SYNCHRONIZATION OF UNDERACTUATED CHAOTIC SYSTEMS WITH APPLICATIONS TO INFORMATION SECURITY

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TESE DE DOUTORADO EM ENGENHARIA ELÉCTRICA

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SINCRONIZAÇÃO DE SISTEMAS CAÓTICOS SUBATUADOS COM APLICAÇÕES PARA SEGURANÇA DA INFORMAÇÃO

JUAN CARLOS GONZÁLEZ GÓMEZ

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TESE DE DOUTORADO SUBMETIDA AO DEPARTAMENTO DE ENGENHARIA ELÉTRICA DA FACULDADE DE TECNOLOGIA DA UNIVERSIDADE DE BRASÍLIA COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE DOUTOR.

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ABSTRACT

Title: Synchronization of Underactuated Chaotic Systems with Applications to Information Security
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Recently, several studies on the synchronization of chaotic systems have been published. Generally, synchronizers in which control is present in all the state equations of the slave system are widely found in the literature. On the other hand, works on secure communication based on minimal synchronization are very rare. Motivated by the improvement possibilities, this doctoral thesis proposes multiple synchronization schemes for chaotic and hyperchaotic systems, where each system has a different proofing methodology, structure, and encryption and decryption process in secure communication. Consequently, each system has a different contribution. Chapter 3 proposes a projective synchronization and antisynchronization scheme based on underactuated control. The scheme is characterized by guaranteeing the boundedness and convergence of the projective synchronization or antisynchronization errors using the Lyapunov theory. Chapter 4 proposes an underactuated synchronization scheme of a hyperchaotic financial system capable of considering bounded disturbances of any nature. The system design uses Lyapunov theory to ensure that the system is robust. Chapter 5 proposes a synchronizer based on a Liu chaotic system. It uses a signal of an underactuated control based on Lyapunov theory to synchronize all states in the presence of perturbations with bounded errors. It has a convenient structure, in conjunction with a non-trivial Lyapunov candidate, to allow synchronization with only a control signal in the second state equation of the slave system. In Chapter 6 of this thesis, the author proposes a simple synchronization scheme for the Chua circuit, considering perturbations, with proportional control in only one of the state equations of the system. In addition, a secure communication scheme with parallel encryption is shown, in which the dimension of the messages is the same as the dimension of the states. Chapter 7 proposes a robust underactuated synchronization of a hyperchaotic Lorenz system. Furthermore, the proposed scheme is applied in secure telecommunications. Finally, Chapter 8 proposes an underactuated synchronization scheme for a perturbed hyperchaotic system using an alternative Lyapunov proof. The proposed approach is minimal in the way that the synchronizer is simplified to the maximum since the control is scalar and acts in only one state. The proposed schemes in all chapters consider the presence of disturbances in all states in the stability analysis.

Keywords: Chaotic and Hyperchaotic Systems, Lyapunov Methods, Systems Synchronyzation, Underactuated Systems.

RESUMO

Título: Sincronização de Sistemas Caóticos Subatuados com Aplicações para Segurança da Informação
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Recentemente, vários estudos sobre a sincronização de sistemas caóticos foram publicados. Geralmente, na literatura, encontram-se sincronizadores onde o controle está presente em todas as equações de estado do sistema escravo. Por outro lado, trabalhos sobre comunicação segura baseada em sincronização mínima são muito raros. Motivado pelas possibilidades de melhoria, esta tese de doutorado propõe múltiplos esquemas de sincronização para sistemas caóticos e hipercaóticos, onde cada sistema possui uma metodologia de prova, estrutura e processo de criptografia e descriptografia distintos na comunicação segura. Consequentemente, cada sistema tem uma contribuição diferente. O capítulo 3 propõe um esquema de sincronização e antissincronização projetiva baseado em controle subatuado. O esquema é caracterizado por garantir a limitação e a convergência dos erros de sincronização ou antissincronização projetiva usando a teoria de Lyapunov. O capítulo 4 propõe um esquema de sincronização subatuado de um sistema financeiro hipercaótico capaz de considerar distúrbios limitados de qualquer natureza. O projeto do sistema usa a teoria de Lyapunov para garantir que o sistema seja robusto. O capítulo 5 propõe um sincronizador baseado em um sistema caótico de Liu. Ele usa um sinal de controle subatuado baseado na teoria de Lyapunov para sincronizar todos os estados na presença de perturbações com erros limitados. O sistema apresenta uma estrutura conveniente, aliada a uma função candidata à Lyapunov não trivial, para permitir a sincronização com apenas um sinal de controle na segunda equação de estado do sistema escravo. No capítulo 6 desta tese, o autor propõe um esquema simples de sincronização para o circuito de Chua, considerando perturbações, com controle proporcional em apenas uma das equações de estado do sistema. Além disso, é mostrado um esquema de comunicação segura com criptografia paralela, em que a dimensão das mensagens é a mesma que a dimensão dos estados. O capítulo 7 propõe uma sincronização subatuada robusta de um sistema hipercaótico de Lorenz. Adicionalmente, o esquema proposto é aplicado em telecomunicações seguras. Finalmente, o capítulo 8 propõe um esquema de sincronização subatuada para um sistema hipercaótico perturbado usando uma prova alternativa de Lyapunov. A abordagem proposta é mínima na forma que o sincronizador é simplificado ao máximo, uma vez que o controle é escalar e atua em apenas um estado. Os esquemas propostos em todos os capítulos consideram a presença de distúrbios em todos os estados na análise de estabilidade.

Palavras-chave: Sistemas Caóticos e Hipercaóticos, Métodos de Lyapunov, Sincronização de Sistemas, Sistemas Subatuados.

SUMMARY

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

V	: Lyapunov function candidate
x	: n-state dimensional vector
$\hat{x}$	: Estimate of the state vector n-dimensional
$\tilde{x}$	: Estimation error of the n-dimensional state vector
u	: m-dimensional admissible input or control vector
$\Re$	: Set of real numbers

Acronyms

P	: Projective Synchronization
MP	: Modified Projective Synchronization
PF	: Projective function synchronization
I	: Identical synchronization
AS	: Antisynchronization
C	: Complete or identical synchronization
IC	: Complete inverse synchronization
AF	: Antiphase synchronization
GA	: Generalized or adaptive synchronization
GI	: Generalized Reverse Synchronization
GP	: Generalized Projective Synchronization
S-δ	: Synchronization δ
PHEC	: Full-state Hybrid Projective Synchronization
CMB	: Combined synchronization
SPA	: Delay synchronization
AFP	: Projective anti-phase synchronization

1 INTRODUCTION

1.1 THESIS MOTIVATION

Recently several studies have been reported in the literature concerning the synchronization of chaotic systems [1]. This interest is motivated, for example, by applications in secure telecommunications [2]. In more detail, consider two chaotic master (transmitter) and slave (receiver) systems, then secure encrypted communication proceeds when the signal to be transmitted is encrypted in the transmitter system and decrypted in the receiver system, which happens when both systems synchronize. Generally, synchronizers where control is present in all state equations of the slave system are widely found in the literature [3–5]. Less frequently, chaotic master-slave synchronization schemes have been found in the literature where control is present in at least two state equations in the slave system [6–8]. Synchronizers based on, for example, adaptive control, sliding, backstepping, etc. have been used in these schemes [3, 5, 7–9]. However, all the works cited have limited application because they assume that the disturbances and the control appear in the same equations of state [1]. In addition, another restrictive assumption that is intended to be mitigated in this work is to consider at least two control signals to synchronize the systems. Motivated by the background, this work proposes a robust scheme for synchronizing a class of underactuated hyperchaotic systems. The proposed method is based on Lyapunov theory, and the initial synchronization error is assumed to be small to simplify the structure of the synchronizer.

1.2 JUSTIFICATION

Chaos theory is widely studied by experts in the field of chaos control. This field can be divided into two research areas: chaos suppression and chaos synchronization. In the last few decades, chaotic synchronization has become a subject of study due to its potential applications in disciplines such as chemical reactions, power converters, aerospace, signal processing, physical lasers, secure communications, satellite systems, and biological systems [10, 11]. Chaotic synchronization consists of making several chaotic systems coincide and converge on the same trajectory after a sufficient time [12].

The general idea of chaotic synchronization used in secure communications is as follows. First, the transmitter encrypts the information using a chaotic system. In turn, the encrypted information is sent over a public communication channel so that it can be received by the receiver. The receiver, in turn, uses synchronization to recover the original message from the encrypted information. A general scheme is shown in Figure 1.1.

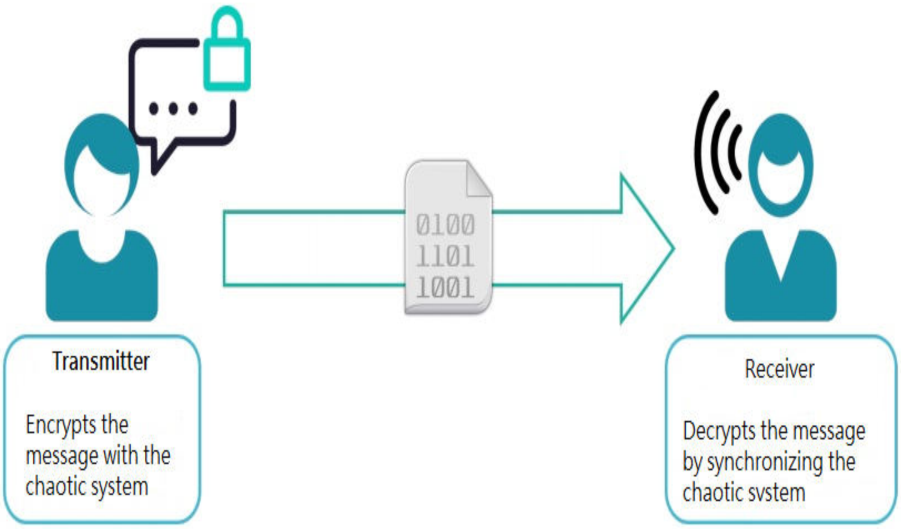

Figure 1.1 – General scheme of secure communication using chaotic synchronization.

1.3 PROBLEM DESCRIPTION

In the scenario where there is a master or controller system and a slave (or response) system, there are several approaches to synchronization, including full synchronization, inverse full synchronization, projective synchronization, generalized projective synchronization, hybrid projective synchronization, antisynchronization, as shown in Figure 1.2, these types will be detailed in section 1.6.3 below, cited in [12–14]. Even different types of synchronization can coexist when synchronizing two systems, as in [13]. The more complex the type of synchronization, the more suitable its application in secure communications [14].

To realize a chaotic synchronization, it is necessary to use some control technique, Figure 1.3 represents some of the control techniques that have been applied, we will cover both integer and fractional order. This thesis addresses the problem of synchronization and secure communication of a generic class of underactuated chaotic systems.

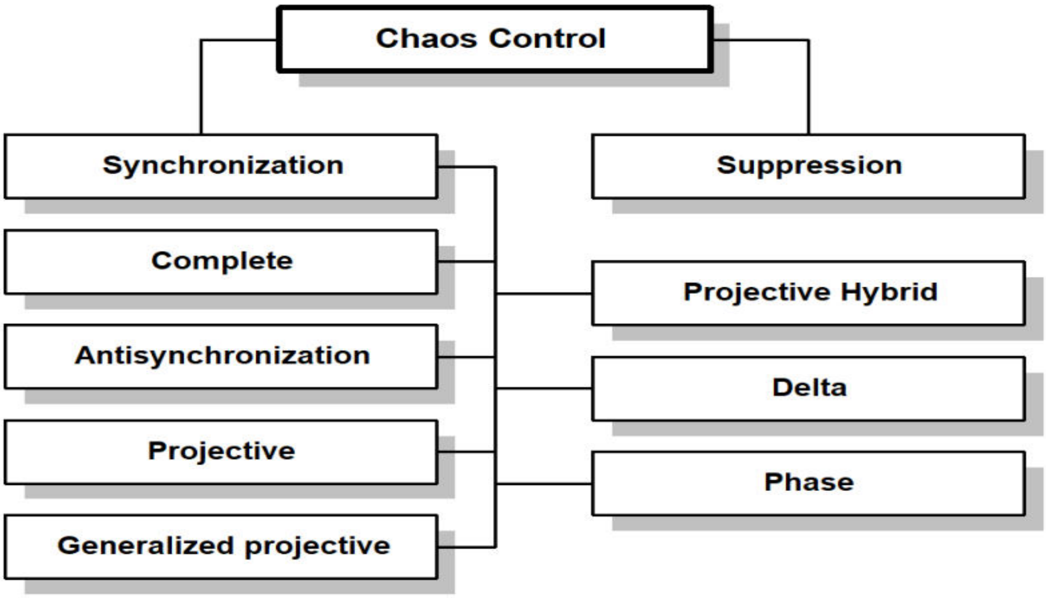

Figure 1.2 – Most commonly used types of synchronization.

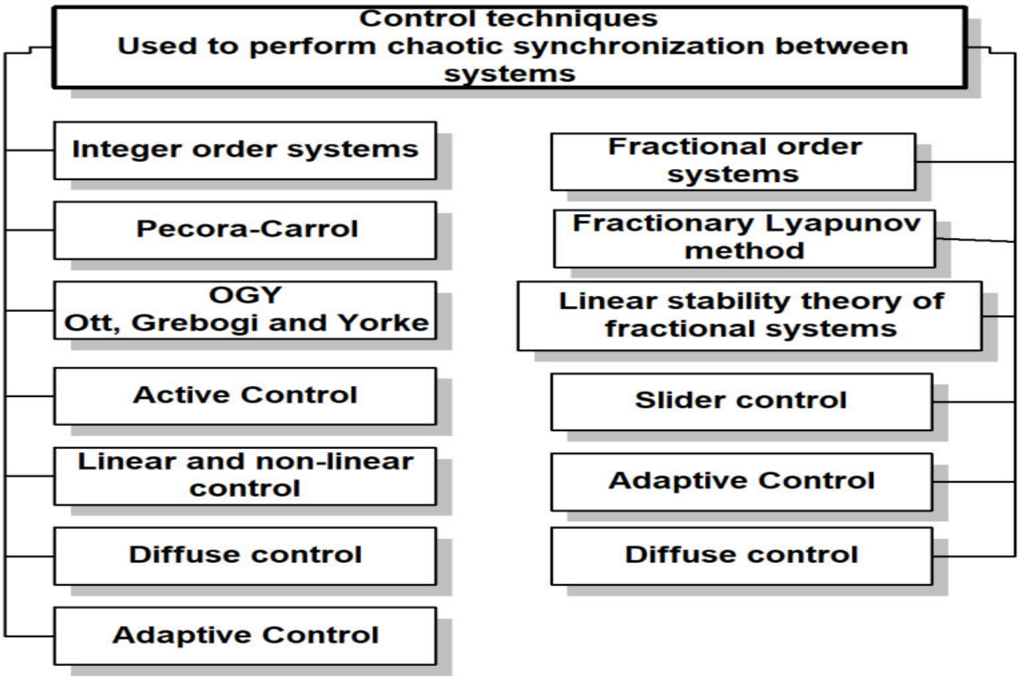

Figure 1.3 – Control techniques used to perform chaotic synchronization between integer or fractional chaotic systems.

1.4 THESIS OBJECTIVES

1.4.1 General objective

- Design of a robust synchronizer using a simple control strategy for a generic class of chaotic systems and considering the presence of limited disturbances in all states.

1.4.2 Specific objectives

- Development of a new proof methodology to be used in synchronization problems for hyperchaotic systems in which disturbances are present in all states and only one-dimensional control is available.
- Application of the synchronization techniques developed for secure communication.
- Perform a stability analysis based on Lyapunov theory to design the synchronizer.
- Evaluate the synchronization performed through simulations in Matlab/Simulink and applications using analog electronics.
- Make a comparison between the proposed algorithms and others in the literature.

1.5 CONTRIBUTIONS OF THE THESIS

1.5.1 Main contribution

- This thesis work presents a synchronization, and secure communication of a generic class of under-actuated chaotic systems, this is one of the main contributions of this doctoral thesis in the area of chaos-based cryptography.
- Proposal of a robust scheme for minimal underactuated synchronization with application to secure communication. Lyapunov stability theory is used to ensure the boundedness of the synchronization error.
- The proposed approach is minimal in that the synchronizer is simplified as much as possible since the control is scalar and proportional. In addition, it considers the presence of disturbances in all states in the stability analysis.
- The slave system does not have to meet $matching\ condition$.
- The proposal of an underactuated projective synchronization and antisynchronization scheme of a chaotic system in which it is theoretically proven that the synchronization error is limited even in the presence of external and internal disturbances.

- Synchronization schemes have been applied for secure communications, where there is encryption and decryption of messages.
- Most of the proposed control schemes are as simple as possible, with a proportional control signal sufficient to synchronize even chaotic and hyperchaotic systems. This feature makes it easy to use these synchronization and encryption schemes in practical applications.
- Validation of the proposed schemes through extensive simulations with Matlab/Simulink software.
- It is shown by simulations that it is possible to validate the state estimation even for chaotic and hyperchaotic systems, and even in the presence of disturbances, demonstrating the robustness of the proposed methods.

1.6 STATE OF THE ART

A review of the literature is presented that covers the main aspects of this research. To begin, we review synchronization techniques applied to underactuated chaotic systems. Even though full synchronization is not the most suitable for secure communication applications, most of the reviewed studies employ it. It is important to work with other types of synchronization, such as projective, generalized projective, antisynchronization, or combined synchronization. Research their advantages and disadvantages compared to full synchronization. Subsequently, the analysis becomes more specific by considering the information in the literature, this work is initially in charge of analyzing some types of synchronization associated with a classical communication scheme with security for chaotic systems of a generic class. Some cases of synchronization of a class of chaotic systems have been studied based on Lyapunov stability theory in the area of chaos-based cryptography, new underactuated synchronization schemes are proposed that are capable of synchronizing chaotic and hyperchaotic systems.

1.6.1 Chaos

Chaos is a concept used to describe the complex behavior of deterministic dynamical systems when it is aperiodic and extremely sensitive to initial conditions, cited by [15]; However, since the studies of Ott, Grebogi, and Yorke (OGY) presented in [16] and by Pecora and Carrol in [17], However, since the studies of Ott, Grebogi y Yorke (OGY) shown in Pecora and Carrol, who first considered methods to control chaos, from then on the scientific community started a quest to find possible applications for chaos. Due to its unpredictable behavior, secure communication emerged as one of its main applications; since then, important advances have been made in chaos theory and it is currently being studied for its applications in control, biomedical engineering, secure communication, optimization, and cryptography, among others [18]. In 1997 they presented a security scheme that combined cryptography and chaos, cited in [19], thus generating a chaotic cryptosystem. Associated with these secure communication schemes are synchronization techniques.

The study of chaotic systems is of great interest because the phenomenon of chaos occurs in important dynamic systems or processes such as in [20–22]. A chaotic system is dynamic and extremely sensitive to its initial conditions, so its behavior is practically unpredictable. This means that the trajectories of the variables (states) of the system, with certain initial conditions, are very different from the trajectories corresponding to a small change in these initial conditions, as in the case of fluid turbulence, in meteorological dynamics, in some biological processes, among others. In [23], a projective synchronization algorithm based on Lyapunov stability theory has been proposed for a Chen system subject to bounded disturbances.

Chaos control consists of designing strategies that allow chaotic systems to be assigned the desired dynamics. There are two basic problems in chaos control: chaos suppression and chaos synchronization. Chaos suppression is the stabilization of the trajectories of a chaotic system around some equilibrium point or in a periodic orbit and is important because the erratic oscillations of a chaotic system are unpredictable and they can cause damage. Chaos suppression is currently being studied in the treatment of heart disease and epilepsy, in laser systems, in mechatronics, and others, shown in [24–27].

Dynamical systems with chaotic behavior are getting more and more attention from researchers as in [28–31]. Thus, the author Strogatz defines chaos as follows.

Definition 2.1: Chaos is noticed when a deterministic system exhibits aperiodic behavior that depends sensitively on the initial conditions, thus making it impossible to predict its future state.

- **Deterministic system**: It implies that the equations of the dynamic system have no random inputs or parameters the erratic behavior of the system stems from its nonlinear dynamics.

- **Aperiodic behavior**: It implies that there are no trajectories in phase space that settle into fixed points or periodic orbits. Furthermore, the trajectories must be bounded; they must not tend to infinity.

- **Sensitivity to initial conditions**: It implies that trajectories that are close together in phase space initially separate exponentially fast in time, the system has a positive Lyapunov exponent.

1.6.2 Synchronization techniques

This section is focused on work that performs synchronization of underactuated chaotic systems where various control techniques have been used to perform synchronization between two chaotic systems. In the current literature, several techniques are presented according to the type of control, method, or laws used, and a brief description of each one. In Figure (1.4), a schematic of the reviewed synchronization techniques is presented.

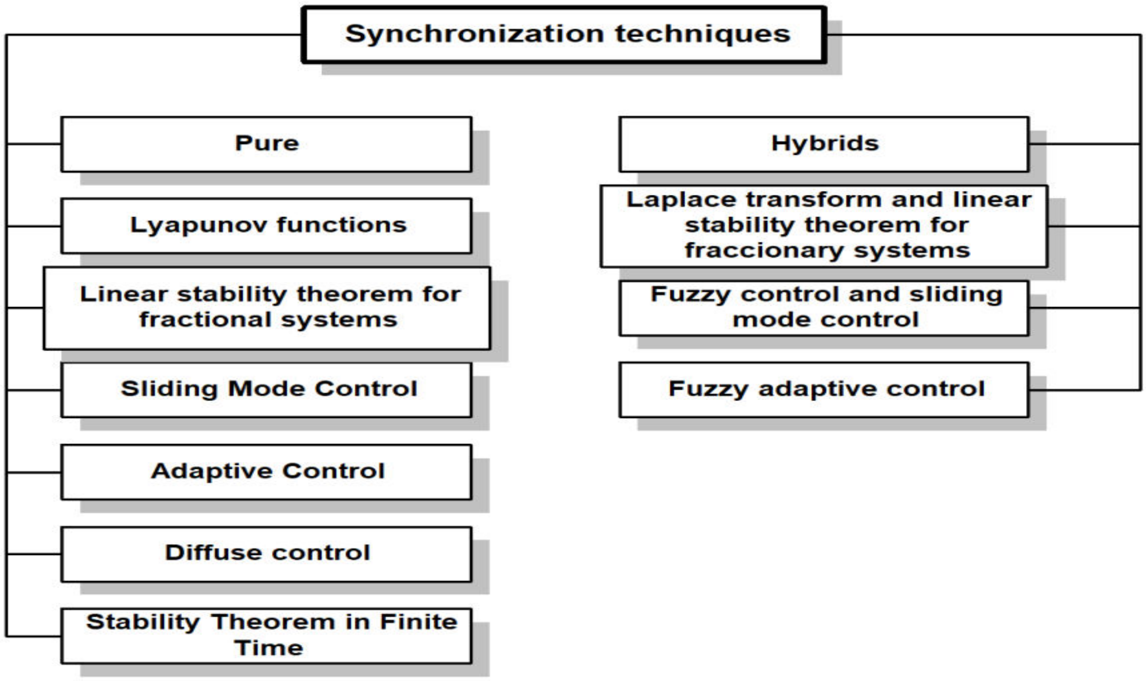

Figure 1.4 – Schematic of synchronization techniques. Pure refers to the use of a single technique. Hybrid refers to the combination of pure techniques.

The synchronization phenomenon can be induced artificially using a control action. Some examples of controlled synchronization can be observed, such as telecommunications systems, whose transmission and reception systems must operate synchronously, autonomous production lines, electrical circuits, and mechanical systems, among others. The key ingredient for the synchronization phenomenon to exist is that there must be a medium, called coupling, through which systems transfer energy to each other until their rhythms are adjusted [32].

Lyapunov functions have been used for control design. These must satisfy the Lyapunov condition to achieve synchronization, cited in [33]. The main challenge in this type of approach is to find the candidates for Lyapunov functions that satisfy the required condition. This control has been employed for full synchronization, inverse complete (IC), antisynchronization (AS), inverse projective hybrid complete (IPHEC), and anti-phase (S-δ) cited in [11, 13, 34–36].

Another widely used approach is based on the linear stability theory of fractional systems. This theory is an extension of the classical theory of linear stability [37]. This control has been used in full synchronization (C), hybrid full projective (PHEC), (IPHEC), generalized (GA), and (S-δ), cited in [10, 13, 34, 35, 38–40].

Recently, papers have been published betting on sliding mode control as in [11, 41], this is a nonlinear control method. It essentially consists of taking the system states over a certain surface in state space and forcing them to evolve over it. Such a surface is called a sliding surface. This type of control has been used in generalized complete (CMB) and robust synchronization, cited by [11,41].

Other studies opt for the combination of Lyapunov functions and linear stability theory of fractional systems, shown in [13,35]. With this hybrid control they achieved full synchronization (C), full inverse (IC), anti-synchronization (AS), and full state hybrid projective (PHEC), cited by [13,35]. Another study proposes a hybrid control whose novelty is to combine a fuzzy control with adaptive sliding mode control. Their objective is to eliminate the difficulties encountered when only sliding mode control is used. In addition, they work with systems that present external disturbances. They have synchronized fully projective (CMB), generalized projective (GP), and antiphase projective (AFP) [42].

An additional proposal is to use fractional linear stability theory together with fuzzy adaptive control [43]. This study provides a generalized fuzzy Takagi Sugeno (T-S) system that can uniformly approximate any continuous function on a compact set with random precision using the Stone-Weierstrass theorem. This hybrid control has been successfully applied to chaotic fractional systems with external disturbances and synchronized saturated inputs in a generalized projective manner (GP) [43].

Adaptive control has also been cited in [44]. It has the ability to adjust to deal with model uncertainties and worked with the assumption of having unknown external disturbances and only performed a full synchronization. Another proposal opts for fuzzy control, which has proven to be an effective control strategy, robust and can work with uncertain parameters, cited by [45–48]. Based on the generalization of the T-S fuzzy model, in [45] provide a stability condition for synchronizing chaotic fractional systems. This control has been used for (C) synchronization, (AS), (G), and (GP) synchronization. Finite-time stability theory for fractional order systems was proposed by [49], studies that opt for this approach use a finite-time stability theorem to stabilize the error of the complete and combined synchronization at a finite time, shown by [50–52].

In addition, we find works that combine some of the previously mentioned controls. In this document we will call them hybrid controls. One such hybrid control combines the Laplace transform with linear stability theory for fractional systems. This proposal consists of applying the Laplace transform to the slave system, performing some operations, and then applying the inverse Laplace transform. Following these transformations, control is derived through linear stability theory of fractional systems. This control has been used to realize (C) synchronization, (G), and (GI) synchronization, according to [34, 53, 54].

Considerable advances have recently been made in the area of chaotic systems. Chaotic systems are deterministic systems with aperiodic nonlinear behavior and are sensitive to initial conditions, as in [55]. A necessary condition for a system to be considered chaotic is that at least one Lyapunov exponent is greater than zero, according to [56]. Chaotic systems have been used in many areas; the following are some of them such as biology [57], chemicals [58], non-linear identification [9, 59], observation [3–5], welding [60] and secure communication [2, 50].

In recent years, several studies have been reported in the literature on the synchronization of chaotic systems, shown in [2]. More specifically, chaotic system synchronization considers two chaotic master and slave systems. The synchronization of chaotic systems was first reported in 1990 by [17]. Many classes of synchronization have since been proposed, such as (AS) [61], delay synchronization (SPA) [62], (P) projective synchronization, cited in [11, 63, 64]. Several articles have been published on the subject of antisynchronization in [65, 66], dedicated to the investigation of the coexistence of (AS) and (C) synchronization of two identical Lu hyperchaotic delay systems through partial variables. Based on Lyapunov stability theory and cited by [67] discussed the (P) synchronization function of the financial function chaotic system with external disturbances. At [68] introduced a new chaotic finance system, which is a model of a complex economic system, and discussed its control using adaptive control theory. At [69] a new secure and chaotic communication system has been announced. Among other issues, in [70] a new 4D hyperchaotic system with two quadratic nonlinearities was introduced and its (A) adaptive synchronization via Lyapunov stability theory was discussed.

Motivated by this information above, in the next section, we review some types of synchronization associated with a classic secure communication scheme, where each of which will be analyzed in later chapters.

1.6.3 Synchronization of Underactuated Chaotic Systems

- **Projective synchronization and antisynchronization.**: Antisynchronization is a kind of synchronization in which its associated errors lie in the addition of the master and slave states, consult [71, 72] for more details. However, most works on this subject consider full-state actuation to have a negative impact on the structure of the synchronizer, which in turn increases the implementation cost and computational burden in analog and digital implementations, respectively.

  Furthermore, different types of synchronizers based on full state control are mainly found in the literature, see for example [5, 9]. Less frequently found in the literature are chaotic master-slave synchronization schemes in which control is present in at least two state equations [6–8]. Synchronization (P) lies in the proportional synchronization of the trajectories of the master and slave systems. The state of the slave system follows the state of the master system by a ratio defined by a scale factor α. This factor defines different types of synchronization, such as: (P) synchronization (α is a non-zero constant), (MP) modified projective synchronization (α is an array); (FP) function projective synchronization (α is a function), (I) identical synchronization ($\alpha = 1$), (ASP) projective antisynchronization (α s negative) and (AF) anti-phase synchronization ($\alpha = -1$). Note that (P) synchronization is valuable because it allows the synchronization of chaotic systems operating at different amplitudes and increases the security of the encoding in a simple form.

In these works, synchronizers based on adaptive control, sliding and backtracking are used , cited in [5–9]. However, they all have the peculiarity of assuming that disturbances and control appear in the same equations of state. Some projective synchronization methods can be found in the literature [73, 74]. However, in [73] was not considered an under-acting control. On the other hand, in [74] was considered an under-actuated control methodology, but no disturbances were considered. Motivated by the above facts, in chapter 3 a projective synchronization and antisynchronization algorithm based on Lyapunov theory for an underactuated chaotic system subject to disturbances is proposed.

- **Hyperchaotic Financial System**: Chaotic systems are deterministic nonlinear systems with aperiodic behavior and sensitive dependence on initial conditions , cited in [55]. Many papers propose new chaotic systems that have been proposed in the literature, all of which are cited in [70, 75, 76]. On the other hand, work related to hyperchaotic systems has increased in recent years. Hyperchaotic systems must have at least two positive Lyapunov exponents, and their dimension must be greater than three, as commented in [77]. Many works propose new hyperchaotic systems that have recently appeared in the literature and are cited in [78–80].

  On the other hand, chaotic and hyperchaotic systems have been applied in many different fields, including economics [79, 81], non-linear identification [59, 82], observation [9, 83], adaptive synchronization [3, 5], welding [84, 85] and secure communication, cited in [50, 75, 86–89].

  Most of the works proposed in the literature also have shortcomings, because in order to perform the synchronization they usually need to control all dimensions of the chaotic system, analyzed in [87–89], that is, they add a control signal in each of the differential equations, another shortcoming is that synchronization algorithms usually do not consider internal and external disturbances in the stability analysis, shown in [73, 90, 91], that may arise due to un-modeled dynamics (due to tolerances and non-optimal component behavior), heating, and electromagnetic noise. Therefore, it is of utmost importance that disturbances are predicted in the analyses done to ensure their robustness in cases of practical application. Therefore, in chapter 4 based on Lyapunov theory, an underactuated synchronization scheme of a hyperchaotic financial system capable of taking into account disturbances of any nature is proposed.

- **Chaotic cryptography with non-trivial candidates**:

  Security is becoming more and more relevant every year. Information systems, databases, distributed systems, and Internet communications are becoming more and more prevalent in the business world. Attacks on information are becoming increasingly sophisticated and commonplace. Thus, many organizations have recognized that it is crucial to have a comprehensive security strategy.

  One option for increasing the level of security lies in using pseudo-random chaos behavior to encrypt information. This form of cryptography has recently spurred a great deal of interest in various industries, cited in [1, 47, 71, 73, 74, 87, 90–93]. However, this is a challenging problem, since the synchronization of chaotic systems, required for secure communication, involves knowledge of different subjects, including nonlinear control and electronic circuit design. In particular, it is crucial to select a suitable candidate Lyapunov function in the synthesis phase to obtain the desired stability and convergence properties for the synchronization algorithm, which may be non-trivial. Synchronization involves forcing the trajectories of two or more systems to behave similarly over time [32, 94] which can be a non-trivial problem.

  Interesting contributions to analog chaos-based cryptography were introduced in [47, 71, 73, 90, 91, 93, 95]. However, a full state synchronization was only considered in [73, 90, 91], with a negative impact on both cost and computational load. At [47, 71], The presence of disturbances in the stability analysis was not considered. At [95], stability analysis in the presence of disturbances was performed, but with a structurally complex synchronizer as in [47, 93].

  Based on the previous facts, in the chapter 5 a chaotic system is proposed by Liu, cited in [96]. However, in contrast to [47, 71, 73, 90, 91, 93, 95], it uses an underactuated control, and the presence of disturbances is considered in the stability analysis to ensure robustness against limited internal and external disturbances.

- **Chaos Synchronization Applied to Parallel Cryptography**:

  Analog cryptography consists of an application of chaotic systems that allow the encryption of information due to its nonlinear and pseudo-random behavior. In secure telecommunications, two chaotic systems are needed to fulfill the communication process, a transmitter (also known as the master) and a receiver (slave). The master system is used to encrypt information, while the slave system is used to decrypt it. Both systems must be synchronized for the communication process to be effective [2].

In [97], a control that uses only one signal was suggested, although they also did not consider the disturbance in the equations. A synchronization method with applicability in Chua's circuit was presented by [98]; However, the complexity of the method limits its practical application. At [99], a communication model with synchronization has been proposed using two channels based on sliding mode control, but also without considering disturbances. At [100], three types of synchronization were proposed where control is applied to only one of the state equations; however, no disturbance was considered in the system equations, this limits the applicability of the system in the presence of internal and external disturbances.

A secure synchronization and communication scheme uses a single control signal and considers that the system disturbance has been studied by [101]. In [102], another successful approach for synchronizing Chua's circuit was presented using only one control signal and considering disturbances. However, these last two approaches used complex synchronizers. At [103], a synchronizer was suggested for Chua's circuit that uses control in each state equation of the system and does not consider the presence of noise. At [104], a synchronizer was developed for Chua's circuit that uses a single control in an extremely simple equation of state. However, without taking into account the disturbance in the system. At [105], a simple synchronizer for the Chua circuit was also proposed, but again without considering disturbances and employing controls in each equation of states; finally, in [106] an underactuated synchronization scheme has been proposed. However, in contrast to our proposal, partial knowledge of the disturbances is considered; therefore, the control signal is adaptive and structurally more complex than a simple proportional control.

Based on the above, all the studies cited above have weaknesses related to both the lack of consideration of disturbances during stability analyses and the complexity of the synchronization schemes. Therefore, in Chapter 6, we propose a simple synchronization scheme for Chua's circuit, considering the disturbances, with a proportional control in only one of the state equations of the system. Furthermore, we show a secure communication scheme with parallel cryptography, in which the dimension of the messages is the same as the dimension of the states.

- **Underactuated 4D-hyper-chaotic system for secure communication in the presence of disturbances**:

  Synchronization of chaotic systems was first reported in 1990, cited by [17]. Muitas classes de sincronização têm sido propostas desde então, tais como a antisincronização [107], delay synchronization [108], projective synchronization [109, 110]. Furthermore, different types of synchronization schemes based on full state control are mainly found in the literature, see for example [5, 111].

It should be noted that there have been few works where the dimension of the control is smaller than the dimension of the dynamic system. However, there are some cases as in [112–114]. Interesting contributions have been proposed in [114–117]. However, disturbances were not considered in the stability and convergence analysis. In addition, many works do not apply the synchronization scheme to ensure communication [114, 116, 118].

1.7 THESIS OVERVIEW

A doctoral thesis is presented that is based on publications. This chapter presents the introduction, motivation, objectives, contributions, and state of the art, in chapter 2 the fundamental basic theoretical concepts that underpin the later chapters of the paper are presented, and Chapters 3-8 include the results of published articles. The doctoral thesis is organized as follows.

In chapter 3, using Lyapunov stability theory, an underactuated projective synchronization and antisynchronization scheme based on proportional control is proposed, and the convergence of the synchronization error to a small neighborhood of the origin even in the presence of disturbances is shown.

After that in chapter 4, a synchronizer for a hyperchaotic financial system is proposed. It is worth noting that unlike most schemes usually found in the literature, the proposed scheme requires only two controls to act on two states in the slave system equations. The proposed scheme has the advantages of being robust against disturbances and being structurally simple, which is interesting because it leads to substantial cost reductions.

In chapter 5, a secure communication scheme is proposed based on the synchronization of a chaotic Liu system with a candidate non-trivial Lyapunov function, which allows the control signal to act only on one state of the slave system. The proposal has the advantages of being robust against disturbances (internal and external) and simple because it leads to significant cost reductions when implemented using analog electronics. A simulation study, which considers the presence of disturbances, is used to validate the theoretical results and show the easy implementation of the proposed approach.

Later in chapter 6, a simple synchronization scheme for Chua's circuit is proposed, considering the disturbances, with proportional control in only one of the system's state equations. Furthermore, we show a secure communication scheme with parallel encryption, in which the dimension of messages is the same as the dimension of states.

In chapter 7, an underactuated synchronization of a hyperchaotic system originating from the Lorenz system is proposed. The main contributions are 1) The underactuated synchronization of a hyperchaotic Lorenz system where disturbances are explicitly considered in the stability analysis; 2) The application of the proposed scheme in secure communications in which the goal is basically to encrypt information using chaotic systems. Thus, it is necessary to use a transmitter system to encrypt sensitive information and a receiver system to reconstruct the encrypted message.

Next in chapter 8, the main proposal of this thesis is described, in which a synchronizer is proposed to apply it to the secure communication of chaotic systems of a generalized class. The main peculiarities of the proposed scheme lie in its robustness against internal or external disturbances and simplicity, which makes it very suitable for applications in secure communication systems.

Finally, Chapter 9 summarizes the theoretical contributions of this thesis work, the results obtained, and suggestions for future research that are also discussed. Appendix A describes the theoretical concepts of general mathematical representations and Lyapunov stability theory that are the basis of the chapters in the study and Appendix C contains the Matlab/Simulink language codes used to generate the figures.

1.8 PUBLICATIONS

During the doctoral period, several articles were published in international congresses, which are shown below.

Articles published in "2021 - 14th IEEE International Conference on Industry Applications (INDUSCON)".

- Projective Synchronization and Antisynchronization of Underactuated Systems.
- Underactuated Synchronization Scheme of a Hyperchaotic Financial System.
- Chaos-based Cryptography Using an Underactuated Synchronizer.

Articles published in "2021 IEEE 5th Colombian Conference on Automatic Control (CCAC)".

- Synchronization of chaos and its application in Parallel Cryptography
- Underactuated 4D-Hyperchaotic System for Secure Communication in the Presence of Disturbances

Paper submitted to a Jornal and published in *Communications in Nonlinear Science and Numerical Simulation (CNSNS)*.

- Minimal Underactuated Synchronization with Applications to Secure Communication.

Paper submitted to a Jornal and accepted for publication in *International Journal of Control, Automation, and Systems (IJCAS)*.

- A Robust Underactuated Synchronizer for a Five-Dimensional Hyperchaotic System: Applications for Secure Communication.

2 TECHNICAL BACKGROUND

In this chapter, I will cover the theoretical background of the general considerations of chaotic systems, synchronization, and secure chaos-based communication that will be used throughout this thesis; my objective is to provide background information on the contents used in this doctoral thesis.

2.1 SYNCHRONIZATION

System synchronization involves taking actions to get two or more systems to enter and remain in a common behavior or at the same rhythm, which has been very interesting since its first discoveries, both for its analysis and for the development of technologies derived from an adequate manipulation of this phenomenon. This synchronization phenomenon can be artificial using a control action. Some examples of controlled synchronization can be observed: telecommunications systems, whose transmission and reception systems must operate synchronously, autonomous production lines, electrical circuits, and mechanical systems, among others. The key ingredient for the synchronization phenomenon to exist is that there must be a medium called coupling, through which systems transfer information to each other until their rhythms coincide [32]. Several areas of knowledge explore the use of controllers, and a case in point is the synchronization of master and slave systems. Cases such as chaotic synchronization, proposed in 1990, emerged to increase the reliability of the communication area with the security [2].

However, in these same fields discussed above, there are particular so-called chaotic systems in which it is not evident to achieve and maintain a state of synchrony. This is because chaotic systems are deterministic dynamical systems in which the evolution of their variables, with certain initial conditions, is very different from the evolution of the variables of the same system with a small change in their initial conditions. This occurs in phenomena such as fluid turbulence, weather systems, mechanical and electrical systems, biological processes, and others. Due to its high sensitivity to initial conditions, it is clear that two isolated chaotic systems, even if they are identical, would not be in synchrony. However, a study in [17] revealed that two identical chaotic systems with a common coupling signal can evolve synchronously. On the other hand, chaos synchronization is the induction of a regime in which two chaotic systems (one called master and one called slave) exhibit identical trajectories ($x_m = x_s$) after introducing some kind of coupling between them, as shown in Figure 2.1.

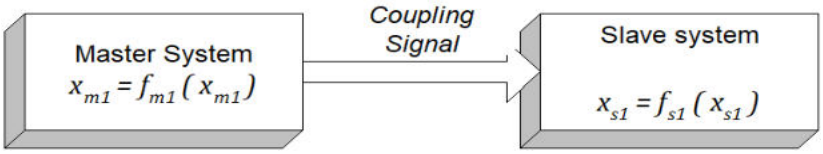

Figure 2.1 – Schematic for synchronizing two systems via a coupling signal [44].

The schemes and techniques in this study have the disadvantage that chaotic systems, even if they are identical replicas, in practice there are imprecisions and uncertainties in the parameters and components of each system, synchrony is not robustly guaranteed, cited by [119]. Currently, research and development of synchronization techniques point in the direction of robust synchronization schemes for identical and different systems, with imprecisions, unknown dynamics, different order, with limited information or partial measurement of their variables, among others. In this regard, it should be that synchronization can be achieved in different degrees or types, these are classified as identical synchronization, phase synchronization, synchronization for backward and forward, generalized, complete, partial, and reduced order, cited in [20, 21, 32, 120–124].

2.1.1 Chaos Synchronization

Chaos synchronization consists of a regime in which two coupled chaotic systems (one called master and the other called slave indicated in Figure 2.1, after called master and the other called slave), after a transition time, exhibit identical chaotic oscillations. This behavior appears in many natural processes, such as the relationship between neurons and the synchronization of the heart and lungs, to name some. Synchronization can be solved, from a control point of view, by projecting the slave system using some technique. There is great interest in the synchronization of chaotic systems since it is desirable to realize important applications for the secure transmission of communications, in services such as military communication links and private companies, financial transactions, and commercial operations with electronic signatures over the Internet, among others. At [125] shows a case of electronic commerce, where it is essential to maintain computer security for Internet buying and selling operations and remote banking, protecting the identity and information of customers and institutions. The motivation for using chaotic systems in information cryptography is due to the unpredictability characteristic of this type of system, which provides a high level of security, cited by [126].

The characteristics of chaotic synchronization are examined in terms of Lyapunov exponents and the effects of the direct Lyapunov method. This method is then used to develop a general approach in chaotic synchronization system projects. Pecora and Carroll [17] state that chaotic systems are systems that, by themselves, appear to defy synchronization. Two identical autonomous systems with the same initial conditions in phase space have trajectories that quickly become uncorrelated even if the two systems map to the same attractor in phase space. When dealing with chaotic synchronization several synchronization models are used, among others [32]. Most chaos-based communications use full synchronization, delay synchronization when the signal offset interval is considered, or generalized synchronization when the receiving system is not identical to the transmitter.

2.1.2 Master-slave synchronization

As already mentioned, Pecora and Carroll's synchronization scheme is known as the "master-slave" system. A master-slave system consists of two chaotic systems described by the same set of differential equations, with the same parameter values. It was shown in [17] that for synchronization to occur, the output of at least one of the differential equations of the first chaotic system must be made available to the second system. Thus, one chaotic system is said to control the other chaotic system by the time series signal generated from one of its differential equations, the master-slave system can also be seen as a transmitter-receiver communication system. One of the necessary conditions for master-slave synchronization to occur is that the slave system must copy the chaotic master system; then the slave system behaves as chaotic.

2.1.3 Types of Synchronization.

Different types of synchronization found in the reviewed literature are collected, which are classified as to the definition of the error. For this end, be the master system x_m = [x_{1m}, x_{2m}, ... x_{nm}]; the slave system x_s = [x_{1s}, x_{2s}, ... x_{ns}] are state vectors and u = [u_1, u_2, ... u_m] is control.

- **Projective Synchronization (P)**: The projective synchronization errors of the system are defined with

$$e = x_{3s} - \alpha x_{3m} \tag{2.1}$$

  where α is a scalar factor, then x_{3s} and x_{3m} are projectively synchronized [13].

- **Antisynchronization (AS)**: If the error is defined as follows

$$e_2 = x_{2m} + x_{2s} \tag{2.2}$$

  then x_{2m} and x_{2s} are antisynchronized [13].

- **Complete or identical synchronization (C)**: If the system error is defined as

$$e_1 = x_{1s} - x_{1m} \tag{2.3}$$

then x_{1s} and x_{1m} are fully synchronized [13].

Complete synchronization: This implies the exact congruence between the vector states of systems that are interacting with each other, either unidirectionally or reciprocally: $v(t) \equiv u(t)$ where these are the state vectors of two different systems. This occurs only in systems with identical elements, each component having the same dynamics and parameters [127–130].

- **Full inverse synchronization (CI)**: If the system error is defined as

$$e_1 = x_{1m} - x_{1s} \tag{2.4}$$

then x_{1m} and x_{1s} are completely inversely synchronized [13].

- **Anti-phase synchronization (AF)**: If the error is defined as follows

$$e_2 = x_{2m} - (-x_{2s}) \tag{2.5}$$

then, x_{2m} and x_{2s} are synchronized in anti-phase [34]. Its concept refers to systems in which phase $\theta(t)$ Its concept refers to systems in which phase 4 fluctuates chaotically and signal amplitude evolves freely and unrelated [131, 132]. Phase synchronization occurs when the instantaneous difference of phases $\theta_0(t)$ and $\theta_1(t)$ of the chaotic signals are time-limited:

$$\mid \theta_1(t) - \theta_0(t) \mid < C \tag{2.6}$$

where C = cte.

- **Generalized or Adaptive Synchronization (GA)**: If the error is defined as

$$e_i = x_{is} - [\beta_i(x_{im})] \tag{2.7}$$

for i = 1, ..., n

Assuming that $n = m$ and $\beta_i : \Re^n \mapsto \Re$ are differentiable functions, then x_{is} and x_{im} are synchronized in a generalized or adaptive form [34, 41]. Adaptive synchronization [127], which is a generalized mode of synchronization.

Generalized synchronization: is characterized by the existence of a functional relationship between the states of the two systems, the receiving system represents a function of the transmitting system, $v(t) = F(u(t))$ [17, 129].

- **Generalized Inverse Synchronization (GI)**: If the error is defined as

$$e_i = \psi_i(x_{is}) - x_{im} \; for \; i = 1, ..., n \tag{2.8}$$

Assuming that n = m and $\psi_i : \Re^n \mapsto \Re$ are differentiable functions, then x_{is} and x_{im} are synchronized in a generalized form [34].

- **Generalized Projective Synchronization (GP)**: If the error is defined as

$$e_i = x_{is} - \lambda_i x_{im} \tag{2.9}$$

Assuming that λ_i is a scalar factor, then x_{is} and x_{im} are projectively synchronized and generalized [42].

- **Synchronization δ (S-δ)**: If the error is defined as

$$e_i = x_s - x_m \tag{2.10}$$

Considers that $\lim_{t \mapsto \infty} ||\text{e(t)}|| \geq \delta$, for some δ > 0, then the systems are said to be synchronized in δ. [10].

- **Full-state Hybrid Projective Synchronization (PHEC)**: If the error is defined as

$$e_3 = x_{4s} - (x_{1m} + 2x_{2m} + 3x_{3m}) \tag{2.11}$$

where n = 3 and m = 4, then x_{4s} synchronized in a hybrid projective form of complete states with x_m [13].

- **Combined Synchronization (CMB)**:The following systems are considered for this synchronization: Dx as the first master system, Dy as the second master system, D $(z + u)$ as the slave system, where x = [x_{1m}, x_{2m}, ... x_{nm}], y = [y_{1m}, y_{2m}, ... y_{nm}] and z = [z_{1s}, z_{2s}, ... z_{ns}] are the state vectors and u = [u_1, u_2, ... u_n] are the controls. If there are three constant matrices Q, W, $E \in \Re^n$ and $E \neq 0$ that satisfy that

$$\lim_{t \mapsto \infty} || \, Q_x + W_y - E_z \, || = 0, \tag{2.12}$$

Then you have the combined synchronization of x_m, y_m and z_s [133].

- **Delay Synchronization (SA)**: Occurs when the interacting systems have practically identical oscillations only shifted by a time interval T, $V(t) \approx u(t + T)$. This synchronization is used when the time interval T comes from the travel time between the transmitter and the receiver.

2.2 SECURE COMMUNICATION BASED ON CHAOS

Today we know that the systems that prevail in nature are the so-called "chaotic systems". Although the word itself suggests disorder, from a scientific point of view, chaos refers to complex dynamic behavior that can be modeled by nonlinear or difference differential equations. The characteristics it has are very particular, such as being sensitive to initial conditions, generating "strange" attractors, and having at least one positive Lyapunov exponent, among others. Secure communication is based on the synchronization of a chaotic circuit that encrypts the transmitted information (master), with a chaotic oscillator to decrypt the encrypted information (slave).

The Pecora-Carroll synchronization scheme has often been described as a "master-slave" system [134, 135]. Essentially, a master-slave system consists of two chaotic systems. The two systems are described by the same set of differential equations, with the same parameter values. It has been shown in [17] that for synchronization to occur, the output of at least one of the coupled differential equations of the first chaotic system must be made available to the second chaotic system, as shown in Figure 2.2 cited in [2]. Thus, one chaotic system is said to drive the other chaotic system by the time series signal generated from one of its differential equations. The first thing that Poincaré developed more than a century ago was the chaotic behavior of continuous dynamical systems. Having mentioned the above, for there to be synchrony between chaotic systems, there needs to be a connection or coupling between them.

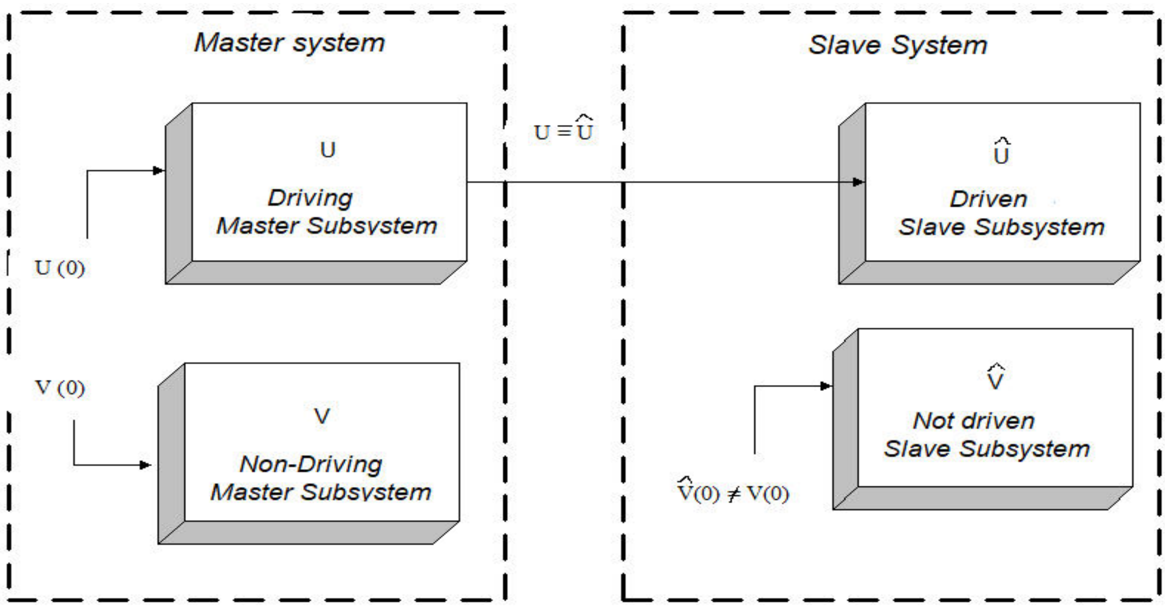

Figure 2.2 – Pecora-Carroll master-slave system divided into subsystems [44].

In 1993 they developed the additive chaotic masking method [92]. In this method, information is added to a state of the chaotic system and sent to another chaotic system, this signal is responsible for the synchronization. Due to the unpredictability of the chaotic behavior, the information is considered encrypted and secure. Also in 1993, the chaotic shift-key method was proposed by Dedieu and collaborators in [136]. In this method, the message consisting of a sequence of bits acted by switching between two Chua circuits in the transmitter; in the receiver, it also consisted of two Chua circuits, the message was recovered by checking which circuit was synchronized; between late 1993 and 1996 two more masking methods emerged. Yang and Chua in [137] introduced the concept of chaotic parameter modulation, where the message to be transmitted was used to modify the system parameters. Consequently, since the development of additive chaotic masking, much of the literature has ignored the presence of disturbances in the systems being synchronized, as in [92, 138–140]. Furthermore, they assume that the systems are entirely structurally known. The next section presents one of the first chaotic communication techniques proposed by [92, 128, 141]. It is based on the synchronization principles of Pecora and Carroll [17].

2.2.1 Chaotic Additive Masking

It involves the transmission of analog signals [128]. This principle involves adding a message signal to a chaotic carrier, signal x, before the transmission of the sum of the two signals occurs. In this way, the slave system at the receiver generates a signal x that is expected to be synchronized with the corresponding master signal x from the transmitter, assuming that noise is close to zero and that the transmitted message can be retrieved, shown in Figure 2.3 which consists of two identical chaotic systems. Chaotic mask x_{mi} represents one of the states of the chaotic transmitter system. The message $m(t)$, which has an amplitude smaller by 20dB to 30dB than x_{mi}, is added to the chaotic mask, giving rise to the transmitted signal $s(t)$. Since the chaotic signal x_{mi} is very complex and $m(t)$ is much smaller than the signal, it is expected that the message cannot be separated from $s(t)$ without someone having the exact knowledge of x_{mi} [142].

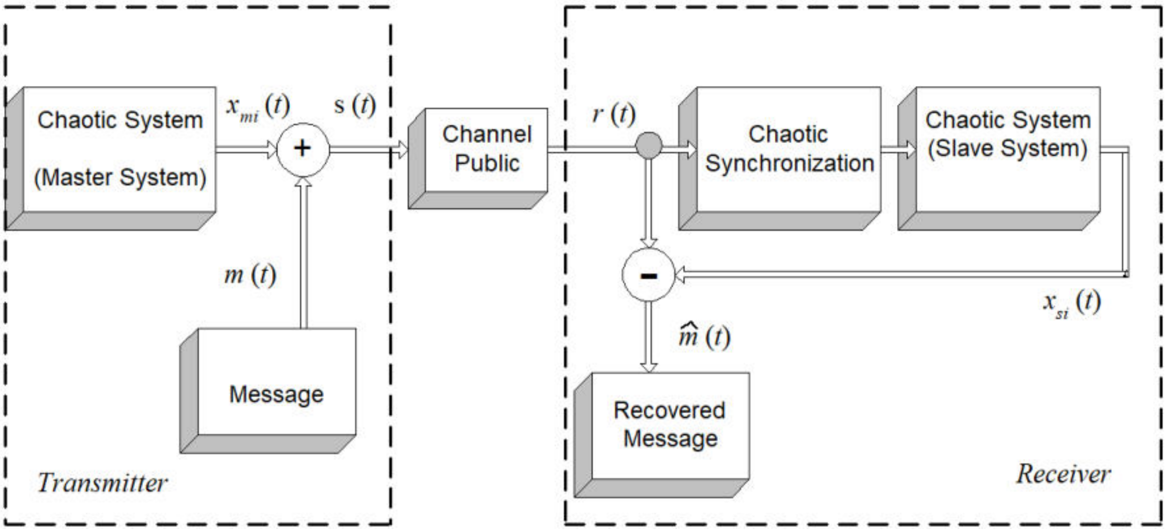

Figure 2.3 – Chaotic additive masking scheme [2].

The next section presents the chaotic modulation methods researched in [143].

2.2.2 Chaotic modulation

Unlike the chaotic masking scheme, whose information is summed directly to some state of the transmitter without the transmitted message influencing its dynamics, chaotic modulation incorporates the message into the dynamic equations by making the transmitter chaotic, was investigated in [143], while chaotic masking is mainly used for analog transmission systems. Since chaotic systems are extremely sensitive to initial conditions and parameters, they have come to be used in secure communication schemes. Different modes of transmitting information signals using chaotic dynamics have already been proposed. The implementation of the chaotic system synchronization scheme in communications consists of a transmitter system (master) generating a chaotic carrier signal, which is modulated by the information signal so that the information is encrypted due to the chaotic characteristic of the carrier. The encryption methods commonly found in the literature that are used for this purpose are shown below.

- **Modulation of chaotic parameters**: The chaotic parameter modulation shown in Figure 2.4 is used to transmit information; this method uses the message signal to change the parameters of the chaotic master system to constantly change the dynamics of the system. At the receiver, a proportional adaptation law is used to estimate the slave system parameters so that the synchronization error approaches a maximum of zero. In this form, the original message is recovered based on the estimated parameters, as investigated in [99].

- **Non-autonomous chaotic modulation**: This method uses the message signal to directly change the trajectories that the system follows on the attractor of the master chaotic system. In this case, the message should not be so much smaller than the required in the additive chaotic masking scheme, as has been shown in [143]. The message is not added only to one state of the chaotic system but is added with the use of an encoding function to all states of the system. The message is retrieved bearing in mind that the receiver has a decoding function, shown in Figure 2.5.

- **Chaotic Cryptosystem**: In this scheme, there is a mix between encryption and synchronization. The message signals $m(t)$ is encrypted before it is added to the chaotic system using signal $k(t)$, which represents one of the states of the master chaotic system, as shown in [144]. The encrypted signal $m_c(t)$ changes the dynamics of the master system, making it even more complex. Considering the possibility of a security breach on the public channel, without the decryption rule there is no mode of obtaining the message. You can note the presence of noise $n(t)$ in the channel. So after synchronization, they will be retrieved at receiver $k(t)$ and $n(t)$, but with some noise. Using both in decryption gives the estimated message signal, as can be seen in Figure 2.6.

- **Chaotic Displacement Switching**: The chaotic switching encryption scheme is shown, which means transmitting a binary signal alternating between two chaotic carriers generated by two different systems, as shown in Figure 2.7, this scheme was developed to transmit digital message signs, was investigated in [2]. The message signal $m(t)$ is used as the input to a multiplexer that has as its output option two chaotic systems of the same structure, but with different parameters. The received signal $r(t)$ leads to synchronization with the slave system and the message is recovered using a low-pass filter and then $thresholding$ on the synchronization error signal $e(t)$.

Subsequently, the signal modulated by any of the above methods is transmitted over a public communication channel, and captured by the receiving (slave) system. This receiving system must be able to synchronize with the master system and have a decryption technique to recover the original information signal. The decryption process is realized using a synchronization error detection process consisting of a low-pass filter and threshold detection, as shown in the schematics and cited in [125].

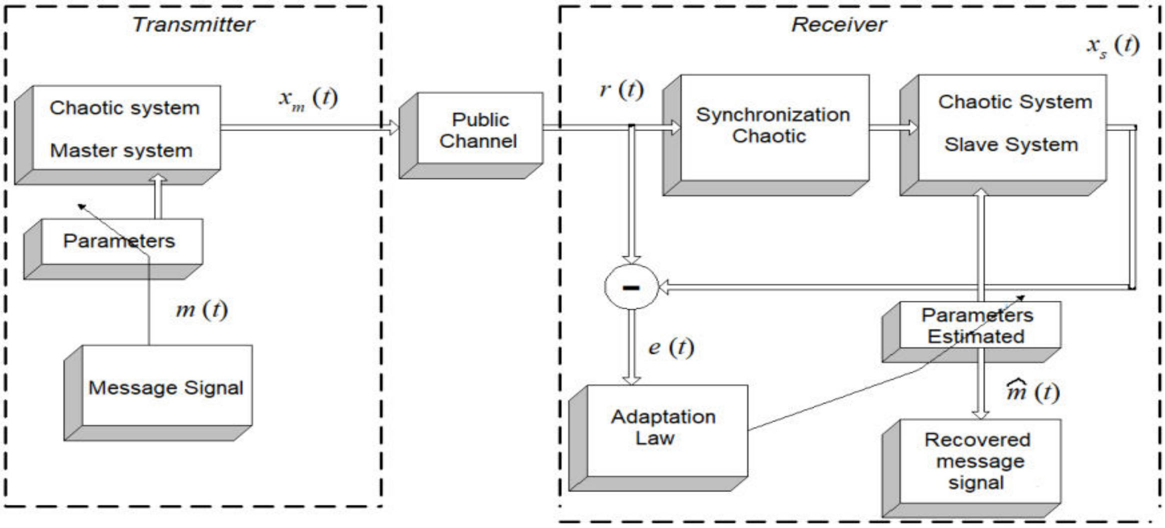

Figure 2.4 – Chaotic Parameter Modulation Scheme [2].

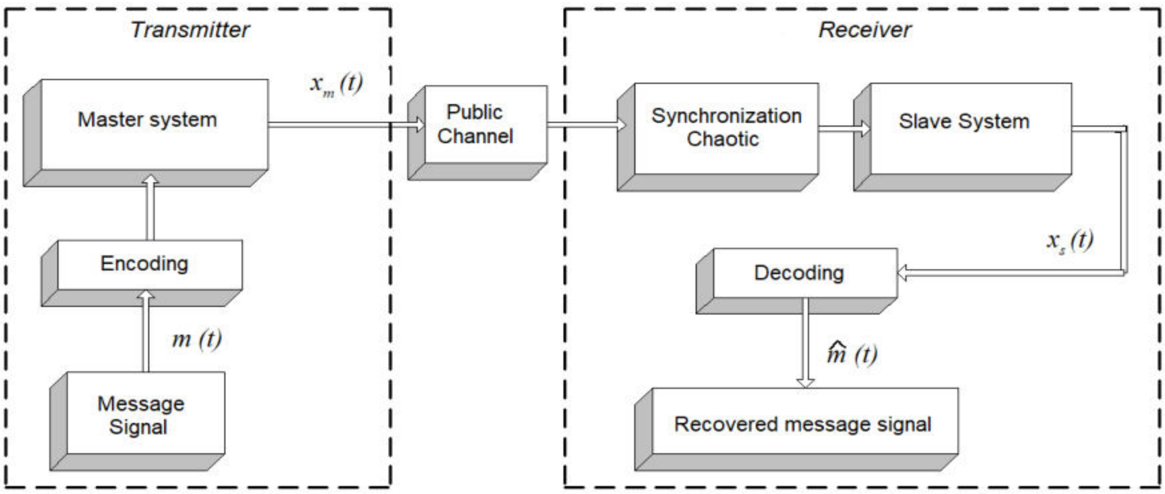

Figure 2.5 – Non-autonomous Chaotic Modulation Scheme [2].

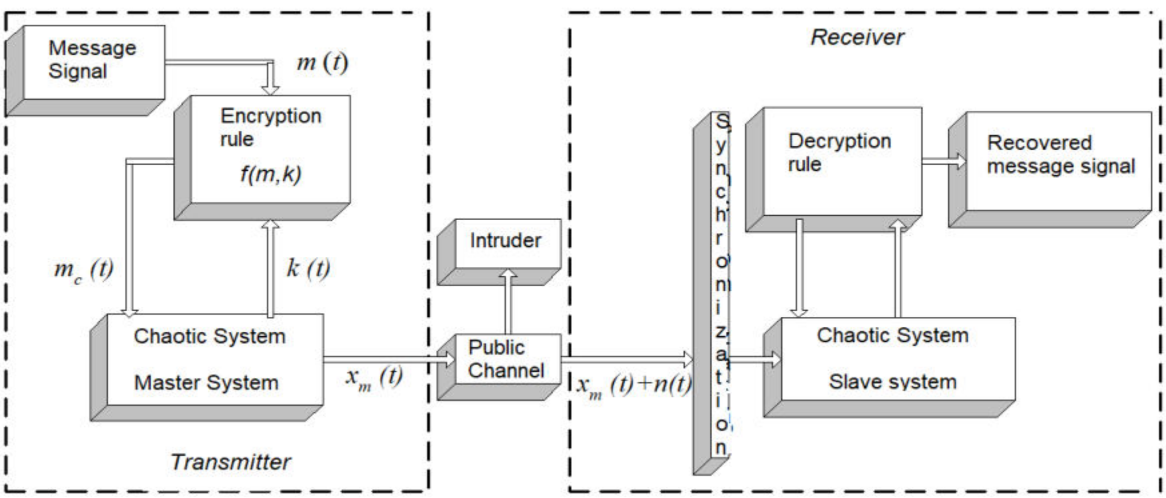

Figure 2.6 – Cryptosystem Scheme [2].

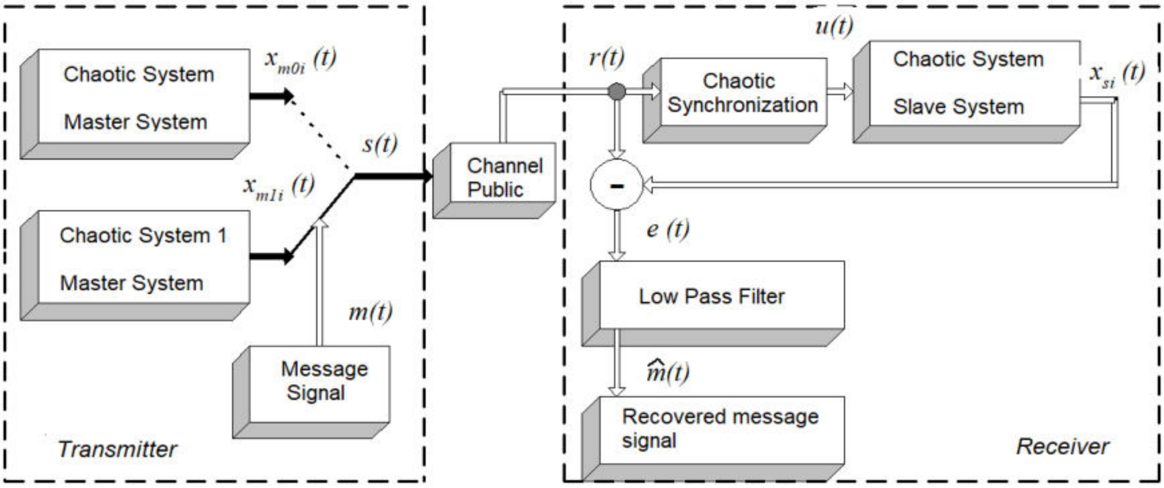

Figure 2.7 – Chaotic Dislocation Switching Scheme [2].

3 PROJECTIVE SYNCHRONIZATION AND ANTISYNCHRONIZATION OF AN UNDERACTUATED SYSTEM BASED ON PROPORTIONAL CONTROL

The research results of this chapter were published as a conference paper entitled "Projective Synchronization and Antisynchronization of Underactuated Systems" in [145]. *This chapter has extended and improved some parts compared to the original paper.*

In this chapter, a simple synchronization and anti-synchronization scheme for a class of chaotic systems is proposed. Based on Lyapunov arguments and proportional control, the convergence of the synchronization error to a small neighborhood of the origin, even in the presence of unmatched disturbances, is shown. The actuation mechanism, which is under-actuated and uncomplicated, and the enhanced performance of the proposed method lie in their main peculiarities. The synchronization and antisynchronization of a three-dimensional chaotic system are accomplished to corroborate the theoretical results and show its straightforward application.

3.1 PROBLEM FORMULATION

Consider the following chaotic system by J. Lü [146]

$$\begin{aligned}
\dot{x} &= dx(t) - y(t)z(t) + c \\
\dot{y} &= -ay(t) + x(t)z(t) \\
\dot{z} &= -bz(t) + x(t)y(t)
\end{aligned} \tag{3.1}$$

where t is the time, $d = ab/(a+b)$, $a = 10$, $b = 4$, and $c = 0$. Note that $x(t)$, $y(t)$, and $z(t)$ are the system states and a, b, and c are non-negative real constants. Based on (3.1), consider the perturbed systems:

$$\begin{aligned}
\dot{x}_m(t) &= dx_m(t) - y_m(t)z_m(t) + c + h_{1m}(t) \\
\dot{y}_m(t) &= -ay_m(t) + x_m(t)z_m(t) + h_{2m}(t) \\
\dot{z}_m(t) &= -bz_m(t) + x_m(t)y_m(t) + h_{3m}(t)
\end{aligned} \tag{3.2}$$

and slave system

$$
\begin{aligned}
\dot{x}_s(t) &= dx_s(t) - y_s(t)z_s(t) + c + h_{1s}(t) + u_1(t) \\
\dot{y}_s(t) &= -ay_s(t) + x_s(t)z_s(t) + h_{2s}(t) + u_2(t) \\
\dot{z}_s(t) &= -bz_s(t) + x_s(t)y_s(t) + h_{3s}(t)
\end{aligned}
\tag{3.3}
$$

Note that $x_m(t)$, $y_m(t)$, and $z_m(t)$ are the states of the master system; $x_s(t)$, $y_s(t)$, and $z_s(t)$ are the states of the slave system; $h_{1m}(t)$, $h_{2m}(t)$, and $h_{3m}(t)$ are disturbances present in the master system; $h_{1s}(t)$, $h_{2s}(t)$, and $h_{3s}(t)$ are disturbances present in the slave system; and $u_1(t)$ and $u_2(t)$ are the control inputs.

The goal is to anti-synchronize or synchronize the systems (3.2) and (3.3), in which the slave system has only two scalar control signals.

Remark 3.1: As is known, system (3.1) is a chaotic system. Hence, its behavior depends strongly on the initial conditions, so the system is sensitive to changes in these conditions. Due to this sensitivity and also the fact that it has aperiodic behavior, synchronization or antisynchronization of chaotic systems is usually considered more challenging than if it were done on other dynamic systems.

Assumption 3.1: Assume that

$$
\begin{aligned}
h_1(t) =& h_{1s}(t) - \alpha_1 \cdot h_{1m}(t) \\
h_2(t) =& h_{2s}(t) - \alpha_2 \cdot h_{2m}(t) \\
h_3(t) =& h_{3s}(t) - \alpha_3 \cdot h_{3m}(t)
\end{aligned}
\tag{3.4}
$$

$\forall t \geq 0$, where α_1, α_2, and α_3 are nonzero constants. The disturbances are assumed to be upper-bounded such as

$$
\begin{aligned}
|h_1(t)| &\leq \bar{h}_1 \\
|h_2(t)| &\leq \bar{h}_2 \\
|h_3(t)| &\leq \bar{h}_3
\end{aligned}
\tag{3.5}
$$

where $\bar{h}_1$,$\bar{h}_2$, and $\bar{h}_3$ are unknown constants.

Remark 3.2: The purpose for presenting systems (3.2) and (3.3) in which disturbances are explicitly considered is to point out that the studied projective synchronization scheme is valid even in the presence of perturbations common in electronic circuits, such as tolerances, non-ideal behavior, heating, and so on.

Fact 3.1: With the boundedness of the system (3.1) [146], it can be established that

$$
\begin{aligned}
|x_m(t)| &\leq \bar{x} \\
|y_m(t)| &\leq \bar{y} \\
|z_m(t)| &\leq \bar{z}
\end{aligned} \tag{3.6}
$$

$\forall t \geq 0$, where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are unknown positive constants.

3.2 ERROR EQUATION AND PROPOSED CONTROL SIGNAL.0

The projective synchronization errors of the system are defined as:

$$
\begin{aligned}
e_1 &= x_s - \alpha_1 x_m \\
e_2 &= y_s - \alpha_2 y_m \\
e_3 &= z_s - \alpha_3 z_m
\end{aligned} \tag{3.7}
$$

Observe that if α_1, α_2, and α_3 are negative, it corresponds to the case of antisynchronization. Based on (3.7), we have

$$
\begin{aligned}
\dot{e}_1 &= \dot{x}_s - \alpha_1 \dot{x}_m \\
\dot{e}_2 &= \dot{y}_s - \alpha_2 \dot{y}_m \\
\dot{e}_3 &= \dot{z}_s - \alpha_3 \dot{z}_m
\end{aligned} \tag{3.8}
$$

Substituting (3.2) and (3.3) into (3.8), we obtain

$$
\begin{aligned}
\dot{e}_1 &= de_1 - e_2 e_3 - \alpha_3 z_m e_2 - \alpha_2 e_3 y_m - y_m z_m(\alpha_2\alpha_3 - \alpha_1) + h_{1s} - \alpha_1 h_{1m} + u_1 \\
\dot{e}_2 &= -ae_2 + e_1 e_3 + \alpha_3 e_1 z_m + \alpha_1 e_3 x_m + x_m z_m(\alpha_1\alpha_3 - \alpha_2) + h_{2s} - \alpha_2 h_{2m} + u_2 \\
\dot{e}_3 &= -be_3 + e_1 e_2 + \alpha_2 e_1 y_m + \alpha_1 e_2 x_m + x_m y_m(\alpha_1\alpha_2 - \alpha_3) + h_{3s} - \alpha_3 h_{3m}
\end{aligned} \tag{3.9}
$$

Once the synchronization errors are defined, an appropriate control signal is required for the slave system to synchronize or anti-synchronize with the master system.

Theorem 3.1: Consider the master and slave systems described in (3.2) and (3.3) and the proportional control laws described by

$$
\begin{aligned}
u_1 &= -\,\psi_1 e_1 \\
u_2 &= -\,\psi_2 e_2
\end{aligned} \tag{3.10}
$$

where ψ_1 and ψ_2 are a user-defined positive constants. It can be established that the synchronization or antisynchronization error converges in finite time to the compact set $\Omega = \{e \in R^3 \mid \|e\| \leq \theta\}$, where θ is a small positive constant.

Proof 3.1: Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\left(2e_1^2 + e_2^2 + e_3^2\right) \tag{3.11}$$

whose derivative is

$$\dot{V} = 2e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 \tag{3.12}$$

Substituting (3.9) into (3.12) we have

$$\begin{aligned}\dot{V} = {} & 2e_1[de_1 - e_2e_3 - \alpha_3e_2z_m - \alpha_2e_3y_m - y_mz_m(\alpha_2\alpha_3 - \alpha_1) + h_{1s} - \alpha_1h_{1m} \\ & + u_1] + e_2[-ae_2 + e_1e_3 + \alpha_3e_1z_m + \alpha_1e_3x_m + x_mz_m(\alpha_1\alpha_3 - \alpha_2) + h_{2s} \\ & - \alpha_2h_{2m} + u_2] + e_3[-be_3 + e_1e_2 + \alpha_2e_1y_m + \alpha_1e_2x_m + x_my_m(\alpha_1\alpha_2 - \alpha_3) \\ & + h_{3s} - \alpha_3h_{3m}]\end{aligned} \tag{3.13}$$

Rearranging and substituting (3.10) into (3.13), one can obtain

$$\begin{aligned}\dot{V} = {} & -e_1^2\left(2\psi_1 - 2d\right) - e_2^2(\psi_2 + a) - be_3^2 + 2e_1(h_{1s} - \alpha_1h_{1m}) \\ & + e_2(h_{2s} - \alpha_2h_{2m}) + e_3(h_{3s} - \alpha_3h_{3m}) - e_1e_3(\alpha_2y_m) - e_1e_2(\alpha_3z_m) \\ & + e_2e_3\left(2\alpha_1x_m\right) - e_1\left[2y_mz_m(\alpha_2\alpha_3 - \alpha_1)\right] + e_2[x_mz_m(\alpha_1\alpha_3 - \alpha_2)] \\ & + e_3[x_my_m(\alpha_1\alpha_2 - \alpha_3)]\end{aligned} \tag{3.14}$$

Based on Young's inequality and applying [147] and applying (3.4) results

$$
\begin{aligned}
&2e_1(h_{1s}-\alpha_1 h_{1m}) \le \sigma_1 e_1^2 + \sigma_1^{-1}\bar{h}_1^2 \\
&e_2(h_{2s}-\alpha_2 h_{2m}) \le 0.5\left(\sigma_2 e_2^2 + \sigma_2^{-1}\bar{h}_2^2\right) \\
&e_3(h_{3s}-\alpha_3 h_{3m}) \le 0.5\left(\sigma_3 e_3^2 + \sigma_3^{-1}\bar{h}_3^2\right) \\
&-e_1 e_3\left(\alpha_2 y_m\right) \le 0.5\left(\bar{y}^2\alpha_2^2 e_1^2 + e_3^2\right) \\
&-e_1 e_2\left(\alpha_3 z_m\right) \le 0.5\left(\bar{z}^2\alpha_3^2 e_1^2 + e_2^2\right) \\
&2e_2 e_3\left(\alpha_1 x_m\right) \le \bar{x}^2\alpha_1^2 e_2^2 + e_3^2 \\
&-2e_1\left[y_m z_m\left(\alpha_2\alpha_3-\alpha_1\right)\right] \le e_1^2 + \left[\bar{y}\bar{z}\left(\alpha_2\alpha_3-\alpha_1\right)\right]^2 \\
&e_2[y_m z_m\left(\alpha_1\alpha_3-\alpha_2\right)] \le 0.5\{\, e_2^2 + \left[\bar{x}\bar{z}\left(\alpha_1\alpha_3-\alpha_2\right)\right]^2\} \\
&e_3[x_m y_m\left(\alpha_1\alpha_2-\alpha_3\right)] \le 0.5\{e_3^2 + \left[\bar{x}\bar{y}\left(\alpha_1\alpha_2-\alpha_3\right)\right]^2\}
\end{aligned}
\tag{3.15}
$$

where σ_1, σ_2, and σ_3 are arbitrary positive constants. The terms $\bar{x}$, $\bar{y}$, $\bar{z}$ are upper bounds for the states of the master system. By using (3.15), (3.14) implies

$$
\begin{aligned}
&\dot{V} \le -e_1^2[2\left(\psi_1-d\right)-\sigma_1-0,5\left(2+\bar{y}^2\alpha_2^2+\bar{z}^2\alpha_3^2\right)]- \\
&e_2^2\left(\psi_2+a-\sigma_2-1-\bar{x}^2\alpha_1^2\right)-e_3^2\left(b-2\right)+[\bar{y}\bar{z}\left(\alpha_2\alpha_3-\alpha_1\right)]^2+ \\
&0,5[\bar{x}\bar{z}\left(\alpha_1\alpha_3-\alpha_2\right)]^2+0,5[\bar{x}\bar{y}\left(\alpha_1\alpha_2-\alpha_3\right)]^2+\frac{\bar{h}_1^2}{\sigma_1}+\frac{\bar{h}_2^2}{2\sigma_2}+\frac{\bar{h}_2^2}{2\sigma_3}
\end{aligned}
\tag{3.16}
$$

Consider that

$$
\begin{aligned}
&\rho_1 = 2\left(\psi_1-d\right)-\sigma_1-0,5\left(2+\bar{y}^2\alpha_2^2+\bar{z}^2\alpha_3^2\right) \\
&\rho_2 = \psi_2+a-\sigma_2-1-\bar{x}^2\alpha_1^2 \\
&\rho_3 = b-\sigma_3-2 \\
&\beta_1 = \frac{\bar{h}_1^2}{\sigma_1}+\frac{\bar{h}_2^2}{2\sigma_2} \\
&\beta_2 = \frac{\bar{h}_2^2}{2\sigma_3} \\
&\beta_3 = [\bar{y}\bar{z}\left(\alpha_2\alpha_3-\alpha_1\right)]^2+0,5[\bar{x}\bar{z}\left(\alpha_1\alpha_3-\alpha_2\right)]^2+0,5[\bar{x}\bar{y}\left(\alpha_1\alpha_2-\alpha_3\right)]^2
\end{aligned}
\tag{3.17}
$$

Substituting (3.17) into (3.16) results

$$
\dot{V} \le -e_1^2\rho_1 - e_2^2\rho_2 - e_3^2\rho_3 + \beta_1 + \beta_2 + \beta_3 \tag{3.18}
$$

Remark 3.3: To analyze this inequality, it is important that ρ_1, ρ_2, and ρ_3 be positive. Note that ρ_1, ρ_2, and ρ_3 depend on ψ_1, ψ_3, σ_1, σ_2, and σ_3. One can choose these parameters to ensure that ρ_1, ρ_2, and ρ_3 are all positive. Defining: $\rho = min\,\{\rho_1,\rho_2,\rho_3\}$ and $\beta=\beta_1+\beta_2+\beta_3$, (3.18) can be written as

$$\dot{V} \leq -\rho \, \|e\|^2 + \beta \tag{3.19}$$

Then, we have $\dot{V} < 0$ as long as $\|e\| > \sqrt{\left(\frac{\beta}{\rho}\right)} := \theta$. Because θ is constant, it can be established that either synchronization or antisynchronization error is bounded. Defining the compact set $\Omega = \{e \in R^3 \mid \ \|e\| \leq \theta\}$, we have that if for any reason $\|e\|$ leave the residual set Ω, $\dot{V}$ becomes negative and forces the convergence of the synchronization error to the residual set Ω, according to (3.19). In other words, if $\dot{V} < 0$ is satisfied, the error norm can only decrease over time. Thus, we conclude that the synchronization error is bounded and converges to a ball with a radius equal to θ.

Remark 3.4: Note that the proof covers any α_1, α_2, and α_3. Theorem 1 shows that both the synchronization and antisynchronization errors are bounded, i.e., the proof of the theorem is valid in both the synchronization and antisynchronization case.

Remark 3.5: From (3.17), it can be seen that σ_1 and σ_2 can be freely chosen by the usuary. This freedom is owing to the fact that ρ_1 and ρ_2 can be arbitrarily adjusted to be positive if ψ_1 and ψ_2 are adjusted as well. Likewise, β_3 can be arbitrarily decreased through the scaling factors. This favorable situation is not the same for β_2, since the maximum value for σ_3 is restricted to 2. However, as can be seen in our simulation, this performance restriction is not important since the value σ_3 is enough to allow a good performance, as far as the residual error is considered, in the presence of unmatched disturbances.

Remark 3.6: Some interesting cases to make $\beta_3 = 0$ in (3.17) are when $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_3 = 1$ (identical synchronization), or $\alpha_1 = -1$, $\alpha_2 = 1$, and $\alpha_3 = -1$ (antisynchronization and synchronization together).

Remark 3.7: It is well-known that the performance of chaos-based cryptography is straightforwardly related to the quality of the synchronizer. Hence, the design of synchronizers with enhanced performance is a hot topic in the literature. In this context, the proposed approach offers a way to increase antisynchronization performance. It was proven that the synchronization and antisynchronization performed together are crucial to increase the performance when unmatched disturbances are present.

3.3 SIMULATION

Matlab/Simulink(2020b) software on a Ryzen 7 1700 computer was used for the simulations with a variable step and ode15s method. It was considered that $a = 10$, $b = 4$, and $c = 0$. The initial conditions were $x_m(0) = [3,\ -4,\ -2]$ and $x_s(0) = [5,\ -5,\ 3]$. For synchronizing the master and slave systems we used the control law (3.10) with $\psi_1 = 10000$, $\psi_2 = 10000$, $\alpha_1 = -1$, $\alpha_2 = 1$, and $\alpha_3 = -1$.

After $t = 10s$ disturbances were introduced in the simulation to check the robustness of the proposed method. Their values are $h_{1m} = cos(4t)$, $h_{2m} = 1.2cos(3t)$, $h_{3m} = 0.1sin(7t)$, $h_{1s} = 1.5sin(4t)$, $h_{2s} = cos(6t)$, and $h_{3s} = 0.2sin(5t)$. Figures (3.1 - 3.3) show the results of the simulation. For better visualization of the errors, Figures (3.4 - 3.6) were also included.

As can be seen from Figures (3.1 - 3.6) the simulations corroborate the theoretical results: the synchronization and antisynchronization errors were close to zero. The reasons for this were:

1) α_1, α_2, and α_3 were chosen as established in Remark 6 to make $\beta_3 = 0$;
2) The considered disturbance h_{3s} in the underactuated state was not huge;
3) ψ_1 and ψ_2 were chosen large enough.

In addition, it can be seen that the synchronizer is robust against matched and unmatched disturbances.

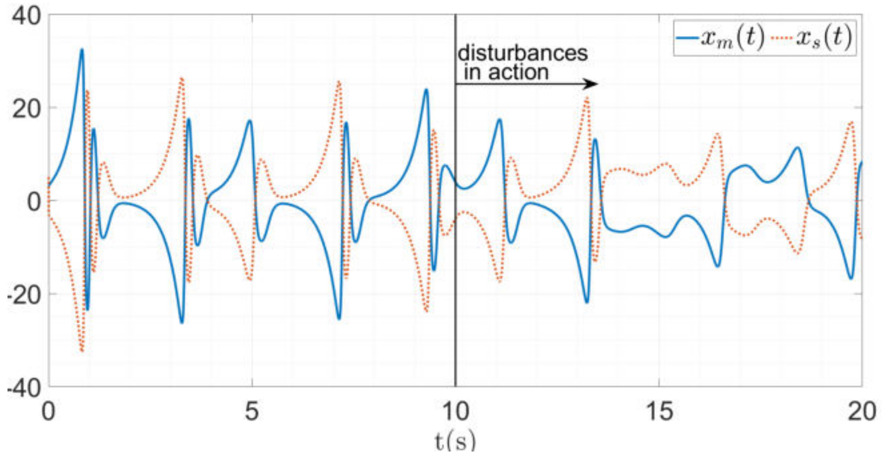

Figure 3.1 – Performance in the antisynchronization of $x_m(t)$ and $x_s(t)$.

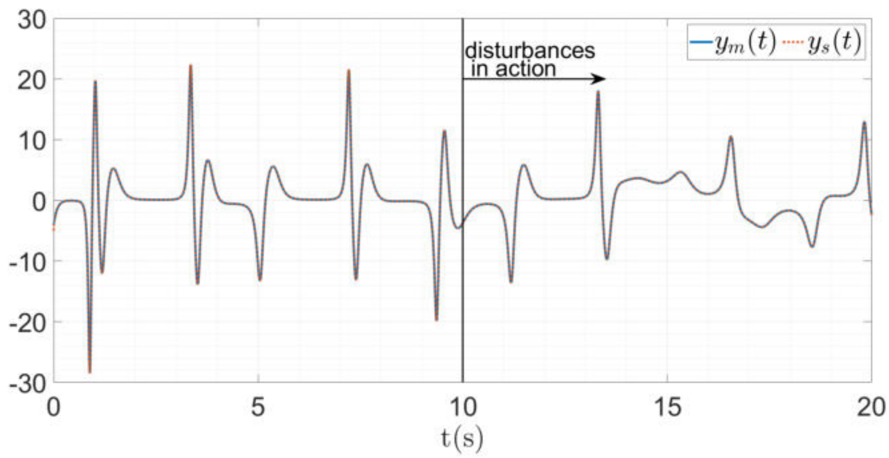

Figure 3.2 – Performance in the antisynchronization of $y_m(t)$ and $y_s(t)$.

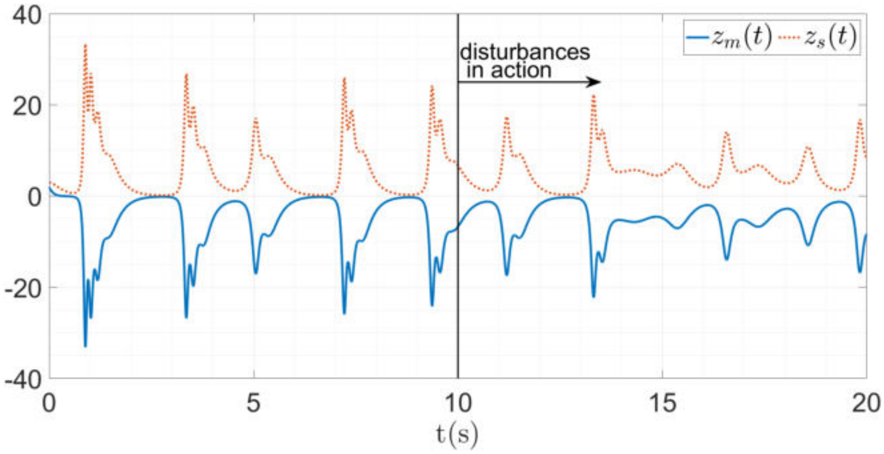

Figure 3.3 – Performance in the antisynchronization of $z_m(t)$ and $z_s(t)$.

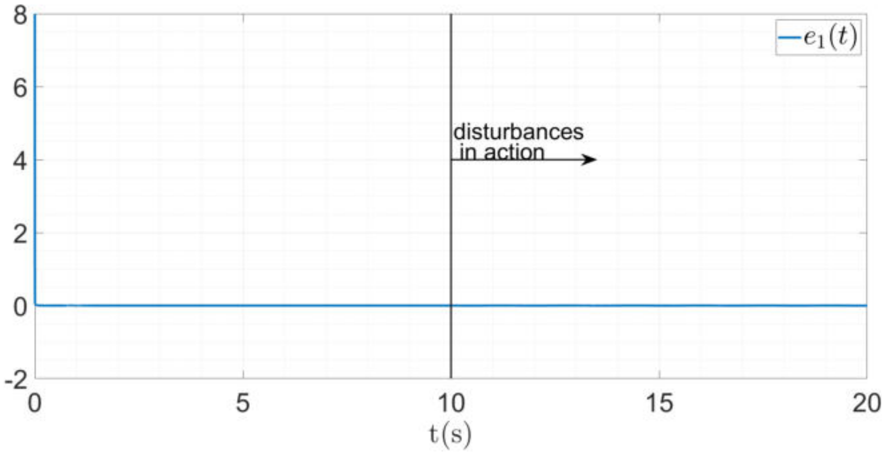

Figure 3.4 – Antisynchronization error $e_1(t)$.

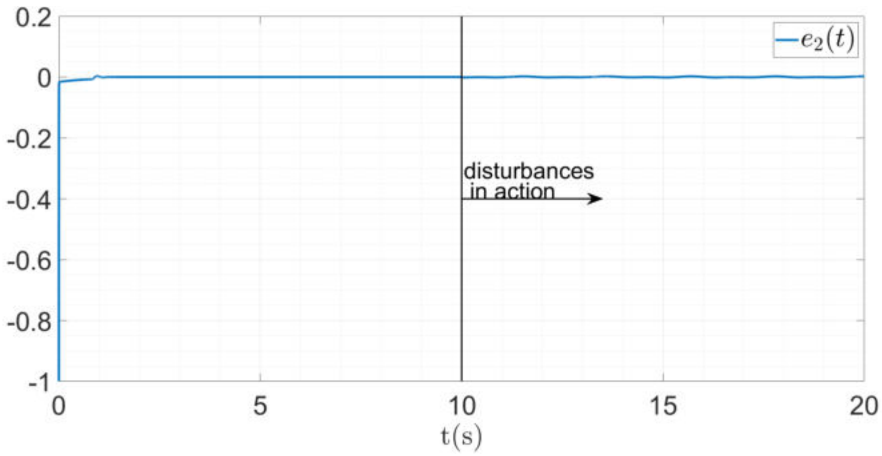

Figure 3.5 – Antisynchronization error $e_2(t)$.

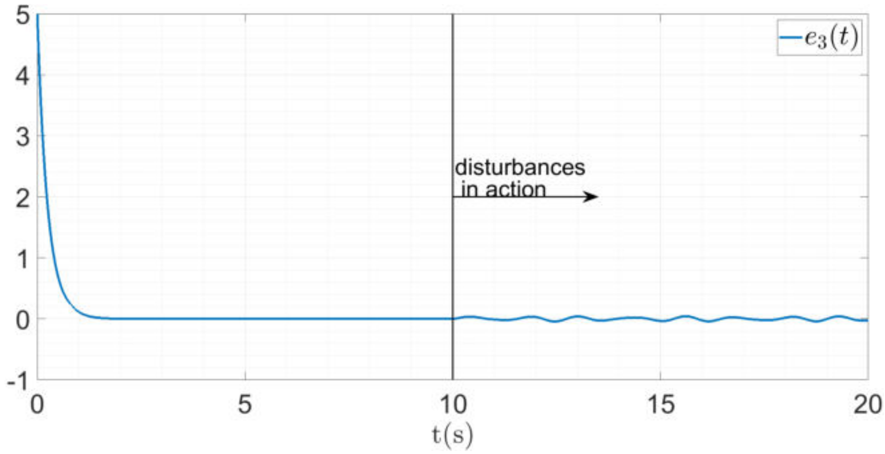

Figure 3.6 – Antisynchronization error $e_3(t)$.

Table 3.1 shows the RMSE for different values of ψ. Small values of ψ correspond to large RMSE and transient. Also, large, ψ corresponds to small RMSE and transient, as shown in Table 3.1 and Figures (3.4 - 3.6).

Table 3.1 – Root mean square of the state errors for t = [0 20] seconds.

Mean square root of the state errors in the proposed algorithm

ψ_1	ψ_2	$e_{1_{rms}}$	$e_{2_{rms}}$	$e_{3_{rms}}$
10	10	2.553946	1.944666	2.170547
100	100	0.595966	0.162219	0.643714
1000	1000	0.272961	0.043283	0.504972
10000	10000	0.094767	0.012270	0.408499

3.4 CONCLUSIONS

This chapter proposes a projective synchronization and antisynchronization algorithm based on Lyapunov theory for an underactuated chaotic system subject to disturbances. It has been proven and validated in a simulation that only two control signals are needed to perform master and slave synchronization and antisynchronization. However, the synchronizer has some limitations, such as the sensitivity of the errors to the disturbances present in the disturbed states.

4 SYNCHRONIZATION OF A HYPERCHAOTIC FINANCIAL SYSTEM

The research results of this chapter were published as a conference paper entitled "Underactuated Synchronization Scheme of a Hyperchaotic Finance System" in [148]. *This chapter has extended and improved some parts compared to the original paper.*

This chapter considers an underactuated synchronization scheme of a hyperchaotic financial system based on Lyapunov analysis, capable of taking into account shocks of any nature. However, it is very rare to find underactuated hyperchaotic systems, and as far as the authors know, no work has been done on the underactuated synchronization of a hyperchaotic system in the literature and that requires only two controls to act on two states in the slave system equations. The proposed scheme has the advantage of being robust against disturbances and structurally simple, which is interesting because it leads to substantial cost reductions. Computational simulations were performed to validate the robustness and simplicity of the proposed scheme.

4.1 PROBLEM FORMULATION

Consider the following chaotic system proposed by H. Yu [149]:

$$\begin{aligned}
\dot{x} &= z(t) + [y(t) - a]x(t) + w(t) \\
\dot{y} &= 1 - by(t) - x^2(t) \\
\dot{z} &= -x(t) - cz(t) \\
\dot{w} &= -dx(t)y(t) - kw(t)
\end{aligned} \tag{4.1}$$

where $a = 0.9$; $b = 0.2$; $c = 1.5$; $d = 0.2$ and $k = 0.17$. This system is proposed by [149]. $x(t)$, $y(t)$, $z(t)$ and $w(t)$ are the states of the system and a, b, c, d and k are real constants. Based on (4.1), consider the following perturbed master system

$$\begin{aligned}
\dot{x}_m &= z_m + (y_m - a)x_m + w_m + h_{1m}(t) \\
\dot{y}_m &= 1 - by_m - x_m^2 + h_{2m}(t) \\
\dot{z}_m &= -x_m - cz_m + h_{3m}(t) \\
\dot{w}_m &= -dx_m y_m - kw_m + h_{4m}(t)
\end{aligned} \tag{4.2}$$

and slave system

$$
\begin{aligned}
\dot{x}_s &= z_s + (y_s - a)x_s + w_s + h_{1s}(t) + u_1 \\
\dot{y}_s &= 1 - by_s - x_s^2 + h_{2s}(t) \\
\dot{z}_s &= -x_s - cz_s + h_{3s}(t) \\
\dot{w}_s &= dx_s y_s - kw_s + h_{4s}(t) + u_2
\end{aligned}
\tag{4.3}
$$

where u_1 and u_2 are the control signals; x_m, y_m, z_m, and w_m are the states of the master system; x_s, y_s, z_s, and w_s are the states of the slave system; h_{1m}, h_{2m}, h_{3m}, and h_{4m} are disturbances present in the master system; and h_{1s}, h_{2s}, h_{3s}, and h_{4s} are disturbances present in the slave system.

The objective is to synchronize systems (4.2) and (4.3), in which the slave system allows only two control signals, that is, acting only in two states. Since system (4.1) is chaotic, its behavior depends heavily on the initial conditions so that the system is sensitive to changes in these initial conditions. Because of this and because it has aperiodic behavior, synchronization of chaotic systems is usually considered more challenging than performed on other dynamical systems. Additionally, the master and slave systems can have the same structure, which, just because they have different initial conditions, will also have different trajectories over time.

Assumption 4.1: Consider that

$$
\begin{aligned}
h_1 &= h_{1s}(t) - h_{1m}(t) \\
h_2 &= h_{2s}(t) - h_{2m}(t) \\
h_3 &= h_{3s}(t) - h_{3m}(t) \\
h_4 &= h_{3s}(t) - h_{3m}(t)
\end{aligned}
\tag{4.4}
$$

$\forall t \geq 0$. The disturbances are bounded such that

$$
\begin{aligned}
|h_1(t)| &\leq \bar{h}_1 \\
|h_2(t)| &\leq \bar{h}_2 \\
|h_3(t)| &\leq \bar{h}_3 \\
|h_4(t)| &\leq \bar{h}_4
\end{aligned}
\tag{4.5}
$$

where $\bar{h}_1$,$\bar{h}_2$, $\bar{h}_3$ and $\bar{h}_4$ are unknown constants.

Remark 4.1: The reason for presenting systems (4.2) and (4.3) in which disturbances are explicitly considered is to emphasize that the proposed synchronization scheme is valid in real situations.

Fact 4.1: Note that the system (4.1) was proved to be bounded and dissipative in [149]. Therefore, it can be stated that

$$\begin{aligned} |x_m(t)| &\leq \bar{x} \\ |y_m(t)| &\leq \bar{y} \\ |z_m(t)| &\leq \bar{z} \\ |w_m(t)| &\leq \bar{w} \end{aligned} \tag{4.6}$$

$\forall t \geq 0$, where $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{w}$ are unknown positive constants.

4.2 SYNCHRONIZATION ERROR EQUATION AND PROPOSED CONTROL SIGNAL

The dynamics of errors can be defined as that derived from synchronization errors

$$\begin{aligned} \dot{e}_1 &= \dot{x}_s - \dot{x}_m \\ \dot{e}_2 &= \dot{y}_s - \dot{y}_m \\ \dot{e}_3 &= \dot{z}_s - \dot{z}_m \\ \dot{e}_4 &= \dot{w}_s - \dot{w}_m \end{aligned} \tag{4.7}$$

Substituting (4.2) and (4.3) in (4.7), and applying (4.4), then

$$\begin{aligned} \dot{e}_1 &= e_3 + e_1 e_2 + e_1(y_m - a) + e_2 x_m + e_4 + h_1 + u_1 \\ \dot{e}_2 &= -b e_2 - e_1^2 - 2 e_1 x_m + h_2 \\ \dot{e}_3 &= -e_1 - c e_3 + h_3 \\ \dot{e}_4 &= -d(e_1 e_2 + e_2 x_m + e_1 y_m) - k e_4 + h_4 + u_2 \end{aligned} \tag{4.8}$$

For the slave system to synchronize correctly with the master system, suitable u_1 and u_2 control signals are required.

Theorem 4.1: Consider the master and slave systems described in (4.2) and (4.3) and the following control law

$$\begin{aligned} u_1 &= -\psi_1 e_1 - \psi_2 e_1^3 \\ u_2 &= -\psi_3 e_4 - \psi_4 e_4^3 \end{aligned} \tag{4.9}$$

where ψ_1, ψ_2, ψ_3 and ψ_4 are positive real parameters arbitrarily chosen by the designer. Thus, the synchronization error converges in finite time to the compact set $\Omega = \{e \in \Re^4 | \parallel e \parallel \leq \theta\}$, where θ is an unknown positive constant.

Proof 4.1: Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \tag{4.10}$$

Deriving the trajectories (4.10) over time

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 \tag{4.11}$$

Substituting (4.8) into (4.11), the result is

$$\begin{aligned}\dot{V} &= e_1\left[e_3 + e_1\left(y_m - a\right) + e_2x_m + e_1e_2 + e_4 + h_1 + u_1\right] \\ &\quad + e_2\left(-be_2 - 2e_1x_m - e_1^2 + h_2\right) + e_3\left(-e_1 - ce_3 + h_3\right) \\ &\quad + e_4\left[-d\left(e_2x_m + e_1y_m + e_1e_2\right) - ke_4 + h_4 + u_2\right]\end{aligned} \tag{4.12}$$

Substituting (4.9) into (4.12), we obtain that

$$\begin{aligned}\dot{V} &= -e_1^2\left(\psi_1 + a - y_m\right) - be_2^2 - ce_3^2 - e_4^2(\psi_3 + k) + e_1h_1 + e_2h_2 + e_3h_3 \\ &\quad + e_4h_4 - de_1e_2e_4 - x_me_1e_2 + e_1e_4(1 - dy_m) - dx_me_2e_4 - \psi_2e_1^4 - \psi_4e_4^4\end{aligned} \tag{4.13}$$

Applying the Young inequality [147], we observed that

$$\begin{aligned}&-y_me_1^2 \le \bar{y}e_1^2 \\ &e_1h_1 \le 0.5(\sigma_1e_1^2 + \sigma_1^{-1}\bar{h}_1^2) \\ &e_2h_2 \le 0.5(\sigma_2e_2^2 + \sigma_2^{-1}\bar{h}_2^2) \\ &e_3h_3 \le 0.5(\sigma_3e_3^2 + \sigma_3^{-1}\bar{h}_3^2) \\ &e_4h_4 \le 0.5(\sigma_4e_4^2 + \sigma_4^{-1}\bar{h}_4^2) \\ &-x_me_1e_2 \le 0.5(\sigma_5\bar{x}^2e_1^2 + \sigma_5^{-1}e_2^2) \\ &(1 - dy_m)e_1e_4 \le 0.5[\sigma_6(1 + d^2\bar{y}^2)e_1^2 + \sigma_6^{-1}e_4^2] \\ &-dx_me_2e_4 \le 0.5(\sigma_7d^2\bar{x}^2e_4^2 + \sigma_7^{-1}e_2^2) \\ &-de_1e_2e_4 \le 0.5d[\sigma_8e_2^2 + 0.25\sigma_8^{-2}(e_1^4 + e_4^4)]\end{aligned} \tag{4.14}$$

where $\sigma_i, i = 1, ..., 8$ are positive parameters arbitrarily chosen, then consider that

$$
\begin{aligned}
\rho_1 &= \psi_1 + a - \bar{y} - 0.5[\sigma_1 + \sigma_5 \bar{x}^2 + \sigma_6(1 + d^2\bar{y}^2)] \\
\rho_2 &= b - 0.5(\sigma_2 + \sigma_5^{-1} + \sigma_7^{-1} + d\sigma_8) \\
\rho_3 &= c - 0.5\sigma_3 \\
\rho_4 &= \psi_3 + k - 0.5(\sigma_4 + \sigma_6^{-1} + \sigma_7 d^2 \bar{x}^2) \\
\rho_5 &= \psi_2 - 2^{-3}\sigma_8^{-2} \\
\rho_6 &= \psi_4 - 2^{-3}\sigma_8^{-2} \\
\beta_c &= 0.5(\sigma_1^{-1}\bar{h}_1^2 + \sigma_4^{-1}\bar{h}_1^2) \\
\beta_n &= 0.5(\sigma_2^{-1}\bar{h}_2^2 + \sigma_3^{-1}\bar{h}_2^2)
\end{aligned} \tag{4.15}
$$

Thus, analyzing (4.13) in the case of $\dot{V}$, and employing (4.15)

$$
\dot{V} \leq -\rho_1 e_1^2 - \rho_2 e_2^2 - \rho_3 e_3^2 - \rho_4 e_4^2 - \rho_5 e_1^4 - \rho_6 e_4^4 + \beta_c + \beta_n \tag{4.16}
$$

Note that $\sigma_i, i = 1, ..., 8$ and ψ_1, ψ_2, ψ_3, and ψ_4 can be chosen in a way that $\rho_j, j = 1, ..., 6$ are positive. So, consider that $\rho := min\{\rho_1, \rho_2, \rho_3, \rho_4\}$ and $\beta := \beta_c + \beta_n$, then (4.16) can be rewritten as

$$
\dot{V} \leq -\rho||e||^2 + \beta \tag{4.17}
$$

Therefore, $\dot{V} < 0$ when $||e|| > \sqrt{\frac{\beta}{\rho}} := \theta$. Since θ is a constant, it can be stated that the synchronization error is bounded. Defining the compact set $\Omega = \{e \in R^4 \,|||e|| \leq \theta\}$, then it can be stated that if for any reason $||e||$ leaves the residual set Ω, $\dot{V}$ becomes negative and forces the convergence of the synchronization error to the residual set Ω, according to (4.17). Thus, it is concluded that the synchronization error is bounded and converges to a ball with a radius equal to θ [59].

Remark 4.2: In (4.17), note that σ_1, σ_4 and σ_6 can be chosen freely, even being able to assume high values. The reason is that ρ_1 and ρ_4 can remain positive as long as the gains from the ψ_1 and ψ_3 controls are correctly adjusted. However, for ρ_2 and ρ_3 to remain positive, the maximum value of σ_2 and σ_3 in our scheme need to be small. Thus, with the control signals' adjustment is possible to arbitrarily make the value β_1 close to zero, while it is impossible to decrease the values of β_2 by adjusting the control signals. This is a standard limitation found in the literature on underactuated synchronization works.

Remark 4.3: It is not possible to affirm the convergence of synchronization errors to zero because of the β value. If β_2 is small, then choosing a small β_1 from the control parameters adjustment, we will have a small β too. Thus, from the controller design parameters' choice, a synchronization error close to zero can be achieved, even in the presence of bounded disturbances in contexts where h_2 and h_3 are small.

4.3 SIMULATION

was made. The Matlab version 2020b on a Ryzen 7 1700 computer was used with the ode15s method with variable steps. The initial conditions were $x_m(0) = \left[\begin{array}{cccc} 1, & 2, & 0.5, & 0.5 \end{array}\right]$ and $x_s(0) = \left[\begin{array}{cccc} -2, & 1.5, & 1, & -0.5 \end{array}\right]$. The parameter was chosen as $\psi_1 = 1000$, $\psi_2 = 100$, $\psi_3 = 1000$ and $\psi_4 = 100$.

In this simulation, disturbances were introduced after forty seconds to show performance better. The disturbances were chosen as $h_{1m} = 0.25cos(10t)$, $h_{2m} = 0.05sin(2t)$, $h_{3m} = 0.03sin(5t)$, $h_{4m} = 0.1cos(7t) + 0.1sin(10t)$, $h_{1s} = 0.2sin(4t)$, $h_{2s} = 0.07cos(5t)$, $h_{3s} = 0.04cos(t)$ and $h_{4s} = 0.25sin(15t)$.

Figures (4.1 - 4.8) show the results of the synchronization. In Figures (4.1 - 4.4) it is shown that the master system has changed after the disturbances, which is not a problem, as disturbances in the master system change its behavior over time. However, even in the presence of disturbances, note that the synchronization occurred satisfactorily.

The Figures (4.5 - 4.8) show that the states where the control is present synchronize perfectly, going to zero over time, even in the presence of disturbances greater than the disturbances present in underactuated states. In Figures (4.6 - 4.7) from the introduction of the disturbances, it can be seen that the disturbances do not go to zero but remain bounded. This is expected since it is known that one of the limitations of the proposed synchronization scheme is the sensitivity to disturbances in states not actuated.

Table 4.1 shows the RMSE for different values of ψ. A small value of ψ corresponds to a large RMSE and transient. Also, large, ψ corresponds to small RMSE and transient, as shown in Table 4.1 and Figures (4.4 - 4.8).

Table 4.1 – Root of the mean square of the state errors for t = [0 20] seconds.

RMSE of state and message synchronization

ψ_1	ψ_2	ψ_3	ψ_4	$e_{1_{rms}}$	$e_{2_{rms}}$	$e_{3_{rms}}$	$e_{4_{rms}}$
1	0.1	1	0.1	0.282143	0.171103	0.165737	0.101571
10	1	10	1	0.215306	0.067242	0.112798	0.085127
100	10	100	10	0.152840	0.075527	0.067677	0.056228
1000	100	1000	100	0.060506	0.079952	0.038423	0.021668
10000	1000	10000	1000	0.019924	0.080653	0.032165	0.007106
100000	10000	100000	10000	0.006364	0.080880	0.031660	0.002269

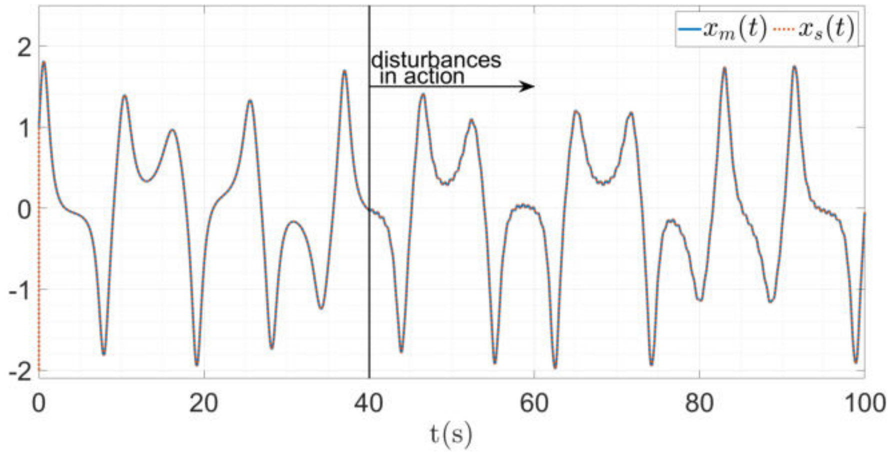

Figure 4.1 – Performance in synchronization of $x_m(t)$ and $x_s(t)$.

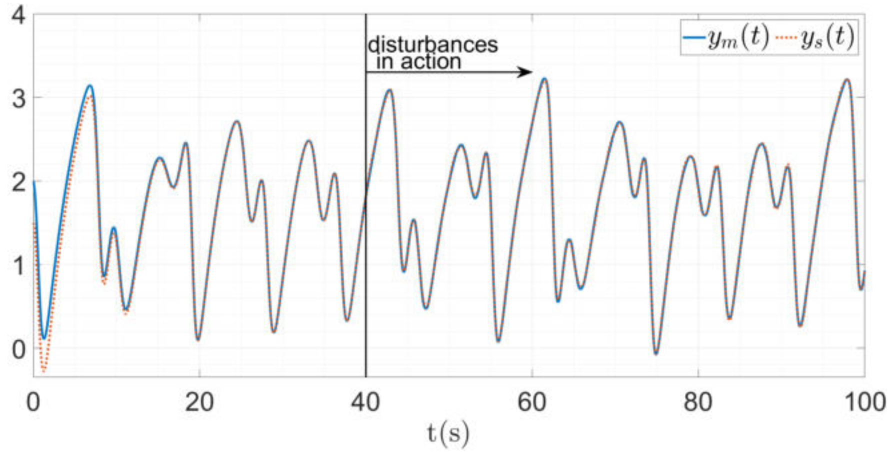

Figure 4.2 – Performance in synchronization of $y_m(t)$ and $y_s(t)$.

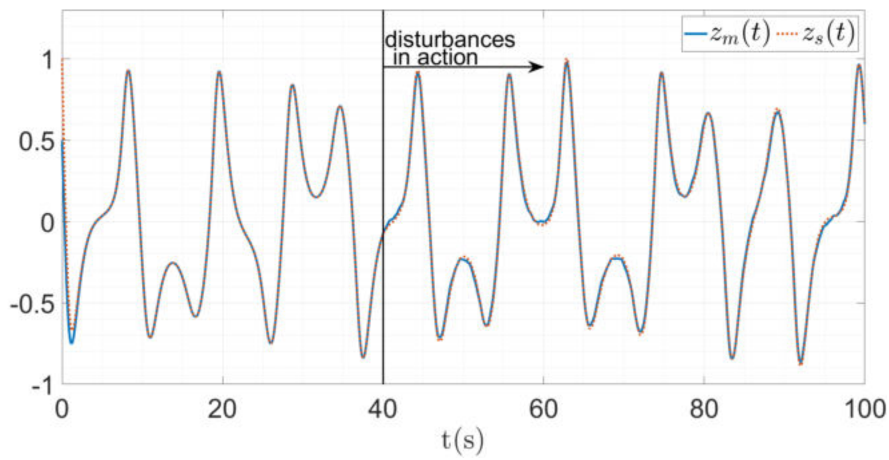

Figure 4.3 – Performance in synchronization of $z_m(t)$ and $z_s(t)$.

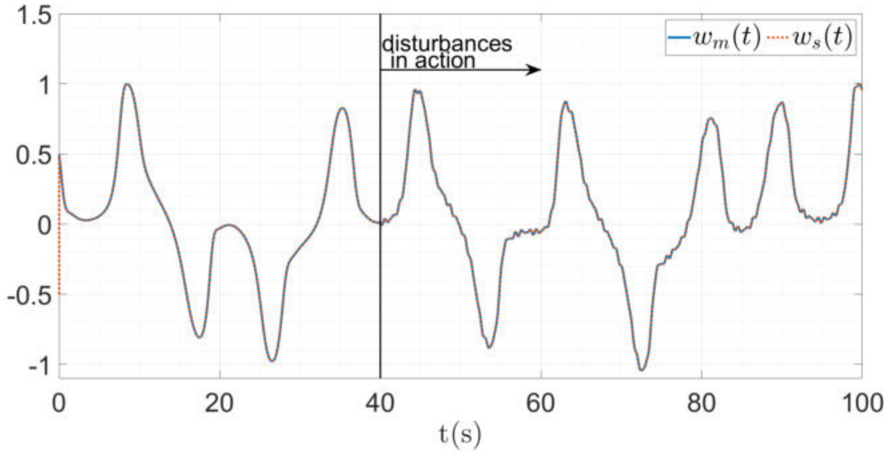

Figure 4.4 – Performance in synchronization of $w_m(t)$ and $w_s(t)$.

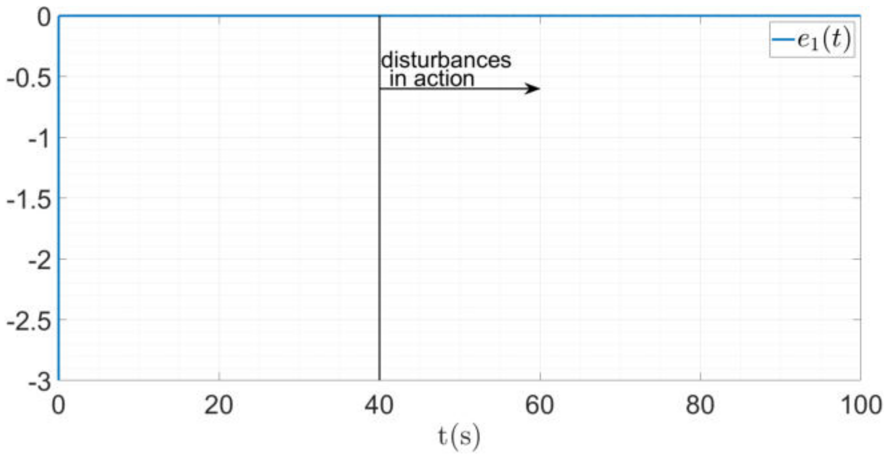

Figure 4.5 – Synchronization error $e_1(t)$.

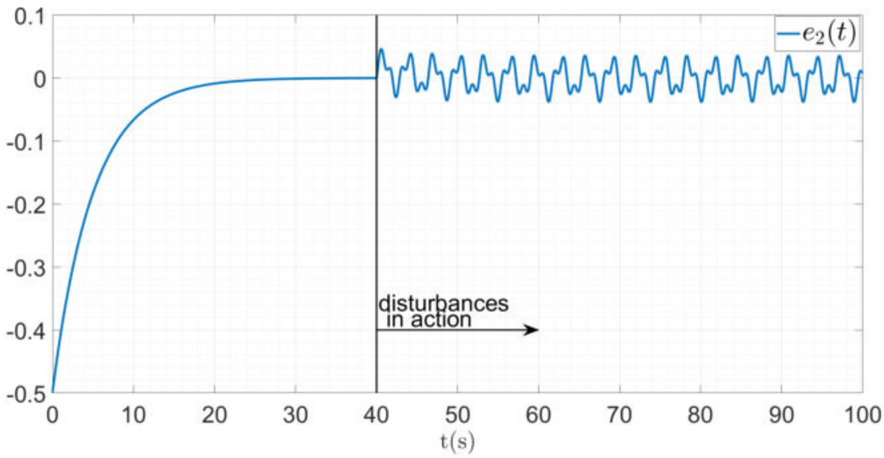

Figure 4.6 – Synchronization error $e_2(t)$.

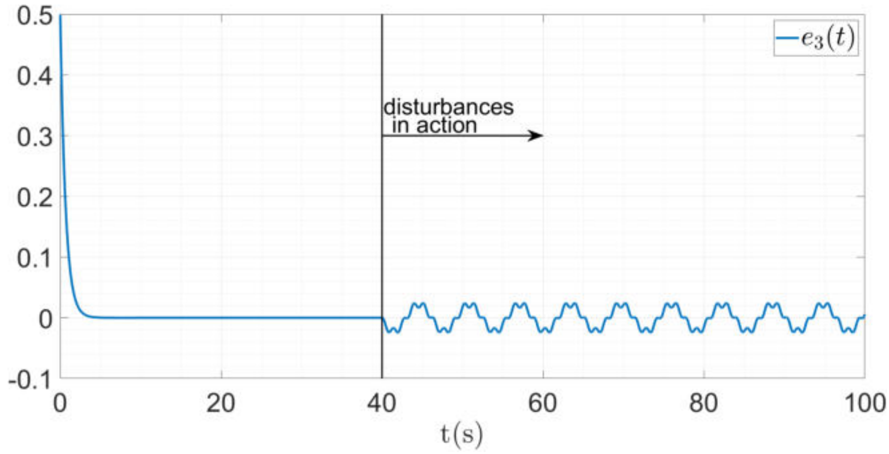

Figure 4.7 – Synchronization error $e_3(t)$.

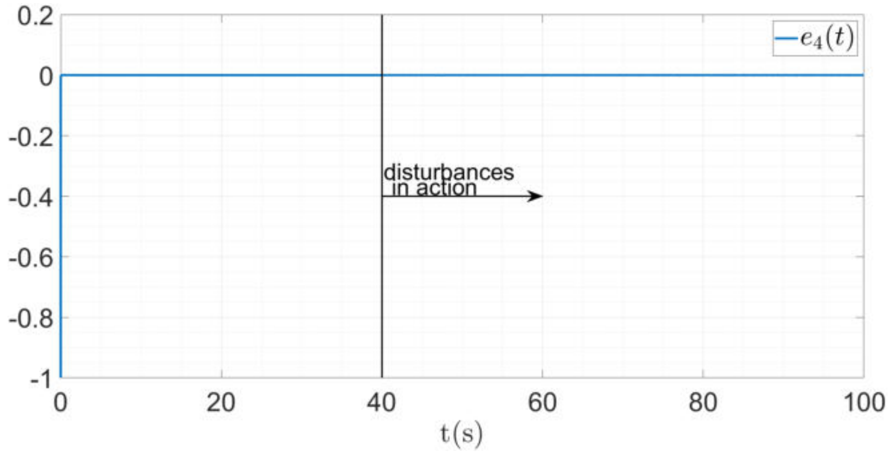

Figure 4.8 – Synchronization error $e_4(t)$.

4.4 CONCLUSION

In this chapter, a synchronization algorithm based on the Lyapunov stability theory has been proposed for a hyperchaotic financial system subject to bounded disturbances. It was shown that for a correct synchronization between master and slave systems, it is a sufficient condition to have control signals present in only two of the system states, and this is the main contribution of the chapter. The main limitations of the chapter are the sensitivity of the synchronization error to disturbances in underactuated states. Another limitation is the need for the control dimension to be two and not one.

5 CHAOS BASED CRYPTOGRAPHY

The research results of this chapter were published as a conference paper entitled "Chaos-based Cryptography Using an Underactuated Synchronizer" in [150]. *This chapter has extended and improved some parts compared to the original paper.*

In this chapter, a secure communication scheme based on the synchronization of a chaotic Liu system with a nontrivial Lyapunov function candidate has been proposed, which allows the control signal to act only on one state of the slave system. The proposal uses an underactuated control and has the advantages of being robust against disturbances (internal and external) and simple, which is essential because it leads to significant cost reductions when implemented using analog electronics. Simulation work, which considers the presence of disturbances, was used to validate the theoretical results and show the easy implementation of the proposed approach.

5.1 PROBLEM FORMULATION

Consider the Liu system [96]:

$$\begin{aligned}\dot{x} &= -ax(t) + ay(t)\\ \dot{y} &= bx(t) - kx(t)z(t)\\ \dot{z} &= -cz(t) + hx^2(t)\end{aligned} \tag{5.1}$$

where t is the time; $x\,(t)$, $y(t)$, and $z(t)$ are the system states; $a = 10$, $b = 40$, $c = 2,5$, $h = 4$, and $k = 1$ are the system parameters. Based on (5.1), let us define the master system as

$$\begin{aligned}\dot{x}_m &= -ax_m + ay_m\\ \dot{y}_m &= bx_m - kx_m z_m\\ \dot{z}_m &= -cz_m + hx_m^2\end{aligned} \tag{5.2}$$

and the slave system as

$$
\begin{aligned}
\dot{x}_s &= -ax_s + ay_s + h_1(t) \\
\dot{y}_s &= bx_s - kx_s z_s + h_2(t) + u \\
\dot{z}_s &= -cz_s + hx_s^2 + h_3(t)
\end{aligned} \tag{5.3}
$$

where x_m, y_m, and z_m are the states of the master system; x_s, y_s, and z_s are the states of the slave system; and h_1, h_2, and h_3 are disturbances present in the slave system.

Remark 5.1: The objective is to synchronize the systems (5.2) and (5.3), in which the slave system allows only one scalar control signal, acting only in one state. The importance of synchronization comes from the fact that the unmasking of messages in secure communication happens when the master and slave systems are synchronized.

Remark 5.2: Since system (5.1) is chaotic, its behavior is highly dependent on the initial conditions, and it also exhibits aperiodic behavior.

Assumption 5.1: It is assumed that the disturbances are bounded. More specifically,

$$
\begin{aligned}
|h_1(t)| &\leq \bar{h}_1 \\
|h_2(t)| &\leq \bar{h}_2 \\
|h_3(t)| &\leq \bar{h}_3
\end{aligned} \tag{5.4}
$$

$\forall t \geq 0$, being $\bar{h}_1$,$\bar{h}_2$, and $\bar{h}_3$ unknown constants.

Remark 5.3: It should be noted that we have added the bounded disturbances in (5.3) to analyze the performance of the proposed method in the presence of perturbations.

Fact 5.1: With the boundedness of the system (5.1) [96], the following inequalities can be established

$$
\begin{aligned}
|x_m(t)| &\leq \bar{x} \\
|y_m(t)| &\leq \bar{y} \\
|z_m(t)| &\leq \bar{z}
\end{aligned} \tag{5.5}
$$

$\forall t \geq 0$, where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are unknown positive constants

5.2 SYNCHRONIZATION ERROR EQUATION AND PROPOSED CONTROL SIGNAL

Let us define the synchronization error equations as

$$\begin{aligned} \dot{e}_1 &= \dot{x}_s - \dot{x}_m \\ \dot{e}_2 &= \dot{y}_s - \dot{y}_m \\ \dot{e}_3 &= \dot{z}_s - \dot{z}_m \end{aligned} \tag{5.6}$$

Substituting (5.2) and (5.3) into (5.6), results

$$\begin{aligned} \dot{e}_1 &= -ae_1 + ae_2 + h_1 \\ \dot{e}_2 &= be_1 - k\left(e_1e_3 + e_1z_m + e_3x_m\right) + h_2 + u \\ \dot{e}_3 &= -ce_3 + h\left(e_1^2 + 2e_1x_m\right) + h_3 \end{aligned} \tag{5.7}$$

Once the error dynamic was defined, the next step is to design a suitable control signal u, which is summarized in the following theorem.

Theorem 5.1: Consider the master and slave systems described in (5.2), (5.3) and the following control law

$$u = -\psi_1 e_2 - \psi_2 e_2^3 \tag{5.8}$$

where ψ_1 e ψ_2 are a user-defined constant. Then, the synchronization error converges in finite time to the compact set $\Omega = \{e(t) \in R^3 \mid V(e) \leq \theta\}$, where θ is a positive small constant, being $\theta := \frac{\beta}{\alpha}$.

Proof 5.1: Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\left(e_1^4 + e_2^2 + e_3^2\right) \tag{5.9}$$

The time-derivative of (5.9) along the error trajectories results in

$$\dot{V} = e_1^3\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 \tag{5.10}$$

Substituting (5.7) into (5.10), gives

$$
\begin{aligned}
\dot{V} &= e_1^3\left(-ae_1+ae_2+h_1\right)+e_2\left[be_1-k\left(e_1e_3+e_1z_m+e_3x_m\right)+h_2+u\right] \\
&\quad + e_3\left[-ce_3+h\left(e_1^2+2e_1x_m\right)+h_3\right]
\end{aligned}
\tag{5.11}
$$

Substituting (5.7) into (5.11), we have

$$
\begin{aligned}
\dot{V} &= -ae_1^4+ae_1^3e_2+e_1^3h_1+be_1e_2-ke_2\left(e_1e_3+e_1z_m+e_3x_m\right) \\
&\quad - e_2\left(\psi e_2+e_2^3\right)+e_2h_2-ce_3^2+he_3\left(e_1^2+2e_1x_m\right)+e_3h_3
\end{aligned}
\tag{5.12}
$$

Noting that
$e_1^3h_1 \leq \frac{1}{4}e_1^4+\left(e_1\bar{h}_1\right)^2 \leq \frac{1}{2}e_1^4+8\left(\bar{h}_1\right)^4$, $e_1e_2\left(b+kz_m\right) \leq \frac{1}{2}\left[e_1^2+e_2^2\left(b+k\bar{z}\right)^2\right]$, $e_2e_3(kx_m)^2 \leq \frac{1}{2}\left[e_2^2(k\bar{x})^2+e_3^2\right]$, $-ke_3e_1e_2 \leq \frac{e_3^2}{2}+\frac{1}{2}(ke_1e_2)^2 \leq \frac{e_3^2}{2}+\frac{1}{4}\left(k^4e_2^4+e_1^4\right)$, $e_2h_2 \leq \frac{1}{2}\left(e_2^2+\bar{h}_2^2\right)$, $he_1^2e_3 \leq \frac{1}{2}\left(e_3^2+h^2e_1^4\right)$, $2he_1e_3x_m \leq \frac{1}{2}e_3^2+2h^2e_1^2\bar{x}^2 \leq \frac{1}{2}e_3^2+\frac{1}{4}\left(e_1^4+8h^4\bar{x}^4\right)$, $e_3h_3 \leq \frac{1}{2}\left(e_3^2+\bar{h}_3^2\right)$, then (5.12) implies

$$
\begin{aligned}
\dot{V} &\leq -e_1^4\left\{a-\left[\frac{3}{2}+\frac{1}{2}h^2\right]\right\}-e_3^2\left(c-2\right)-e_2^2\left\{\psi-\frac{1}{2}\left[\left(b+k\bar{z}\right)^2+\left(k\bar{x}\right)^2+\frac{k^4}{2}+1\right]\right\} \\
&\quad + \frac{1}{2}\left[16\left(\bar{h}_1\right)^4+2\bar{h}_2^2+4h^4\bar{x}^4+\bar{h}_3^2\right]
\end{aligned}
\tag{5.13}
$$

Choosing a suitable ψ such that $\psi-\frac{1}{2}\left[\left(b+kz_m\right)^2+\left(kx_m\right)^2+\frac{k^4}{2}+1\right] \geq \alpha$, and considering that $\beta=\frac{1}{2}\left[16\left(\bar{h}_1\right)^4+2\bar{h}_2^2+4h^4\bar{x}^4+\bar{h}_3^2\right]$, and $0.5 \geq \alpha \geq 0.1$, we can conclude

$$
\dot{V} \leq -\alpha V+\beta
\tag{5.14}
$$

This implies that $\dot{V}<0$ when $V>\frac{\beta}{\alpha}:=\theta$. Since θ is a constant, it can be established that V and, hence, the synchronization error is bounded. If for any reason e leaves the residual set Ω, $\dot{V}$ becomes negative definite and forces the convergence of e to the residual set Ω. In addition, it can be concluded that the convergence is in a finite time owing to the particular form of (5.13) [1], This concludes the proof.

By using Lemma 3.2.4 [1], it can be established that:

$$
\begin{aligned}
&V(t) \leqslant e^{-\alpha(t-t_0)}V(t_0) \\
&\forall t \geq t_0 \geqslant 0
\end{aligned}
\tag{5.15}
$$

Assuming that $t_0=0$, then $V(t) \leqslant V(0)e^{-\alpha t}$

Remark 5.4: It was impossible to design a synchronizer based on a scalar control using a trivial candidate (a quadratic function of the synchronization errors) and standard Lyapunov arguments. However, this obstacle is overcome here by choosing a nontrivial candidate (5.8). This selection allows us to decrease the complexity of the synchronizer. Hence, the proposed synchronizer uses uncomplicated feedback based on a scalar control.

5.3 SIMULATION

Matlab version 2020b on a Ryzen 7 1700 computer was used for the simulations. The initial conditions were $x_m(0) = \begin{bmatrix} 0.2, & -0.3, & 0.4 \end{bmatrix}$ and $x_s(0) = \begin{bmatrix} 20, & -30, & 100 \end{bmatrix}$. For synchronizing the master and slave systems we used the control law (5.7) with $\psi = 100$.

Figures (5.1 - 5.3) show the results of the synchronization. Note that the slave system follows the master to make the synchronization errors close to zero. Although control is present only in the second state, all three states show satisfactory synchronization responses. To check the robustness of the proposed method, the bounded disturbances $h_1 = 1.5\text{sen}(2t)$, $h_2 = 2\text{cos}(3t)$, and $h_3 = \text{sen}(4t)$ were introduced at $t = 10s$. Note in Figures (5.1 - 5.3) that the performance of the proposed synchronizer is practically not affected by the presence of disturbances, even in the unmatched case.

Finally, the message was encoded and decoded using the proposed approach. Figures (5.3 - 5.6) show the results of our simulations. The chosen message was $msg = msg_1 + msg_2$, where $msg_1 = sin(2t) + 0.5sin(8t) + 0.3cos(20t)$ and msg_2 is a square signal of amplitude 1 and period 2π. This message was added to the first state of the master system.

The Figures (5.3 - 5.6) show that the synchronizer is robust, even when the introduced unmatched disturbances are present. Furthermore, from Figure (5.4), it can be observed that the encoded message differs from the original message, which ensures its confidentiality. Note also from Figures (5.5 - 5.6) that the difference between the original and recovered message is practically nonexistent. Furthermore, it is possible to improve the message reconstruction by increasing ψ. It is also possible to decrease the time of convergence of the synchronization error by chaotic circuit scaling.

Table 5.1 shows the RMSE for different values of ψ. A small value of ψ corresponds to a large RMSE and transient. In addition, ψ corresponds to small transient RMSE, as shown in Table 5.1 and in Figure 5.6.

Table 5.1 – Root of the mean square of the errors of the difference between the recovered and original messages for t = [0 20] seconds.

RMSE of the errors of the recovered and original messages.					
ψ_1	10	100	1000	10000	100000
e_{rms}	3.638192	3.078710	1.722255	1.103631	0.976269

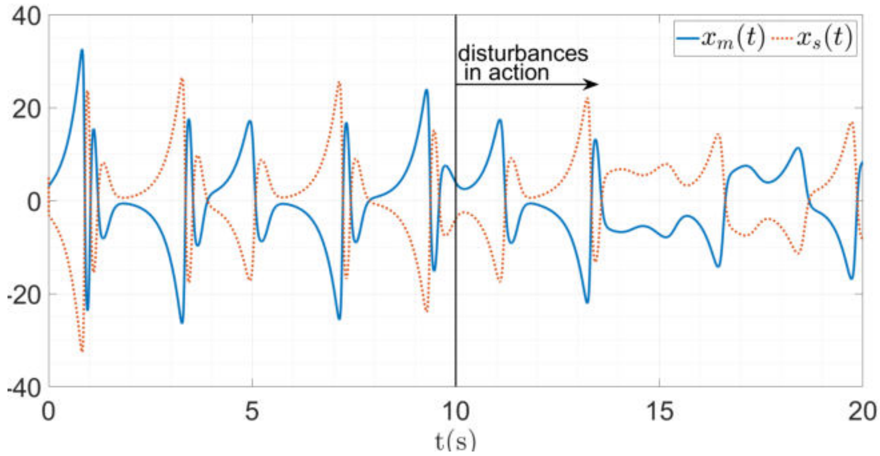

Figure 5.1 – Performance in the synchronization of $x_m(t)$ and $x_s(t)$.

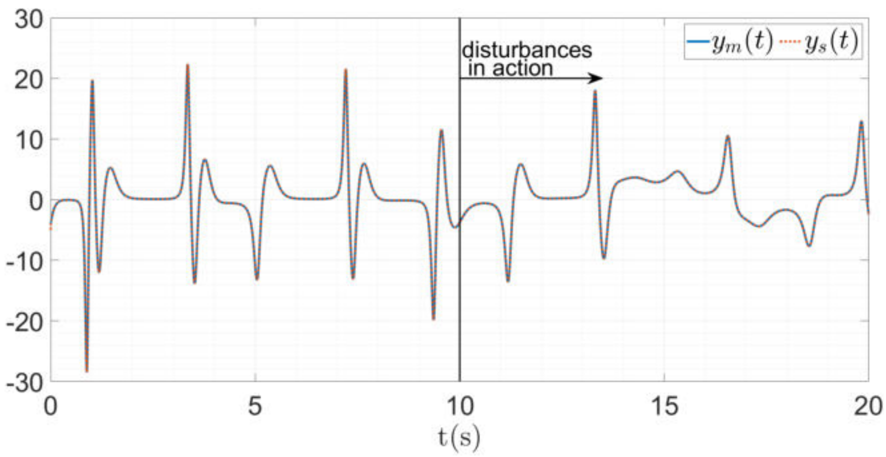

Figure 5.2 – Performance in the synchronization of $y_m(t)$ and $y_s(t)$.

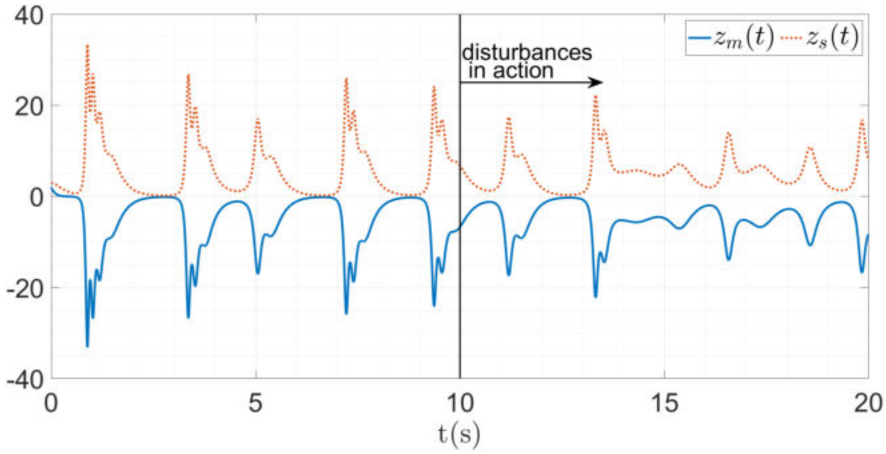

Figure 5.3 – Performance in the synchronization of $z_m(t)$ and $z_s(t)$.

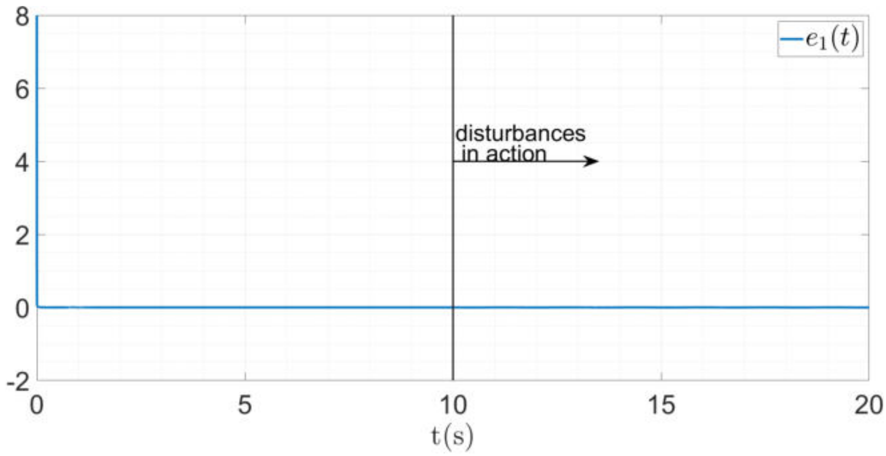

Figure 5.4 – Original and encrypted messages.

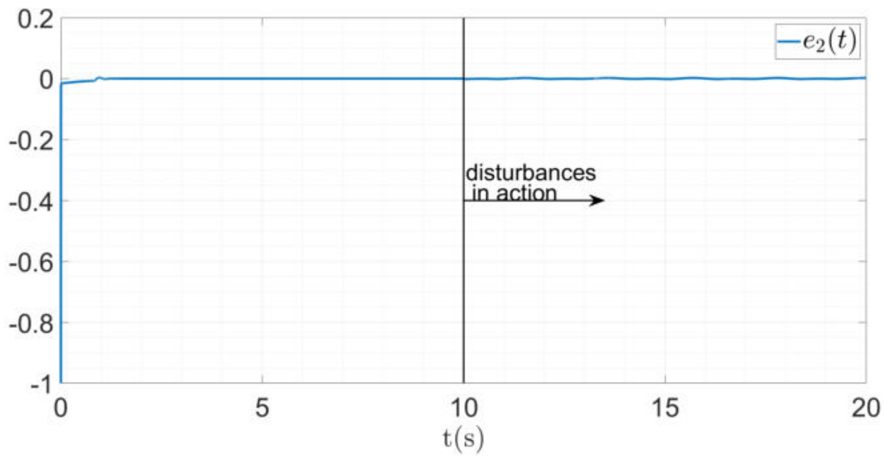

Figure 5.5 – Original and decoded messages.

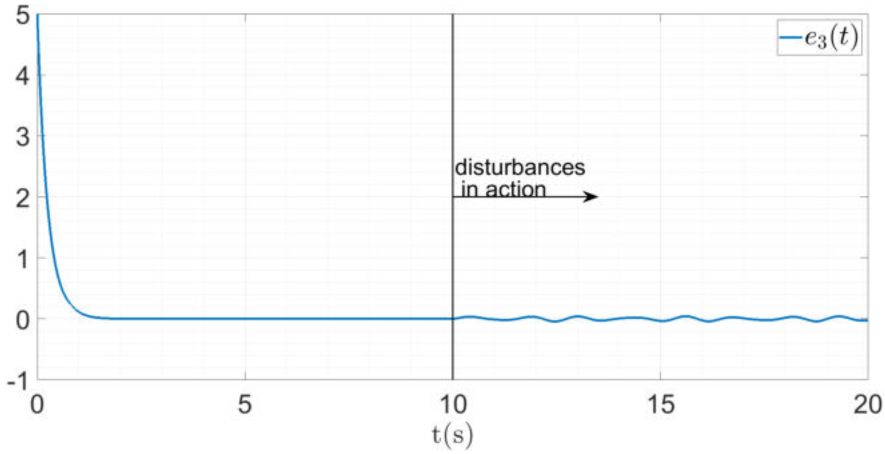

Figure 5.6 – Difference between the retrieved and original messages.

5.4 CONCLUSIONS

In this chapter, a synchronization algorithm based on the Lyapunov stability theory has been proposed for a Liu system subject to bounded disturbances. It was proved and validated via simulations that it is necessary to use the control signal in only one of the states to achieve the complete synchronization of the master and slave systems, which is the main advantage of the proposed method. The system presents as limitations the synchronization error's sensitivity to the presence of disturbances in non-actuated states. The proposed methods' effectiveness to encode and restore the message both theoretically and in the simulations was also demonstrated.

6 CHAOS SYNCHRONIZATION AND ITS APPLICATION IN PARALLEL CRYPTOGRAPHY

The research results of this chapter were published as a conference paper entitled "Chaos Synchronization and its Application in Parallel Cryptography" in [151]. *This chapter has extended and improved some parts compared to the original paper*.

This chapter considers an underactuated synchronization scheme for the Chua system with application in secure analog communication. Based on Lyapunov's theory, a control scheme is presented, which in contrast to the most commonly found in literature, uses a proportional control signal in only one of the state equations of the slave system. The main advantages of the proposed secure communication scheme are its simplicity and robustness against internal and external disturbances. Disturbances were considered in the stability analysis. These peculiarities are of great significance in practical applications. To validate the proposed approach, we considered a synchronization of two chaotic Chua circuits in the presence of disturbances.

6.1 PROBLEM STATEMENT

Consider the following nonlinear system [106], [152]:

$$\begin{aligned} C_1\frac{\partial V_1}{\partial t} &= \frac{V_2 - V_1}{R} - f(V_1) + I_0 \\ C_2\frac{\partial V_2}{\partial t} &= \frac{V_1 - V_2}{R} + I_L \\ \frac{\partial I_L}{\partial t} &= \frac{-V_2}{L} - \frac{R_L I_L}{L} \end{aligned} \tag{6.1}$$

where t is the time.

The system (6.1) is obtained by applying Kirchhoff's laws to Chua's circuit in which V_1 and V_2 are the voltage across the capacitor C_1 and C_2, I_L is the current flowing through the inductor $L{\cdot}f(V_N) = f(V_1)$, $f(V_1) = m_0 V_1 + \frac{1}{2}(m_1 - m_0)[|V_1 + B_P \mid - \mid V_1 - B_P \;|]$ expresses the current characterized by the nonlinear resistance of Chua's circuit. Constants m_0, m_1, and B_P are $-7.87{\cdot}10^{-4}$ S, $-1.4357{\cdot}10^{-3}$ S, and 1 V, respectively; constant R_L, which defines the internal resistance of the circuit's inductor is 2 Ω; C_1, C_2, L, R, and I_0 are

15 nF, 150 nF, 10 mH, 1 kΩ, and 100 μA, respectively. For more details see [106], [152].

From system (6.1), changing the original variables to state variables $x(t) = V_1$, $y(t) = V_2$, and $z(t) = I_L$ and adopting some changes [106], we get the master system

$$\begin{aligned} \dot{x}_m(t) &= m[y_m(t) - x_m(t)] + \theta_1 x_m(t) + \theta_2[|x_m(t) + \beta_P| - |x_m(t) - \beta_P|] + d \\ \dot{y}_m(t) &= x_m(t) - y_m(t) + z_m(t) \\ \dot{z}_m(t) &= -\beta y_m(t) - \gamma z_m(t) \end{aligned} \tag{6.2}$$

and the slave system is

$$\begin{aligned} \dot{x}_s(t) &= m[y_s(t) - x_s(t)] + \theta_1 x_s(t) + d + h_{1s}(t) + \theta_2[|x_s(t) + \beta_P| - |x_s(t) - \beta_P|] \\ &\quad + u(t) \\ \dot{y}_s(t) &= x_s(t) - y_s(t) + z_s(t) + h_{2s}(t) \\ \dot{z}_s(t) &= -\beta y_s(t) - \gamma z_s(t) + h_{3s}(t) \end{aligned} \tag{6.3}$$

where $m = 10$, $\theta_1 = 7.87$, $\theta_2 = 3.23$, $\beta_P = 15$, and $\gamma = 0.03$. The state variables of the master system are $x_m(t)$, $y_m(t)$, and $z_m(t)$; the state variables of the slave system are, $x_s(t)$, $y_s(t)$, and $z_s(t)$; the slave system's disturbance are $h_1(t)$, $h_2(t)$, and $h_3(t)$; and the control signal is $u(t)$.

This study aims to synchronize systems (6.2) and (6.3), independent from the disturbance and initial conditions, where the slave system will only be influenced by one control signal, which can be found in the first state equation.

Remark 6.1: As the system (6.1) is assumed to be chaotic, its behavior is aperiodic and sensitive to initial conditions. Thus, changes in the values of initial conditions will affect the behavior of the system over time.

Assumption 6.1: We assume that the disturbances are bounded. More precisely, if:

$$\begin{aligned} |h_{1s}(t)| &\leq \bar{h}_1 \\ |h_{2s}(t)| &\leq \bar{h}_2 \\ |h_{3s}(t)| &\leq \bar{h}_3 \end{aligned} \tag{6.4}$$

$\forall t \geq 0$, being $\bar{h}_1$, $\bar{h}_2$, and $\bar{h}_3$ unknown positive constants..

Remark 6.2: System (6.3) shows explicit disturbance is rare to find in the literature. This allows us to evaluate these uncertainties over the boundedness and convergence of the synchronization errors. Disturbances are inevitable in practical implementations because of the components' tolerance, environmental conditions, and electromagnetic noise, among others.

Fact 6.1 [106]: Once the system is bounded (6.2), then

$$\begin{aligned} |x_m(t)| &\leq \bar{x} \\ |y_m(t)| &\leq \bar{y} \\ |z_m(t)| &\leq \bar{z} \end{aligned} \tag{6.5}$$

$\forall t \geq 0$, where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are unknown positive constants.

6.2 SYNCHRONIZATION ERROR AND PROPOSED SIGNAL CONTROL

Synchronization errors of the system are defined by

$$\begin{aligned} e_1(t) &= x_s(t) - x_m(t) \\ e_2(t) &= y_s(t) - y_m(t) \\ e_3(t) &= z_s(t) - z_m(t) \end{aligned} \tag{6.6}$$

Then, dynamic equations of the errors can be obtained from the time-derivative of (6.6), by using systems (6.2) and (6.3). Thus

$$\begin{aligned} \dot{e}_1 &= m(e_2 - e_1) + \theta_1 e_1 + g_s - g_m + h_{1s} + u \\ \dot{e}_2 &= -e_2 + e_1 + e_3 + h_{2s} \\ \dot{e}_3 &= -\gamma e_3 - \beta e_2 + h_{3s} \end{aligned} \tag{6.7}$$

where $g_m = \theta_2(|x_m + \beta_P| - |x_m - \beta_P|)$ and $g_s = \theta_2(|x_s + \beta_P| - |x_s - \beta_P|)$.

Theorem 6.1: Consider the master and slave system described in (6.2) and (6.3) and the proportional control law defined by

$$u = -\psi e_1 \tag{6.8}$$

where ψ is a positive constant defined by the designer. Then, the synchronization error converges in finite time to the compact set $\Omega = \{e \in \Re^3 \mid ||e|| \leq \theta\}$, $\theta := \sqrt{\frac{\zeta}{\rho}}$, $\zeta = \zeta_1 + \zeta_2$, $\rho := min\{\rho_1, \rho_2, \rho_3\}$, and

$$
\begin{aligned}
&\rho_1 = \psi + m - \theta_1 - 0.5\sigma_1 - 1 - m^2 \\
&\rho_2 = 0.75 - 0.5\sigma_2 \\
&\rho_3 = \gamma\beta^{-1} - 0.5\sigma_3 \\
&\zeta_1 = 0.5\sigma_1^{-1}(\bar{h}_1^2 + 16\theta_2^2) \\
&\zeta_2 = 0.5(\sigma_2^{-1}\bar{h}_2^2 + \sigma_3^{-1}\beta^{-2}\bar{h}_2^2)
\end{aligned}
\tag{6.9}
$$

where σ_i $(i = 1, ..., 3)$ are arbitrary positive constants.

Proof 6.1: Consider the following Lyapunov function candidate

$$
V(t) = 0.5\left[e_1^2(t) + e_2^2(t) + \beta^{-1}e_3^2(t)\right] \tag{6.10}
$$

Deriving (6.10) based on the time over the error trajectory, results in:

$$
\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + \beta^{-1}e_3\dot{e}_3 \tag{6.11}
$$

Replacing (6.7) in (6.11), we get

$$
\begin{aligned}
\dot{V} = {} & e_1[m(e_2 - e_1) + \theta_1 e_1 + g_s - g_m + h_{1s} + u] + \\
& e_2\left(-e_2 + e_1 + e_3 + h_{2s}\right) + \frac{e_3}{\beta}\bigg(- \gamma e_3 - \beta e_2 + h_{3s}\bigg)
\end{aligned}
\tag{6.12}
$$

Replacing (6.8) in (6.12), we get

$$
\begin{aligned}
\dot{V} = {} & -e_1^2(\psi + m - \theta_1) - e_2^2 - \frac{\gamma e_3^2}{\beta} + e_1 h_{1s} + e_2 h_{2s} \\
& + \frac{e_3 h_{3s}}{\beta} + e_1(g_s - g_m) + e_1 e_2(1 + m)
\end{aligned}
\tag{6.13}
$$

We analyze the case in which $\dot{V} \leq 0$, so that (6.13) is analyzed in the case of inequality. Notice that $-2\beta_P \leq |x_s + \beta_P| - |x_s - \beta_P| \leq 2\beta_P$, $-2\beta_P \leq |x_m + \beta_P| - |x_m - \beta_P| \leq 2\beta_P$, and from Young's inequalities [147]:

$$
\begin{aligned}
&e_1(h_1 + 4\theta_2) \leq 0.5[\sigma_1 e_1^2 + \sigma_1^{-1}(\bar{h}_1^2 + 16\theta_2^2)] \\
&e_2 h_2 \leq 0.5(\sigma_2 e_2^2 + \sigma_2^{-1}\bar{h}_2^2) \\
&\beta^{-1}e_3 h_3 \leq 0.5(\sigma_3 e_3^2 + \sigma_3^{-1}\beta^{-2}\bar{h}_3^2) \\
&e_1 e_2(1 + m) \leq e_1^2(1 + m^2) + 0.25e_2^2
\end{aligned}
\tag{6.14}
$$

By using (6.14), (6.13) implies

$$\dot{V} = -e_1^2(\psi + m - \theta_1 - 0.5\sigma_1 - 1 - m^2) - e_2^2(0.75 - 0.5\sigma_2) - \\ e_3^2(\gamma\beta^{-1} - 0.5\sigma_3) + 0.5[\sigma_1^{-1}(\bar{h}_1^2 + 16\theta_2^2) + \sigma_2^{-1}\bar{h}_2^2 + \sigma_3^{-1}\beta^{-2}\bar{h}_2^2] \tag{6.15}$$

Replacing (6.9) in (6.15), results

$$\dot{V} \leq -e_1^2\rho_1 - e_2^2\rho_2 - e_3^2\rho_3 + \zeta_1 + \zeta_2 \tag{6.16}$$

Observe that ψ and σ_i $(i = 1, ..., 3)$ can be selected so that $\rho_j > 0$ $(j = 1, ..., 3)$. Thus, (6.16) can be written as

$$\dot{V} \leq -\rho \left|\left|e\right|\right|^2 + \zeta \tag{6.17}$$

Based on (6.17) we reach the situation where $\dot{V} < 0$, when $||e|| > \theta$, being θ constant. Hence, it can be stated that the synchronization error is bounded. Therefore, if under any circumstance $||e||$ leaves the compact set Ω, $\dot{V}$ becomes negative definite and forces the convergence of the synchronization error to a ball with radius equal θ, as shown in (6.17), [1].

Remark 6.3: It can be seen from the proof that bounded disturbances are being considered in the Lyapunov stability analysis. Note that the synchronization errors are bounded even with a simple underactuated proportional control, in contrast to [106].

Remark 6.4: It can be seen that σ_1 can be freely chosen by the designer, and thus the value of ζ_1. This freedom is because ρ_1 can be arbitrarily adjusted to be positive if ψ is adjusted as well.

Remark 6.5: Because of the structure of ρ_1 and ρ_2, σ_2 and σ_3 can not be freely chosen. Since the maximum value for σ_2 and σ_3 are restricted, thus there are restrictions to the choice of ζ_2.

6.3 SIMULATION

A computer simulation to validate the theoretical results was performed in a computer with Ryzen 7 1700 processor and Matlab 2020b Software with the Ode45 integration method. For the master system the following initial conditions were considered $x_m(0) = 0.2$, $y_m(0) = 0.2$, and $z_m(0) = 0.2$. For the slave system $x_s(0) = 0.3$, $y_s(0) = 0$, and $z_s(0) = -0.2$. White Gaussian noises were used as disturbances so that h_1, h_2, and h_3 have the power of noise of $-3dBW$, $-10dBW$, and $-3dBW$, respectively. The synchronization of the systems (6.2) and (6.3) is achieved through the implementation of the control law (6.8) and by establishing the parameter $\psi = 1000$.

In Figures (6.1 - 6.6), the synchronization results using Matlab are displayed. Based on the results shown in these figures, it is possible to notice that the master and slave systems get approximated. The results allow us to affirm that the synchronization error is small. The results show that even though the control is applied to only one of the states (x), synchronization is effectively achieved in the three states. However, synchronization errors are not equal to zero due to the disturbances. This result was expected since, in Theorem 1, it was not proved that the synchronization errors converge to zero. It was proved that they are bounded.

Applying the proposed scheme to secure communication we have as the transmitted messages $m_1(t) = 0.2square(5t)$, $m_2(t) = 0.05cost(0.5t) + 0.025sin(10t)$, and $m_3(t) = 0.1sin(20t) + 0.2sin(8t)$. In this scheme, the master systems encoded the message and the slave system decoded the message. Note that the amplitude of the messages represents approximately 5% of the maximum amplitude of the master states. Assume that the encoded messages are $s_1(t) = x_m(t)+m_1(t)$, $s_2(t) = y_m(t)+m_2(t)$, and $s_3(t) = z_m(t)+m_3(t)$. The decoded messages $\hat{m}_1(t) = s_1(t) - x_s(t)$, $\hat{m}_2(t) = s_2(t) - y_s(t)$, and $\hat{m}_3(t) = s_3(t) - z_s(t)$ and the message errors are $\tilde{m}_i(t) = \hat{m}_i(t) - m_i(t)$ $(i = 1, ..., 3)$. A natural consequence of the synchronization errors being bounded is that the message errors are also bounded. In this case, due to having three simultaneous messages being transmitted in the scheme, it is a parallel encryption system.

Figures (6.7 - 6.9) show the original, decoded, and encoded messages. We can see that encrypted messages are different from the original messages. In addition, the retrieved messages are very close to the original messages. Figures 6.10 - 6.12 show that the errors in the messages are small. These results were expected since it was theoretically shown that synchronization errors are bounded.

Table 6.1 shows the RMSE for different values of ψ. A small value of ψ corresponds to a large RMSE and transient. In addition, ψ corresponds to small transient RMSE, as shown in Table 6.1 and in Figure (6.4 - 6.6) and (6.10 - 6.12).

Table 6.1 – RMSE of state and message synchronization for t = [0 20] seconds

RMSE of state synchronization and message errors in the proposed algorithm.

ψ_1	e_{s1}	e_{s2}	e_{s3}	e_{m1}	e_{m2}	e_{m3}
10	0.062867	0.070627	0.277385	0.214112	0.081983	0.322913
100	0.004	0.037418	0.129306	0.200408	0.056139	0.209161
1000	0.002104	0.037522	0.124638	0.199730	0.056667	0.205866
10000	0.001214	0.036046	0.122733	0.199750	0.055300	0.204911
100000	0.000385	0.035271	0.121752	0.199786	0.054588	0.204505

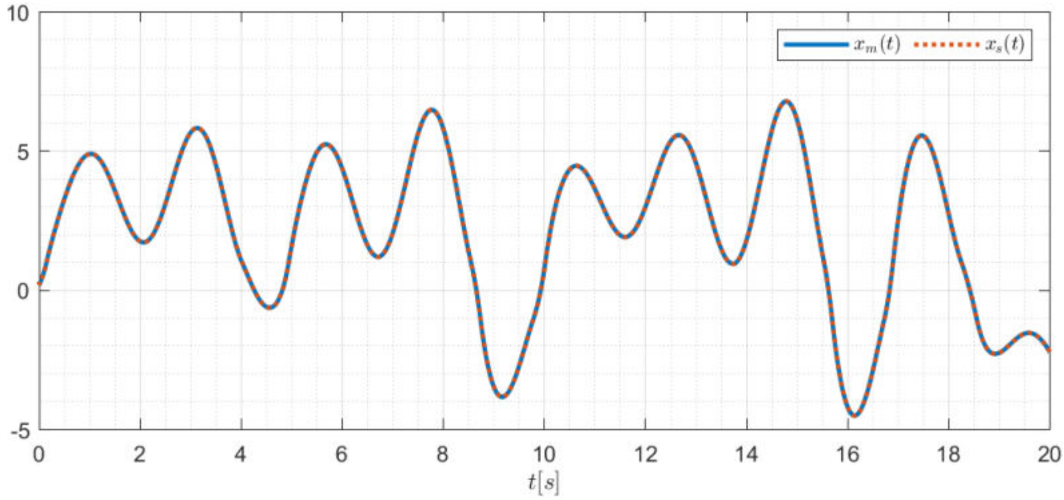

Figure 6.1 – Synchronization of x state.

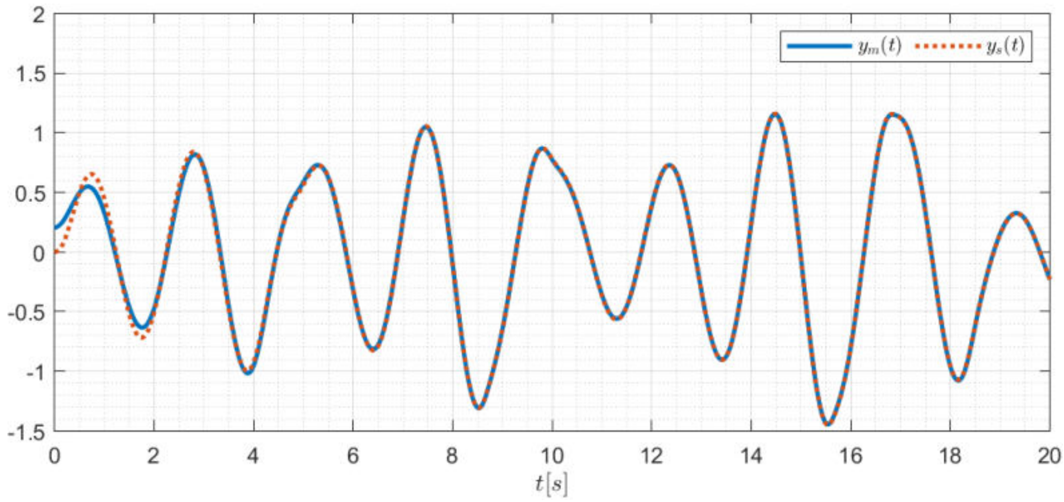

Figure 6.2 – Synchronization of y state.

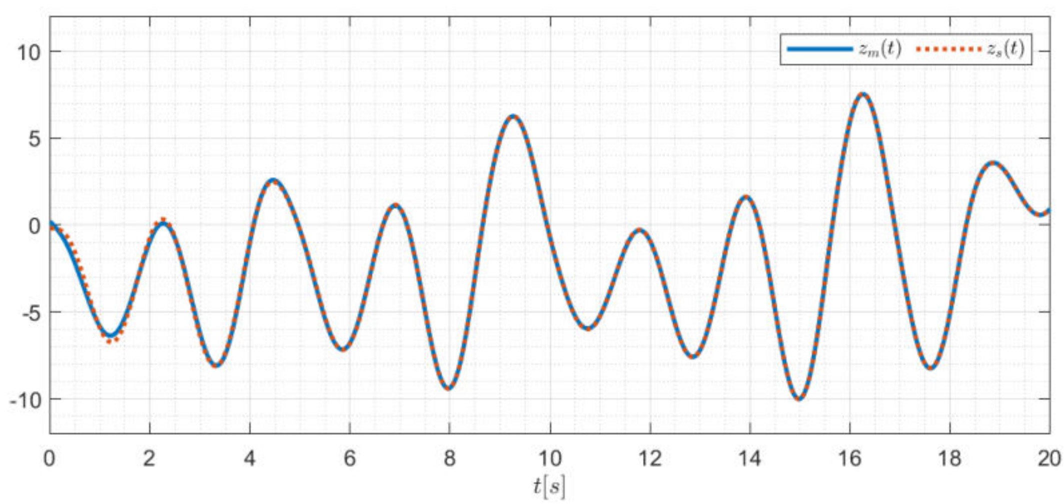

Figure 6.3 – Synchronization of z state.

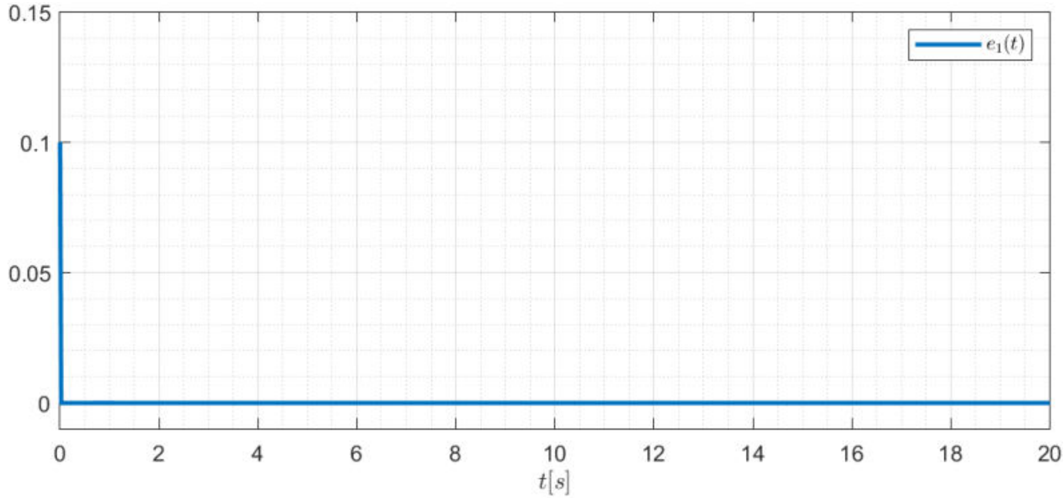

Figure 6.4 – Synchronization error of x state.

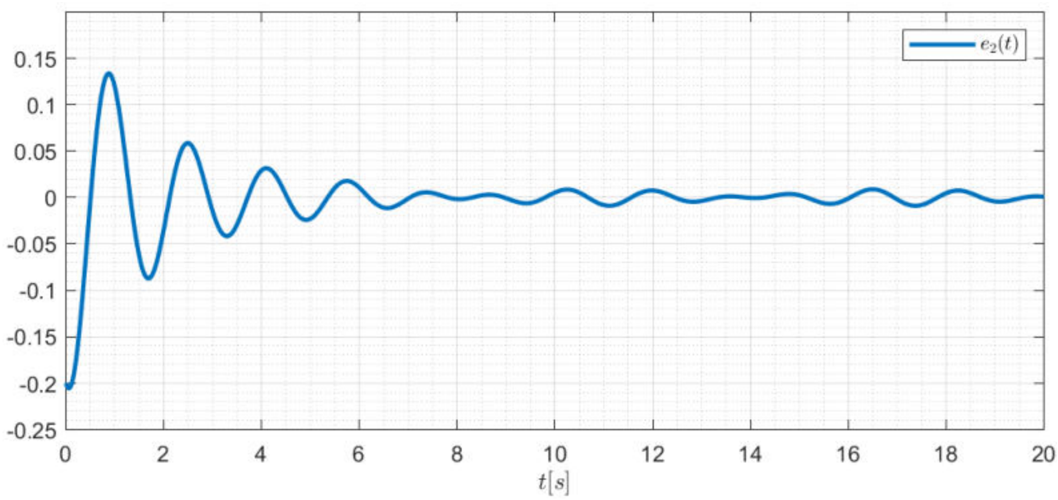

Figure 6.5 – Synchronization error of y state.

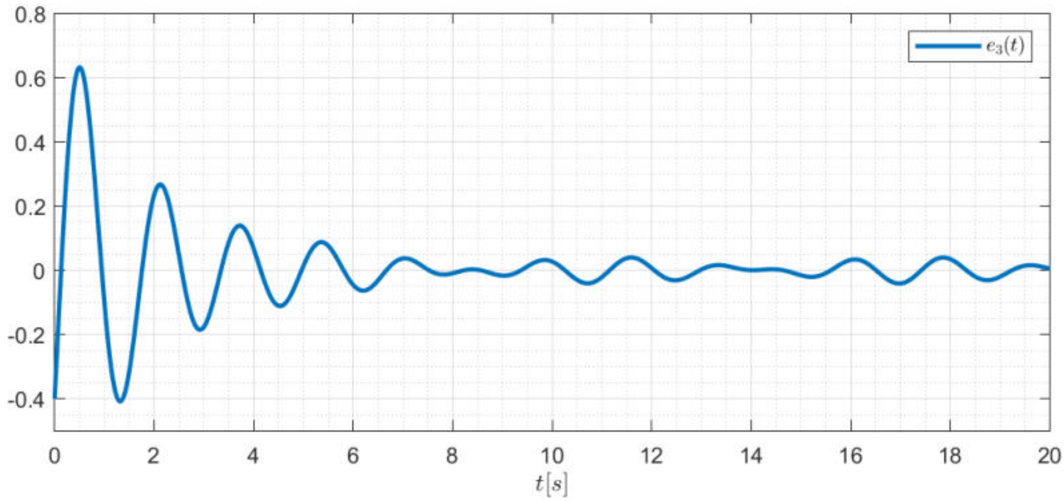

Figure 6.6 – Synchronization error of z state.

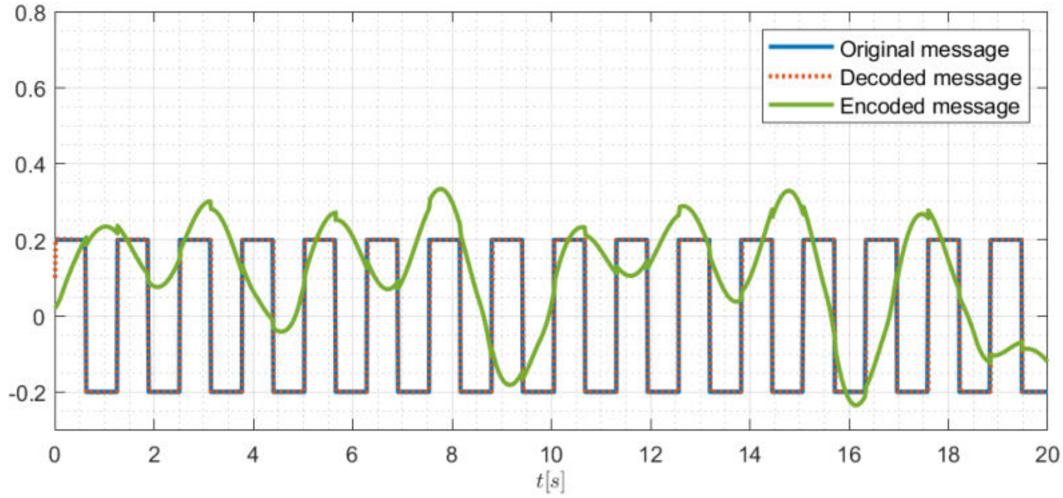

Figure 6.7 – Encoded and decoded messages $(m_1(t), \hat{m}_1(t))$.

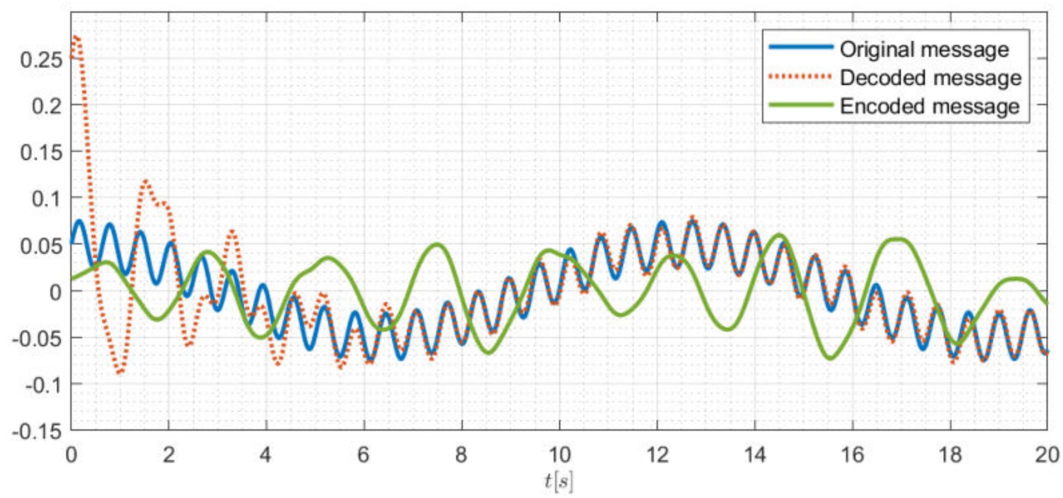

Figure 6.8 – Encoded and decoded messages $(m_2(t), \hat{m}_2(t))$.

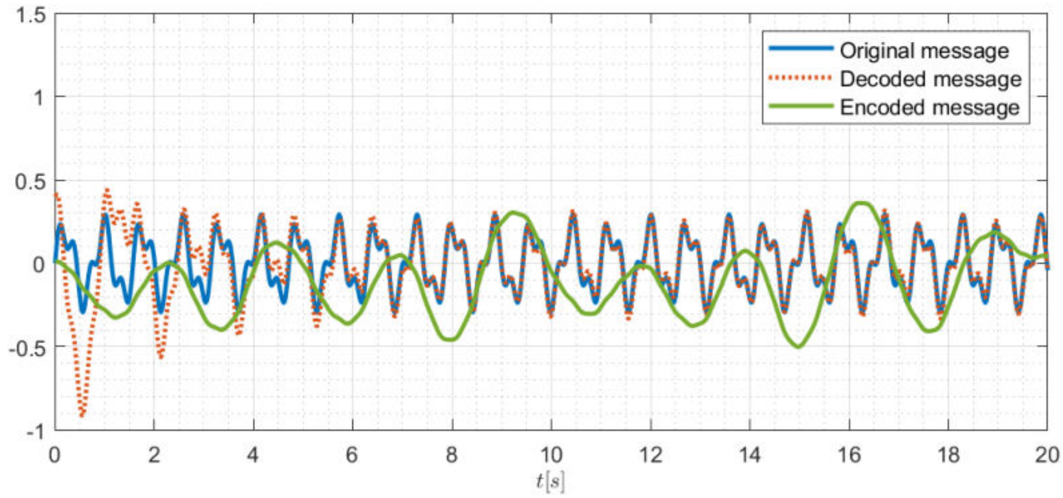

Figure 6.9 – Encoded and decoded messages $(m_3(t), \hat{m}_3(t))$.

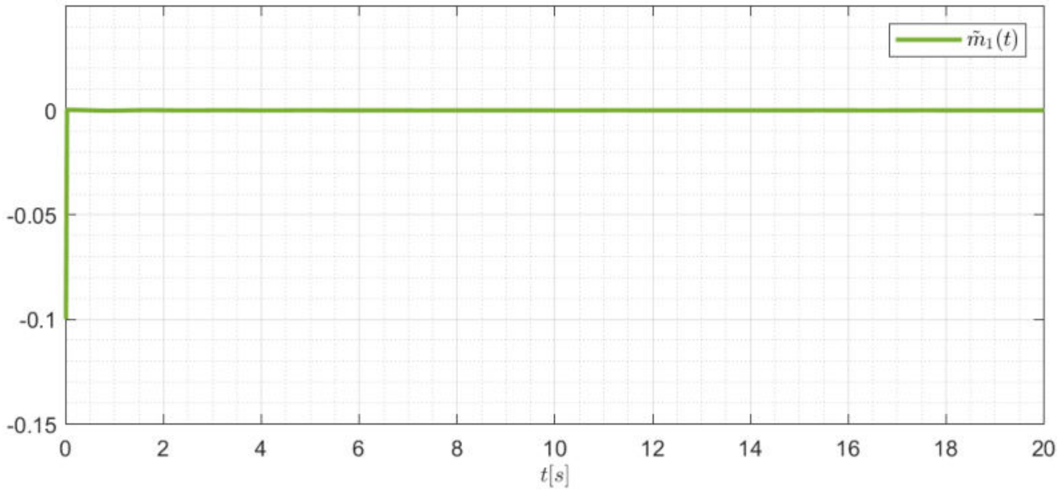

Figure 6.10 – Message error 1.

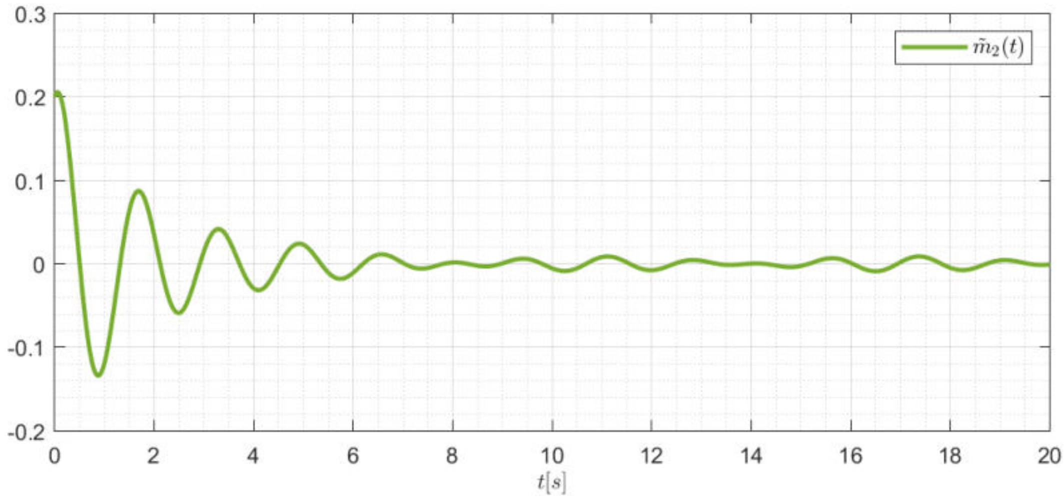

Figure 6.11 – Message error 2.

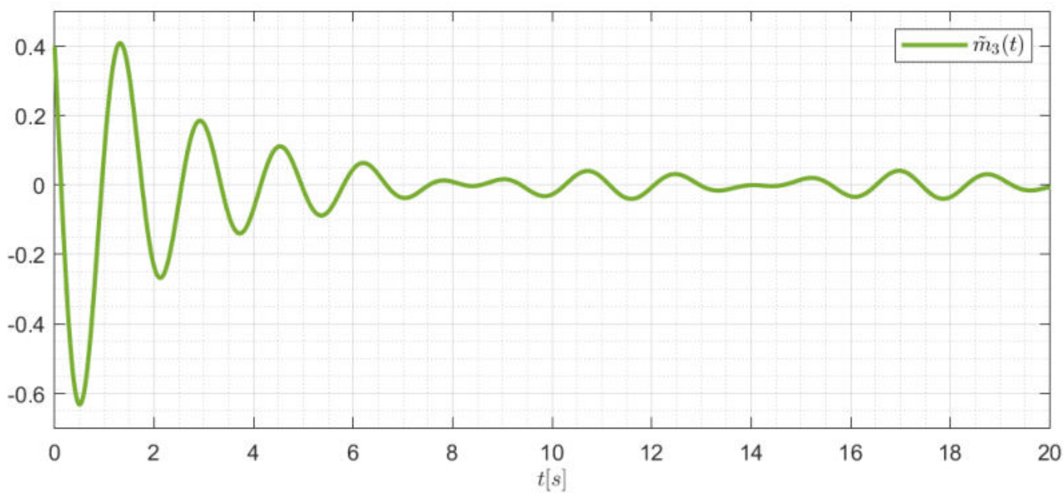

Figure 6.12 – Message error 3.

6.4 CONCLUSION

This chapter proposes an underactuated synchronization scheme of Chua's chaotic system and subject to disturbances based on Lyapunov's stability theory. The main advantages of the proposed scheme are its simplicity, as it is based on proportional control, and considering the presence of disturbance in the stability analysis. A secure communication scheme is also suggested based on this proposal to emphasize its applicability in practical situations. We also prove that only one control signal is enough to synchronize the master and slave system through Lyapunov's theory. The efficacy of the control signal is validated through computer simulations using Matlab, and the results are a typical realization.

7 UNDERACTUATED 4D-HYPERCAOTIC SYSTEM FOR SECURE COMMUNICATION.

The research results of this chapter were published as a conference paper entitled "Underactuated 4D-Hyperchaotic System for Secure Communication in the Presence of Disturbances" in [153]. *This chapter has extended and improved some parts compared to the original paper*.

This chapter deals with a secure communication scheme based on the synchronization of an underactuated 4D hyperchaotic system. It is proven based on Lyapunov analysis that the synchronization errors are bounded even with the consideration of disturbances in the stability analysis. The proposed scheme requires only that control acts in two of the state equations of the slave system. The scheme has the advantage to be robust against bounded internal and external disturbances. Computational simulations have been done, including a secure telecommunication scheme, to validate the robustness and simplicity of the method.

7.1 PROBLEM FORMULATION

Consider the following hyperchaotic Lorenz system [114]:

$$\begin{cases} \dot{x}(t) = ay(t) - ax(t) + w(t) \\ \dot{y}(t) = cx(t) - dy(t) - x(t)z(t) \\ \dot{z}(t) = x(t)y(t) - bz(t) \\ \dot{w}(t) = -y(t)z(t) - w(t) \end{cases} \tag{7.1}$$

where t is the time. Based on (7.1), the master system can be defined as

$$\begin{cases} \dot{x}_m(t) = ay_m(t) - ax_m(t) + w_m(t) \\ \dot{y}_m(t) = cx_m(t) - y_m(t) - x_m z_m(t) \\ \dot{z}_m(t) = x_m(t)y_m(t) - bz_m(t) \\ \dot{w}_m(t) = -y_m(t)z_m(t) - w_m(t) \end{cases} \tag{7.2}$$

and the slave system can be defined as

$$
\begin{aligned}
\dot{x}_s(t) &= ay_s(t) - ax_s(t) + w_s(t) + h_1(t) \\
\dot{y}_s(t) &= cx_s(t) - y_s(t) - x_s(t)z_s(t) + h_2(t) + u_1(t) \\
\dot{z}_s(t) &= x_s y_s(t) - bz_s(t) + h_3(t) \\
\dot{w}_s(t) &= -y_s(t)z_s(t) - w_s(t) + h_4(t) + u_2(t)
\end{aligned}
\tag{7.3}
$$

where $x(t)$, $y(t)$, $z(t)$, and $w(t)$ are the state variables of the system (7.1); $x_{1m}(t)$, $x_{2m}(t)$, $x_{3m}(t)$, and $x_{4m}(t)$ are state variables of the master system; and $x_{1s}(t)$, $x_{2s}(t)$, $x_{3s}(t)$, and $x_{4s}(t)$ are state variables of the slave system. The system parameters are $a = 10$, $b = 8/3$, $c = 28$, and $d = 1$. The disturbances of the slave system are $h_1(t)$, $h_2(t)$, $h_3(t)$, and $h_4(t)$; and $u_1(t)$ and $u_2(t)$ are the control signals. The objective is the synchronization of the systems (7.2) and (7.3).

Remark 7.1: It should be noted that system (7.1) is hyperchaotic [114]; Then its behavior depends on the initial conditions so that the system is sensitive to changes in these initial conditions. In addition, in our approach, we consider (7.2) and (7.3) are different due to the presence of disturbances and control signals in the actuated states.

Remark 7.2: The purpose of presenting a system (7.3) in which disturbances are explicitly considered is to emphasize that the studied synchronization scheme is valid even in the presence of bounded disturbances. For example, applications in analog electronics could happen, and changes may appear due to heating, component tolerances, or electromagnetic noise, among others.

Hypothesis 7.1: It is assumed that the disturbances are bounded. More precisely

$$
\begin{aligned}
|h_1(t)| &\leq \bar{h}_1 \\
|h_2(t)| &\leq \bar{h}_2 \\
|h_3(t)| &\leq \bar{h}_3 \\
|h_4(t)| &\leq \bar{h}_4
\end{aligned}
\tag{7.4}
$$

$\forall t \geq 0$, where $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$, and $\bar{h}_4$ are unknown constants.

Fact 7.1 [114]: The states of the master system are bounded, i.e.,

$$
\begin{aligned}
|x_m(t)| &\leq \bar{x} \\
|y_m(t)| &\leq \bar{y} \\
|z_m(t)| &\leq \bar{z} \\
|w_m(t)| &\leq \bar{w}
\end{aligned}
\tag{7.5}
$$

$\forall t \geq 0$, where $\bar{x}$, $\bar{y}$, $\bar{z}$, and $\bar{w}$ are unknown constants.

7.2 SYNCHRONIZATION ERROR EQUATION AND PROPOSED CONTROL SIGNAL

The synchronization error can be defined as:

$$\begin{cases} e_1(t) = x_s(t) - x_m(t) \\ e_2(t) = y_s(t) - y_m(t) \\ e_3(t) = z_s(t) - z_m(t) \\ e_4(t) = w_s(t) - w_m(t) \end{cases} \tag{7.6}$$

The error dynamic equation is obtained taking the time-derivative of (7.6) and using (7.2) and (7.3). Therefore,

$$\begin{aligned} \dot{e}_1 &= -ae_1 + ae_2 + e_4 + h_1 \\ \dot{e}_2 &= -de_2 + ce_1 - e_1e_3 - e_1z_m - e_3x_m + h_2 + u_1 \\ \dot{e}_3 &= -be_3 + e_1e_2 + e_1y_m + e_2x_m + h_3 \\ \dot{e}_4 &= -e_4 - e_2e_3 - e_2z_m - e_3y_m + h_4 + u_2 \end{aligned} \tag{7.7}$$

Theorem 7.1: Consider the master and slave system described in (7.2) and (7.3) and the control laws defined by

$$\begin{aligned} u_1(t) &= -\psi_1 e_2(t) - \psi_2 e_2^3(t) \\ u_2(t) &= -\psi_3 e_4(t) - \psi_4 e_4^3(t) \end{aligned} \tag{7.8}$$

where $\psi_i > 0$ $(i = 1, ..., 4)$ and their values are user-definable. Then, the synchronization error converges in finite time to the compact set $\Omega = \{e \in \Re^4 \mid ||e|| \leq \theta\}$, being $\theta := \sqrt{\frac{\beta}{\rho}}$, $\beta = \beta_1 + \beta_2$, $\rho := min\{\rho_1, \rho_2, \rho_3, \rho_4\}$, and

$$\begin{aligned} \rho_1 &= a\gamma - 0.5\left(\sigma_1 + 2 + 4\bar{y}^2\alpha_2^2\right) \\ \rho_2 &= \psi_1 + d - 0.5\left(\sigma_2 + c^2 + a^2\gamma^2 + 2\bar{z}^2\right) \\ \rho_3 &= b - 0.5\sigma_3 - \frac{3}{8} \\ \rho_4 &= \psi_3 + 1 - 0.5\left(\sigma_4 + \gamma^2 + \bar{z}^2 + 4\bar{y}^2\right) \\ \beta_1 &= \frac{\bar{h}_2^2}{2\sigma_2} + \frac{\bar{h}_4^2}{2\sigma_4} \\ \beta_2 &= \frac{\gamma^2\bar{h}_1^2}{2\sigma_1} + \frac{\bar{h}_1^2}{2\sigma_3} \end{aligned} \tag{7.9}$$

where γ and σ_i $(i = 1, ..., 4)$ are arbitrary positive constants.

Proof 7.1: Consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2}[\gamma e_1^2(t) + e_2^2(t) + e_3^2(t) + e_4^2(t)] \tag{7.10}$$

Deriving (7.10) along the trajectories errors about time, we obtain

$$\dot{V} = \gamma e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 \tag{7.11}$$

Substituting (7.7) and (7.8) at (7.11), we have

$$\begin{aligned} \dot{V} = &-\gamma a e_1^2 - e_2^2(d + \psi_1) - b e_3^2 - e_4^2(1 + \psi_3) + \gamma e_1 h_1 \\ &+ e_2 h_2 + e_3 h_3 + e_4 h_4 + e_1 e_2 (c - a\gamma - z_m) + \gamma e_1 e_4 \\ &+ e_1 e_3 y_m - e_2 e_3 e_4 - e_2 e_4 z_m - e_3 e_4 y_m - \psi_2 e_2^4 - \psi_4 e_4^4 \end{aligned} \tag{7.12}$$

On the other hand, using Young's inequality [147], the result is

$$\begin{aligned} &\gamma e_1 h_1 \leq 0.5(\sigma_1 e_1^2 + \gamma^2 \sigma_1^{-1} \bar{h}_1^2);\ e_2 h_2 \leq 0.5(\sigma_2 e_2^2 + \sigma_2^{-1} \bar{h}_2^2) \\ &e_3 h_3 \leq 0.5(\sigma_3 e_3^2 + \sigma_3^{-1} \bar{h}_3^2);\ e_4 h_4 \leq 0.5(\sigma_4 e_4^2 + \sigma_4^{-1} \bar{h}_4^2) \\ &e_1 e_2 (c - a\gamma - z_m) \leq 0.5[e_1^2 + e_2^2(c^2 + a^2\gamma^2 + \bar{z}^2)] \\ &\gamma e_1 e_4 \leq 0.5(e_1^2 + \gamma^2 e_4^2);\ y_m e_1 e_3 \leq 2\bar{y}^2 e_1^2 + \frac{e_3^2}{8} \\ &z_m e_2 e_4 \leq 0.5\bar{z}(e_2^2 + e_4^2);\ y_m e_3 e_4 \leq \frac{e_3^2}{8} + 2\bar{y}^2 e_4^2 \\ &e_2 e_3 e_4 \leq \frac{e_3^2}{8} + \frac{e_2^4}{2} + \frac{e_4^4}{2} \end{aligned} \tag{7.13}$$

By using (7.9) and (7.13), (7.12) implies

$$\dot{V} \leq -\rho_1 e_1^2 - \rho_2 e_2^2 - \rho_3 e_3^2 - \rho_4 e_4^2 - \rho_5 e_2^4 - \rho_6 e_4^4 + \beta_1 + \beta_2 \tag{7.14}$$

Since γ, $\psi_i (i = 1, .., 4)$ and $\sigma_j (j = 1, ..., 4)$ are user-defined, they can be chosen so that ρ_1, ρ_2, ρ_3, and ρ_4 are positive. Thus, there will be a positive ρ such that

$$\dot{V} \leq -\rho\ \|e\|^2 + \beta \tag{7.15}$$

Based on the compact set Ω and based on (7.15), the situation where $\dot{V} < 0$ occurs when $\|e\| > \sqrt{\frac{\beta}{\rho}} = \theta$. Since θ is constant, it can be stated that the synchronization error is bounded. That is, if in any case $\|e\|$ leaves the compact set Ω, then $\dot{V}$ becomes negative definite and forces the convergence of the synchronization error to the residual set Ω [114]:

Remark 7.3: It can be seen from the proof that bounded disturbances are being considered in the stability analysis, unlike [114]. Thus, the selection of certain controller design parameters can lead to a small finite-time synchronization error, even in the presence of bounded disturbances.

Remark 7.4: It can be seen that σ_2 and σ_4 can be freely chosen by the usuary. This freedom is because ρ_2 and ρ_4 can be arbitrarily adjusted to be positive if ψ_1 and ψ_2 are adjusted as well. Thus, β_1 can be arbitrarily decreased through the scaling factors. This favorable situation is not the same for β_2, since the maximum value for σ_1 and σ_3 is restricted.

7.3 SIMULATION

Systems (7.2) - (7.3) and control law (7.8) were implemented in a computer simulation to validate the theoretical results. The simulations were performed on a computer with Ryzen 7 1700 processor and with Matlab 2020b by the Ode45 integration method. The initial conditions in the master system are $x_m(0) = [0.6; 1; -0.2; -0.4]$ and in the slave system are $x_s(0) = [0.2; -0.5; 0.1; 0]$. The disturbances are white Gaussian noises implemented by using the (WGN) Matlab function so that h_1, h_2, h_3, and h_4 have the power of noise of $1.46dBW$, $3dBW$, $1.46dBW$, and $11.46dBW$, respectively.

Control parameters were chosen as $\psi_1 = 100$, $\psi_2 = 10$, $\psi_3 = 100$, and $\psi_4 = 10$. The Figures (7.1 - 7.4) show the results of the synchronization. The figures show that although the control signal is placed in only two of the state equations (state y and w), even so, the synchronization occurs satisfactorily for the four states. In the Figures (7.5 - 7.8) it is possible to see the difference between the states of the slave and master systems so that it can be said that the synchronization errors are small.

The proposed scheme was also applied to secure communication problems. A scheme that functions as a message encoder and decoder was simulated to analyze the effectiveness and robustness of control in transmitting two messages placed in non-actuated states. The messages used are $m_1(t) = 0.3sin(10t) + 0.45sin(30t)$ and $m_2(t) = 0.15sin(20t) + 0.3sin(2t)$ that are added to state x_m and z_m respectively. The amplitude of the messages represents approximately 4% of the maximum amplitude of the x_m and z_m signals because if it is too high, it will be easily noticeable when analyzing the signal, x_m and if it is too small, it will be recovered with too much sensitivity to disturbances. Assume that the encrypted message is represented by s_1 and s_2, which are equal to $s_1 = x_m + m_1$ and $s_2 = z_m + m_2$. The message is recovered by subtracting the s_1 sign from the x_s sign and the s_2 sign from the z_s.The message errors $\tilde{m}_1$ and $\tilde{m}_2$ are equal to the recovered messages minus the original messages.

The recovery of the messages and the encoded message can be seen in Figures (7.9 - 7.10), where it can be seen that there is almost perfect message recovery, showing the efficiency of the controller. Also, note that the form of the encoded message is very different from the original message. Figures (7.11 - 7.12) show the message errors. As can be seen, the performance of the secure telecommunication scheme shows a result with small errors, which is to be expected since what has been shown theoretically is that synchronization errors are bounded.

A Tabela 7.1 shows the RMSE for different values of ψ. Small ψ value corresponds to large RMSE and transient. Also, large, ψ corresponds to small RMSE and transient, as shown in Table 7.1 and in Figures (7.5 - 7.8) and (7.11 - 7.12).

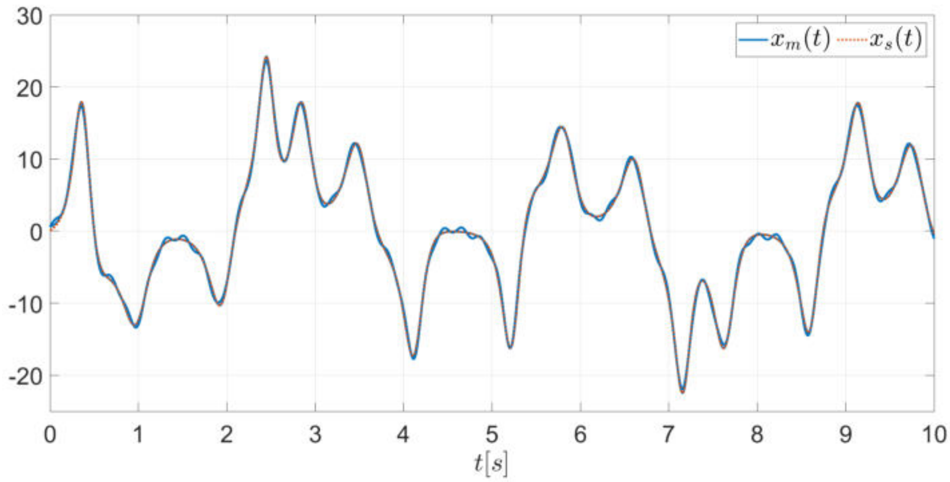

Figure 7.1 – x state synchronization.

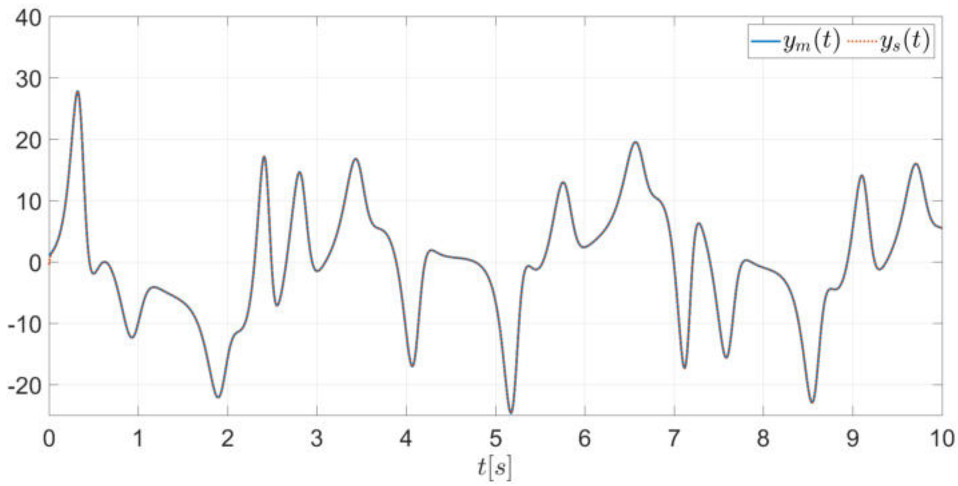

Figure 7.2 – y state synchronization.

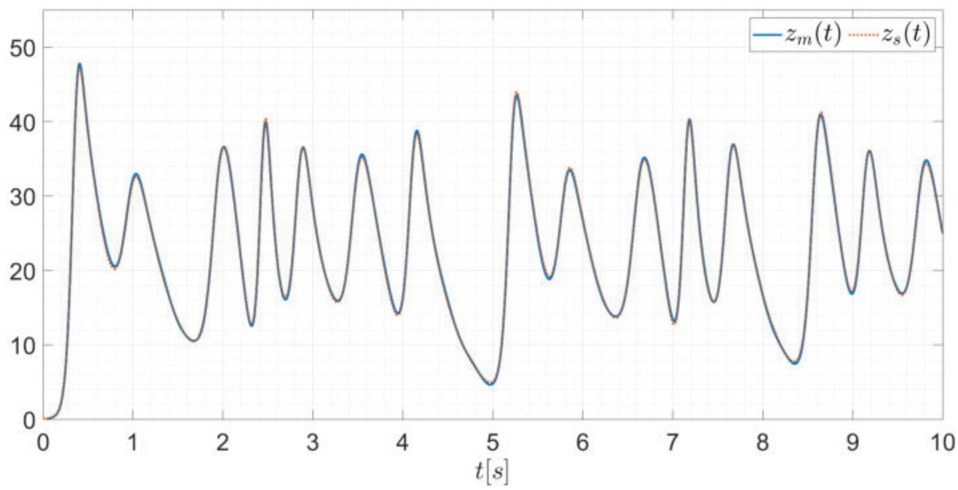

Figure 7.3 – z state synchronization.

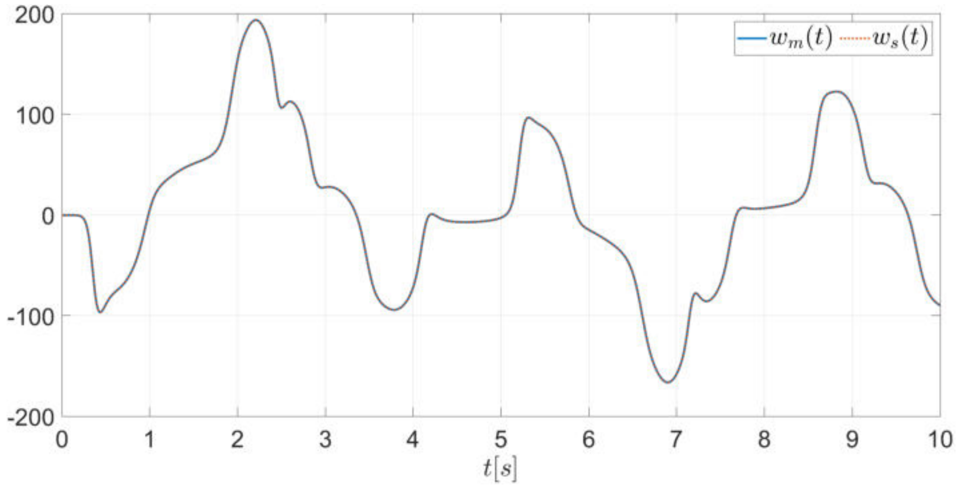

Figure 7.4 – w state synchronization.

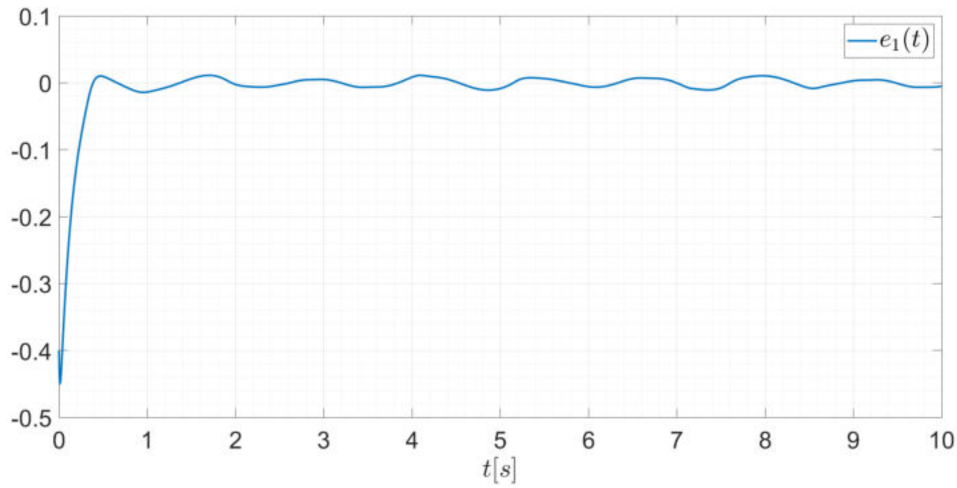

Figure 7.5 – x state synchronization error.

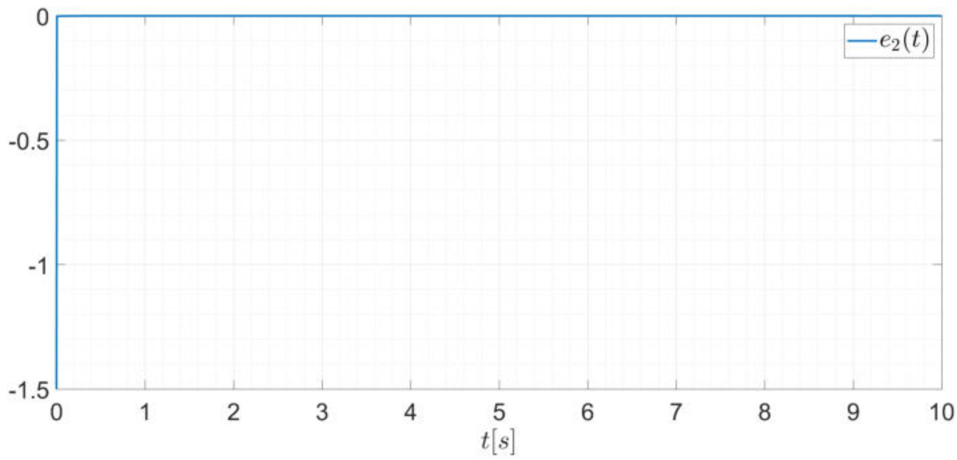

Figure 7.6 – y state synchronization error.

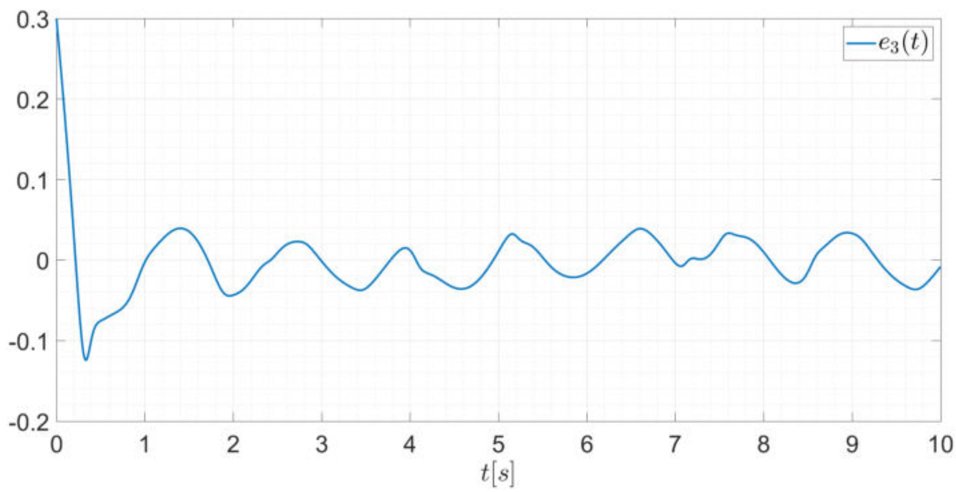

Figure 7.7 – z state synchronization error.

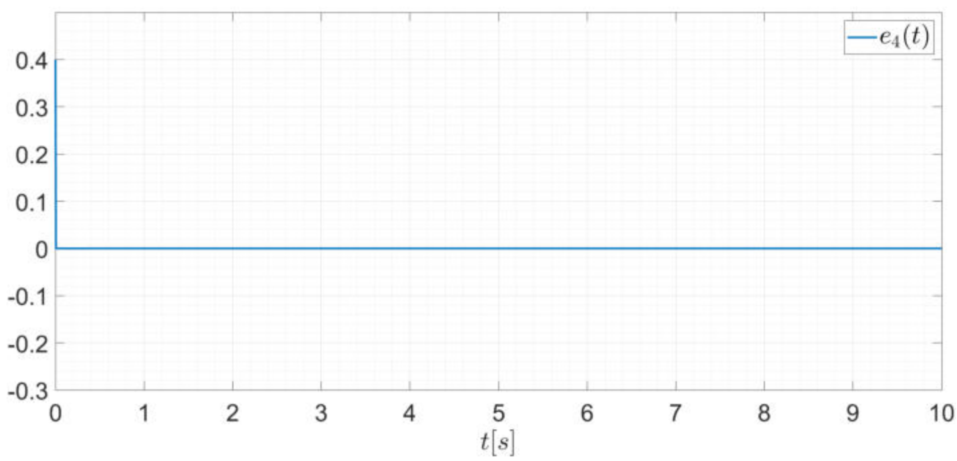

Figure 7.8 – w state synchronization error.

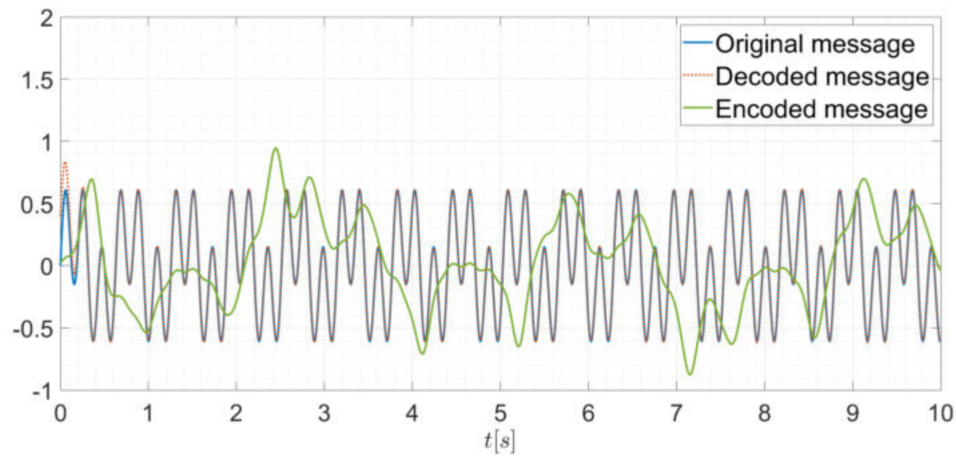

Figure 7.9 – Encoded and decoded messages $(m_1(t), \hat{m}_1(t))$.

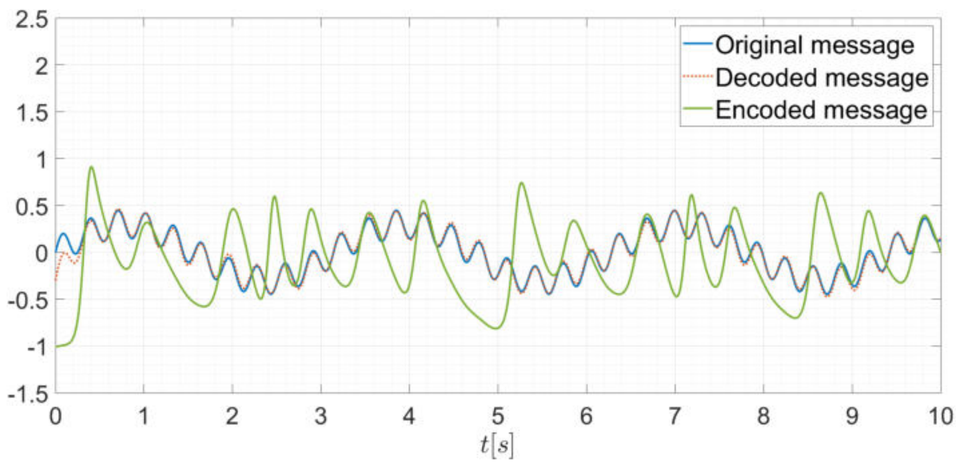

Figure 7.10 – Encoded and decoded messages $(m_2(t), \hat{m}_2(t))$.

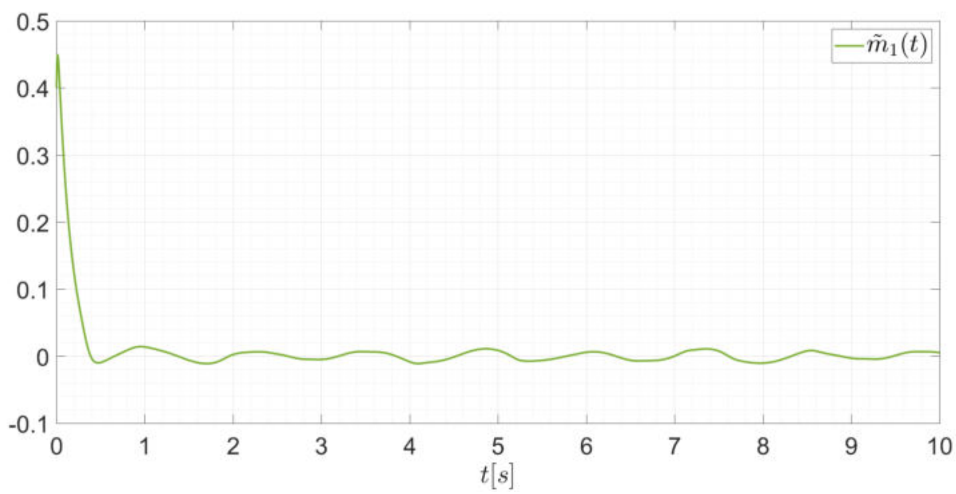

Figure 7.11 – Message error 1.

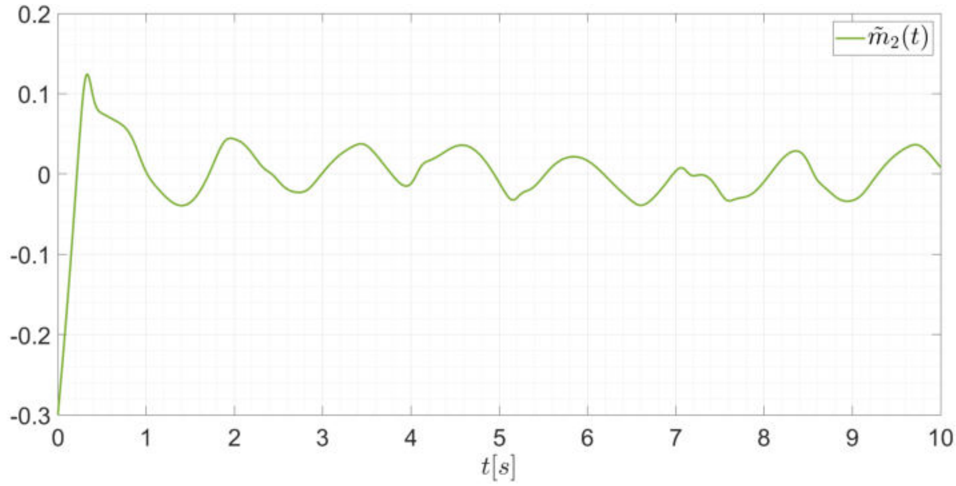

Figure 7.12 – Message error 2.

Table 7.1 – RMSE of state and message synchronization for t = [0 20] seconds.

RMSE of state synchronization and message errors in the proposed algorithm

ψ_1	ψ_2	ψ_3	ψ_4	e_{s1}	e_{s2}	e_{s3}	e_{s4}	e_{m1}	e_{m2}
10	1	10	1	0.272708	0.401611	0.536434	0.461409	0.272708	0.536434
100	10	100	10	0.050932	0.066544	0.048278	0.027194	0.050932	0.048278
1000	100	1000	100	0.053566	0.062984	0.061886	0.028084	0.053566	0.061886
10000	1000	10000	1000	0.040419	0.037034	0.050518	0.016739	0.040419	0.050518
100000	10000	100000	10000	0.029200	0.013716	0.040987	0.008500	0.029200	040987

7.4 CONCLUSION

In this chapter, a synchronization algorithm based on the Lyapunov stability theory has been proposed for a hyperchaotic system subject to bounded disturbances. It has been proven and validated via simulations that it is sufficient to employ the control signal in only two states to achieve complete synchronization of the master and slave systems. A simple and robust system for secure communication has also been simulated. An application of the proposed method was also implemented using message encoding and restoration. As a differential about the literature, disturbances are assumed in all states and, therefore, in the stability analysis.

8 MINIMAL UNDERACTUATED SYNCHRONIZATION OF CHAOTIC SYSTEMS

The research outcomes of this chapter have been submitted to the CNSNS (Communications in Nonlinear Science and Numerical Simulation). This paper was done in joint work with other authors: Kevin Herman Muraro Gularte, Hiago dos Santos Rabelo, José Alfredo Ruiz Vargas, The name of the submitted work is "Minimal Underactuated Synchronization with Applications to Secure Communication".

This chapter deals with the synchronization of a class of underactuated hyperchaotic systems. The proposed method is based on Lyapunov theory, and the initial synchronization error is assumed to be small to simplify the synchronizer structure. The proposed synchronization method ensures the convergence of the synchronization error to a neighborhood of the origin, even employing a scalar proportional control law and in the presence of full-state disturbances. A comparison study is accomplished to depict the advantages of the proposed method. Furthermore, a secure communication system based on the proposed synchronization approach is implemented using analog electronics to show a typical application.

8.1 INTRODUCTION

Chaos refers to an important nonlinear behavior found in the real world and has been extensively studied in recent years. Chaotic systems are deterministic nonlinear systems that have an aperiodic behavior and show sensitive dependence on initial conditions [55]. A necessary condition for a system to be chaotic is that at least one Lyapunov exponent is positive [154]. The first chaotic model was introduced by Lorenz in 1963 [123]. Since then, many other relevant chaotic systems have been proposed. See for instance, the seminal works by Rössler [155], Chen [156], Sprott [157], and Lü [158]; and, more recently, the interesting contributions in [70, 76, 115].

On the other hand, hyperchaos has attracted the attention of a lot of researchers in recent years. Hyperchaotic systems must have at least two positive Lyapunov exponents, and their dimension must be higher than three [77]. Rössler introduced hyperchaos in 1979 [159]. Other important hyperchaotic models appeared in the following years, such as those by Chua [154], Chen [160], Lü, [161], Lorenz [162]; and, more recently, [78, 80, 81, 86, 116, 163–178].

Furthermore, since hyperchaotic systems have a more unpredictable behavior than those chaotic [179], these kinds of systems have motivated a lot of applications in secure communication [75, 86, 165, 180–185]. Chaotic and hyperchaotic systems have also been used in many contexts, such as, for instance, biology [186], economy [79, 81, 187], image encryption [188–194], neural computing [9,59,82,195–198], optics [199] and robotics [200].

Chaos synchronization consists of the adjustment of the dynamic of a master (transmitter) and a slave (receiver) chaotic system so that their trajectories converge over time. Pecora and Carrol introduced the first work on this subject in 1990 [17]. Nowadays, there are various types of synchronization found in the literature, such as antisynchronization [76, 201], lag synchronization [62, 202], projective synchronization [109, 110, 184], hybrid synchronization [175], predefined-time synchronization [203] and so on. Applications of hyperchaos-based synchronization include cryptosystems for video/audio streaming [204], image encryption [205, 206], and secure communication using analog circuits [180, 184]. Other potential applications can be found in [207].

Despite all the advances in this field, there are several drawbacks. For example, interesting contributions were proposed in [43, 175, 187, 208, 209] where different hyperchaotic synchronizers were proposed based on the Lyapunov theory. However, disturbances were not considered in the analysis, with a negative impact on the robustness of these methods. Perturbations can occur due to various factors, such as tolerance, heating, non-ideal behavior, and electromagnetic noise. Also, fully actuated schemes based on Lyapunov theory are usually considered in the literature [3,5,11,43,118,182,208,210–213]. On the other hand, when underactuated schemes are considered, the control law is not scalar [184,187,214–216], which makes the control law complex. For example, robust control schemes, based on slide mode and backstepping control, respectively, were utilized in [173,215,216] and [217] to synchronize underactuated hyperchaotic systems. Nevertheless, in [173, 214–217], the controls are structurally complex.

In [167], a synchronization scheme based on a new hyperchaotic system, Lyapunov theory, and adaptive control was proposed. The system proposed in [167] is fully actuated to show that the synchronization error is bounded. In addition, the synchronization structure is nine-dimensional and therefore complex. To the best of our knowledge, an underacted synchronization scheme with only a proportional control signal for the system [167] has not been proposed in the literature. So, it would be desirable to propose a synchronizer system that: i) only uses one proportional control signal, thus facilitating the application; ii) considers disturbances in the Lyapunov analysis to make the system robust; iii) uses an alternative proof methodology based on Lyapunov theory to ensure that the synchronization error is bounded.

Motivated by the previous facts, this work proposes a robust scheme for synchronizing a class of underactuated hyperchaotic systems. To simplify the synchronizer, the initial condition on the synchronization error is assumed to be small, which can be easily implemented by setting the initial conditions of the synchronization error to zero when the hyperchaotic systems do not have a trivial equilibrium point. Lyapunov theory is used to ensure the boundedness of the synchronization error. The stability proof is possible due to the particular structure of the system, small initial conditions, and control law. More precisely, this paper presents the following contributions.

1) An underactuated synchronization scheme for a class of perturbed hyperchaotic systems based on [167] is proposed, in contrast to [3, 5, 11, 43, 118, 182, 210–213].

2) The proposed approach is minimal in the way that the synchronizer is simplified to the maximum since the control is scalar and proportional, in contrast to [3, 5, 118, 213, 215, 216, 218].

3) The proposed scheme considers the presence of disturbances in all states in the stability analysis, in contrast to [43, 175, 187, 208, 209]. It is worth mentioning that the presence of disturbances in the analysis ensures the robustness of the method.

4) The proposed scheme is applied to secure communication, in contrast to [43, 175, 203, 208–210, 217, 219–222].

The rest of this work is organized as follows. Preliminaries are presented in Section 8.2. In Section 8.3, the problem and main assumptions are introduced. The synchronization error and secure communication procedure are introduced, respectively, in Section 8.4 and Section 8.5. In Section 8.6, the main contribution of the paper is presented, the stability proof of the proposed minimal synchronization method. In Section 8.7, a comparison study with another work in the literature and an application are presented. Finally, the conclusions of this work are outlined in Section 8.8.

8.2 PRELIMINARIES

Let the two-norm (Euclidian norm) [223] of a vector $e \in \Re^n$ be denoted by $||e|| = \sqrt{\sum_{i=1}^{n} e_i^2}$. By defining that a_1 and a_2 are positive variables, consider the following Young inequality [224]

$$2a_1a_2 \leq a_1^2 + a_2^2 \tag{8.1}$$

Definition 8.1 [223]. The solutions of $\dot{x} = f(x,t)$, with $x(t_0) = x_0$ and $x \in \Re^n$, are uniformly ultimately bounded (u.u.b.) (with bound B) if there exists a $B > 0$ and if corresponding to any $\delta > 0$ and $t_0 \in \Re+$, there exists a $T = T(\delta) > 0$ (independent of t_0) such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < B$ for all $t \geq t_0 + T$.

8.3 PROBLEM FORMULATION

Consider the following master hyperchaotic system [167]

$$\begin{cases} \dot{x}_m = a\left(y_m - x_m\right) \\ \dot{y}_m = cx_m - y_m - x_m z_m + w_m - d \\ \dot{z}_m = -bz_m + x_m y_m \\ \dot{w}_m = my_m + w_m - nx_m^3 \end{cases} \tag{8.2}$$

where $a = 10$, $b = 2$, $c = 28$, $d = 0.1$, $m = 27$, and $n = 0.5$; and x_m, y_m, z_m, and w_m are the states variables of the master system (8.2). Based on the system (8.2), we defined a class of underactuated hyperchaotic slave systems as

$$\begin{cases} \dot{x}_s = a\left(y_s - x_s\right) + h_1 \\ \dot{y}_s = cx_s - y_s - x_s z_s + w_s - d + h_2 \\ \dot{z}_s = -bz_s + x_s y_s + h_3 \\ \dot{w}_s = my_s + w_s - nx_s^3 + h_4 + u \end{cases} \tag{8.3}$$

where x_s, y_s, z_s, w_s are the state variables of the slave system (8.3); u is a scalar control to be determined afterwards; and h_1, h_2, h_3, h_4 are disturbances.

This work proposes the synchronization of (8.2) and (8.3) by using the scalar control u, even in the presence of full-state disturbances.

Remark 8.1: Since system (8.2) is hyperchaotic [167], it follows that

$$\begin{aligned} |x_m(t)| &\le \bar{x} \\ |y_m(t)| &\le \bar{y} \\ |z_m(t)| &\le \bar{z} \\ |w_m(t)| &\le \bar{w} \end{aligned} \tag{8.4}$$

$\forall t \ge 0$, being $\bar{x}$, $\bar{y}$, $\bar{z}$, and $\bar{w}$ positive constants.

Assumption 8.1: The disturbances are bounded. More exactly,

$$|h_i(t)| \le \bar{h}_i, \forall t \ge 0, (i = 1, ..., 4) \tag{8.5}$$

where $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$, and $\bar{h}_4$ are positive constants.

8.4 SYNCHRONIZATION ERROR

The synchronization error is defined as

$$\begin{aligned} e_1 &= x_s - x_m \\ e_2 &= y_s - y_m \\ e_3 &= z_s - z_m \\ e_4 &= w_s - w_m \end{aligned} \tag{8.6}$$

By using (8.2), (8.3), and the time-derivative of (8.6), the synchronization error equations can be defined as

$$\begin{aligned} \dot{e}_1 &= -ae_1 + ae_2 + h_1 \\ \dot{e}_2 &= -e_2 + ce_1 - e_1e_3 - z_me_1 - x_me_3 + e_4 + h_2 \\ \dot{e}_3 &= -be_3 + e_1e_2 + y_me_1 + x_me_2 + h_3 \\ \dot{e}_4 &= e_4 + me_2 - n(e_1^3 + 3x_me_1^2 + 3x_m^2e_1) + h_4 + u \end{aligned} \tag{8.7}$$

8.5 SECURE COMMUNICATION

In addition to synchronization, we are interested in secure communication. For additional details about this topic, refer to [2]. Motivated by [2], let us define

$$\tilde{m}_i(t) = -e_i(t), \forall t \geq 0 \tag{8.8}$$

where $\tilde{m}_i = \hat{m}_i - m_i$, $\tilde{m}_i$ are the message reconstruction errors, $\hat{m}_i$ are the decoded messages, and m_i are the messages before codification for $i = 1, ..., 4$.

Assumption 8.2: It is assumed that

$$|m_i(t)| \leq \bar{m}_i, \forall t \geq 0,\ (i = 1, ..., 4) \tag{8.9}$$

where $\bar{m}_1$, $\bar{m}_2$, $\bar{m}_3$, and $\bar{m}_4$ are positive constants. From equation (8.8), it can be concluded that

$$\hat{m}_i = m_i - e_i,\ (i = 1, ..., 4) \tag{8.10}$$

Remark 8.2: Note in (8.8) that if the synchronization error is bounded, then the message error is also bounded. Thus, it is sufficient to prove that the synchronization error is bounded for the message error and decoded message to be bounded.

8.6 UNDERACTUATED SYNCHRONIZATION AND LYAPUNOV ANALYSIS

This section proposes a minimal underactuated synchronization scheme based on a proportional scalar control. The design is based on an exhaustive Lyapunov analysis for a simple synchronization based on the domination of positive terms on the right-hand side of the time-derivative of a Lyapunov function candidate [223]. The initial condition of the synchronization error is assumed to be sufficiently small to simplify the structure of the synchronizer. The main result of the paper is presented in the following.

Theorem 8.1:

Consider the master and slave systems described by (8.2)-(8.3) and the proportional control law defined by

$$u = -\psi e_4 \tag{8.11}$$

If,

$$||e(0)|| \in [0, \alpha) \tag{8.12}$$

$$\psi > \delta_1 \tag{8.13}$$

$$\beta \leq \delta_2 \tag{8.14}$$

Then, the synchronization error is bounded and converges to the compact set $\Omega_1 = \{e \in \Re^4 \mid \ ||e|| \leq \theta\}$,

where $\alpha = \sqrt{\frac{\rho_9+\sqrt{\rho_9^2-4\rho_{10}\beta}}{2\rho_{10}}}$ is a sufficiently small constant, $\theta = \sqrt{\frac{\rho_9-\sqrt{\rho_9^2-4\rho_{10}\beta}}{2\rho_{10}}}$, $\delta_1 = 0.5[\sigma_4 + 9(\sigma_{11}^{-1}\bar{x}^2 + \sigma_9^{-1}\bar{x}^4) + \sigma_{10}^{-1}(m+1)^2]$, $\delta_2 = \frac{\rho_9^2}{4\rho_{10}}$, $\beta = \beta_u + \beta_n + \beta_c$, $\beta_u = 0.5\sigma_4^{-1}\bar{h}_4^2$, $\beta_n = 0.5\left(\sigma_1^{-1}\bar{h}_1^2 + \sigma_2^{-1}\bar{h}_2^2 + \sigma_3^{-1}\bar{h}_3^2\right)$, $\beta_c = \frac{\sigma_5(a^2+c^2+\bar{z}^2)}{4} + \frac{\sigma_7\bar{y}^2}{4}$, $\rho_1 = a - 0.5(\sigma_1 + n^2\sigma_9)$, $\rho_2 = 1-0.5(\sigma_2+\sigma_{10})$, $\rho_3 = c - 0.5\sigma_3$, $\rho_4 = \psi - \delta_1$, $\rho_5 = 0.5(\sigma_5^{-2}\sigma_6 + \sigma_7^{-2}\sigma_8 + n^2\sigma_{11} + n\sigma_{12} + n\sigma_{13})$, $\rho_6 = 0.5\sigma_5^{-2}\sigma_6^{-1}$, $\rho_7 = 0.5\sigma_7^{-2}\sigma_8^{-1}$, $\rho_8 = \frac{n}{8\sigma_{12}^2\sigma_{13}}$, $\rho_9 = min\ \{\rho_1, \rho_2, \rho_3, \rho_4\}$, $\rho_{10} = max\ \{\rho_5, \rho_6, \rho_7, \rho_8\}$, $\sigma_i (i = 1, ..., 13)$ are positive constants, and ψ is the control gain.

Proof 8.1: Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2\right) \tag{8.15}$$

The time-derivative of (8.15) along the trajectories of (8.7) results

$$\begin{aligned}\dot{V} &= e_1\left(-ae_1 + ae_2 + h_1\right) + e_2(-e_2 + ce_1 - e_1e_3 - z_me_1 \\ &\quad - x_me_3 + e_4 + h_2) + e_3(-be_3 + e_1e_2 + y_me_1 + x_me_2 + h_3) \\ &\quad + e_4[e_4 + me_2 - n(e_1^3 + 3x_me_1^2 + 3x_m^2e_1) + h_4 + u]\end{aligned} \tag{8.16}$$

By using (8.11) in (8.16), after some manipulations, we have

$$\begin{aligned}\dot{V} &= -ae_1^2 - e_2^2 - be_3^2 - (\psi - 1)e_4^2 + h_1e_1 + h_2e_2 \\ &\quad + h_3e_3 + h_4e_4 - ne_1^3e_4 - 3nx_me_1^2e_4 + e_1e_2(a + c \\ &\quad - z_m) + y_me_1e_3 - 3nx_m^2e_1e_4 + (m+1)e_2e_4\end{aligned} \tag{8.17}$$

On the other hand, by using Young's inequality, the following expressions are true

$$\begin{aligned}&e_1h_1 \leq 0.5(\sigma_1e_1^2 + \sigma_1^{-1}\bar{h}_1^2); e_2h_2 \leq 0.5(\sigma_2e_2^2 + \sigma_2^{-1}\bar{h}_2^2) \\ &e_3h_3 \leq 0.5(\sigma_3e_3^2 + \sigma_3^{-1}\bar{h}_3^2); e_4h_4 \leq 0.5(\sigma_4e_4^2 + \sigma_4^{-1}\bar{h}_4^2) \\ &e_1e_2(a + c - z_m) \leq \frac{\sigma_5(a^2 + c^2 + \bar{z}^2)}{4} + \frac{\sigma_6e_1^4}{2\sigma_5^2} + \frac{e_2^4}{2\sigma_5^2\sigma_6} \\ &y_me_1e_3 \leq \frac{\sigma_7\bar{y}^2}{4} + \frac{\sigma_8e_1^4}{2\sigma_7^2} + \frac{e_3^4}{2\sigma_7^2\sigma_8} \\ &-3nx_m^2e_1e_4 \leq 0.5(\sigma_9n^2e_1^2 + 9\sigma_9^{-1}\bar{x}^4e_4^2) \\ &(m+1)e_2e_4 \leq 0.5[\sigma_{10}e_2^2 + \sigma_{10}^{-1}(m+1)^2e_4^2] \\ &-3nx_me_1^2e_4 \leq 0.5(\sigma_{11}n^2e_1^4 + 9\sigma_{11}^{-1}\bar{x}^2e_4^2) \\ &-ne_1^3e_4 \leq \frac{ne_1^4(\sigma_{12} + \sigma_{13})}{2} + \frac{ne_4^4}{8\sigma_{12}^2\sigma_{13}}\end{aligned} \tag{8.18}$$

By employing (8.18) in (8.17), it can be concluded that

$$\dot{V} \leq -\rho_1e_1^2 - \rho_2e_2^2 - \rho_3e_3^2 - \rho_4e_4^2 + \rho_5e_1^4 + \rho_6e_2^4 + \rho_7e_3^4 + \rho_8e_4^4 + \beta_u + \beta_n + \beta_c \tag{8.19}$$

which implies

$$\dot{V} \leq -\rho_9||e||^2 + \rho_{10}||e||^4 + \beta \tag{8.20}$$

Hence, $\dot{V} < 0$ as long as $e \in \Omega_2$, where

$$\Omega_2 = \left\{ e \in \Re^4 \mid \theta < ||e|| < \alpha \right\} \tag{8.21}$$

Let us define $\Omega_3 = \Omega_1 \cup \Omega_2$, then $e(0) \in \Omega_3$ by (8.12). Note that (8.14) ensures the existence of a non-null set Ω_2. In case that $e(0) \in \Omega_2$, the synchronization error can only decrease until $e(t) \in \Omega_1$ since $\dot{V} < 0$. In case that $e(0) \in \Omega_1$, the synchronization error will remain in the region Ω_1 forever due to the continuity of the states and $\dot{V} < 0$ in Ω_2. Hence, the synchronization error e is uniformly bounded [223]. Figure 8.1 shows the main sets. The proof is complete.

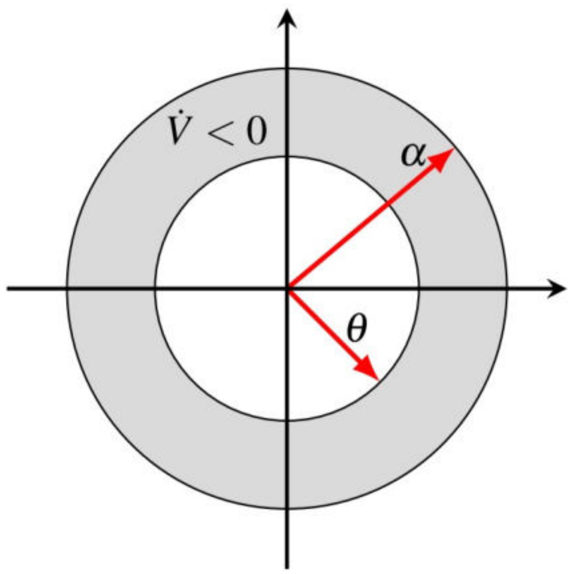

Figure 8.1 – Bounded sets.

Remark 8.3: From a theoretical point of view, the main innovation of our work lies in the proposition of a stability-proof methodology based on an invariant annulus to make the residual synchronization error bounded and arbitrarily small. The existence of the annulus is assured when (8.13)-(8.14) are satisfied. On the other hand, minor errors can be obtained when the control gain (8.11) is sufficiently large and the initial errors belong to the annulus.

Remark 8.4: Note that it is not possible to design a synchronizer for [167] using a scalar control and based exclusively on the direct Lyapunov method. Basically, in (8.20), it is not possible to make $\dot{V} < 0$ due to the particular structure of [167]. However, the synchronization error can be made bounded using (8.11)-(8.12) because the first term on the right-hand side of (8.20) will dominate the second term, and this is only possible with a small initial synchronization error.

Remark 8.5: Condition (8.12) can be easily satisfied when the chaotic master and slave circuits are initialized in zero (for example, discharging energy storage devices). This is possible in chaotic systems that do not have the origin as an equilibrium point. Note that from (8.2) and (8.3) there is no trivial equilibrium point due to the presence of the constant d

in the second state. Examples of other chaotic systems that have no trivial equilibrium points can be found in [81, 214, 225–230].

Remark 8.6: By replacing the parameters $m = 3.08571$ and $n = 4.06808$ in (8.13), (8.13) can be rewritten as

$$\psi > 0.5[\sigma_4 + 9(\sigma_{11}^{-1}\bar{x}^2 + \sigma_9^{-1}\bar{x}^4) + 16.7\sigma_{10}^{-1}] \tag{8.22}$$

For instance, based on the simulation in the next section, it can be assumed that $\bar{x} = 2.2$, $\sigma_4 = 10$, $\sigma_9 = 30$, $\sigma_{10} = 1$, and $\sigma_{11} = 1$, then $\psi \geq 38.7$. In general, the adjustment of the control gains ψ can be made by a trial and error procedure depending on the requirements of the user.

Remark 8.7 Condition (8.14) can be written as

$$\sum_{i=1}^{4} \frac{\bar{h}_i^{\,2}}{\sigma_i} + \frac{(a^2 + c^2 + \bar{z}^2)\sigma_5}{2} + \frac{\bar{y}^2\sigma_7}{2} \leq \frac{\rho_9^2}{2\rho_{10}} \tag{8.23}$$

Notice that (8.23) prevents the upper bounds for h_1, h_2, and h_3 from being arbitrarily high when the right-hand side of (8.23) remains small. In fact, from the definitions after (8.14), we can conclude that, at most, $\sigma_1 < 2a$, $\sigma_2 < 2$, $\sigma_3 < 2c$, and $\sigma_4 < 2\psi$. Then, since σ_4^{-1}, σ_5, and σ_7 on the left-hand side of (8.23) can be adjusted arbitrarily small, for instance, via a high gain control, it may be necessary to have small upper bounds for the unmatched disturbances h_1, h_2, and h_3 to satisfy (8.23). In practice, it is enough to use a high-gain controller. In addition, disturbances can be decreased, for example, by employing operational amplifiers and multipliers with high precision; and capacitors and resistors with small tolerances.

8.7 SIMULATIONS

Simulations are presented to validate the theoretical results and show the performance and application of the proposed method. For all simulations, Matlab/Simulink ®, R2020a, in a platform Windows 10, i5-3330, was used. Other simulation parameters were: ODE15s solver with variable-step, and relative tolerance of 10^{-8}. Two cases were considered in this section: 1) a comparison with another work in the literature, and 2) an implementation of the proposed scheme by using analog electronics. The purpose is to show a comparative performance with another work in the literature and the implementation of the proposed method via analog electronics. In both simulation cases, the system parameters were $a = 10$, $b = 2$, $c = 28$, $d = 0.1$, $m = 27$, $n = 0.5$, and $\psi = 10^4$. The initial conditions were

$x_m(0) = -2.9$, $y_m(0) = 3.8$, $z_m(0) = 4.7$, and $w_m(0) = -1.2$ for the master system, and $x_s(0) = 12.4$, $y_s(0) = -7.5$, $z_s(0) = 10.2$, and $w_s(0) = 3.4$ for the slave system.

8.7.1 Comparison with [167]

Table 8.1 – Comparison of control laws.

Control law in [167]	**Proposed control law**
$u_x = -\hat{a}(t)(e_y - e_x) - k_x e_x$ $u_y = -\hat{c}(t)e_x + e_y + x_2 z_2 - x_1 z_1 - e_w - k_y e_y$ $u_z = \hat{b}(t)e_z - x_2 y_2 + x_1 y_1 - k_z e_z$ $u_w = -\hat{m}(t)e_y - e_w + \hat{n}(t)(x_2^3 - x_1^3) - k_w e_w$ $\dot{\hat{a}}(t) = e_x(e_y - e_x)$ $\dot{\hat{b}}(t) = -e_z^2$ $\dot{\hat{c}}(t) = e_x e_y$ $\dot{\hat{m}}(t) = e_y e_w$ $\dot{\hat{n}}(t) = -e_w(x_2^3 - x_1^3)$	$u = -\psi e_4$

In [167], a new hyperchaotic system was introduced; however, disturbances were not considered in the stability analysis. Also, a fully actuated adaptive controller was used to obtain finite-time synchronization. In Figure (8.2-8.9), we show the performance comparison between the proposed method and that in [167] for a time interval of 15 seconds. The disturbances $h_1 = 0.1sin(5t)$, $h_2 = 0.1cos(3t)$, $h_3 = 0.3cos(5t)$, and $h_4 = 50[sin(2t) + 0.4sin(10t)]$ were considered at $t = 7s$ to check the robustness of both methods.

Note that the main difference between both methods (Figure (8.2-8.9) lies in the transient and robustness. The system proposed in [167] has a faster synchronization because it has the control applied in all slave states. The synchronization using the proposed method (8.3) synchronizes more quickly only in the fourth state, in which the control is present, and it is robust against disturbances arising at $t = 7s$.

Observe that the performance of the proposed method and that of [167] are similar. However, the proposed scheme is simpler. Table 8.1 shows the structural difference between them. The proposed approach requires fewer components when implemented using analog electronics and reduces the computational burden in digital applications.

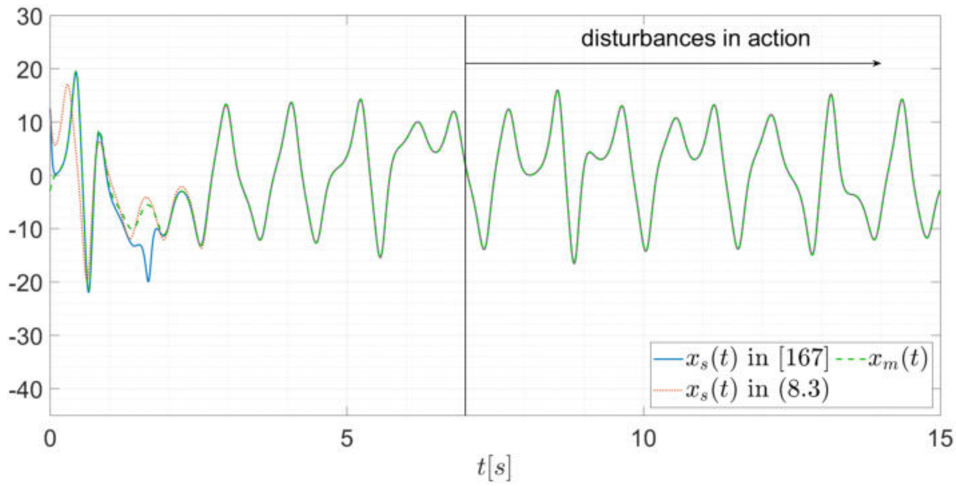

Figure 8.2 – Performance comparison between the proposed approach and that in [167] of the first state.

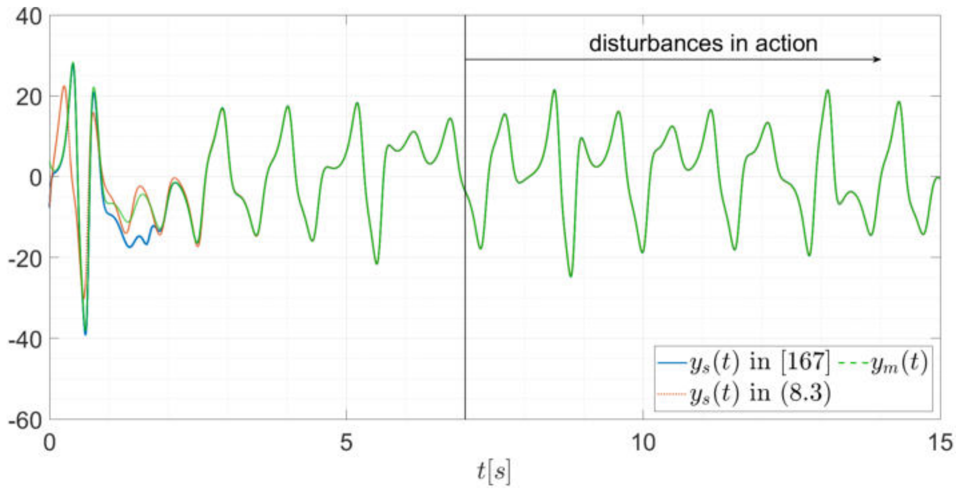

Figure 8.3 – Performance comparison between the proposed approach and that in [167] of the second state.

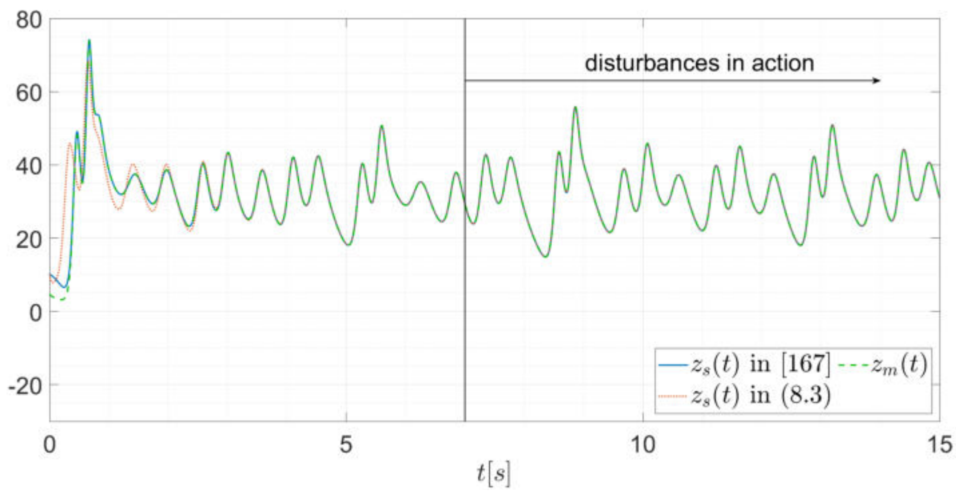

Figure 8.4 – Performance comparison between the proposed approach and that in [167] of the third state.

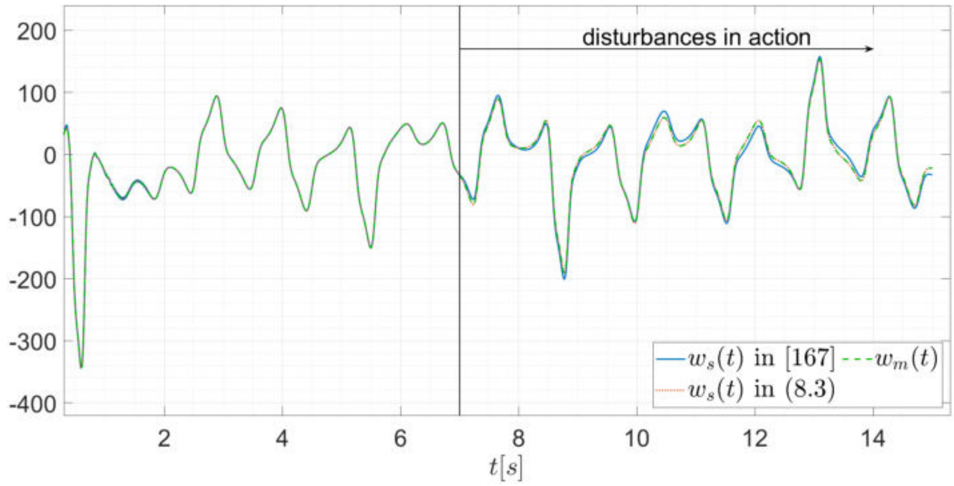

Figure 8.5 – Performance comparison between the proposed approach and that in [167] of the fourth state.

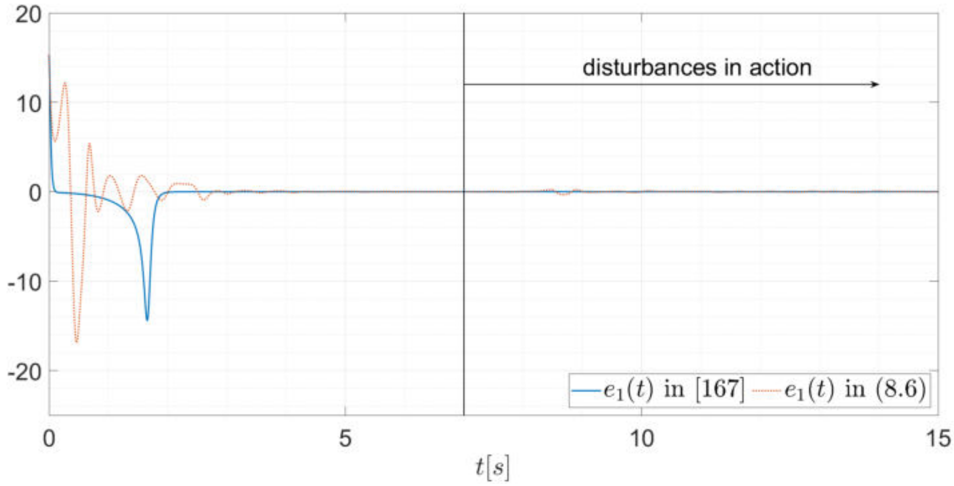

Figure 8.6 – Performance comparison between the proposed approach and that in [167], First state error.

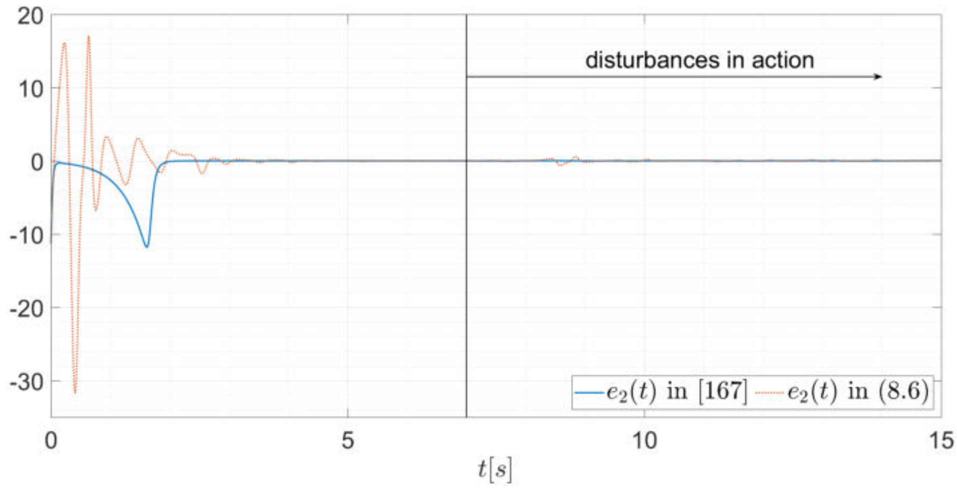

Figure 8.7 – Performance comparison between the proposed approach and that in [167], second state error.

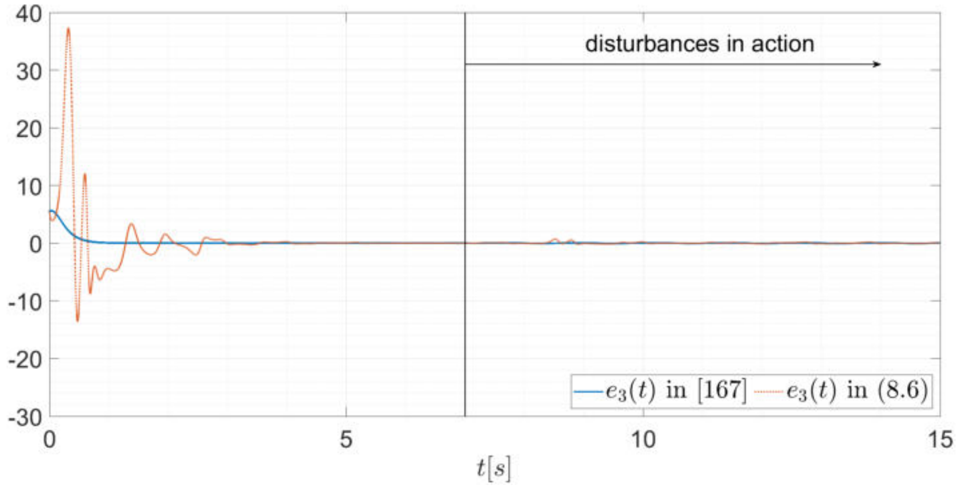

Figure 8.8 – Performance comparison between the proposed approach and that in [167], third state error.

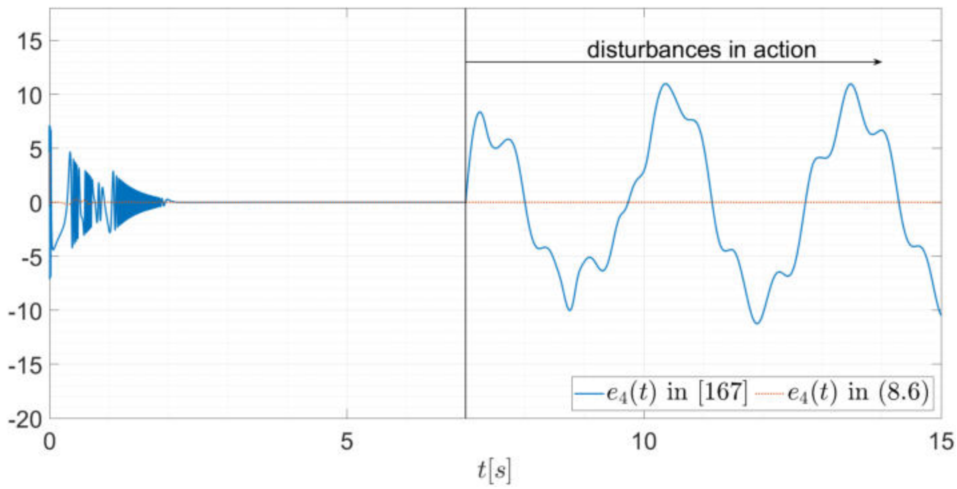

Figure 8.9 – Performance comparison between the proposed approach and that in [167], fourth state error.

8.7.2 Implementation Example

The implementation was performed in the following stages: 1) preliminary simulation to verify the synchronization performance and state upper bounds, 2) amplitude and time scaling to enable circuit implementation, 3) circuit design, and 4) circuit simulation using Matlab/Simulink®.

The results of the preliminary simulation are shown in Figure (8.10-8.14). It can be concluded that the synchronization performance is satisfactory since the errors converge to a neighborhood of zero.

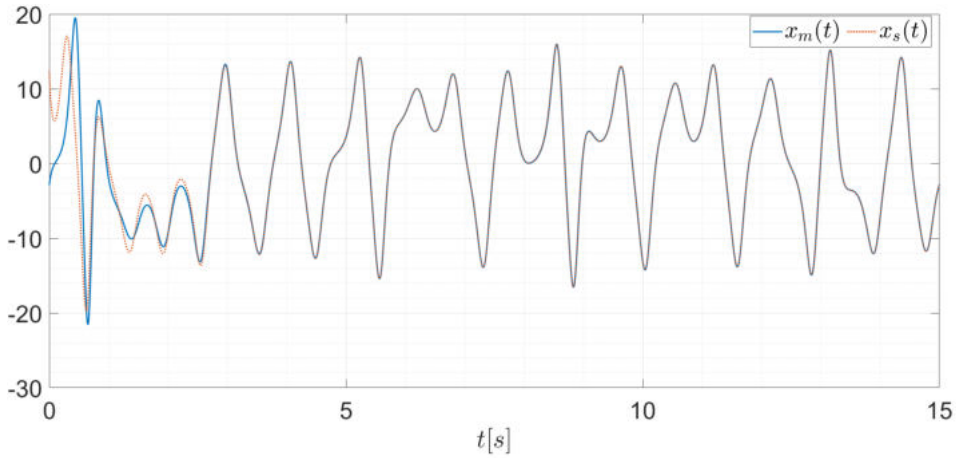

Figure 8.10 – Synchronization performance of the non-scaled systems (8.2) and (8.3), x_m and x_s.

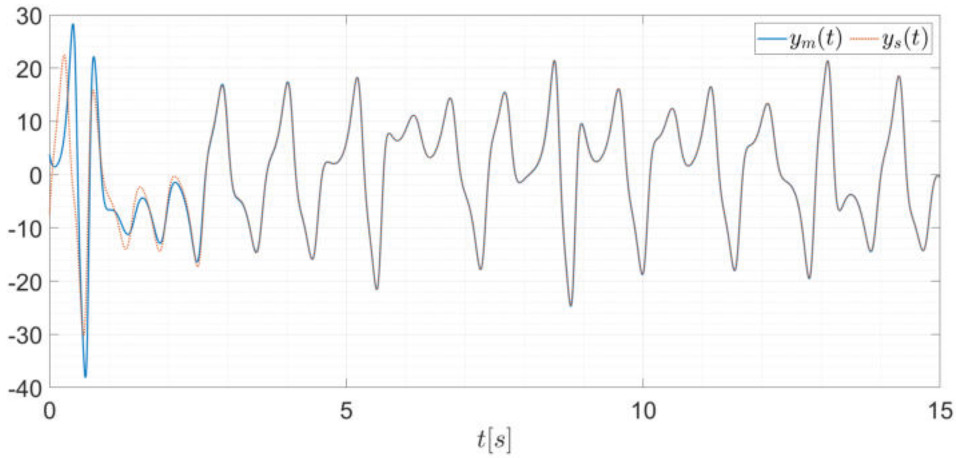

Figure 8.11 – Synchronization performance of the non-scaled systems (8.2) and (8.3), y_m and y_s.

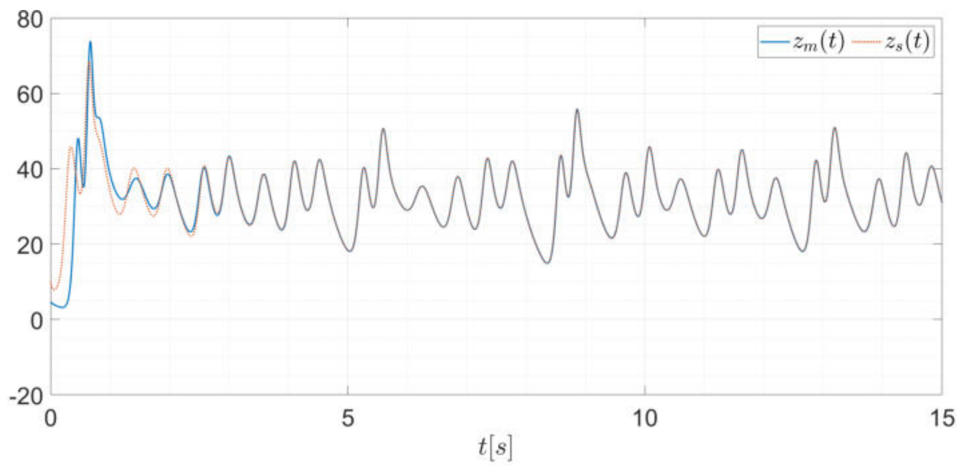

Figure 8.12 – Synchronization performance of the non-scaled systems (8.2) and (8.3), z_m and z_s.

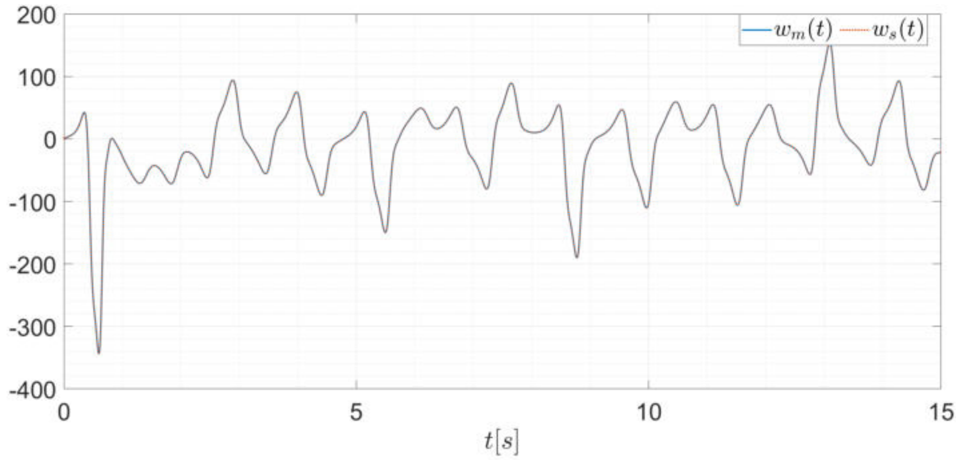

Figure 8.13 – Synchronization performance of the non-scaled systems (8.2) and (8.3), w_m and w_s.

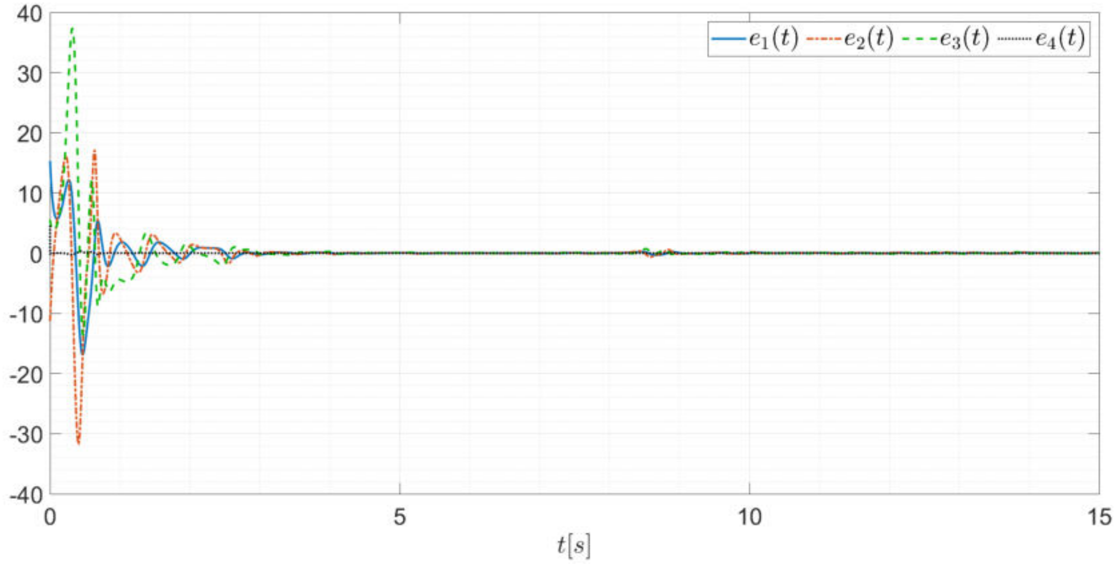

Figure 8.14 – Synchronization performance of the non-scaled systems (8.2) and (8.3), e_1, e_2, e_3 and e_4.

For the scaling phase, consider the scaled states $\mathcal{X} = \frac{x}{11.25}$, $\mathcal{Y} = \frac{y}{20}$, $\mathcal{Z} = \frac{z}{37.5}$, $\mathcal{W} = \frac{w}{175}$. Then, the magnitude scaling system results

$$\begin{cases} \dot{\mathcal{X}} &= 17.7777\mathcal{Y} - 10\mathcal{X} \\ \dot{\mathcal{Y}} &= 15.75\mathcal{X} - \mathcal{Y} - 21.09375\mathcal{X}\mathcal{Z} + 8.75\mathcal{W} - 0.005 \\ \dot{\mathcal{Z}} &= -2\mathcal{Z} + 6\mathcal{X}\mathcal{Y} \\ \dot{\mathcal{W}} &= 3.08571\mathcal{Y} + \mathcal{W} - 4.06808\mathcal{X}^3 + \kappa u \end{cases} \tag{8.24}$$

where κ is a constant defined as 0 for the master and 1 for the slave system. The frequency scaling is performed by dividing the left-hand side of (8.24) by 1000. Hence, we obtain the scaled system that will be used for implementation.

$$\begin{cases} \dot{X} &= 17777.78Y - 10000X \\ \dot{Y} &= 15750X - 1000Y - 21093.75XZ + 8750W - 5 \\ \dot{Z} &= -2000Z + 6000XY \\ \dot{W} &= 3085.71Y + 1000W - 4068.08X^3 + 1000\kappa u \end{cases} \tag{8.25}$$

Figuras (8.15 - 8.20) show the electronic circuits associated with (8.25), where OA1 is an operational amplifier block (OPA228), I1 is an inverter block, and M is an analog multiplier block (AD633JNC). Figure 8.15 depict the control block. It is possible to observe the simplicity of the control block defined by an adder circuit with $R_{13} = R_{14}$ and gain $\frac{R_{15}}{R_{13}}$. Figure 8.16 shows the message and encoded message block in which the master states are added to the message. The state blocks, Figures (8.17-8.20), are different for the master and slave systems due to the initial conditions, control, and disturbances. The initial conditions are represented by the voltages on the capacitors, see Figures (8.17 - 8.20) for further details. Rewriting (8.25) in terms of electronic components, we obtain

$$\begin{cases} \frac{dvc_1}{dt} &= \frac{1}{R_1C_1}vc_2 - \frac{1}{R_2C_1}vc_1 \\ \frac{dvc_2}{dt} &= \frac{1}{R_3C_2}vc_1 - \frac{1}{R_4C_2}vc_2 - \frac{1}{R_5C_2}vc_1vc_2 - \frac{1}{R_6C_2}vc_4 + \frac{1}{R_7C_2}V_d \\ \frac{dvc_3}{dt} &= -\frac{1}{R_8C_3}vc_3 + \frac{1}{R_9C_3}vc_1vc_3 \\ \frac{dvc_4}{dt} &= \frac{1}{R_{10}C_4}vc_2 + \frac{1}{R_{11}C_4}vc_4 - \frac{1}{R_{12}C_4}vc_1^3 + \kappa\zeta \end{cases} \tag{8.26}$$

where $\zeta = \frac{1}{RC_4}(vc_{4s} - vc_{4m})$; $R_1 \ldots R_{18}$, $C_1 \ldots C_4$ in Figures (8.15 - 8.20) have their values described in Table 8.2; and the voltages vc_1, vc_2, vc_3, and vc_4 are defined as being the states X, Y, Z, and W, respectively. The nominal values of the components are shown in Table 8.2.

□

In Figures (8.21) and (8.23), the analog multiplier blocks and operational amplifier blocks are depicted. It is important to note that the blocks were constructed considering the datasheet specifications (input bias and input offset voltage) of the OPA228 and AD633JNZ to model their imperfections. We consider offset voltages $S_4 = S_5 = 50$mV for the multiplier block in Figure 8.21, and an output defined as $O2 = \{\xi \cdot [(X1 - X2) \cdot (Y1 - Y2)/\upsilon] - (Z1 - Z2)\}$, with $\xi = 1$ and $\upsilon = 10$. For the operational amplifier OPA228, we consider an offset voltage $S_3 = 200\mu$V and leakage current $S_1 = S_2 = 10$nA.

The inverter block is shown in Figure 8.22. It was constructed using resistors with $R_1 = R_2 = 1k\Omega$ and an OA1 block (OPA228).

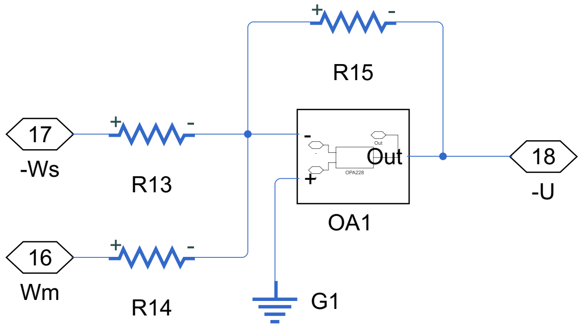

Figure 8.15 – Control block.

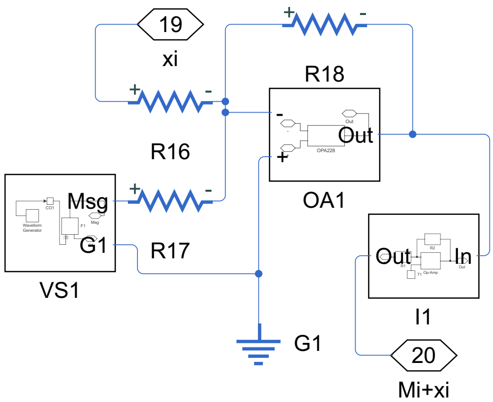

Figure 8.16 – Message and encoded message block, where VS1 (Voltage Source) is the message, xi is the master state, and Mi+xi is the encoded message (i = 1,2,3,4)

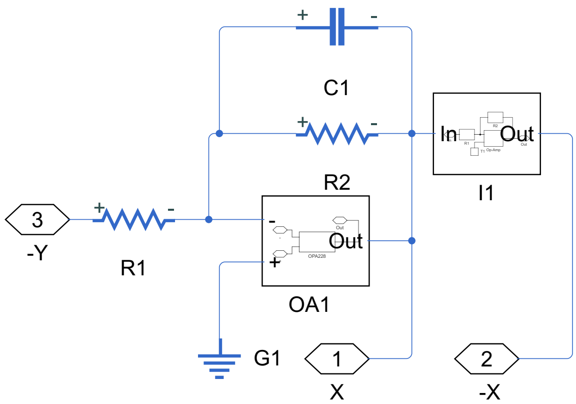

Figure 8.17 – X state block.

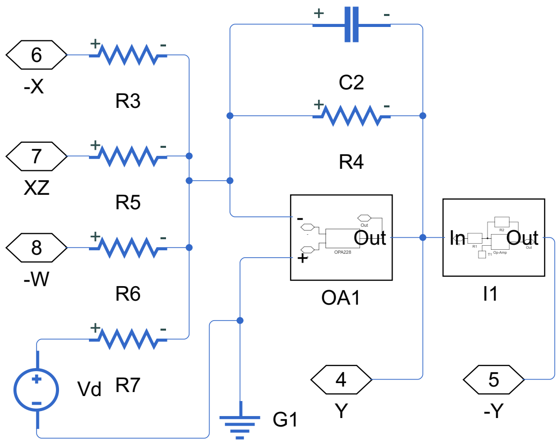

Figure 8.18 – Y state block.

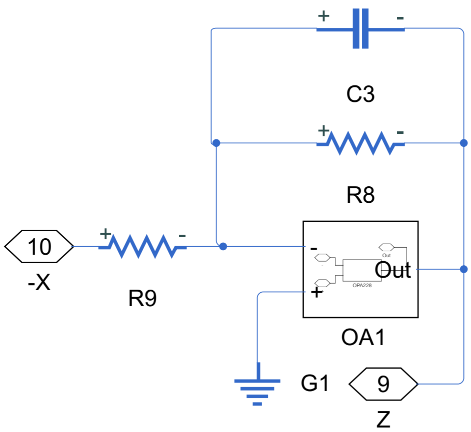

Figure 8.19 – Z state block.

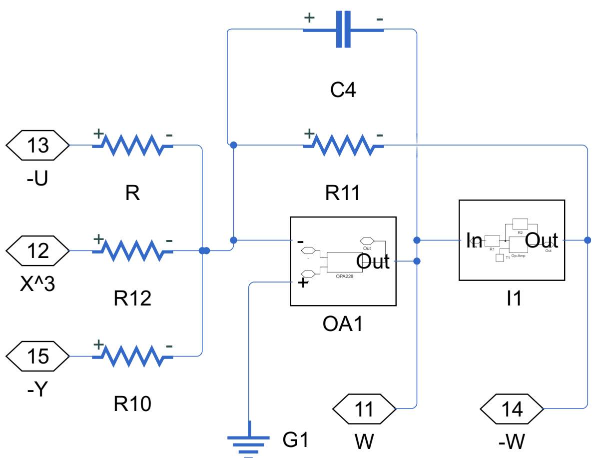

Figure 8.20 – W state block.

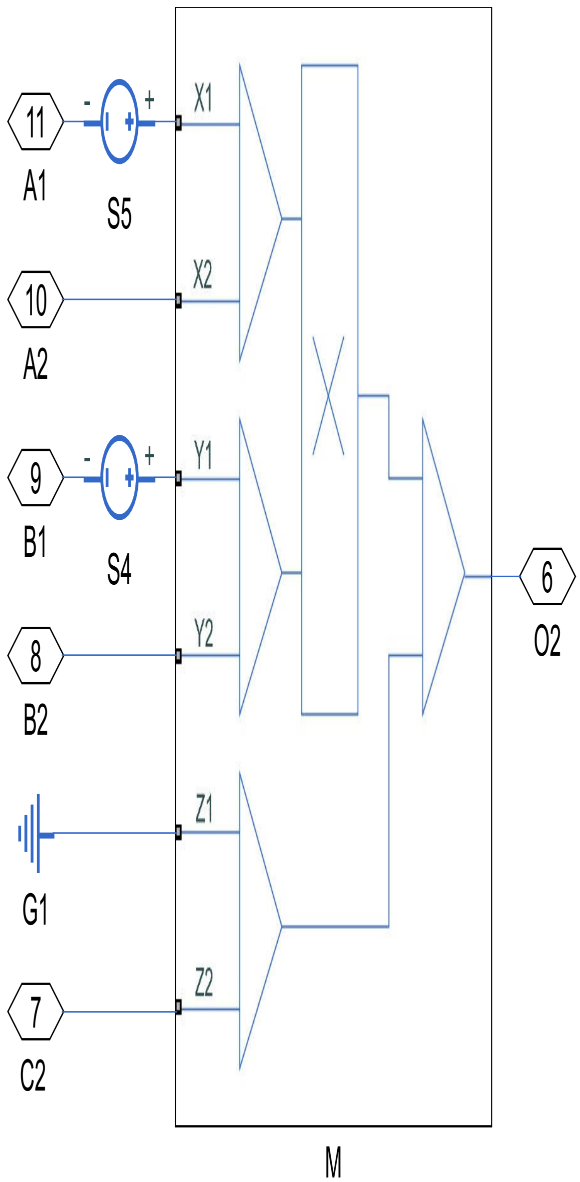

Figure 8.21 – Analog multiplier block (AD633JNZ).

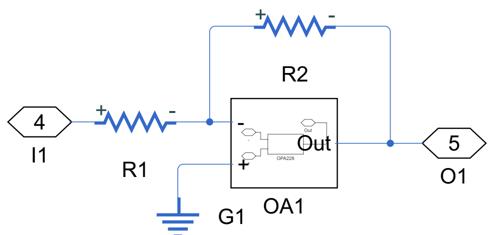

Figure 8.22 – Inverter block, where OA1 is an operational amplifier block (OPA228).

Table 8.2 – Electronic components used in Figures (8.15-8.20).

Parameter	Meaning	Value	Standard Value
R	Resistance	1kΩ	1kΩ
R_1	Resistance	5.62kΩ	5.49kΩ
R_2	Resistance	10kΩ	10kΩ
R_3	Resistance	6.349kΩ	6.34kΩ
R_4	Resistance	100kΩ	100kΩ
R_5	Resistance	4.74kΩ	4.7kΩ
R_6	Resistance	11.43kΩ	11.5kΩ
R_7	Resistance	10MΩ	10MΩ
R_8	Resistance	50kΩ	50kΩ
R_9	Resistance	16.667kΩ	16.7kΩ
R_{10}	Resistance	32.41kΩ	32.80kΩ
R_{11}	Resistance	100kΩ	100kΩ
R_{12}	Resistance	24.58kΩ	24.3kΩ
R_{13}, R_{14}, R_{15}	Resistance	1kΩ	1kΩ
R_{16}, R_{17}, R_{18}	Resistance	1kΩ	1kΩ
C_1, C_2, C_3, C_4	Capacitance	10nF	10nF
V_d	Voltage	2V	2V

To control the output gain of the multiplier AD633JNZ in Figure 8.21, the resistors $R_3 = 1k\Omega$ and $R_4 = 9k\Omega$ are chosen, as shown in the Figure 8.24.

The simulation is performed following the tolerance and environment conditions contained in Table 8.3.

Table 8.3 – Circuit Parameters.

Parameter	Value
Resistor tolerance	0.1%
Capacitance tolerance	0.1%
Maximum voltage resistor	25V
Maximum voltage capacitor	25V
Maximum power rating resistor	0.25W
Maximum power rating capacitor	40W
Temperature simulation	20°C
Noise sampling time	0.001s

Note that in Figure (8.25-8.32), the slave system trajectories converge to the master system trajectories, and, consequently, the synchronization errors converge to a neighborhood of the origin. The disturbances are due, mainly, to the imperfections of the components and the tolerances of the devices considered in the simulation.

To apply the synchronization algorithm for secure communication, the messages m_1, m_2, m_3, and m_4 were defined as: 1) m_1 is a filtered square wave with an amplitude of 0.1V and period of 1.25ms; 2) m_2 is a filtered sawtooth wave with amplitude 0.06V and period of

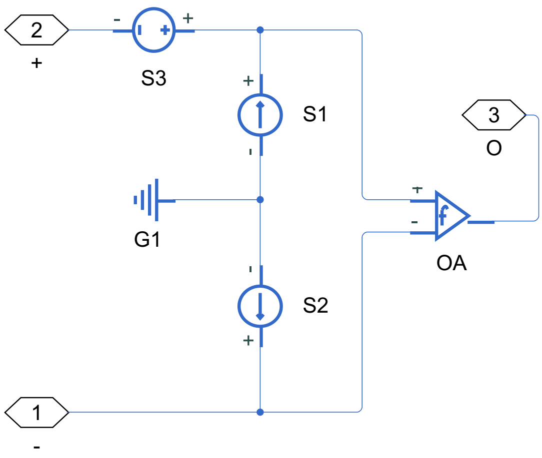

Figure 8.23 – Operational amplifier block (OPA228), where OA is an ideal operational amplifier.

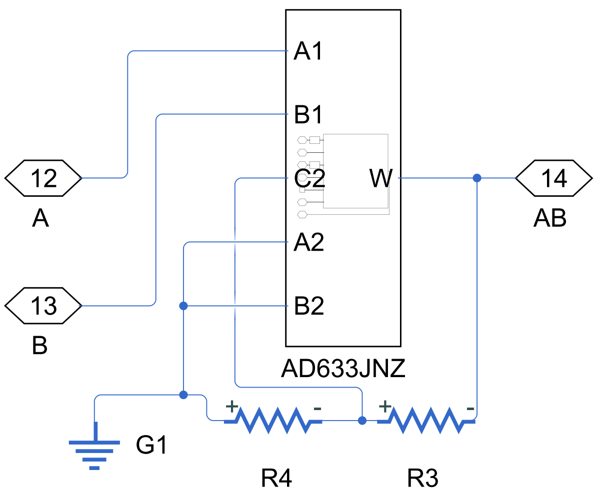

Figure 8.24 – The multiplier block using the AD633JNZ CI. The block M is an analog multiplier AD633JNZ.

1.25ms, and 3) m_3 and m_4 are filtered random bit sequences of amplitude 0.04V and period of 1.25ms. The filtered messages were obtained using a first-order filter with a transfer function given by

$$\frac{50000}{s + 50000} \tag{8.27}$$

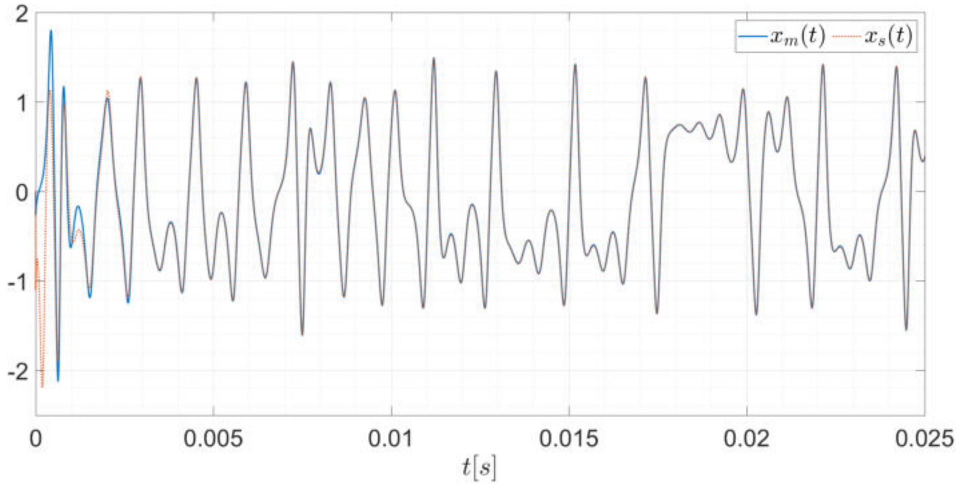

Figure 8.25 – Synchronization of the scaled master-slave system using circuital simulation. X_m, X_s.

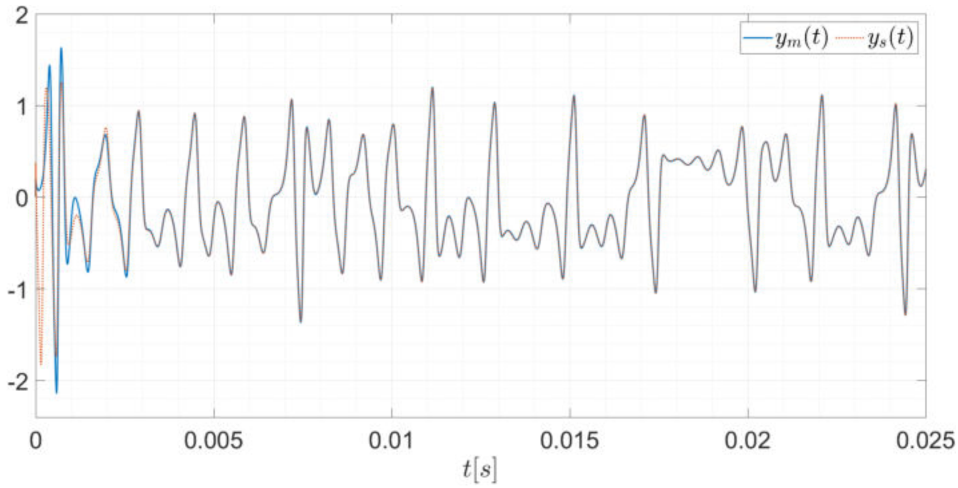

Figure 8.26 – Synchronization of the scaled master-slave system using circuital simulation. Y_m, Y_s.

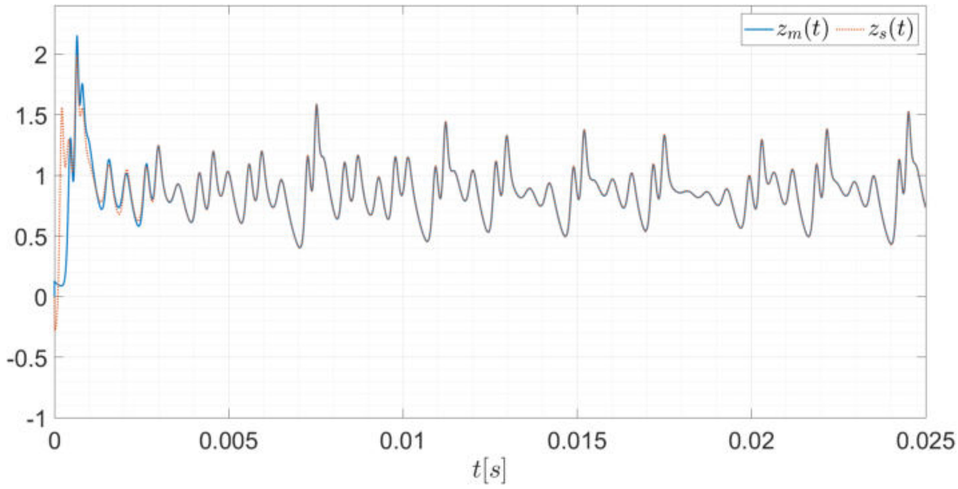

Figure 8.27 – Synchronization of the scaled master-slave system using circuital simulation. Z_m, Z_s.

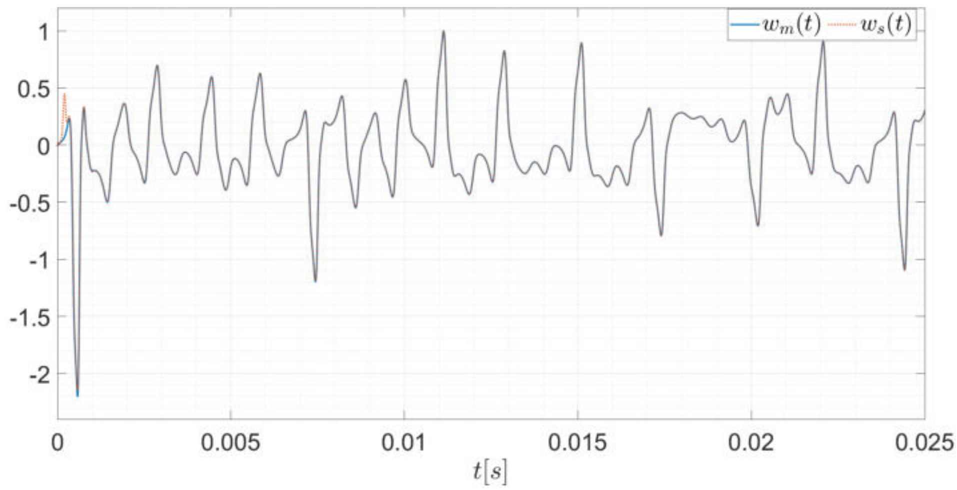

Figure 8.28 – Synchronization of the scaled master-slave system using circuital simulation. W_m, W_s.

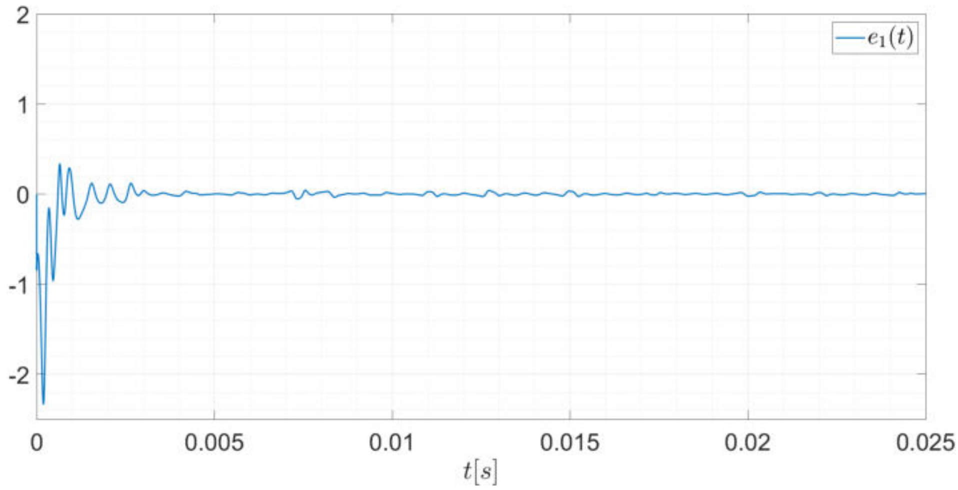

Figure 8.29 – Scaled master-slave system synchronization error using circuital simulation, $e_1(t)$.

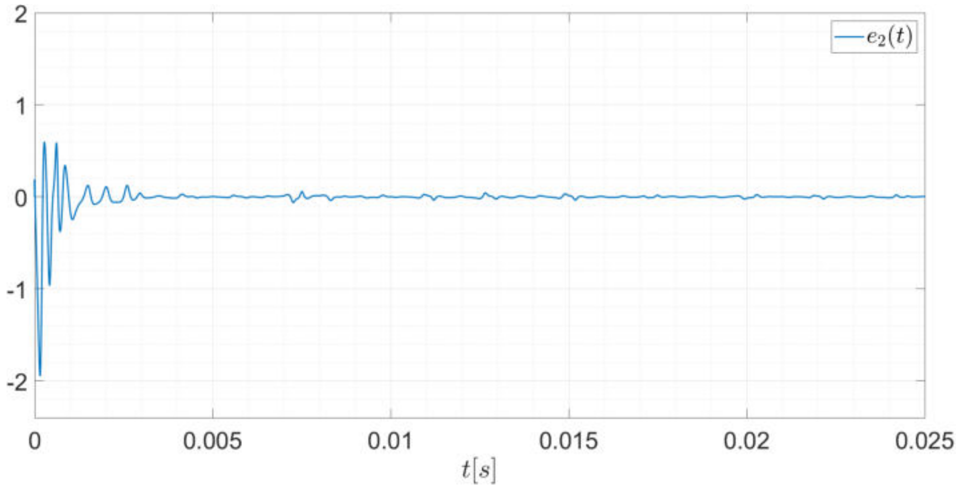

Figure 8.30 – Scaled master-slave system synchronization error using circuital simulation, $e_2(t)$.

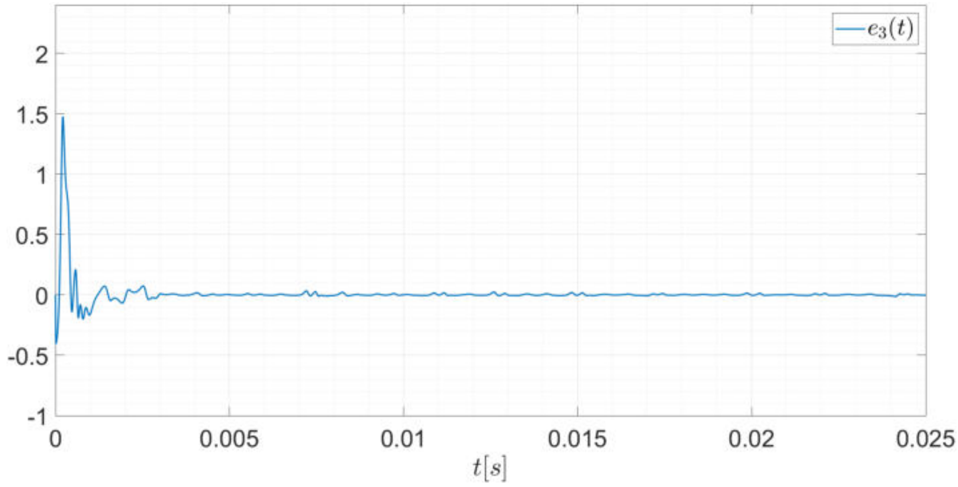

Figure 8.31 – Scaled master-slave system synchronization error using circuital simulation, $e_3(t)$.

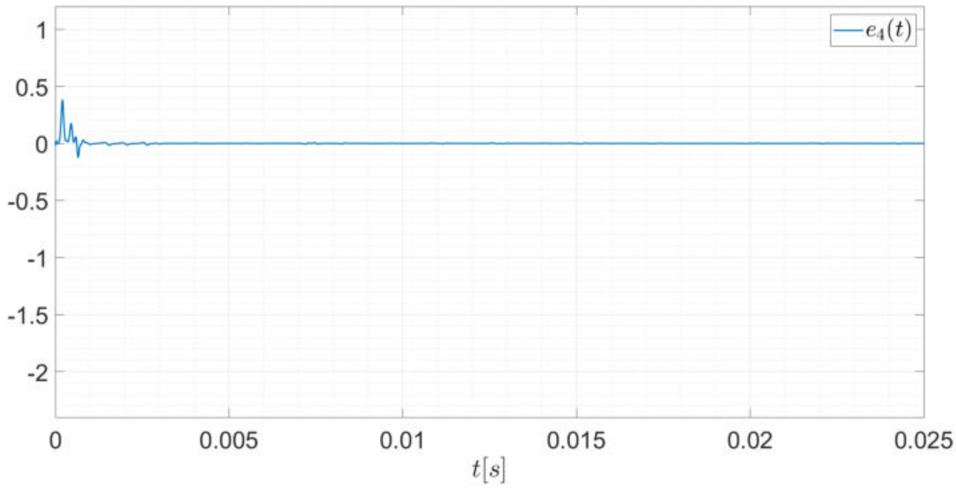

Figure 8.32 – Scaled master-slave system synchronization error using circuital simulation $e_4(t)$.

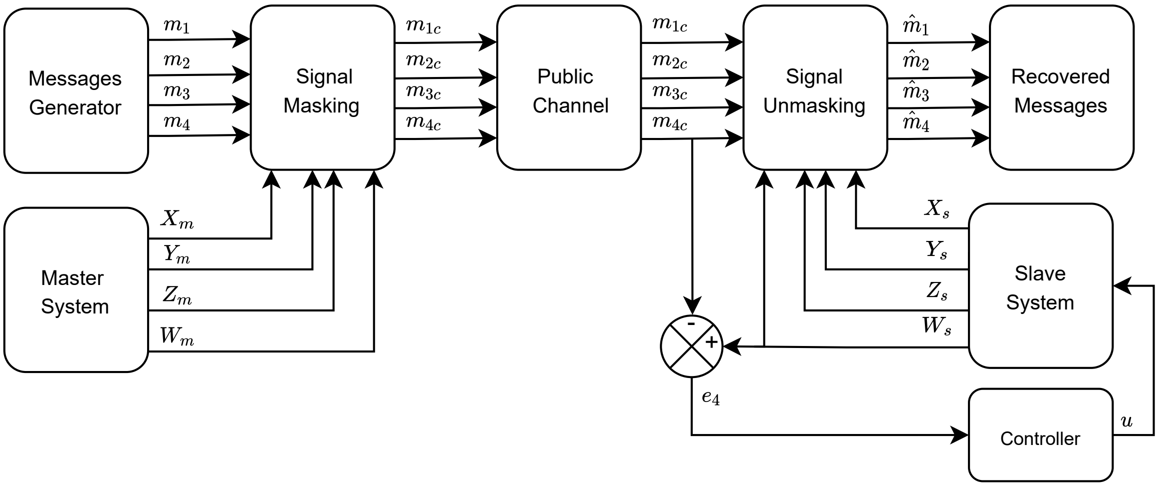

Figure 8.33 – Block diagram of the secure communication system.

Being these filtered messages encoded by the master system according to Figure 8.33. The encoded messages are defined as m_{1c}, m_{2c}, m_{3c}, and m_{4c}. In Figure (8.34-8.37), the original and encrypted messages are compared. Figure (8.38 - 8.45) shows the original and decoded messages. As expected, the decryption errors are small, even in the presence of non-modeled dynamics and bounded disturbances due to the considered simulation parameters in Table 8.3.

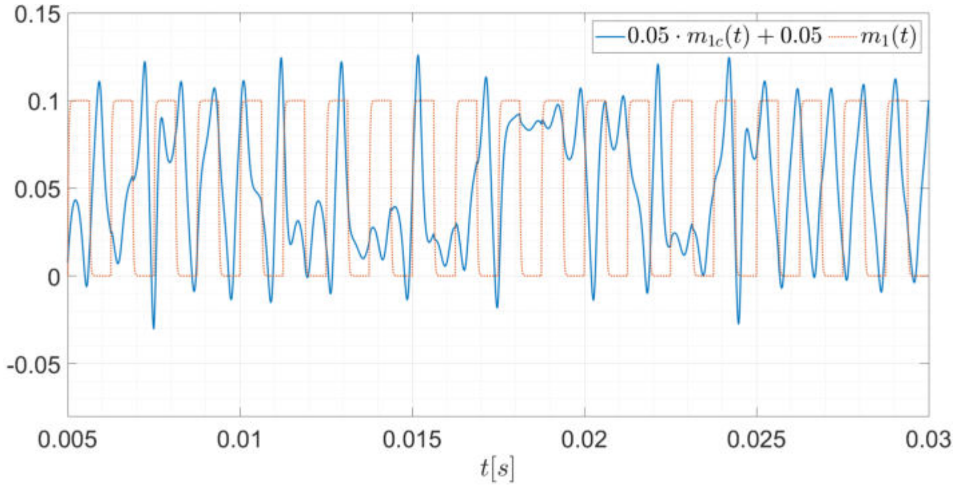

Figure 8.34 – Comparison between the original and encrypted messages with the conditions given in the Table (8.3), m_1, m_{1c}.

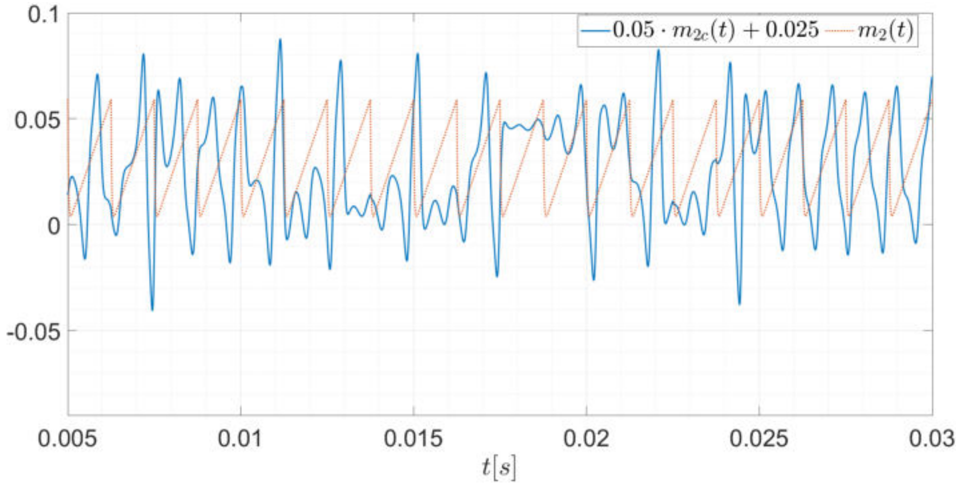

Figure 8.35 – Comparison between the original and encrypted messages with the conditions given in the Table (8.3), m_2, m_{2c}.

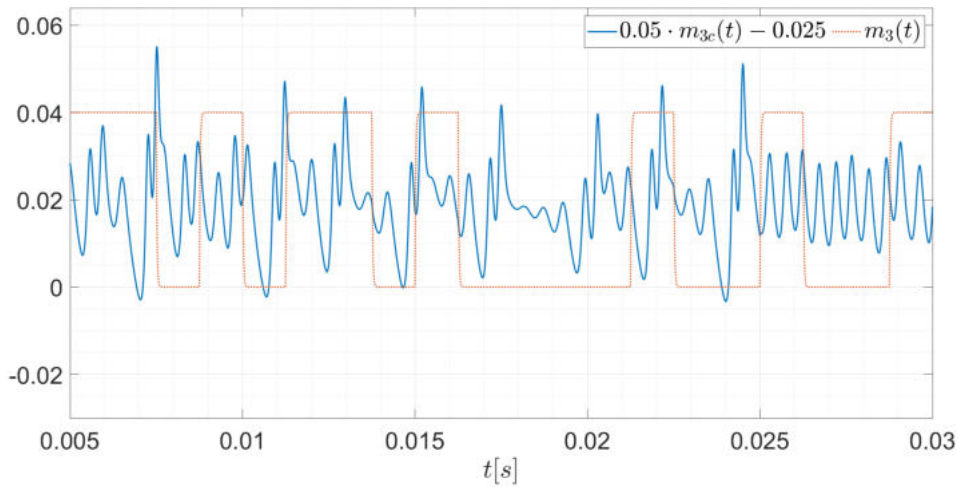

Figure 8.36 – Comparison between the original and encrypted messages with the conditions given in the Table (8.3), m_3, m_{3c}.

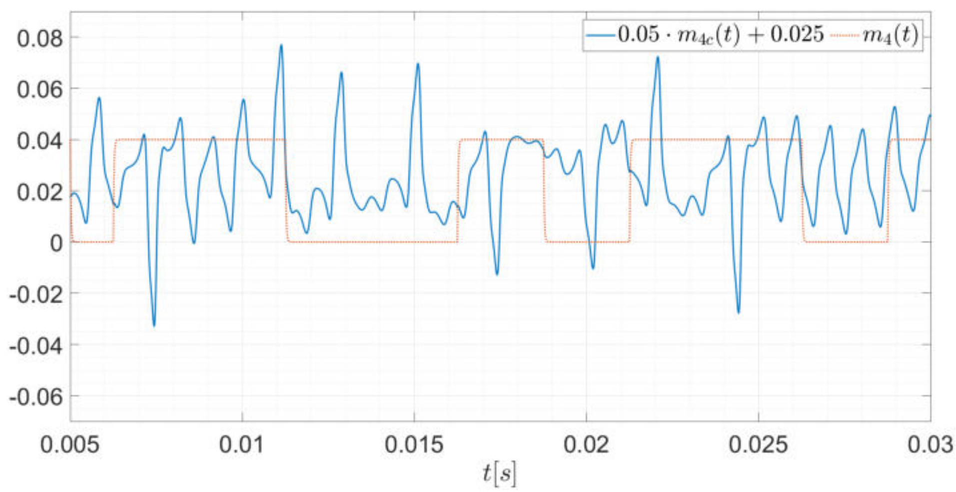

Figure 8.37 – Comparison between the original and encrypted messages with the conditions given in the Table (8.3), m_4, m_{4c}.

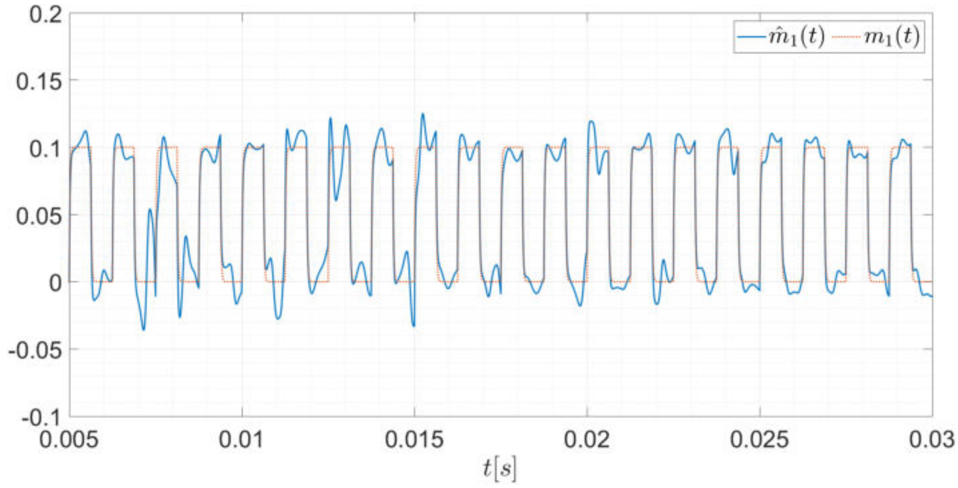

Figure 8.38 – Comparison between the recovered message and the original message. ($\hat{m}_1$ and m_1)

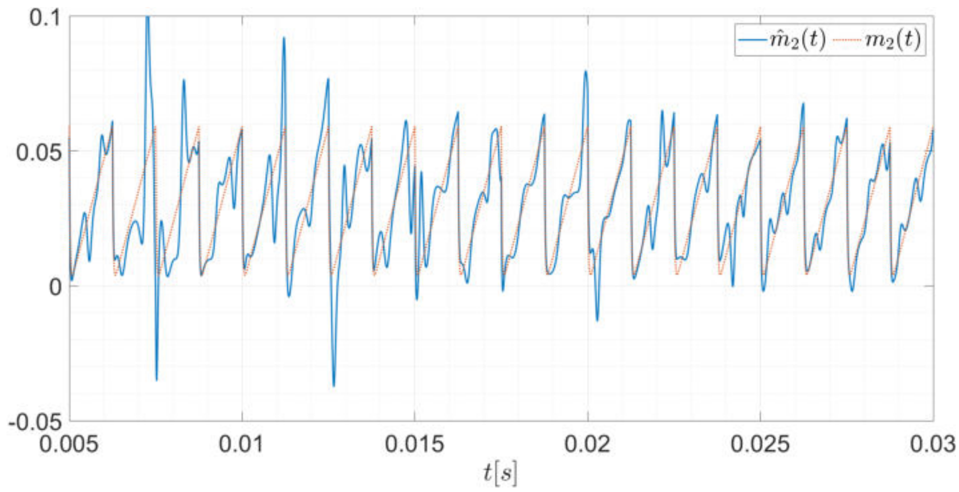

Figure 8.39 – Comparison between the recovered message and the original message. $\hat{m}_2$ and m_2.

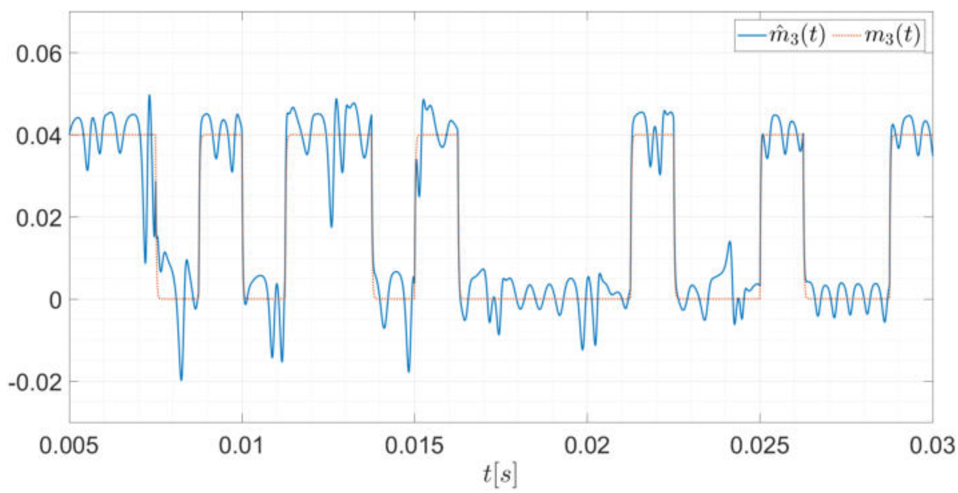

Figure 8.40 – Comparison between the recovered message and the original message. $\hat{m}_3$ and m_3.

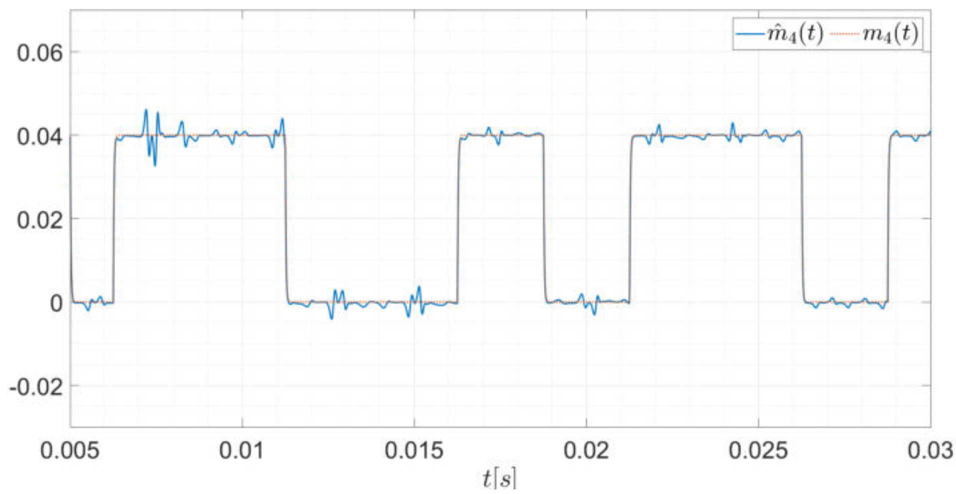

Figure 8.41 – Comparison between the recovered message and the original message. $\hat{m}_4$ and m_4.

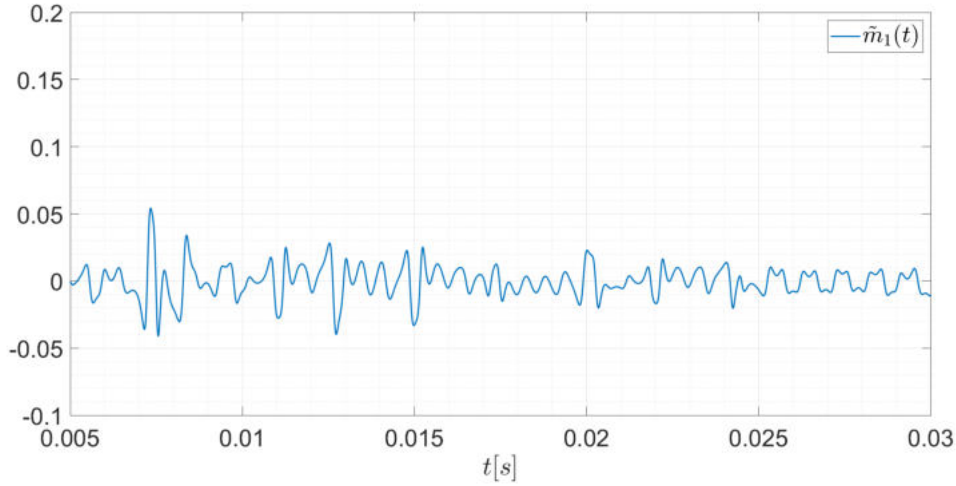

Figure 8.42 – Errors in the recovery of encrypted messages, $\tilde{m}_1$.

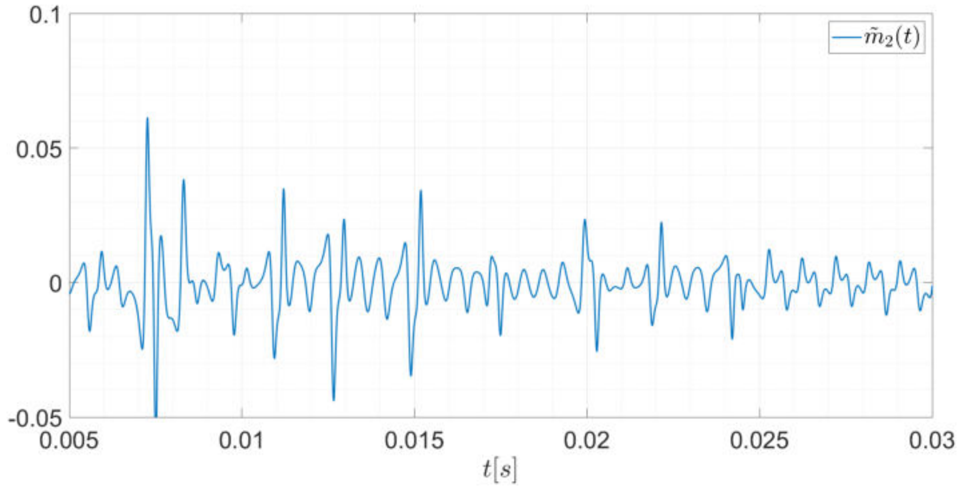

Figure 8.43 – Errors in the recovery of encrypted messages, $\tilde{m}_2$.

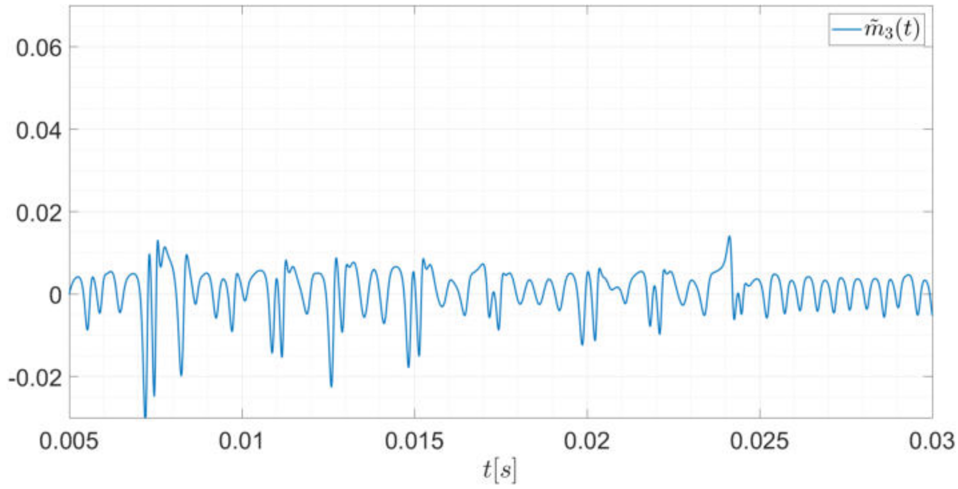

Figure 8.44 – Errors in the recovery of encrypted messages, $\tilde{m}_3$.

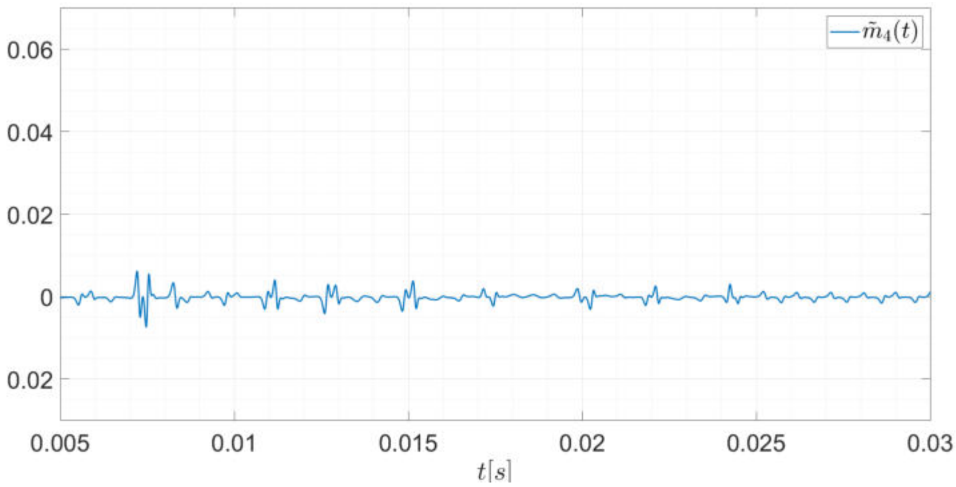

Figure 8.45 – Errors in the recovery of encrypted messages, $\tilde{m}_4$.

8.8 CONCLUSIONS

A minimal synchronization method for a class of hyperchaotic systems has been proposed in this chapter. Based on Lyapunov theory, a scalar control law has been introduced to make the residual synchronization error small, even in the presence of full-state disturbances. The design considers that the initial synchronization error is sufficiently small to simplify the structure of the synchronizer. This condition can be satisfied when the hyperchaotic systems do not have trivial equilibrium points. The main advantage of the proposed scheme is its simplicity and robustness. A comparison study has been accomplished to show the peculiarities of the proposed method. Besides, an secure communication example using analog electronics has been considered to show a typical application. The main drawback of the work is the impossibility of applying the proof methodology to systems with trivial equilibrium points. Another limitation is that preliminary knowledge of the master system is required to design the proposed synchronizer. Future works include more sophisticated synchronization algorithms based on complex strategies such as backstepping, sliding mode control, and online approximators.

9 CONCLUSIONS

This work different synchronization schemes were studied as a basis to then propose a generic communication system based on hyperchaotic systems of a generic class.

9.1 CHAPTERS CONCLUSIONS

In chapter 3, a projective synchronization and anti-synchronization algorithm based on Lyapunov theory was proposed for an underactuated chaotic system subject to disturbances where only two control signals are required to perform master and slave synchronization and anti-synchronization. The synchronizer has some limitations, such as the sensitivity of the errors to the disturbances present in the disturbed states. Underactuated control is one of the contributions of this work since it is rare in the literature to use the control signal in only one state equation for the synchronization case. Computational simulations were performed using Matlab and Simulink to validate the simplicity of the proposed scheme.

In chapter 4, a synchronization algorithm based on Lyapunov stability theory was proposed for a hyperchaotic financial system subject to limited disturbances. It was shown that for a correct synchronization between master and slave systems, it is a sufficient condition to have control signals present in only two of the system states, and this is the main contribution of the work. The main limitations of the work are the sensitivity of the synchronization error to disturbances in underactuated states. Another limitation is the need for the control dimension to be two and not one. Computational simulations were performed using Matlab to validate the designed scheme.

In chapter 5, A synchronization algorithm based on the Lyapunov stability theory was proposed for a Liu system subject to bounded disturbances. It has been proven and validated through computer simulations using Matlab and Simulink that it is necessary to use the control signal in only one of the states to achieve complete synchronization of the master and slave systems, which is the main advantage of the proposed method. The system has limitations in the sensitivity of the synchronization error to the presence of disturbances in non-actuated states. The effectiveness of the proposed method to encode and restore the message in both theory and simulations has also been demonstrated.

In chapter 6, an under-actuated synchronization scheme for Chua's chaotic system and subject to disturbances based on Lyapunov stability theory is proposed. The main advantages of the proposed scheme are its simplicity since it is based on proportional control, and considering the presence of disturbances in the stability analysis. A secure communication scheme is also suggested based on this proposal to emphasize its applicability in practical situations. We also prove that only one control signal is sufficient to synchronize the master and slave system through Lyapunov theory. The effectiveness of the control signal is validated through computer simulations using Matlab.

In chapter 7, a secure communication scheme based on the synchronization of an underactuated 4D hyperchaotic system is proposed. It is proven, based on Lyapunov analysis, that synchronization errors are bound even with the consideration of disturbances in the stability analysis. The proposed scheme only requires the control to act on two of the slave system's state equations. The scheme has the advantage of being robust against limited internal and external disturbances. Computational simulations were performed using Matlab to validate the robustness of the proposed scheme.

Finally, chapter 8 presents the main contribution of the thesis: a minimal synchronization method for a class of hyperchaotic systems. Based on Lyapunov theory, a scalar control law was introduced to make the residual synchronization error small, even in the presence of full-state perturbations. The design considers that the initial synchronization error is small enough to simplify the structure of the synchronizer. The main advantage of the proposed scheme is its simplicity and robustness. A comparative study was conducted to show the peculiarities of the proposed method. In addition, an example of secure communication using analog electronics was considered to show a typical application. Computational simulations were performed using Matlab and Simulink to validate the robustness and simplicity of the proposed method.

9.2 FUTURE WORK

- Incorporate more sophisticated synchronization algorithms based on complex strategies such as sliding mode control, online approximators, nonlinear damping, and control based on *integrator backsteeping* to improve synchronization performance and hence the fidelity of the encrypted message reconstruction.
- Reduce implementation costs or limit the computational load to minimize the resource usage of FPGAs.
- Validate the developed schemes using Matlab/Simulink by performing intensive simulations and using FPGAs.
- Propose robust, underactuated synchronization schemes applied to high-dimensional hyperchaotic systems.

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APPENDIX

A LYAPUNOV STABILITY THEORY

A.1 PRELIMINARY MATHEMATICS

This section provides some fundamental mathematical concepts that are necessary for the chapters studied.

A.1.1 Vector norms

DEFINITION 1 Let $x \in X \subset \Re^n$ be a n-dimensional vector. The p-norm of f is defined by

$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{1/p} \tag{A.1}$$

for

$$p \in [1, \infty)$$

Thus, by denoting $p = 1, 2, \infty$, the corresponding normed spaces are called L_1, L_2, L_∞, respectively. In this thesis usually, it is used the case where $p = 2$:

$$\|x\| = \|x\|_2 = \sqrt{\sum_i |x_i|^2} \tag{A.2}$$

$$\|x\|^2 = x^T x, \; x \in \Re^{1 \times n} \tag{A.3}$$

By defining a real constant α, using the vector $x \in \Re^n$, and considering $y \in Y \subset \Re^n$ a n-dimensional vector, the followings properties are true:

$$||x|| \geq 0 \tag{A.4}$$

$$||\alpha {\cdot} x|| = |\alpha| {\cdot} ||x|| \tag{A.5}$$

$$||x + y|| \leq ||x|| + ||y|| \tag{A.6}$$

More information can be found at [231, 232]

A.1.2 Lyapunov Stability Theory

We present in this section some concepts about Lyapunov stability theory. The following definitions and theorem were extracted from [223].

A.1.3 Stability Principles

We consider systems described by ordinary differential equations of the form

$$\dot{x} = f(t, x), \; x(t_0) = x_0 \tag{A.7}$$

where $x \in \Re^n$, $f : \tau \times B(r)$, $\tau = [t_0, \infty)$, and $B(r) = \{x \in \Re^n | \, \|x\| < r\}$. We assume that f is of such nature that for every $x_0 \in B(r)$ and every $t_0 \in \Re^+$, (A.6) possesses one and only one solution $x(t; t_0; x_0)$.

DEFINITION 2 A state x_e is said to be an **equilibrium state** of the system described by (A.7) if

$$f(t, x_e) \equiv 0 \, para \; toda \;\; t \geq t_0 \tag{A.8}$$

DEFINITION 3 An equilibrium state x_e is called an **isolated equilibrium state** if there exists a constant $r > 0$ such that $B(x_e, r) := \{x | \, \|x - x_e\| < r\}$ contains no equilibrium state of (A.7) other than x_e.

DEFINITION 4 The equilibrium state x_e is said to be **stable(in the sense of Lyapunov)** if for arbitrary t_0 and $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_0)$ such that $|x_0 - x_e| < \delta$ implies $|x(t; t_0; x_0) - x_e|$ for all $t \geq t_0$.

DEFINITION 5 The equilibrium state x_e is said to be **uniformly stable (u.s)** if it is stable and if $\delta(\varepsilon, t_0)$ in Definition 4 does not depend on t_0.

DEFINITION 6 The equilibrium state x_e is said to be **asymptotically stable (a.s)** if (i) it is stable, and (ii) there exists a $\delta(t_0)$ such that $|x_0 - x_e| < \delta(t_0)$ implies $lim_{t\to\infty}|x(t; t_0; x_0) - x_e| = 0$. If condition (ii) is satisfied, then the equilibrium state x_e is said to be **attractive**.

DEFINITION 7 The set of all $x_0 \in \Re^n$ such that $x(t; t_0; x_0) \to x_e$ as $t \to \infty$ for some $t_0 \geq 0$ is called the **region of attraction** of the equilibrium state x_e.

DEFINITION 8 The equilibrium state x_e is said to be **uniformly asymptotically stable (u.a.s)** if (i) it is uniformly stable, (ii) for every $\varepsilon > 0$ and any $t_0 \in \Re^+$, there exist a $\delta_0 > 0$ independent of t_0 and ε and a $T(\varepsilon) > 0$ independent of t_0, such that $|x(t; t_0; x_0) - x_e| < \varepsilon$ for all $t > t_0 + T(\varepsilon)$ whenever $|x_0 - x_e| < \delta_0$.

Definition 9 The equilibrium state x_e is **exponentially stable (e.s)** if there exists an $\alpha > 0$, and for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\|x(t; t_0; x_0) - x_e\| < \varepsilon e^{-\alpha(t-t_0)} \; para\ toda\ \ t \geq t_0 \tag{A.9}$$

whenever $\|x_0 - x_e\| < \delta(\varepsilon)$.

Definition 10 The equilibrium state x_e is said to be **unstable** if it is not stable. (A.6) have a unique solution for each $x_0 \in \Re^n$ and $t_0 \in \Re^+$, we need the following definitions for the global characterization of solutions.

.

Definition 11 A solution $x(t; t_0; x_0)$ of (A.6) is **bounded** if there exists a $\beta > 0$ such that $|x(t; t_0; x_0)| < \beta$ for all $t > t_0$, where β may depend on each solution.

Definition 12 The solutions of (A.6) are **uniformly bounded (u.b)** if for any $\alpha > 0$ and $t_0 \in \Re^+$, there exists a $\beta = \beta(\alpha)$ independent of t_0 such that if $|x_0| < \alpha$, then $|x(t; t_0; x_0)| < \beta$ for all $t > t_0$.

Definition 13 The solutions of (A.6) are **uniformly ultimately bounded (u.u.b)** (with bound B) if there exists a $B > 0$ and if corresponding to any $\alpha \geq 0$ and $t_0 \in \Re^+$, there exists a $T = T(\alpha) > 0$ (independent of t_0) such that $|x_0| < \alpha$ implies $|x(t; t_0; x_0)| < B$ for all $t > t_0 + T$.

Definition 14 If $x(t; t_0; x_0)$ is a solution of $\dot{x} = f(t, x)$, then the trajectory $x(t; t_0; x_0)$ is said to be **stable (u.s., a.s., u.a.s., e.s., unstable)** if the equilibrium point $z_e = 0$ of the differential equation

$$\dot{z} = f(t, z + x(t; t_0; x_0)) - f(t, x(t; t_0; x_0)) \tag{A.10}$$

é **estável (u.d., a.e., u.a.e., e.e., instável, respectivamente)**.

A.1.4 Lyapunov's Direct Method

The stability properties of the equilibrium state or solution of (A.6) can be studied by using the direct method of Lyapunov (also known as Lyapunov's second method). The objective of this method is to answer questions of stability by using the form of $f(t, x)$ in (A.6) rather than the explicit knowledge of the solutions. We start with the following definitions.

Definition 15 A continuous function $\varphi : [0, r] \to \Re^+$ (or a continuous function $\varphi : [0, \infty) \to \Re^+$) is said to belong to **class** K, i.e., $\varphi \in K$, if

(i) $\varphi(0) = 0$.

(ii) φ is strictly increasing on $[0, r]$ (or on $[0, \infty)$).

Definition 16 A continuous function $\varphi : [0, \infty) \to \Re^+$ is said to belong to **class** KR, i.e., $\varphi \in KR$, if

(i) $\varphi(0) = 0$.

(ii) φ is strictly increasing on $[0, \infty)$.

(iii) $lim_{r\to\infty}\varphi(r) = \infty$.

Definition 17 Two functions $\varphi_1, \varphi_2 \in K$ defined on $[0, r]$ (or on $[0, \infty]$) are said to be **of the same order of magnitude** if there exist positive constants k_1, k_2, such that

$$k_1\varphi_1(r_1) \leq \varphi_2(r_1) \leq k_2\varphi_2(r_1), \forall r_1 \in [0, r]\,(or \forall r_1 \in [0, \infty]) \tag{A.11}$$

Definition 18 A function $V(t, x) : \Re^+ \times B(r) \to \Re$ with $V(t, 0) = 0, \forall t \in \Re^+$ is **positive definite** if there exists a continuous function $\varphi \in K$ such that $V(t, x) \geq \varphi(|x|), \forall t \in \Re^+$, $x \in B(r)$ and some $r > 0$. $V(t, x)$ is called **negative-definite** if $-V(t, x)$ is positive definite.

Definition 19 A function $V(t, x) : \Re^+ \times B(r) \to \Re$ with $V(t, 0) = 0, \forall t \in \Re^+$ is said to be **positive(negative) semidefinite** if $V(t, x) \geq 0 (V(t, x) \leq 0)$, for all $t \in \Re^+$ and $x \in B(r)$ for some $r > 0$.

Definition 20 A function $V(t, x) : \Re^+ \times B(r) \to \Re$, with $V(t, 0) = 0, \forall t \in \Re^+$ is said to be **decrescent** if there exists $\varphi \in K$ such that $|V(t, x)| \leq \varphi(|x|), \forall t \geq 0$ and $\forall x \in B(r)$ for some $r > 0$.

Definition 21 A function $V(t, x) : \Re^+ \times \Re^r \to \Re$ with $V(t, 0) = 0, \forall t \in \Re^+$ is said to be **radially unbounded** if there exists $\varphi \in KR$ such that $V(t, x) \geq \varphi(|x|)$ for all $x \in \Re^n$ and $t \in \Re^+$.

It is clear from the Definition (21) that if $V(t, x)$ is radially unbounded, it is also positive definite for all $x \in \Re^n$, but the converse is not true.

Let us assume (without loss of generality) that $x_e = 0$ is an equilibrium point of (A.6) e definir $\dot{V}$ para ser a derivada temporal da função $V(t, x)$ ao longo da solução de (A.6), assim

$$\dot{V} = \frac{\partial V}{\partial t} + (\nabla V)^T f(t, x) \tag{A.12}$$

where $\nabla V = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, ..., \frac{\partial V}{\partial x_n}\right]^T$ is the gradient of V with respect to x. The second method of Lyapunov is summarized by the following theorem.

THEOREM A.1.1 Suppose there exists a positive definite function $V(t,x) : \Re^+ \times B(r) \to \Re$ for some $r > 0$ with continuous first-order partial derivatives with respect to x, t, and $V(t,0) = 0, \forall t \in \Re^+$. Then, the following statements are true:

(i) If $\dot{V} \leq 0$, then $x_e = 0$ is **stable**.

(ii) If V is decrescent and $\dot{V} \leq 0$, then $x_e = 0$ is **uniformly stable**.

(iii) If V is decrescent and $\dot{V} < 0$, then x_e is **uniformly asymptotically stable**.

(iv) If V is decrescent and there exist $\varphi_1, \varphi_2, \varphi_3 \in K$ of the same order of magnitude such that

$$\varphi_1(|x|) \leq V(t,x) \leq \varphi_2(|x|), V(t,x) \leq -\varphi_3(|x|) \tag{A.13}$$

for all $x \in B(r)$ and $t \in \Re^+$, then $x_e = 0$ is exponentially stable.

In the above theorem, the state x is restricted to be inside the ball $B(r)$ for some $r > 0$. Therefore, the results (i) to (iv) of Theorem A.1.1 are referred to as local results.

THEOREM A.1.2 Assume that (A.7) possesses unique solutions for all $x_0 \in \Re^n$. If there exists a function $V(t,x)$ defined on $|x| \geq R$ (where R may be large) and $t \in [0,\infty)$ with continuous first-order partial derivatives with respect to x, t and if there exist $\varphi_1, \varphi_2 \in KR$ such that

(i) $\varphi_1(|x|) \leq V(t,x) \leq \varphi_2(|x|)$

(ii) $\dot{V}(t,x) \leq 0$ for all $|x| \geq R$ and $t \in [0,\infty)$, then, the solutions of (A.6) are uniformly bounded. If in addition there exists $\varphi_3 \in K$ defined on $[0,\infty)$ and

(iii) $\dot{V}(t,x) \leq -\varphi_3(|x|)$ for all $|x| \geq R$ and $t \in [0,\infty)$ then, the solutions of (A.6) are uniformly ultimately bounded.

A.1.5 Young inequality

By defining that a and b are positive variables, $p > 1$ and $q > 1$, consider the following Young inequality for products

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{A.14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Assuming p = 2 and q = 2, then

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \tag{A.15}$$

Note that this expression can also be deduced from inequality $(a-b)^2 \geq 0$. This is important because, in the case of $p=2$ and $q=2$, (A.15) is also correct for any negative real value of a and b. Rewriting (A.15) results

$$\left(a\sqrt{\sigma}\right)\left(\frac{b}{\sqrt{\sigma}}\right) \leq \frac{a^2\sigma}{2} + \frac{b^2}{2\sigma} \tag{A.16}$$

being $\sigma > 0$. More information can be obtained in [224].

B COMPUTATIONAL CODES USING MATLAB TO VALIDATE THE PROPOSED METHOD

B.1 CODES FOR SIMULATIONS IN CHAPTER 3

B.1.1 Simulink Plant for the Projective Synchronization and Antisynchronization of an Underactuated System Based on Proportional Control, corresponding to Figures 3.1-3.6 and the Table 3.1.

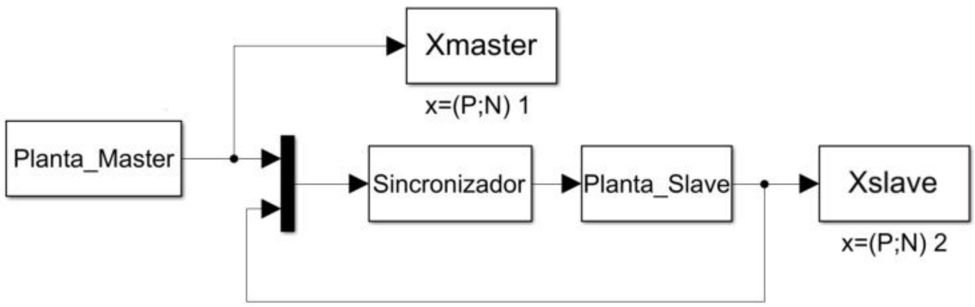

Figure B.1 – Simulink Plant.

Listing B.1 – Planta-Master.m

```
    function [sys,x0,str,ts] = Planta_Master(t,x,u,flag)

a=-10;  %Constants
b=-4;
c=0;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Initialization %
   %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 3; %Number of constants states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 3;    %Number of outputs
    sizes.NumInputs = 0;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
```

```
    sys = simsizes(sizes);
    x0=[3 -4 2]; %Initial Conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Directives   %
   %%%%%%%%%%%%%%
case 1,      %Sistem
  sys = [-(a*b/(a + b))*x(1) - x(2)*x(3) + c;
        a*x(2) + x(1)*x(3);
        b*x(3) + x(1)*x(2)] + disturb(x,u,t);
   %%%%%%%%%
   % Outputs %
   %%%%%%%%%
case 3,
   sys = x;
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; % Does nothing
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end


function disturb = disturb(x,u,t)
if t>=10
    disturb= [1*cos(4*t); %4
              1.2*cos(3*t);   %6
              0.1*sin(7*t)];  %3
else
    disturb=0; %Until t=10 secs the disturb is null
end
```

Listing B.2 – Planta-Slave.m

```
function [sys,x0,str,ts] = Planta_Slave(t,x,u,flag)

a=-10;  %Constants
b=-4;
c=0;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Initialization %
   %%%%%%%%%%%%%%%%%
case 0,
```

```
  sizes = simsizes;
    sizes.NumContStates = 3; %Number of constants states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 3;     %Number of outputs
    sizes.NumInputs = 3;      %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=[5 -5 3]; %Initial Conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Directives   %
   %%%%%%%%%%%%%%
case 1,      %Sistem
  sys = [-(a*b/(a + b))*x(1) - x(2)*x(3) + c + u(1);
        a*x(2) + x(1)*x(3) + u(2);
        b*x(3) + x(1)*x(2)+ u(3)] + disturb(x,u,t);
   %%%%%%%%%%
   % Outputs %
   %%%%%%%%%%
case 3,
   sys = x;
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; %Does nothing
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end

function disturb = disturb(x,u,t)
if t>=10
    disturb= [1.5*sin(4*t); %4
              1*cos(6*t);    %6
              0.2*sin(5*t)];  %3
else
    disturb=0; %Until t=10 secs the disturb is null
end
```

Listing B.3 – Sincronizador.m

```
function [sys,x0,str,ts] = Sincronizador(t,x,u,flag)

psi = 10000;

switch flag,
```

```
   %%%%%%%%%%%%%%%%%
   % Initialization %
   %%%%%%%%%%%%%%%%%
case 0,

  sizes = simsizes;
    sizes.NumContStates  = 3;   %Number of constant states
    sizes.NumDiscStates  = 0;   %Number of discrete states
    sizes.NumOutputs     = 3;   %Number of outputs
    sizes.NumInputs      = 6;   %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=zeros(3,1);              %Initial conditions
  x0(1)=0;
    x0(2)=0;
    x0(3)=0;
        str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Directives   %
   %%%%%%%%%%%%%%
case 1,  %here would be estimators of the weights of a neural net if
    there were, in this case there are not
      sys = [0; 0; 0];
   %%%%%%%%%%
   % Outputs %
   %%%%%%%%%%
  case 3,    %controller
      sys =  [-1*psi*(u(4) + u(1));
              -1*psi*(u(5) - u(2));
              -0*psi*(u(6) + u(3))];
    case {2,4,9},
    sys = [];

 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end
```

Listing B.4 – Graficos.m

```
%Running this file --> automatically shows the graphics of the  %
    simulation and saves it in the %folder in png format (you could choose
     jpg format too)
clc
fsize=30;

fSize = 30;
```

```
lSize = 2;
axesSize = 30;
dvlsize = 2;
dhlsize = 2;

%Figure 1
fig=figure;
plot(t,Xmaster(:,1),t, Xslave(:,1),':','LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize', 30);

xlabel('Time (s)','Fontsize',fsize);
ylabel('$$x_{m}(t), x_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow to the current axes
pa.X = [10 13.5];            % the location of the arrow
pa.Y = [25 25];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,32,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,28,'in action','Fontsize',fSize) % write a text on top of the
    arrow

h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG1.png');
%close(fig)

%Figure 2
fig=figure;
plot(t,Xmaster(:,2),t, Xslave(:,2),':','LineWidth',3);
set(0,'DefaultAxesFontSize',30);
```

```
grid on
grid minor
set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$y_{m}(t), y_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [20 20];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,25,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,22,'in action','Fontsize',fSize) % write a text on top of the
    arrow

h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG2.png');
%close(fig)

%Figure 3
fig=figure;
plot(t,Xmaster(:,3),t, Xslave(:,3),':','LineWidth',3);set(0,'
    DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize',30);

xlabel('Time (s)','Fontsize',fsize);
ylabel('$$z_{m}(t), z_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
```

```
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [25 25];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,32,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,28,'in action','Fontsize',fSize) % write a text on top of the
    arrow

h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG3.png');
%close(fig)

%Figure 4
fig=figure;
aux1 = Xmaster(:,1) + Xslave(:,1);
plot(t,aux1,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor
set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{1}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [4 4];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;
```

```
text(10.05,4.65,'disturbances','Fontsize',fSize) % write a text on top of
     the arrow
text(10.2,4.25,'in action','Fontsize',fSize) % write a text on top of the
     arrow

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG4.png');
%close(fig)

%Figure 5
fig=figure;
aux2 = Xslave(:,2) - Xmaster(:,2);
plot(t,aux2,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{2}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [-0.4 -0.4];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,-0.32,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(10.2,-0.37,'in action','Fontsize',fSize) % write a text on top of
    the arrow

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG5.png');
%close(fig)

%Figure 6
fig=figure;
```

```
aux3 = Xmaster(:,3) + Xslave(:,3);
plot(t,aux3,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{3}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [2 2];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,2.4,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,2.15,'in action','Fontsize',fSize) % write a text on top of the
     arrow

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG6.png');
%close(fig)
```

Listing B.5 – Graficos-sem-mensagem-estilo-3.m

```
fonte = 10;
fSize = 12;
dhlsize = 2;
dvlsize = 2;
largura_linha = 2;
color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];

addpath('./Figuras/Figuras_3_sem_mensagem_estilo_3/');

local = 'Figuras/Figuras_3_sem_mensagem_estilo_3';
format = 'png';
format2 = 'epsc';
```

```
eps = '.eps';

nome_1 = '/FIG_3_1';
nome_2 = '/FIG_3_2';
nome_3 = '/FIG_3_3';
nome_4 = '/FIG_3_4';
nome_5 = '/FIG_3_5';
nome_6 = '/FIG_3_6';

local_1 = append(local, nome_1);
local_2 = append(local, nome_2);
local_3 = append(local, nome_3);
local_4 = append(local, nome_4);
local_5 = append(local, nome_5);
local_6 = append(local, nome_6);
set(0,'DefaultAxesFontSize', fonte);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.011]);
%close(figure(1));
figure(1);
subplot(1,1,1);
plot(t,Xmaster(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,1),':','Color',color2,'LineWidth',largura_linha);
%ylim([-15 25])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [-34 -34];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,-26.5,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,-30.5,'em ação','Fontsize',fSize) % write a text on top of the
    arrow
```

```
xlabel("$t[s]$",'Interpreter','latex')
legend("$x_m(t)$","$x_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_1, format);
saveas(gcf, append(local_1, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.011]);
%close(figure(2));
figure(2);
subplot(1,1,1);
plot(t,Xmaster(:,2),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,2),':','Color',color2,'LineWidth', largura_linha);
%ylim([-1.5 2])
YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [19 19];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,25.4,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,21.7,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$y_m(t)$","$y_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_2, format);
saveas(gcf, append(local_2, eps), format2);
```

```
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.011]);
%close(figure(3));
figure(3);
subplot(1,1,1);
plot(t,Xmaster(:,3),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,3),':','Color',color2,'LineWidth', largura_linha);
%ylim([-12 12])
YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [-34 -34];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,-26.5,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,-30.5,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$z_m(t)$","$z_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_3, format);
saveas(gcf, append(local_3, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.035
    0.011]);
%close(figure(4));
figure(4);
subplot(1,1,1);
e1 = Xslave(:,1) + Xmaster(:,1);
plot(t,e1,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
```

```
%ylim([-0.01 0.15])
YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [4 4];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,5.1,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,4.5,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$e_1(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_4, format);
saveas(gcf, append(local_4, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.011]);
%close(figure(5));
figure(5);
subplot(1,1,1);
e2 = Xslave(:,2) - Xmaster(:,2);
plot(t,e2,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
%ylim([-0.25 0.2])
YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
```

```
pa.Y = [-0.6 -0.6];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,-0.47,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,-0.54,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$e_2(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_5, format);
saveas(gcf, append(local_5, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.03
    0.011]);
%close(figure(6));
figure(6);
subplot(1,1,1);
e3 = Xslave(:,3) + Xmaster(:,3);
plot(t,e3,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
%ylim([-0.5 0.8])
YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [2 2];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,2.65,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(10.2,2.3,'em ação','Fontsize',fSize) % write a text on top of the
    arrow
```

```
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_3(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_6, format);
saveas(gcf, append(local_6, eps), format2);
%close(figure(1));


x1=(t);
y1=(e1);
y2=(e2);
y3=(e3);
%y4=(aux4);
C=polyfit(x1,y1,1);
n=length(x1);
fx1=polyval(C,x1);
ecm1=sqrt(sum((fx1-y1).^2)/n)
fprintf(1,'%f\n',ecm1)


y2=(e2);
C=polyfit(x1,y2,1);
ecm2=sqrt(sum((fx1-y2).^2)/n)
fprintf(1,'%f\n',ecm2)

y3=(e3);
C=polyfit(x1,y3,1);
ecm3=sqrt(sum((fx1-y3).^2)/n)
fprintf(1,'%f\n',ecm3)
```

B.2 CODES FOR SIMULATIONS IN CHAPTER 4

B.2.1 Simulink Plant for the Synchronization of a Hyperchaotic Financial System, corresponding to Figures 4.1-4.8 and the Table 4.1.

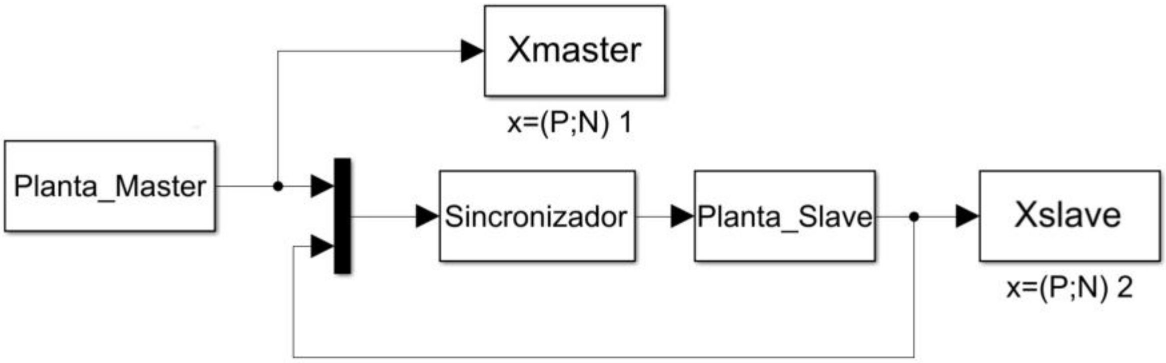

Figure B.2 – Simulink Plant.

Listing B.6 – Planta-Master.m

```
function [sys,x0,str,ts] = Planta_Master(t,x,u,flag)

a=0.9;  %Constans
b=0.2;
c=1.5;
d=0.2;
k=0.17;

d1 =0;  %Disturbances
d2=0;
d3=0;
d4=0;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Initialization %
   %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 4; %Number of constant states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 4;    %Number of outputs
    sizes.NumInputs = 0;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
```

```
    x0=[1 2 0.5 0.5]; %Initial conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Diretives  %
   %%%%%%%%%%%%%%
case 1,      %Sistem
  sys = [x(3) + (x(2) - a)*x(1) + x(4);
        1 - b*x(2) - x(1)*x(1);
        -x(1) - c*x(3);
        -d*x(1)*x(2) - k*x(4)] + disturb(x,u,t);
   %%%%%%%%%
   % Output %
   %%%%%%%%%
case 3,
   sys = x;
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; % Does nothing
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end

function disturb = disturb(x,u,t)
if t>=40
    disturb= [0.25*cos(10*t); %4
              0.05*sin(2*t);
              0.03*sin(5*t);%6
              0.1*cos(7*t) + 0.1*sin(10*t)];  %3
else
    disturb=0; %Until t=10 secs the disturb is null
end
```

Listing B.7 – Planta-Slave.m

```
function [sys,x0,str,ts] = Planta_Slave(t,x,u,flag)

a=0.9;  %Constants
b=0.2;
c=1.5;
d=0.2;
k=0.17;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Inicialization %
```

```
  %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 4; %Number of constant states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 4;    %Number of outputs
    sizes.NumInputs = 4;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=[-2 1.5 1 -0.5]; %Initial conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Diretives  %
   %%%%%%%%%%%%%%
case 1,      %Sistem
  sys = [x(3)+(x(2)-a)*x(1) + x(4) + u(1);
        1 - b*x(2)-x(1)*x(1) + u(2);
        -x(1)-c*x(3) + u(3);
        -d*x(1)*x(2) - k*x(4) + u(4)] + disturb(x,u,t);
   %%%%%%%%%%
   % Outputs %
   %%%%%%%%%%
case 3,
   sys = x;
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; % Does nothing
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end

function disturb = disturb(x,u,t)
if t>=40
    disturb= [0.2*sin(4*t); %4
              0.07*cos(5*t);
              0.04*cos(t);%6
              0.25*sin(15*t)];  %3
else
    disturb=0; %Until t=10 secs the disturb is null
end
```

Listing B.8 – Sincronizador.m

```
function [sys,x0,str,ts] = Sincronizador(t,x,u,flag)
```

```
psi1 = 100000;
psi2 = 10000;
psi3 = 100000;
psi4 = 10000;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Inicialization %
   %%%%%%%%%%%%%%%%%
case 0,

  sizes = simsizes;
    sizes.NumContStates   = 4;   %Number of constant states
    sizes.NumDiscStates   = 0;   %Number of discrete states
    sizes.NumOutputs      = 4;   %Number of outputs
    sizes.NumInputs       = 8;   %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=zeros(4,1);               %Initial conditions
  x0(1)=0;
    x0(2)=0;
    x0(3)=0;
    x0(4)=0;
        str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Diretives  %
   %%%%%%%%%%%%%%
case 1,  %here would be estimators of the weights of a neural net if
   there were, in this case there aren't
      sys = [0;
             0;
             0;
             0];
   %%%%%%%%%%
   % Outputs %
   %%%%%%%%%%
  case 3,    %controller

       sys = [ -1*(psi1*(u(5) - u(1)) + psi2*(u(5) - u(1))^3);
               -0*(psi2*(u(6) - u(2)));
               -0*(psi3*(u(7) - u(3)));
               -1*(psi3*(u(8) - u(4)) + psi4*(u(8) - u(4))^3)];

    case {2,4,9},
```

```
    sys = [];

 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end
```

Listing B.9 – Graficos.m

```
%Running this file automatically shows the graphics of the simulation and
     saves it in the folder in %png format (could be chosen jpg format too
    )
clc
fsize=30;

fSize = 30;
lSize = 2;
axesSize = 30;
dvlsize = 2;
dhlsize = 2;

%Figure 1
fig=figure;
plot(t,Xmaster(:,1),t, Xslave(:,1),':','LineWidth',3);set(0,'
    DefaultAxesFontSize',30);
grid on
grid minor
set(0,'DefaultAxesFontSize', 30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$x_{m}(t), x_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];            % the location of arrow
pa.Y = [1.5 1.5];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,1.85,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(41.2,1.65,'in action','Fontsize',fSize) % write a text on top of the
     arrow
```

```
h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

ylim([-2.1 2.5]);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG1.png');
close(fig)


%Figure 2
fig=figure;
plot(t,Xmaster(:,2),t, Xslave(:,2),':','LineWidth',3);set(0,'
    DefaultAxesFontSize',30);
grid on
grid minor
h=legend('Master','Slave','Location','southeast');
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$y_{m}(t), y_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [3.3 3.3];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,3.65,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(41.2,3.45,'in action','Fontsize',fSize) % write a text on top of the
     arrow

h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

ylim([-0.35 4]);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG2.png');
close(fig)
```

```
%Figure 3
fig=figure;
plot(t,Xmaster(:,3),t, Xslave(:,3),':','LineWidth',3);set(0,'
    DefaultAxesFontSize',30);
grid on
grid minor
set(0,'DefaultAxesFontSize', 30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$z_{m}(t), z_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [0.95 0.95];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,1.13,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(41.2,1.02,'in action','Fontsize',fSize) % write a text on top of the
     arrow

h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

ylim([-1 1.3]);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG3.png');
close(fig)

%Figure 4
fig=figure;
plot(t,Xmaster(:,4),t, Xslave(:,4),':','LineWidth',3);set(0,'
    DefaultAxesFontSize',30);
grid on
grid minor
set(0,'DefaultAxesFontSize', 30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$w_{m}(t), w_{s}(t)$$','Interpreter','Latex','Fontsize',fsize)
```

```
YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
   ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [1.1 1.1];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,1.3,'disturbances','Fontsize',fSize) % write a text on top of
   the arrow
text(41.2,1.18,'in action','Fontsize',fSize) % write a text on top of the
    arrow

h=legend('Master','Slave','Location','northeast');
set(h,'FontSize',fsize);

ylim([-1.1 1.5]);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG4.png');
close(fig)

%Figure 5
fig=figure;
aux1 = Xslave(:,1) - Xmaster(:,1);
plot(t,aux1,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{1}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
   ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
```

```
pa.X = [40 60];            % the location of arrow
pa.Y = [-0.6 -0.6];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,-0.35,'disturbances','Fontsize',fSize) % write a text on top of
     the arrow
text(41.2,-0.5,'in action','Fontsize',fSize) % write a text on top of the
     arrow

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG5.png');
close(fig)

%Figure 6
fig=figure;
aux2 = Xslave(:,2) - Xmaster(:,2);
plot(t,aux2,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{2}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [-0.4 -0.4];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,-0.355,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(41.2,-0.38,'in action','Fontsize',fSize) % write a text on top of
    the arrow

ylim([-0.5 0.1]);
```

```
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG6.png');
close(fig)

%Figure 7
fig=figure;
aux3 = Xslave(:,3) - Xmaster(:,3);
plot(t,aux3,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor

set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{3}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [0.3 0.3];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,0.35,'disturbances','Fontsize',fSize) % write a text on top of
    the arrow
text(41.2,0.32,'in action','Fontsize',fSize) % write a text on top of the
     arrow

ylim([-0.1 0.5]);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG7.png');
close(fig)

%Figure 8
fig=figure;
aux4 = Xslave(:,4) - Xmaster(:,4);
plot(t,aux4,'LineWidth',3);
set(0,'DefaultAxesFontSize',30);
grid on
grid minor
```

```
set(0,'DefaultAxesFontSize',30);
xlabel('Time (s)','Fontsize',fsize);
ylabel('$$e_{4}(t)$$','Interpreter','Latex','Fontsize',fsize)

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [-0.4 -0.4];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(40.2,-0.315,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(41.2,-0.365,'in action','Fontsize',fSize) % write a text on top of
    the arrow

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG8.png');
close(fig)
```

Listing B.10 – Graficos-sem-mensagem-estilo-3.m

```
fonte = 10;
fSize = 12;
dhlsize = 2;
dvlsize = 2;
largura_linha = 2;
color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];

addpath('./Figuras/Figuras_3_sem_mensagem_estilo_3/');

local = 'Figuras/Figuras_3_sem_mensagem_estilo_3';
format = 'png';
format2 = 'epsc';
eps = '.eps';

nome_1 = '/FIG_3_1';
nome_2 = '/FIG_3_2';
nome_3 = '/FIG_3_3';
```

```
nome_4 = '/FIG_3_4';
nome_5 = '/FIG_3_5';
nome_6 = '/FIG_3_6';
nome_7 = '/FIG_3_7';
nome_8 = '/FIG_3_8';

local_1 = append(local, nome_1);
local_2 = append(local, nome_2);
local_3 = append(local, nome_3);
local_4 = append(local, nome_4);
local_5 = append(local, nome_5);
local_6 = append(local, nome_6);
local_7 = append(local, nome_7);
local_8 = append(local, nome_8);
set(0,'DefaultAxesFontSize', fonte);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.03
    0.016]);
%close(figure(1));
figure(1);
subplot(1,1,1);
plot(t,Xmaster(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,1),':','Color',color2,'LineWidth',largura_linha);
ylim([-2 3])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');   % store the arrow information in pa
pa.Parent = gca;            % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [1.8 1.8];
pa.LineWidth  = dhlsize;    % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,2.35,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,2.05,'em ação','Fontsize',fSize) % write a text on top of the
    arrow
```

```
xlabel("$t[s]$",'Interpreter','latex')
legend("$x_m(t)$","$x_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_1, format);
saveas(gcf, append(local_1, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.03
    0.016]);
%close(figure(2));
figure(2);
subplot(1,1,1);
plot(t,Xmaster(:,2),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,2),':','Color',color2,'LineWidth', largura_linha);
ylim([-1 5])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];            % the location of arrow
pa.Y = [3.5 3.5];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,4.15,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,3.8,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$y_m(t)$","$y_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_2, format);
saveas(gcf, append(local_2, eps), format2);
```

```
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.016]);
%close(figure(3));
figure(3);

subplot(1,1,1);
plot(t,Xmaster(:,3),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,3),':','Color',color2,'LineWidth', largura_linha);
ylim([-1 1.5])
xlabel("$t[s]$",'Interpreter','latex')

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];            % the location of arrow
pa.Y = [1 1];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,1.25,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,1.13,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

legend("$z_m(t)$","$z_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_3, format);
saveas(gcf, append(local_3, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.016]);
%close(figure(4));
figure(4);
subplot(1,1,1);
plot(t,Xmaster(:,4),'-','Color',color1,'LineWidth', largura_linha);
grid on
```

```
grid minor
hold on;
plot(t, Xslave(:,4),':','Color',color2,'LineWidth', largura_linha);
ylim([-1 1.5])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');   % store the arrow information in pa
pa.Parent = gca;            % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [1 1];
pa.LineWidth  = dhlsize;    % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,1.27,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,1.13,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$w_m(t)$","$w_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_4, format);
saveas(gcf, append(local_4, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.03
    0.016]);
%close(figure(5));
%figure(5);
subplot(1,1,1);
e1 = Xslave(:,1) - Xmaster(:,1);
plot(t,e1,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-4 1])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
```

```
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];            % the location of arrow
pa.Y = [-3 -3];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,-2.45,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,-2.75,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$e_1(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_5, format);
saveas(gcf, append(local_5, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.016]);
%close(figure(6));
figure(6);
subplot(1,1,1);
e2 = Xslave(:,2) - Xmaster(:,2);
plot(t,e2,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.6 0.2])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];            % the location of arrow
pa.Y = [-0.4 -0.4];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;
```

```
text(40.5,-0.31,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,-0.36,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$e_2(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_6, format);
saveas(gcf, append(local_6, eps), format2);

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.016]);
%close(figure(7));
figure(7);
subplot(1,1,1);
e3 = Xslave(:,3) - Xmaster(:,3);
plot(t,e3,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.2 0.6])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');   % store the arrow information in pa
pa.Parent = gca;            % associate the arrow the the current axes
pa.X = [40 60];             % the location of arrow
pa.Y = [0.2 0.2];
pa.LineWidth  = dhlsize;    % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,0.29,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,0.24,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$e_3(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
```

```
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_7, format);
saveas(gcf, append(local_7, eps), format2);
close(figure(1));

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.016]);
%close(figure(8));
figure(8);
subplot(1,1,1);
e4 = Xslave(:,4) - Xmaster(:,4);
plot(t,e4,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-1.1 0.3])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([40, 40], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [40 60];            % the location of arrow
pa.Y = [-0.7 -0.7];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(40.5,-0.56,'distúrbios','Fontsize',fSize) % write a text on top of
    the arrow
text(41,-0.63,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

xlabel("$t[s]$",'Interpreter','latex')
legend("$e_4(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_8, format);
saveas(gcf, append(local_8, eps), format2);
%close(figure(1));


x1=(t);
```

```
y1=(e1);
y2=(e2);
y3=(e3);
y4=(e4);
C=polyfit(x1,y1,1);
n=length(x1);
fx1=polyval(C,x1);
ecm1=sqrt(sum((fx1-y1).^2)/n)
fprintf(1,'%f\n',ecm1)


y2=(e2);
C=polyfit(x1,y2,1);
ecm2=sqrt(sum((fx1-y2).^2)/n)
fprintf(1,'%f\n',ecm2)

y3=(e3);
C=polyfit(x1,y3,1);
ecm3=sqrt(sum((fx1-y3).^2)/n)
fprintf(1,'%f\n',ecm3)

y4=(e4);
C=polyfit(x1,y4,1);
ecm4=sqrt(sum((fx1-y4).^2)/n)
fprintf(1,'%f\n',ecm4)
```

B.3 CODES FOR SIMULATIONS IN CHAPTER 5

B.3.1 Synchronization of a Cryptosystem Based on the Synchronization of a Chaotic Liu System, corresponding to Figures 5.1-5.6 and the Table 5.1.

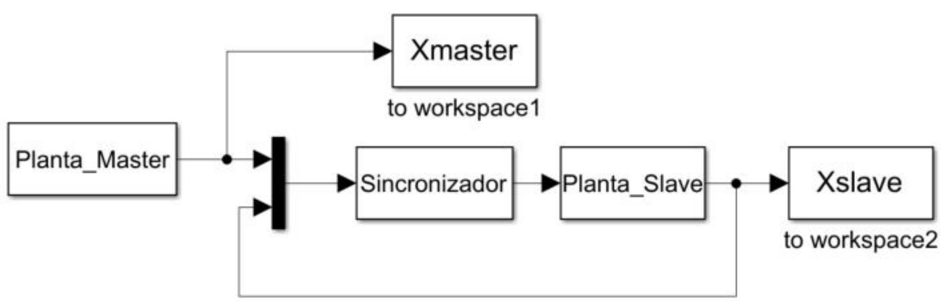

Figure B.3 – Planta Simulink.

Listing B.11 – Planta-Master.m

```
function [sys,x0,str,ts] = Planta_Master(t,x,u,flag)
```

```
a = 10;
b = 40;
c = 2.5;
h = 4;
k = 1;
msg = 1*sin(2*t) +  0.5*sin(8*t) + 0.3*cos(20*t) +  1*square(t);

switch flag,
   %%%%%%%%%%%%%%%%%
   % Inicialization %
   %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 3; %Number of constant states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 3;    %Number of outputs
    sizes.NumInputs = 0;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=[0.2 -0.3 0.4]; Initial conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Diretives  %
   %%%%%%%%%%%%%%
case 1,      %Sistema
      sys = [-a*x(1) + a*x(2);
             b*x(1) - k*x(1)*x(3);
             -c*x(3) + h*x(1)^2];
   %%%%%%%%%
   % Output %
   %%%%%%%%%
case 3,
   sys = [x(1) + msg; x(2); x(3)];
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; % Does nothing
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end
```

Listing B.12 – Planta-Slave.m

```
function [sys,x0,str,ts] = Planta_Slave(t,x,u,flag)
```

```
a = 10;
b = 40;
c = 2.5;
h = 4;
k = 1;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Inicialization %
   %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 3; %Number of constant states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 3;    %Number of outputs
    sizes.NumInputs = 3;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=[20 -30 100]; %Initial conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Diretives  %
   %%%%%%%%%%%%%%
case 1,      %Sistema
      sys = [-a*x(1) + a*x(2) + u(1);
             b*x(1) - k*x(1)*x(3) + u(2);
             -c*x(3) + h*x(1)^2 + u(3)] + disturb(x,u,t);
   %%%%%%%%%%
   % Outputs %
   %%%%%%%%%%
case 3,
   sys = x;
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; %Does nothing
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end

function disturb = disturb(x,u,t)
if t>=10
    disturb= [1.5*sin(2*t); %4
              2*cos(3*t);   %6
```

```
              1*sin(4*t)];  %3
else
    disturb=0; %Until t=10 secs the disturb is null
end
```

Listing B.13 – Sincronizador.m

```
function [sys,x0,str,ts] = Sincronizador(t,x,u,flag)

psi = 10;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Inicialization %
   %%%%%%%%%%%%%%%%%
case 0,

  sizes = simsizes;
    sizes.NumContStates   = 3;   %Number of constant states
    sizes.NumDiscStates   = 0;   %Number of discrete states
    sizes.NumOutputs      = 3;   %Number of outputs
    sizes.NumInputs       = 6;   %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=zeros(3,1);               %Initial conditions
  x0(1) = 0;
    x0(2) = 0;
    x0(3) = 0;
        str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Diretives  %
   %%%%%%%%%%%%%%
case 1,  %here would be estimators of the weights of a neural net if
   there were, in this case there aren't
      sys = [0;
             0;
             0];
   %%%%%%%%%%
   % Outpunts %
   %%%%%%%%%%
  case 3,    %controller
      sys =  [-0*(psi*(u(4) - u(1)));
              -1*(psi*(u(5) - u(2))) - (u(5) - u(2))^3;
              -0*(psi*(u(6) - u(3)))];
    case {2,4,9},
    sys = [];
```

```
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end
```

Listing B.14 – Graficos.m

```
%Running this file --> automatically shows the graphs of %the simulation
    and saves it in the folder in png %format(you could choose jpg format
    too)
clc
fsize=36;
msg1 = 1*sin(2*t) +  0.5*sin(8*t) + 0.3*cos(20*t);
msg2 = 1*square(t);
msg = msg1 + msg2;

fSize = 36;
lSize = 2;
axesSize = 36;
dvlsize = 2;
dhlsize = 2;

%Figure 5
fig=figure;
plot(t,msg,t, Xmaster(:,1),':','LineWidth',3);
grid on
grid minor
h=legend('Original Message','Encrypted Message','Location','northeast');
set(h,'FontSize',fsize);
set(0,'DefaultAxesFontSize', 30);

xlabel('Time (s)','Fontsize',fsize);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG5.png');
%close(fig)

%Figure 6
fig=figure;
aux = Xmaster(:,1) - Xslave(:,1);
plot(t,msg,t,aux,':','LineWidth',3);
grid on
grid minor
set(0,'DefaultAxesFontSize', 30);
ylim([-20 10]);
xlabel('Time (s)','Fontsize',fsize);

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
```

```
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [-17 -17];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,-14.2,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(10.2,-15.7,'in action','Fontsize',fSize) % write a text on top of
    the arrow
h=legend('Original Message','Recovered Message','Location','northeast');
set(h,'FontSize',fsize);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG6.png');
close(fig)

%Figure 7
fig=figure;
aux2 = aux - msg;
plot(t,aux2,'LineWidth',3);
grid on
grid minor
set(0,'DefaultAxesFontSize', 30);
xlabel('Time(s)','Fontsize',fsize);
ylabel('Message error','Fontsize',fsize);

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 13.5];            % the location of arrow
pa.Y = [-10 -10];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 20;
pa.HeadLength = 20;

text(10.05,-7.8,'disturbances','Fontsize',fSize) % write a text on top of
```

```
    the arrow
text(10.2,-9,'in action','Fontsize',fSize) % write a text on top of the
   arrow

set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,'FIG7.png');
close(fig)
```

```
x1=(t);
y1=(aux2);
C=polyfit(x1,y1,1);
n=length(x1);
fx1=polyval(C,x1);
ecm1=sqrt(sum((fx1-y1).^2)/n)
fprintf(1,'%f\n',ecm1)
```

Listing B.15 – Graficos-com-mensagem-estilo-3.m

```
fonte = 10;
fSize = 12;
largura_linha = 2;
dhlsize = 2;
dvlsize = 2;
color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];
color3 = [0.4660 0.6740 0.1880];

msg1 = 1*sin(2*t) +  0.5*sin(8*t) + 0.3*cos(20*t);
msg2 = 1*square(t);
msg = msg1 + msg2;

addpath('./Figuras/Figuras_6_com_mensagem_estilo_3/');

local = 'Figuras/Figuras_6_com_mensagem_estilo_3';
format = 'png';
format2 = 'epsc';
eps = '.eps';

nome_1 = '/FIG_6_1';
nome_2 = '/FIG_6_2';
nome_3 = '/FIG_6_3';
nome_4 = '/FIG_6_4';
nome_5 = '/FIG_6_5';
nome_6 = '/FIG_6_6';
nome_7 = '/FIG_6_7';
```

```
nome_8 = '/FIG_6_8';
nome_9 = '/FIG_6_9';
nome_10 = '/FIG_6_10';
nome_11 = '/FIG_6_11';
nome_12 = '/FIG_6_12';
nome_13 = '/FIG_6_13';
nome_14 = '/FIG_6_14';
nome_15 = '/FIG_6_15';
nome_16 = '/FIG_6_16';
nome_17 = '/FIG_6_17';
nome_18 = '/FIG_6_18';

local_1 = append(local, nome_1);
local_2 = append(local, nome_2);
local_3 = append(local, nome_3);
local_4 = append(local, nome_4);
local_5 = append(local, nome_5);
local_6 = append(local, nome_6);
local_7 = append(local, nome_7);
local_8 = append(local, nome_8);
local_9 = append(local, nome_9);
local_10 = append(local, nome_10);
local_11 = append(local, nome_11);
local_12 = append(local, nome_12);
local_13 = append(local, nome_13);
local_14 = append(local, nome_14);
local_15 = append(local, nome_15);
local_16 = append(local, nome_16);
local_17 = append(local, nome_17);
local_18 = append(local, nome_18);
set(0,'DefaultAxesFontSize', fonte);

%Figure 1
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.013]);
%close(figure(1));
figure(1);
subplot(1,1,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, Xmaster(:,1),':','Color',color2,'LineWidth', largura_linha);
%ylim([-0.3 0.7])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
```

```
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [-17 -17];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,-11.7,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(10.2,-14.5,'in action','Fontsize',fSize) % write a text on top of
    the arrow

h=legend('Original message', 'Mensagem criptografada','Location','
    northeast');
set(h,'FontSize',fonte);

xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_7, format);
saveas(gcf, append(local_7, eps), format2);

%Figure 2
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.013]);
%close(figure(2));
figure(2);
subplot(1,1,1);
aux = Xmaster(:,1) - Xslave(:,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux,':','Color',color2,'LineWidth', largura_linha);
ylim([-20 12])

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;
```

```
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [-17 -17];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
pa.HeadLength = 10;

text(10.05,-13.5,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(10.2,-15.5,'em ação','Fontsize',fSize) % write a text on top of the
    arrow

h=legend('Original message', 'Mensagem recuperada','Location','northeast'
    );
set(h,'FontSize',fonte);

xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_10, format);
saveas(gcf, append(local_10, eps), format2);

%Figure 3
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.07
    0.013]);
%close(figure(3));
figure(3);
subplot(1,1,1);
aux = Xmaster(:,1) - Xslave(:,1);
aux2 = aux - msg;
plot(t,aux2,'Color',color1,'LineWidth', largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([10, 10], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',dvlsize)
    ;

pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [10 14];            % the location of arrow
pa.Y = [-17 -17];
pa.LineWidth  = dhlsize;   % make the arrow bolder for the figure
pa.HeadWidth  = 10;
```

```
pa.HeadLength = 10;

text(10.05,-14.2,'disturbances','Fontsize',fSize) % write a text on top
    of the arrow
text(10.2,-15.8,'in action','Fontsize',fSize) % write a text on top of
    the arrow

xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('Message error','Fontsize',fonte)
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_16, format);
saveas(gcf, append(local_16, eps), format2);
%close(figure(1));




x1=(t);
y1=(aux2);
C=polyfit(x1,y1,1);
n=length(x1);
fx1=polyval(C,x1);
ecm1=sqrt(sum((fx1-y1).^2)/n)
fprintf(1,'%f\n',ecm1)
```

B.4 CODES FOR SIMULATIONS IN CHAPTER 6

B.4.1 Chaos Synchronization and its Application in Parallel Cryptography, Corresponding Figures 6.1-6.12 and the Table 6.1.

Listing B.16 – Principal.m

```
clear   %clears previous variables
clc     %clears what was written in the terminal

%Amplitude Scaling
x_fator = 1;        %\bar{x} = x/x_fator , being "\bar{x}" the new value
    and "x" the old value
y_fator = 1;        %\bar{x} = x/x_fator , being"\bar{x}" the new value
    and "x" the old value
z_fator = 1;        %\bar{x} = x/x_fator , being "\bar{x}"the new value
    and "x" the old value
%Frequency Scaling
freq_fator = 1;    %The higher, the faster the simulation runs

h = 1; %Presence or not of disturbances, h =0 if no, h=1 if yes
```

```
%Initial conditions, only essential rule is that master and slave states
    are different.
x0_mestre =  [0.2 0.2 0.2];
x0_escravo = [0.3 0 -0.2];


psi = 10000;      %Underactuated control gain    $

t_fim = 20;     %Time at which the simulation ends

options = odeset('RelTol', 1e-10,...    %simulation settings
                 'AbsTol', 1e-10,...
                 'MaxStep',0.001);

addpath('./Arquivos/');

Simulate
%Graphs
```

Listing B.17 – Sistema.m

```
function equation = Sistema(x, y, z)
%alfa = 0.9346;
%beta1 = 0.15;
%beta2 = 1.5;
%gamma = 0.1;

%C1 = 15;
%C2 = 150;
%m0 = -0.7879;
%m1 = -1.4357;
%L = 10;
%R = 1070;
%Rl = 2;
%Bp = 1;

m = 10;
teta1 = 7.87;
teta2 = 3.23;
gamma = 0.03;
d = 1;
Bp = 1;
beta = 15;

%equation =  [(y - x)*(alfa/beta1) - m0*x/beta1 - 0.5*(m1 - m0)*(abs(x +
    Bp) - abs(x - Bp))/beta1;    %put here the structure of your dynamic
    system
 %            (x - y)*(alfa/beta2) + z/beta2;
```

```
  %           -y/gamma - Rl*z/gamma];

equation =  [m*(y - x) + teta1*x + teta2*(abs(x + Bp) - abs(x - Bp)) + d;
       %put here the structure of your dynamic system
             x - y + z;
             - beta*y - gamma*z];
end
```

Listing B.18 – Mensagens.m

```
function msg = Mensagens(t)
msg = [0.2*square(5*t);   %put here the structure of your dynamic system
       0.05*cos(0.5*t) + 0.025*sin(10*t);
       0.1*sin(20*t) + 0.2*sin(8*t)];
end
%Messages sent (must be at most 5% of the maximum value reached by the
   state)
%Understand as msg1 a message present in the first state, for example
%Do not put messages in states where control is present
```

Listing B.19 – Simular.m

```
t_ciclo = [0 t_fim ];   %put the initial and final simulation time in a
   vector

amp_f(1) = x_fator;
amp_f(2) = y_fator;
amp_f(3) = z_fator;

x0(1) = x0_mestre (1)/amp_f(1);  %passing the initial conditions to
   scaled variables
x0(2) = x0_mestre (2)/amp_f(2);
x0(3) = x0_mestre (3)/amp_f(3);
x0(4) = x0_escravo(1)/amp_f(1);
x0(5) = x0_escravo(2)/amp_f(2);
x0(6) = x0_escravo(3)/amp_f(3);

[t, x] = ode45(@Esquema, t_ciclo, x0, options, psi, h, amp_f, freq_fator)
   ;
%runs the simulation and saves the results
aux = size(t);
msg = zeros(aux(1), 3);

for i = 1:aux(1)
    tempo = t(i,1);
    msg_aux = Mensagens(tempo);
    msg(i,:) = msg_aux';
end
```

```
Xmaster_sem_msg = x(:, 1:3);
Xmaster = x(:, 1:3) + msg(:, 1:3);
Xslave = x(:, 4:6);
%Repassed the results to vectors simpler to print on graphics %
   Xmaster_without_msg as the name %suggests is the master state without
   the presence of the message %Xmaster is the encrypted or %encoded
   message (with the presence of the message)
clearvars -except t Xmaster_sem_msg Xmaster Xslave msg


function y = Esquema(t, x, psi, h, amp_f, freq_fator)

eq_mestre =  Sistema(amp_f(1)*x(1),...
                     amp_f(2)*x(2),...
                     amp_f(3)*x(3));    %master system equation
eq_escravo =  Sistema(amp_f(1)*x(4),...
                      amp_f(2)*x(5),...
                      amp_f(3)*x(6));   %slave system equation

eq_mestre(1) = freq_fator*eq_mestre(1)/amp_f(1);
eq_mestre(2) = freq_fator*eq_mestre(2)/amp_f(2);
eq_mestre(3) = freq_fator*eq_mestre(3)/amp_f(3);

eq_escravo(1) = eq_escravo(1)/amp_f(1);
eq_escravo(2) = eq_escravo(2)/amp_f(2);
eq_escravo(3) = eq_escravo(3)/amp_f(3);

y(1:3, 1) = eq_mestre;
y(4:6, 1) = freq_fator*(eq_escravo + controle(x, psi)) + h*disturb(t);
%The result of the master and slave systems

end

function controle = controle(x, psi)
controle = [-psi*(x(4) - x(1));                              %control
   structure
            0;
            0];
end

function disturb = disturb(t)   %disturbances, if you want to put them
if t>=0                         %Start of disturbances starting at 5
   seconds
    disturb = [0.1*sin(2*t);
               0.01*cos(3*t);
               0.02*sin(4*t)];
```

```
else
    disturb = 0; %Until t=10 seconds the disturbances are null
end
end
```

Listing B.20 – Graficos.m

```
clc

addpath('./Arquivos/');
addpath('./Figuras/');

Graficos_sem_mensagem_estilo_3
Graficos_com_mensagem_estilo_3
```

Listing B.21 – Graficos-com-mensagem-estilo-3.m

```
fonte = 10;
largura_linha = 2;
color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];
color3 = [0.4660 0.6740 0.1880];

addpath('./Figuras/Figuras_6_com_mensagem_estilo_3/');

local = 'Figuras/Figuras_6_com_mensagem_estilo_3';
format = 'png';
format2 = 'epsc';
eps = '.eps';

nome_1 = '/FIG_6_1';
nome_2 = '/FIG_6_2';
nome_3 = '/FIG_6_3';
nome_4 = '/FIG_6_4';
nome_5 = '/FIG_6_5';
nome_6 = '/FIG_6_6';
nome_7 = '/FIG_6_7';
nome_8 = '/FIG_6_8';
nome_9 = '/FIG_6_9';
nome_10 = '/FIG_6_10';
nome_11 = '/FIG_6_11';
nome_12 = '/FIG_6_12';
nome_13 = '/FIG_6_13';
nome_14 = '/FIG_6_14';
nome_15 = '/FIG_6_15';
nome_16 = '/FIG_6_16';
nome_17 = '/FIG_6_17';
nome_18 = '/FIG_6_18';
```

```
local_1 = append(local, nome_1);
local_2 = append(local, nome_2);
local_3 = append(local, nome_3);
local_4 = append(local, nome_4);
local_5 = append(local, nome_5);
local_6 = append(local, nome_6);
local_7 = append(local, nome_7);
local_8 = append(local, nome_8);
local_9 = append(local, nome_9);
local_10 = append(local, nome_10);
local_11 = append(local, nome_11);
local_12 = append(local, nome_12);
local_13 = append(local, nome_13);
local_14 = append(local, nome_14);
local_15 = append(local, nome_15);
local_16 = append(local, nome_16);
local_17 = append(local, nome_17);
local_18 = append(local, nome_18);
set(0,'DefaultAxesFontSize', fonte);

%Figure 1
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.03
    0.012]);
close(figure(1));
figure(1);
subplot(1,1,1);
plot(t,Xmaster(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,1),':','Color',color2,'LineWidth',largura_linha);
ylim([-5 10])
xlabel("$t[s]$",'Interpreter','latex')
legend("$x_m(t)$","$x_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_1, format);
saveas(gcf, append(local_1, eps), format2);

%Figure 2
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(2));
figure(2);
subplot(1,1,1);
```

```
plot(t,Xmaster(:,2),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,2),':','Color',color2,'LineWidth', largura_linha);
ylim([-1.5 1.8])
xlabel("$t[s]$",'Interpreter','latex')
legend("$y_m(t)$","$y_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_2, format);
saveas(gcf, append(local_2, eps), format2);

%Figure 3
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.012]);
close(figure(3));
figure(3);
subplot(1,1,1);
plot(t,Xmaster(:,3),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,3),':','Color',color2,'LineWidth', largura_linha);
ylim([-12 12])
xlabel("$t[s]$",'Interpreter','latex')
legend("$z_m(t)$","$z_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_3, format);
saveas(gcf, append(local_3, eps), format2);

%Figure 4
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(4));
figure(1);
subplot(1,1,1);
e1 = Xslave(:,1) - Xmaster(:,1);
plot(t,e1,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.3 0.4])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_1(t)$",'Interpreter','latex','Location','northeast','
```

```
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_4, format);
saveas(gcf, append(local_4, eps), format2);

%Figure 5
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(5));
figure(5);
subplot(1,1,1);
e2 = Xslave(:,2) - Xmaster(:,2);
plot(t,e2,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.3 0.2])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_2(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_5, format);
saveas(gcf, append(local_5, eps), format2);

%Figure 6
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(1));
figure(6);
subplot(1,1,1);
e3 = Xslave(:,3) - Xmaster(:,3);
plot(t,e3,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.5 1])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_3(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_6, format);
saveas(gcf, append(local_6, eps), format2);

%Figure 7
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
```

```
close(figure(7));
figure(7);
subplot(1,1,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, 0.05*Xmaster(:,1),':','Color',color2,'LineWidth', largura_linha);
ylim([-0.3 0.7])
h=legend('Mensagem original', 'Messagem codificada','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_1(t), 0.05{\cdot}s_1(t)$$','Interpreter','Latex','Fontsize',
    fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_7, format);
saveas(gcf, append(local_7, eps), format2);

%Figure 8
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.06
    0.012]);
close(figure(8));
figure(8);
subplot(1,1,1);
plot(t,msg(:,2),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, 0.05*Xmaster(:,2),':','Color',color2,'LineWidth', largura_linha);
ylim([-0.08 0.15])
h=legend('Mensagem original', 'Messagem codificada','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_2(t), 0.05{\cdot}s_2(t)$$','Interpreter','Latex','Fontsize',
    fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_8, format);
saveas(gcf, append(local_8, eps), format2);

%Figure 9
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(9));
```

```
figure(9);
subplot(1,1,1);
plot(t,msg(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, 0.05*Xmaster(:,3),':','Color',color2,'LineWidth', largura_linha);
ylim([-0.6 0.9])
h=legend('Mensagem original', 'Messagem codificada','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_3(t), 0.05{\cdot}s_3(t)$$','Interpreter','Latex','Fontsize',
    fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_9, format);
saveas(gcf, append(local_9, eps), format2);

%Figure 10
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(10));
figure(10);
subplot(1,1,1);
aux = Xmaster(:,1) - Xslave(:,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux,':','Color',color2,'LineWidth', largura_linha);
ylim([-0.25 0.5])
h=legend('Mensagem original', 'Mensagem decodificada','Location','
    northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_1(t), \hat{m}_1(t)$$','Interpreter','Latex','Fontsize',fonte)
    ;
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_10, format);
saveas(gcf, append(local_10, eps), format2);

%Figure 11
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.06
    0.012]);
close(figure(11));
```

```
figure(11);
subplot(1,1,1);
aux2 = Xmaster(:,2) - Xslave(:,2);
plot(t,msg(:,2),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux2,':','Color',color2,'LineWidth', largura_linha);
ylim([-0.15 0.3])
h=legend('Mensagem original', 'Mensagem decodificada','Location','
    northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_2(t), \hat{m}_2(t)$$','Interpreter','Latex','Fontsize',fonte)
    ;
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_11, format);
saveas(gcf, append(local_11, eps), format2);

%Figure 12
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(12));
figure(12);
subplot(1,1,1);
aux3 = Xmaster(:,3) - Xslave(:,3);
plot(t,msg(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux3,':','Color',color2,'LineWidth', largura_linha);
ylim([-1 1])
h=legend('Mensagem original', 'Mensagem decodificada','Location','
    northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_3(t), \hat{m}_3(t)$$','Interpreter','Latex','Fontsize',fonte)
    ;
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_12, format);
saveas(gcf, append(local_12, eps), format2);

%Figure 13
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
```

```
close(figure(13));
figure(13);
subplot(1,1,1);
aux4 = Xmaster(:,1) - Xslave(:,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux4,':','Color',color2,'LineWidth', largura_linha);
plot(t, 0.05*Xmaster(:,1),'-','Color',color3,'LineWidth', largura_linha);
ylim([-0.3 0.8])
h=legend('Mensagem original', 'Mensagem decodificada', 'Messagem
    codificada','Location','northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_1(t), \hat{m}_1(t), 0.05{\cdot}s_1(t)$$','Interpreter','Latex
    ','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_13, format);
saveas(gcf, append(local_13, eps), format2);

%Figure 14
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.06
    0.012]);
close(figure(14));
figure(14);
subplot(1,1,1);
aux5 = Xmaster(:,2) - Xslave(:,2);
plot(t,msg(:,2),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux5,':','Color',color2,'LineWidth', largura_linha);
plot(t, 0.05*Xmaster(:,2),'-','Color',color3,'LineWidth', largura_linha);
ylim([-0.15 0.3])
h=legend('Original message', 'Mensagem decodificada', 'Messagem
    codificada','Location','northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_2(t), \hat{m}_2(t), 0.05{\cdot}s_2(t)$$','Interpreter','Latex
    ','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_14, format);
saveas(gcf, append(local_14, eps), format2);
```

```
%Figure 15
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
close(figure(15));
figure(15);
subplot(1,1,1);
aux6 = Xmaster(:,3) - Xslave(:,3);
plot(t,msg(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux6,':','Color',color2,'LineWidth', largura_linha);
plot(t, 0.05*Xmaster(:,3),'-','Color',color3,'LineWidth', largura_linha);
ylim([-1 1.5])
h=legend('Original message', 'Mensagem decodificada', 'Messagem
    codificada','Location','northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_3(t), \hat{m}_3(t), 0.05{\cdot}s_3(t)$$','Interpreter','Latex
    ','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_15, format);
saveas(gcf, append(local_15, eps), format2);

%Figure 16
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.06
    0.012]);
%close(figure(16));
figure(16);
subplot(1,1,1);
aux7 = Xmaster(:,1) - Xslave(:,1);
aux0 = aux7 - msg(:,1);
plot(t,aux0,'Color',color3,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.15 0.05])
h=legend('$$\tilde{m}_1(t)$$','Interpreter','Latex','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('Message error','Fontsize',fonte)
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_16, format);
saveas(gcf, append(local_16, eps), format2);
```

```
%Figure 17
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(17));
figure(17);
subplot(1,1,1);
aux8 = Xmaster(:,2) - Xslave(:,2);
aux00 = aux8 - msg(:,2);
plot(t,aux00,'Color',color3,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.2 0.3])
h=legend('$$\tilde{m}_2(t)$$','Interpreter','Latex','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('Message error','Fontsize',fonte)
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_17, format);
saveas(gcf, append(local_17, eps), format2);
%close(figure(1));

%Figure 18
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(18));
figure(18);
subplot(1,1,1);
aux9 = Xmaster(:,3) - Xslave(:,3);
aux000 = aux9 - msg(:,3);
plot(t,aux000,'Color',color3,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.7 0.5])
h=legend('$$\tilde{m}_3(t)$$','Interpreter','Latex','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('Message error','Fontsize',fonte)
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_18, format);
saveas(gcf, append(local_18, eps), format2);
%close(figure(1));

x1=(t);
```

```
y1=(aux7);
y2=(aux8);
y3=(aux9);
C=polyfit(x1,y1,1);
n=length(x1);
fx1=polyval(C,x1);
ecm1=sqrt(sum((fx1-y1).^2)/n)
fprintf(1,'%f\n',ecm1)

y2=(aux8);
C=polyfit(x1,y2,1);
ecm2=sqrt(sum((fx1-y2).^2)/n)
fprintf(1,'%f\n',ecm2)

y3=(aux9);
C=polyfit(x1,y3,1);
ecm3=sqrt(sum((fx1-y3).^2)/n)
fprintf(1,'%f\n',ecm3)
```

B.5 CODES FOR SIMULATIONS IN CHAPTER 7.

B.5.1 Underactuated 4D-Hypercaotic System for Secure Communication, Corresponding to the Figures 7.1-7.12 and the Table 7.1

.

Listing B.22 – Principal.m

```
clear   %clears previous variables
clc     %clears what was written in the terminal

%Amplitude Scaling
x_fator = 1;       %\bar{x} = x/x_fator , being "\bar{x}" the new value
    and "x" the old value
y_fator = 1;       %\bar{y} = y/y_fator , being "\bar{x}" the new value
    and "x" the old value
z_fator = 1;       %\bar{z} = z/z_fator , being "\bar{x}" the new value
    and "x" the old value
w_fator = 1;       %\bar{w} = w/w_fator , being "\bar{x}" the new value
    and"x" the old value
%Frequency Scaling
freq_fator = 1;   %The higher it is, the faster the simulation happens

h = 1; %Ppresence or not of disturbances, h =0 if no, h=1 if yes
%Initial conditions, the only essential rule is that master and slave
    states are different.
x0_mestre =  [0.6 1 -0.2 -0.4];
```

```
x0_escravo = [0.2 -0.5 0.1 0];

psi_1 = 100;      %Gain of the underactuated control    $
psi_2 = 10;
psi_3 = 100;
psi_4 = 10;
t_fim = 10;     %Time at which the simulation ends

options = odeset('RelTol', 1e-10,...    %simulation settings
                 'AbsTol', 1e-10,...
                 'MaxStep',0.001);

addpath('./Arquivos/');

Simulate
%Graphis
```

Listing B.23 – Sistema.m

```
function equation = Sistema(x, y, z, w)
a = 10;
b = 8/3;
c = 28;
d = 1;
equation =  [a*y - a*x + w;    %put here the structure of your dynamic
    system
             c*x - d*y - x*z;
             x*y - b*z;
             -y*z - w];
end
```

Listing B.24 – Mensagens.m

```
function msg = Mensagens(t)
msg = [0.3*(sin(10*t) + 1.5*sin(30*t));   %coloque aqui a estrutura do
    seu sistema dinâmico
       0;
       0.15*sin(20*t) + 0.3*sin(2*t);
       0];
end
%Messages sent (must be at most 5% of the maximum value reached by the
    state) Understand as msg1 a %message present in the first state, for
    example Do not put messages in states where control is %present
```

Listing B.25 – Simular.m

```
t_ciclo = [0 t_fim ];   %put the initial and final simulation time in a
    vector
psi = [psi_1 psi_2 psi_3 psi_4];
```

```
amp_f(1) = x_fator;
amp_f(2) = y_fator;
amp_f(3) = z_fator;
amp_f(4) = w_fator;

x0(1) = x0_mestre (1)/amp_f(1);  %passing the initial conditions to
    scaled variables
x0(2) = x0_mestre (2)/amp_f(2);
x0(3) = x0_mestre (3)/amp_f(3);
x0(4) = x0_mestre (4)/amp_f(4);
x0(5) = x0_escravo(1)/amp_f(1);
x0(6) = x0_escravo(2)/amp_f(2);
x0(7) = x0_escravo(3)/amp_f(3);
x0(8) = x0_escravo(4)/amp_f(4);

[t, x] = ode45(@Esquema, t_ciclo, x0, options, psi, h, amp_f, freq_fator)
    ;
%runs the simulation and saves the results
aux = size(t);
msg = zeros(aux(1), 4);

for i = 1:aux(1)
    tempo = t(i,1);
    msg_aux = Mensagens(tempo);
    msg(i,:) = msg_aux';
end

Xmaster_sem_msg = x(:, 1:4);
Xmaster = x(:, 1:4) + msg(:, 1:4);
Xslave = x(:, 5:8);
%Repassed the results to vectors simpler to print on graphics %
    Xmaster_without_msg as the name %suggests is the master state without
    the presence of the message %Xmaster is the encrypted or %encoded
    message (with the presence of the message)
clearvars -except t Xmaster_sem_msg Xmaster Xslave msg


function y = Esquema(t, x, psi, h, amp_f, freq_fator)

eq_mestre =  Sistema(amp_f(1)*x(1),...
                     amp_f(2)*x(2),...
                     amp_f(3)*x(3),...
                     amp_f(4)*x(4));    %master system equation
eq_escravo =  Sistema(amp_f(1)*x(5),...
                      amp_f(2)*x(6),...
                      amp_f(3)*x(7),...
```

```
                      amp_f(4)*x(8));   %slave system equation

eq_mestre(1) = freq_fator*eq_mestre(1)/amp_f(1);
eq_mestre(2) = freq_fator*eq_mestre(2)/amp_f(2);
eq_mestre(3) = freq_fator*eq_mestre(3)/amp_f(3);
eq_mestre(4) = freq_fator*eq_mestre(4)/amp_f(4);

eq_escravo(1) = eq_escravo(1)/amp_f(1);
eq_escravo(2) = eq_escravo(2)/amp_f(2);
eq_escravo(3) = eq_escravo(3)/amp_f(3);
eq_escravo(4) = eq_escravo(4)/amp_f(4);

y(1:4, 1) = eq_mestre;
y(5:8, 1) = freq_fator*(eq_escravo + controle(x, psi)) + h*disturb(t);
%The result of the master and slave systems

end

function controle = controle(x, psi)
controle = [0;                            %control structure
            -1*(psi(1)*(x(6) - x(2)) + psi(2)*(x(6) - x(2))^3);
            0;
            -1*(psi(3)*(x(8) - x(4)) + psi(4)*(x(8) - x(4))^3)];
end

function disturb = disturb(t)   %disturbances, if you want to place them
if t>=0                         %Start of disturbances starting at 5
    seconds
    disturb = [0.1*sin(5*t);
               0.25*cos(3*t);
               0.15*cos(5*t);
               0.2*sin(t)];
else
    disturb = 0; %Until t=10 seconds the disturbances are null
end
end
```

Listing B.26 – Graficos.m

```
clc

addpath('./Arquivos/');
addpath('./Figuras/');

Graficos_sem_mensagem_estilo_3
Graficos_com_mensagem_estilo_3

%Hint for older Matlabs, if you are having difficulty displaying style 2
```

```
    and 3 graphs, prioritize %style 1 graphs Style 2 and 3 graphs make it
    easier to remove whitespace, but they often do not %work on older
    Matlab systems
```

Listing B.27 – Graficos-com-mensagem-estilo-3.m

```
fonte = 10;
largura_linha = 2;
color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];
color3 = [0.4660 0.6740 0.1880];

addpath('./Figuras/Figuras_6_com_mensagem_estilo_3/');

local = 'Figuras/Figuras_6_com_mensagem_estilo_3';
format = 'png';
format2 = 'epsc';
eps = '.eps';

nome_1 = '/FIG_6_1';
nome_2 = '/FIG_6_2';
nome_3 = '/FIG_6_3';
nome_4 = '/FIG_6_4';
nome_5 = '/FIG_6_5';
nome_6 = '/FIG_6_6';
nome_7 = '/FIG_6_7';
nome_8 = '/FIG_6_8';
nome_9 = '/FIG_6_9';
nome_10 = '/FIG_6_10';
nome_11 = '/FIG_6_11';
nome_12 = '/FIG_6_12';
nome_13 = '/FIG_6_13';
nome_14 = '/FIG_6_14';
nome_15 = '/FIG_6_15';
nome_16 = '/FIG_6_16';

local_1 = append(local, nome_1);
local_2 = append(local, nome_2);
local_3 = append(local, nome_3);
local_4 = append(local, nome_4);
local_5 = append(local, nome_5);
local_6 = append(local, nome_6);
local_7 = append(local, nome_7);
local_8 = append(local, nome_8);
local_9 = append(local, nome_9);
local_10 = append(local, nome_10);
local_11 = append(local, nome_11);
local_12 = append(local, nome_12);
```

```
local_13 = append(local, nome_13);
local_14 = append(local, nome_14);
local_15 = append(local, nome_15);
local_16 = append(local, nome_16);
set(0,'DefaultAxesFontSize', fonte);

%Figure 1
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.012]);
%close(figure(1));
figure(1);
subplot(1,1,1);
plot(t,Xmaster(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,1),':','Color',color2,'LineWidth',largura_linha);
ylim([-25 30])
xlabel("$t[s]$",'Interpreter','latex')
legend("$x_m(t)$","$x_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_1, format);
saveas(gcf, append(local_1, eps), format2);

%Figure 2
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.04
    0.012]);
%close(figure(2));
figure(2);
subplot(1,1,1);
plot(t,Xmaster(:,2),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,2),':','Color',color2,'LineWidth', largura_linha);
ylim([-25 40])
xlabel("$t[s]$",'Interpreter','latex')
legend("$y_m(t)$","$y_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_2, format);
saveas(gcf, append(local_2, eps), format2);

%Figure 3
```

```
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.035
    0.012]);
%close(figure(3));
figure(3);
subplot(1,1,1);
plot(t,Xmaster(:,3),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,3),':','Color',color2,'LineWidth', largura_linha);
ylim([0 55])
xlabel("$t[s]$",'Interpreter','latex')
legend("$z_m(t)$","$z_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_3, format);
saveas(gcf, append(local_3, eps), format2);

%Figure 4
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.055
    0.012]);
%close(figure(4));
figure(4);
subplot(1,1,1);
plot(t,Xmaster(:,4),'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
hold on;
plot(t, Xslave(:,4),':','Color',color2,'LineWidth', largura_linha);
%  ylim([-0.5 5])
xlabel("$t[s]$",'Interpreter','latex')
legend("$w_m(t)$","$w_s(t)$",'Interpreter','latex','Location','northeast'
    ,'Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_4, format);
saveas(gcf, append(local_4, eps), format2);

%Figure 5
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(5));
figure(5);
subplot(1,1,1);
e1 = Xslave(:,1) - Xmaster(:,1);
plot(t,e1,'-','Color',color1,'LineWidth', largura_linha);
```

```
grid on
grid minor
%  ylim([-0.05 0.05])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_1(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_5, format);
saveas(gcf, append(local_5, eps), format2);

%Figure 6
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(6));
figure(6);
subplot(1,1,1);
e2 = Xslave(:,2) - Xmaster(:,2);
plot(t,e2,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
%  ylim([-0.05 0.05])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_2(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_6, format);
saveas(gcf, append(local_6, eps), format2);

%Figure 7
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(7));
figure(7);
subplot(1,1,1);
e3 = Xslave(:,3) - Xmaster(:,3);
plot(t,e3,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-1 1])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_3(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_7, format);
```

```
saveas(gcf, append(local_7, eps), format2);

%Figure 8
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.05
    0.012]);
close(figure(8));
figure(8);
subplot(1,1,1);
e4 = Xslave(:,4) - Xmaster(:,4);
plot(t,e4,'-','Color',color1,'LineWidth', largura_linha);
grid on
grid minor
ylim([-0.3 0.5])
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_4(t)$",'Interpreter','latex','Location','northeast','
    Orientation','horizontal')
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_8, format);
saveas(gcf, append(local_8, eps), format2);

%Figure 9
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(9));
figure(9);
subplot(1,1,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, 0.04*Xmaster(:,1),':','Color',color2,'LineWidth', largura_linha);
ylim([-1 2])
h=legend('Mensagem original', 'Mensagem codificada','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_1(t), 0.04{\cdot}s_1(t)$$','Interpreter','Latex','Fontsize',
    fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_9, format);
saveas(gcf, append(local_9, eps), format2);

%Figure 10
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
```

```
%close(figure(10));
figure(10);
subplot(1,1,1);
plot(t,msg(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, 0.04*Xmaster(:,3) - 1,':','Color',color2,'LineWidth',
    largura_linha);
ylim([-1.5 2])
h=legend('Mensagem original', 'Mensagem codificada','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_2(t), 0.04{\cdot}s_2(t) - 1$$','Interpreter','Latex','
    Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_10, format);
saveas(gcf, append(local_10, eps), format2);

%Figure 11
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(11));
figure(11);
subplot(1,1,1);
aux = Xmaster(:,1) - Xslave(:,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux,':','Color',color2,'LineWidth', largura_linha);
ylim([-1 1.5])
h=legend('Mensagem original', 'Mensagem decodificada','Location','
    northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_1(t), \hat{m}_1(t)$$','Interpreter','Latex','Fontsize',fonte)
    ;
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_11, format);
saveas(gcf, append(local_11, eps), format2);

%Figure 12
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
```

```
    0.012]);
%close(figure(12));
figure(12);
subplot(1,1,1);
aux = Xmaster(:,3) - Xslave(:,3);
plot(t,msg(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux,':','Color',color2,'LineWidth', largura_linha);
ylim([-0.8 1])
h=legend('Mensagem original', 'Mensagem decodificada','Location','
    northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_2(t), \hat{m}_2(t)$$','Interpreter','Latex','Fontsize',fonte)
    ;
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_12, format);
saveas(gcf, append(local_12, eps), format2);

%Figure 13
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(13));
figure(13);
subplot(1,1,1);
aux2 = Xmaster(:,1) - Xslave(:,1);
plot(t,msg(:,1),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux2,':','Color',color2,'LineWidth', largura_linha);
plot(t, 0.04*Xmaster(:,1),'-','Color',color3,'LineWidth', largura_linha);
ylim([-1 2])
h=legend('Mensagem original', 'Mensagem decodificada', 'Mensagem
    codificada','Location','northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_1(t), \hat{m}_1(t), 0.04{\cdot}s_1(t)$$','Interpreter','Latex
    ','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_13, format);
saveas(gcf, append(local_13, eps), format2);
```

```
%Figure 14
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(14));
figure(14);
subplot(1,1,1);
aux3 = Xmaster(:,3) - Xslave(:,3);
plot(t,msg(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on;
plot(t, aux3,':','Color',color2,'LineWidth', largura_linha);
plot(t, 0.04*Xmaster(:,3) - 1,'-','Color',color3,'LineWidth',
    largura_linha);
ylim([-1.5 2.5])
h=legend('Mensagem original', 'Mensagem decodificada', 'Mensagem
    codificada','Location','northeast');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('$$m_2(t), \hat{m}_2(t), 0.04{\cdot}s_2(t) - 1$$','Interpreter','
    Latex','Fontsize',fonte);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_14, format);
saveas(gcf, append(local_14, eps), format2);

%Figure 15
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(15));
figure(15);
subplot(1,1,1);
aux = Xmaster(:,1) - Xslave(:,1);
aux4 = aux - msg(:,1);
plot(t,aux4,'Color',color3,'LineWidth', largura_linha);
grid on
grid minor
h=legend('$$\tilde{m}_1(t)$$','Interpreter','Latex','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('Message error','Fontsize',fonte)
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_15, format);
saveas(gcf, append(local_15, eps), format2);
```

```
%Figure 16
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.14 0.02], [0.045
    0.012]);
%close(figure(16));
figure(16);
subplot(1,1,1);
aux = Xmaster(:,3) - Xslave(:,3);
aux5 = aux - msg(:,3);
plot(t,aux5,'Color',color3,'LineWidth', largura_linha);
grid on
grid minor
h=legend('$$\tilde{m}_2(t)$$','Interpreter','Latex','Location','northeast
    ');
set(h,'FontSize',fonte);
xlabel('$$t[s]$$','Interpreter','Latex','Fontsize',fonte);
ylabel('Message error','Fontsize',fonte)
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,8])
saveas(gcf, local_16, format);
saveas(gcf, append(local_16, eps), format2);
%close(figure(1));
```

B.6 CODES FOR SIMULATIONS IN CHAPTER 8

B.6.1 Minimal Underactuated Synchronization of Chaotic Systems.

B.6.2 Simulink Plant used for simulations corresponding to Figures (8.2-8.14); (8.25-8.32) and (8.34-8.45).

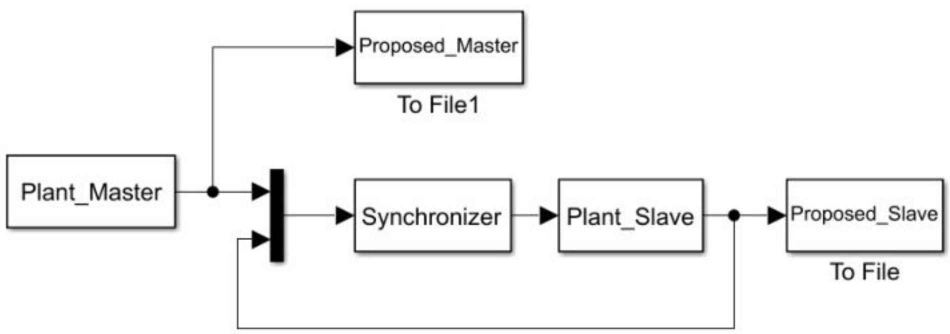

Figure B.4 – Comparative scheme to show the peculiarities of the proposed method.

.

Listing B.28 – PlantaMaster.m

```
function [sys,x0,str,ts] = Plant_Master(t,x,u,flag)
```

```
a = 10;
b = 2;
c = 28;
d = 0.1;
m = 27;
n = 0.5;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Initialization %
   %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 4; %Number of constants states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 4;    %Number of outputs
    sizes.NumInputs = 0;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
   x0=[-2.9 3.8 4.7 -1.2]; %Initial conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Directives  %
   %%%%%%%%%%%%%%
case 1,      %System
       sys = [a*(x(2) - x(1));
              (c*x(1) - x(2) - x(1)*x(3) + x(4) - d);
              (-b*x(3) + x(1)*x(2));
              (m*x(2) + x(4) - n*(x(1)^3))];

   %%%%%%%%%%%
   % Outputs %
   %%%%%%%%%%%
case 3,
   %sys = [x(1)+ 0.1*sin(8*t); x(2)+ 0.1*sin(9*t); x(3)+ 0.1*sin(10*t); x
    (4)+ 0.1*square(10*t)];
   sys = [x(1); x(2); x(3); x(4)];
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; % It does not do anything
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
```

```
end
```

Listing B.29 – PlantaSlave.m

```
function [sys,x0,str,ts] = Plant_Slave(t,x,u,flag)

a = 10;
b = 2;
c = 28;
d = 0.1;
m = 27;
n = 0.5;

switch flag,
   %%%%%%%%%%%%%%%%%
   % Initialization %
   %%%%%%%%%%%%%%%%%
case 0,
  sizes = simsizes;
    sizes.NumContStates = 4; %Number of constants states
    sizes.NumDiscStates = 0; %Number of discrete states
    sizes.NumOutputs = 4;    %Number of outputs
    sizes.NumInputs = 4;     %Number of inputs
    sizes.DirFeedthrough = 1;
    sizes.NumSampleTimes = 1;
    sys = simsizes(sizes);
    x0=[12.4 -7.5 10.2 3.4]; %Initial conditions
  str=[];
  ts=[0 0];
   %%%%%%%%%%%%%%
   % Directives %
   %%%%%%%%%%%%%%
case 1,      %System
       sys = [a*(x(2) - x(1)) + u(1);
              c*x(1) - x(2) - x(1)*x(3) + x(4) - d + u(2);
              -b*x(3) + x(1)*x(2) + u(3);
              m*x(2) + x(4) - n*(x(1)^3) + u(4)] + disturb(x,u,t);

   %%%%%%%%%%%
   % Outputs %
   %%%%%%%%%%%
case 3,
   sys = x;
   %%%%%%%%%
   %  End  %
   %%%%%%%%%
case {2,4,9},
    sys = []; % It does not do anything
```

```
 otherwise
   error(['unhandled flag = ',num2str(flag)]);
end

function disturb = disturb(x,u,t)
if t>=7
    disturb = [0.1*sin(5*t); 0.1*cos(3*t); 0.3*cos(5*t); 50*(sin(2*t) +
   0.4*sin(10*t))]; %disturbance
else
    disturb = 0; %Until t=10 secs the disturb is null
end
```

Listing B.30 – Graphs-comparacao.m

```
% -------------------------------------------- 'Scheme proposed '
   --------------------------------------------
% Proposed_Master(:,1) = Master time
% Proposed_Master(:,2) until Proposed_Master(:,5) = Xm, Ym, Zm and Wm
   states, respectivaly, from the master system
%
% Proposed_Slave(:,1) = Slave time
% Proposed_Slave(:,2) until Proposed_Slave(:,5) = Xs, Ys, Zs and Ws
   states, respectivaly, from the slave system

clc
format = 'png';
addpath('./Figuras/');
local_1 = 'Figuras/Comparison';

fonte = 10;
fonte_letrinhas = 8;
largura_linha = 1.5;

load('Article2015_Master.mat');
Article2015_Master = Article2015_Master';

load('Article2015_Slave.mat');
Article2015_Slave = Article2015_Slave';

load('Proposed_Master.mat');
Proposed_Master = Proposed_Master';

load('Proposed_Slave.mat');
Proposed_Slave = Proposed_Slave';

% % % % % % % % % % % % % % % % % % % % % % % % %
% % % % % % % Differential Equation % % % % % % %
% % % % % % % % % % % % % % % % % % % % % % % % %
```

```
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.064 0.006], [0.045
    0.012]);

color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];



%=====================================================
%%%%%%%%%%%%%%%%%% single figures%%%%%%%%%%%%%%%
%%====================================================

format = 'png';
addpath('./Figuras/');
local_2 = 'Figuras/fig_2';
local_3 = 'Figuras/fig_3';
local_4 = 'Figuras/fig_4';
local_5 = 'Figuras/fig_5';
local_6 = 'Figuras/fig_6';
local_7 = 'Figuras/fig_7';
local_8 = 'Figuras/fig_8';
local_9 = 'Figuras/fig_9';

fonte = 20;
fonte_letrinhas = 24;
largura_linha = 1.5;

%close(figure(5));
figure(5);
%Figure 5
%subplot(4,2,1)
plot(Article2015_Slave(:,1), Article2015_Slave(:,2), '-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1), Proposed_Slave(:,2), ':','Color',color2,'
    LineWidth',largura_linha);
hold on
plot(Article2015_Master(:,1), Article2015_Master(:,2),'--','Color',
    [0,0.7,0],'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
```

```
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];            % the location of arrow
pa.Y = [21 21];
pa.LineWidth  = 1;    % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
%text(7.2,25,'disturbances in action','Fontsize',8) % write a text on top
     of the arrow
text(7.2,25,'distúrbios em ação','Fontsize',8) % write a text on top of
    the arrow

ylim([-45, 30]);
xlim([0, 15]);

(t)$','Interpreter','latex','Location','southeast','NumColumns',2)
legend('$x_{s}(t)$ em [1]','$x_{s}(t)$ em (8.3)','$x_{m}(t)$','
    Interpreter','latex','Location','northeast','NumColumns',2)
title('(a)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_2, format);
saveas(gcf,'Figuras/fig_2.eps','epsc');
%close(figure(2));on', [0.5, -0.36, 0],'fontname','times','FontSize',
    fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')


%close(figure(6));
figure(6);
%Figure 6
%Error 1
plot(Article2015_Slave(:,1), Article2015_Slave(:,2) - Article2015_Master
    (:,2), '-','Color',color1,'LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1), Proposed_Slave(:,2) - Proposed_Master(:,2), ':'
    ,'Color',color2,'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
```

```
pa.X = [7 14];              % the location of arrow
pa.Y = [12 12];
pa.LineWidth  = 1;    % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
text(7.2,15,'disturbances in action','Fontsize',8) % write a text on top
    of the arrow

(6)','Interpreter','latex','Location','southeast','Orientation','
    horizontal')
legend('$e_{1}(t)$ in [1]','$e_{1}(t)$ in (8.6)','Interpreter','latex','
    Location','northeast','Orientation','horizontal')
title('(b)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-25, 20]);
xlim([0, 15]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_3, format);
saveas(gcf,'Figuras/fig_6.eps','epsc');


%close(figure(7));
figure(7);
%Figure 7
plot(Article2015_Slave(:,1), Article2015_Slave(:,3), '-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),  Proposed_Slave(:,3), ':','Color',color2,'
    LineWidth',largura_linha);
plot(Article2015_Master(:,1),  Article2015_Master(:,3),'--','Color',
    [0,0.7,0],'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];             % the location of arrow
pa.Y = [29 29];
pa.LineWidth  = 1;    % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
```

```
text(7.2, 35,'distúrbios em ação','Fontsize',8) % write a text on top of
    the arrow
(t)$','Interpreter','latex','Location','southeast','NumColumns',2)
legend('$y_{s}(t)$ em [1]','$y_{s}(t)$ em (8.3)','$y_{m}(t)$','
    Interpreter','latex','Location','northeast','NumColumns',2)
title('(c)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-60, 40]);
xlim([0, 15]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_4, format);
saveas(gcf,'Figuras/fig_3.eps','epsc');

figure(8);
plot(Article2015_Slave(:,1), Article2015_Slave(:,3) - Article2015_Master
    (:,3), '-','Color',color1,'LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1), Proposed_Slave(:,3) - Proposed_Master(:,3),':',
    'Color',color2,'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];             % the location of arrow
pa.Y = [11.5 11.5];
pa.LineWidth  = 1;   % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
text(7.2,15,'distúrbios em ação','Fontsize',8) % write a text on top of
    the arrow

legend('$e_{2}(t)$ em [1]','$e_{2}(t)$ em (8.6)','Interpreter','latex','
    Location','northeast','Orientation','horizontal')
title('(d)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-35, 20]);
xlim([0, 15]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
```

```
saveas(gcf,local_5, format);
saveas(gcf,'Figuras/fig_7.eps','epsc');

figure(9);
plot(Article2015_Slave(:,1), Article2015_Slave(:,4), '-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),  Proposed_Slave(:,4), ':','Color',color2,'
    LineWidth',largura_linha);
plot(Article2015_Master(:,1),  Article2015_Master(:,4),'--','Color',
    [0,0.7,0],'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];            % the location of arrow
pa.Y = [63 63];
pa.LineWidth  = 1;    % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
text(7.2, 70,'distúrbios em ação','Fontsize',8) % write a text on top of
    the arrow

legend('$z_{s}(t)$ em [1]','$z_{s}(t)$ em (8.3)','$z_{m}(t)$','
    Interpreter','latex','Location','northeast','NumColumns',2)
title('(e)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-30, 80]);
xlim([0, 15]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_6, format);
saveas(gcf,'Figuras/fig_4.eps','epsc');

figure(10);
plot(Article2015_Slave(:,1), Article2015_Slave(:,4) - Article2015_Master
    (:,4), '-','Color',color1,'LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1), Proposed_Slave(:,4) - Proposed_Master(:,4), ':'
    ,'Color',color2,'LineWidth',largura_linha);
grid on
```

```
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];             % the location of arrow
pa.Y = [31 31];
pa.LineWidth  = 1;    % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
text(7.2, 35,'disturbances in action','Fontsize',8) % write a text on top
     of the arrow

legend('$e_{3}(t)$ em [1]','$e_{3}(t)$ in (8.6)','Interpreter','latex','
    Location','northeast','Orientation', 'horizontal')
title('(f)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-30, 40]);
xlim([0, 15]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_7, format);
saveas(gcf,'Figuras/fig_8.eps','epsc');

figure(11);
plot(Article2015_Slave(:,1), Article2015_Slave(:,5), '-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),  Proposed_Slave(:,5), ':','Color',color2,'
    LineWidth',largura_linha);
plot(Article2015_Master(:,1),  Article2015_Master(:,5),'--','Color',
    [0,0.7,0],'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];             % the location of arrow
pa.Y = [170 170];
```

```
pa.LineWidth  = 1;   % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
text(7.2, 205,'distúrbios em ação','Fontsize',8) % write a text on top of
     the arrow
legend('$w_{s}(t)$ em [1]','$w_{s}(t)$ em (8.3)','$w_{m}(t)$','
    Interpreter','latex','Location','northeast','NumColumns',2)
title('(g)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
xlim([0, 15]);
ylim([-420, 240]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_8, format);
saveas(gcf,'Figuras/fig_9.eps','epsc');

figure(12);
plot(Article2015_Slave(:,1), Article2015_Slave(:,5) - Article2015_Master
    (:,5), '-','Color',color1,'LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1), Proposed_Slave(:,5) - Proposed_Master(:,5), ':'
    ,'Color',color2,'LineWidth',largura_linha);
grid on
grid minor

YL = get(gca, 'ylim'); %plot the vertical line
YR = YL(2) - YL(1);
YL = [YL(1) - 1000 * YR, YL(2) + 1000 * YR];
line([7, 7], YL, 'YLimInclude', 'off', 'Color','k','LineWidth',1);
pa = annotation('arrow');  % store the arrow information in pa
pa.Parent = gca;           % associate the arrow the the current axes
pa.X = [7 14];             % the location of arrow
pa.Y = [13 13];
pa.LineWidth  = 1;   % make the arrow bolder for the figure
pa.HeadWidth  = 8;
pa.HeadLength = 8;
text(7.2,15,'disturbances in action','Fontsize',8) % write a text on top
    of the arrow

legend('$e_{4}(t)$ em [1]','$e_{4}(t)$ em (8.6)','Interpreter','latex','
    Location','northeast','Orientation', 'horizontal')
title('(h)', 'Units', 'normalized', 'Position', [0.5, -0.36, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
xlim([0, 15]);
ylim([-20, 18]);
```

```
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_9, format);
saveas(gcf,'Figuras/fig_9.eps','epsc');
%close(figure(12));
```

Listing B.31 – Graphs-desempenho da sincronizacao.m

```
% This script is responsasible for generating the Figure 1 at the article
    .

clc
format = 'png';
addpath('./Figuras/');
local_1 = 'Figuras/Fig1';
local_2 = 'Figuras/Fig1_e1_e2_e3_e4';
local_3 = 'Figuras/sincro_x';
local_4 = 'Figuras/sincro_y';
local_5 = 'Figuras/sincro_z';
local_6 = 'Figuras/sincro_w';
fonte = 10;
fonte_letrinhas = 8;
largura_linha = 1.5;
set(0,'DefaultAxesFontSize',fonte);


load("Proposed_Master.mat");
Proposed_Master = Proposed_Master';
load("Proposed_Slave.mat");
Proposed_Slave = Proposed_Slave';

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.075 0.006], [0.038
    0.030]);

color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];


%%%%%%%%%%%individual figures%%%%%%%%%%%%%%%%%%%%%

%close(figure(1));
figure(2);
%Figure 2
plot(Proposed_Master(:,1),Proposed_Master(:,2),'-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),Proposed_Slave(:,2),':','Color',color2,'
    LineWidth',largura_linha);
```

```
grid on
grid minor
legend("$x_m(t)$","$x_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
title('(a)', 'Units', 'normalized', 'Position', [0.5, -0.28, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-30 20]);
saveas(gcf,local_3, format);
saveas(gcf,'Figuras/sincro_x.eps','epsc');
%close(figure(2));


%close(figure(3));
figure(3);
%Figure 3
plot(Proposed_Master(:,1),Proposed_Master(:,3),'-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),Proposed_Slave(:,3),':','Color',color2,'
    LineWidth',largura_linha);
grid on
grid minor
legend("$y_m(t)$","$y_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(b)', 'Units', 'normalized', 'Position', [0.5, -0.28, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-40 30]);
saveas(gcf,local_4, format);
saveas(gcf,'Figuras/sincro_y.eps','epsc');
%close(figure(3));


%close(figure(4));
figure (4);
%Figure 4
plot(Proposed_Master(:,1),Proposed_Master(:,4),'-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),Proposed_Slave(:,4),':','Color',color2,'
    LineWidth',largura_linha);
grid on
grid minor
legend("$z_m(t)$","$z_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
title('(c)', 'Units', 'normalized', 'Position', [0.5, -0.28, 0],'fontname
```

```
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-20 80]);
saveas(gcf,local_5, format);
saveas(gcf,'Figuras/sincro_z.eps','epsc');
%close(figure(4));


%close(figure(5));
figure (5);
%Figure 5
plot(Proposed_Master(:,1),Proposed_Master(:,5),'-','Color',color1,'
    LineWidth',largura_linha);
hold on
plot(Proposed_Slave(:,1),Proposed_Slave(:,5),':','Color',color2,'
    LineWidth',largura_linha);
grid on
grid minor
legend("$w_m(t)$","$w_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
title('(d)', 'Units', 'normalized', 'Position', [0.5, -0.28, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
ylim([-400 200]);
saveas(gcf,local_6, format);
saveas(gcf,'Figuras/sincro_w.eps','epsc');
figure (6);
subplot(1,2,[1 2])
e1 = Proposed_Slave(:,2) - Proposed_Master(:,2);
e2 = Proposed_Slave(:,3) - Proposed_Master(:,3);
e3 = Proposed_Slave(:,4) - Proposed_Master(:,4);
e4 = Proposed_Slave(:,5) - Proposed_Master(:,5);
plot(Proposed_Master(:,1),e1,Proposed_Master(:,1),e2,'-.','LineWidth'
    ,1.8);
hold on
plot(Proposed_Master(:,1),e3,'--', 'Color', [0,0.7,0],'LineWidth',1.8);
plot(Proposed_Master(:,1),e4,'k:','LineWidth',1.8);
grid on
grid minor
title('(e)', 'Units', 'normalized', 'Position', [0.5, -0.28, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_1(t)$","$e_2(t)$","$e_3(t)$","$e_4(t)$",'Interpreter','latex',
    'Location','southeast','Orientation','horizontal')
ylim([-40 40]);

set(gcf,'renderer','painters','units','centimeters','position'
```

```
    ,[5,5,17,20])
saveas(gcf,local_2, format);
saveas(gcf,'Figuras/Fig1_e1_e2_e3_e4.eps','epsc');
%close(figure(6));
```

Listing B.32 – Graphs-Comparacao entre mensagens.m

```
%% Description of DataCompSimulWithMSG:
% DataCompSimulWithMSG has the rows and collumns inverted, so it is
% necessary to use the transpose of it

load('Data.mat');
Data = DataCompSimulWithMSG';

% --- Description --- :
% Data(:,1)  : Time
% Data(:,2)  até Data(:,5): Xmaster
% Data(:,6)  até Data(:,9): Xslave
% Data(:,10) até Data(:,13): Msg
% Data(:,14)  até Data(:,17): Xmaster + Msg

% Graphics
clc
format = 'png';
addpath('./Figuras/');
local_1 = 'Figuras/Fig3';
local_2 = 'Figuras/Fig5';
local_3 = 'Figuras/Fig4';

local_4 = 'Figuras/Fig6';
local_5 = 'Figuras/Fig7';
local_6 = 'Figuras/Fig8';
local_7 = 'Figuras/Fig9';
local_8 = 'Figuras/Fig10';
local_9 = 'Figuras/Fig11';
local_10 = 'Figuras/Fig12';
local_11 = 'Figuras/Fig13';

local_12 = 'Figuras/Fig14';
local_13 = 'Figuras/Fig15';
local_14 = 'Figuras/Fig16';
local_15 = 'Figuras/Fig17';

local_16 = 'Figuras/Fig18';
local_17 = 'Figuras/Fig19';
local_18 = 'Figuras/Fig20';
local_19 = 'Figuras/Fig21';
local_20 = 'Figuras/Fig22';
```

```
local_21 = 'Figuras/Fig23';
local_22 = 'Figuras/Fig24';
local_23 = 'Figuras/Fig25';
local_24 = 'Figuras/Fig26';

fonte = 20;
fonte_letrinhas = 24;
largura_linha = 1.5;
set(0,'DefaultAxesFontSize',fonte);

% % % % % % % % % % % % % % % % % % % % % % % %%
% Real Components Simulation - no message
% This part will generate the Figure 5 at the article
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.065 0.006], [0.023
    0.030]);

color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];



%%%%%%%%%%%%%%%%SINGLE FIGURES%%%%%%%%%%%%%%%%

% % % % % % % % % % % % % % % % % % % % % % % %%
% Real Components Simulation - no message
% This part will generate the Figure 5 at the article
subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.065 0.006], [0.023
    0.030]);

color1 = [0 0.4470 0.7410];
color2 = [0.8500 0.3250 0.0980];

figure(6);
plot(Data(:,1),Data(:,2),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on
plot(Data(:,1),Data(:,6),':','Color',color2,'LineWidth',largura_linha);
legend("$x_m(t)$","$x_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(a)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-2.5, 2]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
```

```
saveas(gcf,local_4, format);
saveas(gcf,'Figuras/Fig6.eps','epsc');
%close(figure(6));



%Error 1
%close(figure(7));
figure(7);
plot(Data(:,1), Data(:,6) - Data(:,2),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
title('(b)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_1(t)$",'Interpreter','latex','Location','southeast','
    Orientation','horizontal')
ylim([-2.5, 2]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_5, format);
saveas(gcf,'Figuras/Fig7.eps','epsc');
%close(figure(7));

figure(8);
plot(Data(:,1),Data(:,3),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on
plot(Data(:,1),Data(:,7),':','Color',color2,'LineWidth',largura_linha);
legend("$y_m(t)$","$y_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(c)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-2.4, 2]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_6, format);
saveas(gcf,'Figuras/Fig8.eps','epsc');
%close(figure(8));
```

```
%Error 2
figure(9);
plot(Data(:,1), Data(:,7) - Data(:,3), '-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
title('(d)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_2(t)$",'Interpreter','latex','Location','southeast','
    Orientation','horizontal')
ylim([-2.4, 2]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_7, format);
saveas(gcf,'Figuras/Fig9.eps','epsc');
%close(figure(9));

figure(10);
plot(Data(:,1),Data(:,4),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on
plot(Data(:,1),Data(:,8),':','Color',color2,'LineWidth',largura_linha);
legend("$z_m(t)$","$z_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(e)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-1, 2.4]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_8, format);
saveas(gcf,'Figuras/Fig10.eps','epsc');
figure(11);
%Error 3
plot(Data(:,1), Data(:,8) - Data(:,4),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
title('(f)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_3(t)$",'Interpreter','latex','Location','southeast','
    Orientation','horizontal')
```

```
ylim([-1, 2.4]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_9, format);
saveas(gcf,'Figuras/Fig11.eps','epsc');
%close(figure(11));

%close(figure(12));
figure(12);
plot(Data(:,1),Data(:,5),'-','Color',color1,'LineWidth',largura_linha);
grid on
grid minor
hold on
plot(Data(:,1),Data(:,9),':','Color',color2,'LineWidth',largura_linha);
legend("$w_m(t)$","$w_s(t)$",'Interpreter','latex','Location','southeast'
    ,'Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(g)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-2.4, 1.2]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_10, format);
saveas(gcf,'Figuras/Fig12.eps','epsc');
%close(figure(12));

%Error 4
figure(13);
plot(Data(:,1), Data(:,9) - Data(:,5),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
title('(h)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
xlabel("$t[s]$",'Interpreter','latex')
legend("$e_4(t)$",'Interpreter','latex','Location','southeast','
    Orientation','horizontal')
ylim([-2.4, 1.2]);
xlim([0, 0.025]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_11, format);
saveas(gcf,'Figuras/Fig13.eps','epsc');
%close(figure(13));
```

```
% % % % % % % % % % % % % % % % % % % % % % % % %
% Real Components Simulation -
% Encrypted vs Original Message
%
% This part will generate the Figure 6 at the article
%%

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.065 0.006], [0.045
    0.10]);

% Data(:,10) até Data(:,13): Msg
% Data(:,14)  até Data(:,17): Xmaster + Msg

%close(figure(14));
figure(14);
plot(Data(:,1),0.05*Data(:,14)+0.05,'-','Color',color1,'LineWidth',
    largura_linha)
hold on
plot(Data(:,1),Data(:,10),':','Color',color2,'LineWidth',largura_linha);
grid on
grid minor
title('(a)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$0.05\cdot m_{1c}(t)+0.05$$','$${m}_{1}(t)$$','Interpreter','
    latex','Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
ylim([-0.08, 0.15]);
xlim([0.005, 0.03]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_12, format);
saveas(gcf,'Figuras/Fig14.eps','epsc');
figure(15);
plot(Data(:,1),0.05*Data(:,15)+0.025,'-','Color',color1,'LineWidth',
    largura_linha)
hold on
plot(Data(:,1),Data(:,11),':','Color',color2,'LineWidth',largura_linha);
grid on
grid minor
title('(b)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$0.05\cdot m_{2c}(t)+0.025$$','$${m}_{2}(t)$$','Interpreter','
    latex','Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
ylim([-0.09, 0.1]);
xlim([0.005, 0.03]);set(gcf,'renderer','painters','units','centimeters','
    position',[5,5,17,20])
```

```
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_13, format);
saveas(gcf,'Figuras/Fig15.eps','epsc');
%close(figure(15));

figure(16);
plot(Data(:,1),0.05*Data(:,16)-0.025,'-','Color',color1,'LineWidth',
    largura_linha)
hold on
plot(Data(:,1),Data(:,12),':','Color',color2,'LineWidth',largura_linha);
grid on
grid minor
title('(c)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$0.05\cdot m_{3c}(t)-0.025$$','$${m}_{3}(t)$$','Interpreter','
    latex','Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
ylim([-0.03, 0.064]);
xlim([0.005, 0.03]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_14, format);
saveas(gcf,'Figuras/Fig16.eps','epsc');
figure(17);
plot(Data(:,1),0.05*Data(:,17)+0.025,'-','Color',color1,'LineWidth',
    largura_linha)
hold on
plot(Data(:,1),Data(:,13),':','Color',color2,'LineWidth',largura_linha);
grid on
grid minor
title('(d)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$0.05\cdot m_{4c}(t)+0.025$$','$${m}_{4}(t)$$','Interpreter','
    latex','Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
ylim([-0.07, 0.09]);
xlim([0.005, 0.03]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_15, format);
saveas(gcf,'Figuras/Fig17.eps','epsc');
%close(figure(17));

% % % % % % % % % % % % % % % % % % % % % % %
% %%%%%%%%%%Real Components Simulation %%%%%%%%%%%-
% Decripted vs Original message
```

```
%
% This part will generate the Figure 7 at the article
%%

subplot = @(m,n,p) Subtightplot (m, n, p, 0.077, [0.065 0.006], [0.045
    0.1]);

%close(figure(18));
figure(18);
e = Data(:,14) - Data(:,6);
plot(Data(:,1),e,'-','Color',color1,'LineWidth',largura_linha)
grid on
grid minor
hold on
plot(Data(:,1),Data(:,10),':','Color',color2,'LineWidth',largura_linha);
legend('$$\hat{m}_{1}(t)$$','$${m}_{1}(t)$$','Interpreter','latex','
    Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(a)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-0.1, 0.2]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_16, format);
saveas(gcf,'Figuras/Fig18.eps','epsc');
%close(figure(18));

close(figure(19));
figure(19);
plot(Data(:,1), (e - Data(:,10)),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
xlabel("$t[s]$",'Interpreter','latex')
title('(b)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$\tilde{m}_{1}(t)$$','Interpreter','latex','Location','southeast
    ','Orientation','horizontal')
ylim([-0.1, 0.2]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_17, format);
saveas(gcf,'Figuras/Fig19.eps','epsc');
%close(figure(19));
```

```
close(figure(20));
figure(20);
e = Data(:,15) - Data(:,7);
plot(Data(:,1),e,'-','Color',color1,'LineWidth',largura_linha)
grid on
grid minor
hold on
plot(Data(:,1),Data(:,11),':','Color',color2,'LineWidth',largura_linha);
legend('$$\hat{m}_{2}(t)$$','$${m}_{2}(t)$$','Interpreter','latex','
    Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(c)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-0.05, 0.1]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_18, format);
saveas(gcf,'Figuras/Fig20.eps','epsc');
%close(figure(20));

figure(21);
plot(Data(:,1), (e - Data(:,11)),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
xlabel("$t[s]$",'Interpreter','latex')
title('(d)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$\tilde{m}_{2}(t)$$','Interpreter','latex','Location','southeast
    ','Orientation','horizontal')
ylim([-0.05, 0.1]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_19, format);
saveas(gcf,'Figuras/Fig21.eps','epsc');
%close(figure(21));

figure(22);
e = Data(:,16) - Data(:,8);
plot(Data(:,1),e,'-','Color',color1,'LineWidth',largura_linha)
grid on
grid minor
hold on
plot(Data(:,1),Data(:,12),':','Color',color2,'LineWidth',largura_linha);
legend('$$\hat{m}_{3}(t)$$','$${m}_{3}(t)$$','Interpreter','latex','
```

```
    Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(e)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-0.03, 0.07]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_20, format);
saveas(gcf,'Figuras/Fig22.eps','epsc');
%close(figure(22));

figure(23);
%Error Message 3
plot(Data(:,1), (e - Data(:,12)),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
xlabel("$t[s]$",'Interpreter','latex')
title('(f)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$\tilde{m}_{3}(t)$$','Interpreter','latex','Location','southeast
    ','Orientation','horizontal')
ylim([-0.03, 0.07]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_21, format);
saveas(gcf,'Figuras/Fig23.eps','epsc');
%close(figure(23));

figure(24);
e = Data(:,17) - Data(:,9);
plot(Data(:,1),e,'-','Color',color1,'LineWidth',largura_linha)
grid on
grid minor
hold on
plot(Data(:,1),Data(:,13),':','Color',color2,'LineWidth',largura_linha);
legend('$$\hat{m}_{4}(t)$$','$${m}_{4}(t)$$','Interpreter','latex','
    Location','southeast','Orientation','horizontal')
xlabel("$t[s]$",'Interpreter','latex')
title('(g)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
ylim([-0.03, 0.07]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
```

```
saveas(gcf,local_22, format);
saveas(gcf,'Figuras/Fig24.eps','epsc');
%close(figure(24));

%Error Message 4
figure(25);
plot(Data(:,1), (e - Data(:,13)),'-','Color',color1,'LineWidth',
    largura_linha);
grid on
grid minor
xlabel("$t[s]$",'Interpreter','latex')
title('(h)', 'Units', 'normalized', 'Position', [0.5, -0.37, 0],'fontname
    ','times','FontSize',fonte_letrinhas,'FontWeight','Normal');
legend('$$\tilde{m}_{4}(t)$$','Interpreter','latex','Location','southeast
    ','Orientation','horizontal')
ylim([-0.03, 0.07]);
xlim([0.005, 0.030]);
set(gcf,'renderer','painters','units','centimeters','position'
    ,[5,5,17,20])
saveas(gcf,local_23, format);
saveas(gcf,'Figuras/Fig25.eps','epsc');
%close(figure(25));


%=========================================================================
%%%%%%%%%%ejes size test%%%%%%%%%%%%%%%%
%=====================================================
%close(figure(26));
figure(26);
% %Figura 26
plot(Data(:,1), (e - Data(:,13)),'-','Color',color1,'LineWidth',2);set(0,
    'DefaultAxesFontSize',24);
grid on
grid minor
h=legend('$$\tilde{m}_{4}(t)$$','Interpreter','latex','Location','
    northeast','Orientation','horizontal')
set(h,'FontSize',fsize);
set(0,'DefaultAxesFontSize', 24);
xlabel("$t[s]$",'Interpreter','latex','Fontsize',fsize)
ylim([-0.03, 0.03]);
xlim([0.005, 0.030]);
set(gcf,'units','normalized','outerposition',[0 0 1 1]);
saveas(gcf,local_24, format);
saveas(gcf,'Figuras/Fig26.eps');
%close(figure(26));
```