) = \frac{1}{2} \int d^3\boldsymbol{x} d^3\boldsymbol{y}\; \overline{f(\boldsymbol{x})}\; \widehat{\omega_{\boldsymbol{p}}}(\boldsymbol{r})\, g(\boldsymbol{y}). \tag{5.1.79}$$

$$(\phi_0[f]\Omega, \pi_0[g]\Omega) = \frac{1}{2} \int d^3\boldsymbol{x} d^3\boldsymbol{y}\; \overline{f(\boldsymbol{x})}\; \hat{f}(\boldsymbol{r})\, g(\boldsymbol{y}). \tag{5.1.80}$$

In (5.1.80), the Fourier transform is of a constant function $f(\boldsymbol{p})$, that is in fact the Dirac delta function $\delta(\boldsymbol{r})$ for $f(\boldsymbol{p}) = 1$. Then using the translation property of the delta function, it simplifies further to yield

$$(\phi_0[f]\Omega, \pi_0[g]\Omega) = \frac{i}{2} \int d^3\boldsymbol{x}\; \overline{f(\boldsymbol{x})} g(\boldsymbol{x}). \tag{5.1.81}$$

Note that (5.1.49) and (5.1.81) are the same. Unfortunately, this is not enough for U_ρ to be unitary. We shall discuss this point in a moment.

Rewriting (5.1.78) and (5.1.79) in spherical coordinates give us

$$(\phi_0[f]\Omega, \phi_0[g]\Omega) = \int \frac{d^3\boldsymbol{x} d^3\boldsymbol{y}}{4\pi^2}\; \overline{f(\boldsymbol{x})} \left(\frac{1}{|\boldsymbol{r}|} \int_0^\infty d|\boldsymbol{p}|\; \frac{|\boldsymbol{p}|}{\omega_{\boldsymbol{p}}} \sin\left(|\boldsymbol{p}|\, |\boldsymbol{r}|\right) \right) g(\boldsymbol{y}). \tag{5.1.82}$$

and

$$(\pi_0[f]\Omega, \pi_0[g]\Omega) = \int \frac{d^3\boldsymbol{x}d^3\boldsymbol{y}}{4\pi^2}\,\overline{f(\boldsymbol{x})}\left(\frac{1}{|\boldsymbol{r}|}\int_0^\infty d|\boldsymbol{p}|\;|\boldsymbol{p}|\,\omega_{\boldsymbol{p}}\sin\left(|\boldsymbol{p}|\,|\boldsymbol{r}|\right)\right)g(\boldsymbol{y}). \tag{5.1.83}$$

Now we need a special function[14], namely modified Bessel function of second kind, also known as Macdonald function. One of the equivalent representations (out of three, see e.g., Ch. VI in [72]) is given by

$$K_n(x) = \frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)\int_0^\infty dt\frac{\cos\left(xt\right)}{x^n\left(t^2+1\right)^{n+\frac{1}{2}}}, \qquad x>0 \tag{5.1.84}$$

with $\Gamma(\cdot)$ being the usual gamma function. Clearly

$$K_0(x) = \int_0^\infty dt\frac{\cos\left(xt\right)}{\sqrt{t^2+1}}, \tag{5.1.85}$$

whereupon taking the derivative with respect to x one finds

$$K_0(x)' = -\int_0^\infty dt\frac{t\sin\left(xt\right)}{\sqrt{t^2+1}}. \tag{5.1.86}$$

Then with $t=\frac{|\boldsymbol{p}|}{m}$ and $x=m|\boldsymbol{r}|$ the last equality gives

$$K_0(m|\boldsymbol{r}|)' = -\frac{1}{m}\int_0^\infty d|\boldsymbol{p}|\frac{|\boldsymbol{p}|\sin\left(|\boldsymbol{p}|\,|\boldsymbol{r}|\right)}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \tag{5.1.87}$$

As a result, (5.1.82) simplifies further to yield

$$\begin{aligned}(\phi_0[f]\Omega, \phi_0[g]\Omega) &= \frac{1}{4\pi^2}\int d^3\boldsymbol{x}d^3\boldsymbol{y}\;\overline{f(\boldsymbol{x})}\left(-\frac{m}{|\boldsymbol{r}|}K_0(m|\boldsymbol{r}|)'\right)g(\boldsymbol{y}) = \\ &= \frac{1}{4\pi^2}\int d^3\boldsymbol{x}d^3\boldsymbol{y}\;\overline{f(\boldsymbol{x})}\left(\frac{m}{|\boldsymbol{r}|}K_1(m|\boldsymbol{r}|)\right)g(\boldsymbol{y}),\end{aligned} \tag{5.1.88}$$

where we also used the recursion relation $K_n(x)' = -\frac{1}{2}\left[K_{n+1}(x)+K_{n-1}(x)\right]$ along with the

[14] Such functions are intrinsically related to group representations. We avoid touching this part of mathematical analysis, as it is not directly related to what we are interested in. Extensive material in this regard can be found in [70, 71]

symmetry $K_n(x) = K_{-n}(x)$. Now let us rewrite (5.1.83) such that

$$\begin{aligned}(\pi_0[f]\Omega, \pi_0[g]\Omega) &= \frac{1}{4\pi^2}\int d^3\boldsymbol{x}d^3\boldsymbol{y}\ \overline{f(\boldsymbol{x})}\left(-\frac{1}{2i|\boldsymbol{r}|}\int_{-\infty}^{\infty} d|\boldsymbol{p}|\ |\boldsymbol{p}|\,\omega_{\boldsymbol{p}}\,e^{-i|\boldsymbol{p}|\,|\boldsymbol{r}|}\right) g(\boldsymbol{y}) = \\ &= \frac{1}{2\pi}\int d^3\boldsymbol{x}d^3\boldsymbol{y}\ \overline{f(\boldsymbol{x})}\left(\frac{i}{2|\boldsymbol{r}|}\widehat{|\boldsymbol{p}|\,\omega_{\boldsymbol{p}}}(|\boldsymbol{r}|)\right) g(\boldsymbol{y}). \qquad (5.1.89)\end{aligned}$$

Note that

$$\widehat{|\boldsymbol{p}|\,\omega_{\boldsymbol{p}}}(|\boldsymbol{r}|) = i\partial_{|\boldsymbol{r}|}\widehat{\omega_{\boldsymbol{p}}}(\boldsymbol{r}) = i\partial_{|\boldsymbol{r}|}\left(m^2 - \partial^2_{|\boldsymbol{r}|}\right)\widehat{\omega_{\boldsymbol{p}}^{-1}}(\boldsymbol{r}), \qquad (5.1.90)$$

with Fourier transform understood in the sense of distribution and

$$\widehat{\omega_{\boldsymbol{p}}^{-1}}(\boldsymbol{r}) = \frac{2m}{2\pi}K_0(m|\boldsymbol{r}|), \qquad (5.1.91)$$

which upon substituting in (5.1.90) immediately produces

$$\widehat{|\boldsymbol{p}|\,\omega_{\boldsymbol{p}}}(|\boldsymbol{r}|) = -\frac{im}{2\pi}\left(2m^2K_1(m|\boldsymbol{r}|) + K_2(m|\boldsymbol{r}|) + K_0(m|\boldsymbol{r}|)\right). \qquad (5.1.92)$$

In summary, we have

$$(\pi_0[f]\Omega, \pi_0[g]\Omega) = \tfrac{1}{4\pi^2}\int d^3\boldsymbol{x}d^3\boldsymbol{y}\ \bar{f}(\boldsymbol{x})\left(\tfrac{m}{2|\boldsymbol{r}|}\left(2m^2K_1(m|\boldsymbol{r}|) + K_2(m|\boldsymbol{r}|) + K_0(m|\boldsymbol{r}|)\right)\right) g(\boldsymbol{y}). \qquad (5.1.93)$$

Now it is clear from (5.1.88) and (5.1.93) that the presence of a mass makes the structure of two point functions more complicated than the massless case. As a result, here U_ρ fails to be even isometric as demonstrated in (5.1.68). Therefore, the approach used in the massless case cannot work here, since U_ρ, the key ingredient for the conformal mapping is missing here. Note that, though abstractly, the unitary equivalence between massive and massless local algebras already exists [73], provided $d = 2, 3$.

5.2 Geometrical Transformations

Our analysis is based on geometrical realization of the modular operators corresponding to various spacetime regions. As such, in this section we shall derive the concerned transformations, which will lead us to the corresponding expressions of modular operators on $\mathcal{H}$, and at the same time serve as a basis for the TT orbits to be considered later in this chapter.

It is important to note that geometrical interpretation of a modular operator exists in limited number of cases, see [19] for the details in this regards. In particular, a family of interconnected spacetime regions, consisting of wedges, lightcones and diamonds exhibit this property. Our aim here is to consider all three cases and discuss the underlying geometrical transformations.

5.2.1 From a Forward Lightcone to a Double Cone

Given the forward light cone V_+, the backward light cone V_-, and canonical orthogonal basis (e_0, e_1, e_2, e_3) in $\mathbb{R}^{1+3}$ one can define the double cone region $O_0 = (V_+ - e_0) \cap (V_- + e_0)$ as shown in Fig. 5.1a. Observe that the time interval between the center of O_0 and the top corner, i.e., the radius of O_0, henceforth the size of O_0, is set to the unity [15], cf. Fig. 5.4a.

Here O_0 will be also referred to as the diamond situated at the origin of the coordinate system. Shifting O_0 by $-e_0$ and taking intersection of the resulting region with $\mathbb{R}e_0$ we find $(O_0 - e_0) \cap \mathbb{R}e_0 = \mathfrak{L}$, as it can be seen in the Fig. 5.1b. In particular,

$$\mathfrak{L} \doteq \Big\{ t \in \mathbb{R} \,\Big|\, -2 < t < 0 \Big\}. \tag{5.2.1}$$

Then using (5.1.14) we define (note that $t < 0$)

$$\rho(\mathfrak{L}) \doteq \left\{ \frac{1}{t} \,\Big|\, \frac{1}{2} < \frac{1}{t} < \infty \right\} \tag{5.2.2}$$

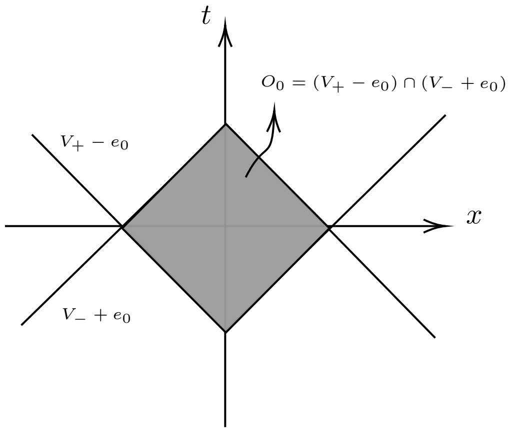
As such, taking the double timelike complement of $\mathfrak{L}$ as defined in (5.1.12) we obtain

$$(\mathfrak{L})^{tt} = O_0 - e_0, \tag{5.2.3}$$

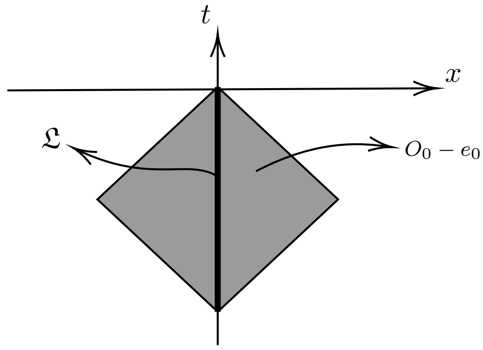
which is basically the diamond O_0 displaced backward by e_0 (see Fig. 5.2a). Then applying inversion ρ on both sides of (5.2.3) and using (5.1.17) we have

$$\rho(O_0 - e_0) = (\rho(\mathfrak{L}))^{tt} = \Big[\big[\rho(\mathfrak{L}) \big]^t \Big]^t. \tag{5.2.4}$$

[15] As a result, the coordinates in our case do not possess a characteristic length, so they are dimensionless. For this reason, (5.1.14) is perfectly fine for the moment. Mind the pitfall: in order to give a physical meaning to any formulas, e.g., modular operators (as we shall see below), the dimensions must be recovered. In that case we shall assign an arbitrary length L to O_0, and accordingly retrieve the dimensions simply by dimensional analysis.



(a) Double cone region



(b) Shifted double cone

Figure 5.1: Schematic diagrams of double cone region O_0 and displaced O_0 in the backward direction by e_0.

In fact,

$$(\rho(\mathfrak{L}))^t = V_- + \frac{1}{2}e_0 \implies (\rho(\mathfrak{L}))^{tt} = V_+ + \frac{1}{2}e_0, \tag{5.2.5}$$

which is clearly demonstrated in Fig. 5.2b.

Finally, comparing (5.2.4) and (5.2.5) we get one of the geometrical transformations that we are interested in:

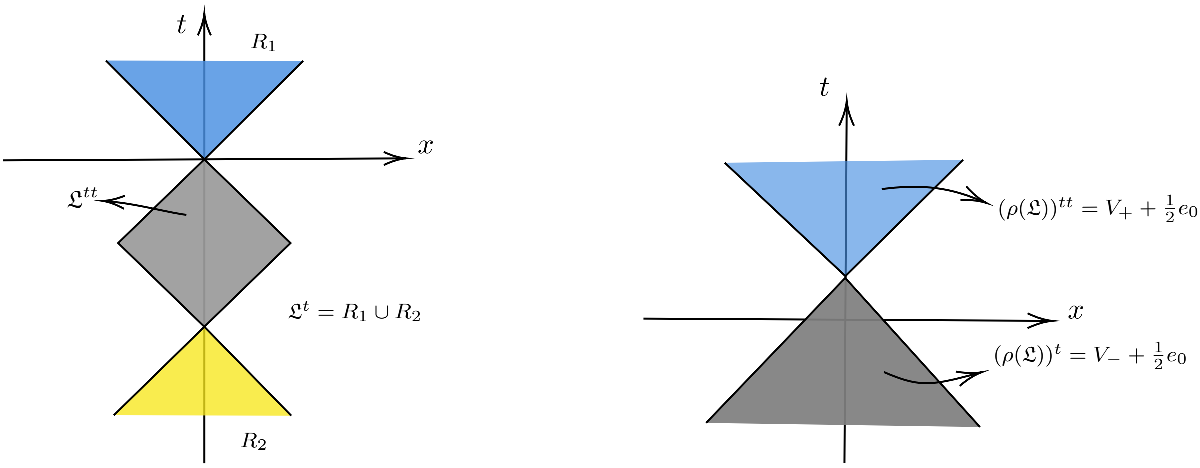
$$O_0 = \rho\left(V_+ + \frac{1}{2}e_0\right) + e_0, \tag{5.2.6}$$

where we used the fact that $\rho(\rho(O)) = O$ for any $O \subset T \cup S$. As we shall see below, the above transformation, from the forward light cone V_+ to the diamond O_0 situated at the origin, plays a very crucial role in deriving a modular operator of Tomita-Takesaki theory for the concerned region.

5.2.2 From a Right Wedge to a Double Cone

An analogue of the geometrical transformations given in (5.2.6) also exists between W_r and O_0, which we shall derive here with all the necessary details. The transformations that we are concerned with takes the form

$$O_0 = \rho\left(W_r + \frac{1}{2}e_1\right) - e_1. \tag{5.2.7}$$



(a) Timelike complements of $\mathfrak{L}$ (b) Timelike complements of $\frac{1}{t}$

Figure 5.2: Schematic diagrams of timelike complements of restricted real-line to the displaced O_0 and inversion of the same, given in (5.2.3) and (5.2.4), respectively.

In order to prove (5.2.7) one requires primarily to translate O_0 by e_1 along the positive x-axis. Then introducing a time slice (Cauchy surface) provides us with a ball - intersection of a double cone and a time slice, which is a disk in $\mathbb{R}^2$, see the fig. 5.3. Making use of the same strategy in $\mathbb{R}^3$ one finds the unit ball consisting of spheres of various radii, on which applying the inversion ρ_0 (restriction of ρ to $\mathbb{R}^3$) we end up with a half space. Finally, taking the double spacelike complements yields the expression that we are looking for.

Let us commence with a sphere centered at the origin C given by $x^2 + y^2 + z^2 = r^2$. We consider the points on such sphere parameterized by the polar coordinates (r, θ_1, θ_2) with $r > 0$, $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$, and write it in terms of a vector as follows.

$$\boldsymbol{x}_0 = (r\,\sin\theta_1\cos\theta_2, r\,\sin\theta_1\sin\theta_2, r\,\cos\theta_1), \tag{5.2.8}$$

whereupon translating by e_1 along the positive x-axis we find,

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{p} = (r\,\sin\theta_1\cos\theta_2 + 1, r\,\sin\theta_1\sin\theta_2, r\,\cos\theta_1). \tag{5.2.9}$$

For which the square of its length is simply given by

$$|\boldsymbol{x}|^2 = 1 + r^2 + 2r\,\sin\theta_1\cos\theta_2. \tag{5.2.10}$$

Let $l_1 \equiv \sin\theta_1\cos\theta_2$, $l_2 \equiv \sin\theta_1\sin\theta_2$ and $l_3 \equiv \cos\theta_1$, then the inversion of the vector given in

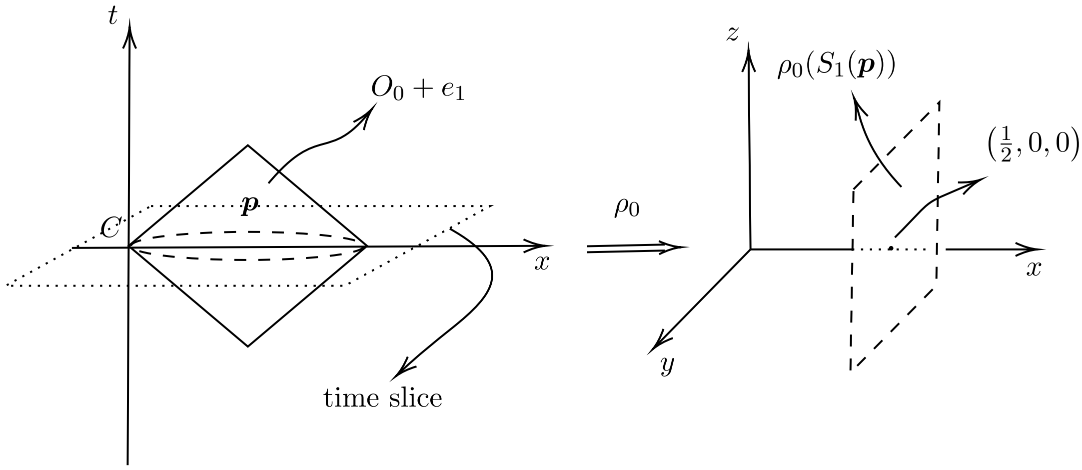


Figure 5.3: Demonstration of introducing time slices (Cauchy surfaces) that intersect the translated diamond $O_0 + e_1$ and give rise to a unit ball in $\mathbb{R}^2$ (i.e. a disk). Analogously in $\mathbb{R}^3$ one obtains a unit ball, which is nothing but the union of concentric spheres of various radii. One of them (outermost) upon applying an inversion ρ_0, maps to the plane. This procedure essentially applies to all such spheres lying in the interior of the unit ball.

(5.2.8) can be written as

$$\rho_0(\boldsymbol{x}) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2} = \left(\frac{rl_1 + 1}{1 + r^2 + 2rl_1}, \frac{rl_2}{1 + r^2 + 2rl_1}, \frac{rl_3}{1 + r^2 + 2rl_1} \right). \tag{5.2.11}$$

Taking $r = 1$ in the expression stated in (5.2.9), we clearly see that it characterizes a unit (2-)sphere $S_1(\boldsymbol{p})$ centered at the point $\boldsymbol{p} = (1, 0, 0)$, which is essentially the boundary of a unit ball $B_1(\boldsymbol{p})$ generated by intersecting the time slice in $\mathbb{R}^3$ and shifted O_0. One of the great circles associated with a unit sphere can be described by fixing $\theta_1 = \frac{\pi}{2}$ in (5.2.9) with $r = 1$, such that $l_1 = \cos\theta_2$, $l_2 = \sin\theta_2$, $l_3 = 0$ and $|\boldsymbol{x}|^2 = 2(1 + l_1)$ that altogether yields the inversion (5.2.11) of a circle under consideration. For example, along with the above settings, choosing $\theta_2 = 0$ in (5.2.8) gives the point $(2, 0, 0)$, which maps to the point $(\frac{1}{2}, 0, 0)$ under the inversion, i.e., $l_2 = l_3 = 0$, $l_1 = 1$ and $|\boldsymbol{x}|^2 = 4$. Furthermore, the concerned circle also passes through the origin C that trivially maps to the point at infinity. Similarly, one might check for the points $(1, -1, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, $(\frac{3}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ and so on — images of these points under inversion are $(\frac{1}{2}, -\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3\sqrt{2}})$ (for this one $\theta_1 = \theta_2 = \frac{\pi}{4}$), respectively, and thus conclude that the mapping of a unit sphere under ρ_0 gives rise to a plane $x = \frac{1}{2}$ (see the Fig. 5.3), which divides $\mathbb{R}^3$ into two half-spaces. As for the ball, for which $r < 1$, all the points map to one of the planes lying parallel to the yz-plane and extending towards the positive x-axis. Let $\mathbb{H}$ denote the half-space containing all these planes. Then just as we did in the case 1, one finds $(B_1(\boldsymbol{p}))^{ss} = O_0 + e_1$. Therefore, we have the following relation:

$$\rho(O_0 + e_1) = \rho(B_1^{ss}) = \rho(B_1)^{ss} = (\mathbb{H})^{ss} = W_r + \frac{1}{2}e_1, \tag{5.2.12}$$

where we generalized ρ_0 to ρ and used (5.1.18), and the last equality follows from the fact that taking the first spacelike complement of a half-space produces the left wedge W_l shifted by $-\frac{1}{2}e_1$ that upon taking the second complement gives W_r translated by $\frac{1}{2}e_1$ along the positive x-axis.

Back to a Forward Lightcone

Equating (5.2.6) and (5.2.7) and solving it for V_+ one easily arrives at

$$V_+ = \rho\left(\rho\left(W_r + \frac{1}{2}e_1\right) - e_1 - e_0\right) - \frac{1}{2}e_0. \tag{5.2.13}$$

This is one of important transformations, which will be useful to demonstrate that in principle, once a modular operator is known for any given region among the ones discussed here, one can derive the rest of them. We shall return to this issue in the next section after having obtained suitable modular operators.

5.2.3 A Translated Double Cone

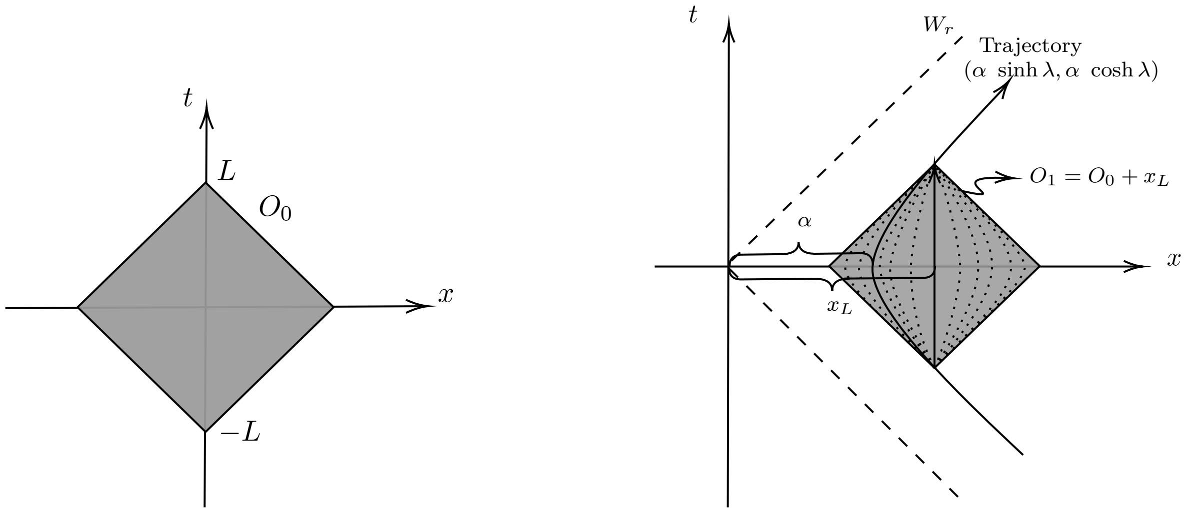
In order to utilize O_0 for our analysis we have to translate it by a suitable distance with its size being arbitrary L. We discuss now the systematic derivation of this tranformation.

As it is well known the orbits of preferred observers in the wedge W_r are hyperbolas. One of such orbits passing through the spacetime point $(0, \alpha)$ is given by[16]

$$x^\mu(\beta) = (\alpha \sinh\beta, \alpha \cosh\beta) \tag{5.2.14}$$

with $\alpha \equiv \frac{1}{a}$ denoting inverse acceleration, which physically amount distance along x-axis, and $\beta = \frac{\tau}{\alpha}$, where τ stands for the proper time of an observer travelling along a given orbit. It is clear that we use a particular observer passing through the spacetime point $(0, \alpha)$, which is enough here to fix the transformations we are interested in. For the temperature related studies, trivial generalization of this trajectory will be needed, see below.

[16] We use shorthand notations, which will be used intensively in subsect. 5.3.2, where the details in this regard can be found.



(a) Double cone of arbitrary size L (b) Translated double cone

Figure 5.4: We consider a double cone (vid. fig. 5.1a) O_0 of arbitrary size L as shown in fig. 5.4a and translate it by x_L with its size unchanged such that the one of the modular trajectories of O_1 passing through the point α coincides to that of W_r.

Now let O_0 be the diamond of size L as shown in fig. 5.4a, which upon translating, say by, x_L, keeping its size fixed, one obtains O_1, i.e.,

$$O_1 = O_0 + x_L, \tag{5.2.15}$$

where x_L is found as follows. Since the translation under consideration is spatial and the size of the diamond is unchanged, one can use the trajectory (5.2.14) for our purpose. In particular, one considers the trajectory intersecting top and bottom of O_1 (see fig. 5.4b), then there exists $\tau_L \in \mathbb{R}$ such that

$$L = \alpha \sinh\left(\frac{\tau_L}{\alpha}\right) \implies \tau_L = \alpha \ln\left(\frac{L}{\alpha} + \sqrt{\frac{L^2}{\alpha^2} + 1}\right). \tag{5.2.16}$$

This proper time upon substituting in the distance formula provides one with the necessary expression. More specifically, one has

$$x_L = \alpha \cosh\left(\frac{\tau_L}{\alpha}\right) = \frac{\alpha}{2}\left(e^{\frac{\tau_L}{\alpha}} + e^{-\frac{\tau_L}{\alpha}}\right) = \sqrt{\alpha^2 + L^2}, \tag{5.2.17}$$

which will play a crucial role in our analysis.

5.3 Modular Dynamics

Having derived the geometrical transformations in the last section, we now move our attention to the corresponding actions on the one-particle Hilbert space $\mathcal{H}$. In order to do so we look for a one parameter unitary group implementing geometrical transformations (sect. 5.2) on the Hilbert space.

5.3.1 Modular Operators

In this subsection we study the modular operators for the right wedge W_r, the double cone O_0 and the forward lightcone V_+. Later, we generalize the modular operator for O_0 to obtain the modular operator for O_1.

The modular operator corresponding to a right wedge (5.1.21) was derived in [29]. It implements the Lorentz boost leaving the right wedge invariant, vid. Thm. 4.1.3[17]:

$$\Delta_{W_r}^{-\frac{i\lambda}{2\pi}} \equiv V_{W_r}(\lambda) = e^{-i\lambda\kappa_1}, \tag{5.3.1}$$

where $\kappa_1 \equiv M_{01}$ is the generator of the Lorentz boost along x_1 direction. (5.3.1) is valid in both cases, massive and massless. For the forward lightcone a similar analysis was carried out in [74] to yield

$$\Delta_{V_+}^{-\frac{i\lambda}{2\pi}} \equiv V_{V_+}(\lambda) = e^{-i\lambda D}, \tag{5.3.2}$$

which, however, is valid only for the massless case. Both of these operators can be rederived using pertinent geometrical transformations, which will be explained in a moment.

Prior knowledge of these operators, in principle at least one of them is enough (see below), along with the geometrical transformations (5.2.6) and (5.2.7), facilitated Hislop and Longo [33] to compute the modular operator for the double cone O_0. We now reproduce their result.

Let $T(\cdot)$ and $S(\cdot)$ be one parameter groups of time translations and space translations acting

[17] Note that the theorem 4.1.3 per se gives the modular operator on the whole space of states, i.e., the Fock space, but here we restrict its action on the one-particle Hilbert space and again denote it by Δ (for all the spacetime regions considered here) to keep track of the constant 2π, which will be recovered again for the one-particle modular operator in sect. 6.1.

on $\mathcal{H}$. Using (5.2.6) one can show that

$$\Delta_{O_0}^{-\frac{i\lambda}{2\pi}} \equiv V_{O_0}(\lambda) = T(1)U_\rho T\left(\frac{1}{2}\right)V_{V_+}(\lambda)T\left(-\frac{1}{2}\right)U_\rho T(-1) = = e^{iP_0}e^{\frac{i}{2}K_0}e^{-i\lambda D}e^{-\frac{i}{2}K_0}e^{-iP_0}. \tag{5.3.3}$$

Since the generators D, K_0 and P_0 are noncommutative in the conformal algebra (5.1.72), one uses the Baker–Campbell–Hausdorff (BCH) formula[18] to obtain

$$V_{O_0}(\lambda) = \exp\left[\frac{i\lambda}{2}\left(L^{-1}K_0 - LP_0\right)\right], \tag{5.3.4}$$

where we recovered L simply by dimensional analysis. This general expression will be used to obtain the modular operator for O_1. Note that the derivation of (5.3.4) depends on the conformal transformations involving an unitary operator U_ρ (see Thm. 5.1.1 and discussion thereafter), which then clearly speaks limitations of this approach, for it being solely geometrical, cannot be used for the massive case. For the latter there are certain evidences that a part of a modular operator involves non-geometrical terms, see e.g. [19]. In fact some efforts to find the explicit modular operator for the massive case can be found in [75–77].

For the forthcoming analysis we are interested in translated O_0. Before we proceed we would like to point out that due to geometrical transformation presented in the previous section one just has to know at least one of the modular operators given above. For example, using (5.2.7) together with (5.3.4) so that[19]

$$V_{W_r}(\lambda) = S\left(-\frac{1}{2}\right)U_\rho S(1)V_{O_0}(\lambda)S(-1)U_\rho S\left(\frac{1}{2}\right) = = e^{-\frac{i}{2}P_1}e^{iK_1}\left(U_\rho V_{O_0}U_\rho\right)e^{-iK_1}e^{\frac{i}{2}P_1}, \tag{5.3.5}$$

where we used (5.1.76) and the fact that $U_\rho U_\rho = 1$. Now as above using BCH formula one

[18] Though the generators that belong to the Lie algebra $\mathfrak{g}$ of the (conformal) Lie group G are unbounded operators, in principle, one can consider the operators with a common dense domain such that arbitrary products of operators, in particular, commutators make sense, then an abstract exponential map $\exp : \mathfrak{g} \to G$ satisfies the usual BCH formula.

[19] Here it is convenient to set $L = 1$, the same goes for the modular operator of V_+.

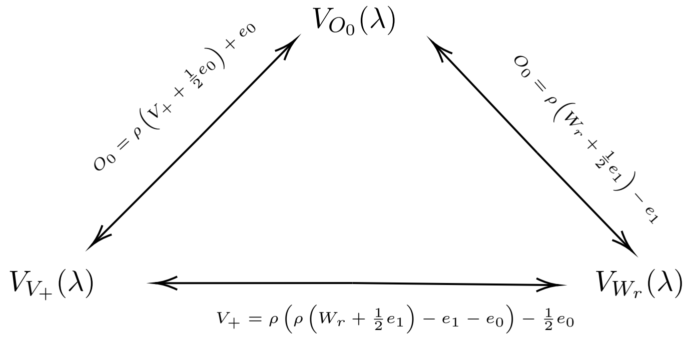


Figure 5.5: Summary diagram depicting modular operator correspondence that is implemented by geometrical transformations associated with O_0, V_+ and W_r.

rederives (5.3.1). Similarly, (5.2.13) and (5.3.1) give

$$\begin{aligned} V_{V_+}(\lambda) &= T\left(-\frac{1}{2}\right) U_\rho T(-1)S(-1)U_\rho S\left(\frac{1}{2}\right) V_{W_r}(\lambda) S\left(-\frac{1}{2}\right) \times \\ &\quad \times U_\rho S(1)T(1)U_\rho T\left(\frac{1}{2}\right) = \\ &= e^{-\frac{i}{2}P_0} e^{-iK_0} e^{-iK_1} e^{\frac{i}{2}P_1} e^{-i\lambda M_{01}} e^{-\frac{i}{2}P_1} e^{iK_1} e^{iK_0} e^{\frac{i}{2}P_0}. \end{aligned} \tag{5.3.6}$$

Whereupon after a bit of algebra one finds (5.3.2). So we started with V_{V_+} to obtain (5.3.4) and again rederived the former as just demonstrated, which justifies the statement given above. To sum up we have the diagram 5.5. Now using (5.2.15) one obtains:

$$\Delta_{O_1}^{-\frac{i\lambda}{2\pi}} \equiv V_{O_1}(\lambda) = e^{ix_L P_1} V_{O_0}(\lambda) e^{-ix_L P_1} = \exp\left[\tfrac{i\lambda}{2L}\left((x_L^2 - L^2)P_0 + K_0 - 2x_L \kappa_1\right)\right]. \tag{5.3.7}$$

5.3.2 Trajectories Induced by Modular Operators

Since the action of spacetime transformations that we are interested in is trivial on the coordinates other than x^0 and x^1, we always work in x^0x^1-plane in $\mathbb{R}^{1+3}$, i.e., $x^2 = x^3 = \text{const}$, which we generally set to 0. As such, we focus only on the x^0 and x^1 components. The metric signature is mostly minus. So, the Lorentzian scalar product is written as $x^2 \equiv x \cdot x = t^2 - |\boldsymbol{x}|^2$.

Let us denote by $\iota(\cdot)$, $s(\cdot)$, $k(\cdot)$, $\varsigma(\cdot)$, $\Lambda(\cdot)$ the action of time translations, space translations (along x_1-direction), temporal part of SCT —special conformal transformation, dilation and boost on $\mathbb{R}^{1+3}$, respectively. They are defined by the following set of mappings: given $x \in \mathbb{R}^{1+3}$, one has (setting, $x^0 = t$ and $x^1 = r$)

$$
\begin{aligned}
&\iota(a) : x \to \iota(a)x = (t + a, r),\\
&s(r') : x \to s(r')x = (t, r + r'),\\
&k(b) : x \to k(b)x = \left(\frac{t - b^0 x^2}{1 - 2b^\mu x_\mu + 2b^2 x^2}, \frac{r - b^1 x^2}{1 - 2b^\mu x_\mu + 2b^2 x^2} \right),\\
&\varsigma(\lambda) : x \to \varsigma(\lambda)x = (e^\lambda t, e^\lambda r),\\
&\Lambda(\theta) : x \to \Lambda(\theta)x = (t\,\mathrm{ch}\theta + r\,\mathrm{sh}\theta, t\,\mathrm{sh}\theta + r\,\mathrm{ch}\theta).
\end{aligned} \tag{5.3.8}
$$

Let $\mathcal{T}$ denote the TT action on $\mathbb{R}^{1+3}$ motivated by the modular operators considered above. First of all let us study the case of a right wedge. In particular, take an arbitrary spacetime point $x = (t, r) \in W_r$ and apply the Lorentz boost $\Lambda(\lambda)$ motivated by (5.3.1) on the point x as follows:

$$
[\mathcal{T}_{W_r} x](\lambda) = \Lambda(\lambda)x = \begin{pmatrix} \mathrm{ch}\lambda & \mathrm{sh}\lambda \\ \mathrm{sh}\lambda & \mathrm{ch}\lambda \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} t\,\mathrm{ch}\lambda + r\,\mathrm{sh}\lambda \\ t\,\mathrm{sh}\lambda + r\,\mathrm{ch}\lambda \end{pmatrix}. \tag{5.3.9}
$$

Similarly, for O_0, using (5.3.4) and an arbitrary point $x \in O_0$ one finds

$$
\begin{aligned}
&[\mathcal{T}_{O_0} x]\,(\lambda) = \iota\,(L)\,k\left(\frac{1}{2L}\right)\varsigma(-\lambda)k\left(-\frac{1}{2L}\right)\iota\,(-L)\,x = \\
&= \left(\frac{L\left(-e^{-2\lambda}\left(x^2-L^2\right)^2+\left(x^2+2tL+L^2\right)^2+4r^2e^{-2\lambda}L^2\right)}{4te^{-\lambda}L\left(x^2\left(e^\lambda-1\right)+\left(e^\lambda+1\right)L^2\right)+e^{-2\lambda}\left(\left(x^2\left(e^\lambda-1\right)+\left(e^\lambda+1\right)L^2\right)^2-4r^2L^2\right)+4t^2L^2}, \right.\\
&\left. \frac{4re^\lambda L^2\left(x^2+2tL+L^2\right)}{\left(x^2\left(e^\lambda-1\right)+\left(e^\lambda+1\right)L^2\right)^2+4L\left(te^\lambda(L^2-x^2)+te^{2\lambda}(x^2+tL+L^2)-Lr^2\right)} \right)
\end{aligned} \tag{5.3.10}
$$

Whereupon using simple trigonometric manipulations one can figure out

$$
[\mathcal{T}_{O_0} x]\,(\lambda) = \left(\frac{L\left(\left(x^2+L^2\right)\mathrm{sh}\lambda+2tL\mathrm{ch}\lambda\right)}{x^2(\mathrm{ch}\lambda-1)+2tL\mathrm{sh}\lambda+L^2(\mathrm{ch}\lambda+1)}, \frac{2rL^2}{x^2(\mathrm{ch}\lambda-1)+2tL\mathrm{sh}\lambda+L^2(\mathrm{ch}\lambda+1)} \right). \tag{5.3.11}
$$

Alternatively using (5.1.74) and (5.1.76), we also have (for $L = 1$)[20]

[20] To obtain the result for generalized L one just has to apply $\varsigma(L)$(dilation) on the final expression. In order to compare it with the later result derived from lightcone coordinates in sect. 5.3.2.2 we keep it here for the special case $L = 1$.

$$
\begin{aligned}
&[\mathcal{T}_{O_0}x]\,|_{L=1}(\lambda) = \iota\,(1)\,\rho\iota\left(\frac{1}{2}\right)\varsigma(\lambda)\iota\left(-\frac{1}{2}\right)\rho\iota\,(-1)\,x = \\
&= \left(\frac{\left(t-\frac{1}{2}\left(e^{\lambda}+1\right)+\frac{x^2}{2}\left(e^{\lambda}-1\right)\right)\left((t-1)^2-r^2\right)}{\left(\frac{1}{2}\left(e^{\lambda}+1\right)-\frac{x^2}{2}\left(e^{\lambda}-1\right)-b_1\right)^2-e^{2\lambda}r^2}+1, \frac{e^{-\lambda}r\left[(t-1)^2-r^2\right]}{\left(\frac{1}{2}\left(1+e^{-\lambda}\right)-\frac{x^2}{2}\left(1-e^{-\lambda}\right)-e^{-\lambda}t\right)^2-r^2}\right).
\end{aligned} \tag{5.3.12}
$$

This basically provides a consistency check. Note that the limits $\lambda \to \{0, \infty, -\infty\}$ in (5.3.10) or (5.3.11) and (5.3.12) yield

$$
\{(t,r),(L,0),(-L,0)\} \quad \text{and} \quad \{(t,r),(1,0),(-1,0)\}\,. \tag{5.3.13}
$$

Physically it means O_0 remains invariant under the geometrical action induced by the associated modular operator. In other words, for any arbitrary parameter $\lambda \in \mathbb{R}$, the translated points under the action of (5.3.4) do not leave O_0. Similarly, using (5.3.7) one derives

$$
\begin{aligned}
[\mathcal{T}_{O_1}(x)]\,(\lambda) &= s(x_L)\mathcal{T}_{O_0}\,(s(-x_L)x) = s(x_L)\mathcal{T}_{O_0}\,(t, r-x_L) = \\
&= \left(\frac{L\left(t^2\mathrm{sh}\lambda+2tL\mathrm{ch}\lambda-\mathrm{sh}\lambda(r-x_L)^2+L^2\mathrm{sh}\lambda\right)}{t^2(\mathrm{ch}\lambda-1)+2tL\mathrm{sh}\lambda-(\mathrm{ch}\lambda-1)(r-x_L)^2+L^2(\mathrm{ch}\lambda+1)},\right. \\
&\quad \left.\frac{2L^2(r-x_L)}{t^2(\mathrm{ch}\lambda-1)+2tL\mathrm{sh}\lambda-(\mathrm{ch}\lambda-1)(r-x_L)^2+L^2(\mathrm{ch}\lambda+1)}+x_L\right).
\end{aligned} \tag{5.3.14}
$$

5.3.2.1 Mapping a Right Wedge to a Double Cone

Here we concentrate on mapping hyperbolic trajectories of W_r to O_0. This will in turn highlight the critical line separating left and right sides of O_0.

The geometrical transformation that we shall be concerned here is given in (5.2.7). Using this with arbitrary space translation parameters u and v, meaning,

$$
O_0 = \rho\,(W_r + ue_1) + ve_1, \tag{5.3.15}
$$

we find

$$
x' = s(v)\rho s(u)x = \left(\frac{r\,\mathrm{sh}\beta}{r^2+u^2+2ru\mathrm{ch}\beta}, \frac{r\,\mathrm{ch}\beta\,(1+2uv)+u+v\left(r^2+u^2\right)}{r^2+u^2+2ru\mathrm{ch}\beta}\right), \tag{5.3.16}
$$

where x stands for any spacetime point within W_r, more precisely[21]

$$x = (r\,\mathrm{sh}\beta, r\,\mathrm{ch}\beta) \tag{5.3.17}$$

with $\beta = \frac{\tau}{\alpha}$. Here our aim is to find r for which the critical vertical line separates the diamond in two equal halves.

In deriving (5.3.16) we also used $\rho(x) = -\frac{x}{x^2}$ and

$$(s(u)x)^2 = (s(u)x)\cdot(s(u)x) = -r^2 - u^2 - 2\alpha u\mathrm{ch}\beta \tag{5.3.18}$$

Note that for W_r parametrization for the TT trajectories taken with respect to the proper time τ is the same as that of modular parameter λ, see below. Then setting $u = \frac{1}{2}$ and $v = -1$ (see (5.2.7)) we obtain

$$x' = \left(\frac{r\,\mathrm{sh}\lambda}{r^2 + r\mathrm{ch}\lambda + \frac{1}{4}}, \frac{1-4r^2}{4r^2+4r\mathrm{ch}\lambda+1}\right). \tag{5.3.19}$$

On the other hand the TT orbit of O_0 (5.3.11) for $\left(0, \frac{1-2r}{1+2r}\right)$ and $L = 1$ yields (to retrieve the expression for general L, one just has to apply $\delta(L)$, i.e., plugging $\left(0, L\frac{1-2r}{1+2r}\right)$ into (5.3.11) does the job)

$$\left[\mathcal{T}_{O_0}\left(0, \frac{1-2r}{1+2r}\right)\right](\lambda) = \left(\frac{4r\,\mathrm{sh}\lambda}{4r^2+4r\,\mathrm{ch}\lambda+1}, \frac{1-4r^2}{4r^2+4r\,\mathrm{ch}\lambda+1}\right). \tag{5.3.20}$$

The last relation is the same as x' given above. Now using the spatial reflection, which exists due to symmetry inside of O_0, one can see that there exists a trajectory

$$\left[\mathcal{T}_{O_0}\left(0, \frac{2r-1}{1+2r}\right)\right](\lambda) = \left(\frac{4r\,\mathrm{sh}\lambda}{4r^2+4r\,\mathrm{ch}\lambda+1}, \frac{4r^2-1}{4r^2+4r\,\mathrm{ch}\lambda+1}\right), \tag{5.3.21}$$

which upon applying reverse transformations gives

$$s\left(-\frac{1}{2}\right)\rho s(1)\left(\mathcal{T}_{O_0}\left(0, \frac{2\alpha-1}{1+2\alpha}\right)\right) = \left(\tfrac{\mathrm{sh}\lambda}{4r}, \tfrac{\mathrm{ch}\lambda}{4r}\right). \tag{5.3.22}$$

It means for any given TT orbit either in the left-half or right-half of O_0 there always exists

[21] Here it is convenient to keep the inverse acceleration r arbitrary, which is, as above, just a distance on x-axis.

the corresponding hyperbola in W_r. The critical line that separates these two sides corresponds to $r = \frac{1}{2}$. This should be compared with the analysis in [35], where the dilation $\delta(2)$, before applying the above transformation, is used for convenience. Then naturally $r = 2$ in that case.

5.3.2.2 Lightcone Coordinate Transformations and Conformal Factors

There is another way of representing the trajectories derived in the previous section, that somewhat makes easier to derive field transformations, as well as the temperature analysis given in the next chapter is convenient to present in this form. The concerned way is using the lightcone coordinates.

We set $z_\pm(\lambda) = t(\lambda) \pm r(\lambda)$ and $z_\pm = t \pm r$, and rewrite the transformations deduced above. We also provide the conformal factors associated with the corresponding field transformations.

In the case of the right wedge W_r, using (5.3.9) with the current settings, one has[22]

$$z_\pm^{W_r}(\lambda) = z_\pm e^{\pm\lambda}. \tag{5.3.23}$$

And for the double O_0, using (5.3.10) or (5.3.11) with $L = 1$ (or (5.3.12)) [33]

$$z_\pm^{O_0}(\lambda) = \frac{1 + z_\pm - e^{-\lambda}(1 - z_\pm)}{1 + z_\pm + e^{-\lambda}(1 - z_\pm)} \equiv \frac{1 + z_\pm \mathrm{cth}\frac{\lambda}{2}}{z_\pm + \mathrm{cth}\frac{\lambda}{2}}. \tag{5.3.24}$$

Similarly, for the region O_1, which is just translated O_0, but this time for general L, using (5.3.14), one finds

$$\begin{aligned} z_\pm^{O_1}(\lambda) &= \frac{-\frac{1}{2}(L \mp x_L + z_\pm)\left(1 \pm L^{-1}x_L\right)\left(1 - e^\lambda\right) + z_\pm}{-\frac{1}{2}\left(-1 + L^{-1}(z_\pm \mp x_L)\right)(1 - e^\lambda) + e^\lambda} \equiv \\ &\equiv \frac{L^2 - x_L^2 + z_\pm\left(L\,\mathrm{cth}\frac{\lambda}{2} \pm x_L\right)}{L\,\mathrm{cth}\frac{\lambda}{2} \mp x_L + z_\pm}. \end{aligned} \tag{5.3.25}$$

The associated conformal factors are now computed using the deduction presented in subsect. 11. In fact, using (5.3.24) one can come up with

$$dz_\pm^{O_0}(\lambda) = \frac{4e^{-\lambda}}{\left(1 + z_\pm \pm e^{-\lambda}(1 - z_\pm)\right)^2} dz_\pm \tag{5.3.26}$$

[22] Here we shall use $z_\pm$ to denote arbitrary spacetime point in all three cases below, since the associated spacetime region is quite clear from the context.

such that the corresponding line element takes the form as

$$ds^2(\lambda) = dz_+(\lambda)dz_-(\lambda) = \frac{16e^{-2\lambda}}{\Big(1+z_+ + e^{-\lambda}(1-z_+)\Big)^2\Big(1+z_- + e^{-\lambda}(1-z_-)\Big)^2} dz_+ dz_- \quad (5.3.27)$$

Whereupon comparing with (5.1.26) one immediately finds the scaling factor $\Lambda(x)$. Since, the scaling dimension is just $\Delta = 1$ for $d = 3$, the conformal factor in this case is given by

$$\gamma_{O_0}(z_+, z_-; \lambda) = 2^2 \Big(1 + z_+ + e^{-\lambda}(1-z_+)\Big)^{-1} \Big(e^{\lambda}(1+z_-) + 1 - z_-\Big)^{-1}. \quad (5.3.28)$$

Similarly, for O_1 one has

$$\begin{aligned}\gamma_{O_1}(z_+, z_-; \lambda) = 2^2 L^2 &\Big(L\Big(1+e^{-\lambda}\Big) + (1-e^{-\lambda})(z_+ - x_L)\Big)^{-1} \\ &\times \Big(L\Big(1+e^{\lambda}\Big) + \Big(e^{\lambda}-1\Big)(z_- + x_L)\Big)^{-1}. \end{aligned} \quad (5.3.29)$$

These are exactly the factors associated with the transformed field (5.1.32) under given transformations. In fact, these are just scale factors coming out of conformal transformations (see (5.1.26) and (5.1.27)). As for the transformations of a test function, one has to find the corresponding Jacobian, which then along with the scale factor found above gives the necessary conformal factor. Note that since the base of a double cone is a unit ball, one requires to use spherical coordinates, which introduces an additional factor (apart from the Jacobian) r^2 in the integral measure, which for the transformations $r \to r(\lambda)$ necessitates to use (5.3.14).

Expressions given in (5.3.28) and (5.3.29) are important in our analysis. These functions are intrinsically associated with the concept of temperature that we shall discuss below.

Note that the above conformal factors are written in terms of the initial spacetime point $z_\pm$. We are interested in studying how the conformal factor, e.g., for O_1, varies across the diamond. This requires a slightly different expression written in terms of an arbitrary spacetime point $z_\pm(\lambda)$. In order to compute the desired conformal factor as a function of arbitrary spacetime point from the diamond O_1, we invert (5.3.25)

$$z_\pm = \frac{2L\Big(\mp x_L + L + z_\pm^{O_1}(\lambda)\Big)}{e^{\lambda}\Big(\pm x_L + L - z_\pm^{O_1}(\lambda)\Big) \mp x_L + L + z_\pm^{O_1}(\lambda)} \pm x_L - L. \quad (5.3.30)$$

Note that

$$\frac{\left(-x_L+L-z_-^{O_1}(\lambda)\right)\left(x_L+L-z_+^{O_1}(\lambda)\right)}{\left(x_L+z_-^{O_1}(\lambda)+L\right)\left(-x_L+L+z_+^{O_1}(\lambda)\right)}=\frac{e^{-2\lambda}\left(-x_L+L-z_-\right)\left(x_L+L-z_+\right)}{\left(x_L+L+z_-\right)\left(-x_L+L+z_+\right)}, \quad (5.3.31)$$

which for the initial point $z_\pm = \pm r$ gives

$$\lambda=\log\left(\sqrt{\frac{\left(x_L+L+z_-^{O_1}(\lambda)\right)\left(-x_L+L+z_+^{O_1}(\lambda)\right)}{\left(-x_L+L-z_-^{O_1}(\lambda)\right)\left(x_L+L-z_+^{O_1}(\lambda)\right)}}\right). \quad (5.3.32)$$

Plugging (5.3.30) and (5.3.32) into (5.3.29) gives the conformal factor as a function of transformed spacetime coordinates (5.3.14)

$$\begin{aligned}\gamma\left(z_+^{O_1}(\lambda), z_-^{O_1}(\lambda)\right) = \quad & \frac{1}{2L^2}\Bigg(L^2-\left(x_L+z_-^{O_1}(\lambda)\right)\left(z_+^{O_1}(\lambda)-x_L\right)+ \\ & +\sqrt{\left(\left(x_L+z_-^{O_1}(\lambda)\right)^2-L^2\right)\left(\left(x_L-z_+^{O_1}(\lambda)\right)^2-L^2\right)}\Bigg). \quad (5.3.33)\end{aligned}$$

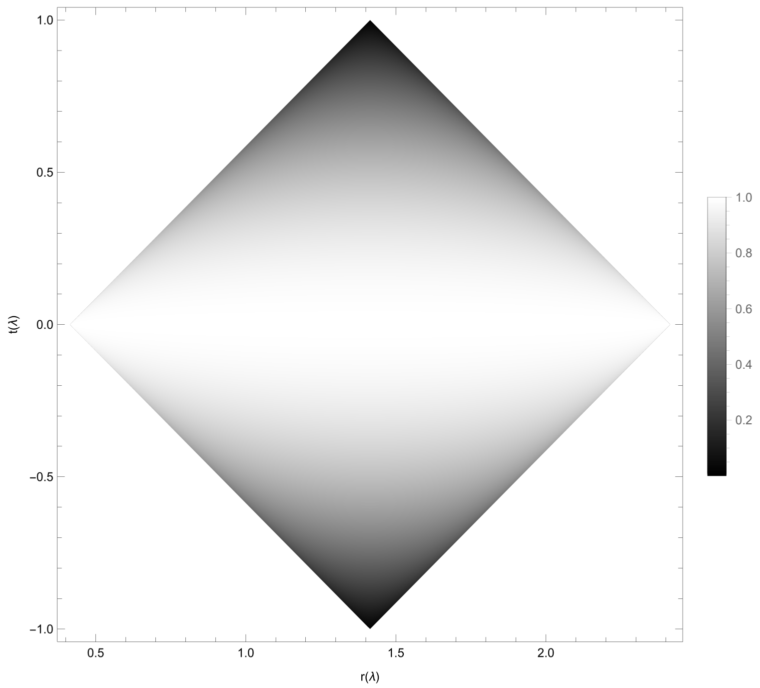


Figure 5.6: Variation of the conformal factor (5.3.34) across the diamond O_1 is shown for $L = \alpha = 1$.

Whereupon substituting $z_{\pm}^{O_1}(\lambda) = t(\lambda) \pm r(\lambda) \equiv x^0(\lambda) \pm x^1(\lambda)$, one arrives at

$$\begin{aligned}\gamma\left(t(\lambda), r(\lambda)\right) = \quad & \frac{1}{2L^2}\Big[-2r(\lambda)\sqrt{L^2+\alpha^2} + 2L^2 + r(\lambda)^2 - t(\lambda)^2 + \alpha^2 + \\ & + \Big(-4r(\lambda)^3\sqrt{L^2+\alpha^2} + 4r(\lambda)\sqrt{L^2+\alpha^2}(t(\lambda)^2-\alpha^2) + 4L^2 r(\lambda)^2 - \\ & -4L^2 t(\lambda)^2 + r(\lambda)^4 - 2r(\lambda)^2\left(t(\lambda)^2 - 3\alpha^2\right) + \left(t(\lambda)^2-\alpha^2\right)^2\Big)^{\frac{1}{2}}\Big] \end{aligned} \tag{5.3.34}$$

Behaviour of this function is shown in the Fig. 5.6. As one can see, the conformal factor is 1 only close to the initial points lying in the vicinity of the center of a diamond and very close to the edges, left and right corners. In other words, an observer moving along the TT orbits of a wedge passing through O_1 cannot distinguish between the TT of W_r and O_1 (or Minkowski spacetime and O_1) in a tiny region, which is precisely the region highlighted by white color in Fig. 5.6. We will have more to say on this in the next chapter. We shall also see that the intrinsic temperature of the diamond O_1 is closely related to the conformal factor (5.3.33).

5.3.2.3 Massless Modular Hamiltonian

Since we now have the conformal factor for the transformations used in obtaining the modular operator for O_0 (5.3.4) or O_1 (5.3.7), it would be convenient to consider the action of the modular Hamiltonian (see below) on the Cauchy data, which was used in [76] to obtain the corresponding action for the massive operator by deformation, though later it turned out to be wrong (see fn. 1 in [77] that requires proof). We shall also comment on one of the outcomes of [77].

Let $\mathrm{Sol}(\mathbb{R}^{1+3})$ be the space of smooth solutions of the wave equation. In particular,

$$\mathrm{Sol}(\mathbb{R}^{1+3}) \doteq \left\{\phi \in C^\infty_{\mathrm{SC}}(\mathbb{R}^{1+3}) \,\middle|\, \Box\phi = 0\right\} = \mathrm{Ker}\Box = C^\infty_0(\mathbb{R}^{1+3})/\Box C^\infty_0(\mathbb{R}^{1+3}), \tag{5.3.35}$$

where $\Box$ is the usual Klein-Gordon operator and SC stands for spatially compact. Meaning, the set $C^\infty_{\mathrm{SC}}(\mathbb{R}^{1+3})$ consists of those smooth functions for which the support is inside of some compact set whose causal complement is a set where these functions vanish. For the details on above equalities we refer to Thm. 3.4.7 in [78].

Any $\phi(x) \in \mathrm{Sol}(R^{1+3})$ takes the form

$$\phi(x) = -\int d^3\boldsymbol{y}\left(\partial_t\Delta_C(x-y)\phi_0(\boldsymbol{y}) + \Delta_C(x-y)\pi_0(\boldsymbol{y})\right). \tag{5.3.36}$$

Using the properties of commutator function $\Delta_C(x-y) = [\phi(x), \phi(y)]$, i.e., $\partial_t \Delta_C(x-y)|_{t=0} = -\delta(\boldsymbol{x}-\boldsymbol{y})$ and $\Delta_C(x-y)|_{t=0} = 0$, one easily confirms that $\phi_0(\boldsymbol{x}) = \phi(x)|_{t=0}$ and $\pi_0(\boldsymbol{x}) = \partial_t \phi(x)|_{t=0}$. We are interested in studying an action of the modular operator V_{O_1}, to be precise, of its generator $H_0 = \frac{1}{i}\left.\frac{dV_{O_1}}{d\lambda}\right|_{\lambda=0}$ or equivalently $H_0 = \log V_{O_1}$, called a modular Hamiltonian.

Now we reproduce the result by Longo-Morsella [76], for the translated O_0, i.e., O_1. In fact, this helps one to recover the similar expression for the wedge W_r as a limiting case and the one for O_0 as a special case, i.e., with $x_L = 0$. The latter though should be done carefully, for which one first requires to retrieve the corresponding expression without α's using (5.2.17).

Let us consider the field transformation law given in (5.1.32). Taking its derivative with respect to the modular parameter λ one easily finds

$$\begin{aligned}(V_{O_1}(\lambda)\phi)(z_+, z_-)' = \; & \gamma\,(z_+, z_-; \lambda)\Big(z_+(\lambda)'\partial_+\phi(z_+(\lambda), z_-(\lambda)) + z_-(\lambda)'\partial_-\phi(z_+(\lambda), z_-(\lambda))\Big) + \\ & + \gamma\,(z_+, z_-; \lambda)'\,\phi(z_+(\lambda), z_-(\lambda)),\end{aligned} \tag{5.3.37}$$

where $\partial_+ = \frac{1}{2}(\partial_t + \partial_r)$, $\partial_- = \frac{1}{2}(\partial_t - \partial_r)$, and $\gamma\,(z_+, z_-; \lambda)$ is the conformal factor (5.3.29). This straightforwardly yields

$$\gamma\,(z_+, z_-; 0) = 1 \quad \text{and} \quad \gamma\,(z_+, z_-; 0)' = -\frac{t}{L}. \tag{5.3.38}$$

As a result, the action we are interested in takes the form

$$\begin{aligned}(H_0\phi)\,(z_+, z_-) \; &= \; \gamma\,(z_+, z_-; 0)'\,\phi(z_+, z_-) + \\ &\quad + \gamma\,(z_+, z_-; 0)\Big(z_+(0)'\partial_+\phi(z_+, z_-) + z_-(0)'\partial_-\phi(z_+, z_-)\Big) = \\ &= \; \frac{1}{L}\left(r x_L - \frac{\alpha^2 + t^2 + r^2}{2}\right)\partial_t\phi + \frac{t}{L}\,(x_L - r)\,\partial_r\phi - \frac{t}{L}\phi.\end{aligned} \tag{5.3.39}$$

where we used the fact that $z_\pm(0) = z_\pm$ with $z_\pm(\lambda) \equiv z_\pm^{O_1}(\lambda)$, see (5.3.25). In particular, one has the following action on Cauchy data:

$$H_0\phi|_{t=0} = \frac{1}{L}\left(r x_L - \frac{\alpha^2 + r^2}{2}\right)\pi_0 \tag{5.3.40}$$

and

$$\partial_t \left(H_0 \phi \right)|_{t=0} = \frac{1}{L} \left(r x_L - \frac{\alpha^2 + r^2}{2} \right) \nabla^2 \phi_0 + \frac{1}{L} \left(x_L - r \right) \partial_r \phi_0 - \frac{1}{L} \phi_0, \tag{5.3.41}$$

In that case, the matrix representation is given by (see also [75])[23]

$$H_0|_{t=0} = \begin{pmatrix} 0 & \frac{r x_L}{L} - \frac{\alpha^2 + r^2}{2L} \\ \left(\frac{r x_L}{L} - \frac{\alpha^2 + r^2}{2L} \right) \nabla^2 + \frac{1}{L} \left(x_L - r \right) \partial_r - \frac{1}{L} & 0 \end{pmatrix} \tag{5.3.42}$$

Note that in the limit $L \to \infty$, one recovers the case of W_r, and for $x_L = 0$, taking into account (5.2.17), one has the concerned operator for O_0.

In [77] it was shown for the massive case, though numerically, that both off diagonal terms in (5.3.42) should depend on mass that is in contrast to the results obtained in [76] (for O_0), where only the term (5.3.41) depends on mass.

[23] Here the off diagonal terms in the modular Hamiltonian is due to the fact that the one-particle modular operator $\delta = \frac{B+1}{B-1}$ [75], where B is unbounded self-adjoint operator, in particular off diagonal matrix, so the modular Hamiltonian for the one-particle Hilbert space, the expansion of $\log \delta$ gives only odd powers of B, which are off diagonals. Therefore, the expression given in (5.3.42) is justified.

Chapter 6

Temperature of a Diamond

In this chapter, after introducing the intrinsic definition of temperature, we reproduce the Unruh temperature for a right wedge. In the sequel, we apply this definition to study the temperature of a diamond, which is either translated or situated at the origin, i.e., $x_L = 0$. The former is more suitable when we study some limiting cases concerning the modular flow of a diamond, and the latter when we consider the ratio of temperatures given with respect to the modular flow of a diamond and that of a wedge.

Before we introduce an intrinsic notion of temperature it is important to recall the classical effect due to Unruh [79], also derived by Davies [80] and Fulling [81], which gives rise to a concept of, roughly speaking, the temperature observed by an uniformly accelerated *eternal* observer inside of a right wedge. Somewhat extensive discussion in this regard can be found in [82]. For comparatively recent progresses in the field we refer to [83].

Let us consider the Rindler spacetime metric[1]

$$ds^2 = dt^2 - dx^2 = e^{\frac{2\xi}{\alpha}} \left(d\tau^2 - d\xi^2\right), \tag{6.0.1}$$

where in the last equality one uses the transformations

$$t = \alpha\, e^{\frac{\xi}{\alpha}} \sinh \frac{\tau}{\alpha} \qquad \text{and} \qquad x = \alpha\, e^{\frac{\xi}{\alpha}} \cosh \frac{\tau}{\alpha} \tag{6.0.2}$$

with $\alpha = \frac{1}{a}$ being an inverse acceleration corresponding to an observer travelling along a hyperbolic trajectory inside of a given wedge. Here (τ, ξ) are the Rindler coordinates that covers

[1] we consider the two dimensional case.

the spacetime region W_r (5.1.21), also called the Rindler wedge, and the edges of this wedge are Rindler horizons (cf. Hawking radiation).

Note that under the rescaling, $ds^2 \to e^{-2\xi/\alpha} ds^2$, the metric (6.0.1) reduces to $d\tau^2 - d\xi^2$. In fact, using the change of coordinates (6.0.2) one can show that

$$\partial_\tau^2 - \partial_\xi^2 = e^{\frac{2\xi}{\alpha}} (\partial_t^2 - \partial_x^2). \tag{6.0.3}$$

Then for a scalar field ϕ, taking into account the Klein-Gordon equation $\Box\,\phi(t,x) = 0$, which is conformally invariant (vid. subsect. 11), one finds

$$e^{\frac{2\xi}{\alpha}} \Box\,\phi(t,x) = \left(\partial_\tau^2 - \partial_\xi^2\right) \phi_R(\tau,\xi) = 0. \tag{6.0.4}$$

In the case of Minkowski spacetime the mod expansion for the field takes the form (here we avoid considering the zero-mod solution, i.e., for $p = 0$, which is absent in four dimensional case anyway)

$$\phi(t,x) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\omega}} \Big(e^{-i(\omega t - px)} a(p) + e^{i(\omega t - px)} a^\dagger(p) \Big), \qquad \omega \equiv |p|, \tag{6.0.5}$$

where $a(p)$ and $a^\dagger(p)$ are the corresponding annihilation and creation operators in the laboratory frame. Similarly, for the Rindler spacetime one has[2]

$$\phi_R(\tau,\xi) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{2\omega_R}} \Big(e^{-i(\omega_R \tau - p\xi)} b(p) + e^{i(\omega_R \tau - p\xi)} b^\dagger(p) \Big), \tag{6.0.6}$$

and $b(p)$ and $b^\dagger(p)$ are the corresponding annihilation and creation operators in an accelerated frame. More specifically, the operators $a(p)$ and $b(p)$ are defined as follows.

Let $|\Omega\rangle$ be the usual Minkowski vacuum and its restriction to W_r so-called the Rindler vacuum $|\Omega_R\rangle$ defined by the action of annihilation operators $a(p)$ and $b(p)$

$$a(p)\,|\Omega\rangle = 0 \qquad \text{and} \qquad b(p)\,|\Omega_R\rangle = 0 \tag{6.0.7}$$

respectively. The above equalities hold for all p, such that $-\infty < p < \infty$.

Clearly the operators $a(p)$ and $b(p)$ are completely different from one and another. Thus

[2] We replace ω by ω_R to distinguish the integral variable, for the Rindler spacetime.

the vacuums $|\Omega\rangle$ and $|\Omega_R\rangle$ are different states. The key here is to use the Bogoliubov-Valatin transformation to establish a connection between these two operators. The procedure is well known, see e.g. [82, 84]. The required expression takes the form

$$b_{\omega_R} = \int_0^\infty d\omega \left(\alpha(\omega, \omega_R) a_\omega + \beta(\omega, \omega_R) a^\dagger_\omega \right), \tag{6.0.8}$$

with the cooefficients

$$\alpha\left(\omega, \omega_R\right) = \left(\frac{\omega}{\omega_R}\right)^{\frac{1}{2}} F(\omega, \omega_R) \quad \text{and} \quad \beta(\omega, \omega_R) = \left(\frac{\omega}{\omega_R}\right)^{\frac{1}{2}} F(-\omega, \omega_R), \tag{6.0.9}$$

where $\omega, \omega_R \in \mathbb{R}_+ \setminus 0$ and

$$F(\pm\omega, \omega_R) = \int_{-\infty}^{\infty} \frac{dz^R_\mp}{2\pi} \exp\bigl(i\omega_R z^R_\pm \mp i\omega z_\pm\bigr) \tag{6.0.10}$$

to be understood in distributional sense, where $z^R_\pm = \tau \pm \xi$ and $z_\pm = t \pm x$. The corresponding relation for $b^\dagger_{\omega_R}$ is easily found by taking Hermitian conjugation of the one given in (6.0.8) and using the identity $F^*(\pm\omega, \omega_R) = F(\mp\omega, -\omega_R)$. Then one finds the vacuum expectation value of a number operator $N_{\omega_R} = b^\dagger_{\omega_R} b_{\omega_R}$ with respect to the Minkowski vacuum that gives

$$\langle N_{\omega_R}\rangle = \langle\Omega| b^\dagger_{\omega_R} b_{\omega_R} |\Omega\rangle = \int_0^\infty d\omega \frac{\omega}{\omega_R} |F(\omega, \omega_R)|^2 = \frac{\mathrm{Vol}}{\exp(2\pi\alpha\omega_R) - 1}. \tag{6.0.11}$$

Then comparing the last equality with Bose-Einstein distribution for the *massless* particle and using $\alpha = \frac{1}{a}$ one immediately finds the well known Unruh temperature:

$$T_{\text{Unruh}} = \frac{a}{2\pi}. \tag{6.0.12}$$

Here we would like to emphasize the fact that the derivation of the Unruh effect explicitly makes use of acceleration for infinite period, since the trajectories under consideration correspond to the Rindler wedge. Having said that we introduce in the next section intrinsic definition of the temperature associated with a specific spacetime region, which contains confined trajectories with acceleration for finite period.

6.1 A Concept of Intrinsic Temperature

In [85], Connes and Rovelli studied that the time, generally considered to be a fundamental physical quantity, depends on a choice of a state. In particular, given a state ω, which naturally determines the flow (see KMS condition, sect. 4.1.1), namely the modular automorphism group σ_λ^ω (def. 2.2.12) associated with a given local algebra $\mathcal{M}$, which in turn provides the theory with a physical time, determined by so-called the modular parameter λ. This parameter is proportional to the flow in spacetime, namely proper time τ, and the proportionality constant is determined by the inverse temperature function. More precisely one has

$$\beta(\lambda) \equiv \frac{1}{T(\lambda)} = \frac{d\tau}{d\lambda}. \tag{6.1.1}$$

Using this Martinetti and Rovelli [34,35] studied the temperature of the diamond O_0 as considered in [33]. In their studies, unfortunately, they relied specifically on a particular trajectory, more precisely, the one shown in fig. 5.4b, which requires explicitly the global observer moving with respect to the dynamics of W_r. In that sense, this approach is very limited as it requires prior knowledge of actual form of trajectories as well as an external observer.

In contrast to their approach, as we promised above, here we introduce the intrinsic definition of temperature in terms of an inverse temperature vector field. In principle, this definition can be used for any spacetime region for which the geometrical TT action is known. Our main goal is to study this concept for the right wedge W_r and the diamond O_1, where for the latter case, setting $x_L = 0$, one easily recovers the corresponding expression for O_0. Before we come to the inverse temperature vector field we discuss general notion of temperature that is related to the KMS condition.

As usual, first of all let us consider the case of a right wedge. Using the TT trajectory passing through an arbitrary point of W_r, i.e., (5.3.9) or (5.3.23), in terms of lightcone coordinates, we have

$$dz_\pm^{W_r}(\lambda) = \pm z_\pm e^{\pm\lambda} d\lambda. \tag{6.1.2}$$

Thus, the proper time interval takes the form

$$d\tau^2 = dz_+^{W_r}(\lambda)dz_-^{W_r}(\lambda) = -(t^2 - r^2)d\lambda^2 = \alpha^2 d\lambda^2, \tag{6.1.3}$$

where we used $t^2 - r^2 = -\alpha^2 = \text{const}$. Clearly, the proper time τ and the modular parameter λ are proportional to each other and the proportionality constant is just an inverse acceleration. Here it would be convenient to rescale $\lambda \to 2\pi\lambda$ so that the modular operator (5.3.1) also rescales accordingly, $\Delta^{-\frac{i\lambda}{2\pi}} \to \Delta^{-i\lambda}$. Then using (6.1.3) one supersedes λ such that the concerned modular automorphism acts as follows.

$$\mathcal{M}(W_r) \ni A \mapsto \sigma^{\omega}_{\frac{2\pi\tau}{\alpha}}(A) = e^{-i\frac{2\pi\tau}{\alpha}\kappa_1} A\, e^{i\frac{2\pi\tau}{\alpha}\kappa_1} \in \mathcal{M}(W_r). \tag{6.1.4}$$

This change in the modular parameter requires to rescale the modular Hamiltonian $\log\Delta^{-i\lambda} = 2\pi\kappa_1$ by α. With regards to the KMS condition (4.1.66), then one immediately finds the vacuum as a thermal equilibrium state at the inverse temperature

$$\beta_{W_r} = \left(T_{W_r}\right)^{-1} = 2\pi\alpha \implies T_{W_r} = \frac{a}{2\pi}. \tag{6.1.5}$$

This is nothing but the Unruh temperature (6.0.12). As stated before, one of our goals here is to demonstrate that the temperature can be derived without referring to the actual form of trajectories whatsoever. Before we do so, we would like to carry out similar computation for the case of O_1.

Using (5.3.25) one can compute

$$dz_{\pm}^{O_1}(\lambda) = \frac{L\left(L^2 - (z_\pm \mp x_L)^2\right)}{2\left(\text{sh}\frac{\lambda}{2}\,(z_\pm \mp x_L) + L\text{ch}\frac{\lambda}{2}\right)^2} d\lambda \equiv \frac{1}{2L}\left(L^2 - \left(z_\pm^{O_1}(\lambda) \mp x_L\right)^2\right) d\lambda \tag{6.1.6}$$

In that case, just as we did for the case of W_r, one finds

$$d\tau^2 = dz_+^{O_1}(\lambda)dz_-^{O_1}(\lambda) = \frac{1}{4L^2}\left(L^2 - \left(z_+^{O_1}(\lambda) - x_L\right)^2\right)\left(L^2 - \left(z_-^{O_1}(\lambda) + x_L\right)^2\right) d\lambda^2. \tag{6.1.7}$$

Therefore,

$$\beta_{O_1} \equiv \beta\left(z_+^{O_1}(\lambda), z_-^{O_1}(\lambda)\right) = \frac{\pi}{L}\sqrt{\left(L^2 - \left(z_+^{O_1}(\lambda) - x_L\right)^2\right)\left(L^2 - \left(z_-^{O_1}(\lambda) + x_L\right)^2\right)}. \quad (6.1.8)$$

As stated before, analogous result for O_0 follows now with $x_L = 0$ in (6.1.8).

6.1.1 The Inverse Temperature Vector Field

Let M be a smooth manifold and let $\gamma\colon [a,b] \subset \mathbb{R} \to M$ be a smooth curve. The tangent space at a point $p \in M$ is denoted by T_pM. Then there exists a vector field $X\colon M \to TM$ such that the *integral curve* γ satisfies $\dot{\gamma}(t) = X(\gamma(t))$. We shall deal with trajectories corresponding to an observer inside of O_1, whose tangent vectors are precisely the vector fields $X(\gamma(t))$.

We want to analyze the trajectories given in (5.3.9) (or 5.3.23) and (5.3.14) (or (5.3.25)). These trajectories are just parametric curves, in fact, similar looking curves with different parametrization. In particular, for the wedge (5.3.9), these are precisely hyperbolas, and for the diamond (5.3.14), they are the trajectories as shown in the Fig. 5.4b, out of which the one passing from the point $(0, \alpha)$ overlaps with a hyperbola of the wedge passing through the same point. This fact was used in [34, 35] to study the temperature of a diamond. We however don't use this form of the trajectories for the following investigation, excluding an analysis for the ratio temperature in sect. 6.1.2.

Taking derivative with respect to the modular parameter λ, one finds the tangent vector for the case of W_r:

$$(\beta_{W_r})_\pm \equiv \frac{dz_\pm^{W_r}(\lambda)}{d\lambda} = \pm z_\pm e^{\pm\lambda}. \quad (6.1.9)$$

Then the temperature in terms of this type of vector we define, independently of the region under consideration, to be of the form

$$T \doteq \frac{1}{2\pi\|\beta\|}, \quad \text{with} \quad \|\beta\| \equiv (|\beta_+\beta_-|)^{\frac{1}{2}}. \quad (6.1.10)$$

Given the geometrical TT action that renders the corresponding trajectories, one finds the tangent vector, whose length is the inverse temperature up to 2π rescaling. As such, one finds

$$T_{W_r} = \frac{a}{2\pi}, \tag{6.1.11}$$

which is the same as we found in (6.1.5). For the diamond O_1, for which the tangent vector follows directly from (6.1.6), one has

$$T_{O_1} \equiv T\Big(z_+^{O_1}(\lambda), z_-^{O_1}(\lambda)\Big) = \frac{L}{\pi\sqrt{\left(L^2 - \left(z_+^{O_1}(\lambda) - x_L\right)^2\right)\left(L^2 - \left(z_-^{O_1}(\lambda) + x_L\right)^2\right)}}. \tag{6.1.12}$$

This temperature can be interpreted as seen by the local observer. As far as, the arbitrary point inside of O_1 is given, the temperature at that point measured by the observer moving with the dynamics provided by the modular operator of O_1 is found. This is valid due to the fact that the derivation of the temperature uses the TT trajectories, thus there is a one to one correspondence between the local temperature and a given spacetime point.

A similar concept using a slightly different approach [36] corroborates our results. The concerned approach is based on the relative entropy between the vacuum and its perturbation. We now briefly discuss it here with a specific example.

Local Temperature: Based on a Relative Entropy Approach

In [36] a concept of local temperature was studied using the second law of thermodynamics. Given the modular theory for a region, say A, of ones interest, the corresponding modular operator, $\Delta_A^{i\lambda}$, provides one with the relative entropy between the Minkowski vacuum Ω and its local perturbation Ω_{pert}. Here the perturbation is understood in the following sense: Let B denote the basis of A, which is basically a Cauchy surface, i.e., $B = A|_{t=0}$. Take any arbitrary point p within B, then there exists a unitary operator U_p localized in a small neighbourhood of a point p, such that $\Omega_{\text{pert}} = U_p\Omega$.

The concerned expression in terms of expectation values of a modular Hamiltonian $K = \log \Delta_A^{i\lambda}$, with respect to Ω and Ω_{pert}, is given by

$$S(\Omega_{\text{pert}}|\Omega) = \langle K\rangle_{\Omega_{\text{pert}}} - \langle K\rangle_{\Omega} = 2\pi P^\mu V_\mu, \tag{6.1.13}$$

where P^μ denotes the four momentum vector and V_μ is the temperature vector we are looking for. In particular, the temperature vector at an arbitrary spacetime point $x = (t, r)$ of a double cone with top and bottom spacetime points a and b, respectively, takes the form [36]

$$V^\mu(x) = \frac{2\pi}{|a-b|^2} \Big((a-x)^\mu(x-b)^\nu(a-b)_\nu + (x-b)^\mu(a-x)^\nu(a-b)_\nu - \\ -(a-b)^\mu(x-b)^\nu(a-x)_\nu\Big), \quad (6.1.14)$$

Let us exemplify the computation for the case of O_1. A straightforward calculation with $a = (L, x_L)$ and $b = (-L, x_L)$ yields

$$\begin{aligned}
&(a-b)^\mu = (2L, 0), \quad |a-b|^2 = (a-b)^\mu(a-b)_\mu = 4L^2, \quad (a-b)^\mu(a-x)_\mu = 2L(L-t), \\
&(a-b)^\mu(x-b)_\mu = 2L(L+t), \quad (x-b)^\mu(a-x)_\mu = L^2 - t^2 + (r-x_L)^2, \\
&(a-x)^\mu(x-b)^\nu(a-b)_\nu = \big(2L(L^2-t^2), -2L(L+t)(r-x_L)\big), \\
&(x-b)^\mu(a-x)^\nu(a-b)_\nu = \big(2L(L^2-t^2), 2L(L-t)(r-x_L)\big), \\
&(a-b)^\mu(x-b)^\nu(a-x)_\nu = \big(2L(L^2-t^2+(r-x_L)^2), 0\big). \qquad (6.1.15)
\end{aligned}$$

As a result, one obtains

$$\begin{aligned}
V^2 = V^\mu V_\mu &= \frac{\pi^2}{L^2}\Big(\left(-L^2 - 2rx_L + x_L^2 + t^2 + r^2\right)^2 - 4t^2\left(r - x_L\right)^2\Big) \equiv \\
&\equiv \frac{\pi^2}{L^2}\Big(L^2 - (t + r - x_L)^2\Big)\Big(L^2 - (t - r + X_L)^2\Big). \qquad (6.1.16)
\end{aligned}$$

Clearly taking the square root it gives inverse of the temperature given in (6.1.12).

6.1.2 The Temperature Ratio

A fact that TT trajectories corresponding to W_r (5.3.9) (or (5.3.23)) and O_1 (5.3.14) (or (5.3.25)) are similar looking parameterized with respect to different dynamics allows one to compare the temperature measured by two different observers travelling along different orbits in respective regions. Here we compare the results for the corresponding temperatures obtained above and find the concerned temperature ratio. This is done using the fact that there is one and only one trajectory inside of W_r that overlaps completely to that of O_1 and intersect top and bottom of O_1, see fig. 5.4b.

Overlapping of the TT trajectories corresponding to W_r and O_1, as shown in the Fig. 5.4b, allows one to find the inverse acceleration α as a function of arbitrary points lying in the region O_1. Here it is convenient to consider first

$$-\alpha^2 = z_+^{W_r}(\lambda) z_-^{W_r}(\lambda) \equiv t(\lambda)^2 - r(\lambda)^2, \tag{6.1.17}$$

which after translating by $-x_L = -\sqrt{L^2+\alpha^2}$ gives

$$\alpha^2 = -\left(z_+^{W_r}(\lambda) - \sqrt{L^2+\alpha^2}\right)\left(z_-^{W_r}(\lambda) + \sqrt{L^2+\alpha^2}\right). \tag{6.1.18}$$

Then one identifies the points of translated W_r to that of O_0, so that $z^{W_r}(\lambda)_\pm \to z^{O_0}(\lambda)_\pm$, whereupon solving for α one arrives at

$$\alpha\left(z_+^{O_0}(\lambda), z_-^{O_0}(\lambda)\right) = \frac{\sqrt{\left(L^2 - \left(z_+^{O_0}(\lambda)\right)^2\right)\left(L^2 - \left(z_-^{O_0}(\lambda)\right)^2\right)}}{z_+^{O_0}(\lambda) - z_-^{O_0}(\lambda)}. \tag{6.1.19}$$

As such, to obtain α with respect to O_1, one just has to translate the radial part of $z_\pm^{O_1}$ by $-x_L$, so that, $z_\pm^{O_1} \mp x_L$. Therefore the corresponding expression for O_1 takes the form

$$\alpha\left(z_+^{O_1}(\lambda), z_-^{O_1}(\lambda)\right) = \frac{\sqrt{\left(L^2 - \left(z_+^{O_1}(\lambda) - x_L\right)^2\right)\left(L^2 - \left(z_-^{O_1}(\lambda) + x_L\right)^2\right)}}{z_+^{O_1}(\lambda) - z_-^{O_1}(\lambda) - 2x_L}. \tag{6.1.20}$$

Now since the temperature for the wedge W_r and the double cone O_1 are given in (6.1.11) and (6.1.12), respectively, the ratio $\frac{T_{W_r}}{T_{O_1}}$ is found to be of the form

$$\frac{T_{W_r}}{T_{O_1}} \equiv \frac{T_{W_r}}{T_{O_1}}\left(z_+^{O_1}(\lambda), z_-^{O_1}(\lambda)\right) = \frac{1}{2L}\left(z_+^{O_1}(\lambda) - z_-^{O_1}(\lambda) - 2x_L\right) = \frac{r\left(z_+^{O_1}(\lambda), z_-^{O_1}(\lambda)\right) - x_L}{L}. \tag{6.1.21}$$

In particular, replacing the radial distance by that written as a function of the modular parameter one has

$$\frac{T_{W_r}}{T_{O_1}}(\lambda) = \frac{r(\lambda) - x_L}{L}, \tag{6.1.22}$$

which dictates the local relative temperature of the diamond O_1 with respect to that measured by the global observer, the one moving along the wedge hyperbolas. Note that $\frac{T_{W_r}}{T_{O_1}}(\lambda) \to 1$ only when the radial distance $(r - x_L)$ is comparable to the size of the diamond O_1, i.e., $(r(\lambda) - x_L) \to L$. In fact, one can easily compute

$$\begin{aligned}
&(-x_L + L - z_-(\lambda))\,(-x_L + L + z_+(\lambda)) - (x_L + L + z_-(\lambda))\,(x_L + L - z_+(\lambda)) = \\
&= \frac{2L^3 \mathrm{csch}^2\frac{\lambda}{2}\,(z_+ - z_- - 2x_L)}{\left(L\mathrm{cth}\frac{\lambda}{2} - x_L + z_+\right)\left(L\mathrm{cth}\frac{\lambda}{2} + x_L + z_-\right)},
\end{aligned} \tag{6.1.23}$$

whereupon one has

$$r(\lambda) - x_L = \frac{L^2 \mathrm{csch}^2\frac{\lambda}{2}\,(r - x_L)}{\left(L\mathrm{cth}\frac{\lambda}{2} - x_L + z_+\right)\left(L\mathrm{cth}\frac{\lambda}{2} + x_L + z_-\right)}. \tag{6.1.24}$$

This is nothing but the spatial part of the orbit (5.3.14). Clearly the numerator of the temperature ratio (6.1.22) corresponds to the translated diamond $O_1 - x_L$.

For the initial point $z_\pm = \pm r$ (6.1.24) reduces to

$$r(\lambda) - x_L = \frac{L^2 \mathrm{csch}^2\frac{\lambda}{2}\,(r - x_L)}{L^2 \mathrm{cth}^2\frac{\lambda}{2} - (r - x_L)^2} \equiv \frac{r - x_L}{1 + \left(1 - \left(\frac{r - x_L}{L}\right)^2\right)\mathrm{sh}^2\frac{\lambda}{2}}. \tag{6.1.25}$$

Now it is quite clear that $(r(\lambda) - x_L) \to L$ holds, only when $\Big(L - (r - x_L)\Big)\mathrm{sh}^2\frac{\lambda}{2} << L$. In that sense a more suitable region is the left or right corner of the diamond O_1, since the limit under consideration gives $r(\lambda) - x_L \to r - x_L$, which is a point on the horizontal line. From the geometry under consideration it is comparable to L, only when it is close to corners of the concerned diamond.

In the above demonstration the distance $r(\lambda) - x_L$ corresponds to all the points in O_0, i.e., $r(\lambda)_{O_0} = r(\lambda)_{O_1} - x_L$. Therefore, one can also talk about similar consequences with respect to O_0.

6.2 Behaviour of the Modular Flow Under Various Limits

Having obtained the modular flow of O_1 in (5.3.7), it is worthwhile to see how does it behave, first as the diamond's size increases, while its position is fixed, and second, as the diamond shifts with an appropriate constraint while its size is being increased. Both of these situations render the following two different scenarios.

6.2.1 Minkowski-like Behaviour

Let us first of recall the modular operator for O_1 given in (5.3.7). Taking $L >> x_L$ and $L >> 1$, keeping x_L fixed, one finds

$$V_{O_1}(\lambda)\Big|_{\{L>>x_L,\, L>>1\}} \longrightarrow \exp\left(-i\frac{i\lambda L}{2}P_0\right). \tag{6.2.1}$$

Carrying out the similar analysis with the orbit (5.3.25), for the initial point $z_\pm = \pm r$, we have

$$z_\pm^{O_1}(\lambda)\Big|_{\{L>>x_L,\, L>>1\}} \longrightarrow \frac{1}{2}L\lambda \pm r, \tag{6.2.2}$$

where we also used $\lambda << 1$ (and $\lambda r << L$). This is the trajectory of an observer moving with respect to the dynamics given by (6.2.1). Here the restriction on λ (along with $\lambda r << L$) is quite important to take into account. Due to which this kind of limiting behaviour is valid only in a very tiny region close to the origin of the diamond O_1, away from the boundaries. As one goes further, the orbit of an observer moving inside of O_1 starts quickly deviating from that provided by (6.2.2).

6.2.2 Wedge-like Behaviour

As above, using (5.3.7), with $x_L^2 - L^2 = \alpha^2$, $L \to \infty$ and $x_L \to \infty$, such that $\frac{x_L}{L} \to 1$, we have

$$V_{O_1}(\lambda)\Big|_{\left\{(L,x_L)\to\infty,\, \frac{x_L}{L}\to 1\right\}} \longrightarrow e^{-i\lambda\kappa_1}, \tag{6.2.3}$$

which obviously coincides with the modular operator for the right wedge W_r (5.3.1). In fact, the orbit (5.3.25) passing through the initial point $z_\pm = \pm r$ in this case takes the form

$$z_\pm^{O_1}(\lambda)\Big|_{\left\{(L,x_L)\to\infty,\, \frac{x_L}{L}\to 1\right\}} \longrightarrow \pm r e^{\pm\lambda}, \tag{6.2.4}$$

which is in fact the same as (5.3.23) with latter being restricted to a initial point. That is what an observer moving with respect to the dynamics provided by the TT action of W_r follows. In the above limiting procedure we have heavily used that the distance on x_1-axis, i.e., r, is much smaller than the size of O_1, which is L. In particular, $\frac{r+L-x_L}{L} \sim \frac{r}{L} << 1$ is quite important here. As a result, one finds the modular flow of O_1 behaving like that of W_r in the left corner of O_1, hence in right corner due to symmetry. That is also for a very short period of physical time, λ.

6.2.3 Comparison with Temperature and Conformal Factor

First of all we note that the temperature of O_1 (6.1.12) is closely related to the conformal factor (5.3.33), where the third term is the same as inverse of T_{O_1} up to a scalar $\frac{2L^3}{\pi}$. Moreover, the limiting cases of the modular flow of O_1 discussed above confirms the density plot shown in fig. 5.6. In particular, the limiting case corresponding to Minkowski spacetime (6.2.2) is found in the center of O_1. That is where the conformal factor naturally equals 1. And the one corresponding to the right wedge W_r (6.2.4) is found in corners, precisely where the conformal factor plot is in agreement. In fact, the temperature ratio (6.1.22) is also 1 only in the left and right corners of the diamond under consideration.

In relation to the temperature of O_1, employing limits discussed in subsections 6.2.1 and 6.2.2 for the expression given in (6.1.12) one trivially finds:

For a wedge[3]

$$T_{O_1}|_{\{(L,x_L)\to\infty,\, \frac{x_L}{L}\to 1\}} \longrightarrow \frac{1}{2\pi\sqrt{-z_+^{W_r}(\lambda) z_-^{W_r}(\lambda)}} = \frac{a}{2\pi}, \tag{6.2.5}$$

where we used (6.1.17).

And for Minkowski spacetime

$$T_{O_1}|_{\{L>>x_L,\, L>>1\}} \longrightarrow 0. \tag{6.2.6}$$

This is precisely what one would expect. But due to our intrinsic definition of temperature this is a trivial consequence of the behaviour of a modular flow discussed above.

A close tie between the temperature and conformal factor highlighted above requires further studies. Analysis carried out in [35] could be used as a guiding principle.

[3] Here one uses the fact that in the limits under consideration arbitrary points, given on TT orbits, of a diamond cannot be distinguished from that of a wedge.

Chapter 7

Conclusion and Outlook

The review part starts with von Neumann algebras and their type classification that could be helpful to the reader to connect the mathematical concepts with the material given in later chapters. Axioms presented in sect. 3.1 play a crucial role in our analysis. A concrete example employing these axioms is provided in sect. 3.2, to reproduce the Reeh–Schlieder theorem 3.2.1 and its converse 3.2.2. Later we have also reviewed the result on the type of local algebras associated with lightlike monotone regions, Thm. 2.3.1. This theorem and Thm. 5.1.1 are the bases of our analysis. In particular, the former one assures the type of underlying local algebras, i.e., type III_1, and the latter, the result due to Hislop and Longo [33], allows one to exploit geometrical transformations between the spacetime regions considered in our studies, namely, a right wedge, a double cone and a forward lightcone. As such, a key ingredient of these transformations, more precisely, the inversion ρ is realized as an unitary operator U_ρ on the Hilbert space without tweaking the fundamental set of axioms. This eventually leads to the modular operator of O_0 (5.3.4) or O_1 (5.3.7).

As a main part of investigation, this thesis proposes an intrinsic definition of temperature in terms of an inverse temperature vector field (6.1.10). Here intrinsic means, as far as the geometrical TT action for a region of ones interest is given, the temperature is computed without referring to a particular trajectory. The proposed vector field is first shown to reproduce the Unruh temperature (6.1.11) for the right wedge W_r. Then we have applied it to compute the temperature of the double cone O_1 (6.1.12) lying inside of a right wedge, or equivalently of a double cone situated at the origin O_0 (with $x_L = 0$) as considered in [33]. Our line of investigation is parallel to an analysis carried out in [36] that uses the relative entropy between the vacuum and its perturbation. In order to prove this we have given an explicit result for the case of O_1, (6.1.16), that coincides with the one derived using our approach (6.1.12).

In the sequel we consider the ratio of temperatures found with respect to the modular flows of a diamond and a wedge. Here is where ones makes use of a particular trajectory passing through a point $(0, \alpha)$ as shown in fig. 5.4b. The concerned trajectory overlaps with the one corresponding to O_1 passing through the same point. The ratio at hand (6.1.22) is found to be 1 only in the left, hence the right, corner of O_1.

In sect. 6.2, we have also provided counterintuitive limits in which the modular flow of a diamond resembles that of Minkowski spacetime and a right wedge. In the former case the stipulated behaviour is found in the center of a diamond, away from the boundaries. In the latter case it is found close to the boundaries, in particular, close to left and right corners. A related discussion on the conformal factor shows a close tie between the computed temperature for O_1 (6.1.12) and the scale factor coming out of the conformal transformations used in our studies. In fact, Fig. 5.6 corroborates the above stipulated behaviour of modular flows, where the conformal factor is expected to take the value 1. Despite in order to explain a reason behind an appearance of the temperature of O_1 in the conformal factor (5.3.33) requires further studies. A starting point in that direction could be an analysis carried out in [35].

It would be worthwhile to apply the definition of intrinsic temperature to study more complicated regions and see the outcomes. In that regard, one can try to investigate the modular flows studied in [86].

Appendix A

Useful Definitions and Facts

Here we outline several central concepts that are used frequently in the main text.

A.1 Operator Topologies on $\mathfrak{B}(\mathcal{H})$

Let $\mathfrak{B}(\mathcal{H})$ denote a set of bounded linear operators acting on a Hilbert space $\mathcal{H}$, and $\mathfrak{B}^*(\mathcal{H})$ the dual of $\mathfrak{B}(\mathcal{H})$. Then we consider topologies defined by the family of seminorms $\{p_i\}_{i\in I}$, which respect the vector space structure of $\mathfrak{B}(\mathcal{H})$. In other words, we are interested in locally convex topologies. For details on this subject matter we refer to [41, 87].

Let $\omega \in \mathfrak{B}^*(\mathcal{H})$ be a continuous linear functional, whose action on $\mathfrak{B}(\mathcal{H})$ is defined by

$$\omega_{\xi,\eta}(A) \doteq (\xi, A\eta)\,,\ A \in \mathfrak{B}(\mathcal{H}). \tag{A.1.1}$$

As such, we refer to a map $p_\omega : \mathfrak{B}(\mathcal{H}) \to \mathbb{R}_+$ defined by

$$p_\omega(A) = |\omega(A)|, \qquad \forall\, A \in \mathfrak{B}(\mathcal{H}), \tag{A.1.2}$$

as a *seminorm*, which we sometime denote as $p_{\xi,\eta}$ or p_ξ, where the latter means entries in the scalar product are the same, i.e., $(\xi, \cdot\xi) = \|\cdot\xi\|^2$.

Now we consider various operator topologies induced by such seminorms, defined in the usual sense, whose base consists of the open balls

$$U_\epsilon(A) = \{B \in \mathfrak{B}(\mathcal{H}) : p_\omega(A - B) < \epsilon\}\,. \tag{A.1.3}$$

Let us first of all consider the strongest/finest topology among the ones given below.

Definition A.1.1. The topology defined by the sup-norm, i.e., for every $A \in \mathfrak{B}(\mathcal{H})$,

$$\|A\| \doteq \sup_{||\xi||_{\mathcal{H}}=1} \{||A\xi||_{\mathcal{H}} : \xi \in \mathcal{H}\}, \tag{A.1.4}$$

is called the *uniform topology.*

This topology is not much of practical interest, rather more coarser ones play a very crucial role in our analysis (see e.g. von Neumann double commutant theorem 2.1.2), which we shall now state.

Definition A.1.2. The topology defined by the family of seminorms,

$$p_{\xi,\eta}(A) = |\omega_{\xi,\eta}(A)| = |(\xi, A\eta)|, \ \xi, \eta \in \mathcal{H} \tag{A.1.5}$$

is called the *weak operator topology* (WOT).

Definition A.1.3. The topology defined by the family of seminorms,

$$p_\xi(A) = |\omega_\xi(A^*A)| = \|A\xi\|^2, \ \forall \xi \in \mathcal{H} \tag{A.1.6}$$

is called the *strong operator topology* (SOT).

From Cauchy–Bunyakovsky–Schwarz inequality follows that $|(\xi, A\eta)| \leq \|A\eta\|\|\xi\|$, meaning SOT is finer than WOT. Note that continuity is preserved as one makes transition from finer topology to coarser topology given on the initial and final spaces, respectively.

We will also need the following slightly finer than SOT and WOT, but coarser than the uniform topology.

Definition A.1.4. Let $\{\xi_n\}_n$ be a sequence such that $\sum_n \|\xi_n\|^2 < \infty$, then one has

$$\sum_n \|A\xi_n\|^2 \leq \|A\|^2 \sum_n \|\xi_n\|^2 < \infty, \ A \in \mathfrak{B}(\mathcal{H}). \tag{A.1.7}$$

As such the topology defined by the seminorms

$$p_{\xi_n}(A) = \left(\sum_n \|A\xi_n\|^2 \right)^{\frac{1}{2}} \tag{A.1.8}$$

is called the σ-SOT.

Definition A.1.5. Let $\{\xi_n\}_n$ and $\{\eta_n\}_n$ be as above, then the topology defined with respect to the seminorms

$$p_{\xi_n,\eta_n}(A) = \left|\sum_n (\eta_n, A\xi_n)\right| \tag{A.1.9}$$

is called the σ-WOT.

If X is a Banach space and X^* its dual, then occasionally we consider the $\sigma(X, X^*)$ topology on X induced by the functionals from X^* and $\sigma(X^*, X)$ on X^* induced by the functionals from separable X, dual of X^*. Natural examples of such topologies are weak topology $\sigma(X, X^*)$ on X and weak* topology $\sigma(X_*, X)$ on the predual[1] $X_* \subset X^*$ that is naturally inherited topology from the dual.

In particular, for $X = \mathfrak{B}(\mathcal{H})$, one has $\mathfrak{B}^*(\mathcal{H}) \supset \mathfrak{B}_*(\mathcal{H}) = L^1(\mathcal{H})$—trace class operators. In that case σ-WOT on $\mathfrak{B}(\mathcal{H})$ is nonother than weak* topology.

We will need the following theorem which is somewhat standard result in topology though.

Theorem A.1.1 (Banach-Alaoglu)**.** *Let X denote a normed space and N a neighbourhood of the origin. Then the unit ball*

$$B = \{A \in X^* : |Ax| \leq 1 \ \forall x \in N\}, \tag{A.1.10}$$

is compact in the weak topology.*

Proof. Thm. 3.15 in [88]. □

A.2 Positive Linear Functionals

Here we consider generalized concepts called weights and traces. Later we discuss states as special instances. See e.g. [41] for further studies.

Definition A.2.1. A *weight* on any given von Neumann algebra $\mathcal{M}$ is the mapping $\omega : \mathcal{M}_+ \to [0, \infty]$ that satisfies the following conditions for every $A, B \in \mathcal{M}_+$ and $\xi \geq 0$:

[1] In general, a predual of a Banach space X is Y whose dual is again X. In particular, for a von Neumann algebra see def. 2.1.9.

1. $\omega(A+B) = \omega(A) + \omega(B)$;

2. $\omega(\xi A) = \xi\omega(A)$,

where $\mathcal{M}_+$ denotes the positive cone of $\mathcal{M}$. A weight is called *finite* if $\omega(\mathbb{1}) < +\infty$, *semi-finite* if the set $\{A \in \mathcal{M}_+ \mid \omega(A) < +\infty\}$ generates $\mathcal{M}$. It is said to be *faithful* if $\omega(A) \neq 0$, for every $0 \neq A \in \mathcal{M}_+$. Furthermore, it is normal if for every bounded increasing net $\{A_n\} \in \mathcal{M}_+$ we have $\omega(\sup A_n) = \sup \omega(A_n)$, i.e., it preserves the supremum. In addition to the properties listed above, if ω satisfies for some unitary operator $U \in \mathcal{M}$ and every $A \in \mathcal{M}$

3. $\omega(UAU^{-1}) = \omega(A)$,

then it is known as a *trace* on $\mathcal{M}$. The definitions of faithfulness, normality, finiteness and semi-finiteness for the trace follow analogously as stated above for the weight.

Note that a finite weight on $\mathcal{M}$ (or on $\mathcal{M}_+$ to be precise) is just a positive linear functional on $\mathcal{M}$. So, from now on former and latter will be treated as synonyms of each other.

Definition A.2.2. A faithful semi-finite normal weight ω is said to be *periodic* if for some $\tau > 0$, $\sigma_\tau^\omega = \mathbb{1}$, and the smallest such τ is known as a *period* of ω. Here σ_τ^ω denotes one parameter modular automorphism group.

Definition A.2.3. A finite weight ω on a unital $\mathcal{M}$ that is normalized, meaning that $\omega(\mathbb{1}) = 1$ is called a *state*. In our analysis, we will need a faithful normal state, which is essentially a finite faithful normal weight.

With ω being the positive linear functional on $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ we define $\omega_\xi \doteq (\xi, A\xi)$, $A \in \mathcal{M}$, $\xi \in \mathcal{H}$. Then we have the following definition for a vector state.

Definition A.2.4. For a given unital von Neumann algebra $\mathcal{M}$, the positive linear functional ω_ξ on $\mathcal{M}$ is called a *vector state*, if it is normalized, i.e. $||\xi|| = 1$ for every unit vector $\xi \in \mathcal{H}$.

Note that normal state is not necessarily a vector state [48].

Definition A.2.5. The Trace-class operators, also known as nuclear operators, acting on $\mathcal{H}$ are defined by the following set

$$L^1(\mathcal{H}) \doteq \{A \in \mathfrak{B}(\mathcal{H}) : \mathrm{Tr}(A) < \infty\}, \tag{A.2.1}$$

which is in fact a predual of $\mathfrak{B}(\mathcal{H})$, see for example Thm. 1.15.3 in [39].

Theorem A.2.1. *Let ω be a state acting on $\mathcal{M} \subset \mathfrak{B}(\mathcal{H})$, then the following conditions are equivalent:*

1. *ω is normal,*
2. *ω is σ-weakly continuous,*
3. *$\exists$ a positive trace-class operator $\rho \in L^1(\mathcal{H})$ with $\mathrm{Tr}(\rho) = 1$, such that*

$$\omega(A) = \mathrm{Tr}(\rho A), \;\; A \in \mathcal{M}. \tag{A.2.2}$$

Proof. For the proof see Thm. 2.4.21 in [40]. □

Theorem A.2.2. *Let ω denote a linear functional on $\mathfrak{B}(\mathcal{H})$, then the following statements are equivalent.*

1. *ω is weakly continuous.*
2. *ω is strongly continuous.*
3. *There exists $\xi_i, \eta_i \in \mathcal{H}$, $i = \overline{1, n}$ such that*

$$\omega(A) = \sum_{i=1}^{n} \left(\xi_i, A\eta_i\right), \tag{A.2.3}$$

for every $A \in \mathfrak{B}(\mathcal{H})$.

Next is the Connes cocycle theorem, for which the proof can be found in Thm. 8.3.3 in [46].

Theorem A.2.3. *Let $\omega_i \in \mathcal{M}^{+}_{*1}$, $i = 1, 2$ be positive faithful normal states, then there exists $\mathbb{R} \ni t \mapsto u_t \in \mathcal{U}(\mathcal{M})$ such that*

1. *$u_{t+s} = u_t \sigma_t^{\omega_1}(u_s)$, $\forall\ t, s \in \mathbb{R}$,*
2. *$\sigma_t^{\omega_1}(\cdot) = u_t^* \sigma_t^{\omega_2}(\cdot) u_t$, $\forall\ t \in \mathbb{R}$,*
3. *There exists a function $\mathcal{F}$ with the properties given in sect. 4.1.1 such that it satisfies (4.1.66) with $\lambda \to t$ and $B = \mathbb{1}$.*

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