UNIVERSIDADE DE BRASÍLIA
INSTITUTO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA

Functional Volterra–Stieltjes Integral Equations

by

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Equações integrais funcionais do tipo Volterra–Stieltjes

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Resumo

Nesta tese, estudamos as equações integrais funcionais do tipo Volterra–Stieltjes dadas por:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} = \phi, \end{cases}$$

onde a integral no lado direito é entendida no sentido de Henstock–Kurzweil–Stieltjes.

Neste trabalho, apresentamos condições suficientes para garantir a existência, unicidade e prolongamento de soluções para esse tipo de equações. Provamos também correspondências entre essas equações e as equações delta integrais funcionais do tipo Volterra em escalas temporais, bem como com as equações funcionais integrais do tipo Volterra–Stieltjes com impulsos. Apresentamos resultados de estabilidade para suas soluções, resultados sobre dependência contínua com respeito aos parâmetros e garantimos a existência de soluções periódicas para essas equações. Os resultados inéditos deste trabalho podem ser encontrados em [31, 33, 32, 46].

Palavras-chave: Equações integrais funcionais; equações de Volterra–Stieltjes; equações integrais com impulsos; equações $\Delta-$integrais em escalas temporais; periodicidade; estabilidade; dependência contínua.

Abstract

In this thesis, we study the functional Volterra–Stieltjes integral equations given by:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} = \phi, \end{cases}$$

where the integral on the right–hand side is taken in the sense of Henstock–Kurzweil–Stieltjes.

In this work, we present sufficient conditions in order to guarantee the existence, uniqueness and prolongation of solutions for this type of equations. We also prove the correspondence between these equations and the functional Volterra delta integral equations on time scales, as well as with the impulsive functional Volterra–Stieltjes integral equations. We present results concerning stability, continuous dependence with respect on parameters and periodicity. The new results can be found in [31, 33, 32, 46].

Key-words: Functional integral equations; Volterra–Stieltjes equations; impulsive integral equations; $\Delta-$integral equations on time scales; periodicity; stability; continuous dependence.

Contents

Introduction

In the decades 20–40, Vito Volterra introduced in the literature an important class of equations given by:

$$x(t) = f(t) + \int_{t_0}^{t} K(t,s)x(s)\mathrm{d}s, \tag{0.0.1}$$

which can encompass many types of equations. The kernel that appears in the above equation allows us to describe many types of phenomena, specially the ones related to the memory, as we will see in this thesis. Throughout the years, several versions of equation (0.0.1) started to appear in the literature in order to improve the descriptions of the phenomena, including for instance integro–differential equations.

The study of Volterra equations is an emerging area of research which possesses interesting mathematical questions and applications, since it can model several natural phenomena such as anomalous diffusion processes, heat conduction with memory and diffusion of fluids in porous media, among others and, therefore, it is an important equation to be studied as it can be verified in the literature. Some few examples can be found in [1, 4, 7, 8, 7, 10, 17, 34].

Let us observe a model describing temperature of one–dimensional bar:

$$\begin{cases} T_t(\xi,t) = T_{\xi\xi}(\xi,t), \quad \xi > 0, \quad t > 0, \\ T_\xi(0,t) = \alpha T(0,t) - q(t), \qquad t > 0, \\ T(\xi,0) = 0, \qquad \xi \geqslant 0, \\ \lim\limits_{\xi\to\infty} T(\xi,t) = 0, \qquad t \geqslant 0, \end{cases} \tag{0.0.2}$$

where $T(\xi,t)$ represents the temperature of one–dimensional bar for $\xi \geqslant 0$ which loses energy at a rate proportional to $T(0,t)$ at the point $\xi = 0$. Assume that an external source generates heat proportional to the function $q(t)$ at this end of the bar, which is insulated at all other parts, with temperature zero at time $t = 0$.

If $\xi = 0$, then it is possible to obtain the solution of this problem by using the following convolution Volterra integral equation

$$x(t) + \int_0^t k(t-s)x(s)ds = f(t), \quad t \geqslant 0, \tag{0.0.3}$$

where

$$x(t) = T(0,t),\; k(t) = \frac{a}{\sqrt{\pi}}t^{-1/2} \text{ and } f(t) = \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}q(s)ds.$$

Therefore, equation (0.0.2) is a particular case of equation (0.0.1), showing the variety of problems that Volterra equations can encompass, including important PDEs. Indeed, using Laplace transforms in (0.0.2), we get the following definition for $T(\xi,t)$:

$$T(\xi,t) = \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}e^{-\xi^2/(4(t-s))}\left(q(s) - \alpha T(0,s)\right)ds, \quad t > 0, \quad \xi \geqslant 0,$$

obtaining the formulation above. This model can be found in [34].

In this thesis, we work with a more general class of equations, called *functional Volterra–Stieltjes integral equations*, that are described as follows:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^t a(t,s)f(x_s,s)\,\mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} = \phi, \end{cases} \tag{0.0.4}$$

where $0 \leqslant t_0 < d$, $r > 0$, $\phi \in G([-r,0],\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$, $a\colon [t_0,d)^2 \to \mathbb{R}$ and $g\colon [t_0,d) \to \mathbb{R}$ is a nondecreasing function, where $t_0 < d \leqslant +\infty$, $x_s\colon [-r,0] \to \mathbb{R}^n$ is given by $x_s(\theta) = x(s+\theta)$ for $s \in [t_0,d)$. Here, $[t_0,d)^2$ denotes the set $[t_0,d) \times [t_0,d)$.

The formulation (0.0.4) is even more general than the one first described by Volterra and as we will see here in this work, it can encompass many types of equations. This fact motivated us to investigate this type of problem. Depending on the definition of a, g and r, equation (0.0.4) may encompass a huge variety of problems, as we describe below:

i) By choosing $a(t,s) = 1, g(s) = s$, $r = 0$ and $\phi \equiv x_0$, we obtain the standard formulation of ODEs:

$$\begin{cases} y(t) = x_0 + \displaystyle\int_{t_0}^t f(y(s),s)\mathrm{d}s, \quad t \geqslant t_0, \\ y(t_0) = x_0. \end{cases}$$

ii) If we choose $a(t,s)=1$ and $g(s)=s$, we obtain the classical functional differential equations:

$$\begin{cases} y(t)=\phi(0)+\displaystyle\int_{t_0}^{t} f(y_s,s)\mathrm{d}s, & t\geqslant t_0,\\ y_{t_0}=\phi. \end{cases}$$

iii) If $a(t,s)=(t-s)^{\alpha-1}/\Gamma(\alpha), g(s)=s$, $r=0$ and $\phi\equiv x_0$, then we are discussing the standard Caputo fractional differential equations (see [17])

$$\begin{cases} y(t)=x_0+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}f(y(s),s)\mathrm{d}s, & t\geqslant t_0,\\ y(t_0)=x_0. \end{cases}$$

The reader will see in later chapters that the functional Volterra–Stieltjes integral equations encompass also integral equations with impulses. We will show how to make a correspondence between the solutions (0.0.4) and the solutions of

$$\begin{cases} x(v)-x(u) &=& \displaystyle\int_{t_0}^{v} a(v,s)f(x_s,s)\mathrm{d}g(s)-\int_{t_0}^{u} a(u,s)f(x_s,s)\mathrm{d}g(s) \text{ for } u,v\in J_k, k\in\mathbb{N}\\ \Delta^{+}x(t_k) &=& I_k(x(t_k)),\quad k=1,\dots,m\\ x_{t_0} &=& \phi, \end{cases} \tag{0.0.5}$$

where $J_0=[t_0,t_1], J_k=(t_k,t_{k+1}]$, for $k=1,\dots,m-1$, $J_m=(t_m,d)$ and $\{t_k\}_{k=1}^{m}$ are the pre-fixed moments of impulses, where each $t_k\in[t_0,d)$ and $d\leqslant\infty$. This fact allows us to investigate impulsive equations "implicitly", by only studying Volterra–Stieltjes integral equations. We will also show in this work that the impulsive fractional differential equations can also be regarded as Volterra–Stieltjes integral equations, even in the case that the order of the derivative α is between 1 and 2, as well as between 0 and 1.

We will also describe a correspondence between the solutions of (0.0.4) and the solutions of the functional Volterra $\Delta-$integrals on time scales:

$$\begin{cases} x(t) &=& x(t_0)+\displaystyle\int_{t_0}^{t} a(t,s)f(x_s^{*},s)\Delta s, \quad t\in[t_0,d)_{\mathbb{T}},\\ x(t) &=& \phi(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}}, \end{cases} \tag{0.0.6}$$

where $\mathbb{T}$ is an arbitrary time scale, $d\in\mathbb{T}\cup\{\infty\}$, $\phi\in G([t_0-r,t_0]_{\mathbb{T}},\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,d)_{\mathbb{T}}\to\mathbb{R}^n$ and $[t_0-r,t_0]_{\mathbb{T}},[t_0,d)_{\mathbb{T}}$ are time scales intervals. These equations play an important role for applications, since the theory of times scales can unify discrete, continuous and "in between" cases (see [13]). To motivate the study of Volterra Δ-integral on time scales, we also present here a version of the famous model in economics that is know as Keynesian-Cross model with "lagged" income, that can be described by a Volterra Δ-integral on time scales, showing the importance of this type of equations. For more details, see Chapter 2.

Moreover, the class of neutral functional differential equations (neutral FDEs, for short) can also be regarded as functional Volterra–Stieltjes integral equations. Indeed, neutral FDEs are usually described by:

$$\dot{x}(t) = Lx_t, \tag{0.0.7}$$

where L is a continuous linear map from $C([-r, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$. Using Riesz Representation Theorem, we can rewrite (0.0.7) as follows:

$$\dot{x} = \int_0^h \mathrm{d}\xi(\theta)x(t-\theta). \tag{0.0.8}$$

Using the initial condition given by $x(\theta) = \varphi(\theta)$ for $-h \leqslant \theta \leqslant 0$, we get

$$\begin{cases} \dot{x}(t) = \displaystyle\int_0^t \xi(\theta)\dot{x}(t-\sigma)\mathrm{d}\theta + g(t) \\ x(0) = \varphi(0), \end{cases} \tag{0.0.9}$$

where $g(t) = \xi(t)\varphi(0) + \int_t^h \mathrm{d}\xi(\theta)\varphi(t-\theta)$.

Integrating (0.0.9), we have

$$x(t) - \varphi(0) = \int_0^t \xi(\theta)[x(t-\theta) - x(0)]\mathrm{d}\theta + \int_0^t g(s)\mathrm{d}s, \tag{0.0.10}$$

which implies $x = x * \xi + f$, where

$$f(t) = \varphi(0) + \int_0^t g(s)\mathrm{d}s - \int_0^t \xi(\theta)\mathrm{d}\theta\varphi(0).$$

Equation (0.0.10) is a type of Volterra integral equations, which shows us that it is possible to rewrite the neutral equation given by (0.0.7) as a Volterra integral equation.

Therefore, it is clear how equation given by (0.0.4) can be general, since it encompasses many type of equations.

Also, in this thesis, we employ the so–called *Henstock–Kurzweil–Stieltjes* integral, which is more general than the Lebesgue–Stieltjes integral. Therefore, all the results obtained here are valid for a more general class of functions than the Lebesgue integrable functions, which allows an oscillating behavior.

This new integral was first defined by the mathematicians Jaroslav Kurzweil [45] in 1957 and Ralph Henstock [39] in 1961, who, independently, formulated an equivalent integral, capable of integrating functions that not even the Lebesgue integral could. The motivation behind the Henstock–Kurzweil integral is the *Kapitza pendulum*, studied at first by Andrew Stephenson [66]. This pendulum had a mass moving around a certain support, like an ordinary pendulum, but also the support itself oscillated at a very high frequency ω. An unusual consequence of this oscillation was the fact that a stable equilibrium position for the mass was exactly *above* the support, as it can be viewed in the following picture.

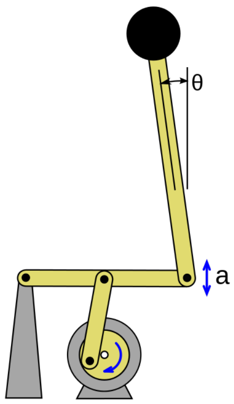


Figure 1: Kapitza's Pendulum
(obtained from Wikipedia website)

In 1951, the physicist Pyotr Kapitza [42, 41] was able to obtain a model that described the movement of this pendulum:

$$\ddot{\theta} = \frac{g}{L}\sin(\theta) - \frac{a\omega^2 \sin(\omega t)}{L}\sin(\theta), \tag{0.0.11}$$

where $g > 0$ is the acceleration of gravity, $L > 0$ is the length of the pendulum, $a > 0$ is the amplitude of the support's vibration and $\theta\colon \mathbb{R} \to \mathbb{R}$ is the angle made by the pendulum with the vertical when the mass is placed upwards. However, because of the high frequency ω, the Lebesgue integral could not be used to find the solution for the aforementioned equation.

The Henstock–Kurweil integral does not only integrate more functions than the Lebesgue integral, but its definition is also simpler. This is very similar to the definition of the Riemann integral, with the difference that instead of asking the partitions to be smaller than a certain constant, it is used a certain function, called *gauge*, which is simply a positive function, to control the size of subintervals of the partitions (for a more complete description of this integral, the reader may refer to [11, 53] and Chapter 1.2 of this thesis).

On the other hand, since (0.0.4) encompasses (0.0.6), we will also present the basis of the theory of time scales. It was in 1988 when Stefan Hilger introduced the concept of a time scale in his PhD thesis [40]. He defined a *time scale* as any nonempty closed subset of $\mathbb{R}$. Hence, both $\mathbb{R}$ and $\mathbb{Z}$ are examples of time scales, but we have more sophisticated examples of time scales such as the Cantor set and the quantum scale, among others.

Studying and solving problems for an arbitrary time scale would give us, as a consequence, solutions for both the discrete and continuous cases, but not only that, since one can construct sets that are not completely continuous nor discrete, but hybrid. In this sense, Hilger ended up unifying the discrete and continuous analysis in a certain way. Therefore, working in an arbitrary time scale, one can understand a great class of different sets and instead of proving the same result for many different cases, one can prove it just once and encompass all those cases. This theory is well described in [13, 14]. Much work has being done concerning the theory of time scales, see for instance [5, 24, 47, 56, 65] and the references therein.

In this work, we investigate deeply the solutions of (0.0.4), its properties and sufficient conditions to ensure its existence, uniqueness, prolongation and boundary value problems. Also, sufficient conditions to ensure its stability are provided via Lyapunov functionals and continuous dependence with respect to the parameters and the correspondence between the solution of (0.0.4) and its analogue in the time scale setting and also, scope of functional Volterra–Stieltjes integral equations with impulses.

The chapters of this thesis are defined in the following manner: in Chapter 1, we give an overview of the theory of regulated functions, the Henstock–Kurzweil–Stieltjes integral and time scale theory. We also present some important definitions and results that will be essential to our purposes. The main references here are [11, 53, 60].

In Chapter 2, we present the correspondences between functional Volterra–Stieltjes integral equations, impulsive functional Volterra–Stieltjes integral equations and functional Volterra Δ-integral equations on time scales that will be used throughout this thesis to ensure that our results proved for functional Volterra–Stieltjes integral equations are also true for impulsive functional Volterra–Stieltjes integral equations and functional Volterra Δ-integral equations on time scales. Also, we present several examples and models that can be described using Volterra–Stieltjes integral equations, illustrating their importance. Further, we justify the generality of our conditions with an example.

In Chapter 3, we present the Volterra–Stieltjes integral equation that will be investigated here, give conditions to guarantee the existence of a unique solution to equation (0.0.4) and prove versions for impulsive functional Volterra–Stieltjes integral equations and functional Volterra $\Delta-$integral equations on time scales. Also, we investigate the existence and uniqueness of maximal solutions to all these equations.

In Chapter 4, we prove results concerning stability, asymptotic stability, uniform stability and exponential stability using Lyapunov functionals for equation (0.0.4). The results presented here generalize the ones found in the literature for measure equations, presenting more general conditions and considering the presence of the delays and kernel in the equation, which turns the techniques more sophisticated to work. In Chapter 5, we present a periodic boundary value problem with respect to (0.0.4) and we prove their analogues for the time scales and impulsive cases. Finally, in Chapter 6, we present some results concerning continuous dependence with respect to the parameters of equation (0.0.4), as well as their analogues to time scales and impulsive cases. The results presented in Chapters 2, 3, 4, 5 and 6 are new and are presented in the papers [31, 33, 32, 46].

Chapter 1

Preliminaries

This chapter is divided in 3 sections. In the first section, we give some basic definitions and results concerning the theory of regulated functions. In the second one, we present some initial concepts and essential results about the Henstock–Kurzweil-Stieltjes integration theory. In the third section, we explore the theory of time scales, presenting the most fundamental definitions and recalling some important concepts and theorems about differentiation and integration on time scales. The main references to this chapter are [11, 26, 53, 60].

The definitions and results presented in this chapter will be very important to prove the main results of this work.

1.1 Regulated functions

We start by recalling the reader about some properties and basic definitions of regulated functions. These properties will be essential to our work, since most of the functions involved in our study are regulated.

Definition 1.1.1 ([26])**.** A function $\varphi\colon [\alpha,\beta] \to \mathbb{R}^n$ is called *regulated*, if the lateral limits

$$\varphi(t^-) = \lim_{s\to t^-} \varphi(s), \quad t \in (\alpha,\beta] \qquad \text{and} \qquad \varphi(t^+) = \lim_{s\to t^+} \varphi(s), \quad t \in [\alpha,\beta)$$

exist. The space of all regulated functions $\varphi\colon [\alpha,\beta] \to \mathbb{R}^n$ will be denoted by $G([\alpha,\beta],\mathbb{R}^n)$.

It is a known fact that $G([\alpha,\beta],\mathbb{R}^n)$ endowed with the usual supremum norm

$$\|\varphi\|_\infty = \sup_{s\in[\alpha,\beta]} \|\varphi(s)\|$$

is a Banach space (see [26]). Let $I \subset \mathbb{R}$ be an interval. We denote by $G(I,\mathbb{R}^n)$ the space of all locally regulated functions $x\colon I \to \mathbb{R}^n$, that is, for each compact interval $[\alpha,\beta] \subset I$, the restriction of x to $[\alpha,\beta]$ belongs to the space $G([\alpha,\beta],\mathbb{R}^n)$.

Remark 1.1.2. If $x \in G(I,\mathbb{R}^n)$ and $[\alpha,\beta] \subset I$, we will use the notation

$$\|x\|_{\infty,[\alpha,\beta]} := \sup_{s\in[\alpha,\beta]} \|x(s)\|$$

to denote the norm of the function x restricted to the interval $[\alpha,\beta]$.

Let $g\colon [\alpha,\beta] \to \mathbb{R}^n$ be a regulated function. We will denote by $\Delta^+ g(t)$ and $\Delta^- g(t)$ the jumps to the right $g(t^+) - g(t)$ and the jumps to the left $g(t) - g(t^-)$, respectively.

Let us also define the variation of a function $f\colon [\alpha,\beta] \to X$, where X is a Banach space, over $[\alpha,\beta]$. By $\|\cdot\|_X$, we denote the norm in X . A set $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{|D|}\}$ is defined to be a partition of $[\alpha,\beta]$ if $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_{|D|=\beta}$. The set of all partitions of $[\alpha,\beta]$ will be denoted by $\mathcal{D}[\alpha,\beta]$.

Definition 1.1.3. We define the *variation* of f over $[\alpha,\beta]$ as

$$\operatorname{var}_\alpha^\beta(f) = \sup_{D\in\mathcal{D}[\alpha,\beta]} \sum_{j=1}^{|D|} \|f(\alpha_j) - f(\alpha_{j-1})\|_X.$$

If $\operatorname{var}_\alpha^\beta(f) < \infty$, then f is said to be a function of *bounded variation* on $[\alpha,\beta]$. We will denote the set of all the functions $f\colon [\alpha,\beta] \to X$ of bounded variaton by $BV([\alpha,\beta],X)$.

We recall that $BV([\alpha,\beta,X]) \subset G([\alpha,\beta],X)$ (see [61]).

The next result gives us an equivalence for the concept of regulated function which comes directly from the definition. With this equivalence in hands, it will become easier to prove some results in the next chapters.

Theorem 1.1.4 (Hönig's Theorem, [53, Theorem 4.15])**.** *The following statements are equivalent:*

(i) $f \in G([\alpha,\beta],\mathbb{R}^n)$;

(ii) *for every* $\varepsilon > 0$*, there is a division of the interval* $[\alpha, \beta]$, $\alpha = s_0 < s_1 < \ldots < s_n = \beta$, *such that for every* $i \in \{1, \ldots, n\}$ *and all* $t, r \in (s_{i-1}, s_i)$*, we have* $\|f(t) - f(r)\| < \varepsilon$.

With the purpose of presenting an analogue of Arzelà–Ascoli Theorem for regulated functions, we give a definition of an *equiregulated set* and then, an equivalence for this definition.

Definition 1.1.5 ([26, Definition 1.3])**.** A set $\mathcal{A} \subset G([\alpha, \beta], \mathbb{R}^n)$ is called *equiregulated*, if for every $\varepsilon > 0$ and $t_0 \in [\alpha, \beta]$, there exists a $\delta > 0$ such that:

(1) if $x \in \mathcal{A}$, $s \in [\alpha, \beta]$ and $t_0 - \delta < s < t_0$, then $\|x(t_0^-) - x(s)\| < \varepsilon$;

(2) if $x \in \mathcal{A}$, $s \in [\alpha, \beta]$ and $t_0 < s < t_0 + \delta$, then $\|x(t_0^+) - x(s)\| < \varepsilon$.

Lemma 1.1.6 ([53, Lemma 4.3.4])**.** *The following statements are equivalent.*

(1) $\mathcal{A} \subset G([\alpha, \beta], \mathbb{R}^n)$ *is equiregulated.*

(2) *For every* $\varepsilon > 0$*, there is a division* $\alpha = s_0 < s_1 < \ldots < s_n = \beta$ *such that for all* $y \in \mathcal{A}$*, all* $i \in \{1, \ldots, n\}$ *and all* $t, s \in (s_{i-1}, s_i)$*, we have* $\|y(t) - y(s)\| < \varepsilon$.

We now are ready to present an Arzelà–Ascoli type theorem for the case of regulated functions. See [53, Theorem 4.3.5] for its proof.

Theorem 1.1.7. *A set* $\mathcal{A} \subset G([\alpha, \beta], \mathbb{R}^n)$ *is relatively compact if, and only if, it is uniformly bounded and equiregulated.*

This type of result will be very important to prove results related to existence and prolongation of solutions. We also point out that there are more general versions of this theorem in the literature, for example, for functions taking value in a Banach space (see [27]), but for our purposes here, it is enough to work with the version presented in Theorem 1.1.7.

1.2 Henstock–Kurzweil–Stieltjes integration

We begin this section with some definitions that are needed for the comprehension of the Henstock–Kurzweil–Stieltjes integral. We also present some classical results in order to give the reader a clearer understanding of this type of integral. For more details, the reader can consult [11, 53, 60].

We also point out that the use of the Henstock–Kurzweil–Stieltjes integral allows us to work with a more general set of functions. This fact occurs because every Lebesgue integrable function is also Henstock–Kurzweil integrable, but the reciprocal is not true, except for positive functions, since a function f is Lebesgue integrable if and only if both f and its absolute value $\|f\|$ are Henstock–Kurzweil integrable (see [11]).

To define the Henstock–Kurzweil–Stieltjes integral, we begin by letting $[\alpha, \beta] \subset \mathbb{R}$, $\alpha < \beta$, be a compact interval. We say that a *tagged partition* of $[\alpha, \beta]$ is a set D of ordered pairs $(\tau_i, [s_{i-1}, s_i])$, where $\alpha = s_0 < s_1 < \ldots < s_{|D|} = \beta$ is a partition of $[\alpha, \beta]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, |D|$, where $|D|$ denotes the cardinality of the set D. We denote the tagged partition just by $D = (\tau_i, [s_{i-1}, s_i])$.

Given a set $B \subseteq [\alpha, \beta]$, we define a *gauge* on B as any function $\delta \colon B \to (0, \infty)$. Given a gauge δ on $[\alpha, \beta]$, we say that a tagged partition $D = (\tau_i, [s_{i-1}, s_i])$ is *δ–fine* if for every $i \in \{1, 2, \ldots, |D|\}$, we have

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

Definition 1.2.1. A function $f \colon [\alpha, \beta] \to \mathbb{R}^n$ is *Henstock–Kurzweil–Stieltjes integrable* on $[\alpha, \beta]$ with respect to a function $g \colon [\alpha, \beta] \to \mathbb{R}$, if there is an element $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a gauge $\delta \colon [\alpha, \beta] \to (0, \infty)$ such that

$$\left\| \sum_{i=1}^{|D|} f(\tau_i) \left(g(s_i) - g(s_{i-1}) \right) - I \right\| < \varepsilon$$

for all δ–fine tagged partition of $[\alpha, \beta]$. In this case, I is called the *Henstock–Kurzweil–Stieltjes* integral of f with respect to g over $[\alpha, \beta]$ and it will be denoted by $\int_\alpha^\beta f(s)\, \mathrm{d}g(s)$, or just by $\int_\alpha^\beta f\, \mathrm{d}g$.

Note that when g is the identity function, that is, $g(s) = s$, we have the classical Henstock–Kurzweil integral.

Lemma 1.2.2 (Cousin's Lemma)**.** *If $I := [a, b]$ is a nondegenerate compact interval in $\mathbb{R}$ and δ is a gauge on I, then there exists a partition of I that is δ-fine.*

A proof of the above result can be found in [11] and it is very important since it ensures that this integral is well–defined.

The classical properties of linearity, additivity with respect to adjacent intervals and integrability on subintervals are all valid for the Henstock–Kurzweil–Stieltjes integral. (see [60] and the references therein).

The following result ensures that the class of regulated functions is Henstock–Kurzweil–Stieltjes integrable with respect to a nondecreasing function. It also gives us an upper bound for the absolute value of the definite integral in an interval.

Theorem 1.2.3 ([60, Corollary 1.34])**.** *Let $f\colon [\alpha,\beta] \to \mathbb{R}^n$ be a regulated function and $g\colon [\alpha,\beta] \to \mathbb{R}$ be a nondecreasing function. Then, the following conditions hold:*

(i) *the integral* $\displaystyle\int_\alpha^\beta f(s)\mathrm{d}g(s)$ *exists;*

(ii) $\displaystyle\left\|\int_\alpha^\beta f(s)\mathrm{d}g(s)\right\| \leqslant \int_\alpha^\beta \|f(s)\|\,\mathrm{d}g(s) \leqslant \|f\|_\infty\,(g(\beta)-g(\alpha))$.

The next inequalities can be easily obtained from the definition of the Henstock–Kurzweil–Stieltjes integral. A version of it for the case of Riemann–Stieltjes integral can be found in [9, Theorem 7.20] and its proof is very similar to the result we present here.

Theorem 1.2.4. *Let $f_1, f_2\colon [\alpha,\beta] \to \mathbb{R}$ be Henstock–Kurzweil–Stieltjes integrable functions on the interval $[\alpha,\beta]$ with respect to a nondecreasing function $g\colon [\alpha,\beta] \to \mathbb{R}$ and such that $f_1(t) \leqslant f_2(t)$, for $t \in [\alpha,\beta]$. Then*

$$\int_\alpha^\beta f_1(s)\mathrm{d}g(s) \leqslant \int_\alpha^\beta f_2(s)\mathrm{d}g(s).$$

Corollary 1.2.5. *Let $f\colon [\alpha,\beta] \to \mathbb{R}$ be a Henstock–Kurzweil–Stieltjes integrable function on the interval $[\alpha,\beta]$ with respect to a nondecreasing function $g\colon [\alpha,\beta] \to \mathbb{R}$ and such that $f(t) \geqslant 0$ for $t \in [\alpha,\beta]$. Then*

(i) $\displaystyle\int_\alpha^\beta f(s)\mathrm{d}g(s) \geqslant 0.$

(ii) *The function $[\alpha,\beta] \ni t \mapsto \int_\alpha^t f(s)\mathrm{d}g(s)$ is nondecreasing.*

The following theorem gives us information about the indefinite Henstock–Kurzweil–Stieltjes integral. It is a special case of [60, Theorem 1.16].

Theorem 1.2.6. *Let $f\colon [\alpha,\beta] \to \mathbb{R}^n$ and $g\colon [\alpha,\beta] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_\alpha^\beta f(s)\mathrm{d}g(s)$ exists. Then the function*

$$h(t) = \int_\alpha^t f(s)\mathrm{d}g(s), \quad t \in [\alpha,\beta]$$

is regulated on $[\alpha,\beta]$ and satisfies

$$\begin{aligned} h(t^+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [\alpha,\beta), \\ h(t^-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (\alpha,\beta]. \end{aligned}$$

Theorem 1.2.7 ([54, Theorem 2.2])**.** *Let $g, g_n \in G([\alpha,\beta],\mathbb{R}^n), f, f_n \in BV([\alpha,\beta], L(\mathbb{R}^n))$ for $n \in \mathbb{N}$. Assume that*

$$\lim_{n\to\infty} \|g_n - g\|_\infty = 0, \quad \lim_{n\to\infty} \|f_n - f\|_\infty = 0 \quad \text{and} \quad \varphi^*: = \sup_{n\in\mathbb{N}} \operatorname{var}_a^b f_n < \infty.$$

Then

$$\lim_{n\to\infty} \left(\sup_{t\in[\alpha,\beta]} \left\| \int_a^t d[f_n]g_n - \int_a^t d[f]g \right\| \right) = 0.$$

The following lemma will be crucial to prove that an impulsive Volterra integral equation can always be transformed to a Volterra integral equation without impulses. This result can be found in [25, Lemma 2.4].

Lemma 1.2.8. *Let $m \in \mathbb{N}$, $\alpha \leqslant t_1 < t_2 < \cdots < t_m \leqslant \beta$. Consider a pair of functions $f\colon [\alpha,\beta] \to \mathbb{R}^n$ and $g\colon [\alpha,\beta] \to \mathbb{R}$, where g is regulated, left–continuous on $[a,b]$, and continuous at $t_1,\dots,t_m$. Let $\tilde f\colon [\alpha,\beta] \to \mathbb{R}^n$ and $\tilde g\colon [\alpha,\beta] \to \mathbb{R}$ be such that $\tilde f(t) = f(t)$ for every $t \in [\alpha,\beta]\backslash\{t_1,\dots,t_m\}$ and $\tilde g - g$ is constant on each of the intervals $[\alpha,t_1]$, $(t_1,t_2],\dots,(t_{m-1},t_m]$, $(t_m,\beta]$. Then the integral $\int_\alpha^\beta \tilde f\,\mathrm{d}\tilde g$ exists if and only if the integral $\int_\alpha^\beta f\,\mathrm{d}g$ exists; in that case, we have*

$$\int_\alpha^\beta \tilde f(s)\,\mathrm{d}\tilde g(s) = \int_\alpha^\beta f(s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<\beta}} \tilde f(t_k)\Delta^+\tilde g(t_k).$$

We end this section by presenting a Substitution–type Theorem for the Henstock–Kurzweil–Stieltjes integral, as well as analogous versions of Gronwall Inequality and Dominated Convergence Theorem for this integral. We also present a way of interchanging the order of integrals.

Theorem 1.2.9 ([53, Theorem 6.55]). *Assume that the function* $h\colon [\alpha,\beta]\to\mathbb{R}$ *is bounded and that the integral* $\int_\alpha^\beta f(s)\mathrm{d}g(s)$ *exists. If one of the integrals*

$$\int\limits_\alpha^\beta h(t)\mathrm{d}\left(\int\limits_\alpha^t f(\xi)\mathrm{d}g(\xi)\right), \quad \int\limits_\alpha^\beta h(t)f(t)\mathrm{d}g(t),$$

exists, then the other one exists as well, in which case the equality below holds

$$\int\limits_\alpha^\beta h(t)\mathrm{d}\left(\int\limits_\alpha^t f(\xi)\mathrm{d}g(\xi)\right) = \int\limits_\alpha^\beta h(t)f(t)\mathrm{d}g(t).$$

Lemma 1.2.10 (Gronwall Inequality [60, Corollary 1.43]). *Let* $g\colon [\alpha,\beta]\to[0,\infty)$ *be a nondecreasing and left–continuous function,* $k>0$ *and* $l\geqslant 0$*. Assume that* $\psi\colon[\alpha,\beta]\to[0,\infty)$ *is bounded and satisfies*

$$\psi(\xi)\leqslant k+l\int_\alpha^\xi \psi(s)\mathrm{d}g(s), \quad \xi\in[\alpha,\beta].$$

Then $\psi(\xi)\leqslant ke^{l(g(\xi)-g(\alpha))}$ *for all* $\xi\in[\alpha,\beta]$*.*

Theorem 1.2.11 (Dominated Convergence Theorem [60, Corollary 1.32]). *Let* $g\colon[\alpha,\beta]\to\mathbb{R}$ *be a nondecreasing function. Assume that the functions* $\varphi_n\colon[\alpha,\beta]\to\mathbb{R}$ *are such that the integral* $\int_\alpha^\beta \varphi_n(s)\,\mathrm{d}g(s)$ *exists for all* $n\in\mathbb{N}$*. Suppose that*

$$\lim_{n\to\infty}\varphi_n(s)=\varphi(s) \text{ for } s\in[\alpha,\beta]$$

and the inequalities

$$\kappa(s)\leqslant\varphi_n(s)\leqslant\omega(s) \text{ for } n\in\mathbb{N}, s\in[\alpha,\beta]$$

hold, where $\omega,\kappa\colon[\alpha,\beta]\to\mathbb{R}$ *are such that the integrals* $\int_\beta^\alpha \kappa(s)\,\mathrm{d}g(s)$ *and* $\int_\alpha^\beta\omega(s)\,\mathrm{d}g(s)$ *exist. Then the integral* $\int_\alpha^\beta\varphi(s)\,\mathrm{d}g(s)$ *exists and*

$$\lim_{n\to\infty}\int_\alpha^\beta\varphi_n(s)\,\mathrm{d}g(s)=\int_\alpha^\beta\varphi(s)\,\mathrm{d}g(s).$$

We finish this section with a result that allows us to interchange the order of integrals. It is interesting to note that since our integral is not necessarily continuous, when we interchange the order of the integrals, it appears a sum with the jumps of the functions related to the Stieltjes integral. This fact brings several complications, turning it much more difficult, when one need to deal with these types of integrals.

Lemma 1.2.12 ([64, Corollary 2.5])**.** *If* $g, h\colon [\alpha,\beta] \to \mathbb{R}$ *are of bounded variation,* $f\colon [\alpha,\beta]\times[\alpha,\beta]$ *is Borel measurable and bounded, then*

$$\int_{\alpha}^{\beta}\left(\int_{\alpha}^{x} f(x,y)\mathrm{d}h(y)\right)\mathrm{d}g(x)$$

$$= \int_{\alpha}^{\beta}\left(\int_{y}^{b} f(x,y)\mathrm{d}g(x)\right)\mathrm{d}h(y) + \sum_{y\in(\alpha,\beta]} f(y,y)\Delta^{-}g(y)\Delta h(y) - \sum_{x\in[\alpha,\beta)} f(x,x)\Delta^{+}h(x)\Delta g(x),$$

with the convention that $\Delta g(\alpha) = \Delta^{+}g(\alpha)$ *and* $\Delta h(b) = \Delta^{-}h(b)$.

1.3 Time Scales Theory

In this section, we begin by giving the definition of a time scale and describing important operators related to a given time scale. Then, we present fundamental results to the theory of time scales. All the results and definitions presented here can be found in [13, 14, 65].

In 1988, in his PhD thesis ([40]), Stefan Hilger introduced the theory of time scales with the aim of unifying discrete and continuous analysis. The intention behind the concept of time scales is that one can obtain results to functions whose domain is an arbitrary time scale, and so the result can be applied to the continuous case, the discrete case and even hybrid cases, depending on how we choose the time scale. This allows us to prove results to a very general class of functions and sets.

1.3.1 Definitions and basic properties

We define a *time scale* as any closed nonempty subset of $\mathbb{R}$ and usually denote an arbitrary time scale by the symbol $\mathbb{T}$. Given $\alpha, \beta \in \mathbb{T}$, we use the notation $[\alpha,\beta]_{\mathbb{T}}$ to denote the set $\{t \in \mathbb{T}\colon \alpha \leqslant t \leqslant \beta\}$, which is called a closed interval in $\mathbb{T}$. Similarly, we define open and half-open intervals in a time scale $\mathbb{T}$.

The next three definitions can be found in [13].

Definition 1.3.1. Given a time scale $\mathbb{T}$, we define the *forward jump operator* $\sigma\colon \mathbb{T}\to\mathbb{T}$ by

$$\sigma(t) = \inf\{s\in\mathbb{T}\colon s > t\}$$

and we define the *backward jump operator* $\rho\colon \mathbb{T}\to\mathbb{T}$ by

$$\rho(t) = \sup\{s\in\mathbb{T}\colon s < t\}.$$

In this definition, we put $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$. This means that $\sigma(t) = t$, if $\mathbb{T}$ has a maximum t and $\rho(s) = s$, if $\mathbb{T}$ has a minimum s.

Definition 1.3.2. Let $\mathbb{T}$ be a time scale and σ and ρ be the forward and backward jump operators, respectively, as defined above.

- If $\sigma(t) > t$, we say that t is *right-scattered*.
- If $\rho(t) < t$, we say that t is *left-scattered*.
- If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, we say that t is *right–dense*.
- If $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say that t is *left–dense*.

If a point t is right-scattered and left-scattered at the same time, we say that t is *isolated*, and if a point t is right–dense and left–dense at the same time, we say that t is *dense*.

Definition 1.3.3. We define the *graininess function* $\mu\colon \mathbb{T} \to [0, \infty)$ by

$$\mu(t) = \sigma(t) - t.$$

Now, define the set $\mathbb{T}^\kappa$ as follows

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty, \\ \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty. \end{cases} \tag{1.3.1}$$

In other words, if $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

In the sequel, we present several definitions related to the extension of an arbitrary time scale $\mathbb{T}$.

Given a time scale $\mathbb{T}$ and a real number $t \leqslant \sup \mathbb{T}$, we define $t^* := \inf\{s \in \mathbb{T}\colon s \geqslant t\}$. This operator was introduced for the first time by Antonín Slavík in [65]. We call the reader's attention to the fact that t^* and $\sigma(t)$ are not necessarily equal, since depending on the chosen time scale, we can have $\sigma(t) \neq t^*$. Also, since $\mathbb{T}$ is closed, we get that $t^* \in \mathbb{T}$. Now, we define the set $\mathbb{T}^*$ as an extension of $\mathbb{T}$ in the following way:

$$\mathbb{T}^* = \begin{cases} (-\infty, \infty), & \text{if } \sup \mathbb{T} = \infty, \\ (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty. \end{cases}$$

Given a function $f\colon \mathbb{T} \to \mathbb{R}^n$, we can extend it to the set $\mathbb{T}^*$ by defining the function $f^*\colon \mathbb{T}^* \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*). \tag{1.3.2}$$

Similarly, given a set $B \subset \mathbb{R}^n$ and a function $f\colon B \times \mathbb{T} \to \mathbb{R}^n$, we define $f^*(x,t) = f(x,t^*)$ for all $x \in B$ and $t \in \mathbb{T}^*$.

Also, given a function $a\colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}^n$, we consider $a^{**}\colon \mathbb{T}^* \times \mathbb{T}^* \to \mathbb{R}^n$ given by

$$a^{**}(t,s) := a(t^*, s^*), \quad (t,s) \in \mathbb{T}^* \times \mathbb{T}^*.$$

Lemma 1.3.4 ([21, Lemma 5.1])**.** *Let $[\alpha,\beta]_{\mathbb{T}}$ be a time scale interval. Let $g\colon [\alpha,\beta] \to \mathbb{R}$ be given by $g(t) = t^*$ for all $t \in [\alpha,\beta]$. Then g satisfies the following conditions:*

(i) *g is a nondecreasing function;*

(ii) *g is left–continuous on $(\alpha,\beta]$.*

1.3.2 Delta derivatives

Definition 1.3.5 ([13, Definition 1.10])**.** Let $f\colon \mathbb{T} \to \mathbb{R}^n$ be a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the vector (if it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\| \leqslant \varepsilon|\sigma(t) - s| \text{ for all } s \in U. \tag{1.3.3}$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) derivative of f at t.

We say that f is *delta differentiable on* $\mathbb{T}^\kappa$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta\colon \mathbb{T}^\kappa \to \mathbb{R}^n$ is called the *delta derivative* of f on $\mathbb{T}^\kappa$.

The next theorem gives us some useful ways of finding the delta derivative of a function.

Theorem 1.3.6 ([13, Theorem 1.16])**.** *Assume that $f\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following statements:*

1. *if f is delta differentiable at t, then f is continuous at t;*

2. *if f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

3. *if t is right–dense, then f is delta differentiable at t if, and only if, the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Now, we state some properties about sum, product and quotient of delta differentiable functions. This theorem and its proof can be found in [13, Theorem 1.20].

Theorem 1.3.7. *Assume $f, g \colon \mathbb{T} \to \mathbb{R}$ are delta differentiable functions at $t \in \mathbb{T}^{\kappa}$. Then:*

1. *The sum $f + g \colon \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *For any constant α, $\alpha f \colon \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

3. *The product $fg \colon \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

4. *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is delta differentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

1.3.3 Delta integrals

We begin this section recalling some concepts that are needed to introduce the definition of delta integrable functions in the sense of Henstock–Kurzweil. For more details, see [56].

Definition 1.3.8 ([13, Definition 1.57])**.** A function $f \colon \mathbb{T} \to \mathbb{R}^n$ is called *regulated* provided its right-sided limits exist (finite) at all right–dense points in $\mathbb{T}$ and its left–sided limits exist (finite) at all left–dense points in $\mathbb{T}$.

Given a set $B \subset \mathbb{R}^n$, the symbol $G([\alpha, \beta]_{\mathbb{T}}, B)$ will be used to denote the set of all regulated functions $f \colon [\alpha, \beta]_{\mathbb{T}} \to B$.

Definition 1.3.9 ([13, Definition 1.58])**.** A function $f\colon \mathbb{T} \to \mathbb{R}^n$ is called *rd-continuous* provided it is continuous at right–dense points in $\mathbb{T}$ and its left–sided limits exist (finite) at left–dense points in $\mathbb{T}$. The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}^n$ will be denoted in this work by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n).$$

Let $\delta = (\delta_L, \delta_R)$ be a pair of nonnegative functions defined on $[\alpha, \beta]_{\mathbb{T}}$. We say that δ is a $\Delta-$*gauge* for $[\alpha, \beta]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ on $(\alpha, \beta]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[\alpha, \beta)_{\mathbb{T}}$, and $\delta_R(t) \geqslant \mu(t)$ for all $t \in [\alpha, \beta)_{\mathbb{T}}$.

A *tagged partition* of $[\alpha, \beta]_{\mathbb{T}}$ consists of division points $s_0, \ldots, s_m \in [\alpha, \beta]_{\mathbb{T}}$ such that $\alpha = s_0 < s_1 < \cdots < s_m = \beta$, and *tags* $\tau_1, \ldots, \tau_m \in [\alpha, \beta]_{\mathbb{T}}$ such that $\tau_i \in [s_{i-1}, s_i]_{\mathbb{T}}$ for every $i \in \{1, \ldots, m\}$. Such a partition is called *δ-fine* if

$$\tau_i - \delta_L(\tau_i) \leqslant s_{i-1} < s_i \leqslant \tau_i + \delta_R(\tau_i), \quad i \in \{1, \ldots, m\}.$$

A function $f\colon [\alpha, \beta]_{\mathbb{T}} \to \mathbb{R}^n$ is called *Henstock–Kurzweil Δ–integrable*, if there exists a vector $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a Δ–gauge δ on $[\alpha, \beta]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{m} f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon$$

for every δ–fine tagged partition of $[\alpha, \beta]_{\mathbb{T}}$. In this case, I is called the *Henstock–Kurzweil Δ–integral* of f over $[\alpha, \beta]_{\mathbb{T}}$ and it will be denoted by $\int_\alpha^\beta f(t)\Delta t$.

Next, we present a class of functions that are Henstock–Kurzweil Δ–integrable on $[\alpha, \beta]_{\mathbb{T}}$.

Theorem 1.3.10 ([56, Corollary 2.7])**.** *Every regulated function f in $[\alpha, \beta]_{\mathbb{T}}$ is Henstock–Kurzweil Δ–integrable on $[\alpha, \beta]_{\mathbb{T}}$.*

The next results are very important for the development of our theory, since with them one can see that it is possible to carry the Henstock–Kurzweil–Stieltjes integral of a function f to its time scale version and also the reciprocal. These results will be essential to us, when proving the correspondence between our main problem and its analogue in the theory of time scales.

Theorem 1.3.11 ([25, Theorem 4.2])**.** *Let $[\alpha,\beta]_{\mathbb{T}}$ be a time scale interval and $f\colon [\alpha,\beta]_{\mathbb{T}} \to \mathbb{R}^n$ be an arbitrary function. Define $g(\tau) = \tau^*$ for every $\tau \in [\alpha,\beta]$. Then the Henstock–Kurzweil Δ–integral $\int_\alpha^\beta f(s)\Delta s$ exists, if and only if, the Henstock–Kurzweil–Stieltjes integral $\int_\alpha^\beta f^*(s)\,\mathrm{d}g(s)$ exists; in this case, both integrals have the same value.*

Lemma 1.3.12 ([25, Lemma 4.4])**.** *Let $\mathbb{T}$ be a time scale such that $\alpha,\beta \in \mathbb{T}$ and $g(t) = t^*$ for every $t \in [\alpha,\beta]$. If $f\colon [\alpha,\beta] \to \mathbb{R}^n$ is such that the Henstock–Kurzweil–Stieltjes integral $\int_c^d f(t)\mathrm{d}g(t)$ exists for every $c,d \in [\alpha,\beta]$, then*

$$\int_c^d f(t)\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\mathrm{d}g(t),$$

for every $\alpha \leqslant c < d < \beta$.

Theorem 1.3.13 ([24, Theorem 4.1])**.** *Let $f\colon \mathbb{T} \to \mathbb{R}^n$ be a function such that the Henstock–Kurzweil Δ–integral $\int_\alpha^\beta f(s)\Delta s$ exists for every $\alpha,\beta \in \mathbb{T}$, $\alpha < \beta$. Choose an arbitrary $\gamma \in \mathbb{T}$ and define*

$$F_1(t) = \int_\gamma^t f(s)\Delta s, \quad t \in \mathbb{T},$$

$$F_2(t) = \int_\gamma^t f^*(s)\,\mathrm{d}g(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$.

Theorem 1.3.14 ([24, Theorem 4.2])**.** *Let $\mathbb{T}$ be a time scale, $g(s) = s^*$ for every $s \in \mathbb{T}^*$, $[\alpha,\beta] \subset \mathbb{T}^*$. Consider a pair of functions $f_1, f_2\colon [\alpha,\beta] \to \mathbb{R}^n$ such that $f_1(s) = f_2(s)$ for every $s \in [\alpha,\beta]_{\mathbb{T}}$. If $\int_\alpha^\beta f_1(s)\,\mathrm{d}g(s)$ exists, then $\int_\alpha^\beta f_2(s)\,\mathrm{d}g(s)$ exists as well and both integrals have the same value.*

Chapter 2

Correspondences among equations

This chapter provides an important motivation to consider Volterra–Stieltjes types of equations, since we will show here that many other types of equations can be regarded as a special case of them.

Before presenting the correspondence, we will present the classes of equations that will be explored in this chapter.

The first type of equation that we will present here is the most important one, that is, *the functional Volterra–Stieltjes integral equations.* These equations play an important role here and they will be the main object of study in this thesis:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{2.0.1}$$

where $0 \leqslant t_0 \leqslant \tau_0 < d$, $r > 0$, $\phi \in G([-r,0],\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$, $a\colon [t_0,d)^2 \to \mathbb{R}$ and $g\colon [t_0,d) \to \mathbb{R}$ is a nondecreasing and left–continuous function, where $t_0 < d \leqslant +\infty$, $x_s\colon [-r,0] \to \mathbb{R}^n$ is given by $x_s(\theta) = x(s+\theta)$ for $s \in [t_0,d)$. Here, $[t_0,d)^2$ denotes the set $[t_0,d) \times [t_0,d)$.

As we already described in the introduction, there are several equations that can be viewed as a particular case of (2.0.1) such as fractional differential equations, functional differential equations, ordinary differential equations, depending on the definitions of a, f and g. Also, as we mentioned int the introduction, there are other types of equations that can also be described as a particular case of these equations, but for that, it is necessary to present a more sophisticated correspondence which is not only given by a very simple definition of a, f and g but requires more elements to describe these correspondences in a precise way. The equations that can be corresponded in this form are the impulsive functional Volterra–Stieltjes integral equations and functional Volterra delta integral equations on time scales.

Let us describe these two equations in the sequel.

We start by introducing the impulsive functional Volterra–Stieltjes integral equations:

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\} \\ t_0 < t_k < t}} a(t,t_k) I_k(x(t_k)) \\ x_{t_0} &= \phi, \end{cases} \tag{2.0.2}$$

where $0 \leqslant t_0 \leqslant \tau_0 < d$, $r > 0$, $\phi \in G([-r,0],\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$, $a\colon [t_0,d)^2 \to \mathbb{R}$ and $g\colon [t_0,d) \to \mathbb{R}$ is a nondecreasing function, where $t_0 < d \leqslant +\infty$, $x_s\colon [-r,0] \to \mathbb{R}^n$ is given by $x_s(\theta) = x(s+\theta)$ for $s \in [t_0,d)$. Here, $[t_0,d)^2$ denotes the set $[t_0,d) \times [t_0,d)$ and $I_k\colon \mathbb{R}^n \to \mathbb{R}^n, k \in \mathbb{N}$, is the impulse operator and $\{t_k\}_{k=1}^m$ are the moments of impulse.

The other equation that will be investigated here is the functional Volterra Δ-integral equation on time scales given by

$$\begin{cases} x(t) &= x(t_0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s^*, s)\Delta s, \quad t \in [t_0,d)_{\mathbb{T}}, \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{2.0.3}$$

where $0 \leqslant t_0 \leqslant \tau_0 < d \leqslant \infty$, $r > 0$, $\phi \in G([-r,0]_{\mathbb{T}},\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d)_{\mathbb{T}} \to \mathbb{R}^n$, $a\colon [t_0,d)_{\mathbb{T}}^2 \to \mathbb{R}$, $x_s\colon [-r,0] \to \mathbb{R}^n$ is given by $x_s(\theta) = x(s+\theta)$ for $s \in [t_0,d)$. Moreover, $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 - r, t_0 \in \mathbb{T}$.

Our goal in this chapter is to obtain correspondences between the solutions of these three types of equations. To achieve this, we will divide this chapter into two sections. In the first one, we will describe the correspondence between the solutions of (2.0.1) and the solutions of (2.0.2) and in the second one, we will describe the correspondence between the solutions of (2.0.1) and the solutions of (2.0.3).

This correspondence will be of great use in the following chapters, since it will allow us to encompass these three types of equations in our results, although proving them just once, for the functional Volterra–Stieltjes integral equations.

We begin by giving below the definition of a solution of equation (2.0.1).

Definition 2.0.1. A function $x\colon [\tau_0 - r, \gamma] \to \mathbb{R}^n$, $\tau_0 < \gamma < d$, is called a *solution* of the equation (2.0.1) on $[\tau_0 - r, \gamma]$ if the following conditions are satisfied:

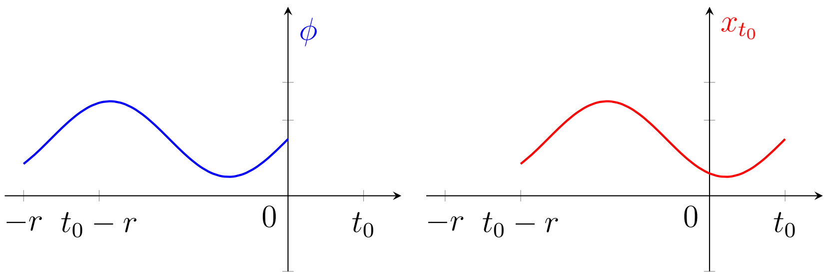
(i) For every $\tau_0 \leqslant t \leqslant \gamma$, the equality

$$x(t) = \phi(0) + \int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s)$$

holds.

(ii) $x(\tau_0 + \theta) = \phi(\theta)$ for all $\theta \in [-r, 0]$, i.e. $x_{\tau_0} = \phi$.

Remark 2.0.2. At this point, it is important to remark that our initial condition $x_{\tau_0} = \phi$ yields that $x(s) = \phi(s - \tau_0)$ for all $s \in [\tau_0 - r, \tau_0]$. Indeed, the equation $x_{\tau_0} = \phi$ is equivalent to $x(\tau_0 + s) = \phi(s)$ for all $s \in [-r, 0]$. Thus, in our problem, the initial condition gives us the behaviour of the solution not only in a single point, but in all the interval $[\tau_0 - r, \tau_0]$.



Remark 2.0.3. We define the norm $\|\cdot\|_\infty$ of a bounded function $x\colon X \to \mathbb{R}^n$, where X is a Banach space as $\|x\|_\infty := \sup\{\|x(a)\|\colon a \in X\}$. When restricting the function x to a subset A of X, we will write $\|x\|_{\infty,A} := \sup\{x(a) : a \in A\}$.

For more details about functional differential equations, please consult [37, 38].

We emphasize here that the initial condition of our problem describes the behaviour of the solution in a interval. Saying that $x_{\tau_0} = \phi$ means that the function x behaves in $[\tau_0 - r, \tau_0]$ as the function ϕ behaves in $[-r, 0]$.

Definition 2.0.4. A function $x\colon [\tau_0 - r, \beta) \to \mathbb{R}^n$, $\tau_0 < \beta \leqslant d$, is called a *solution* of the equation (2.0.1) on $[\tau_0 - r, \beta)$ if, for each $\tau_0 < \alpha < \beta$, the restriction of x to $[\tau_0 - r, \alpha]$ is a solution of the equation (2.0.1).

The results obtained in this chapter are new and are contained in [32]. They will be very useful throughout this work. Using the correspondences here described, we will be able to "transport" the results obtained for functional Volterra–Stieltjes integral equations to the impulsive and time scales cases.

2.1 Impulsive functional Volterra–Stieltjes integral equations

In this section, our goal is to investigate a class of equations called *impulsive functional Volterra–Stieltjes integral equations* and to show that this class of equations represents a special case of the functional Volterra–Stieltjes integral equations. In order to establish this fact, we follow some ideas from [6, 25].

Our attention will be focused on the case of pre–assigned moments of impulses. The case of state-dependent impulses is more complicated and will not be treated here. However, as far as we know, it is still an open question how to relate these equations with other types of equations and it is a very interesting question to answer.

Thus, let us assume that $\{t_k\}_{k=1}^m$ are moments of impulses and each $t_k \in [t_0, d)$, for $d \leqslant \infty$. Suppose also that the condition $\Delta^+ x(t_k) = I_k(x(t_k))$, where $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$ is the impulse operator, is satisfied for each $k = 1, \ldots, m$. Therefore, consider the following equation

$$\begin{aligned}
x(v) - x(u) &= \int_{t_0}^{v} a(v,s) f(x_s, s)\,\mathrm{d}g(s) - \int_{t_0}^{u} a(u,s) f(x_s, s)\,\mathrm{d}g(s), \quad \text{for } u, v \in J_k,\ k \in \mathbb{N},\\
\Delta^+ x(t_k) &= I_k(x(t_k)), \quad k = 1, \ldots, m,\\
x_{t_0} &= \phi,
\end{aligned}$$

where $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, \ldots, m$, and $J_m = (t_m, d)$. Notice that the value of both integrals

$$\int_{t_0}^{v} a(v,s) f(x_s, s)\,\mathrm{d}g(s) \quad \text{and} \quad \int_{t_0}^{u} a(u,s) f(x_s, s)\,\mathrm{d}g(s),$$

where $u, v \in J_k$, do not change if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k (see [25]). Moreover, assume the following conditions:

(A1) The function $g\colon [t_0, d) \to \mathbb{R}$ is nondecreasing and left–continuous on (t_0, d).

(A2) The function $a\colon [t_0, d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable and regulated with respect to the second variable.

(A3) The Henstock–Kurzweil–Stieltjes integral $\int_{\tau_1}^{\tau_2} a(t,s)f(x_s,s)\mathrm{d}g(s)$ exists for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, all $x \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$, $t \in [t_0, d)$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

(A4) There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M\colon [t_0, d) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} (c_2 a(\tau_2, s) + c_1 a(\tau_1, s))\, f(x_s, s)\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)|\, M(s)\mathrm{d}g(s),$$

for all $x \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$, all $c_1, c_2 \in \mathbb{R}$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

(A5) There exists a locally regulated function $L\colon [t_0, d) \to \mathbb{R}^+$ such that for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)]\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, L(s)\, \|x_s - z_s\|_\infty\, \mathrm{d}g(s),$$

for all $x, z \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$, and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

Assume also that a is continuous with respect to the first variable at $\{t_k\}_{k=1}^m$ and that g is continuous on the moments of impulse t_k for $k = 1, 2, \ldots, m$. Under these assumptions, our problem can be rewritten as

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s)f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\} \\ t_0 < t_k < t}} a(t, t_k) I_k(x(t_k)) \\ x_{t_0} &= \phi. \end{cases} \tag{2.1.1}$$

Since g is continuous at t_k for each $k = 1, \ldots, m$ and a is continuous with respect to the first variable at $t_k, k = 1, \ldots, m$, by the same arguments that will be used in the proof of Lemma 3.2.5, we obtain that the function

$$t \mapsto \int_{t_0}^{t} a(t,s)f(x_s,s)\,\mathrm{d}g(s)$$

is continuous at $t_1, \ldots, t_m$ and, therefore, $\Delta^+ x(t_k) = I_k(x(t_k))$ for every $k \in \{1, \ldots, m\}$. It will be very important to obtain the analogue results for impulsive functional Volterra–Stieltjes integral equations just knowing the results for functional Volterra–Stieltjes integral equations.

In the next result, we describe how we can transfer the conditions on impulsive functional Volterra–Stieltjes integral equation to the conditions on functional Volterra–Stieltjes integral equation.

Lemma 2.1.1. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < d$, $I_1, \ldots, I_m \colon \mathbb{R}^n \to \mathbb{R}^n$, $a \colon [t_0, d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and locally bounded on $[t_0, d)^2$. Assume that $g \colon [t_0, d) \to \mathbb{R}$ is a left–continuous and nondecreasing function. Let $f \colon G([-r, 0], \mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$ be an arbitrary function. Define $\tilde{f} \colon G([-r, 0], \mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$ by*

$$\tilde{f}(y, \tau) = \begin{cases} f(y, \tau), & \tau \in [t_0, d) \backslash \{t_1, \ldots, t_m\}, \\ I_k(y(0)), & \tau = t_k, \ k \in \{1, \ldots, m\}, \end{cases}$$

and define $\tilde{g} \colon [t_0, d) \to \mathbb{R}$ by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [t_0, t_1], \\ g(\tau) + k, & \tau \in (t_k, t_{k+1}], \ k \in \{1, \ldots, m-1\}, \\ g(\tau) + m, & \tau \in (t_m, d). \end{cases}$$

Then the following statements hold.

1. *The function $\tilde{g}$ is nondecreasing and left–continuous.*

2. *If the Henstock–Kurzweil–Stieltjes integral*

$$\int_{u_1}^{u_2} a(t, s) f(x_s, s) \mathrm{d}g(s)$$

exists for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $t \in [t_0, d)$ and $\tau_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$. Then the Henstock–Kurzweil–Stieltjes integral

$$\int_{u_1}^{u_2} a(t, s) \tilde{f}(x_s, s) \mathrm{d}\tilde{g}(s)$$

exists for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $t \in [t_0, d)$ and $\tau_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$.

3. *If there exists a locally Henstock–Kurzweil–Stieltjes integrable function* $M_1\colon [t_0, d) \to \mathbb{R}^+$ *with respect to* g *such that for each compact interval* $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, *we have*
$$\left\| \int_{u_1}^{u_2} b(u_2, s) f(x_s, s) \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} M_1(s) \left| b(u_2, s) \right| \mathrm{d}g(s),$$
for all $x \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$, $b \in G_2([\tau_0, \tau_0+\sigma]^2, \mathbb{R})$ *and* $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$, *and there exists a constant* $M_2 > 0$ *such that*
$$\|I_k(x)\| \leqslant M_2$$
for every $k \in \{1, \dots, m\}$ *and* $x \in \mathbb{R}^n$. *Then there exists a locally Henstock–Kurzweil–Stieltjes integrable function* $M\colon [t_0, d) \to \mathbb{R}^+$ *with respect to* $\tilde{g}$ *such that for each compact interval* $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, *we have*
$$\left\| \int_{u_1}^{u_2} b(u_2, s) \tilde{f}(x_s, s) \mathrm{d}\tilde{g}(s) \right\| \leqslant \int_{u_1}^{u_2} M(s) \left| b(u_2, s) \right| \mathrm{d}\tilde{g}(s),$$
for all $x \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$, $b \in G_2([\tau_0, \tau_0+\sigma]^2, \mathbb{R})$ *and* $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$.

4. *If there exists a regulated function* $L_1\colon [t_0, d) \to \mathbb{R}^+$ *such that*
$$\left\| \int_{u_1}^{u_2} a(u_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_1(s) \left| a(u_2, s) \right| \|x_s - z_s\|_\infty \mathrm{d}g(s),$$
for all $x, z \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$ *and* $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$ *and there exists a constant* $L_2 > 0$ *such that*
$$\|I_k(x) - I_k(z)\| \leqslant L_2 \|x - z\|$$
for every $k \in \{1, \dots, m\}$ *and* x, $z \in \mathbb{R}^n$. *Then there exists a locally regulated function* $L\colon [t_0, d) \to \mathbb{R}^+$ *such that for each compact interval* $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, *we have*
$$\left| \int_{u_1}^{u_2} a(u_2, s)[\tilde{f}(x_s, s) - \tilde{f}(z_s, s)] \mathrm{d}\tilde{g}(s) \right| \leqslant \int_{u_1}^{u_2} L(s) \left| a(u_2, s) \right| \|x_s - z_s\|_\infty \mathrm{d}\tilde{g}(s),$$
for all $x, z \in G([\tau_0 - r, \tau_0+\sigma], \mathbb{R}^n)$ *and* $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$.

Proof. The first statement is an immediate consequence from the definition of g. Also,
$$\tilde{g}(v) - \tilde{g}(u) \geqslant g(v) - g(u) \tag{2.1.2}$$

whenever $t_0 \leqslant u \leqslant v < d$. Notice that the second statement follows by combining item 1 and the hypotheses from $\tilde{f}$ and a together with Lemma 1.2.8.

In order to prove the third statement, let $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, $x \in G([\tau_0-r, \tau_0+\sigma], \mathbb{R}^n)$, $b \in G_2([\tau_0, \tau_0+\sigma]^2, \mathbb{R})$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0+\sigma$. From Lemma 1.2.8, we obtain

$$\begin{aligned}\int_{u_1}^{u_2} b(u_2, s)\tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s) &= \int_{u_1}^{u_2} b(u_2, s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} b(u_2, t_k)\tilde{f}(x_{t_k}, t_k)\Delta^+\tilde{g}(t_k)\\ &= \int_{u_1}^{u_2} b(u_2, s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} b(u_2, t_k) I_k(x(t_k))\Delta^+\tilde{g}(t_k)\end{aligned}$$

and, therefore,

$$\left\|\int_{u_1}^{u_2} b(u_2, s)\tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s)\right\| \leqslant \int_{u_1}^{u_2} M_1(s)\,|b(u_2, s)|\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b(u_2, t_k)|\,\Delta^+\tilde{g}(t_k). \tag{2.1.3}$$

Using (2.1.2) and the definition of the Henstock–Kurzweil–Stieltjes integral, we have

$$\int_{u_1}^{u_2} M_1(s)\,|b(u_2, s)|\,\mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} M_1(s)\,|b(u_2, s)|\,\mathrm{d}\tilde{g}(s) \leqslant \int_{u_1}^{u_2} M(s)\,|b(u_2, s)|\,\mathrm{d}\tilde{g}(s),$$

where $M(s) := 1 + M_2 + M_1(s)$ for all $s \in [t_0, t_0+\sigma]$. In particular

$$\int_{u_1}^{u_2} M_1(s)\,|b(u_2, s)|\,\mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} M(s)\,|b(u_2, s)|\,\mathrm{d}\tilde{g}(s). \tag{2.1.4}$$

On the other hand, we observe that the function

$$h(t) := \int_{t_0}^{t} M(s)\,|b(u_2, s)|\,\mathrm{d}\tilde{g}(s), \quad t \in [\tau_0, \tau_0+\sigma],$$

is nondecreasing and $\Delta^+ h(t_k) = M(t_k)\,|b(u_2, t_k)|\,\Delta^+\tilde{g}(t_k)$ for $k \in \{1, \dots, m\}$. Hence

$$\begin{aligned}&\sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M_2\,|b(u_2, t_k)|\,\Delta^+\tilde{g}(t_k) \leqslant \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} M(t_k)\,|b(u_2, t_k)|\,\Delta^+\tilde{g}(t_k)\\ &= \sum_{\substack{k\in\{1,\dots,m\},\\ u_1\leqslant t_k<u_2}} \Delta^+ h(t_k) \leqslant h(u_2) - h(u_1) = \int_{u_1}^{u_2} M(s)\,|b(u_2, s)|\,\mathrm{d}\tilde{g}(s).\end{aligned} \tag{2.1.5}$$

Now, by (2.1.3), (2.1.4) and (2.1.5), we get

$$\left\|\int_{u_1}^{u_2} b(u_2, s)\tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s)\right\| \leqslant 2\int_{u_1}^{u_2} M(s)\,|b(u_2, s)|\,\mathrm{d}\tilde{g}(s), \tag{2.1.6}$$

proving Condition 3.

To prove the fourth statement, let $t_0 \leqslant u_1 \leqslant u_2 \leqslant t_0 + \sigma$. Using Lemma 1.2.8 again, we obtain

$$\begin{aligned}
&\left\| \int_{u_1}^{u_2} a(u_2, s) \left(\tilde{f}(x_s, s) - \tilde{f}(z_s, s) \right) \mathrm{d}\tilde{g}(s) \right\| \\
&\leqslant \int\limits_{u_1}^{u_2} L_1(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \ldots, m\}, \\ u_1 \leqslant t_k < u_2}} L_2 \left| a(u_2, t_k) \right| \| x(t_k) - z(t_k) \| \, \Delta^+ \tilde{g}(t_k).
\end{aligned} \tag{2.1.7}$$

Using (2.1.2) and the definition of the Henstock–Kurzweil–Stieltjes integral, we see that

$$\begin{aligned}
\int\limits_{u_1}^{u_2} L_1(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \mathrm{d}g(s) &\leqslant \int\limits_{u_1}^{u_2} L_1(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \mathrm{d}\tilde{g}(s) \\
&\leqslant \int\limits_{u_1}^{u_2} L(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \mathrm{d}\tilde{g}(s),
\end{aligned} \tag{2.1.8}$$

where $L(s) := 1 + L_2 + L_1(s)$ for all $s \in [t_0, t_0 + \sigma]$.

Next, we observe that the function

$$\gamma(t) := \int_{t_0}^{t} L(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \, \mathrm{d}\tilde{g}(s) \quad t \in [\tau_0, \tau_0 + \sigma],$$

is nondecreasing and $\Delta^+ \gamma(t_k) = L(t_k) \left| a(u_2, t_k) \right| \| x_{t_k} - z_{t_k} \|_\infty \Delta^+ \tilde{g}(t_k)$,for $k \in \{1, \ldots, m\}$. Therefore

$$\begin{aligned}
\sum_{\substack{k \in \{1, \ldots, m\}, \\ u_1 \leqslant t_k < u_2}} L_2 \left| a(u_2, t_k) \right| \| x_{t_k} - z_{t_k} \|_\infty \Delta^+ \tilde{g}(t_k) \quad &\leqslant \quad \sum_{\substack{k \in \{1, \ldots, m\}, \\ u_1 \leqslant t_k < u_2}} \Delta^+ \gamma(t_k) \\
&\leqslant \quad \gamma(u_2) - \gamma(u_1) \\
&= \quad \int_{u_1}^{u_2} L(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \, \mathrm{d}\tilde{g}(s)
\end{aligned} \tag{2.1.9}$$

and it follows from (2.1.7) and (2.1.9) that

$$\left\| \int_{u_1}^{u_2} a(u_2, s) \left(\tilde{f}(x_s, s) - \tilde{f}(z_s, s) \right) \mathrm{d}\tilde{g}(s) \right\| \leqslant 2 \int_{u_1}^{u_2} L(s) \left| a(u_2, s) \right| \| x_s - z_s \|_\infty \, \mathrm{d}\tilde{g}(s),$$

proving the result. $\square$

The following theorem describes a strong relation existent between impulsive Volterra–Stieltjes integral equations and Volterra–Stieltjes integral equations without impulses. It follows some ideas from [6].

Theorem 2.1.2. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < d$, $I_1, \ldots, I_m \colon \mathbb{R}^n \to \mathbb{R}^n$ and $f \colon G([-r,0],\mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$. Assume that $g \colon [t_0, d) \to \mathbb{R}$ is a regulated and left–continuous function which is continuous at $t_1, \ldots, t_m$, $a \colon [t_0, d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable, locally bounded on $[t_0, d)^2$ and, continuous with respect to first variable at $t_1, \ldots, t_m$. Define $\tilde{f} \colon G([-r,0],\mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$ by*

$$\tilde{f}(y,\tau) = \begin{cases} f(y,\tau), & \tau \in [t_0,d)\backslash\{t_1,\ldots,t_m\}, \\ I_k(y(0)), & \tau = t_k, \ k \in \{1,\ldots,m\}, \end{cases}$$

and $\tilde{g} \colon [t_0, d) \to \mathbb{R}$ by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [t_0, t_1], \\ g(\tau) + k, & \tau \in (t_k, t_{k+1}], \ k \in \{1,\ldots,m-1\}, \\ g(\tau) + m, & \tau \in (t_m, d). \end{cases}$$

Then $x \colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, $[t_0, t_0+\sigma] \subset [t_0, d)$, is a solution of

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\}, \\ t_k < t}} a(t,t_k) I_k(x(t_k)) \\ x_{t_0} &= \phi \end{cases} \tag{2.1.10}$$

if, and only if, $x \colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a solution of

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) \tilde{f}(x_s, s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} &= \phi. \end{cases} \tag{2.1.11}$$

Proof. It is clear from the definition of $\tilde{g}$ that $\Delta^+\tilde{g}(t_k) = 1$ for every $k \in \{1,\ldots,m\}$. Also by hypotheses, the integral $\int_{t_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s)$ exists for all $t \in [t_0, t_0+\sigma]$ and for every $x \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$. This implies by the definition of $\tilde{f}$ and $\tilde{g}$, that the functions $\tilde{f}$ and $\tilde{g}$ inherit the conditions from f and g (see Lemma 2.1.1). Therefore, the integral

$\int_{t_0}^t a(t,s)\tilde{f}(x_s,s)\,\mathrm{d}\tilde{g}(s)$ exists. According to Lemma 1.2.8 and by the definition of $a, \tilde{f}$ and $\tilde{g}$, we have

$$\begin{aligned}\int_{t_0}^t a(t,s)\tilde{f}(x_s,s)\,\mathrm{d}\tilde{g}(s) &= \int_{t_0}^t a(t,s)f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} a(t,t_k)\tilde{f}(x_{t_k},t_k)\Delta^+\tilde{g}(t_k)\\ &= \int_{t_0}^t a(t,s)f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} a(t,t_k)I_k(x(t_k)),\end{aligned}$$

i.e. the right–hand sides of (2.1.11) and (2.1.10) are indeed identical. □

We will present some examples to illustrate the impulsive functional Volterra–Stieltjes integral equations. They are inspired in the ones found in [35].

Example 2.1.3. Let M be a constant and $y, z \in G([-r,0],\mathbb{R}^n)$. Then the IVP:

$$\begin{cases} x' + Mx = y_t, \quad t \neq t_k, k = 1,\dots,m\\ \Delta^+ x|_{t=t_k} = I_k(z(t_k)), \quad k = 1,\dots,m\\ x_0 = \phi \end{cases} \tag{2.1.12}$$

has a unique solution given by

$$x(t) = \phi(0)e^{-Mt} + \int_0^t e^{-M(t-s)}y_s + \sum_{0<t_k<t} e^{-M(t-t_k)}I_k(z(t_k)). \tag{2.1.13}$$

Indeed, suppose that (2.1.13) is satisfied. Then, clearly $x_0 = \phi$ and

$$\Delta^+ x(t_k) = x(t_k^+) - x(t_k) = e^{-M(t_k-t_k)}I_k(z(t_k)) = I_k(z(t_k)).$$

Also, by (2.1.13), we have:

$$x(t)e^{Mt} = \phi(0) + \int_0^t e^{Ms}y_s\mathrm{d}s + \sum_{0<t_k<t} e^{Mt_k}I_k(z(t_k)).$$

Therefore, differentiating, we get for $t \neq t_k$ that $(x'(t) + Mx(t))e^{Mt} = e^{Mt}y(t)$, which implies that x satisfies (2.1.12).

Example 2.1.4. Now, let us consider the following IVP:

$$\begin{cases} x' = f(t,z_t) - M(x-z), \quad t =\neq t_k\\ \Delta x|_{t=t_k} = I_k(z(t_k))\\ x_0 = \phi. \end{cases} \tag{2.1.14}$$

Statement: The solution of (2.1.14) can be rewritten as the following Volterra–Stieltjes functional integral equations:

$$x(t) = \phi(0)e^{-Mt} + \int_0^t e^{-M(t-s)}[f(s,z_s) + Mz(s)]\mathrm{d}s + \sum_{0\leqslant t_k<t} e^{-M(t-t_k)} I_k(z(t_k)).$$

Define $x = Bz$. Then, we get:

$$Bz(t) = \phi(0)e^{-Mt} + \int_0^t e^{-M(t-s)}[f(s,z_s) + Mz(s)]\mathrm{d}s + \sum_{0<t_l<t} e^{-M(t-t_k)} I_k(z(t_k)).$$

Using the same ideas as before, it is possible to show that x is a solution of the IVP below:

$$\begin{cases} x' = f(t, x_t), \quad t \neq t_k \\ \Delta^+ x|_{t=t_k} = I_k(x(t_k)) \\ x_0 = \phi \end{cases} \tag{2.1.15}$$

if and only if $x = Bz = z$, that is, x is a solution of

$$x(t) = \phi(0)e^{-Mt} + \int_0^t e^{-M(t-s)}[f(s,x_s) + Mx(s)]\mathrm{d}s + \sum_{0<t_k<t} e^{-M(t-t_k)} I_k(x(t_k)).$$

With this, we conclude the statement.

The fact that functional Volterra–Stieltjes integral equations encompass impulsive functional Volterra–Stieltjes integral equations implies that these first equations also encompass the impulsive fractional functional differential equations.

Indeed, consider the following fractional functional differential equation with impulses:

$$\begin{cases} {}^C\mathcal{D}_{0,t}x(t) := \ {}^C\mathcal{D}_t^{\alpha}x(t) = f(t, x_t), \quad t \in J' := J\backslash\{t_1, \ldots, t_m\}, J := [0, T] \\ \Delta^+ x(t_k) = I_k(x(t_k)), k = 1, \ldots, m \\ x|_{[-r,0]} = \phi, \end{cases} \tag{2.1.16}$$

where ${}^C\mathcal{D}_t^{\alpha}$ is the Caputo derivative, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $f \colon J \times G([-r, 0]) \to \mathbb{R}^n$, $I_k \colon \mathbb{R}^n \to \mathbb{R}^n$.

In [36], the authors showed that if $0 < \alpha < 1$, then a solution of (2.1.16) satisfies the following integral form:

$$x(t)=\begin{cases}\varphi(t), \quad t\in[-r,0]\\ \varphi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\mathrm{d}s, \quad t\in[0,t_1]\\ \varphi(0)+I_1(x(t_1^-))+)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\mathrm{d}s, \quad t\in(t_1,t_2]\\ \vdots\\ \varphi(0)+\sum_{k=1}^m I_k(x(t_k^-))+)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\mathrm{d}s, \quad t\in(t_m,T]\end{cases} \tag{2.1.17}$$

A careful examination shows that (2.1.17) is a special case of impulsive functional Volterra–Stieltjes integral equations.

On the other hand, notice that even in the case that $1<\alpha<2$, these Volterra–Stieltjes integral equations can encompass such equations. Indeed, consider the following fractional impulsive equation (without delays):

$$\begin{cases}{}^C\mathcal{D}_t^\alpha u(t)=f(t,u(t)), \quad t\in J'=J\backslash\{t_1,\dots,t_m\}\\ \Delta u(t_k)=y_k, \quad k=1,\dots,m\\ \Delta u'(t_k)=\overline{y}_k, \quad k=1,\dots,m\\ u(0)=0, \quad u'(1)=0,\end{cases} \tag{2.1.18}$$

where $1<\alpha<2$.

In [68], the authors prove that (2.1.18) is equivalent to the following integral formulation:

$$u(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))\mathrm{d}s\\ \quad -\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m\overline{y}_k\right)t, \quad t\in[0,t_1)\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))\mathrm{d}s+\overline{y}_1(t-t_1)+y_1\\ \quad -\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m\overline{y}_k\right)t, \quad t\in(t_1,t_2]\\ \vdots\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m\overline{y}_i(t-t_i)+\sum_{i=1}^k y_i\\ \quad -\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m\overline{y}_k\right)t, \quad t\in(t_k,t_{k+1}], k=1,\dots,m\end{cases}$$

Therefore, we can see that it is a type of Volterra–Stieltjes integral equation with impulses, motivating our study of these equations.

2.2 Functional Volterra delta integral equations on time scales

In this section, our goal is to investigate functional Volterra Δ–integral equations on time scales, providing results concerning existence and uniqueness of solutions and existence of maximal solutions for these equations. Also, our goal is to show a relation between the functional Volterra–Stieltjes integral equations and functional Volterra delta integral equations on time scales. Regarding the notation and previous results used in this section, the reader may want to consult Section 1.3 and the references mentioned there.

Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 - r, t_0 \in \mathbb{T}$. In this section, we consider the functional Volterra Δ–integral equation on time scales given by

$$\begin{cases} x(t) &=& x(t_0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s^*, s)\Delta s, \quad t \in [t_0, d)_{\mathbb{T}}, \\ x(t) &=& \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{2.2.1}$$

where $d \in \mathbb{T} \cup \{\infty\}$, $\phi \in G([t_0 - r, t_0]_{\mathbb{T}}, \mathbb{R}^n)$, $f \colon G([-r, 0], \mathbb{R}^n) \times [t_0, d)_{\mathbb{T}} \to \mathbb{R}^n$.

To motivate the investigations of equation (2.2.1), let us consider a simple model from economics in the time scale setting that is known as *Keynesian–Cross model with "lagged" income*. See [67] for more details.

Let us consider a simple closed economy. Also, consider the following notation:

D: aggregate demand;

y: aggregate income;

C: aggregate consumption;

I: aggregate investment;

G: government spending.

With these variables in hands, consider the following model given by the equations below:

$$D(t) = C(t) + I + G \tag{2.2.2}$$

$$C(t) = C_0 + cy(t) \tag{2.2.3}$$

$$y^{\Delta} = \delta[D^{\sigma} - y], \quad t \geqslant t_0, \tag{2.2.4}$$

where $\delta < 1$ is a positive constant interpreted as the "speed of adjustment term", C_0 and

C are non-negative constants and $t_0 \geqslant 0$. Let us assume for simplicity that G and I are constants and the current consumption depends on current income.

This model is a generalization for time scales of the classical Keynesian–Cross model for the discrete case.

Combining (2.2.2) and (2.2.3) in (2.2.4), we get:

$$y^{\Delta} = \delta[C_0 + cy^{\sigma} + I + G - y] := h(t, y, y^{\sigma}).$$

Using the simple useful formula $y^{\sigma} = y + \mu y^{\Delta}$ and assuming $1 - \delta c\mu(t) \neq 0$, we get

$$y^{\Delta} = \delta[C_0 + c(y + \mu y^{\Delta}) + I + G - y]$$

$$\Longrightarrow y^{\Delta} - \delta c\mu y^{\Delta} = \delta C_0 + cy\delta + I\delta + G\delta - y\delta$$

$$\Longrightarrow y^{\Delta}(1 - \delta c\mu) = \delta C_0 + cy\delta + I\delta + G\delta - y\delta.$$

Therefore,

$$y^{\Delta} = \frac{\delta y(c-1)}{1 - \delta c\mu} + \frac{\delta(C_0 + I + G)}{1 - \delta c\mu}.$$

It can be rewritten as follows:

$$y^{\Delta} = f(t)y + g(t).$$

Using the Variation Constant Formula, we get:

$$y(t) = e_f(t, a)\left[y(t_0) + \int_{t_0}^{t} \frac{g(s)}{e_f(\sigma(s), t_0)}\Delta s\right], \quad t \geqslant t_0 \tag{2.2.5}$$

whenever f is a regressive function. Therefore, a careful examination shows that (2.2.5) is a type of Volterra delta integral of the form:

$$y(t) = h(t) + \int_{t_0}^{t} a(t, s)g(s)\Delta s,$$

showing the generality of this kind of equation, as well as motivating its study, since it is possible to use them to investigate many types of problems, including important models.

In the next result, we establish a relationship between the solutions of the functional Volterra Δ–integral equation on time scales and the solutions of functional Volterra–Stieltjes integral equation. We follow some ideas from [24].

Theorem 2.2.1. *Let $[t_0-r,t_0+\eta]_{\mathbb{T}}$ be a time scale interval, $t_0 \in \mathbb{T}$, $[t_0,t_0+\eta]_{\mathbb{T}} \subset [t_0,d)_{\mathbb{T}}$, $a\colon [t_0,d)^2_{\mathbb{T}} \to \mathbb{R}$ and $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,d)_{\mathbb{T}} \to \mathbb{R}^n$. Define $g\colon [t_0,d) \to \mathbb{R}$ by $g(s)=s^*$, for every $s \in [t_0,d)$. If $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional Volterra Δ–integral equation on time scales*

$$\begin{cases} x(t) &= x(t_0)+\displaystyle\int_{t_0}^{t} a(t,s)f(x^*_s,s)\Delta s, \quad t\in[t_0,t_0+\eta]_{\mathbb{T}},\\ x(t) &= \phi(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}}, \end{cases} \tag{2.2.6}$$

Then $x^*\colon [t_0-r,t_0+\eta] \to \mathbb{R}^n$ is a solution of the functional Volterra–Stieltjes integral equation

$$\begin{cases} x(t) &= x(t_0)+\displaystyle\int_{t_0}^{t} a^{**}(t,s)f^*(x_s,s)\,\mathrm{d}g(s)\\ x_{t_0} &= \phi^*_{t_0}. \end{cases} \tag{2.2.7}$$

Conversely, if $y\colon [t_0-r,t_0+\eta] \to \mathbb{R}^n$ satisfies the equation (2.2.7), then it must have the form $y=x^*$, where $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the equation (2.2.6).

Proof. Suppose that $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies equation (2.2.6). Then

$$x(t) \quad = \quad x(t_0)+\int_{t_0}^{t} a(t,s)f(x^*_s,s)\Delta s, \quad t\in[t_0,t_0+\eta]_{\mathbb{T}}.$$

Proceeding as in [25], we have

$$x^*(t) \quad = \quad x^*(t_0)+\int_{t_0}^{t} a^{**}(t,s)f^*(x^*_s,s)\mathrm{d}g(s),$$

which proves the first part.

Conversely, assume that y satisfies (2.2.7). If $t\in[t_0,t_0+\eta]\backslash\mathbb{T}$, then g is constant on $[t,t^*]$ and, therefore, $y(t)=y(t^*)$. It follows that $y=x^*$, where $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}} \to \mathbb{R}^n$ is the restriction of y to $[t_0-r,t_0+\eta]_{\mathbb{T}}$. By reversing our previous reasoning, we conclude that x is a solution of (2.2.6). ☐

From now on, we will assume the following conditions concerning the functions $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,d)_{\mathbb{T}} \to \mathbb{R}^n$ and $a\colon [t_0,d)^2_{\mathbb{T}} \to \mathbb{R}$.

(C1) The function $a\colon [t_0,d)^2_{\mathbb{T}} \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and rd–continuous with respect to the first variable.

(C2) The Henstock–Kurzweil Δ–integral

$$\int_{s_1}^{s_2} a(\tau, s) f(x_s, s)\Delta s$$

exists for each time scale interval $[s_0, s_0+\delta]_{\mathbb{T}} \subset [t_0, d)_{\mathbb{T}}$, $x \in G([s_0 - r, s_0+\delta], \mathbb{R}^n)$, $\tau \in [s_0, s_0+\delta]_{\mathbb{T}}$ and $s_1, s_2 \in [s_0, s_0+\delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(C3) There exists a locally Henstock–Kurzweil Δ–integrable function $M_1 \colon [t_0, d)_{\mathbb{T}} \to \mathbb{R}^+$ such that for each time scale interval $[s_0, s_0+\delta]_{\mathbb{T}} \subset [t_0, d)_{\mathbb{T}}$, we have

$$\left\| \int_{s_1}^{s_2} (c_1 a(s_2, s) + c_2 a(s_1, s)) f(x_s, s)\Delta s \right\| \leqslant \int_{s_1}^{s_2} M_1(s) \left| c_1 a(s_2, s) + c_2 a(s_1, s) \right| \Delta s,$$

for all $x \in G([s_0 - r, s_0+\delta], \mathbb{R}^n)$, $c_1, c_2 \in \mathbb{R}$ and $s_1, s_2 \in [s_0, s_0+\delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(C4) There exists a locally regulated function $L_1 \colon [t_0, d)_{\mathbb{T}} \to \mathbb{R}^+$ such that for each time scale interval $[s_0, s_0+\delta]_{\mathbb{T}} \subset [t_0, d)_{\mathbb{T}}$, we have

$$\left\| \int_{s_1}^{s_2} a(s_2, s)[f(x_s, s) - f(z_s, s)]\Delta s \right\| \leqslant \int_{s_1}^{s_2} L_1(s) \left| a(s_2, s) \right| \left\| x_s - z_s \right\|_\infty \Delta s,$$

for all $x, z \in G([s_0 - r, s_0+\delta], \mathbb{R}^n)$ and $s_1, s_2 \in [s_0, s_0+\delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

The next result will show how to transfer the conditions on functional Volterra Δ-integral equations to functional Volterra–Stieltjes integral equations.

Lemma 2.2.2. *Assume that $t_0, d \in \mathbb{T}$, and that d is left dense. Moreover, let $f \colon G([-r, 0], \mathbb{R}^n) \times [t_0, d)_{\mathbb{T}} \to \mathbb{R}^n$ and $a \colon [t_0, d)_{\mathbb{T}}^2 \to \mathbb{R}$ be arbitrary functions. Define the functions $g(s) := s^*$ for $s \in [t_0, d)$, $f^*(\psi, s) := f(\psi, s^*)$ for $s \in [t_0, d)$ and $\psi \in G([-r, 0], \mathbb{R}^n)$ and $a^{**}(t, s) := a(t^*, s^*)$ for $t, s \in [t_0, d)$. Then the following statements hold.*

1. *If $a \colon [t_0, d)_{\mathbb{T}}^2 \to \mathbb{R}$ satisfies condition (C1), then the function $a^{**} \colon [t_0, d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and locally bounded on $[t_0, d)^2$.*

2. *If $f \colon G([-r, 0], \mathbb{R}^n) \times [t_0, d)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies condition (C2), then the Henstock–Kurzweil–Stieltjes integral*

$$\int_{u_1}^{u_2} a^{**}(t, s) f^*(x_s, s)\mathrm{d}g(s)$$

exists for each compact interval $[\tau_0, \tau_0 + \eta] \subset [t_0, d)$, $x \in G([\tau_0 - r, \tau_0 + \eta], \mathbb{R}^n)$, $t \in [\tau_0, \tau_0 + \eta]$ *and* $\tau_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \eta$.

3. *If* $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, d)_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies condition (C3), then* $f^*\colon G([-r, 0], \mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$ *satisfies the condition*

$$\left\| \int_{u_1}^{u_2} (c_1 a^{**}(u_2, s) + c_2 a^{**}(u_1, s)) f^*(x_s, s) \mathrm{d}g(s) \right\|$$
$$\leqslant \int_{u_1}^{u_2} M_1^*(s) \left| c_1 a^{**}(u_2, s) + c_2 a^{**}(u_1, s) \right| \mathrm{d}g(s),$$

for each compact interval $[\tau_0, \tau_0 + \eta] \subset [t_0, d)$, $x \in G([\tau_0 - r, \tau_0 + \eta], \mathbb{R}^n)$, $\tau_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \eta$ *and* $c_1, c_2 \in \mathbb{R}$.

4. *If* $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, d)_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies condition (C4), then* $f^*\colon G([-r, 0], \mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$ *satisfies the condition*

$$\left\| \int_{u_1}^{u_2} a^{**}(u_2, s)[f^*(x_s, s) - f^*(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_1^*(s) \left| a^{**}(u_2, s) \right| \left\| x_s - z_s \right\|_\infty \mathrm{d}g(s),$$

for each compact interval $[\tau_0, \tau_0 + \eta] \subset [t_0, d)$, $x, z \in G([\tau_0 - r, \tau_0 + \eta], \mathbb{R}^n)$ *and* $\tau_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \eta$.

Proof. It is clear from the definition of a^{**} that it is nondecreasing with respect to the first variable if a is nondecreasing with respect to the first variable.

On the other hand, it is easy to check that a^{**} is locally bounded on $[t_0, d)^2$, since a is locally bounded on $[t_0, d)^2_{\mathbb{T}}$.

Now, we show that a^{**} is regulated with respect to the second variable. In fact, let $t \in [t_0, d)$ be arbitrary, let us show that $a^{**}(t, \cdot)$ is regulated on each compact interval $[\alpha, \beta] \subset [t_0, d)$. Indeed, let $s_0 \in (\alpha, \beta]$ and consider two cases: $s_0 \in \mathbb{T}$ and otherwise.
If $s_0 \in \mathbb{T}$ is such that it is left–dense, then

$$\mathbb{R} \ni \lim_{s \to s_0^-} a(t^*, s) = \lim_{s \to s_0^-} a(t^*, s^*) = \lim_{s \to s_0^-} a^{**}(t, s).$$

If $s_0 \in \mathbb{T}$ is such that it is left–scattered, then

$$\mathbb{R} \ni a(t^*, s_0) = \lim_{s \to s_0^-} a(t^*, s^*) = \lim_{s \to s_0^-} a^{**}(t, s).$$

Finally, if $s_0 \notin \mathbb{T}$, then

$$\mathbb{R} \ni a(t^*, s_0^*) = \lim_{s \to s_0^-} a(t^*, s^*) = \lim_{s \to s_0^-} a^{**}(t, s).$$

Hence, $\lim_{s \to s_0^-} a^{**}(t, s)$ exists for all $s_0 \in (\alpha, \beta]$. Analogously, we can prove that $\lim_{s \to s_0^+} a^{**}(t, s)$ exists for all $s_0 \in [\alpha, \beta)$, proving item 1.

To prove the second statement, consider an arbitrary compact interval $[\tau_0, \tau_0 + \eta] \subset [t_0, d)$, $x \in G([\tau_0 - r, \tau_0 + \eta], \mathbb{R}^n)$ and $t \in [\tau_0, \tau_0 + \eta]$. Let $u_1, u_2 \in [\tau_0, \tau_0 + \eta]$ be given with $u_1 \leqslant u_2$. Then $t^*, u_1^*, u_2^* \in [\tau_0^*, (\tau_0 + \eta)^*]$. Therefore, by hypothesis (C2), the integral $\int_{u_1^*}^{u_2^*} a(t^*, s) f(x_s, s) \Delta s$ exists. Then, by Theorems 1.3.11, 1.3.14 and Lemma 1.3.12, we have

$$\int_{u_1^*}^{u_2^*} a(t^*, s) f(x_s, s) \Delta s = \int_{u_1^*}^{u_2^*} a(t^*, s^*) f(x_{s^*}, s^*) \mathrm{d}g(s) = \int_{u_1^*}^{u_2^*} a^{**}(t, s) f(x_{s^*}, s^*) \mathrm{d}g(s). \tag{2.2.8}$$

Since $a^{**}(t, s) f(x_{s^*}, s^*) = a^{**}(t, s) f(x_s, s^*)$, for all $s \in [u_1^*, u_2^*]_{\mathbb{T}}$, by Theorem 1.3.14 and Lemma 1.3.12, we get

$$\int_{u_1^*}^{u_2^*} a^{**}(t, s) f(x_{s^*}, s^*) \mathrm{d}g(s) = \int_{u_1^*}^{u_2^*} a^{**}(t, s) f(x_s, s^*) \mathrm{d}g(s) = \int_{u_1}^{u_2} a^{**}(t, s) f^*(x_s, s) \mathrm{d}g(s). \tag{2.2.9}$$

Now, according to (2.2.8) and (2.2.9), we obtain that the last integral exists and, in this case,

$$\int_{u_1}^{u_2} a^{**}(t, s) f^*(x_s, s) \mathrm{d}g(s) = \int_{u_1^*}^{u_2^*} a(t^*, s) f(x_s, s) \Delta s. \tag{2.2.10}$$

Now, let us prove the third statement. Indeed, let $x \in G([\tau_0 - r, \tau_0 + \eta], \mathbb{R}^n)$, $c_1, c_2 \in \mathbb{R}$ and $u_1, u_2 \in [\tau_0, \tau_0 + \eta]$ with $u_1 \leqslant u_2$. Then, by (2.2.10), (C3), Theorems 1.3.11, 1.3.14 and Lemma 1.3.12, we have

$$\begin{aligned} \left\| \int_{u_1}^{u_2} (c_1 a^{**}(u_2, s) + c_2 a^{**}(u_1, s)) f^*(x_s, s) \mathrm{d}g(s) \right\| &= \left\| \int_{u_1^*}^{u_2^*} (c_1 a(u_2^*, s) + c_2 a(u_1^*, s)) f(x_s, s) \Delta s \right\| \\ \leqslant \int_{u_1^*}^{u_2^*} M_1(s^*) \left| c_1 a(u_2^*, s^*) + c_2 a(u_1^*, s^*) \right| \mathrm{d}g(s) &= \int_{u_1}^{u_2} M_1^*(s) \left| c_1 a^{**}(u_2, s) + c_2 a^{**}(u_1, s) \right| \mathrm{d}g(s), \end{aligned}$$

obtaining the desired result.

Finally, for the fourth statement, let $t_0 \leqslant u_1 \leqslant u_2 \leqslant t_0 + \eta$, then using (2.2.10), (C4), Theorems 1.3.11, 1.3.14 and Lemma 1.3.12, we get

$$\begin{aligned}
\left\| \int_{u_1}^{u_2} a^{**}(u_2, s)[f^*(x_s, s) - f^*(z_s, s)]\mathrm{d}g(s) \right\| &= \left\| \int_{u_1^*}^{u_2^*} a(u_2^*, s)[f(x_s, s) - f(z_s, s)]\Delta s \right\| \\
\leqslant \int_{u_1^*}^{u_2^*} L_1(s^*)\, |a(u_2^*, s^*)|\, \|x_{s^*} - z_{s^*}\|_\infty \,\mathrm{d}g(s) &= \int_{u_1}^{u_2} L_1^*(s)\, |a^{**}(u_2, s)|\, \|x_s - z_s\|_\infty \,\mathrm{d}g(s),
\end{aligned}$$

for all $x, z \in G([t_0 - r, t_0 + \eta], \mathbb{R}^n)$, proving the result. □

We want to point out that Volterra integral on time scales play an important role for applications, being used to describe several phenomena, specially when one considers the discrete time scales such as $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = h\mathbb{Z}$. Many authors have investigated this type of equations proving several properties for their solutions such as existence, uniqueness, stability, asymptotic behavior of the solutions, among others (see [3, 30, 43, 52, 57, 59, 58, 62, 63, 18] and the references therein).

However, all the results presented here for these equations are more general, since we require less regularity for the involved functions a and f in our equation. Also, in the chapter about stability, the conditions on the Lyapunov functionals are more general than the classical ones found in the literature.

This fact motivates us to consider this type of correspondence between functional Volterra–Stieltjes integral equations and functional Volterra–Stieltjes delta integral equations on time scales, since the results that are obtained from the application of this correspondence are usually more general and allow us to describe important models in a precise way.

Chapter 3

Existence, uniqueness and continuation of solutions

In this chapter, we prove results concerning existence and uniqueness of solutions for the following functional Volterra–Stieltjes integral equation:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} = \phi. \end{cases} \tag{3.0.1}$$

Also, we prove results concerning continuation of solutions of (3.0.1). These results are crucial to investigate the asymptotic behavior of the solutions, such as stability.

Note that this equation encompasses many other types of equations depending on how we choose our functions a, f and g and if we take $r = 0$ or $r > 0$, as explained in the introduction, and also in Chapter 2.

The results presented here are fundamental to prove the results of the next chapters, since we will be investigating the properties of the solutions of these equations. All the results of this chapter are new and are contained in [32].

We divide this chapter in 4 sections. In the first one, we prove the existence and uniqueness of solutions of our integral equation. In the second section, we show that our equation admits a unique maximal solution and give sufficient conditions under which the interval of existence of the solution is unbounded. In the third one, we prove the analogue results for impulsive functional Volterra–Stieltjes integral equations. In the fourth section, we prove analogue results for functional Volterra–Stieltjes Δ-integral equations on time scales.

3.1 Existence and uniqueness of solutions

In this section, our goal is to prove local existence and uniqueness of solutions of the following type of integral equation:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{3.1.1}$$

where $0 \leqslant t_0 \leqslant \tau_0 < d$, $r > 0$, $\phi \in G([-r,0],\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$, $a\colon [t_0,d)^2 \to \mathbb{R}$ and $g\colon [t_0,d) \to \mathbb{R}$ is a nondecreasing function, where $t_0 < d \leqslant +\infty$, $x_s\colon [-r,0] \to \mathbb{R}^n$ is given by $x_s(\theta) = x(s+\theta)$ for $s \in [t_0,d)$. Here, $[t_0,d)^2$ denotes the set $[t_0,d) \times [t_0,d)$.

Throughout this text, we will assume that the integral in the right–hand side exists in the sense of Henstock–Kurzweil–Stieltjes with respect to g, and thus, the integral equation given by (3.1.1) makes sense and is well–defined.

Let $t_0 \in \mathbb{R}$ and $r > 0$. Given $x \in G([t_0 - r, +\infty),\mathbb{R}^n)$ and $t \geqslant t_0$, let $x_t\colon [-r,0] \to \mathbb{R}^n$ be defined as usual by

$$x_t(\theta) := x(t+\theta),$$

for all $\theta \in [-r,0]$. See [38] for details.

The following result ensures that if $x \in G([t_0 - r, +\infty),\mathbb{R}^n)$, then $x_t \in G([-r,0],\mathbb{R}^n)$ for all $t \geqslant t_0$. This property will be very important to our purposes.

Lemma 3.1.1. *Let $x \in G([t_0 - r, +\infty),\mathbb{R}^n)$ and $t \geqslant t_0$ be given. Then $x_t \in G([-r,0],\mathbb{R}^n)$.*

Proof. Let $\tau \in (-r,0]$ be fixed. We will show that $\lim\limits_{s\to\tau-} x_t(s)$ exists. Indeed, since $t_0 - r < t + \tau$ and $x \in G([t_0 - r, +\infty),\mathbb{R}^n)$, the limit $L := \lim\limits_{\xi\to(t+\tau)-} x(\xi)$ exists. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ (we can take $-r < \tau - \delta$) such that

$$\|x(\xi) - L\| < \varepsilon, \quad \text{for all } \xi \in (t+\tau-\delta, t+\tau).$$

This implies that

$$\|x(s+t) - L\| < \varepsilon, \quad \text{for all } s \in (\tau-\delta, \tau).$$

Consequently,

$$\|x_t(s) - L\| < \varepsilon, \quad \text{for all } s \in (\tau - \delta, \tau),$$

obtaining the desired result. The existence of $\lim_{s\to\tau+} x_t(s)$ for $\tau \in [-r, 0)$ can be proved similarly. □

As a consequence, we obtain immeadiately the next result.

Corollary 3.1.2. *If $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$, then for each compact interval $[\alpha, \beta] \subset [t_0, +\infty)$, the function $s \mapsto \|x_s\|_{\infty,[\alpha,\beta]}$ is regulated on $[\alpha, \beta]$.*

Throughout this thesis, we will use the symbol $G_2([t_0, d)^2, \mathbb{R})$ to denote the set of all functions $b\colon [t_0, d)^2 \to \mathbb{R}$ that are regulated with respect to the second variable, that is, for any fixed $t \in [t_0, d)$, the function

$$b(t, \cdot)\colon s \in [t_0, d) \longmapsto b(t, s) \in \mathbb{R}$$

is regulated.

From Definitions 2.0.1 or 2.0.4, it is not possible to infer much information about the properties of the function $x\colon [\tau_0 - r, \alpha] \to \mathbb{R}^n$ which is a solution of (3.1.1). Nevertheless, we assume the following conditions for which it is possible to get more specific information about the solutions of equation (3.1.1), and it will allow us to ensure its existence and uniqueness:

(A1) The function $g\colon [t_0, d) \to \mathbb{R}$ is nondecreasing and left–continuous on (t_0, d).

(A2) The function $a\colon [t_0, d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable and regulated with respect to the second variable.

(A3) The Henstock–Kurzweil–Stieltjes integral $\int_{\tau_1}^{\tau_2} a(t, s) f(x_s, s)\mathrm{d}g(s)$ exists for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $t \in [t_0, d)$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

(A4) There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M\colon [t_0, d) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} \left(c_2 a(\tau_2, s) + c_1 a(\tau_1, s)\right) f(x_s, s)\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)|\, M(s)\mathrm{d}g(s),$$

for all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, all $c_1, c_2 \in \mathbb{R}$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

(A5) There exists a locally regulated function $L\colon [t_0, d) \to \mathbb{R}^+$ such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, L(s) \, \|x_s - z_s\|_\infty \, \mathrm{d}g(s),$$

for all $x, z \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

Remark 3.1.3. Notice that both the integrals

$$\int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, L(s) \|x_s - z_s\|_\infty \mathrm{d}g(s) \quad \text{and} \quad \int_{\tau_1}^{\tau_2} |b(\tau_2, s)| \, M(s) \mathrm{d}g(s)$$

exist. Indeed, by Corollary 3.1.2, $s \mapsto \|x_s - z_s\|_\infty$ is regulated on $[\tau_1, \tau_2]$. On the other hand, since $s \mapsto |a(\tau_2, s)|$ and $s \mapsto L(s)$ are regulated on $[\tau_1, \tau_2]$, we have that the function $s \mapsto |a(\tau_2, s)| \, L(s) \, \|x_s - z_s\|_\infty$ is regulated on $[\tau_1, \tau_2]$. Hence, by the properties of this integral, it follows that

$$\int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, L(s) \|x_s - z_s\|_\infty \mathrm{d}g(s)$$

exists. For the second integral, note that the function $[\tau_1, \tau_2] \ni s \mapsto |b(\tau_2, s)|$ is bounded and that $[\tau_1, \tau_2] \ni t \mapsto \int_{\tau_1}^{t} M(s) \mathrm{d}g(s)$ is a nondecreasing function. Then, similarly, the integral $\int_{\tau_1}^{\tau_2} |b(\tau_2, s)| \, \mathrm{d} \left(\int_{\tau_1}^{s} M(\xi) \mathrm{d}g(\xi) \right)$ exists and, therefore, by Theorem 1.2.9, the integral $\int_{\tau_1}^{\tau_2} |b(\tau_2, s)| \, M(s) \mathrm{d}g(s)$ exists.

Remark 3.1.4. We point out that in condition (A4), it is necessary to consider a general kernel $b \in G_2([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$. It is not enough to have this condition only for the kernel $a \in G_2([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$, since we will need to estimate linear combinations of the kernel a applied to different values on $[\tau_0, \tau_0 + \sigma]^2$. However, this condition is adequated and it is expected when we are dealing with Volterra–Stieltjes integral equations.

Remark 3.1.5. A first look at the conditions (A1)–(A5) seems to be very general and with no motivation behind, however the next example shows a reason to require Carathéodory-type conditions on the indefinite integral instead of simply imposing conditions to the integrand. This example can be found in [15].

Example 3.1.6. Consider the functions $\varphi, \lambda \colon [0,1] \to \mathbb{R}$ defined by:

$$\varphi(t) = \begin{cases} \dfrac{(-1)^{k+1}2^k}{k}, & \text{if } t \in [d_{k-1}, d_k), k \in \mathbb{N} \\ 0, & \text{if } t = 1, \end{cases}$$

and

$$\lambda(t) = \begin{cases} \dfrac{2^k}{k}, & \text{if } t \in [d_{k-1}, d_k), k \in \mathbb{N} \\ 0, & \text{if } t = 1, \end{cases}$$

where $d_k = 1 - \frac{1}{2^k}, k \in \mathbb{N}$. Since the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges, it follows that φ is Riemann improper integrable function over $[0,1]$.

Indeed, let us prove this fact. For it, we present the proof found in [11, Example 2.7].

Let $\sum_{k=1}^{\infty} a_k$ be any convergent series in $\mathbb{R}$ and let A be its limit. Let $c_n := 1 - \frac{1}{2^n}, n \in \mathbb{N}_0$ so that $c_0 = 0, c_1 = \frac{1}{2}, c_2 = \frac{3}{4}, c_3 = \frac{7}{8}, \ldots$. We define the function $h \colon [0,1] \to \mathbb{R}$ by:

$$h(t) = \begin{cases} 2^k a_k, & \text{if } t \in [c_{k-1},, c_k), k \in \mathbb{N} \\ 0, & \text{if } t = 1. \end{cases}$$

Let us prove that h is Riemann improper integrable function over $[0,1]$ and that $\int_0^1 h dt = A = \sum_{k=1}^{\infty} a_k$. With this in hands, we will show the previous statement.

We start by remarking that the length of the interval $[c_{k-1}, c_k]$ is $\frac{1}{2^k}$. It implies that if the integral exists, then

$$\sum_{k=1}^{\infty} 2^k a_k \frac{1}{2^k} = \sum_{k=1}^{\infty} a_k = A.$$

In order to prove the integrability of h on $[0,1]$ with integral A, we need to choose an appropriate gauge.

As explained previously, the main advantage of this integral is the fact that we can “caliber” the gauge in an appropriate way in order to avoid the discontinuities and the points that our function does not behave well. Therefore, here in this case, using this fact, we need to choose a gauge that forces the points 1 and c_k for sufficiently small $k \in \mathbb{N}$ to be tags, since these points are the ones for which the function jumps.

We start by taking $M \geqslant \sup\{|a_k| : k \in \mathbb{N}\}$ and $M \geqslant 1$. Given $\varepsilon > 0$ with $\varepsilon \leqslant 1$, let $m(\varepsilon) \in \mathbb{N}$ be such that if $m \geqslant m(\varepsilon)$, then

$$|a_m| \leqslant \varepsilon \quad \text{and} \quad \left| \sum_{k=m}^{\infty} a_k \right| \leqslant \varepsilon.$$

This fact follows from the convergence of the series. Now, define $E := \{c_k : k \in \mathbb{N}\} \cup \{1\}$ and define the gauge δ_ε on $[0,1]$ by

$$\delta_\varepsilon = \begin{cases} \dfrac{1}{2}\mathrm{dist}(t,E), \text{ for } t \in [0,1]\backslash E \\ \dfrac{\varepsilon}{4^{k+1}M}, \text{ for } t = c_k, k \in \mathbb{N} \\ \dfrac{1}{2}m(\varepsilon), \text{ for } t = 1. \end{cases}$$

Consider that $\mathcal{P} = \{(t_i, [x_{i-1}, x_i])\}_{i=1}^{n}$ is a $\delta_\varepsilon-$fine partition of $[0,1]$. We can also suppose that $c_1 = \frac{1}{2} \leqslant x_{n-1} < 1$. Therefore, it follows from these facts that the point $t = 1$ is the tag for the final subinterval $[x_{n-1}, 1]$ in $\mathcal{P}$.

Let $\mu = \inf\{k \in \mathbb{N} : x_{n-1} \leqslant c_k\}$ be such that $c_k < x_{n-1}$ for $k = 0, 1, \ldots, \mu - 1$. The fact the $\mathcal{P}$ is $\delta_\varepsilon-$fine implies that

$$1 - \frac{1}{2^{m(\varepsilon)}} = 1 - \delta_\varepsilon(1) \leqslant x_{n-1} \leqslant c_\mu = 1 - \frac{1}{2^\mu},$$

whence we have $m(\varepsilon) \leqslant \mu$.

By the properties of δ_ε, each c_k in $[0, x_{n-1}] \subset [0, c_\mu]$ is a tag for any subinterval in $\mathcal{P}$ that contains this point. Also, it is possible to assume that each such point c_k is a tag for two consecutive subintervals in $\mathcal{P}$. Therefore, we have two cases to consider.

- **Case 1:** $x_{n-1} = c_\mu$

  For each $k = 1, \ldots, \mu$, we let the contribution T_k to $S(h; \mathcal{P})$ corresponding to the subintervals $[c_{k-1}, x_r], \ldots [x_s, c_k]$. The last of these subintervals has tag at c_k, where $h(c_k) = 2^{k+1}a_{k+1}$. All the other tags $t_r, \ldots, t_{s-1}$ satisfy $h(t_i) = 2^k a_k$. It implies that:

  $$T_k = 2^k a_k(x_s - c_{k-1}) + 2^{k+1}a_{k+1}(c_k - x_s).$$

  On the other hand,

  $$x_s - c_{k-1} = (x_s - c_k) + (c_k - c_{k-1}) = (x_s - c_k) + \frac{1}{2^k}.$$

  It implies that:

  $$T_k = 2^k a_k \frac{1}{2^k} + (2^{k+1}a_{k+1} - 2^k a_k)(c_k - x_s).$$

  Therefore,

  $$|T_k - a_k| \leqslant 2^k 3M \frac{\varepsilon}{4^{k+1}M} \leqslant \frac{\varepsilon}{2^k}.$$

Considering the fact that the contribution to $S(h;\mathcal{P})$ due to $[x_{n-1},1]$ is $h(1)(1-x_{n-1})=0$, it follows that $S(h;\mathcal{P})=\sum_{k=1}^{\mu}T_k$.

Thus,

$$|S(h;\mathcal{P})-A|\leqslant\left|\sum_{k=1}^{\mu}T_k-\sum_{k=1}^{\mu}a_k\right|+\left|\sum_{k=1}^{\mu}a_k\right|\leqslant\sum_{k=1}^{\mu}|T_k-a_k|+\varepsilon\leqslant\sum_{k=1}^{\mu}\frac{\varepsilon}{2^k}+\varepsilon\leqslant 2\varepsilon.$$

This implies that if $\mathcal{P}$ is $\delta_\varepsilon-$fine and if $x_{n-1}=c_\mu=\frac{1}{2^\mu}$, then $|S(h;\mathcal{P})-A|\leqslant 2\varepsilon$, and we have the desired result for this case.

- **Case 2:** $x_{n-1}<c_\mu$

  In this case, note that the subintervals in $\mathcal{P}$ immediately preceding $[x_{n-1},1]$ have the form: $[c_{\mu-1},x_r],\dots,[x_{n-2},x_{n-1}]$ and the value of h at all of the tags for this intervals is $2^\mu a_\mu$. From this, we get that the contribution T_μ to $S(h;\mathcal{P})$ from these intervals is:

$$T_\mu=2^\mu a_\mu(x_{n-1}-c_{\mu-1}).$$

  On the other hand, we have $c_{\mu-1}<x_{n-1}<c_\mu$. This implies that $0<x_{n-1}-c_{\mu-1}<c_\mu-c_{\mu-1}=\frac{1}{2^\mu}$, so that

$$T_\mu\leqslant 2^\mu|a_\mu|\frac{1}{2^\mu}.$$

  Thus,

$$S(h;\mathcal{P})=\sum_{k=1}^{\mu-1}T_k+T_\mu+0,$$

  which implies that

$$|S(h;\mathcal{P})-A|\leqslant\left|\sum_{k=1}^{\mu-1}T_k-\sum_{k=1}^{\mu-1}a_k\right|+|T_\mu|+\left|\sum_{k=\mu}^{\infty}a_k\right|\leqslant 3\varepsilon,$$

  getting the desired result for this case.

Therefore, we have that the statement is proved.

From this fact, we get that φ is Riemann improper integrable over $[0,1]$ and

$$\int_0^1\varphi(s)\mathrm{d}s=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}.$$

On the other hand, the integral of λ over $[0,1]$ is not finite. This means that it is not Henstock–Kurzweil integrable over $[0,1]$ and hence, neither Lebesgue integrable. Since

$\lambda(t) = |\varphi(t)|$ for every $t \in [0,1]$, it implies that φ is not Lebesgue integrable, due to the fact that it is not absolutely integrable.

Now, let us define $f\colon G([-r,0],\mathbb{R}) \times [0,1] \to \mathbb{R}$ by $f(\xi, s) = \varphi(s)$ for every $(\xi, s) \in G([-r,0],\mathbb{R}) \times [0,1]$, where φ is given by

$$\varphi(t) = \begin{cases} \dfrac{(-1)^{k+1}2^k}{k}, & \text{if } t \in [d_{k-1}, d_k), k \in \mathbb{N} \\ 0, & \text{if } t = 1, \end{cases}$$

and $r > 0$. Also, suppose that $a(t,s) \equiv 1$ for all $t, s \in [0,1]$. Define $g(s) = s$. Thus, by the definition, we get that f is independent of the first variable. By the previous statement, for every $x \in G([-r,1],\mathbb{R})$, the mapping $s \mapsto f(x_s, s) = \varphi(s)$ is Henstock–Kurzweil–Stieltjes integrable with respect to g over $[0,1]$ and

$$\left|\int_0^1 f(x_s,s)\mathrm{d}s\right| = \left|\int_0^1 \varphi(s)\mathrm{d}s\right| = \left|\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\right| < \int_0^1 M(s)\mathrm{d}s = \left|\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\right| + 1, \quad (3.1.2)$$

where $M(s) = \left|\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\right| + 1$ for every $s \in [0,1]$. On the other hand, we point out that (3.1.2) does not imply that

$$|f(x_s,s)| \leqslant M(s) = \left|\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\right| + 1$$

for every $s \in [0,1]$ and $x \in G([-r,1],\mathbb{R})$. Otherwise, we would have:

$$|\varphi(s)| = |f(x_s,s)| \leqslant \left|\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\right| + 1$$

for every $s \in [0,1]$. This fact would imply that φ is a Lebesgue integrable function, which is a contradiction.

Therefore, this example motivates us to consider more general conditions to our functions such as the conditions presented by (A1)–(A5). Notice also that these types of assumptions allow that the involved functions a, f and g do not behave suitable, being appropriate to describe important phenomena in a more precise way.

When the right–hand side of (3.1.1) satisfies the above mentioned conditions, the solution $x\colon [\tau_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ of (3.1.1) is a regulated function on $[\tau_0 - r, t_0 + \sigma]$, as it will be proved in the next lemma.

Lemma 3.1.7. *Assume* $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,d)\to\mathbb{R}^n$ *satisfies conditions (A3) and (A4),* $a\colon[t_0,d)^2\to\mathbb{R}$ *satisfies condition (A2) and* $g\colon[t_0,d)\to\mathbb{R}$ *satisfies condition (A1). If* $x\colon[t_0-r,t_0+\sigma]\to\mathbb{R}^n$*,* $t_0+\sigma<d$*, is a solution of the equation* (3.1.1)*, then* x *is a regulated function on* $[t_0-r,t_0+\sigma]$.

Proof. Suppose that $x\colon[t_0-r,\alpha]\to\mathbb{R}^n$ is a solution of equation (3.1.1). Let us prove that x is regulated on $[t_0-r,\alpha]$.

Step 1. $x|_{[t_0-r,t_0]}$ is a regulated function.

Indeed, let $\tau\in(t_0-r,t_0]$. Then $\tau-t_0\in(-r,0]$. Since $\phi\in G([-r,0],\mathbb{R}^n)$, $\lim\limits_{\theta\to(\tau-t_0)-}\phi(\theta)$ exists and is given by

$$\lim_{\theta\to(\tau-t_0)-}\phi(\theta)=\lim_{\theta\to(\tau-t_0)-}\phi(\theta+t_0-t_0)=\lim_{s\to\tau-}\phi(s-t_0)=\lim_{s\to\tau-}x_{t_0}(s-t_0)=\lim_{s\to\tau-}x(s),$$

which implies that $\lim\limits_{s\to\tau-}x(s)$ exists for $\tau\in(t_0-r,t_0]$. Similarly, we can prove that $\lim\limits_{s\to\tau+}x(s)$ exists for $\tau\in[t_0-r,t_0)$.

Step 2. $x|_{[t_0,t_0+\sigma]}$ is a regulated function.

In fact, for $t_0\leqslant\tau_1\leqslant\tau_2\leqslant t_0+\sigma$, by conditions (A2), (A3), (A4), we have

$$\begin{aligned}\|x(\tau_2)-x(\tau_1)\|&=\Big\|\int_{t_0}^{\tau_2}a(\tau_2,s)f(x_s,s)\,\mathrm{d}g(s)-\int_{t_0}^{\tau_1}a(\tau_1,s)f(x_s,s)\,\mathrm{d}g(s)\Big\|\\&\leqslant\Big\|\int_{\tau_1}^{\tau_2}a(\tau_2,s)f(x_s,s)\,\mathrm{d}g(s)\Big\|+\Big\|\int_{t_0}^{\tau_1}\Big(a(\tau_2,s)-a(\tau_1,s)\Big)f(x_s,s)\,\mathrm{d}g(s)\Big\|\\&\leqslant\int_{\tau_1}^{\tau_2}|a(\tau_2,s)|\,M(s)\,\mathrm{d}g(s)+\int_{t_0}^{\tau_1}|a(\tau_2,s)-a(\tau_1,s)|\,M(s)\,\mathrm{d}g(s).\end{aligned}\tag{3.1.3}$$

By (A2), a is nondecreasing with respect to the first variable and there exists $c:=\sup\limits_{(t,s)\in[t_0,t_0+\sigma]^2}|a(t,s)|$. Thus, $|a(\tau_2,s)|\leqslant c$, for $s\in[t_0,t_0+\sigma]$. Using this fact, we have

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}|a(\tau_2,s)|\,M(s)\,\mathrm{d}g(s)+\int_{t_0}^{\tau_1}|a(\tau_2,s)-a(\tau_1,s)|\,M(s)\,\mathrm{d}g(s)\\&\leqslant\int_{\tau_1}^{\tau_2}cM(s)\,\mathrm{d}g(s)+\int_{t_0}^{\tau_1}(a(\tau_2,s)-a(\tau_1,s))M(s)\,\mathrm{d}g(s)\end{aligned}$$

$$\leqslant \int_{\tau_1}^{\tau_2} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} (a(\tau_2, s) - a(\tau_1, s))M(s)\,\mathrm{d}g(s). \tag{3.1.4}$$

Combining (3.1.3) and (3.1.4), we get

$$\|x(\tau_2) - x(\tau_1)\| \leqslant \int_{\tau_1}^{\tau_2} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} (a(\tau_2, s) - a(\tau_1, s))M(s)\,\mathrm{d}g(s). \tag{3.1.5}$$

Define $h\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ by

$$h(t) := \int_{t_0}^{t} cM(s)\,\mathrm{d}g(s) + \int_{t_0}^{t_0+\sigma} a(t, s)M(s)\,\mathrm{d}g(s), \tag{3.1.6}$$

for every $t \in [t_0, t_0 + \sigma]$. Since M is a Henstock–Kurzweil–Stieltjes integrable function with respect to g on $[t_0, t_0 + \sigma]$, $\int_{t_0}^{t} cM(s)\,\mathrm{d}g(s)$ exists for all $t \in [t_0, t_0 + \sigma]$. On the other hand, similarly as in Remark 3.1.3, we can prove that $\int_{t_0}^{t_0+\sigma} a(t, s)M(s)\,\mathrm{d}g(s)$ exists for all $t \in [t_0, t_0 + \sigma]$. Then, h is well–defined and is a nondecreasing function. Also, using (3.1.5) and (3.1.6), we have

$$\|x(\tau_2) - x(\tau_1)\| \leqslant h(\tau_2) - h(\tau_1), \tag{3.1.7}$$

for all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$. Now, by (3.1.7) and by the fact that h is a nondecreasing function, the lateral limits

$$\lim_{s\to\tau^+} x(s) \text{ for } \tau \in [t_0, t_0 + \sigma)$$

and

$$\lim_{s\to\tau^-} x(s) \text{ for } \tau \in (t_0, t_0 + \sigma]$$

exist. This implies that $x|_{[t_0,t_0+\sigma]}$ is regulated, proving the result. □

In the sequel, we recall the classic Schauder Fixed–Point Theorem, which will be important to our purposes.

Theorem 3.1.8 ([38, Lemma 2.4])**.** *(Schauder Fixed–Point Theorem) Let $(E, \|\cdot\|)$ be a normed vector space, S a nonempty convex and closed subset of E. Assume that $T\colon S \to S$ is a continuous function such that $T(S)$ is relatively compact. Then T has a fixed point in S.*

Now, we state the main theorem of this section, which gives us sufficient conditions in order to guarantee the existence and uniqueness of a local solution of (3.1.1). The proof of this result is similar to the one found in [6] with the necessary adaptations, but we will write it here for the reader's convenience. We call the reader's attention that in [6], the result was proved for the case without delays.

Theorem 3.1.9. *Assume $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,d)\to\mathbb{R}^n$ satisfies conditions (A3), (A4) and (A5), $a\colon[t_0,d)^2\to\mathbb{R}$ satisfies condition (A2) and $g\colon[t_0,d)\to\mathbb{R}$ satisfies condition (A1). Then for all $\tau_0\in[t_0,d)$ and all $\phi\in G([-r,0],\mathbb{R}^n)$, there exists a $\sigma>0$ and a unique solution $x\colon[\tau_0-r,\tau_0+\sigma]\to\mathbb{R}^n$ of the initial value problem:*

$$\begin{cases} x(t) &= \phi(0)+\displaystyle\int_{\tau_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s) \\ x_{\tau_0} &= \phi. \end{cases} \tag{3.1.8}$$

Proof. Let us start by proving the existence.

Existence. Consider the set

$$H_\phi:=\{\varphi\in G([\tau_0-r,\tau_0+\sigma],\mathbb{R}^n)\colon \varphi_{\tau_0}=\phi\}.$$

The idea now is to construct an operator $T\colon H_\phi\to H_\phi$ that satisfies all hypotheses of Schauder's Fixed Point Theorem and with it, to obtain that equation (3.1.8) possesses a solution.

Assertion 1. The set H_ϕ is nonempty.

In fact, define $\Gamma\colon[\tau_0-r,\tau_0+\sigma]\to\mathbb{R}^n$ by

$$\Gamma(t):=\begin{cases}\phi(t-\tau_0), & t\in[\tau_0-r,\tau_0] \\ \phi(0), & t\in[\tau_0,\tau_0+\sigma].\end{cases}$$

Let $\tau\in[\tau_0-r,\tau_0+\sigma)$ and consider two cases: $\tau\in[\tau_0,\tau_0+\sigma)$ and otherwise.

If $\tau\in[\tau_0,\tau_0+\sigma)$, then $\lim\limits_{s\to\tau^+}\Gamma(s)$ exists, since $\Gamma|_{[\tau_0,\tau_0+\sigma)}$ is a constant function.

If $\tau\in[\tau_0-r,\tau_0)$, then $\tau-\tau_0\in[-r,0)$. Now, since $\phi\in G([-r,0],\mathbb{R}^n)$, the limit $\lim\limits_{\eta\to(\tau-\tau_0)^+}\phi(\eta)$ exists and

$$\lim_{\eta\to(\tau-\tau_0)^+}\phi(\eta)=\lim_{\eta\to(\tau-\tau_0)^+}\phi(\eta+\tau_0-\tau_0)=\lim_{s\to\tau^+}\phi(s-\tau_0)=\lim_{s\to\tau^+}\Gamma(s),$$

proving that the last limit exists. Analogously, we can show that $\lim\limits_{s\to\tau^-}\Gamma(s)$ exists for $\tau\in(\tau_0-r,\tau_0+\sigma]$. This implies that $\Gamma\in G([\tau_0-r,\tau_0+\sigma],\mathbb{R}^n)$. On the other hand, given $\theta\in[-r,0]$, we have $\theta+\tau_0\in[\tau_0-r,\tau_0]$. Also, for $\theta\in[-r,0]$, it follows

$$\Gamma_{\tau_0}(\theta):=\Gamma(\theta+\tau_0)=\phi(\theta+\tau_0-\tau_0)=\phi(\theta),$$

which implies that $\Gamma_{\tau_0} = \phi$ and, therefore, $\Gamma \in H_\phi$, proving Assertion 1.

Now, define the operator $T\colon H_\phi \to H_\phi$ by

$$(Tx)(t) = \begin{cases} \phi(t-\tau_0), & t \in [\tau_0 - r, \tau_0] \\ \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\mathrm{d}g(s), & t \in [\tau_0, \tau_0 + \sigma]. \end{cases} \tag{3.1.9}$$

Assertion 2. T is well–defined.

In order to show it, let us prove that $T(H_\phi) \subseteq H_\phi$. Let $x \in H_\phi$ be fixed. Using the same arguments from the proof of Lemma 3.1.7, we can show that Tx is regulated on $[\tau_0 - r, \tau_0 + \sigma]$. Thus, for $\theta \in [-r, 0]$, we have:

$$(Tx)_{\tau_0}(\theta) := (Tx)(\theta + \tau_0) = \phi(\theta + \tau_0 - \tau_0) = \phi(\theta),$$

which implies $(Tx)_{\tau_0} = \phi$. Hence $Tx \in H_\phi$.

Assertion 3. H_ϕ is a convex and closed set.

Let $\psi, \varphi \in H_\phi$ be given. Then for $\xi \in [0,1]$, it follows from the properties of regulated functions that $(1-\xi)\psi + \xi\varphi \in G([\tau_0 - r, \tau_0 + \sigma])$ and for $\theta \in [-r, 0]$, we get

$$(1-\xi)\psi_{\tau_0}(\theta) + \xi\varphi_{\tau_0}(\theta) = (1-\xi)\phi(\theta) + \xi\phi(\theta) = \phi(\theta),$$

which implies that H is convex.

Now, let us prove that H_ϕ is closed. Assume that $\varphi_k \in H_\phi$, $k \in \mathbb{N}$, is a sequence which converges in $G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ to a certain function φ. Given $\theta \in [-r, 0]$, we have

$$\varphi_{\tau_0}(\theta) = \varphi(\tau_0 + \theta) = \lim_{k\to\infty} \varphi_k(\tau_0 + \theta) = \lim_{k\to\infty} (\varphi_k)_{\tau_0}(\theta) = \phi(\theta),$$

that is, $\varphi_{\tau_0} = \phi$, proving the assertion.

Assertion 4. $\mathcal{A} := T(H_\phi) = \{Tx \colon x \in H_\phi\}$ is relatively compact.

We will show that $\mathcal{A}$ is uniformly bounded and equiregulated. Indeed, let $y \in \mathcal{A}$ be arbitrary, then there exists $x \in H_\phi$ such that $y = Tx$. Let $t \in [\tau_0 - r, \tau_0]$, then

$$\|(Tx)(t)\| = \|\phi(t - \tau_0)\| \leqslant \sup_{\theta \in [-r,0]} \|\phi(\theta)\| = \|\phi\|_\infty. \tag{3.1.10}$$

On the other hand, for $t \in [\tau_0, \tau_0 + \sigma]$, by condition (A4), Theorem 1.2.4 and Corollary 1.2.5, we obtain

$$\begin{aligned}
\|(Tx)(t)\| &\leqslant \|\phi(0)\| + \left\| \int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s) \right\| \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{t} |a(t,s)| M(s)\, \mathrm{d}g(s) \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{t} cM(s)\, \mathrm{d}g(s) \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{\tau_0+\sigma} cM(s)\, \mathrm{d}g(s) = \|\phi\|_\infty + \beta, \qquad (3.1.11)
\end{aligned}$$

where $\beta = \int_{\tau_0}^{\tau_0+\sigma} cM(s)\, \mathrm{d}g(s) < \infty$ in view of condition (A4).

Combining (3.1.10) and (3.1.11), we conclude that

$$\|y\|_\infty = \|Tx\|_\infty \leqslant K,$$

where $K := \|\phi\|_\infty + \beta$ does not depend on $y \in \mathcal{A}$. Thus, the set $\mathcal{A}$ is uniformly bounded.

Next we show that $\mathcal{A}$ is equiregulated. In fact, let an arbitrary $\varepsilon > 0$ be given. Since the function $[\tau_0 - r, \tau_0] \ni t \mapsto \phi(t - \tau_0)$ belongs to the set $G([\tau_0 - r, \tau_0], \mathbb{R}^n)$, by Höning's Theorem (Theorem 1.1.4), there is a division of $[\tau_0 - r, \tau_0]$ given by $\tau_0 - r = \alpha_0 < \alpha_1 < \ldots < \alpha_k = \tau_0$ such that

$$\|y(t) - y(s)\| = \|(Tx)(t) - (Tx)(s)\| = \|\phi(t - \tau_0) - \phi(s - \tau_0)\| < \varepsilon,$$

for all $t, s \in (\alpha_{i-1}, \alpha_i)$, $i \in \{1, \ldots, k\}$. On the other hand, by conditions (A2), (A3), (A4), Theorem 1.2.4, Corollary 1.2.5 and using the same arguments as in Step 2 of the proof of Lemma 3.1.7, we can prove that, for $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$,

$$\begin{aligned}
\|y(\tau_2) - y(\tau_1)\| &= \|(Tx)(\tau_2) - (Tx)(\tau_1)\| \\
&= \left\| \int_{\tau_0}^{\tau_2} a(\tau_2, s) f(x_s, s)\, \mathrm{d}g(s) - \int_{\tau_0}^{\tau_1} a(\tau_1, s) f(x_s, s)\, \mathrm{d}g(s) \right\| \\
&\leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\tau_0+\sigma} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s).
\end{aligned}$$

This gives

$$\|y(\tau_2) - y(\tau_1)\| \leqslant |h(\tau_2) - h(\tau_1)|, \qquad (3.1.12)$$

for all $y \in \mathcal{A}$ and all $\tau_2, \tau_1 \in [\tau_0, \tau_0 + \sigma]$, where $h \colon [\tau_0, \tau_0 + \sigma] \to \mathbb{R}$ is given by

$$h(t) := \int_{\tau_0}^{t} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\tau_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s), \qquad (3.1.13)$$

for every $t \in [\tau_0, \tau_0 + \sigma]$.

Since h clearly is a nondecreasing function on $[\tau_0, \tau_0+\sigma]$ (and, therefore, $h \in G([\tau_0, \tau_0+\sigma], \mathbb{R})$), then again by Höning's Theorem (Theorem 1.1.4), there is a division of $[\tau_0, \tau_0+\sigma]$ given by $\tau_0 = \xi_0 < \xi_1 < \ldots < \xi_m = \tau_0 + \sigma$ such that

$$|h(t) - h(s)| < \varepsilon,$$

for all $t, s \in (\xi_{i-1}, \xi_i)$ and $i \in \{1, \ldots, m\}$. Using this fact together with (3.1.12), we have

$$\|y(t) - y(s)\| < \varepsilon,$$

for all $y \in \mathcal{A}$, $t, s \in (\xi_{i-1}, \xi_i)$ and $i \in \{1, \ldots, m\}$. Now, define

$$\gamma_i = \begin{cases} \alpha_i, & i \in \{0, \ldots, k\} \\ \xi_{i-k}, & i \in \{k+1, k+2, \ldots, k+m\}. \end{cases}$$

Obviously, $\tau_0 - r = \gamma_0 < \gamma_1 < \ldots < \gamma_{k+m} = \tau_0 + \sigma$ is a division of $[\tau_0 - r, \tau_0 + \sigma]$ and

$$\|y(t) - y(s)\| < \varepsilon,$$

for arbitrary $y \in \mathcal{A}$, $t, s \in (\gamma_{i-1}, \gamma_i)$ and $i \in \{1, \ldots, k+m\}$. Hence by Lemma 1.1.6, $\mathcal{A}$ is equiregulated. Therefore, by Theorem 1.1.7, $\mathcal{A}$ is relatively compact, proving the assertion.

Assertion 5. T is continuous.

Let $x, z \in H_\phi$ be given. Then, for $t \in [\tau_0 - r, \tau_0]$, we have

$$\|(Tx)(t) - (Tz)(t)\| = \|\phi(t - \tau_0) - \phi(t - \tau_0)\| = 0.$$

On the other hand, for $t \in [\tau_0, \tau_0 + \sigma]$, by condition (A5), Theorem 1.2.4 and Corollary 1.2.5, we get

$$\begin{aligned}
\|(Tx)(t) - (Tz)(t)\| &= \left\| \int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s) - \int_{\tau_0}^{t} a(t,s) f(z_s, s)\, \mathrm{d}g(s) \right\| \\
&= \left\| \int_{\tau_0}^{t} a(t,s)(f(x_s, s) - f(z_s, s))\, \mathrm{d}g(s) \right\| \\
&\leqslant \int_{\tau_0}^{t} |a(t,s)| L(s) \|x_s - z_s\|_\infty \, \mathrm{d}g(s)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{\tau_0}^{t} \|x_s - z_s\|_\infty cL(s)\,\mathrm{d}g(s)\\
&\leqslant \int_{\tau_0}^{\tau_0+\sigma} \|x_s - z_s\|_\infty cL(s)\,\mathrm{d}g(s)\\
&\leqslant \|x - z\|_\infty \left(\int_{\tau_0}^{t_0+\sigma} cL(s)\,\mathrm{d}g(s)\right),
\end{aligned}
$$

since $\|x_s - z_s\|_\infty \leqslant \|x - z\|_\infty$ for $\tau_0 \leqslant t \leqslant \tau_0 + \sigma$. These arguments together with the fact that, by condition (A5), $\int_{\tau_0}^{t_0+\sigma} cL(s)\,\mathrm{d}g(s) := \gamma < \infty$, imply that T is continuous.

Finally, all the hypotheses from Schauder Fixed–Point Theorem (Theorem 3.1.8) are satisfied. Then, we have that T has a fixed point in H_ϕ. By the definition of the operator T given by (3.1.9), we conclude that (3.1.8) has a solution $x\colon [\tau_0 - r, \tau_0 + \sigma] \to \mathbb{R}^n$.

It remains to ensure the uniqueness of the solution.

Uniqueness. Assume that $x, z\colon [\tau_0 - r, \tau_0 + \sigma] \to \mathbb{R}^n$ are two solutions of equation (3.1.8). It is clear that $x(t) = z(t) = \phi(t - \tau_0)$ for all $t \in [\tau_0 - r, \tau_0]$. Keeping in mind condition (A5) and Theorem 1.2.4, we have for $t \in [\tau_0, \tau_0 + \sigma]$

$$
\begin{aligned}
\|x(t) - z(t)\| &= \left\| \int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s) - \int_{\tau_0}^{t} a(t,s) f(z_s, s)\,\mathrm{d}g(s) \right\|\\
&= \left\| \int_{\tau_0}^{t} a(t,s)(f(x_s, s) - f(z_s, s))\,\mathrm{d}g(s) \right\|\\
&\leqslant \int_{\tau_0}^{t} |a(t,s)| L(s) \|x_s - z_s\|_\infty\,\mathrm{d}g(s)\\
&\leqslant c\,\|L\|_{\infty,[\tau_0,\tau_0+\sigma]} \int_{\tau_0}^{t} \|x_s - z_s\|_\infty\,\mathrm{d}g(s).
\end{aligned}
$$

Using the fact that

$$
\|x_s - z_s\|_\infty = \sup_{\theta \in [-r,0]} \|x(s+\theta) - z(s+\theta)\| = \sup_{\eta \in [s-r,s]} \|x(\eta) - z(\eta)\|,
$$

we get

$$
\|x(t) - z(t)\| \leqslant c\,\|L\|_{\infty,[\tau_0,\tau_0+\sigma]} \int_{\tau_0}^{t} \sup_{\eta \in [s-r,s]} \|x(\eta) - z(\eta)\|\,\mathrm{d}g(s). \tag{3.1.14}
$$

Since the right–hand side of (3.1.14) is nondecreasing, we have

$$
\sup_{\tau \in [t-r,t]} \|x(\tau) - z(\tau)\| \leqslant c\,\|L\|_{\infty,[\tau_0,\tau_0+\sigma]} \int_{\tau_0}^{t} \sup_{\eta \in [s-r,s]} \|x(\eta) - z(\eta)\|\,\mathrm{d}g(s),
$$

and, therefore,

$$\sup_{\tau\in[t-r,t]} \|x(\tau)-z(\tau)\| \leqslant \varepsilon + c\,\|L\|_{\infty,[\tau_0,\tau_0+\sigma]} \int_{\tau_0}^{t} \sup_{\eta\in[s-r,s]} \|x(\eta)-z(\eta)\| \,\mathrm{d}g(s),$$

for every $\varepsilon > 0$. Then, by the Gronwall's inequality for the Henstock–Kurzweil–Stieltjes integral (Theorem 1.2.10), we obtain

$$\sup_{\tau\in[t-r,t]} \|x(\tau)-z(\tau)\| \leqslant \varepsilon e^{c\|L\|_{\infty,[\tau_0,\tau_0+\sigma]}(g(t)-g(\tau_0))}.$$

Since $\|x(t)-z(t)\| \leqslant \sup_{\tau\in[t-r,t]} \|x(\tau)-z(\tau)\|$, we have

$$\|x(t)-z(t)\| \leqslant \varepsilon e^{c\|L\|_{\infty,[\tau_0,\tau_0+\sigma]}(g(t)-g(\tau_0))}.$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $x(t) = z(t)$ for all $t \in [\tau_0, \tau_0+\sigma]$. Hence $x = z$, proving the uniqueness of the solution. □

Remark 3.1.10. If $a(t,s) \equiv 1$, then equation (3.1.8) reduces to the usual measure functional differential equation given by:

$$\begin{cases} x(t) &=& \phi(0) + \displaystyle\int_{\tau_0}^{t} f(x_s,s)\mathrm{d}g(s) \\ x_{\tau_0} &=& \phi \end{cases} \tag{3.1.15}$$

Results concerning existence and uniqueness for this type of equations were obtained in [24], using the correspondence between (3.1.15) and generalized ODEs. Also, the conditions presented in [24] are stronger than the ones presented here for the function f, allowing us to get a more general result. A careful examination at the conditions assumed by function f shows that it is required that f be bounded by a constant instead of its integral be bounded by as a function as we require here.

On the other hand, considering $a(t,s) = k(t-s)$ for every $(t,s) \in \mathrm{Dom}(a)$ and $g(t) = t$ for every $t \in [t_0, d)$, equation (3.1.8) reduces to the usual functional Volterra integral equations of convolution type given by

$$\begin{cases} x(t) &=& \phi(0) + \displaystyle\int_{\tau_0}^{t} k(t-s)f(x_s,s)\mathrm{d}s \\ x_{\tau_0} &=& \phi. \end{cases} \tag{3.1.16}$$

The results presented here are more general than the ones found in the literature for this type of equation (see [34]). The same applies for more general kernels, such as $k(t-s) = (t-s)^{\alpha-1}/\Gamma(\alpha)$, which transforms equation (3.1.16) in a fractional functional differential equation.

Remark 3.1.11. Notice that if

$$\int_{t_0}^{t_0+\sigma} cL(s)\mathrm{d}g(s) < 1,$$

then it is possible to show the existence and uniqueness of solutions using Banach's Fixed Point Theorem, since in this case, one can show that the operator T defined previously is a contraction.

Now, we present a concrete example of our main result considering as external nonlinear force the function $f(\psi, s) = e^{-\gamma s}e^{-\sin\psi(-1)}$, as well as the coupling of the Maxwell and power type materials in the kernel $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\delta t}$ for the case $1 < \alpha < 2$ and $\delta = 0$. We were inspired by the example found in [8].

Example 3.1.12. We consider the following integral equation

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{\tau_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\gamma s} e^{-\sin x(s-1)}\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} &= \phi, \end{cases} \tag{3.1.17}$$

where $\gamma > 0$, $1 < \alpha < 2$, $\phi \in G([-r, 0], \mathbb{R})$, $g\colon [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing and left–continuous function and Γ is the gamma function. Taking $a\colon [t_0, +\infty)^2 \to \mathbb{R}$ and $f\colon G([-r, 0], \mathbb{R}) \times [t_0, +\infty) \to \mathbb{R}$ given, respectively, by

$$a(t,s) := \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } \ s \leqslant t \\ 0, & \text{if } \ t < s \end{cases}$$

and

$$f(\psi, s) := e^{-\gamma s}e^{-\sin\psi(-1)}, \quad (\psi, s) \in G([-r, 0], \mathbb{R}) \times [t_0, +\infty),$$

we have that (3.1.17) is in the form of (3.1.8) with $d = +\infty$. Observe that, since $x_s(-1) = x(s-1)$, we get $f(x_s, s) = e^{-\gamma s}e^{-\sin x(s-1)}$.

Let us show that conditions (A1)–(A5) are all satisfied. It is clear that g satisfies condition (A1) by definition.

Note that for any $t \in [t_0, +\infty)$, the function $s \mapsto a(t, s)$ is regulated on $[\alpha, \beta]$, for each compact interval $[\alpha, \beta] \subset [t_0, +\infty)$. On the other hand, given $s \in [t_0, +\infty)$ fixed, we shall prove that $a(\cdot, s)$ is nondecreasing. For it, we will consider three cases.

Case 1. Let $t_1, t_2 \in [t_0, +\infty)$ be such that $s \leqslant t_1 < t_2$. Then

$$a(t_1, s) = \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \leqslant \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} = a(t_2, s).$$

Case 2. Let $t_1, t_2 \in [t_0, +\infty)$ be such that $t_1 < t_2 < s$. Then $a(t_1, s) = a(t_2, s) = 0$.

Case 3. Let $t_1, t_2 \in [t_0, +\infty)$ be such that $t_1 < s < t_2$. Then

$$a(t_1, s) = 0 \leqslant \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} = a(t_2, s).$$

In any case, we have that if $t_1, t_2 \in [t_0, +\infty)$ are such that $t_1 < t_2$, then $a(t_1, s) \leqslant a(t_2, s)$. Also, clearly, a is bounded on any compact rectangle $[\tau_0, \tau_0 + \sigma]^2 \subset [t_0, +\infty)^2$.

Note that if $[\tau_0, \tau_0 + \sigma] \subset [t_0, +\infty)$, $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R})$, $t \in [t_0, +\infty)$ and $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$, then it follows that $[\tau_1, \tau_2] \ni s \mapsto a(t, s) f(x_s, s)$ is a regulated function on $[\tau_1, \tau_2]$. This implies the existence of $\int_{\tau_1}^{\tau_2} a(t, s) f(x_s, s) \mathrm{d}g(s)$, proving that condition (A3) is satisfied.

Now, let us prove that (A4) is satisfied. Define $M\colon [t_0, +\infty) \to \mathbb{R}^+$ by $M(s) = e^{1-\gamma s}$, for $s \in [t_0, +\infty)$. Observe that M is a locally Henstock–Kurzweil–Stieltjes integrable function with respect to g and

$$\begin{aligned}
\left| \int_{\tau_1}^{\tau_2} c_1 a(\tau_2, s) + c_2 a(\tau_1, s) f(x_s, s) \mathrm{d}g(s) \right| &\overset{\substack{Thm\, 1.2.3 \\ \downarrow}}{\leqslant} \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2, s) + c_2 a(\tau_1, s)| \, |f(x_s, s)| \, \mathrm{d}g(s) \\
&= \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2, s) + c_2 a(\tau_1, s)| \left| e^{-\gamma s} e^{-\sin x(s-1)} \right| \mathrm{d}g(s) \\
&\leqslant \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2, s) + c_2 a(\tau_1, s)| \, e^{-\gamma s} \, e \, \mathrm{d}g(s) \\
&= \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2, s) + c_2 a(\tau_1, s)| \, M(s) \mathrm{d}g(s),
\end{aligned}$$

for $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R})$, $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$ and $c_1, c_2 \in \mathbb{R}$, getting (A4).

Finally, let us show that (A5) is also satisfied. Define $L\colon [t_0, +\infty) \to \mathbb{R}^+$ by $L(s) = e^{1-\gamma s}$, for $s \in [t_0, +\infty)$. It is clear that L is a locally regulated function and for $x, y \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R})$ and $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$, we get

$$\begin{aligned}
\left| \int_{\tau_1}^{\tau_2} a(\tau_2, s) \left[f(x_s, s) - f(y_s, s) \right] \mathrm{d}g(s) \right| &\overset{\substack{Thm\, 1.2.3 \\ \downarrow}}{\leqslant} \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, |f(x_s, s) - f(y_s, s)| \, \mathrm{d}g(s) \\
&= \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \left| e^{-\gamma s} \left(e^{-\sin x_s(-1)} - e^{-\sin y_s(-1)} \right) \right| \mathrm{d}g(s)
\end{aligned}$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, e^{-\gamma s} e\, |x_s(-1) - y_s(-1)|\, \mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, L(s)\, |x_s - y_s|_\infty\, \mathrm{d}g(s),$$

giving us condition (A5), where the third inequality follows from the estimates given by the Mean Value Theorem.

Therefore, f, a and g satisfy all the hypotheses of Theorem 3.1.9. Thus, there exists a $\sigma > 0$ such that equation (3.1.17) has a unique solution on $[\tau_0 - r, \tau_0 + \sigma]$.

We finish this section with another example, which is completely new in the literature.

Example 3.1.13. Consider the following integral equation

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{\tau_0}^{t} \beta(t)\eta(s)e^{\alpha s}e^{\gamma\cos(x_s(\theta))}\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} &= \phi, \end{cases} \tag{3.1.18}$$

where

$$g(s) = \begin{cases} s, & s \in [0,1], \\ s+1, & s \in (1,d), \quad d > 1, \end{cases}$$

$\eta\colon [t_0, d) \to \mathbb{R}^+$ is a regulated function, $\beta\colon [t_0, d) \to \mathbb{R}$ is a nondecreasing function, $\gamma > 0$ and $\alpha < 0$. Define the following functions

$$\begin{aligned} a(t,s)\colon \quad [t_0, d) \times [t_0, d) &\to \quad \mathbb{R} \\ (t,s) &\mapsto \beta(t)e^{\alpha s}, \end{aligned}$$

and for $\theta \in [-r, 0]$,

$$\begin{aligned} f(\psi,s)\colon \quad G([-r,0],\mathbb{R}) \times [t_0, d) &\to \quad \mathbb{R} \\ (\psi,s) &\mapsto \eta(s)e^{-\gamma\cos(\psi(\theta))}. \end{aligned}$$

By definition, it is clear that g satisfies (A1) and a satisfies (A2).

Since $\beta(t)\eta(s)e^{\alpha s}e^{-\gamma\cos(x(s+\theta))}$ is regulated for all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R})$, $t \in [t_0, d)$, $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, the integral

$$\int_{\tau_1}^{\tau_2} e^{\alpha(t-s)}\eta(s)e^{\gamma\cos(x_s(\theta))}\mathrm{d}g(s)$$

exists in the sense of Henstock–Kurzweil–Stieltjes for all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$, getting (A3).

To show (A4), notice that $0 \leqslant \eta(s)e^{-\gamma \cos(x_s(\theta))} \leqslant \eta(s)e^{\gamma}$. Defining $M(s) := \eta(s)e^{\gamma}$ for $s \in [t_0, d)$, we have that M is Henstock–Kurzweil–Stieltjes integrable with respect to g and

$$\left| \int_{\tau_1}^{\tau_2} c_1 a(\tau_2, s) + c_2 a(\tau_1, s) f(x_s, s) \mathrm{d}g(s) \right| \leqslant \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2, s) + c_2 a(\tau_1, s)| M(s) \mathrm{d}g(s),$$

for $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R})$, $c_1, c_2 \in \mathbb{R}$ and $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$, proving (A4).

Let $L\colon [t_0, d) \to \mathbb{R}^+$ be defined as $L(s) = \gamma e^{\gamma}\eta(s)$. Notice that by the Mean Value Theorem, $|e^{-\gamma \cos(u)} - e^{-\gamma \cos(v)}| \leqslant \gamma e^{\gamma}|u - v|$. Hence

$$\begin{aligned}
\left| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right| &= \left| \int_{\tau_1}^{\tau_2} a(\tau_2, s)\eta(s)[e^{-\gamma \cos(x_s(\theta))} - e^{-\gamma \cos(z_s(\theta))}] \mathrm{d}g(s) \right| \\
&\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\eta(s)e^{\gamma}\gamma |x_s - z_s|_\infty \mathrm{d}g(s) \\
&= \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| L(s) |x_s - z_s|_\infty \mathrm{d}g(s),
\end{aligned}$$

proving condition (A5). Therefore, all the hypotheses of Theorem 3.1.9 are satisfied, then there exists a $\sigma > 0$ such that equation (3.1.18) has a unique solution on $[\tau_0 - r, \tau_0 + \sigma]$.

3.2 Existence and uniqueness of maximal solutions

In this section, we are interested to investigate under which conditions we can ensure the existence and uniqueness of maximal solutions of the following equation:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t, s) f(x_s, s) \mathrm{d}g(s), & t \geqslant \tau_0, \\ x_{\tau_0} = \phi, \end{cases} \tag{3.2.1}$$

where $\tau_0 \geqslant t_0$, $\phi \in G([-r, 0], \mathbb{R}^n)$, $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$, $a\colon [t_0, +\infty)^2 \to \mathbb{R}$ and $g\colon [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing function and the integral in the right–hand side is understood in the sense of Henstock–Kurzweil–Stieltjes.

We are now interested about maximal solutions since we intend to investigate the asymptotic behaviour of the solutions of (3.2.1), such as stability results. From now on, we assume the conditions (A1)–(A5) for the case $d = +\infty$, which can be read as follows:

(B1) The function $g\colon [t_0, +\infty) \to \mathbb{R}$ is nondecreasing and left–continuous on $(t_0, +\infty)$.

(B2) The function $a\colon [t_0, +\infty)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and, locally bounded on $[t_0, +\infty)^2$.

(B3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{\tau_1}^{\tau_2} a(t,s) f(x_s, s) \mathrm{d}g(s)$$

exists, for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, +\infty)$, all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $t \in [t_0, +\infty)$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

(B4) There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M\colon [t_0, +\infty) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, +\infty)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} b(\tau_2, s) f(x_s, s) \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |b(\tau_2, s)| \, M(s) \mathrm{d}g(s),$$

for all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, all $b \in G_2([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

(B5) There exists a locally regulated function $L\colon [t_0, +\infty) \to \mathbb{R}^+$ such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, +\infty)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \, L(s) \|x_s - z_s\|_\infty \mathrm{d}g(s),$$

for all $x, z \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ and all $\tau_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \tau_0 + \sigma$.

Definition 3.2.1. (Prolongation to the right) Let $\tau_0 \geqslant t_0$, $\phi \in G([-r, 0], \mathbb{R}^n)$ and $x\colon J \to \mathbb{R}^n$, $J \subset [t_0 - r, +\infty)$, be a solution of (3.2.1) on the interval J with $\tau_0 - r = \min J$. The solution $y\colon \widehat{J} \to \mathbb{R}^n$, $\widehat{J} \subset [t_0 - r, +\infty)$ with $\tau_0 - r = \min \widehat{J}$, of (3.2.1) is called a *prolongation to the right* of x, if $J \subset \widehat{J}$ and $x(t) = y(t)$ for all $t \in J$. If $J \subsetneq \widehat{J}$, then y is called a *proper prolongation* of x to the right.

Definition 3.2.2. (Maximal solution) Let $\tau_0 \geqslant t_0$, $\phi \in G([-r,0],\mathbb{R}^n)$. A solution $y\colon I \to \mathbb{R}^n$, $I \subset [t_0 - r, +\infty)$ and I is such that $\tau_0 - r = \min I$, of the equation (3.2.1), is called *maximal*, if there is no proper prolongation of y to the right. In this case, I is called the *maximal interval of existence* of y.

The proof of the next result is very similar to the proof of the uniqueness of solution (see Theorem 3.1.9) and thus, we will omit it here.

Lemma 3.2.3. *Assume $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ satisfies the conditions (B3), (B4) and (B5), $a\colon [t_0,+\infty)^2 \to \mathbb{R}$ satisfies condition (B2) and $g\colon [t_0,+\infty) \to \mathbb{R}$ satisfies condition (B1). Let $\tau_0 \geqslant t_0$, $\phi \in G([-r,0],\mathbb{R}^n)$ and consider the equation* (3.2.1). *If $x\colon J_x \to \mathbb{R}^n$ and $y\colon J_y \to \mathbb{R}^n$ are solutions of* (3.2.1), *where J_x and J_y are intervals such that $\tau_0 - r = \min J_x = \min J_y$, then $x(t) = y(t)$ for all $t \in J_x \cap J_y$.*

Next, we present the main theorem of this section.

Theorem 3.2.4. *Suppose $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ satisfies conditions (B3), (B4) and (B5), $a\colon [t_0,+\infty)^2 \to \mathbb{R}$ satisfies condition (B2) and $g\colon [t_0,+\infty) \to \mathbb{R}$ satisfies condition (B1). Then, for every $\tau_0 \geqslant t_0$ and $\phi \in G([-r,0],\mathbb{R}^n)$, there exists a unique maximal solution $x\colon I \to \mathbb{R}^n$ of the equation* (3.2.1), *where I is a nondegenerate interval with $\tau_0 - r = \min I$. Also, $I = [\tau_0 - r, \omega)$, with $\omega \leqslant +\infty$.*

Proof. Let $\tau_0 \geqslant t_0$ and $\phi \in G([-r,0],\mathbb{R}^n)$ be fixed. Firstly, we will show the existence of a maximal solution.

Existence. Consider the set

$$S := \{x\colon J_x \to \mathbb{R}^n\colon J_x \text{ is an interval such that } \tau_0 - r = \min J_x \text{ and}$$

$$x \text{ is a solution of the equation (3.2.1)}\}$$

The set S is nonempty by the local existence and uniqueness of solution given in Theorem 3.1.9.

Define $I := \bigcup\limits_{y \in S} J_y$ and $x\colon I \to \mathbb{R}^n$ by the relation $x(t) = y(t)$, where $y \in S$ and $t \in J_y$. Note that if y and z belong to S, then $y(s) = z(s)$, for all $s \in J_y \cap J_z$, by Lemma 3.2.3. Thus, we conclude that x is well–defined. Note that I is an interval with $\tau_0 - r = \min I$

(since I is union connected with a common point) and x is a maximal solution of the equation (3.2.1), proving the existence of a maximal solution.

It remains to ensure the uniqueness of the maximal solution.

Uniqueness. Assume $x_1 \colon I_1 \to \mathbb{R}^n$ and $x_2 \colon I_2 \to \mathbb{R}^n$ are two maximal solutions of the equation (3.2.1), where I_1, I_2 are intervals such that $\tau_0 - r = \min I_1 = \min I_2$. Hence, by Lemma 3.2.3, we have

$$x_1(t) = x_2(t), \quad \text{for all } t \in I_1 \cap I_2. \tag{3.2.2}$$

Since $\tau_0 - r = \min I_1 = \min I_2$, we have only one of the following possibilities:

1) $I_1 \subsetneq I_2$

2) $I_2 \subsetneq I_1$

3) $I_1 = I_2$.

We will show that the only possibility is (3). Any other ones lead a contradiction. Indeed, without loss generality, we assume that $I_1 \subsetneq I_2$, then $I_1 \cap I_2 = I_1$ and, therefore, by (3.2.2), we have $x_1(t) = x_2(t)$ for all $t \in I_1$. It implies that $x_2|_{I_1} = x_1$ and $I_1 \subsetneq I_2$, i.e., x_2 is a proper prolongation of x_1, that is assumed to be maximal, which is a contradiction. Hence $I_2 = I_1$ and $x_1(t) = x_2(t)$, for all $t \in I_1$, that is, $x_1 = x_2$.

Finally, let us prove that the interval of existence of the maximal solution must be right–open.

Let $\varphi^{max} \colon I \to \mathbb{R}^n$ be the maximal solution of

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s) \mathrm{d}g(s), \qquad t \geqslant \tau_0, \\ x_{\tau_0} &= \phi, \end{cases} \tag{3.2.3}$$

where I is an interval with $\tau_0 \in I$ and $\min I = \tau_0 - r$.

It is clear that $I \subset [t_0 - r, +\infty)$. Define $\omega := \sup I$. Hence $\omega \leqslant +\infty$. If $\omega = +\infty$, the result follows immediately. Suppose that $\omega < +\infty$.

Assertion 1. $\omega \notin I$.

Let us assume that $\omega \in I$, that is, $I = [\tau_0, \omega]$. Define $\gamma \colon [t_0, +\infty) \to \mathbb{R}^n$ by

$$\gamma(t) := \phi(0) + \int_{\tau_0}^{\omega} a(t,s) f(\varphi_s^{max}, s) \mathrm{d}g(s). \tag{3.2.4}$$

Notice that by definition $\gamma(\omega) = \varphi^{max}(\omega)$. On the other hand, consider the following problem

$$\begin{cases} y(t) &= \gamma(t) + \displaystyle\int_{\omega}^{t} a(t,s) f(y_s,s)\mathrm{d}g(s), \qquad t \geqslant \omega, \\ y_\omega &= \varphi_\omega^{max}. \end{cases} \tag{3.2.5}$$

Assertion 2. There exists a local solution $y\colon [\omega - r, \omega + \eta] \to \mathbb{R}^n$ of (3.2.5).

Indeed, let $\eta > 0$. Consider the set

$$\mathcal{H} := \{\psi \in G([\omega - r, \omega + \eta], \mathbb{R}^n) \colon \psi_\omega = \varphi_\omega^{max}\}.$$

Clearly $\mathcal{H}$ is nonempty, closed and convex. Now, define the operator $T\colon \mathcal{H} \to \mathcal{H}$ by

$$(Tx)(t) = \begin{cases} \varphi_\omega^{max}(t-\omega), & t \in [\omega - r, \omega] \\ \gamma(t) + \displaystyle\int_{\omega}^{t} a(t,s) f(x_s,s)\mathrm{d}g(s), & t \in [\omega, \omega + \eta], \end{cases} \tag{3.2.6}$$

where $\gamma(t)$ is the function defined by (3.2.4).

Assertion 3. T is well–defined.

Let $x \in \mathcal{H}$ be fixed. We need to prove that $T(\mathcal{H}) \subset \mathcal{H}$. Therefore, we start by proving that Tx is regulated on $[\omega - r, \omega + \eta]$. In order to do this, we divide the proof in two steps.

Step 1. The restriction of Tx to $[\omega - r, \omega]$ is regulated.

Indeed, let $\tau \in (\omega - r, \omega]$, then $\tau - \omega \in (-r, 0]$. Now, since $\varphi_\omega^{max} \in G([-r,0],\mathbb{R}^n)$, $\lim\limits_{\theta \to (\tau-\omega)^-} \varphi_\omega^{max}(\theta)$ exists and

$$\lim_{\theta \to (\tau-\omega)^-} \varphi_\omega^{max}(\theta) = \lim_{\theta \to (\tau-\omega)^-} \varphi_\omega^{max}(\theta + \omega - \omega) = \lim_{s \to \tau^-} \varphi_\omega^{max}(s - \omega) = \lim_{s \to \tau^-} (Tx)(s),$$

which implies that $\lim\limits_{s \to \tau^-} (Tx)(s)$ exists for $\tau \in (\omega - r, \omega]$. Similarly, we can prove that $\lim\limits_{s \to \tau^+} (Tx)(s)$ exists for $\tau \in [\omega - r, \omega)$.

Step 2. The restriction of Tx to $[\omega, \omega + \eta]$ is regulated.

In fact, for $\omega \leqslant \tau_1 \leqslant \tau_2 \leqslant \omega + \eta$, by conditions (B2), (B3), (B4), Theorem 1.2.4, Corollary 1.2.5 and the definition of γ (given by (3.2.4)), we have

$$\begin{aligned} &\|(Tx)(\tau_2) - (Tx)(\tau_1)\| \\ &= \left\|\gamma(\tau_2) - \gamma(\tau_1) + \int_{\omega}^{\tau_2} a(\tau_2,s) f(x_s,s)\,\mathrm{d}g(s) - \int_{\omega}^{\tau_1} a(\tau_1,s) f(x_s,s)\,\mathrm{d}g(s)\right\| \\ &\leqslant \|\gamma(\tau_2) - \gamma(\tau_1)\| + \left\|\int_{\omega}^{\tau_2} a(\tau_2,s) f(x_s,s)\,\mathrm{d}g(s) - \int_{\omega}^{\tau_1} a(\tau_1,s) f(x_s,s)\,\mathrm{d}g(s)\right\| \end{aligned}$$

$$\leqslant \Big\| \int_{\tau_0}^{\omega} \Big(a(\tau_2, s) - a(\tau_1, s)\Big) f(\varphi_s^{max}, s)\, \mathrm{d}g(s) \Big\|$$

$$+ \Big\| \int_{\omega}^{\tau_1} a(\tau_2, s) f(x_s, s)\, \mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} a(\tau_2, s) f(x_s, s)\, \mathrm{d}g(s) - \int_{\omega}^{\tau_1} a(\tau_1, s) f(x_s, s)\, \mathrm{d}g(s) \Big\|$$

$$\leqslant \int_{\tau_0}^{\omega} |a(\tau_2, s) - a(\tau_1, s)|\, M(s)\, \mathrm{d}g(s)$$

$$+ \Big\| \int_{\tau_1}^{\tau_2} a(\tau_2, s) f(x_s, s)\, \mathrm{d}g(s) \Big\| + \Big\| \int_{\omega}^{\tau_1} \Big(a(\tau_2, s) - a(\tau_1, s)\Big) f(x_s, s)\, \mathrm{d}g(s) \Big\|$$

$$\leqslant \int_{\tau_0}^{\omega} |a(\tau_2, s) - a(\tau_1, s)|\, M(s)\, \mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, M(s)\, \mathrm{d}g(s)$$

$$+ \int_{\omega}^{\tau_1} |a(\tau_2, s) - a(\tau_1, s)|\, M(s)\, \mathrm{d}g(s)$$

$$= \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, M(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\tau_1} |a(\tau_2, s) - a(\tau_1, s)|\, M(s)\, \mathrm{d}g(s).$$

By condition (B2), a is nondecreasing with respect to the first variable, and there exists $c := \sup_{(t,s)\in[\omega,\omega+\eta]^2} |a(t,s)|$. Hence

$$\int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, M(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\tau_1} |a(\tau_2, s) - a(\tau_1, s)|\, M(s)\, \mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\tau_1} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\omega+\eta} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s),$$

that is,

$$\|Tx(\tau_2) - Tx(\tau_1)\| \leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\omega+\eta} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s). \qquad (3.2.7)$$

Define $h\colon [\omega, \omega+\eta] \to \mathbb{R}$ by

$$h(t) := \int_{\tau_0}^{t} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\omega+\eta} a(t, s) M(s)\, \mathrm{d}g(s), \qquad (3.2.8)$$

for every $t \in [\omega, \omega+\eta]$. In view of the Henstock–Kurzweil–Stieltjes integrability of the function M with respect to the function g on $[\tau_0, \omega+\eta]$, the integral $\int_{\tau_0}^{t} cM(s)\, \mathrm{d}g(s)$ exists for all $t \in [\omega, \omega+\eta]$. In a similar way, we can prove that $\int_{\tau_0}^{\omega+\eta} a(t,s) M(s)\, \mathrm{d}g(s)$ exists for all $t \in [\omega, \omega+\eta]$. Thus, h is well–defined and is a nondecreasing function. Also, using (3.2.7) and (3.2.8), we have

$$\|(Tx)(\tau_2) - (Tx)(\tau_1)\| \leqslant h(\tau_2) - h(\tau_1), \qquad (3.2.9)$$

for all $\omega \leqslant \tau_1 \leqslant \tau_2 \leqslant \omega + \eta$. Now, by (3.2.9) and by the fact that h is a nondecreasing function, both the lateral limits

$$\lim_{s\to\tau^+}(Tx)(s) \text{ for } \tau \in [\omega, \omega+\eta) \quad \text{and} \quad \lim_{s\to\tau^-}(Tx)(s) \text{ for } \tau \in (\omega, \omega+\eta]$$

exist. This implies that the restriction of Tx to $[\omega, \omega+\eta]$ is regulated, proving Step 2.

Also, notice that for $\theta \in [-r, 0]$, we have $\theta + \omega \in [\omega - r, \omega]$ and, therefore,

$$(Tx)_\omega(\theta) = (Tx)(\theta+\omega) = \varphi_\omega^{max}(\theta+\omega-\omega) = \varphi_\omega^{max}(\theta),$$

which implies $(Tx)_\omega = \varphi_\omega^{max}$. Hence $Tx \in \mathcal{H}$, proving the Assertion 3.

Assertion 4. $\mathcal{A} := T(\mathcal{H}) = \{Tx\colon x \in \mathcal{H}\}$ is relatively compact.

We will show that $\mathcal{A}$ is uniformly bounded and equiregulated. Indeed, let $y \in \mathcal{A}$ be arbitrary, then there exists $x \in \mathcal{H}$ such that $y = Tx$. Let $t \in [\omega - r, \omega]$, then

$$\|(Tx)(t)\| = \|\varphi_\omega^{max}(t-\omega)\| \leqslant \sup_{\theta\in[-r,0]} \|\varphi_\omega^{max}(\theta)\| = \|\varphi_\omega^{max}\|_\infty. \tag{3.2.10}$$

On the other hand, for $t \in [\omega, \omega+\eta]$, by condition (A4), Theorem 1.2.4 and Corollary 1.2.5, we obtain

$$\begin{aligned}
\|(Tx)(t)\| &\leqslant \|\gamma(t)\| + \left\|\int_\omega^t a(t,s)f(x_s,s)\,\mathrm{d}g(s)\right\| \\
&= \left\|\phi(0) + \int_{\tau_0}^\omega a(t,s)f(\varphi_s^{max},s)\mathrm{d}g(s)\right\| + \left\|\int_\omega^t a(t,s)f(x_s,s)\,\mathrm{d}g(s)\right\| \\
&\leqslant \|\phi(0)\| + \int_{\tau_0}^\omega |a(t,s)|M(s)\mathrm{d}g(s) + \int_\omega^t |a(t,s)|M(s)\,\mathrm{d}g(s) \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^\omega cM(s)\,\mathrm{d}g(s) + \int_\omega^t cM(s)\,\mathrm{d}g(s) \\
&= \|\phi\|_\infty + \int_{\tau_0}^t cM(s)\,\mathrm{d}g(s) \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{\omega+\eta} cM(s)\,\mathrm{d}g(s).
\end{aligned} \tag{3.2.11}$$

Combining (3.2.10) and (3.2.11), we conclude that

$$\|y\|_\infty = \|Tx\|_\infty \leqslant K,$$

where $K := \max\left\{\|\varphi_\omega^{max}\|_\infty, \|\phi\|_\infty + \int_{\tau_0}^{\omega+\eta} cM(s)\,\mathrm{d}g(s)\right\}$ does not depend on $y \in \mathcal{A}$. Thus, the set $\mathcal{A}$ is uniformly bounded.

Next, we show that $\mathcal{A}$ is equiregulated. In fact, let an arbitrary $\varepsilon > 0$ be given. Since the function $[\omega - r, \omega] \ni t \mapsto \varphi_\omega^{max}(t - \omega)$ belongs to the set $G([\omega - r, \omega], \mathbb{R}^n)$, we can use Höning's Theorem to guarantee the existence of a division of the $[\omega - r, \omega]$ given by $\omega - r = \alpha_0 < \alpha_1 < \ldots < \alpha_k = \omega$ such that

$$\|y(t) - y(s)\| = \|(Tx)(t) - (Tx)(s)\| = \|\varphi_\omega^{max}(t - \omega) - \varphi_\omega^{max}(s - \omega)\| < \varepsilon,$$

for all $t, s \in (\alpha_{i-1}, \alpha_i)$, $i \in \{1, \ldots, k\}$. On the other hand, by conditions (B2), (B3), (B4), Theorem 1.2.4 and Corollary 1.2.5, we can prove that for $\omega \leqslant \tau_1 \leqslant \tau_2 \leqslant \omega + \eta$,

$$\begin{aligned}
\|y(\tau_2) - y(\tau_1)\| &= \|(Tx)(\tau_2) - (Tx)(\tau_1)\| \\
&= \left\|\gamma(\tau_2) - \gamma(\tau_1) + \int_\omega^{\tau_2} a(\tau_2, s) f(x_s, s)\, \mathrm{d}g(s) - \int_\omega^{\tau_1} a(\tau_1, s) f(x_s, s)\, \mathrm{d}g(s)\right\| \\
&= \left\|\int_{\tau_0}^{\omega} (a(\tau_2, s) - a(\tau_1, s)) f(\varphi_s^{max}, s) \mathrm{d}g(s)\right\| + \left\|\int_{\omega}^{\tau_1} (a(\tau_2, s) - a(\tau_1, s)) f(x_s, s) \mathrm{d}g(s)\right\| \\
&+ \left\|\int_{\tau_1}^{\tau_2} a(\tau_2, s) f(x_s, s) \mathrm{d}g(s)\right\| \\
&\leqslant \int_{\tau_0}^{\tau_1} (a(\tau_2, s) - a(\tau_1, s)) M(s) \mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} cM(s) \mathrm{d}g(s) \\
&\leqslant \int_{\tau_1}^{\tau_2} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\omega+\eta} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s).
\end{aligned}$$

This gives

$$\|y(\tau_2) - y(\tau_1)\| \leqslant |h(\tau_2) - h(\tau_1)|, \tag{3.2.12}$$

for all $y \in \mathcal{A}$ and all $\tau_2, \tau_1 \in [\omega, \omega + \eta]$, where $h \colon [\omega, \omega + \eta] \to \mathbb{R}$ is given by

$$h(t) := \int_{\tau_0}^{t} cM(s)\, \mathrm{d}g(s) + \int_{\tau_0}^{\omega+\eta} a(t, s) M(s)\, \mathrm{d}g(s), \tag{3.2.13}$$

for every $t \in [\omega, \omega + \eta]$, which is a nondecreasing function on $[\omega, \omega + \eta]$ (and, therefore, $h \in G([\omega, \omega + \eta], \mathbb{R})$), then again by Höning's Theorem, there is a division of $[\omega, \omega + \eta]$ given by $\omega = \xi_0 < \xi_1 < \ldots < \xi_m = \omega + \eta$ such that $|h(t) - h(s)| < \varepsilon$, for all $t, s \in (\xi_{i-1}, \xi_i)$ and $i \in \{1, \ldots, m\}$. Using this fact together with (3.2.12), we have $\|y(t) - y(s)\| < \varepsilon$, for all $y \in \mathcal{A}$, $t, s \in (\xi_{i-1}, \xi_i)$ and $i \in \{1, \ldots, m\}$. Now, define

$$\gamma_i = \begin{cases} \alpha_i, & i \in \{0, \ldots, k\} \\ \xi_{i-k}, & i \in \{k+1, k+2, \ldots, k+m\}. \end{cases}$$

Obviously, $\omega - r = \gamma_0 < \gamma_1 < \ldots < \gamma_{k+m} = \omega + \eta$ is a division of $[\omega - r, \omega + \eta]$ and

$$\|y(t) - y(s)\| < \varepsilon,$$

for arbitrary $y \in \mathcal{A}$, $t, s \in (\gamma_{i-1}, \gamma_i)$ and $i \in \{1, \ldots, k+m\}$. Hence by Lemma 1.1.6, $\mathcal{A}$ is equiregulated. Therefore, $\mathcal{A}$ is relatively compact, proving the assertion.

Assertion 5. T is continuous.

Let $x, z \in H$ be given. Then, for $t \in [\omega - r, \omega]$, we have

$$\|(Tx)(t) - (Tz)(t)\| = \|\varphi_\omega^{max}(t - \omega) - \varphi_\omega^{max}(t - \omega)\| = 0.$$

On the other hand, for $t \in [\omega, \omega + \eta]$, by condition (B5), Theorem 1.2.4 and Corollary 1.2.5, we get

$$\begin{aligned}
\|(Tx)(t) - (Tz)(t)\| &= \left\| \int_\omega^t a(t,s) f(x_s, s)\, \mathrm{d}g(s) - \int_\omega^t a(t,s) f(z_s, s)\, \mathrm{d}g(s) \right\| \\
&= \left\| \int_\omega^t a(t,s)(f(x_s, s) - f(z_s, s))\, \mathrm{d}g(s) \right\| \\
&\leqslant \int_\omega^t |a(t,s)| L(s) \|x_s - z_s\|_\infty \, \mathrm{d}g(s) \\
&\leqslant \int_\omega^t \|x_s - z_s\|_\infty c L(s)\, \mathrm{d}g(s) \\
&\leqslant \int_\omega^{\omega+\eta} \|x_s - z_s\|_\infty c L(s)\, \mathrm{d}g(s) \\
&\leqslant \|x - z\|_\infty \left(\int_\omega^{\omega+\eta} c L(s)\, \mathrm{d}g(s) \right),
\end{aligned}$$

since $\|x_s - z_s\|_\infty \leqslant \|x - z\|_\infty$ for $\omega \leqslant s \leqslant \omega + \eta$. These arguments imply that T is continuous.

Since all the hypotheses of Schauder Fixed–Point Theorem (Theorem 3.1.8) are satisfied, we have that T has a fixed point in $\mathcal{H}$. By the definition of the operator T given by (3.2.6), we conclude that the equation (3.2.5) possesses a solution $y\colon [\omega - r, \omega + \eta] \to \mathbb{R}^n$. Thus, the assertion is true.

Now, define $u\colon [\tau_0 - r, \omega + \eta] \to \mathbb{R}^n$ by

$$u(t) = \begin{cases} \varphi^{max}(t), & t \in [\tau_0 - r, \omega] \\ y(t), & t \in (\omega, \omega + \eta]. \end{cases}$$

Note that, for $\theta \in [-r, 0]$, we have $\theta + \tau_0 \in [\tau_0 - r, \tau_0] \subset [\tau_0 - r, \omega]$ and, therefore,

$$u_{\tau_0}(\theta) = u(\theta + \tau_0) = \varphi^{max}(\theta + \tau_0) = \varphi^{max}_{\tau_0}(\theta) = \phi(\theta),$$

that is, $u_{\tau_0} = \phi$. If $t \in (\omega, \omega + \eta]$, we have

$$\begin{aligned} u(t) &= y(t) \\ &= \gamma(t) + \int_{\omega}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s) \\ &= \phi(0) + \int_{\tau_0}^{\omega} a(t,s) f(\varphi^{max}_s, s) \mathrm{d}g(s) + \int_{\omega}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s). \end{aligned} \tag{3.2.14}$$

Note that for $\theta \in [-r, 0]$ and $s \in [\tau_0, \omega]$, we get $\theta + s \in [\tau_0 - r, \omega]$ and, therefore,

$$\varphi^{max}_s(\theta) = \varphi^{max}(\theta + s) = u(\theta + s) = u_s(\theta),$$

that is, $\varphi^{max}_s = u_s$ for all $s \in [\tau_0, \omega]$. From this, we get

$$\int_{\tau_0}^{\omega} a(t,s) f(\varphi^{max}_s, s) \mathrm{d}g(s) = \int_{\tau_0}^{\omega} a(t,s) f(u_s, s) \mathrm{d}g(s). \tag{3.2.15}$$

On the other hand, since y is a solution of the IVP (3.2.5), we have $y_\omega = \varphi^{max}_\omega$, that is, $y(\theta + \omega) = \varphi^{max}(\theta + \omega)$ for all $\theta \in [-r, 0]$, which implies

$$y(\xi) = \varphi^{max}(\xi) \qquad \text{for all } \xi \in [\omega - r, \omega]. \tag{3.2.16}$$

Now, for $\theta \in [-r, 0]$, $s \in [\omega, t]$ and $t \in (\omega, \omega + \eta]$, we get $\omega - r \leqslant \theta + s \leqslant t \leqslant \omega + \eta$. In particular, $\theta + s \in [\omega - r, \omega + \eta] = [\omega - r, \omega] \cup (\omega, \omega + \eta]$. Now, consider two cases:

(i) If $\theta + s \in [\omega - r, \omega]$, then by (3.2.16), $y(\theta + s) = \varphi^{max}(\theta + s) = u(\theta + s)$ and, therefore, $y_s = u_s$.

(ii) If $\theta + s \in (\omega, \omega + \eta]$, then according to the definition of u, $y(\theta + s) = u(\theta + s)$ and, therefore, $y_s = u_s$.

Hence, $y_s = u_s$ for all $s \in [\omega, t]$ and all $t \in (\omega, \omega + \eta]$. From this, we get

$$\int_{\omega}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s) = \int_{\omega}^{t} a(t,s) f(u_s, s) \mathrm{d}g(s). \tag{3.2.17}$$

Thus, by (3.2.14), (3.2.15) and (3.2.17), we have

$$u(t) = \phi(0) + \int_{\tau_0}^{\omega} a(t,s) f(u_s, s) \mathrm{d}g(s) + \int_{\omega}^{t} a(t,s) f(u_s, s) \mathrm{d}g(s) \tag{3.2.18}$$

$$= \phi(0) + \int_{\tau_0}^{t} a(t,s) f(u_s, s) \mathrm{d}g(s) \tag{3.2.19}$$

for all $t \in (\omega, \omega + \eta]$. This implies that u is a solution of (3.2.1). It is easy to see that u is a proper prolongation of φ^{max}, which is assumed to be maximal. Therefore, we have a contradiction. Hence $\omega \notin I$ and $I = [\tau_0 - r, \omega)$, getting the desired result. □

The following result will be crucial to prove that the maximal solution is defined on $[\tau_0 - r, +\infty)$.

Lemma 3.2.5. *Assume $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ satisfies the conditions (B3) and (B4), $a\colon [t_0,+\infty)^2 \to \mathbb{R}$ satisfies condition (B2) and $g\colon [t_0,+\infty) \to \mathbb{R}$ satisfies condition (B1). If a is left–continuous with respect to the first variable, then for each $x \in G([\tau_0 - r, \beta], \mathbb{R}^n)$, $t_0 \leqslant \tau_0 < \beta$, the function*

$$[\tau_0, \beta] \ni t \mapsto \int_{\tau_0}^{t} a(t,s) f(x_s, s) dg(s)$$

is left–continuous on $(\tau_0, \beta]$, that is,

$$\lim_{t \to \eta^-} \int_{\tau_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) = \int_{\tau_0}^{\eta} a(\eta,s) f(x_s,s)\,\mathrm{d}g(s), \quad \eta \in (\tau_0, \beta].$$

Proof. Suppose that f, g and a satisfy the assumptions above. Using the same arguments as Step 2 of the proof of Lemma 3.1.7, we can prove that

$$\left\| \int_{\tau_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) - \int_{\tau_0}^{\tau} a(\tau,s) f(x_s,s)\,\mathrm{d}g(s) \right\| \leqslant |h(t) - h(\tau)|\,, \tag{3.2.20}$$

for all $t, \tau \in [\tau_0, \beta]$, where h is given by

$$h(t) := \int_{\tau_0}^{t} cM(s)\,\mathrm{d}g(s) + \int_{\tau_0}^{\beta} a(t,s) M(s)\,\mathrm{d}g(s), \quad t \in [\tau_0, \beta].$$

Here $c := \sup\limits_{(t,s) \in [\tau_0,\beta]^2} |a(t,s)|$. Notice that every point in $(\tau_0 - r, \beta]$ at which the function h is left–continuous is a left–continuity point of the function $t \mapsto \int_{\tau_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s)$.

In order to prove that h is left–continuous on $(\tau_0 - r, \beta]$, we will prove two statements.

Statement 1. $h_1(t) := \int_{\tau_0}^{t} cM(s)\,\mathrm{d}g(s)$, $t \in [t_0, t_0 + \sigma]$, is left–continuous on $(\tau_0 - r, \beta]$. Indeed, since g is left–continuous $(\tau_0 - r, \beta]$, by Lemma 1.2.6, $h_1(t) := \int_{\tau_0}^{t} cM(s)\,\mathrm{d}g(s)$ is left–continuous $(\tau_0 - r, \beta]$, proving statement 1.

Statement 2. $h_2(t) := \int_{\tau_0}^{\beta} a(t,s) M(s)\,\mathrm{d}g(s)$, $t \in [\tau_0, \beta]$, is left–continuous on $(\tau_0 - r, \beta]$.

Let $\eta \in (\tau_0 - r, \beta]$ and $(\tau_n)_{n\in\mathbb{N}} \subset (\tau_0, \eta]$ such that $\tau_n \overset{n\to\infty}{\to} \eta$. Define the sequence of functions

$$\varphi_n(s) := a(\tau_n, s)M(s), \qquad s \in [\tau_0, \beta], \tag{3.2.21}$$

and $\varphi\colon [\tau_0, \beta] \to \mathbb{R}$ by

$$\varphi(s) := a(\eta, s)M(s), \qquad s \in [\tau_0, \beta].$$

Since $a(\cdot, s)$ is left–continuous at η and $(\tau_n)_{n\in\mathbb{N}} \subset (\tau_0, \eta]$ is such that $\tau_n \overset{n\to\infty}{\to} \eta$, we have $\lim\limits_{n\to\infty} a(\tau_n, s) = a(\eta, s)$ and, therefore,

$$\lim_{n\to\infty} \varphi_n(s) = \lim_{n\to\infty} a(\tau_n, s)M(s) = a(\eta, s)M(s) = \varphi(s),$$

that is,

$$\lim_{n\to\infty} \varphi_n(s) = \varphi(s), \quad s \in [\tau_0, \beta].$$

According to condition (B3), $\int_{\tau_0}^{\beta} a(\tau_n, s)M(s)\,\mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. Using this fact together with the (3.2.21), we get

$$\int_{\tau_0}^{\beta} \varphi_n(s)\,\mathrm{d}g(s) \text{ exists for all } n \in \mathbb{N}.$$

On the other hand, for all $t \in [\tau_0, \beta]$, we have

$$|\varphi_n(t)| = |a(\tau_n, t)M(t)| \leqslant c\,|M(t)| = cM(t), \text{ for all } n \in \mathbb{N}.$$

This implies that $\kappa(t) \leqslant \varphi_n(t) \leqslant \omega(t), t \in [\tau_0, \beta]$, where $\omega(t) := cM(t)$ and $\kappa(t) = -cM(t)$. Also, observe that the integrals $\int_{\tau_0}^{\beta} \kappa(s)\,\mathrm{d}g(s)$ and $\int_{\tau_0}^{\beta} \omega(s)\,\mathrm{d}g(s)$ exist, since M is a locally Henstock–Kurzweil–Stieltjes integrable function. Since all the hypotheses of Theorem 1.2.11 are satisfied, we obtain

$$\lim_{n\to\infty} \int_{\tau_0}^{\beta} \varphi_n(s)\,\mathrm{d}g(s) = \int_{t_0}^{\beta} \varphi(s)\,\mathrm{d}g(s),$$

that is,

$$\lim_{n\to\infty} h_2(\tau_n) = h_2(\eta).$$

Hence, the function h_2 is left–continuous at η, for each $\eta \in (\tau_0, \beta]$, obtaining Statement 2.

Now, by Statements 1, 2 and the fact that $h(t) = h_1(t) + h_2(t)$, it follow that h is left–continuous on $(\tau_0, \beta]$. Using this fact together with (3.2.20), we have that the function $[\tau_0 - r, \beta] \ni t \mapsto \int_{\tau_0}^{t} a(t, s)f(x_s, s)\mathrm{d}g(s)$ is left–continuous on $(\tau_0 - r, \beta]$. $\square$

The next result provides conditions in order to ensure that the maximal solution is defined on $[\tau_0 - r, +\infty)$.

Theorem 3.2.6. *Assume* $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty)\to\mathbb{R}^n$ *satisfies the conditions (B3), (B4) and (B5),* $a\colon [t_0,+\infty)^2\to\mathbb{R}$ *satisfies condition (B2) and* $g\colon [t_0,+\infty)\to\mathbb{R}$ *satisfies condition (B1). Suppose* $\tau_0 \geqslant t_0$, $\phi\in G([-r,0],\mathbb{R}^n)$ *and* $x\colon [\tau_0-r,\omega)\to\mathbb{R}^n$ *is the maximal solution of the equation* (3.2.1). *If* a *is left–continuous with respect to the first variable, then* $\omega = +\infty$.

Proof. Suppose that the conclusion of the theorem is not true, i.e., $\omega < +\infty$.

Assertion 1. The limit $\lim\limits_{t\to\omega^-} x(t)$ exists.

By conditions (B2), (B3), (B4), Theorem 1.2.4, Corollary 1.2.5 and using the same arguments as in Step 2 of the proof of Lemma 3.1.7, we can prove that, for any $\tau_0 \leqslant u \leqslant t < \omega$, we have

$$\begin{aligned}
\|x(t)-x(u)\| &= \Big\| \int_{\tau_0}^{t} a(t,s)f(x_s,s)\,\mathrm{d}g(s) - \int_{\tau_0}^{u} a(u,s)f(x_s,s)\,\mathrm{d}g(s)\Big\| \\
&\leqslant \Big\| \int_{u}^{t} a(t,s)f(x_s,s)\,\mathrm{d}g(s)\Big\| + \Big\| \int_{\tau_0}^{u} \Big(a(t,s)-a(u,s)\Big) f(x_s,s)\,\mathrm{d}g(s)\Big\| \\
&\leqslant \int_{u}^{t} cM(s)\,\mathrm{d}g(s) + \int_{\tau_0}^{u} (a(t,s)-a(u,s))M(s)\,\mathrm{d}g(s) \\
&\leqslant \int_{u}^{t} cM(s)\,\mathrm{d}g(s) + \int_{\tau_0}^{\omega} (a(t,s)-a(u,s))M(s)\,\mathrm{d}g(s).
\end{aligned}$$

It implies that

$$\|x(t)-x(u)\| \leqslant |h(t)-h(u)|, \quad \text{for all } t,u\in[\tau_0,\omega), \tag{3.2.22}$$

where $h\colon [\tau_0,+\infty)\to\mathbb{R}$ is given by

$$h(\xi) := \int_{\tau_0}^{\xi} cM(s)\,\mathrm{d}g(s) + \int_{\tau_0}^{\omega} a(\xi,s)M(s)\,\mathrm{d}g(s), \tag{3.2.23}$$

for all $\xi\in[\tau_0,+\infty)$. Taking into account that M is a locally Henstock–Kurzweil–Stieltjes integrable function and using the same arguments as in the Remark 3.1.3, we infer that the integrals on the right–hand side of (3.2.23) are well–defined. Now, since $\omega\in(\tau_0,+\infty)$ and h is a nondecreasing function (which follows from definition), $\lim\limits_{t\to\omega^-} h(t)$ exists. Thus, given $\varepsilon > 0$, by the Cauchy Condition, there exists $\delta > 0$ (we can take $\tau_0 < \omega - \delta$) such that

$$|h(t)-h(s)| < \varepsilon, \qquad \text{for all } t,s\in(\omega-\delta,\omega). \tag{3.2.24}$$

Then, by (3.2.22) and (3.2.24), the following inequality holds

$$\|x(t) - x(u)\| \leqslant |h(t) - h(u)| < \varepsilon,$$

for every $t, u \in (\omega - \delta, \omega)$. Then, again by the Cauchy Condition, $\lim\limits_{t\to\omega^-} x(t)$ exists. Define $y\colon [\tau_0 - r, \omega] \to \mathbb{R}^n$ by

$$y(\tau) = \begin{cases} x(\tau), & \tau \in [\tau_0 - r, \omega) \\ \lim\limits_{t\to\omega^-} x(t), & \tau = \omega. \end{cases} \tag{3.2.25}$$

Obviously, $y \in G([\tau_0 - r, \omega], \mathbb{R}^n)$. By Lemma 3.2.5, we have

$$\lim_{t\to\omega^-} \int_{\tau_0}^{t} a(t,s) f(y_s, s)\,\mathrm{d}g(s) = \int_{\tau_0}^{\omega} a(\omega, s) f(y_s, s)\,\mathrm{d}g(s). \tag{3.2.26}$$

Therefore, we get

$$\begin{aligned} x(t) &= \phi(0) + \int_{\tau_0}^{t} a(t,s) f(x_s, s)\mathrm{d}g(s) \\ &= \phi(0) + \int_{\tau_0}^{t} a(t,s) f(y_s, s)\mathrm{d}g(s), \quad t \in [\tau_0, \omega), \end{aligned}$$

then

$$\lim_{t\to\omega^-} x(t) = \phi(0) + \lim_{t\to\omega^-} \int_{\tau_0}^{t} a(t,s) f(y_s, s)\mathrm{d}g(s).$$

Hence, by (3.2.25) and (3.2.26), we obtain

$$y(\omega) = \phi(0) + \int_{\tau_0}^{\omega} a(\omega, s) f(y_s, s)\mathrm{d}g(s).$$

Thus y is a solution of the equation (3.2.1) and also, it is a proper prolongation of x, which is assumed to be maximal. Therefore, we have a contradiction. Hence $\omega = +\infty$. □

3.3 Existence and uniqueness of solutions of impulsive equations

In this section, we use the previous results to ensure the existence and uniqueness of solutions for impulsive functional Volterra–Stieltjes integral equations, using the correspondence presented in Chapter 2.

Theorem 3.3.1. *Let $m \in \mathbb{N}$ and $t_0 \leqslant t_1 < \cdots < t_m < d$. Assume that $g\colon [t_0, d) \to \mathbb{R}$ is a regulated left–continuous function which is continuous at $t_1, \ldots, t_m$, $a\colon [t_0, d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable, locally bounded on $[t_0, d)^2$ and, continuous with respect to first variable at $t_1, \ldots, t_m$. Also, suppose that $I_1, \ldots, I_m\colon \mathbb{R}^n \to \mathbb{R}^n$ and $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, d) \to \mathbb{R}^n$ satisfy the following conditions:*

1. *The integral $\int_{u_1}^{u_2} a(t,s) f(x_s, s) \mathrm{d}g(s)$ exists in the sense of Henstock–Kurzweil–Stieltjes, for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$, all $x \in G([\tau_0-r, \tau_0+\sigma], \mathbb{R}^n)$, $t \in [t_0, d)$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$.*

2. *There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M_1\colon [t_0, d) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, d)$, we have*
$$\left\| \int_{u_1}^{u_2} b(u_2, s) f(x_s, s) \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} M_1(s) \left| b(u_2, s) \right| \mathrm{d}g(s),$$
for all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $b \in G_2([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$, and there exists a constant $M_2 > 0$ such that
$$\|I_k(x)\| \leqslant M_2$$
for every $k \in \{1, \ldots, m\}$ and $x \in \mathbb{R}^n$.

3. *There exists a regulated function $L_1\colon [t_0, d) \to \mathbb{R}^+$ such that*
$$\left\| \int_{u_1}^{u_2} a(u_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_1(s) \left| a(u_2, s) \right| \|x_s - z_s\|_\infty \mathrm{d}g(s),$$
for all $x, z \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$ and there exists a constant $L_2 > 0$ such that
$$\|I_k(x) - I_k(y)\| \leqslant L_2 \left| x - y \right|$$
for every $k \in \{1, \ldots, m\}$ and x, $y \in \mathbb{R}^n$.

Then for all $\phi \in G([-r,0],\mathbb{R}^n)$ *there exist a* $\sigma > 0$ *and a unique solution* $x\colon [t_0-r, t_0+\sigma] \to \mathbb{R}^n$ *of the initial value problem:*

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<t}} a(t,t_k) I_k(x(t_k)) \\ x_{t_0} &= \phi. \end{cases} \tag{3.3.1}$$

Proof. Let $\phi \in G([-r,0],\mathbb{R}^n)$ be given. Define $\tilde{f}\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$ by

$$\tilde{f}(y,\tau) = \begin{cases} f(y,\tau), & \tau \in [t_0,d)\backslash\{t_1,\ldots,t_m\}, \\ I_k(y(0)), & \tau = t_k,\ k \in \{1,\ldots,m\}, \end{cases}$$

and $\tilde{g}\colon [t_0,d) \to \mathbb{R}$ by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [t_0,t_1], \\ g(\tau)+k, & \tau \in (t_k,t_{k+1}],\ k \in \{1,\ldots,m-1\}, \\ g(\tau)+m, & \tau \in (t_m,d). \end{cases}$$

According to Lemma 2.1.1, we see that the functions $\tilde{f}$, $\tilde{g}$, and a satisfy all the hypotheses of Theorem 3.1.9. Hence, there exist $\sigma > 0$ and a unique solution $x\colon [t_0-r,t_0+\sigma] \to \mathbb{R}^n$ of the functional Volterra–Stieltjes integral equation

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} &= \phi. \end{cases}$$

Now, by Theorem 2.1.2, the function x is also a unique solution of the impulsive functional Volterra–Stieltjes integral equation

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<t}} a(t,t_k) I_k(x(t_k)) \\ x_{t_0} &= \phi \end{cases}$$

proving the desired result. □

The next result gives us sufficient conditions to ensure the existence and uniqueness of maximal solution of impulsive functional Volterra–Stieltjes integral equation.

Theorem 3.3.2. *Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that $g\colon [t_0, \infty) \to \mathbb{R}$ is a regulated left–continuous function which is continuous at $\{t_k\}_{k=1}^{\infty}$, $a\colon [t_0, \infty)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable, locally bounded on $[t_0, \infty)^2$ and, continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Also, suppose that $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{N}$, and $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, \infty) \to \mathbb{R}^n$ satisfy the following conditions:*

1. *The integral $\int_{u_1}^{u_2} a(t, s) f(x_s, s) \mathrm{d}g(s)$ exists in the sense of Henstock–Kurzweil–Stieltjes for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, \infty)$, all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $t \in [t_0, \infty)$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$.*

2. *There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M_1\colon [t_0, \infty) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, \infty)$, we have*
$$\left\| \int_{u_1}^{u_2} b(u_2, s) f(x_s, s) \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} M_1(s) \left| b(u_2, s) \right| \mathrm{d}g(s),$$
for all $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$, $b \in G_2([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$, and there exists a constant $M_2 > 0$ such that
$$\|I_k(x)\| \leqslant M_2$$
for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$.

3. *There exists a regulated function $L_1\colon [t_0, \infty) \to \mathbb{R}^+$ such that*
$$\left\| \int_{u_1}^{u_2} a(u_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_1(s) \left| a(u_2, s) \right| \|x_s - z_s\|_{\infty} \mathrm{d}g(s),$$
for all $x, z \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ and $t_0 \leqslant u_1 \leqslant u_2 \leqslant \tau_0 + \sigma$ and there exists a constant $L_2 > 0$ such that
$$\|I_k(x) - I_k(y)\| \leqslant L_2 \|x - y\|$$
for every $k \in \mathbb{N}$ and x, $y \in \mathbb{R}^n$.

Then for all $\phi \in G([-r,0],\mathbb{R}^n)$ *there exists a unique maximal solution* $x\colon I \to \mathbb{R}^n$ *of the initial value problem:*

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} a(t,t_k) I_k(x(t_k)), \quad t \geqslant t_0 \\ x_{t_0} = \phi, \end{cases} \tag{3.3.2}$$

where $I = [t_0 - r, \omega), \omega \leqslant \infty$.

Proof. Let $\phi \in G([-r,0],\mathbb{R}^n)$ be given. Define $\tilde{f}\colon G([-r,0],\mathbb{R}^n) \times [t_0,\infty) \to \mathbb{R}^n$ by

$$\tilde{f}(y,\tau) = \begin{cases} f(y,\tau), & \tau \in [t_0,\infty)\backslash\{t_k\}_{k=1}^{\infty}, \\ I_k(y(0)), & \tau = t_k, \quad k \in \mathbb{N}, \end{cases}$$

and $\tilde{g}\colon [t_0,\infty) \to \mathbb{R}$ by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [t_0,t_1], \\ g(\tau) + k, & \tau \in (t_k,t_{k+1}], \quad k \in \mathbb{N}. \end{cases}$$

According to Lemma 2.1.1, we see that the functions $\tilde{f}$, $\tilde{g}$, and a satisfy all the hypotheses of Theorem 3.2.4. Therefore, there exists a unique maximal solution $x\colon I \to \mathbb{R}^n$, $I = [t_0 - r, \omega)$, of the functional Volterra–Stieltjes integral equation

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} = \phi. \end{cases} \tag{3.3.3}$$

Now, by Theorem 2.1.2, the function x is also a unique solution of the impulsive functional Volterra–Stieltjes integral equation

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) + \sum_{k\in\mathbb{N};t_k<t} a(t,t_k) I_k(x(t_k)) \\ x_{t_0} = \phi. \end{cases} \tag{3.3.4}$$

It is clear that x is a maximal solution of (3.3.4), otherwise there would be $y\colon J \to \mathbb{R}^n$, $I \subsetneq J$, such that y is a solution of (3.3.4). However, Theorem 2.1.2 would imply that y is a solution of (3.3.3), contradicting the maximality of x.

□

3.4 Existence and uniqueness of solutions Δ-integral equations on time scales

In this section, we use the previous results to ensure the existence and uniqueness of solutions for functional Volterra Δ-integral equations on time scales. We begin by recalling the assumed conditions on Section 2.2.

(C1) The function $a\colon [t_0, d)_{\mathbb{T}}^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and rd–continuous with respect to the first variable.

(C2) The Henstock–Kurzweil Δ–integral

$$\int_{s_1}^{s_2} a(\tau, s) f(x_s, s) \Delta s$$

exists for each time scale interval $[s_0, s_0 + \delta]_{\mathbb{T}} \subset [t_0, d)_{\mathbb{T}}$, $x \in G([s_0 - r, s_0 + \delta], \mathbb{R}^n)$, $\tau \in [s_0, s_0 + \delta]_{\mathbb{T}}$ and $s_1, s_2 \in [s_0, s_0 + \delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(C3) There exists a locally Henstock–Kurzweil Δ–integrable function $M_1\colon [t_0, d)_{\mathbb{T}} \to \mathbb{R}^+$ such that for each time scale interval $[s_0, s_0 + \delta]_{\mathbb{T}} \subset [t_0, d)_{\mathbb{T}}$, we have

$$\left\| \int_{s_1}^{s_2} (c_1 a(s_2, s) + c_2 a(s_1, s)) f(x_s, s) \Delta s \right\| \leqslant \int_{s_1}^{s_2} M_1(s) \left| c_1 a(s_2, s) + c_2 a(s_1, s) \right| \Delta s,$$

for all $x \in G([s_0 - r, s_0 + \delta], \mathbb{R}^n)$, $c_1, c_2 \in \mathbb{R}$ and $s_1, s_2 \in [s_0, s_0 + \delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(C4) There exists a locally regulated function $L_1\colon [t_0, d)_{\mathbb{T}} \to \mathbb{R}^+$ such that for each time scale interval $[s_0, s_0 + \delta]_{\mathbb{T}} \subset [t_0, d)_{\mathbb{T}}$, we have

$$\left\| \int_{s_1}^{s_2} a(s_2, s) [f(x_s, s) - f(z_s, s)] \Delta s \right\| \leqslant \int_{s_1}^{s_2} L_1(s) \left| a(s_2, s) \right| \left\| x_s - z_s \right\|_{\infty} \Delta s,$$

for all $x, z \in G([s_0 - r, s_0 + \delta], \mathbb{R}^n)$ and $s_1, s_2 \in [s_0, s_0 + \delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

The next result gives sufficient conditions to ensure the existence and uniqueness of solutions of functional Volterra Δ-integral equations on time scales.

Theorem 3.4.1. *Let $[t_0-r,t_0]_{\mathbb{T}}$ be a time scale interval and let $d\in\mathbb{T}$ be a left dense point such that $d>t_0$. Assume $a\colon [t_0,d)^2_{\mathbb{T}}\to\mathbb{R}$ satisfies condition (C1), and $f\colon G([-r,0],\mathbb{R}^n)\times [t_0,d)_{\mathbb{T}}\to\mathbb{R}^n$ satisfies conditions (C2)–(C4). Then, for all $\phi\in G([t_0-r,t_0]_{\mathbb{T}},\mathbb{R}^n)$ there exists $\eta>0$ such that $\eta\geqslant\mu(t_0)$ and $t_0+\eta\in\mathbb{T}$, and a function $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}}\to\mathbb{R}^n$ which is a unique solution of the functional Volterra Δ–integral equation on time scales given by*

$$\begin{cases} x(t) &= x(t_0)+\displaystyle\int_{t_0}^{t} a(t,s)f(x_s^*,s)\Delta s, \quad t\in[t_0,t_0+\eta]_{\mathbb{T}},\\ x(t) &= \phi(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}}. \end{cases} \tag{3.4.1}$$

Proof. Define the functions $g(s)=s^*$ for $s\in[t_0,d)$, $f^*(\psi,s)=f(\psi,s^*)$ for $s\in[t_0,d)$ and $\psi\in G([-r,0],\mathbb{R}^n)$ and $a^{**}(t,s)=a(t^*,s^*)$ for $t,s\in[t_0,d)$. Using the hypotheses and Lemma 2.2.2, we get that f^*, a^{**}, $\phi^*_{t_0}$ and g satisfy all conditions of Theorem 3.9. Hence, there exists $\beta>0$ and a unique solution $y\colon [t_0-r,t_0+\beta]\to\mathbb{R}^n$ of the functional Volterra–Stieltjes integral equation

$$\begin{cases} y(t) &= y(t_0)+\displaystyle\int_{t_0}^{t} a^{**}(t,s)f^*(y_s,s)\,\mathrm{d}g(s)\\ y_{t_0} &= \phi^*_{t_0}. \end{cases} \tag{3.4.2}$$

If t_0 is right–dense, then there exists $\tau\in\mathbb{T}$ such that $t_0<\tau<t_0+\beta$. Define $\eta:=\tau-t_0$. Notice that $\eta>0$ and $t_0+\eta=\tau\in\mathbb{T}$. Since $[t_0-r,t_0+\eta]\subset[t_0-r,t_0+\beta]$, $y|_{[t_0-r,t_0+\eta]}$ is also a solution of (3.4.2) (on $[t_0-r,t_0+\eta]$). Then, by Theorem 2.2.1, $y|_{[t_0-r,t_0+\eta]}=x^*$, where $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}}\to\mathbb{R}^n$ is a solution of the equation (3.4.1). Again by Theorem 2.2.1, we conclude that x is the unique solution of the functional Volterra Δ–integral equation on time scales (3.4.1).

If t_0 is right–scattered, then without loss of generality, we can assume that $\eta\geqslant\mu(t_0)$; otherwise, let $y(\sigma(t_0))=\phi(t_0)+f(\phi^*_{t_0},t_0)\mu(t_0)$ to obtain a solution defined on $[t_0-r,t_0+\mu(t_0)]_{\mathbb{T}}$. Then, as the same way as before, by Theorem 2.2.1, $y|_{[t_0-r,t_0+\eta]}=x^*$, where $x\colon [t_0-r,t_0+\eta]_{\mathbb{T}}\to\mathbb{R}^n$ is a solution of Equation (3.4.1). Again by Theorem 2.2.1, we conclude that x is the unique solution of the functional Volterra Δ–integral equation on time scales (3.4.1). □

The next result ensures that there exists a unique maximal solution of the functional Volterra Δ-integral equation on time scales.

Theorem 3.4.2. *Let $\mathbb{T}$ be a time scale such that* $\sup \mathbb{T} = \infty$ *and* $t_0 - r, t_0 \in \mathbb{T}$. *Let* $[t_0 - r, t_0]_{\mathbb{T}}$ *be a time scale interval. Assume* $a\colon [t_0, \infty)^2_{\mathbb{T}} \to \mathbb{R}$ *satisfies condition (C1), where* $d = \infty$, *and* $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies conditions (C2)–(C4), where* $d = \infty$. *Then, for all* $\phi \in G([t_0 - r, t_0]_{\mathbb{T}}, \mathbb{R}^n)$ *there exists a function* $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$, $\omega \leqslant \infty$, *which is a unique maximal solution of the functional Volterra* Δ*–integral equation on time scales given by*

$$\begin{cases} x(t) &= x(t_0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s^*, s) \Delta s, \quad t \in [t_0, t_0 + \omega^*)_{\mathbb{T}}, \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{cases} \tag{3.4.3}$$

Also, if $\omega < \infty$, *then* $\omega \in \mathbb{T}$ *and* ω *is left-dense.*

Proof. Define the functions $g(s) = s^*$ for $s \in [t_0, \infty)$, $f^*(\psi, s) = f(\psi, s^*)$ for $s \in [t_0, \infty)$ and $\psi \in G([-r, 0], \mathbb{R}^n)$ and $a^{**}(t, s) = a(t^*, s^*)$ for $t, s \in [t_0, \infty)$. Using the hypotheses and Lemma 2.2.2, we get that f^*, a^{**}, $\phi^*_{t_0}$ and g satisfy all conditions of Theorem 3.2.4. Hence, there exists $\omega \leqslant \infty$ and a unique solution $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$ of the functional Volterra–Stieltjes integral equation

$$\begin{cases} y(t) &= y(t_0) + \displaystyle\int_{t_0}^{t} a^{**}(t,s) f^*(y_s, s)\, \mathrm{d}g(s) \\ y_{t_0} &= \phi^*_{t_0}. \end{cases} \tag{3.4.4}$$

Let us consider two cases:

Case 1: $\omega = \infty$

By Theorem 2.2.1, $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$ must have the form $y - x^*$, where $x\colon [t_0 - r, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional Volterra $\Delta-$integral equation on time scales (3.4.3). Clearly, x is a maximal solution of the functional Volterra $\Delta-$integral equation on time scales (3.4.3).

Case 2: $\omega < \infty$

Assertion 1: $\omega \in \mathbb{T}$

Suppose that $\omega \notin \mathbb{T}$ and define $B := \{s \in \mathbb{T} : s < \omega\}$. Clearly, B is nonempty, since $t_0 \in B$. Since $\omega \notin \mathbb{T}$, $B = \mathbb{T} \cap (-\infty, \omega]$ and thus, this implies that B is a closed subset of $\mathbb{R}$.

Denote $\beta := \sup B$. Since B is closed, $\beta \in B$. By the definition of B, $\beta \leqslant \omega$, but since $\omega \notin \mathbb{T}$, we have that $\beta < \omega$.

By the definition of g, it follows that g is constant on $(\beta, \omega]$ and, therefore,

$$\int_s^t a(t,s)f(y_s,s)\mathrm{d}g(s) = 0$$

for all $t, s \in (\beta, \omega]$.

Let $\sigma \in (\beta, \omega)$ be fixed and define a function $u\colon [t_0 - r, \omega) \to \mathbb{R}^n$ by

$$u(t) = \begin{cases} \phi(t), & t \in [t_0 - r, t_0] \\ y(t), & t \in [t_0, \omega) \\ y(\sigma), & t = \omega. \end{cases} \tag{3.4.5}$$

Note that u is well-defined and $u|_{[t_0,\omega)} = y$.

Assertion 1.1: The function u defined by (3.4.5) is a solution of the functional Volterra–Stieltjes integral equation (3.4.4) on $[t_0 - r, \omega]$.

Clearly, u satisfies the initial condition of (3.4.4) by definition.

Now, let $s_1, s_2 \in [t_0, \omega]$ be such that $s_1 \in [t_0, \omega)$ and $s_2 = \omega$. Then,

$$\begin{aligned} u(s_2) - u(s_1) &= y(\sigma) - y(s_1) \\ &= \int_{s_1}^{\sigma} a^{**}(t,s)f^*(y_s,s)\mathrm{d}g(s) \\ &= \int_{s_1}^{\sigma} a^{**}(t,s)f^*(y_s,s)\mathrm{d}g(s) + \int_{\sigma}^{\omega} a^{**}(t,s)f^*(y_s,s)\mathrm{d}g(s) \\ &= \int_{s_1}^{\omega} a^{**}(t,s)f^*(y_s,s)\mathrm{d}g(s) \\ &= \int_{s_1}^{s_2} a^{**}(t,s)f^*(u_s,s)\mathrm{d}g(s), \end{aligned} \tag{3.4.6}$$

which implies that:

$$u(s_2) - u(s_1) = \int_{s_1}^{s_2} a^{**}(t,s)f^*(u_s,s)\mathrm{d}g(s)$$

for all $s_1, s_2 \in [t_0, \omega]$ such that $s_1 \in [t_0, \omega)$ and $s_2 = \omega$.

It remains to check only the case where $s_1, s_2 \in [t_0, \omega)$. This follows immediately from the definition of u and from the fact the y is a solution of (3.4.4).

Then, u is a solution of (3.4.4) on $[t_0 - r, \omega]$. Therefore, it implies that u is a proper prolongation of $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$ to the right, which contradicts the fact that y is the maximal solution of the functional Volterra–Stieltjes integral equation (3.4.4).

From this, we conclude that $\omega \in \mathbb{T}$ and we prove the Assertion 1.

Assertion 2: ω is left-dense

Suppose the contrary, that is, $\rho(\omega) = \sup\{s \in \mathbb{T} : s < \omega\} < \omega$.

By the definition of g, we get that g is constant on $(\rho(\omega), \omega]$. Hence, arguing the same way as in the proof of Assertion 1, taking $\beta = \rho)(\omega)$, we can prove that there exists a proper prolongation of $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$ to the right, which contradicts the fact that y is the maximal solution of (3.4.4). Thus, we have that ω is left-dense.

Now, note that by Theorem 2.2.1, $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$ must have the form $y = x^*$ where $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional Volterra Δ-integral equation on time scales (3.4.3).

Assertion 3: $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ is a maximal solution of (3.4.3).

Suppose the contrary, that is, let $z\colon J_{\mathbb{T}} \to \mathbb{R}^n$ be a proper prolongation of $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ to the right. Then, without loss of generality, consider $J_{\mathbb{T}} = [t_0 - r, \omega]_{\mathbb{T}}$. Since $z\colon [t_0 - r, \omega]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of (3.4.3), Theorem 2.2.1 implies that $z^*\colon [t_0 - r, \omega] \to \mathbb{R}^n$ is a solution of functional Volterra–Stieltjes integral equation (3.4.4). On the other hand, notice that $z^*|_{[t_0 - r, \omega)} = y$. It implies that $z^*\colon [t_0 - r, \omega] \to \mathbb{R}^n$ is a proper prolongation of $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$, which contradicts the fact that y is the maximal solution of the functional Volterra–Stieltjes integral equation (3.4.4). Hence, it follows that $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ is a maximal solution of functional Volterra Δ-integral on time scales (3.4.3), proving the Assertion 3.

Now, it remains to prove the uniqueness of the maximal solution x. Suppose that $v\colon L_{\mathbb{T}} \to \mathbb{R}^n$ is also a maximal solution of (3.4.3).

Assertion 4: $x(t) = v(t)$ for all $t \in [t_0 - r, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$.

Indeed, by Theorem 2.2.1, $v^*\colon L \to \mathbb{R}^n$ is a solution of the functional Volterra–Stieltjes integral equation (3.4.4). On the other hand, $y\colon [t_0 - r, \omega) \to \mathbb{R}^n$ is the maximal solution of (3.4.4). It implies that $y(t) = v^*(t)$ for every $t \in [t_0 - r, \omega) \cap L$.

In particular, since $[t_0 - r, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}} = [t_0 - r, \omega) \cap L \cap \mathbb{T} \subset [t_0 - r, \omega) \cap L$, we have that $y(t) = v^*(t)$ for all $[t_0 - r, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$, which implies that, for $t \in [t_0 - r, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$,

$$x(t) = x(t^*) = x^*(t) = y(t) = v^*(t) = v(t^*) = v(t),$$

that is, $x(t) = v(t)$ for all $t \in [t_0 - r, \omega)_{\mathbb{T}} \cap L_{\mathbb{T}}$.

This concludes the proof of Assertion 4.

Now, define $\lambda\colon E_{\mathbb{T}} \to \mathbb{R}^n$, $E = [t_0 - r, \omega) \cup L$, by:

$$\lambda(t) = \begin{cases} x(t), & t \in [t_0 - r, \omega)_{\mathbb{T}} \\ v(t), & t \in L_{\mathbb{T}}. \end{cases} \tag{3.4.7}$$

By Assertion 4, λ is well defined. Clearly, λ is a solution of (3.4.4) and the time scales intervals $[t_0 - r, \omega)_{\mathbb{T}}$ and $L_{\mathbb{T}}$ are contained in $E_{\mathbb{T}}$. Also:

$$\begin{cases} \lambda|_{[t_0 - r, \omega)_{\mathbb{T}}} = x \\ \lambda|_{L_{\mathbb{T}}} = v. \end{cases} \tag{3.4.8}$$

Since x and v are maximal solutions of (3.4.3), it follows that $E_{\mathbb{T}} = [t_0 - r, \omega)_{\mathbb{T}} = L_{\mathbb{T}}$ and $\lambda(t) = x(t) = v(t)$ for all $t \in E_{\mathbb{T}}$, that is, $x(t) = v(t)$ for all $t \in E_{\mathbb{T}} = [t_0 - r, \omega)_{\mathbb{T}} = L_{\mathbb{T}}$, proving the uniqueness of x. $\square$

As an immediate consequence, we obtain the following corollary:

Corollary 3.4.3. *Let* $\mathbb{T}$ *be a time scale such that* $\sup \mathbb{T} = \infty$. *Let* $[t_0 - r, t_0]_{\mathbb{T}}$ *be a time scale interval. Assume* $a\colon [t_0, \infty)^2 \to \mathbb{R}$ *satisfies condition (C1), where* $d = \infty$ *and* $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies conditions (C2)–(C4), where* $d = \infty$. *Let* $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ *be the maximal solution of the functional Volterra* Δ*-integral equation o time scales* (3.4.3) *(ensured by Theorem 3.4.2). If each point of* $\mathbb{T}$ *is left scattered, then* $\omega = \infty$.

Proof. Suppose the contrary, that is, $\omega < \infty$. Then, Theorem 3.4.2 implies that $\omega \in \mathbb{T}$ and ω is left-dense, which contradicts the hypothesis that each point of $\mathbb{T}$ is left-scattered. $\square$

Chapter 4

Stability of solutions

We begin this chapter by considering our usual functional Volterra–Stieltjes integral equation:

$$\begin{cases} x(t) & = \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s)\mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} & = \phi, \end{cases} \tag{4.0.1}$$

where $r \geqslant 0$, $t_0 < d \leqslant \infty$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$ is regulated, $g\colon [t_0,d) \to \mathbb{R}$ is nondecreasing and left–continuous, $a\colon [t_0,d)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable and left–continuous with respect to the second variable and $\phi \in G([-r,0],\mathbb{R}^n)$. To ensure that the problem (4.0.1) makes sense, we assume that the Henstock–Kurzweil–Stieltjes integral, which appears in the right–hand side, $\int_{\tau_1}^{\tau_2} a(t,s)f(x_s,s)\mathrm{d}g(s)$ exists for each compact interval $[\tau_1,\tau_2] \subset [t_0,d)$, for all $x \in G([t_0-r,d),\mathbb{R}^n)$, $t \in [t_0,d)$ and all $t_0 \leqslant \tau_1 \leqslant \tau_2 < d$.

In this chapter, we will investigate four types of stability for our equation: stability, asymptotic stability, uniform stability and exponential stability. We will use Lyapunov functionals to study these types of stability. In what follows, we will adopt the notation $B(0,\xi)$ to denote the set $\{x \in \mathbb{R}^n \ : \|x\| < \xi\}$ for some positive real number ξ.

Now, we present the definitions of stability, asymptotic stability, uniform stability and uniform asymptotic stability for equation (4.0.1). We will denote by $x(t) = x(t,t_0,\phi)$, $t \in [t_0-r,+\infty)$, the unique solution of (4.0.1) and x_t by $x_t(t_0,\phi)$. The existence of the solution is guaranteed by Theorem 3.2.4.

Definition 4.0.1. The trivial solution $x \equiv 0$ of (4.0.1) is said to be:

- *stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$, such that for all $\phi \in B(0, \delta) \subset G([-r, 0], \mathbb{R}^n)$, $x_t(t_0, \phi) \in B(0, \varepsilon) \subset G([-r, 0], \mathbb{R}^n)$ for all $t \geqslant t_0$.

- *asymptotically stable* if it is stable and there exists $\eta > 0$, such that if $\phi \in B(0, \eta) \subset G([-r, 0], \mathbb{R}^n)$, then $x(t_0, \phi)(t) \to 0$ when $t \to \infty$.

- *uniformly stable* if it is stable with $\delta > 0$ independent of t_0.

- *uniformly asymptotically stable* if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists $T = T(\varepsilon) \geqslant 0$ such that if $t_0 \geqslant 0$ and $\phi \in B(0, \delta_0) \subset G([-r, 0], \mathbb{R}^n)$, then $x_t(t_0, \phi) \in B(0, \varepsilon) \subset G([-r, 0], \mathbb{R}^n)$ for all $t \in [t_0, w(t_0, \phi)] \cap [t_0 + T, \infty)$.

In order to Definition 4.0.1 to make sense, we assume that $f(0, t) \equiv 0$. With these assumptions, $x \equiv 0$ is a solution of (4.0.1).

The investigations made by A. M. Lyapunov more than a hundred years ago are still very important and relevant in many different problems. The first publication of Lyapunov concerning stability of motion of systems with a finite number of degrees of freedom were in 1888. Four years later, Lyapunov presented a rigorous definition of stability, which was part of his PhD thesis entitled *"The General Problem of Stability of Motion"*.

Until nowadays, the work of many mathematicians receives his influence and to understand the stability of solutions of certain equation is among the most studied topics in the last years, and his techniques are applied in most of cases.

On the other hand, the investigations concerning stability for Stieltjes integral equations type are very recent. First, because these equations started to be studied in the last few years and the second reason for that comes from the fact that it is not easy to deal with Stieltjes integral equations, since their solutions are not continuous in most of the cases. Therefore, the discontinuities which appear can complicate a lot the study of the dynamic of the solution. Also, since the solution of a Stieltjes integral equation does not need to be differentiable, this turns the problem even more complicated.

However, although the difficulties behind such problem, some authors investigated the classical concepts of stability for these equations by using Lyapunov functionals (see [22, 28]). As a consequence, they obtained more general results than the ones found in the literature, that could allow that the Lyapunov functional to be very general.

All the results presented here, even the ones that employ Lyapunov functionals, are more general than the ones presented previously in the literature. It follows directly from the generality of the equation that we are considering.

We divide this Chapter into two sections. The results of the first section investigate many types of stability using Lyapunov functionals. They are inspired by the results of [28] and can be found in [33, 46]. In the second section, we show that the theorem present in the first section are also valid when dealing with impulsive functional Volterra–Stieltjes integral equations.

4.1 Lyapunov's Second Method

Here, we consider the particular case of (4.0.1) given by

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s)\mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi, \end{cases} \tag{4.1.1}$$

where $r \geqslant 0$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,\infty) \to \mathbb{R}^n$ is such that $f(0,t) \equiv 0$ for all $t \in [t_0,\infty)$, $a\colon [t_0,\infty) \times [t_0,\infty) \to \mathbb{R}$ is regulated with respect to the second variable, $g\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing left–continuous function, $\phi \in G([-r,0],\mathbb{R}^n)$ and the integral in the right–hand side is understood on the sense of Henstock–Kurzweil–Stieltjes.

Below, we recall important definitions.

Definition 4.1.1. We say that $V\colon [t_0,\infty) \times \overline{B}_\rho \to \mathbb{R}$ is a *Lyapunov functional* with respect to (4.1.1), where $\overline{B}_\rho = \{y \in G([-r,0],\mathbb{R}^n) : \|y\| \leqslant \rho\}$, $\rho > 0$, if the following conditions are satisfied:

(V1) For every solution x of (4.1.1), the function from $[t_0,\infty)$ to $\mathbb{R}$ defined by $t \mapsto V(t,x_t)$ is continuous;

(V2) For every $(t_0,\phi) \in \mathbb{R} \times \overline{B}_\rho$, the function defined by $t \mapsto V(t, x_t(t_0,\phi))$ is nonincreasing, where x is the unique maximal solution of (4.1.1) with initial condition $x_{t_0} = \phi$.

With these definitions in hands, we are ready to state and prove our first result of stability, which ensures that the trivial solution of (4.1.1) is stable.

Theorem 4.1.2. *Let* $t_0 \in \mathbb{R}$ *and suppose that there exists a Lyapunov functional* $V\colon [t_0,\infty)\times \overline{B}_\rho \to \mathbb{R}$, $\overline{B}_\rho = \{y \in G([-r,0],\mathbb{R}^n) : \|y\| \leqslant \rho\}$, $\rho > 0$, *with respect to* (4.1.1) *such that*

1. $V(t,0) = 0$ *for all* $t \in [t_0,\infty)$;

2. *If* $x\colon [t_0 - r,\infty) \to \mathbb{R}^n$ *is a solution of* (4.1.1), *then* $\alpha(\|x_t\|) \leqslant V(t,x_t)$ *for all* $t \in [t_0,\infty)$, *where* $\alpha\colon [0,\infty) \to [0,\infty)$ *is an increasing function such that* $\alpha(0) = 0$ *and* $\lim_{s\to\infty} \alpha(s) = \infty$.

Then the trivial solution $x \equiv 0$ *of* (4.1.1) *is stable.*

Proof. Let $\varepsilon > 0$. We want to show that there exists $\delta > 0$ such that if $\phi \in B(0,\delta)$, then the solution x of (4.1.1) exists for all $t \geqslant t_0$ and $x_t(t_0,\phi) \in B(0,\varepsilon)$ for all $t \geqslant t_0$. The existence is ensured by Condition 2 and the properties of the Lyapunov functional.

According to item (V1) from the Definition 4.1.1 and by hypothesis, given $\alpha(\varepsilon) > 0$, there exists $\delta := \delta(t_0,\varepsilon) > 0$ such that if $\|x\|_\infty < \delta$, then

$$V(t_0,x_{t_0}) < \alpha(\varepsilon). \tag{4.1.2}$$

Let $\phi \in B(0,\delta) \subset G([-r,0],\mathbb{R}^n)$. By hypothesis and (4.1.2), for all $t \in [t_0,\infty)$, we have

$$\alpha(\|x_t(t_0,\phi)\|) \leqslant V(t,x_t) \leqslant V(t_0,x_{t_0}) = V(t_0,\phi) < \alpha(\varepsilon), \tag{4.1.3}$$

where the second inequality follows from the fact that $t \mapsto V(t,x_t)$ is nonincreasing. Since α is an increasing function, $\|x_t(t_0,\phi)\| < \varepsilon$ for all $t \geqslant t_0$, as desired, and the result follows. □

The next result gives sufficient conditions to ensure that the trivial solution of (4.1.1) is asymptotically stable.

Theorem 4.1.3. *Let* $t_0 \in \mathbb{R}$ *and suppose there exists a Lyapunov functional* $V\colon [t_0,\infty)\times \overline{B}_\rho \to \mathbb{R}$ *with respect to* (4.1.1) *and a function* $\alpha\colon [0,\infty) \to [0,\infty)$ *that satisfy all the conditions of Theorem 4.1.2. Suppose also that, for all nonextendable solution* $x\colon [t_0 - r,\infty) \to \mathbb{R}^n$ *of* (4.1.1), *we have*

$$V(t,x_t) - V(s,x_s) \leqslant -\int_s^t \widetilde{\alpha}(V(\xi,x_\xi))\mathrm{d}\gamma(\xi) \tag{4.1.4}$$

for all $t, s \in [t_0, \infty)$ *with* $t \geqslant s$, *where* $\gamma\colon [t_0, \infty) \to \mathbb{R}$ *is a nondecreasing function such that* $\lim_{t\to\infty} \gamma(t) = \infty$ *and* $\widetilde{\alpha}\colon [t_0, \infty) \to [t_0, \infty)$ *is an increasing function such that* $\widetilde{\alpha}(0) = 0$ *and* $\lim_{s\to\infty} \widetilde{\alpha}(s) = \infty$. *Then the trivial solution* $x \equiv 0$ *of* (4.1.1) *is asymptotically stable.*

Proof. Since all the hypotheses of Theorem 4.1.2 are satisfied, it follows that $x \equiv 0$ is stable. Thus there exists $\eta > 0$ satisfying Definition 4.0.1 such that $\|\phi\| < \eta$ and let $x_t := x_t(t_0, \phi)$.

Notice that for $t \in [t_0, \infty)$, the function $t \to V(t, x_t)$ is nonincreasing and since x is a solution of (4.1.1), $\alpha(\|x_t\|) \leqslant V(t, x_t)$ for all $t \in [t_0, \infty)$, which implies that $V(t, x_t) \geqslant 0$. Therefore, there exists $z \geqslant 0$ such that $\lim_{t\to\infty} V(t, x_t) = z$. If $z > 0$, then, by (4.1.4), we get

$$\begin{aligned} 0 \leqslant V(t, x_t) &\leqslant V(t_0, x_{t_0}) - \int_{t_0}^{t} \widetilde{\alpha}(V(s, x_s))\mathrm{d}\gamma(s) \\ &\leqslant V(t_0, x_{t_0}) - \int_{t_0}^{t} \widetilde{\alpha}(z)\mathrm{d}\gamma(s) \\ &= V(t_0, x_{t_0}) - \widetilde{\alpha}(z)[\gamma(t) - \gamma(t_0)]. \end{aligned} \tag{4.1.5}$$

For sufficiently large t, the right–hand side of (4.1.5) is negative, leading us to a contradiction. Hence, we conclude that $z = 0$.

Thus, since $0 \leqslant \alpha(\|x_t\|) \leqslant V(t, x_t)$, we have $\lim_{t\to\infty} \alpha(\|x_t\|) = 0$, which implies $\lim_{t\to\infty} x(t) = 0$, by the property of α, proving the theorem. □

The next result provides sufficient conditions to ensure that the trivial solution of (4.1.1) is uniformly stable.

Theorem 4.1.4. *Let* $V\colon [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}^n$, $0 < \rho < c$, *be a Lyapunov functional. Assume also that* V *satisfies the following condition:*

(H) There exist two continuous increasing functions $\alpha, \beta\colon [0, \infty) \to [0, \infty)$ *satisfying* $\alpha(0) = 0 = \beta(0)$ *such that for every solution* $x\colon [t_0 - r, \infty) \to \overline{B_\rho}$ *of equation* (4.1.1), *we have*

$$\beta(\|x_t\|_\infty) \leqslant V(t, x_t) \leqslant \alpha(\|x_t\|_\infty), \tag{4.1.6}$$

for all $t \geqslant t_0$.

Then the trivial solution $x \equiv 0$ *of the equation* (4.1.1) *is uniformly stable.*

Proof. Let $t_0 \geqslant 0$ and $\varepsilon \geqslant 0$. Since $\alpha(0) = 0$, α is increasing and $\alpha|_{[0,\varepsilon]}$ is uniformly continuous, there exists $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$ such that $\alpha(\delta) < \beta(\varepsilon)$.

Suppose $\phi \in B_\rho$ and the maximal solution of x of (4.1.4) satisfies $\|\phi\|_\infty < \delta$. We need to show that $\|x_t\|_\infty < \varepsilon$ for all $t \in [t_0, \omega)$. Since V is a Lyapunov functional, then

$$V(t, x_t) \leqslant V(t_0, x_{t_0})$$

for all $t \in [t_0, \omega)$.

Hence, by hypotheses, for every $t \in [t_0, \omega)$, we get

$$\beta(\|x_t\|_\infty) \leqslant V(t, x_t) \leqslant V(t_0, x_{t_0}) \leqslant \alpha(\|x_{t_0}\|_\infty) \leqslant \alpha(\delta) \leqslant \beta(\varepsilon).$$

Since β is an increasing function, we get $\|x_t\|_\infty < \varepsilon$ for all $t \in [t_0, \omega)$, getting the desired result. □

Now, let us define *exponential stability* of the trivial solution of (4.1.1).

Definition 4.1.5. We say that the trivial solution of (4.1.1) is *exponentially stable* if there exist constants $\rho, a, b > 0$ such that if $t \geqslant t_0$ and $\|\phi\|_\infty \in B(0, \rho) \subset G([-r, 0], \mathbb{R}^n)$, then

$$\|x_t(t_0, \phi)\|_\infty < a\|\phi\|_\infty e^{-b(t-t_0)} \tag{4.1.7}$$

for all $t \geqslant t_0$.

The next result gives us conditions that will ensure that the trivial solution of (4.1.1) is exponentially stable.

Theorem 4.1.6. *Suppose that there exist positive constants σ, β, α, k and a Lyapunov functional $V\colon [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}$ with respect to equation* (4.1.1) *such that*

1. $\sigma\|\phi\|_\infty^k \leqslant V(t, \phi) \leqslant \beta\|\phi\|_\infty^k$ *for all* $\phi \in \overline{B_\rho}$ *and* $t \geqslant t_0$.

2. *For every maximal solution* $x(t) = x(t, s_0, \psi)$ *with* $(s_0, \psi) \in \Omega$ *of* (4.1.1), *we have*

$$V(t, x_t) - V(s, x_s) \leqslant -\alpha \int_s^t V(\xi, x_\xi)\mathrm{d}\xi$$

for all $t, s \in [s_0, \infty)$ *with* $t \leqslant s$.

Then the trivial solution $y \equiv 0$ *of* (4.1.1) *is exponentially stable.*

Proof. Let $V\colon [t_0,\infty)\times\overline{B_\rho}\to\mathbb{R}$ be a Lyapunov functional satisfying Conditions 1 and 2 of Theorem 4.1.6. Note that if $x\colon [t_0,\infty)\to\mathbb{R}^n$ is a solution of (4.1.1), we have

$$\sigma\|x_t\|_\infty^k \leqslant V(t,x_t)\leqslant \beta\|x_t\|_\infty^k \ \ \forall \ \ t\geqslant t_0,$$

which implies that Conditions 1 and 2 of Theorem 4.1.2 are fulfilled. Thus, the trivial solution $x\equiv 0$ is stable.

Let $s_0\geqslant t_0, \phi\in\overline{B_\delta}$, where $0<\delta<\rho$ is chosen by the stability of x, and let $x(\cdot)=x(\cdot,s_0,\phi)$ be the maximal solution of (4.1.1) defined on $[s_0,\infty)$.

Combining the second condition of this theorem with the fact that $t\mapsto V(t,x_t)$ is a nonincreasing function, for every $s_0\leqslant\theta_1\leqslant\theta_2<\infty$, we estimate

$$V(\theta_1,x_{\theta_1})-V(\theta_2,x_{\theta_2})\geqslant\alpha\int_{\theta_1}^{\theta_2}V(\xi,x_\xi)\mathrm{d}\xi\geqslant\alpha\int_{\theta_1}^{\theta_2}V(\theta_2,x_{\theta_2})\mathrm{d}\xi=\alpha V(\theta_2,x_{\theta_2})(\theta_2-\theta_1)$$

From this, we get

$$V(\theta_1,x_{\theta_1})\geqslant(1+\alpha(\theta_2-\theta_1))V(\theta_2,x_{\theta_2}) \tag{4.1.8}$$

for all $s_0\leqslant\theta_1\leqslant\theta_2<\infty$.

On the other hand, we claim that

$$V(s+s_0,x_{s+s_0})<e^{-\alpha s}V(s_0,\phi) \tag{4.1.9}$$

for all $s\in[0,\infty)$.

In fact, let $s\in[0,\infty)$ be given and let n be an arbitrary fixed natural number. Define

$$\tau_i:=\frac{is}{n}+s_0 \ \text{ for all } \ i\in\{1,2,\ldots,n\}.$$

Note that

$$s_0=\tau_0<\tau_1<\cdots<\tau_n=s+s_0 \ \text{ and } \ \tau_i-\tau_{i-1}=\frac{s}{n}.$$

Using this fact together with (4.1.8), we have

$$V(\tau_{i-1},x_{\tau_{i-1}})\geqslant(1+\alpha(\tau_i-\tau_{i-1}))V(\tau_i,x_{\tau_i})=\left(1+\frac{\alpha s}{n}\right)V(\tau_i,x_{\tau_i})$$

for all $i\in\{1,2,\ldots,n\}$. Hence, using a recursive argument, we obtain

$$V(\tau_0,x_{\tau_0})\geqslant\left(1+\frac{\alpha s}{n}\right)^n V(\tau_n,x_{\tau_n}). \tag{4.1.10}$$

Since $V(\tau_0, x_{\tau_0}) = V(s_0, \phi)$ and $V(\tau_n, x_{\tau_n}) = V(s + s_0, x_{s+s_0})$, we conclude that $V(s_0, \phi) \geqslant \left(1 + \frac{\alpha s}{n}\right)^n V(s + s_0, x_{s+s_0})$ for all $s \geqslant 0$ and all $n \in \mathbb{N}$. Therefore, as n tends to ∞, we obtain that $V(s_0, \phi) \geqslant e^{\alpha s} V(s + s_0, x_{s+s_0})$, getting the claim, which implies that

$$V(t, x_t) \leqslant e^{-\alpha(t-s_0)} V(s_0, \phi) \quad \text{for all} \quad t \in [s_0, \infty).$$

According to Condition 1 of this Theorem,

$$\sigma \|x_t\|_\infty^k \leqslant V(t, x_t) \text{ and } V(s_0, \phi) \leqslant \beta \|\phi\|_\infty^k.$$

Therefore, we get that

$$\sigma \|x_t\|_\infty^k \leqslant V(t, x_t) \leqslant e^{-\alpha(t-s_0)} V(s_0, \phi) \leqslant e^{-\alpha(t-s_0)} \beta \|\phi\|_\infty^k$$

$$\|x_t\|_\infty^k \leqslant \left(\frac{\beta}{\sigma}\right) \|\phi\|_\infty^k e^{-\alpha(t-s_0)},$$

which leads to

$$\|x_t\|_\infty \leqslant \left(\frac{\beta}{\sigma}\right)^{\frac{1}{k}} \|\phi\|_\infty e^{\frac{-\alpha}{k}(t-s_0)}, \tag{4.1.11}$$

proving that the trivial solution is indeed exponentially stable. □

To finish this section, we present an example to illustrate Theorem 4.1.6. This example was inspired by [28].

Example 4.1.7. Let $\{t_k\}_{k=1}^\infty$ be moments of impulses such that $t_k < t_{k+1}$ for $k \in \mathbb{N}$. Consider the following Volterra–Stieltjes integral equation:

$$x(t) = x(0) + \int_0^t a(t, s) f(x_s, s) \mathrm{d}g(s), \tag{4.1.12}$$

where $g(t) = g(0) + t + \sum_{j=1}^\infty \chi_{(t_j, \infty)}(t)$, $t \in [0, \infty)$ and χ is the characteristic function. The function $f\colon G([-r, 0], [1, \infty)) \times [0, \infty) \to \mathbb{R}$ is given by:

$$f(x_t, t) = \begin{cases} (\sin(\ln(t+1)) - 2) x_t, \text{ if } t \neq t_k \\ x_t(0), \text{ if } t = t_k; \end{cases}$$

the function $a\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$ is given by

$$a(t, s) = \begin{cases} e^{-\alpha}, \text{ if } t \neq t_k \\ e^{-\beta(t - t_{k-1})} \gamma_k, \text{ if } t = t_k, \end{cases}$$

where $\alpha, \beta > 0$ and $-1 \leqslant \gamma_k \leqslant 0$.

Thus by Theorem 2.1.2, our Volterra–Stieltjes integral equation (4.1.12) can be rewritten as an impulsive Volterra equation given by:

$$\begin{cases} x(t) = x(0) + \displaystyle\int_0^t e^{-\alpha(t-s)}(\sin(\ln(s+1)) - 2)x_s \mathrm{d}s, & t \neq t_k \\ \Delta^+ x(t) = e^{\beta(t-t_{k-1})}\gamma_k x(t), & t = t_k. \end{cases} \tag{4.1.13}$$

Assuming that x is differentiable a.e., the integral equation above can be rewritten as the following impulsive differential equation a.e.:

$$\begin{cases} x'(t) = e^{-\alpha(t-s)}(\sin(\ln(s+1)) - 2)x_s \mathrm{d}s, & t \neq t_k \\ \Delta^+ x(t) = e^{-\beta(t-t_{k-1})}\gamma_k x(t), & t = t_k. \end{cases} \tag{4.1.14}$$

Defining $V(t, x_t) = \frac{(x_t(0))^2}{2} = \frac{(x(t))^2}{2}$, we have that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = xx' = x(\sin(\ln(t+1)) - 2)x_t \leqslant -1 < 0.$$

For $t = t_k$, we get

$$\begin{aligned} V(t_k, x_{t_k}) &= \frac{(x(t_k^+))^2}{2} = \frac{(x(t_k) + I_k(x(t_k)))^2}{2} \\ &= \frac{(x(t_k) + e^{-\beta(t_k - t_{k-1})}\gamma_k x(t_k))^2}{2} \\ &= \frac{((1 + e^{-\beta(t_k - t_{k-1})}\gamma_k)x(t_k))^2}{2} \\ &\leqslant (1 + e^{-\beta(t_k - t_{k-1})}\gamma_k)^2 V(t_k, x(t_k)) \leqslant V(t_k, x(t_k)), \end{aligned}$$

since $\gamma_k < 0$.

Therefore, all the assumptions of Theorem 4.1.6 are satisfied, which implies that the trivial solution $x \equiv 0$ of (4.1.12) is exponentially stable.

4.2 Lyapunov's Second Method for impulsive functional Volterra–Stieltjes integral equations

In this section we will use the correspondence between the functional Volterra–Stieltjes integral equations and the impulsive functional Volterra Stieltjes integral equations to obtain the results about stability for the impulsive case.

Throughout this section, assume that $\{t_k\}_{k=1}^{\infty}$ are the moments of impulses in $[t_0, \infty)$ such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Let us recall the definitions of the functions $\tilde{f}$ and $\tilde{g}$:

Define $\tilde{f}\colon G([-r,0],\mathbb{R}^n) \times [t_0,\infty) \to \mathbb{R}^n$ by

$$\tilde{f}(y,\tau) = \begin{cases} f(y,\tau), & \tau \in [t_0,d)\backslash\{t_k\}_{k=1}^{\infty}, \\ I_k(y(0)), & \tau = t_k, \quad k \in \mathbb{N}, \end{cases}$$

and $\tilde{g}\colon [t_0,\infty) \to \mathbb{R}$ by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [t_0,t_1], \\ g(\tau)+k, & \tau \in (t_k,t_{k+1}], \quad k \in \mathbb{N}. \end{cases}$$

Consider here the following functional Volterra–Stieltjes integral equations with impulses

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\mathbb{N} \\ t_0<t_k<t}} a(t,t_k) I_k(x(t_k)) \\ x_{t_0} &= \phi, \end{cases} \tag{4.2.1}$$

and assume that $I_k(0) = 0$ for each $k \in \mathbb{N}$.

Theorem 4.2.1. *Let $t_0 \in \mathbb{R}$ and suppose that there exists a Lyapunov functional $V\colon [t_0,\infty)\times \overline{B}_\rho \to \mathbb{R}$, $\overline{B}_\rho = \{y \in G([-r,0],\mathbb{R}^n) : \|y\| \leqslant \rho\}$, $\rho > 0$, with respect to (4.2.1) such that*

1. *$V(t,0) = 0$ for all $t \in [t_0,\infty)$;*
2. *If $x\colon [t_0 - r,\infty) \to \mathbb{R}^n$ is a solution of (4.2.1), then $\alpha(\|x_t\|) \leqslant V(t,x_t)$ for all $t \in [t_0,\infty)$, where $\alpha\colon [0,\infty) \to [0,\infty)$ is an increasing function such that $\alpha(0) = 0$ and $\lim_{s\to\infty} \alpha(s) = \infty$.*

Then the trivial solution $x \equiv 0$ of (4.2.1) is stable.

Proof. By Theorem 2.1.2, (4.2.1) has a solution if, and only if,

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} &= \phi \end{cases} \tag{4.2.2}$$

has a solution and, in this case, the solutions are the same.

By Theorem 4.1.2, the trivial solution of (4.2.2) is stable, that is, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$, such that for all $\phi \in B(0, \delta) \subset G([-r, 0], \mathbb{R}^n)$, the solution x exists for every $t \geqslant t_0$ and $x_t(t_0, \phi) \in B(0, \varepsilon) \subset G([-r, 0], \mathbb{R}^n)$ for all $t \geqslant t_0$.

Since the solutions of (4.2.2) and (4.2.1) are equal for all $t \in [t_0, \infty)$, it follows they have the same properties, which means the trivial solution of (4.2.1) is stable. □

The proof of the following theorems are very similar to the proof of Theorem 4.2.1, since the same argument is used in all of them, that is, the correspondence between the equations and the known results for functional Volterra–Stieltjes integral equations. In order not to tire the reader, we will omit them.

Theorem 4.2.2. *Let $t_0 \in \mathbb{R}$ and suppose there exists a Lyapunov functional $V\colon [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}$ with respect to (4.2.1) and a function $\alpha\colon [0, \infty) \to [0, \infty)$ that satisfy all the conditions of Theorem 4.1.2. Suppose also that, for all nonextendable solution $x\colon [t_0 - r, \infty) \to \mathbb{R}^n$ of (4.2.1), we have*

$$V(t, x_t) - V(s, x_s) \leqslant -\int_s^t \widetilde{\alpha}(V(\xi, x_\xi))\mathrm{d}\gamma(\xi) \tag{4.2.3}$$

for all $t, s \in [t_0, \infty)$ with $t \geqslant s$, where $\gamma\colon [t_0, \infty) \to \mathbb{R}$ is a nondecreasing function such that $\lim\limits_{t\to\infty} \gamma(t) = \infty$ and $\widetilde{\alpha}\colon [t_0, \infty) \to [t_0, \infty)$ is an increasing function such that $\widetilde{\alpha}(0) = 0$ and $\lim_{s\to\infty} \widetilde{\alpha}(s) = \infty$.

Then the trivial solution $x \equiv 0$ of (4.2.1) is asymptotically stable.

Theorem 4.2.3. *Let $V\colon [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}^n$, $0 < \rho < c$, be a Lyapunov functional. Assume also that V satisfies the following condition:*

(H) There exist two continuous increasing functions $\alpha, \beta\colon [0, \infty) \to [0, \infty)$ satisfying $\alpha(0) = 0 = \beta(0)$ such that for every solution $x\colon [t_0 - r, \infty) \to B_\rho$ of equation (4.2.1), we have

$$\beta(\|x_t\|_\infty) \leqslant V(t, x_t) \leqslant \alpha(\|x_t\|_\infty), \tag{4.2.4}$$

for all $t \geqslant t_0$.

Then the trivial solution $x \equiv 0$ of the equation (4.2.1) is uniformly stable.

Theorem 4.2.4. *Suppose that there exist positive constants σ, β, α, k and a Lyapunov functional $V\colon [t_0, \infty) \times \overline{B_\rho} \to \mathbb{R}$ with respect to equation (4.2.1) such that*

1. *$\sigma\|\phi\|_\infty^k \leqslant V(t,\phi) \leqslant \beta\|\phi\|_\infty^k$ for all $\phi \in \overline{B_\rho}$ and $t \geqslant t_0$.*

2. *For every maximal solution $x(t) = x(t,s_0,\psi)$ with $(s_0,\psi) \in \Omega$ of (4.2.1), we have*

$$V(t,x_t) - V(s,x_s) \leqslant -\alpha \int_s^t V(\xi,x_\xi)\mathrm{d}\xi$$

   for all $t,s \in [s_0,\infty)$ with $t \leqslant s$.

Then the trivial solution $y \equiv 0$ of (4.2.1) is exponentially stable.

Remark 4.2.5. It is also possible to translate all the results to their analogues in the time scale case.

Chapter 5

Periodic boundary value problem

In this chapter, we consider the following periodic boundary value problem:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_0^t a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \in [0,\omega] \\ x_0 = \phi, \\ x(0) = x(\omega) \end{cases} \tag{5.0.1}$$

where $\phi \in G([-r,0],\mathbb{R}^n)$, $f\colon G([-r,0],\mathbb{R}^n) \times [0,\omega] \to \mathbb{R}^n$, $a\colon [0,\omega]^2 \to \mathbb{R}$ and $g\colon [0,\omega] \to \mathbb{R}$ is nondecreasing and left continuous on $[0,\omega]$. Here, $[0,\omega]^2$ denotes the set $[0,\omega]\times[0,\omega]$.

The goal of this chapter is to seek solutions for this problem. We investigate sufficient conditions on the functions a, f and g in order to guarantee the existence of a solution of this type of problem. All the results of this chapter are new and can be found in [31]. We divide this chapter in three sections. In the first section, we study the existence of solutions of the periodic boundary value problem (5.0.1). In the second section, we present a correspondence between (5.0.1) and its analogue with impulses. In the third section, we prove the analogue results to Δ-integral equations on time scales.

The motivation to investigate this kind of problems for Volterra–Stieltjes equations came initially from [23]. In this paper, the authors study the existence of periodic solutions for generalized ordinary differential equations (GODEs for short) and they apply their results to other types of equations such as measure equation without delays.

It is important to emphasize that that are other papers in this direction such as [16, 48, 49, 50, 51], and this type of problem has been attracting the attention by several researchers, since periodicity appears quite natural in most of phenomena.

5.1 Solutions for periodic boundary value problems

We begin this section by assuming the following conditions:

(H1) The function $g\colon [0,\omega] \to \mathbb{R}$ is nondecreasing and left–continuous on $(0,\omega]$.

(H2) The function $a\colon [0,\omega]^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable and regulated with respect to the second variable.

(H3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{\tau_1}^{\tau_2} a(t,s)f(x_s,s)\mathrm{d}g(s)$$

exists, for all $x \in G([-r,\omega],\mathbb{R}^n)$, all $t \in [0,\omega]$ and all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \omega$.

(H4) There exists a Henstock–Kurzweil–Stieltjes integrable function $M\colon [0,\omega] \to \mathbb{R}^+$ with respect to g such that

$$\left\| \int_{\tau_1}^{\tau_2} \left(\beta_2 a(\tau_2,s) + \beta_1 a(\tau_1,s)\right) f(x_s,s)\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} \left|\beta_2 a(\tau_2,s) + \beta_1 a(\tau_1,s)\right| M(s)\mathrm{d}g(s),$$

for all $x \in G([-r,\omega],\mathbb{R}^n)$, all $\beta_1,\beta_2 \in \mathbb{R}$ and all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \omega$.

(H5) There exists a regulated function $L\colon [0,\omega] \to \mathbb{R}^+$ with respect to g such that

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2,s)[f(x_s,s) - f(z_s,s)]\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} \left|a(\tau_2,s)\right| L(s)\|x_s - z_s\|_\infty \mathrm{d}g(s),$$

for all $x, z \in G([-r,\omega],\mathbb{R}^n)$ and all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \omega$.

Remark 5.1.1. It is easy to check that if a satisfies condition (H2), then a is bounded in $[0,\omega]^2$.

For further purposes, let us define the following constant:

$$\eta := \sup_{(t,s)\in[0,\omega]^2} \left|a(t,s)\right|, \tag{5.1.1}$$

which is well–defined by Remark 5.1.1.

The next result can be found in [44] and will be essential to prove the existence of solutions for the main problem of this section.

Theorem 5.1.2 (Krasnosel'skiĭ Fixed Point Theorem)**.** *Let X be a Banach space and Y be a nonempty convex and closed subset of X. Let $\mathcal{G}_1, \mathcal{G}_2 \colon Y \to X$ be two operators satisfying*

(i) *if $u, v \in Y$, then $\mathcal{G}_1 u + \mathcal{G}_2 v \in Y$.*

(ii) *$\mathcal{G}_1$ is a contraction on Y.*

(iii) *$\mathcal{G}_2$ is compact and continuous on Y.*

Then, there exists $z \in Y$ such that $\mathcal{G}_1 z + \mathcal{G}_2 z = z$.

The following corollary is a consequence of the previous theorem with $Y = X$.

Corollary 5.1.3. *Let X be a Banach space. Let $\mathcal{G}_1, \mathcal{G}_2 \colon X \to X$ be two operators such that $\mathcal{G}_1$ satisfies condition (ii) of Theorem 5.1.2 and $\mathcal{G}_2$ satisfies condition (iii) of Theorem 5.1.2. Then, there exists $x \in X$ such that $\mathcal{G}_1 x + \mathcal{G}_2 x = x$.*

Next, we exhibit the main result of this section. By using Theorem 5.1.2 (Krasnosel'skiĭ Fixed Point Theorem), we obtain a criterion to guarantee the existence of at least one solution for the periodic boundary value problem (5.0.1). In order to use Theorem 5.1.2, we will consider $Y = X = \mathbb{R}^n \times G([-r, \omega], \mathbb{R}^n)$.

Theorem 5.1.4. *Consider the periodic boundary value problem:*

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_0^t a(t,s) f(x_s, s)\, \mathrm{d}g(s), \quad t \in [0, \omega] \\ x_0 = \phi \\ x(0) = x(\omega) \end{cases} \tag{5.1.2}$$

and assume that conditions (H1)–(H5) are satisfied. Moreover, suppose that the following condition holds

$$\eta \, \|L\|_\infty \, (g(\omega) - g(0)) < 1,$$

where η is given by (5.1.1) *and L is given in (H5). Then the periodic boundary value problem* (5.1.2) *has at least one solution.*

Proof. Let us consider the Banach space $\mathbb{R}^n \times G([-r, \omega], \mathbb{R}^n)$ endowed with the norm

$$\|(\beta, x)\|_{\mathbb{R}^n, \infty} = \|\beta\| + \|x\|_\infty, \qquad \text{for} \ \ (\beta, x) \in \mathbb{R}^n \times G\left([0, \omega], \mathbb{R}^n\right),$$

where $\|\cdot\|$ is any norm in $\mathbb{R}^n$ and $\|\cdot\|_\infty$ is given by $\|x\|_\infty = \sup_{t\in[-r,\omega]} \|x(t)\|$.

Consider operators $\mathcal{W}\colon \mathbb{R}^n \times G([-r,\omega],\mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{Q}\colon G([-r,\omega],\mathbb{R}^n) \to G([-r,\omega],\mathbb{R}^n)$ defined by

$$\mathcal{W}(\beta,x) := \beta + \int_0^\omega a(\omega,s)f(x_s,s)\,\mathrm{d}g(s)$$

and

$$\mathcal{Q}(x)(t) = \begin{cases} \phi(t), & t \in [-r,0] \\ \phi(0) + \displaystyle\int_0^t a(t,s)f(x_s,s)\mathrm{d}g(s), & t \in [0,\omega], \end{cases}$$

for all $x \in G([-r,\omega],\mathbb{R}^n)$, all $t \in [-r,\omega]$ and all $\beta \in \mathbb{R}^n$.

By condition (H3), the integral $\int_0^\omega a(\omega,s)f(x_s,s)\,\mathrm{d}g(s)$ exists in the sense of Henstock–Kurzweil–Stieltjes. Thus, $\mathcal{W}(\beta,x) \in \mathbb{R}^n$. On the other hand, using the same arguments as in Lemma 3.1.7, we can prove that the function $\mathcal{Q}(x)$ is regulated on $[-r,\omega]$, that is, $\mathcal{Q}(x) \in G([-r,\omega],\mathbb{R}^n)$.

Now, consider the operators $\mathcal{G}_1, \mathcal{G}_2\colon \mathbb{R}^n \times G([-r,\omega],\mathbb{R}^n) \longrightarrow \mathbb{R}^n \times G([-r,\omega],\mathbb{R}^n)$ defined for $(\beta,x) \in \mathbb{R}^n \times G([-r,\omega],\mathbb{R}^n)$ by

$$\mathcal{G}_1(\beta,x) = (0_{\mathbb{R}^n}, \mathcal{Q}(x)) \quad \text{and} \quad \mathcal{G}_2(\beta,x) = (\mathcal{W}(\beta,x), 0_G),$$

where $0_{\mathbb{R}^n} := (0,\dots,0)$ and $0_G\colon [-r,\omega] \to \mathbb{R}^n$ is given by $0_G(t) = 0_{\mathbb{R}^n}$ for $t \in [-r,\omega]$.

Statement 1. If (β,x) is a fixed point of the operator $\mathcal{G}_1 + \mathcal{G}_2$, then x is a solution of the periodic boundary value problem (5.1.2).

Suppose that $(\mathcal{G}_1 + \mathcal{G}_2)(\beta,x) = (\beta,x)$. Thus,

$$\int_0^\omega a(\omega,s)f(x_s,s)\,\mathrm{d}g(s) = 0 \tag{5.1.3}$$

and, for all $t \in [-r,\omega]$, we obtain

$$x(t) = \begin{cases} \phi(t), & t \in [-r,0] \\ \phi(0) + \displaystyle\int_0^t a(t,s)f(x_s,s)\mathrm{d}g(s), & t \in [0,\omega]. \end{cases} \tag{5.1.4}$$

In view of (5.1.4), x is a solution of the functional Volterra–Stieltjes integral equation

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_0^t a(t,s)f(x_s,s)\,\mathrm{d}g(s), & t \in [0,\omega] \\ x_0 = \phi. \end{cases}$$

On the other hand, taking $t = \omega$ in (5.1.4) and using (5.1.3), we get $x(0) = x(\omega)$, proving Statement 1.

Statement 2. $\mathcal{G}_1$ is a contraction.

Initially, note that

$$\|\mathcal{G}_1(\beta, x) - \mathcal{G}_1(\alpha, z)\|_{\mathbb{R}^n,\infty} = \|(0_{\mathbb{R}^n}, \mathcal{Q}(x) - \mathcal{Q}(z))\|_{\mathbb{R}^n,\infty} = \sup_{t\in[-r,\omega]} \|\mathcal{Q}(x)(t) - \mathcal{Q}(z)(t)\| .$$

Let $x, z \in G([-r, \omega], \mathbb{R}^n)$ be given. Then, for $t \in [-r, 0]$, we get

$$\|\mathcal{Q}(x)(t) - \mathcal{Q}(z)(t)\| = \|\phi(0) - \phi(0)\| = 0,$$

and for $t \in [0, \omega]$, by (H5), (5.1.1), Theorem 1.2.4 and Corollary 1.2.5, we have

$$\begin{aligned}
\|(\mathcal{Q}(x)(t) - \mathcal{Q}(z)(t)\| &= \left\|\int_0^t a(t,s)(f(x_s,s) - f(z_s,s))\,\mathrm{d}g(s)\right\| \\
&\leqslant \int_0^t |a(t,s)|L(s)\|x_s - z_s\|_\infty \,\mathrm{d}g(s) \\
&\leqslant \int_0^t \eta\,\|L\|_\infty\,\|x_s - z_s\|_\infty \,\mathrm{d}g(s) \\
&\leqslant \int_0^\omega \eta\,\|L\|_\infty\,\|x_s - z_s\|_\infty \,\mathrm{d}g(s) \\
&\leqslant \eta\,\|L\|_\infty\,(g(\omega) - g(0))\,\|x - z\|_\infty . \qquad (5.1.5)
\end{aligned}$$

Since by hypothesis $\eta\,\|L\|_\infty\,(g(\omega) - g(0)) < 1$, we have that $\mathcal{G}_1$ is a contraction. Notice that for $s \in [0, \omega]$, we obtain

$$\begin{aligned}
\|x_s - z_s\|_\infty &= \sup_{\theta\in[-r,0]} \|x(s+\theta) - z(s+\theta)\| \\
&= \sup_{\xi\in[s-r,s]} \|x(\xi) - z(\xi)\| \\
&\leqslant \sup_{\xi\in[-r,\omega]} \|x(\xi) - z(\xi)\| \\
&= \|x - z\|_\infty,
\end{aligned}$$

which shows the inequality (5.1.5).

Statement 3. $\mathcal{G}_2$ is continuous.

Note that, for $x, z \in G([-r, \omega], \mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{R}^n$, we have

$$\|\mathcal{G}_2(\beta, x) - \mathcal{G}_2(\alpha, z)\|_{\mathbb{R}^n,\infty} = \|\,(\mathcal{W}(\beta, x) - \mathcal{W}(\alpha, z), 0_G)\,\|_{\mathbb{R}^n,\infty}$$

$$= \|\mathcal{W}(\beta, x) - \mathcal{W}(\alpha, z)\| \,.$$

Hence, by (H5), Theorem 1.2.4 and the fact that $\|x_s - z_s\|_\infty \leqslant \|x - z\|_\infty$, we obtain

$$\begin{aligned}
\|\mathcal{W}(\beta, x) - \mathcal{W}(\alpha, z)\| &= \left\| \beta - \alpha + \int_0^\omega a(\omega, s)\left(f(x_s, s) - f(z_s, s)\right) \mathrm{d}g(s) \right\| \\
&\leqslant \|\beta - \alpha\| + \int_0^\omega |a(\omega, s)|\, L(s)\, \|x_s - z_s\|_\infty \,\mathrm{d}g(s) \\
&\leqslant \|\beta - \alpha\| + \left(\int_0^\omega |a(\omega, s)|\, L(s)\,\mathrm{d}g(s) \right) \|x - z\|_\infty \\
&\leqslant \left(1 + \int_0^\omega |a(\omega, s)|\, L(s)\,\mathrm{d}g(s) \right) \|(\beta, x) - (\alpha, z)\|_{\mathbb{R}^n, \infty} \,.
\end{aligned}$$

The last inequality follows from the fact that

$$\|\beta - \alpha\| \leqslant \|(\beta, x) - (\alpha, z)\|_{\mathbb{R}^n,\infty} \quad \text{and} \quad \|x - z\|_\infty \leqslant \|(\beta, x) - (\alpha, z)\|_{\mathbb{R}^n,\infty} \,.$$

From this estimate, we get that $\mathcal{G}_2$ is continuous.

Statement 4. $\mathcal{G}_2$ is compact.
Let $B = B_1 \times B_2 \subset \mathbb{R}^n \times G([-r, \omega], \mathbb{R}^n)$ be bounded. Then, there exist constants $C_1 > 0$ and $C_2 > 0$ such that $\|\beta\| \leqslant C_1$, for all $\beta \in B_1$ and $\|x\|_\infty \leqslant C_2$, for every $x \in B_2$.

The goal is to show that $\mathcal{G}_2(B)$ is relatively compact in $\mathbb{R}^n \times G([-r, \omega], \mathbb{R}^n)$. In fact, notice that

$$\mathcal{G}_2(B) = \{(\mathcal{W}(\beta, x), 0_G) : \beta \in B_1,\ x \in B_2\} = \mathcal{W}(B) \times \{0_G\} \,.$$

Since $\{0_G\}$ is relatively compact in $G([-r, \omega], \mathbb{R}^n)$, it is sufficient to prove that $\mathcal{W}(B)$ is relatively compact in $\mathbb{R}^n$.

Let $(\beta, x) \in B$. Then, by (H4), we get

$$\begin{aligned}
\|\mathcal{W}(\beta, x)\| &= \left\| \beta + \int_0^\omega a(\omega, s) f(x_s, s)\,\mathrm{d}g(s) \right\| \\
&\leqslant \|\beta\| + \int_0^\omega |a(\omega, s)|\, M(s)\,\mathrm{d}g(s) \\
&\leqslant C_1 + \int_0^\omega |a(\omega, s)|\, M(s)\,\mathrm{d}g(s).
\end{aligned}$$

Thus, $\mathcal{W}(B)$ is bounded, which implies that $\overline{\mathcal{W}(B)}$ is also bounded. Now, since $\overline{\mathcal{W}(B)}$ is bounded and closed in $\mathbb{R}^n$, we conclude that $\overline{\mathcal{W}(B)}$ is compact on $\mathbb{R}^n$.

Since all hypotheses of Krasnosel'skiĭ Fixed Point Theorem (Theorem 5.1.2) are satisfied, we conclude that $\mathcal{G}_1 + \mathcal{G}_2$ has a fixed point $(\beta, x) \in \mathbb{R}^n \times G([-r, \omega], \mathbb{R}^n)$.

Therefore, by Statement 1, $x\colon [-r,\omega] \to \mathbb{R}^n$ is a solution of the periodic boundary value problem (5.1.2) and this completes the proof. $\square$

5.2 Periodic impulsive boundary value problems

In this section, our goal is to translate Theorem 5.1.4 to impulsive functional Volterra–Stieltjes integral equations. At first, we start by describing the correspondence between the two boundary value problems. Then, using this correspondence and the proved result, we get our result to impulsive functional Volterra–Stieltjes integral equations. Clearly, it will follow very similar to the proof presented in Section 2.1 with obvious adaptations that we will describe here.

We start by considering the periodic impulsive boundary value problem below:

$$\begin{aligned} x(v) - x(u) &= \int_0^v a(v,s) f(x_s,s)\,\mathrm{d}g(s) - \int_0^u a(u,s) f(x_s,s)\,\mathrm{d}g(s), \quad \text{for } u,v \in J_k,\ k \in \mathbb{N}, \\ \Delta^+ x(t_k) &= I_k(x(t_k)), \quad k = 1,\dots,m, \\ x_0 &= \phi \\ x(0) &= x(\omega), \end{aligned}$$

where $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1,\dots,m$, and $J_m = (t_m,\omega]$. The same as it was remarked before, the value of both integrals

$$\int_0^v a(v,s) f(x_s,s)\,\mathrm{d}g(s) \quad \text{and} \quad \int_0^u a(u,s) f(x_s,s)\,\mathrm{d}g(s),$$

where $u,v \in J_k$, do not change if we replace g by a function $\tilde{g}$ such that $g-\tilde{g}$ is a constant function on J_k (see [25]). Rewriting the above equations as

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_0^t a(t,s) f(x_s,s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\} \\ 0<t_k<t}} a(t,t_k) I_k(x(t_k)), \quad t \in [0,\omega] \\ x_0 = \phi \\ x(0) = x(\omega). \end{cases} \tag{5.2.1}$$

Now, we define the functions $\tilde{f}$ and $\tilde{g}$ the same way as in Section 2.1, that is: $\tilde{f}\colon G([-r,0],\mathbb{R}^n)\times [0,\omega] \to \mathbb{R}^n$ is defined by

$$\tilde{f}(y,\tau) = \begin{cases} f(y,\tau), & \tau \in [0,\omega]\backslash\{t_1,\dots,t_m\}, \\ I_k(y(0)), & \tau = t_k,\ k \in \{1,\dots,m\}, \end{cases}$$

and $\tilde{g}\colon [0,\omega] \to \mathbb{R}$ is defined by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [0, t_1], \\ g(\tau) + k, & \tau \in (t_k, t_{k+1}],\ k \in \{1, \ldots, m-1\}, \\ g(\tau) + m, & \tau \in (t_m, \omega]. \end{cases}$$

Theorem 5.2.1. *Let $m \in \mathbb{N}$, $0 \leqslant t_1 < \cdots < t_m < d$, $I_1, \ldots, I_m\colon \mathbb{R}^n \to \mathbb{R}^n$ and $f\colon G([-r,0],\mathbb{R}^n) \times [0,\omega] \to \mathbb{R}^n$. Assume that $g\colon [0,\omega] \to \mathbb{R}$ is a nondecrasing left–continuous function which is continuous at $t_1, \ldots, t_m$, $a\colon [0,\omega]^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and continuous with respect to first variable at $t_1, \ldots, t_m$ and that conditions (H3)–(H5) are satisfied. Assume also that the function $L\colon [0,\omega] \to \mathbb{R}^+$ in Lemma 2.1.1 is such that*

$$\tilde{\eta}\, \|L\|_\infty \,(\tilde{g}(\omega) - \tilde{g}(0)) < 1,$$

where η is given by (5.1.1)*. Then, equation* (5.2.1) *has a solution on* $[0,\omega]$.

Proof. By Theorem 2.1.2, equation (5.2.1) has a solution on $[0,\omega]$ if, and only if, equation

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_0^t a(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} = \phi \\ x(0) = x(\omega) \end{cases} \tag{5.2.2}$$

has a solution, and, in this case, the solution is the same for both equations.

By Lemma 2.1.1, we know that the functions $\tilde{g}, a$ and $\tilde{f}$ satisfy conditions (H1)–(H5) and we have that $\eta\, \|L\|_\infty \,(\tilde{g}(\omega) - \tilde{g}(0)) < 1$, thus the solution of (5.2.2) is guaranteed by Theorem 5.1.4. Therefore, the result follows. □

5.3 Periodic boundary value problem on time scales

In this section, our goal is to prove the analogue of Theorem 5.1.4 for the following periodic boundary value problem on time scale:

$$\begin{cases} x(t) = x(0) + \displaystyle\int_0^t a(t,s)f(x_s,s)\Delta s \\ x_{t_0} = \phi(t_0) \\ x(0) = x(\omega), \end{cases} \tag{5.3.1}$$

where $\mathbb{T}$ is a $\omega-$periodic time scale, that is, $t \in \mathbb{T}$ implies $t \pm \omega \in \mathbb{T}$. Also, $0 \in \mathbb{T}$.

We point out that there are many different concepts of periodicity on time scales, but here we will consider the classical one, that is, the one which requires the additive property of the time scale. In a certain way, it excludes interesting time scales such as quantum scales, some hybrid ones, but on the other hand, there are several time scales that satisfy this property and it is interesting to consider these cases such as $\mathbb{T} = \mathbb{Z}, h\mathbb{Z}, \mathbb{R}, \bigcup_{k=1}^{\infty}[a_k, b_k]$, among others.

Now, consider the main theorem of this section.

Theorem 5.3.1. *Let $[-r, \omega]_{\mathbb{T}}$ be a time scale interval such that $0 \in \mathbb{T}$. Consider the periodic boundary value problem:*

$$\begin{cases} x(t) = x(0) + \displaystyle\int_0^t a(t,s) f(x_s^*, s)\Delta s \\ x_{t_0} = \phi(t_0) \\ x(0) = x(\omega), \end{cases} \tag{5.3.2}$$

where $\mathbb{T}$ is a $\omega-$periodic time scale, that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. Assume also that the following conditions hold.

(C1) *The function $a\colon [0,\omega]_{\mathbb{T}}^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and rd–continuous with respect to the first variable.*

(C2) *The Henstock–Kurzweil Δ–integral*

$$\int_{s_1}^{s_2} a(\tau, s) f(x_s, s)\Delta s$$

exists for $x \in G([0,\omega], \mathbb{R}^n)$, $\tau \in [0,\omega]_{\mathbb{T}}$ and $s_1, s_2 \in [0,\omega]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(C3) *There exists a locally Henstock–Kurzweil Δ–integrable function $M_1\colon [0,\omega]_{\mathbb{T}} \to \mathbb{R}^+$ such that*

$$\left\| \int_{s_1}^{s_2} (c_1 a(s_2, s) + c_2 a(s_1, s)) f(x_s, s)\Delta s \right\| \leqslant \int_{s_1}^{s_2} M_1(s) \left| c_1 a(s_2, s) + c_2 a(s_1, s) \right| \Delta s,$$

for all $x \in G([0,\omega], \mathbb{R}^n)$, $c_1, c_2 \in \mathbb{R}$ and $s_1, s_2 \in [0,\omega]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(C4) *There exists a locally regulated function* $L_1\colon [0,\omega]_{\mathbb{T}} \to \mathbb{R}^+$ *such that*

$$\left\| \int_{s_1}^{s_2} a(s_2,s)[f(x_s,s) - f(z_s,s)]\Delta s \right\| \leqslant \int_{s_1}^{s_2} L_1(s)\,|a(s_2,s)|\,\|x_s - z_s\|_\infty\,\Delta s,$$

for all $x, z \in G([0,\omega],\mathbb{R}^n)$ *and* $s_1, s_2 \in [0,\omega]_{\mathbb{T}}$, $s_1 \leqslant s_2$.

Moreover, suppose that

$$\eta\,\|L_1^*\|_\infty\,\omega < 1,$$

where $\eta := \sup_{t,s\in[0,\omega]} a^{**}(t,s)$ *and* L_1 *is given in (C4). Then the periodic boundary value problem* (5.1.2) *has at least one solution.*

Proof. Let $g\colon [0,\omega] \to \mathbb{R}$ be defined as $g(s) = s^*$. Notice that, since $0, \omega \in \mathbb{T}$, $g(0) = 0^* = 0$ and $g(\omega) = \omega^* = \omega$, hence $g(\omega) - g(0) = \omega$. This fact together with Lemma 2.2.2 and the Conditions (C1)–(C4) imply that the functions a^{**} and f^* satisfy the Conditions of Theorem 5.1.4. Thus, the equation

$$\begin{cases} x(t) = x(0) + \displaystyle\int_0^t a^{**}(t,s) f^*(x_s,s)\mathrm{d}g(s) \\ x_{t_0} = \phi^*_{t_0} \\ x(0) = x(\omega), \end{cases} \tag{5.3.3}$$

has a solution $y\colon [-r,\omega] \to \mathbb{R}^n$. Using Theorem 2.2.1, we get that y must be of the form $y = x^*$, where $x\colon [-r,\omega]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of (5.3.2), as we desired.

□

Chapter 6

Continuous dependence with respect to parameters

In this chapter, our goal is to investigate under which conditions the solutions of a sequence of equations

$$\begin{cases} x(t) &= \phi_n(0) + \displaystyle\int_{t_0}^{t} a_n(t,s) f_n(x_s,s)\mathrm{d}g_n(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi_n, \end{cases} \tag{6.0.1}$$

$n \in \mathbb{N}$, converge to a solution of the equation

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s,s)\mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi, \end{cases} \tag{6.0.2}$$

where $\phi_n, \phi \in G([-r,0],\mathbb{R}^n)$, $a_n, a\colon [t_0,t_0+\sigma]^2 \to \mathbb{R}$, $\sigma > 0$, $f, f_n\colon G([-r,0],\mathbb{R}^n) \times [t_0,t_0+\sigma] \to \mathbb{R}^n$ and $g_n, g\colon [t_0,t_0+\sigma] \to \mathbb{R}$. Also, using the correspondence between functional Volterra–Stieltjes integral equations and impulsive functional Volterra–Stieltjes integral equations as well as the correspondence between the first one and the functional Volterra delta integral equations on time scales, we prove the analogue results for these types of equations.

Results concerning convergence of a sequence of problems was already investigated by several researchers, specially the ones related to time scales theory. In 2004, the article [19] considered families of dynamic equations on time scales given by:

$$x^{\Delta} = f(t,x)$$

subject to the initial condition $x(t_0) = x_0$ over different time scales. The main goal of this paper was to investigate the behavior of the solutions with the same initial value problems over different time scales in order to understand phenomena such as bifurcations and the asymptotic behavior for variable time scales. Also, they proved that the limit of the solutions over convergent sequences of time scales converges to a solution over the limiting time scale.

In [29], Garay and Hilger discussed the continuous dependence of solutions of a dynamic equation in its integral form given by:

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\Delta_{\mathbb{T}}(s),$$

investigating this equation in “the space of graphs”. In other words, they replaced an analytical concept (the distance of functions) by a geometric concept (the distance of curves), bringing several interesting results.

On the other hand, in 2008, Adamec [2] investigated the same type of problem considering the usual “distance of functions”. To prove his results, he employed the method of Euler polygons, having a good approach.

In 2009, Esty and Hilger [20] investigated about the convergence of solutions of dynamic equations on time scales considering the Fell topology instead of Hausdorff topology, bringing interesting remarks and results about that, and also justifying the use of this topology in the framework of time scales.

In 2013, Bohner, Federson and Mesquita [12] extended these results for a more general class of functions, called *measure functional differential equations* and using the relation between them and impulsive measure functional differential equations and impulsive functional dynamic equations on time scales, they proved results concerning the convergence of the solutions of a sequence of equations to the solution of the limiting problem for all these equations.

In this work, we consider a more general equation, the so-called functional Volterra–Stieltjes integral equations, which has a kernel in its formulation. Also, the conditions that appear in our results are more general than the ones assumed in [12], even in the case that the kernel $a \equiv 1$, allowing that the involved functions have many discontinuities.

Results concerning the convergence of solutions as the one presented here are very important to researchers that work with numerical analysis and numerical simulations, since we translate the results for equations on time scales and it is very useful to employ discrete equations defined on $h\mathbb{Z}$, $h \in \mathbb{N}$, to investigate the approximations of solutions of continuous equations defined on $\mathbb{R}$.

All the results in this chapter are new and can be found in [33].

Throughout this chapter, we also assume the following conditions on functions a, a_n, f, f_n, g and g_n, $n \in \mathbb{N}$:

(B1) The functions $f, f_n\colon G([-r,0],\mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ are regulated with respect to the second variable, for $\sigma > 0$.

(B2) The functions $g, g_n\colon [t_0, t_0+\sigma] \to \mathbb{R}$ are nondecreasing and left–continuous on $(t_0, t_0+\sigma], \sigma > 0$.

(B3) The functions $a, a_n\colon [t_0, t_0+\sigma]^2 \to \mathbb{R}$ are nondecreasing with respect to the first variable and regulated with respect to the second variable, $\sigma > 0$.

(B4) The Henstock–Kurzweil–Stieltjes integrals

$$\int_{\tau_1}^{\tau_2} a(t,s) f(y_s,s)\mathrm{d}g(s) \text{ and } \int_{\tau_1}^{\tau_2} a_n(t,s) f_n(y_s,s)\mathrm{d}g_n(s)$$

exist for all $y \in G([t_0-r, t_0+\sigma],\mathbb{R}^n)$, $t \in [t_0, t_0+\sigma]$, $n \in \mathbb{N}$ and all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$.

(B5) For each $n \in \mathbb{N}$, there exist regulated functions $M, M_n\colon [t_0, t_0+\sigma] \to \mathbb{R}$ such that

$$\left\| \int_{\tau_1}^{\tau_2} (c_1 a(\tau_2,s) + c_2 a(\tau_1,s)) f(y_s,s)\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} (c_1 a(\tau_2,s) + c_2 a(\tau_1,s)) M(s)\mathrm{d}g(s)$$

and

$$\left\| \int_{\tau_1}^{\tau_2} (c_{1_n} a_n(\tau_2,s) + c_{2_n} a_n(\tau_1,s)) f_n(y_s,s)\mathrm{d}g_n(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} (c_{1_n} a_n(\tau_2,s) + c_{2_n} a_n(\tau_1,s)) M_n(s)\mathrm{d}g_n(s),$$

for all $y \in G([t_0-r, t_0+\sigma],\mathbb{R}^n)$, all $c_1, c_2, c_{1_n}, c_{2_n} \in \mathbb{R}$ and all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$.

(B6) There exist regulated functions $L, L_n\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2,s)[f(y_s,s) - f(z_s,s)]\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)| L(s) \left\| y_s - z_s \right\|_\infty \mathrm{d}g(s)$$

and

$$\left\|\int_{\tau_1}^{\tau_2} a_n(\tau_2,s)[f_n(y_s,s)-f_n(z_s,s)]\mathrm{d}g_n(s)\right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|L_n(s)\,\|y_s-z_s\|_\infty\,\mathrm{d}g_n(s),$$

for all $y,z \in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$, and all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$.

The next theorem gives us conditions in order to guarantee that the solutions of a sequence of IVPs converges to the solution of the limiting IVP. We follow some ideas from [60, Theorem 8.2].

Theorem 6.0.1. *For each $n \in \mathbb{N}$, let $r > 0$ and $\phi_n \in G([-r,0],\mathbb{R}^n)$. Consider the following sequence of equations*

$$\begin{cases} x(t) &= \phi_n(0) + \int_{t_0}^{t} a_n(t,s)f_n(x_s,s)\mathrm{d}g_n(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi_n, \end{cases} \tag{6.0.3}$$

where $\sigma > 0, f_n\colon G([-r,0],\mathbb{R}^n) \times [t_0,t_0+\sigma] \to \mathbb{R}^n, a_n\colon [t_0,t_0+\sigma]\times[t_0,t_0+\sigma] \to \mathbb{R}, g_n\colon [t_0,t_0+\sigma] \to \mathbb{R}$.

Moreover, assume that

$$\begin{aligned} \lim_{n\to\infty} a_n(t,s) &= a(t,s) \text{ uniformly on } [t_0,t_0+\sigma]^2, \\ \lim_{n\to\infty} f_n(x_s,s) &= f(x_s,s) \text{ uniformly on } G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma], \\ \lim_{n\to\infty} g_n(s) &= g(s) \text{ uniformly on } [t_0,t_0+\sigma], \\ \lim_{n\to\infty} \phi_n(t) &= \phi(t) \text{ uniformly on } [-r,0], \end{aligned}$$

where $f_n, f\colon G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma] \to \mathbb{R}^n$, $a_n, a\colon [t_0,t_0+\sigma]\times[t_0,t_0+\sigma] \to \mathbb{R}$, $g_n, g\colon [t_0,t_0+\sigma] \to \mathbb{R}$ and $\phi_n,\phi\colon [-r,0] \to \mathbb{R}^n$. Assume also that the functions ϕ and ϕ_n are regulated and a, f, g, a_n, f_n and g_n satisfy the conditions (B1)–(B6) for each $n \in \mathbb{N}$, and that the sequences $\{a_n(t,s)\}_{n\in\mathbb{N}}$, $\{M_n(s)\}_{n\in\mathbb{N}}$ and $\{L(s)\}_{n\in\mathbb{N}}$ are, each of them, uniformly bounded by the positve constants $\tilde{A}, \tilde{M}$ and $\tilde{L}$, respectively, for $t,s \in [t_0,t_0+\sigma]$.

Let $x_n\colon [t_0-r,t_0+\sigma] \to \mathbb{R}^n$ be the unique solution of (6.0.3) and assume that

$$\lim_{n\to\infty} x_n(t) = x(t), \quad t \in [t_0-r,t_0+\sigma].$$

Then $x\colon [t_0-r,t_0+\sigma] \to \mathbb{R}^n$ is the unique solution of

$$\begin{cases} x(t) &= \phi(0) + \int_{t_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi. \end{cases} \tag{6.0.4}$$

Proof. Notice that, if $t \in [t_0 - r, t_0]$, then $x_n(t) = \phi_n(t - t_0)$. By hypothesis, $\phi_n(t - t_0)$ converges uniformly to $\phi(t - t_0)$ on $[t_0 - r, t_0]$ and $x_n(t)$ converges to $x(t)$ on $[t_0 - r, t_0 + \sigma]$. By the uniqueness of the limit, we get that $x(t) = \phi(t - t_0)$, and hence $x_{t_0} = \phi$.

Since $g_n(s) \to g(s)$ uniformly on $[t_0, t_0 + \sigma]$, it follows that $g \to g_n$ pointwisely on $[t_0, t_0 + \sigma]$. This fact implies that $g_n(t_0) \to g(t_0)$ and $g_n(t_0 + \sigma) \to g(t_0 + \sigma)$. As a consequence, the sequences $\{g_n(t_0)\}_{n=1}^{\infty}$ and $\{g_n(t_0 + \sigma))\}_{n=1}^{\infty}$ are bounded. Hence, the function g_n is of bounded variation on $[t_0, t_0 + \sigma]$. More precisely, there exists a constant $G \geqslant 0$ such that $V_{t_0}^{t_0+\sigma}(g_n) = g_n(t_0 + \sigma) - g_n(t_0) \leqslant G$ for all $n \in \mathbb{N}$, where $V_a^b(g)$ denotes the total variation of a function g on the interval $[a, b]$.

Therefore, this fact implies that the integral

$$\int_{t_0}^{t_0+\sigma} a(t,s) f(x_s, s) \mathrm{d}(g_n - g)(s)$$

exists in the sense of Henstock–Kurzweil-Stieltjes, since $s \mapsto f(x_s, s)$ and $s \mapsto a(t, s)$ are regulated and $g_n - g$ is BV. Thus, by Theorem 1.2.7, we have that

$$\lim_{n\to\infty} \int_{t_0}^{t} a(t,s) f(x_s, s) \mathrm{d}(g_n - g)(s) = 0$$

uniformly with respect to $t \in [t_0, t_0 + \sigma]$. Hence, for an arbitrary $\varepsilon > 0$, there exists $\overline{n} \in \mathbb{N}$ such that

$$\left\| \int_{t_0}^{t} a(t,s) f(x_s, s) \mathrm{d}(g_n - g)(s) \right\| \leqslant \varepsilon, \text{ for all } n \geqslant \overline{n} \text{ and } t \in [t_0, t_0 + \sigma].$$

Assume now that $t \in [t_0, t_0 + \sigma]$ and fix $\varepsilon > 0$. By the uniform convergence, there exists $\tilde{n}$ sufficiently large such that for $n > \tilde{n}$, we have:

$$|a_n(t,s) - a(t,s)| < \varepsilon, \quad (t,s) \in [t_0, t_0 + \sigma]^2 \tag{6.0.5}$$

$$\|f_n(x_s, s) - f(x_s, s)\| < \varepsilon, \quad (x, s) \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \tag{6.0.6}$$

$$|g_n(s) - g(s)| < \varepsilon, \quad s \in [t_0, t_0 + \sigma] \tag{6.0.7}$$

$$\|\phi_n(s) - \phi(s)\| < \varepsilon. \quad s \in [t_0, t_0 + \sigma]. \tag{6.0.8}$$

Suppose $y\colon [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is a solution of (6.0.4), we shall prove that $y = x$.

For $t \in [t_0, t_0 + \sigma]$, we have

$$x_n(t) = \phi_n(0) + \int_{t_0}^{t} a_n(t,s) f_n(x_s, s) \mathrm{d}g_n(s)$$

and

$$y(t) = \phi(0) + \int_{t_0}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s).$$

Therefore, for $t \in [t_0, t_0 + \sigma]$ and $n > \max\{\overline{n}, \tilde{n}\}$, we have

$$\|x_n(t) - y(t)\| = \left\| \phi_n(0) + \int_{t_0}^{t} a_n(t,s) f_n((x_n)_s, s) \mathrm{d}g_n(s) - \phi(0) - \int_{t_0}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s) \right\|$$

$$\overset{(6.0.8)}{\leqslant} \varepsilon + \left\| \int_{t_0}^{t} a_n(t,s) f_n((x_n)_s, s) \mathrm{d}g_n(s) - \int_{t_0}^{t} a_n(t,s) f_n(y_s, s) \mathrm{d}g_n(s) \right\|$$

$$+ \left\| \int_{t_0}^{t} a_n(t,s) f_n(y_s, s) \mathrm{d}g_n(s) - \int_{t_0}^{t} a(t,s) f_n(y_s, s) \mathrm{d}g_n(s) \right\|$$

$$+ \left\| \int_{t_0}^{t} a(t,s) f_n(y_s, s) \mathrm{d}g_n(s) - \int_{t_0}^{t} a(t,s) f(y_s, s) \mathrm{d}g_n(s) \right\|$$

$$+ \left\| \int_{t_0}^{t} a(t,s) f(y_s, s) \mathrm{d}g_n(s) - \int_{t_0}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s) \right\|.$$

Using conditions (B5) and (B6), the estimates (6.0.5), (6.0.6) and Theorem 1.2.7, we obtain, for $n > \max\{\overline{n}, \tilde{n}\}$ and $t \in [t_0, t_0 + \sigma]$,

$$\|x_n(t) - y(t)\|$$

$$\leqslant \int_{t_0}^{t} |a_n(t,s)| L_n(s) \|x_{n_s} - y_s\|_\infty \mathrm{d}g_n(s) + \varepsilon \int_{t_0}^{t} M_n(s) \mathrm{d}g_n(s) + \varepsilon \left\| \int_{t_0}^{t} a(t,s) \mathrm{d}g_n(s) \right\| + 2\varepsilon$$

$$\leqslant \int_{t_0}^{t} |a_n(t,s)| L_n(s) \|x_{n_s} - y_s\|_\infty \mathrm{d}g_n(s) + \varepsilon(\tilde{M}G + \tilde{A}G + 2).$$

Therefore,

$$\begin{aligned} \|x_n(t) - y(t)\| &\leqslant \int_{t_0}^{t} |a_n(t,s)| L_n(s) \sup_{\theta \in [-r,0]} \|x_n(s+\theta) - y(s+\theta)\| \mathrm{d}g_n(s) \\ &\qquad + \varepsilon(\tilde{M}G + \tilde{A}G + 2) \\ &= \int_{t_0}^{t} |a_n(t,s)| L_n(s) \sup_{\theta \in [s-r,s]} \|x_n(\theta) - y(\theta)\| \mathrm{d}g_n(s) + \varepsilon(\tilde{M}G + \tilde{A}G + 2) \\ &\leqslant \tilde{A}\tilde{L} \int_{t_0}^{t} \sup_{\theta \in [s-r,s]} \|x_n(s) - y(s)\| \mathrm{d}g_n(s) + \varepsilon(\tilde{M}G + \tilde{A}G + 2). \end{aligned} \tag{6.0.9}$$

Since the right–hand side of (6.0.9) is nondecreasing, we have

$$\sup_{\eta\in[t-r,t]}\|x_n(\eta)-y(\eta)\| \leqslant \tilde{A}\tilde{L}\int_{t_0}^{t}\sup_{\theta\in[s-r,s]}\|x_n(s)-y(s)\|\mathrm{d}g_n(s)+\varepsilon(\tilde{M}G+\tilde{A}G+2) \quad (6.0.10)$$

Using Gronwall Inequality for Stieltjes integrals (Lemma 1.2.10), we obtain

$$\|x_n(t)-y(t)\| \leqslant \varepsilon(\tilde{M}G+\tilde{A}G+2)e^{\tilde{A}\tilde{L}G}, \quad (6.0.11)$$

and the result follows. □

In the sequel, we state the well-known Helly's First Choice Theorem. It will be important to prove our next result.

Theorem 6.0.2 ([55, Helly's First Choice Theorem]). *Let an infinite family of functions $F=\{f(x)\}$ be defined on the segment $[a,b]$. If all functions of the family and the total variation of all functions of the family are bounded by a single number*

$$\|f(x)\| \leqslant K, \quad var_a^b(f) \leqslant K,$$

then there exists a sequence $\{f_n(x)\}$ in the family F which converges at every point of $[a,b]$ to some function $\phi(x)$ of finite variation.

The next result on continuous dependence allows us to construct a sequence of functions formed by solutions of a sequence of problems that converges to our solution of a limiting problem. It is a type of "inverse" problem. This result is very useful to prove nonperiodic averaging principles, since we need to deal with some convergences and continuous dependence results to get those results (see [60]). The proof of the next result is inspired by [60, Theorem 8.6].

Theorem 6.0.3. *For each $n\in\mathbb{N}$, let $r>0$, $\phi_n\in G([-r,0],\mathbb{R}^n)$ and consider the sequence of problems:*

$$\begin{cases} x(t) &= \phi_n(0)+\displaystyle\int_{t_0}^{t}a_n(t,s)f_n(x_s,s)\mathrm{d}g(s), \quad t\geqslant t_0 \\ x_{t_0} &= \phi_n \end{cases} \quad (6.0.12)$$

where $\sigma>0, f_n\colon G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma]\to\mathbb{R}^n, a_n\colon[t_0,t_0+\sigma]\times[t_0,t_0+\sigma]\to\mathbb{R}, g\colon[t_0,t_0+\sigma]\to\mathbb{R}$ satisfy conditions (B1)–(B6), with $M_n(t)=M(t), L_n(t)=L(t)$ for all $n\in\mathbb{N}$ and $t\in[t_0,t_0+\sigma]$.

Let us also assume that $\{a_n\}_{n=1}^{\infty}$ *is uniformly bounded in compact sets and that there exist functions* $a\colon [t_0, t_0+\sigma]\times[t_0,t_0+\sigma]\to\mathbb{R}$ *and* $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma]\to\mathbb{R}^n$ *satisfying conditions (B1)–(B6) such that:*

$$\begin{aligned}
\lim_{n\to\infty}\phi_n &= \phi \text{ uniformly on } [-r,0],\\
\lim_{n\to\infty}a_n(t,s) &= a(t,s) \text{ uniformly on } [t_0,t_0+\sigma]^2,\\
\lim_{n\to\infty}f_n(x_s,s) &= f(x_s,s) \text{ uniformly on } G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma],
\end{aligned}$$

for every $x\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$, $t,s\in[t_0,t_0+\sigma]$.

Let $x_0\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ *be the unique solution of*

$$\begin{cases} x(t) = \phi(0)+\displaystyle\int_{t_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s), & t\geqslant t_0\\ x_{t_0} = \phi. \end{cases} \tag{6.0.13}$$

Then there exists a sequence of solutions $x_n\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ *of* (6.0.12) *such that* $x_n\to x$ *as* $n\to\infty$.

Proof. The existence of solutions of (6.0.12) is guaranteed by Theorem 3.1.9, since all hypotheses are satisfied. Thus, it remains to prove that the sequence of solutions $\{x_n\}_{n=1}^{\infty}$ of (6.0.12) converges to x_0.

Since $\{a_n\}_{n=1}^{\infty}$ is uniformly bounded in compact sets, there exists a regulated function $\hat{a}\colon [t_0,t_0+\sigma]\times[t_0,t_0+\sigma]\to\mathbb{R}$, nondecreasing with respect to the first variable such that $\hat{a}(t,s)\geqslant a_n(t,s)$ for every $(t,s)\in[t_0,t_0+\sigma]\times[t_0,t_0+\sigma]$ and $n\in\mathbb{N}$.

Define the function $h\colon [t_0,t_0+\sigma]\to\mathbb{R}$ by

$$h(t)\colon = \int_{t_0}^{t} cM(s)\mathrm{d}g(s)+\int_{t_0}^{t_0+\sigma}\hat{a}(t,s)M(s)\mathrm{d}g(s). \tag{6.0.14}$$

The function h is nondecreasing, since g is nondecreasing and for $s,t\in[t_0,t_0+\sigma]$, $s>t$ $\|x_n(s)-x_n(t)\|\leqslant h(s)-h(t)$ by assumptions (B1)–(B6) and using the fact that x_n is a solution of (6.0.12). Therefore, it follows that x_n is of bounded variation and the total variation of x_n, $V_{t_0}^{t_0+\sigma}(x_n)$, satisfies $V_{t_0}^{t_0+\sigma}(x_n)\leqslant h(t_0+\sigma)-h(t_0)$ for all $n\in\mathbb{N}$.

Since $\phi_n\to\phi$ uniformly and, for each $n\in\mathbb{N}$, ϕ_n is a regulated function, it follows that ϕ is also regulated (because it is the uniform limit of regulated functions). Hence, there exists a positive constant K such that $\|\phi_n(t)\|\leqslant K$ and $\|\phi(t)\|\leqslant K$ for all $t\in[-r,0]$ and $n\in\mathbb{N}$. Therefore, for $t\geqslant t_0$

$$
\begin{aligned}
\|x_n(t)\| &= \|\phi_n(0)\| + \left\| \int_{t_0}^{t} a_n(t,s) f_n(x_{n_s}, s) \mathrm{d}g(s) \right\| \\
&\leqslant K + \int_{t_0}^{t} AM(s) \mathrm{d}g(s) \\
&\leqslant K + \int_{t_0}^{t_0} +\sigma AM(s) \mathrm{d}g(s),
\end{aligned}
$$

where A is a uniform bound of $\{a_n\}_{n=1}^{\infty}$.

Notice also that if $t \in [t_0 - r, t_0]$, we can write $t = t_0 + \theta$, where $\theta \in [-r, 0]$. Hence, in this case, $\|x_n(t)\| = \|x_n(t_0 + \theta)\| = \|(x_n)_{t_0}(\theta)\| = \|\phi_n(\theta)\| \leqslant K$, $\theta \in [-r, 0]$

Since the function M is regulated, it is also integrable in the sense of Henstock–Kurzweil–Stieltjes, then, for each $n \in \mathbb{N}$, the functions $\{x_n\}_{n=1}^{\infty}$ is uniformly bounded on $[t_0, t_0 + \sigma]$. Hence, by Helly's First Choice Theorem (Theorem 6.0.2), $\{x_n\}_{n=1}^{\infty}$ has a subsequence that converges to a function y. By Theorem 6.0.1 and from the uniqueness of solutions, we have that y must be x_0, and we finish the proof of this Theorem. □

6.1 Continuous dependence on impulsive equations

In this section, we will show that we can use the correspondence obtained in Section 2.1 to obtain the analogues of Theorem 6.0.1 and Theorem 6.0.3 also hold for impulsive functional Volterra–Stieltjes integral equations.

In this section, let the functions f, f_n, a, a_n, g and g_n be defined as in the first part of this chapter. Also, let us assume that conditions (B1)–(B6) are satisfied. We will also consider the functions $\tilde{f}, \tilde{f}_n, \tilde{g}$ and $\tilde{g}_n$ which will be defined in the same manner that it was done in Section 2.1, that is:

- $\tilde{f} \colon G([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ by

$$
\tilde{f}(y, \tau) = \begin{cases} f(y, \tau), & \tau \in [t_0, t_0 + \sigma) \backslash \{t_1, \dots, t_m\}, \\ I_k(y(0)), & \tau = t_k,\ k \in \{1, \dots, m\}, \end{cases}
$$

- $\tilde{g}\colon [t_0, t_0+\sigma] \to \mathbb{R}$ be given by

$$\tilde{g}(\tau) = \begin{cases} g(\tau), & \tau \in [t_0, t_1], \\ g(\tau) + k, & \tau \in (t_k, t_{k+1}],\ k \in \{1, \dots, m-1\}, \\ g(\tau) + m, & \tau \in (t_m, t_0+\sigma]. \end{cases}$$

The functions $\tilde{f}_n$ and $\tilde{g}_n$ are defined analogously. The next theorem shows that we can obtain the analogous result as in Theorem 6.0.1 to impulsive functional Volterra–Stieltjes integral equations.

Theorem 6.1.1. *For each $n \in \mathbb{N}$, let $r > 0$ and $\phi_n \in G([-r, 0], \mathbb{R}^n)$. Consider the following sequence of equations*

$$\begin{cases} x(t) &= \phi_n(0) + \displaystyle\int_{t_0}^{t} a_n(t,s) f_n(x_s, s) \mathrm{d}g_n(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ t_0 < t_k < t}} a_n(t, t_k) I_k(x(t_k)), \quad t \geqslant t_0 \\ x_{t_0} &= \phi_n, \end{cases} \tag{6.1.1}$$

where $\{t_k\}_{k=1}^{m}$ are the moments of impulses and each $t_k \in [t_0, t_0+\sigma]$.

Moreover, assume that

$$\begin{aligned} \lim_{n\to\infty} a_n(t,s) &= a(t,s) \text{ uniformly on } [t_0, t_0+\sigma]^2, \\ \lim_{n\to\infty} f_n(x_s,s) &= f(x_s,s) \text{ uniformly on } G([-r,0],\mathbb{R}^n) \times [t_0, t_0+\sigma], \\ \lim_{n\to\infty} g_n(s) &= g(s) \text{ uniformly on } [t_0, t_0+\sigma], \\ \lim_{n\to\infty} \phi_n(t) &= \phi(t) \text{ uniformly on } [-r, 0], \end{aligned}$$

where $f_n, f\colon G([-r,0],\mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$, $a_n, a\colon [t_0, t_0+\sigma] \times [t_0, t_0+\sigma] \to \mathbb{R}$, $g_n, g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ and $\phi_n, \phi\colon [-r, 0] \to \mathbb{R}^n$. Assume also that the functions ϕ and ϕ_n are regulated and a, f, g, a_n, f_n and g_n satisfy the conditions (B1)–(B6) for each $n \in \mathbb{N}$, and that the sequences $\{a_n(t,s)\}_{n\in\mathbb{N}}$, $\{M_n(s)\}_{n\in\mathbb{N}}$ and $\{L(s)\}_{n\in\mathbb{N}}$ are, each of them, uniformly bounded by constants $\tilde{A}, \tilde{M}$ and $\tilde{L}$, respectively, for $t, s \in [t_0, t_0+\sigma]$.

Let $x_n\colon [t_0 - r, t_0+\sigma] \to \mathbb{R}^n$ be the unique solution of (6.1.1) and assume that

$$\lim_{n\to\infty} x_n(t) = x(t), \quad t \in [t_0 - r, t_0+\sigma].$$

Then $x\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *is the unique solution of*

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ t_0 < t_k < t}} a(t, t_k) I_k(x(t_k)), \quad t \geqslant t_0 \\ x_{t_0} &= \phi. \end{cases} \tag{6.1.2}$$

Proof. Using Theorem 2.1.2 and by hypotheses, we know that x_n is also the unique solution of

$$\begin{cases} x(t) &= \phi_n(0) + \displaystyle\int_{t_0}^{t} a_n(t,s) \tilde{f}_n(x_s, s) \mathrm{d}\tilde{g}_n(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi_n, \end{cases} \tag{6.1.3}$$

for each $n \in \mathbb{N}$.

In view of the definitions of $a_n, \tilde{f}_n$ and $\tilde{g}_n$, it is clear that these functions converge to $a, \tilde{f}$ and $\tilde{g}$, respectively, uniformly. Therefore, we can use Theorem 6.0.1 to obtain that x is the solution of

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) \tilde{f}(x_s, s) \mathrm{d}\tilde{g}(s), \quad t \geqslant t_0 \\ x_{t_0} &= \phi, \end{cases} \tag{6.1.4}$$

since $\lim_{n\to\infty} x_n(t) = x(t), \quad t \in [t_0 - r, t_0 + \sigma]$.

Using Theorem 2.1.2 once again, we conclude that x is the unique solution of

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ t_0 < t_k < t}} a(t, t_k) I_k(x(t_k)), \quad t \geqslant t_0 \\ x_{t_0} &= \phi, \end{cases} \tag{6.1.5}$$

as desired. □

Proceeding in a similar manner as in the previous theorem, we can also obtain an analogue result to Theorem 6.0.3 for the corresponding equations in the impulsive case.

Theorem 6.1.2. *For each* $n \in \mathbb{N}$, *let* $r > 0$, $\phi_n \in G([-r, 0], \mathbb{R}^n)$ *and consider the sequence of problems:*

$$\begin{cases} x(t) &= \phi_n(0) + \displaystyle\int_{t_0}^{t} a_n(t,s) f_n(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ t_0 < t_k < t}} a_n(t, t_k) I_k(x(t_k)), \quad t \geqslant t_0 \\ x_{t_0} &= \phi_n \end{cases} \tag{6.1.6}$$

where $\{t_k\}_{k=1}^m$ *are the moments of impulses and each* $t_k \in [t_0, t_0+\sigma]$*. Also,* $f_n\colon G([-r,0],\mathbb{R}^n)\times [t_0,t_0+\sigma]\to\mathbb{R}^n$, $a_n\colon [t_0,t_0+\sigma]\times[t_0,t_0+\sigma]\to\mathbb{R}$, $g\colon [t_0,t_0+\sigma]\to\mathbb{R}$ *satisfy conditions (B1)–(B6), with* $M_n(t)=M(t), L_n(t)=L(t)$ *for all* $n\in\mathbb{N}$ *and* $t\in[t_0,t_0+\sigma]$*. Let us also assume that* $\{a_n\}_{n=1}^\infty$ *is uniformly bounded in compact sets and that there exist functions* $a\colon [t_0,t_0+\sigma]\times[t_0,t_0+\sigma]\to\mathbb{R}$ *and* $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma]\to\mathbb{R}^n$ *satisfying conditions (B1)–(B6) such that:*

$$\begin{aligned}
\lim_{n\to\infty}\phi_n &= \phi \text{ uniformly on } [-r,0],\\
\lim_{n\to\infty}a_n(t,s) &= a(t,s) \text{ uniformly on } [t_0,t_0+\sigma]^2,\\
\lim_{n\to\infty}f_n(x_s,s) &= f(x_s,s) \text{ uniformly on } G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma],
\end{aligned}$$

for every $x\in G([t_0-r,t_0+\sigma],\mathbb{R}^n)$, $t,s\in[t_0,t_0+\sigma]$*.*

Let $x_0\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ *be the unique solution of*

$$\begin{cases} x(t) = \phi(0)+\displaystyle\int_{t_0}^t a(t,s)f(x_s,s)\mathrm{d}g(s)+\sum_{\substack{k\in\{1,\dots,m\}\\ t_0<t_k<t}} a(t,t_k)I_k(x(t_k)), & t\geqslant t_0\\ x_{t_0} = \phi. \end{cases} \tag{6.1.7}$$

Then there exists a sequence of solutions $x_n\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ *of* (6.1.6) *such that* $x_n\to x_0$ *as* $n\to\infty$*.*

Proof. By Theorem 2.1.2 and by hypotheses, x_0 is also the unique solution of

$$\begin{cases} x(t) = \phi(0)+\displaystyle\int_{t_0}^t a(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s), & t\geqslant t_0\\ x_{t_0} = \phi. \end{cases} \tag{6.1.8}$$

By hypothesis and the definitions of $a, a_n, \tilde{f}, \tilde{f}_n, \tilde{g}$ and $\tilde{g}_n$, it follows that $a_n\to a$, $\tilde{f}_n\to\tilde{f}$ and $\tilde{g}_n\to\tilde{g}$ uniformly.

Hence, by Theorem 6.0.3, there exists a sequence of solutions $x_n\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ of

$$\begin{cases} x(t) = \phi_n(0)+\displaystyle\int_{t_0}^t a_n(t,s)\tilde{f}_n(x_s,s)\mathrm{d}\tilde{g}(s), & t\geqslant t_0\\ x_{t_0} = \phi_n \end{cases} \tag{6.1.9}$$

such that $x_n\to x_0$. By Theorem 2.1.2, once again, x_n, for each $n\in\mathbb{N}$, is also a solution of the equation (6.1.6), which concludes the proof.

□

6.2 Continuous dependence on time scales

In this section, we want to prove analogues results for a sequence of functional Volterra–Stieltjes Δ- integral equations on time scales.

Assume the following conditions concerning the functions $f\colon G([-r,0]_{\mathbb{T}},\mathbb{R}^n)\times[t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^n$ and $a\colon [t_0,t_0+\sigma]_{\mathbb{T}}^2\to\mathbb{R}$.

(C1) The function $a\colon [t_0,t_0+\sigma]_{\mathbb{T}}^2\to\mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and rd–continuous.

(C2) The Henstock–Kurzweil Δ–integral

$$\int_{s_1}^{s_2} a(\tau,s)f(x_s,s)\Delta s$$

exists for each time scale interval $[s_1,s_2]_{\mathbb{T}}\subset[t_0,t_0+\sigma]_{\mathbb{T}}$, $x\in G([t_0-r,t_0+\sigma]_{\mathbb{T}},\mathbb{R}^n)$, $\tau\in[t_0,t_0+\sigma]_{\mathbb{T}}$.

(C3) There exists a Henstock–Kurzweil Δ-integrable function $M_1\colon [t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^+$ such that

$$\left\|\int_{s_1}^{s_2}(c_1a(s_2,s)+c_2a(s_1,s))f(x_s,s)\Delta s\right\|\leqslant\int_{s_1}^{s_2}M_1(s)\left|c_1a(s_2,s)+c_2a(s_1,s)\right|\Delta s,$$

for all $x\in G([t_0-r,t_0+\sigma]_{\mathbb{T}},\mathbb{R}^n)$, $c_1,c_2\in\mathbb{R}$ and $s_1,s_2\in[t_0,t_0+\sigma]_{\mathbb{T}}, s_1\leqslant s_2$.

(C4) There exists a regulated function $L_1\colon [t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^+$ such that

$$\left|\int_{s_1}^{s_2}a(s_2,s)[f(x_s,s)-f(z_s,s)]\Delta s\right|\leqslant\int_{s_1}^{s_2}L_1(s)\left|a(s_2,s)\right|\|x_s-z_s\|_\infty\Delta s,$$

for all $x,z\in G([t_0-r,t_0+\sigma]_{\mathbb{T}},\mathbb{R}^n)$ and $s_1,s_2\in[t_0,t_0+\sigma]_{\mathbb{T}}$, $s_1\leqslant s_2$.

Theorem 6.2.1. *Let $\mathbb{T}_n$ be a sequence of time scales. For some $\sigma>0$ and each $n\in\mathbb{N}$, assume that $t_0,t_0+\sigma\in\mathbb{T}$. Consider the following sequence of equations:*

$$\begin{cases} x(t)= & \phi(0)+\displaystyle\int_{t_0}^{t}a_n(t,s)f_n(x_s^*,s)\Delta s, \quad t\in[t_0,t_0+\sigma]_{\mathbb{T}_n}\\ x(t)= & \phi(t),\ t\in[t_0-r,t_0]_{\mathbb{T}_n},\end{cases} \tag{6.2.1}$$

where $\phi_n\in G([t_0-r,t_0]_{\mathbb{T}_n},\mathbb{R}^n)$, $f_n\colon G([-r,0]_{\mathbb{T}_n},\mathbb{R}^n)\times[t_0,t_0+\sigma]_{\mathbb{T}_n}\to\mathbb{R}^n$ and $a_n\colon [t_0,t_0+\sigma]_{\mathbb{T}_n}\times[t_0,t_0+\sigma]_{\mathbb{T}_n}\to\mathbb{R}$. Also define the functions $g_n\colon [t_0,t_0+\sigma]\to\mathbb{T}_n$ by $g_n(s)=s^*$ and $g\colon [t_0,t_0+\sigma]\to\mathbb{T}$ by $g(s)=s^*$.

Moreover, assume that

$$\begin{aligned}
\lim_{n\to\infty}\phi_n^*(t) &= \phi^*(t) \text{ uniformly on } [t_0-r,t_0], && (6.2.2)\\
\lim_{n\to\infty}a_n^{**}(t,s) &= a^{**}(t,s) \text{ uniformly on } [t_0,t_0+\sigma], && (6.2.3)\\
\lim_{n\to\infty}f_n^*(x_s,s) &= f^*(x_s,s) \text{ uniformly on } G([-r,0],\mathbb{R}^n)\times[t_0,t_0+\sigma], && (6.2.4)\\
\lim_{n\to\infty}g_n(s) &= g(s) \text{ uniformly on } [t_0,t_0+\sigma] && (6.2.5)
\end{aligned}$$

where $f\colon G([-r,0]_{\mathbb{T}},\mathbb{R}^n)\times[t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^n$, $a\colon [t_0,t_0+\sigma]_{\mathbb{T}}\times[t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}$, $g\colon [t_0,t_0+\sigma]_{\mathbb{T}}\to\mathbb{T}$ *and the functions* a, f, a_n *and* f_n *satisfy the conditions (C1)–(C4) for each* $n\in\mathbb{N}$. *Let* $x_n\colon [t_0-r,t_0+\sigma]_{\mathbb{T}_n}\to\mathbb{R}^n$ *be the solution of*

$$\begin{cases} x(t) &= \phi_n(0)+\displaystyle\int_{t_0}^{t} a_n(t,s)f_n(x_s^*,s)\Delta s \quad t\in[t_0,t_0+\sigma]_{\mathbb{T}_n}\\ x(t) &= \phi_n(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}_n}\end{cases} \tag{6.2.6}$$

and assume that

$$\lim_{n\to\infty}x_n(t)=x(t). \tag{6.2.7}$$

Then $x\colon [t_0-r,t_0+\sigma]_{\mathbb{T}}\to\mathbb{R}^n$ *is the solution of*

$$\begin{cases} x(t) &= \phi(0)+\displaystyle\int_{0}^{t} a(t,s)f(x_s^*,s)\Delta s \quad t\in[t_0,t_0+\sigma]_{\mathbb{T}}\\ x(t) &= \phi(t), \quad t\in[t_0-r,t_0]_{\mathbb{T}}.\end{cases} \tag{6.2.8}$$

Proof. First, we call the reader's attention to the fact that $\phi_n^*(t)=\phi_n(t^*)$, where $t^*=\inf\{\xi\in\mathbb{T}_n:\xi\geqslant t\}$, whereas $\phi^*(t)=\phi(t^*)$, where $t^*=\inf\{\xi\in\mathbb{T}:\xi\geqslant t\}$. Analogously, we define the functions a^{**}, a_n^{**}, f^*, and f_n^*.

Notice that, by Theorem 2.2.1, x_n^* is a solution of

$$\begin{cases} x(t) &= x(t_0)+\displaystyle\int_{t_0}^{t} a_n^{**}(t,s)f_n^*(x_s,s)\,\mathrm{d}g_n(s) \quad t\in[t_0,t_0+\sigma]\\ x_{t_0} &= (\phi_n^*)_{t_0}.\end{cases} \tag{6.2.9}$$

Since for all $t\in[t_0-r,t_0+\sigma]_{\mathbb{T}}$, $\lim_{n\to\infty}x_n(t)=x(t)$, it follows that

$$\lim_{n\to\infty}x_n^*(t)=x^*(t),$$

where $x^*\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$ and $x_n^*\colon [t_0-r,t_0+\sigma]\to\mathbb{R}^n$. Hence, by Theorem 6.0.1, we obtain that x^* is a solution of

$$\begin{cases} x(t) &= x(t_0) + \displaystyle\int_{t_0}^{t} a^{**}(t,s) f^*(x_s^*, s) \mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma] \\ x_{t_0} &= (\phi^*)_{t_0}. \end{cases} \tag{6.2.10}$$

Using Theorem 2.2.1 again, we obtain that $x\colon [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of

$$\begin{cases} x(t) &= x(t_0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s^*, s) \Delta s, \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}}, \\ x(t) &= \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \end{cases} \tag{6.2.11}$$

as desired, proving the result. $\square$

Theorem 6.2.2. *Let $\mathbb{T}$ be a time scale and let $\mathbb{T}_n$ be a sequence of time scales such that $t_0, t_0+\sigma \in \mathbb{T}$ and $t_0, t_0+\sigma \in \mathbb{T}_n$. For each $n \in \mathbb{N}$, consider the following sequence of equations:*

$$\begin{cases} x(t) = & \phi_n(0) + \displaystyle\int_{t_0}^{t} a_n(t,s) f_n(x_s^*, s) \Delta s, \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}_n} \\ x(t) = & \phi_n(t), t \in [t_0 - r, t_0]_{\mathbb{T}_n}, \end{cases} \tag{6.2.12}$$

where $\phi_n \in G([t_0 - r, t_0], \mathbb{R}^n)$, $f_n\colon G([-r,0]_{\mathbb{T}_n}, \mathbb{R}^n) \times [t_0, t_0+\sigma]_{\mathbb{T}_n} \to \mathbb{R}^n$ and $a_n\colon [t_0, t_0+\sigma]^2_{\mathbb{T}_n} \to \mathbb{R}$. Also define the functions $g_n\colon [t_0, t_0+\sigma] \to \mathbb{T}_n$ by $g_n(s) = s^*$ and $g\colon [t_0, t_0+\sigma] \to \mathbb{T}$ by $g(s) = s^*$.

Moreover, assume that, for $t, s \in \mathbb{T}$

$$\lim_{n\to\infty} \phi_n^*(t) = \phi^*(t) \text{ uniformly on } [t_0 - r, t_0], \tag{6.2.13}$$

$$\lim_{n\to\infty} a_n^{**}(t,s) = a^{**}(t,s) \text{ uniformly on } [t_0, t_0+\sigma]^2, \tag{6.2.14}$$

$$\lim_{n\to\infty} f_n^*(x_s, s) = f^*(x_s, s) \text{ uniformly on } G([-r,0], \mathbb{R}^n) \times [t_0, t_0+\sigma], \tag{6.2.15}$$

$$\lim_{n\to\infty} g_n(s) = g(s) \text{ uniformly on } [t_0, t_0+\sigma], \tag{6.2.16}$$

where $f\colon G([-r,0], \mathbb{R}^n) \times [t_0, t_0+\sigma]_{\mathbb{T}} \to \mathbb{R}^n, a\colon [t_0, t_0+\sigma]^2_{\mathbb{T}} \to \mathbb{R}$, $g\colon [t_0, t_0+\sigma] \to \mathbb{T}$. Assume also that the functions a, f, a_n and f_n satisfy the conditions (C1)–(C4) for each $n \in \mathbb{N}$ and that there exists a nondecreasing left continuous function $\hat{g}\colon \mathbb{R} \to \mathbb{R}$, such that the conditions (B5) and (B6) can be rewritten as:

(B5) *For each $n \in \mathbb{N}$, there exist regulated functions $M, M_n\colon [t_0, t_0+\sigma] \to \mathbb{R}$ such that*

$$\left\| \int_{\tau_1}^{\tau_2} (c_1 a(\tau_2, s) + c_2 a(\tau_1, s)) f(y_s, s) \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} (c_1 a(\tau_2, s) + c_2 a(\tau_1, s)) M(s) \mathrm{d}g(s)$$

and

$$\left\| \int_{\tau_1}^{\tau_2} (c_{1_n} a_n(\tau_2, s) + c_{2_n} a_n(\tau_1, s)) f_n(y_s, s) \mathrm{d}g_n(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} (c_{1_n} a_n(\tau_2, s) + c_{2_n} a_n(\tau_1, s)) M_n(s) \mathrm{d}\hat{g}(s),$$

for all $y \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, *all* $c_1, c_2, c_{1_n}, c_{2_n} \in \mathbb{R}$ *and all* $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$.

(B6) *There exist regulated functions* $L, L_n \colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ *such that*

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(y_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| L(s) \|y_s - z_s\|_\infty \mathrm{d}g(s)$$

and

$$\left\| \int_{\tau_1}^{\tau_2} a_n(\tau_2, s)[f_n(y_s, s) - f_n(z_s, s)] \mathrm{d}g_n(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| L_n(s) \|y_s - z_s\|_\infty \mathrm{d}\hat{g}(s),$$

for all $y, z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, *and all* $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$.

In this case, let $x \colon [t_0, t_0 + \sigma]_{\mathbb{T}} \to \mathbb{R}^n$ *be the solution of*

$$\begin{cases} x(t) & = \phi(0) + \displaystyle\int_0^t a(t, s) f(x_s^*, s) \Delta s \quad t \in [t_0, t_0 + \sigma]_{\mathbb{T}} \\ x(t) & = \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{cases} \tag{6.2.17}$$

Then there exists a sequence of solutions $x_n \colon [t_0 - r, t_0 + \sigma]_{\mathbb{T}_n} \to \mathbb{R}^n$ *of* (6.2.12) *such that* $x_n \to x$ *as* $n \to \infty$.

Proof. Notice that, by Theorem 2.2.1, $x^* \colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a solution of

$$\begin{cases} x(t) & = & x(t_0) + \displaystyle\int_{t_0}^t a^{**}(t, s) f^*(x_s, s) \, \mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma] \\ x_{t_0} & = & \phi_{t_0}^*. \end{cases} \tag{6.2.18}$$

Consider the following sequence of problems:

$$\begin{cases} x^*(t) & = & x^*(t_0) + \displaystyle\int_{t_0}^t a_n^{**}(t, s) f_n^*(x_s^*, s) \, \mathrm{d}g_n(s), \quad t \in [t_0, t_0 + \sigma] \\ x_{t_0}^* & = & (\phi_n^*)_{t_0}. \end{cases} \tag{6.2.19}$$

By Theorem 6.0.3, the uniform limits and considering the function h defined by

$$h(t) \colon = \int_{t_0}^t c M(s) \mathrm{d}\hat{g}(s) + \int_{t_0}^{t_0 + \sigma} \hat{a}(t, s) M(s) \mathrm{d}\hat{g}(s), \tag{6.2.20}$$

we conclude that the sequence of solutions of (6.2.19) converges to the solution of (6.2.18).

Now, using Theorem 2.2.1 once again to go back to the original sequence of problems on time scales, we obtain that the sequence of solutions of (6.2.12) converges to the solution of (6.2.17), proving the desired result. □

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