TESE DE DOUTORADO

Dynamic output feedback for Takagi-Sugeno fuzzy systems subjected to inexact premise variables matching

Tássio Melo Linhares

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RESUMO EXPANDIDO

Título: Realimentação dinâmica de saída de sistemas fuzzy Takagi-Sugeno sujeitos à correspondência inexata de variáveis premissas
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Este trabalho apresenta novas condições de projeto de controladores de realimentação dinâmica de saída de ordem completa para sistemas fuzzy Takagi-Sugeno (T-S) contínuos e discretos no tempo. O controlador de saída fuzzy pode ter uma quantidade de regras e um conjunto de funções de pertinência diferente do modelo T-S da planta permitindo a seleção de variávies premissas usadas pelo controlador. Essa caracaterística permite lidar com cenários importantes presentes em muitas aplicações: variáveis premissas totalmente ou parcialmente não medidas ou medidas com imprecisão. O principal aspecto da metodologia proposta é apresentar condições em que os ganhos do controlador são independentes das variáveis premissas que não podem ser medidas, permitindo maior flexibilidade para o projetista em cenários reais. As condições são expressas como inequações matriciais lineares combinadas com parâmetros escalares que fornecem graus de liberdade extra. A metodologia de controle proposta também lida com incertezas de modelo, saturação da entrada e uso de funções de Lyapunov fuzzy para sistemas T-S discretos na busca de condições de estabilidade locais e estimação do domínio de atração da origem do sistema. A efetividade e aplicabilidade das metodologias propostas são verificadas através de exemplos numéricos.

Palavras-chave: Fuzzy Takagi-Sugeno; Realimentação Dinâmica de Saída, Desigualdades matriciais lineares, Correspondência imperfeita de variáveis premissas.

ABSTRACT

This work presents new design conditions of full-order dynamic output feedback controllers for continuous and discrete-time Takagi-Sugeno (T-S) fuzzy systems. The fuzzy output controller can have a different number of fuzzy rules and a different set of membership functions from the T-S model allowing the selection of the premise variables used by the controller. This feature handles important scenarios present in many practical applications: immeasurable or imprecise measurement of premise variables. The central aspect of the proposed methodology is to present conditions where the control gains are independent of the premise variables that cannot be measured, allowing flexibility for the designer in a realistic output feedback context. The design conditions are expressed as linear matrix inequality (LMI) relaxations combined with scalar parameters that provide extra degrees of freedom. The proposed control methodology also deals with model uncertainties, input saturation, and fuzzy Lyapunov functions for discrete-time T-S systems in search of local stability conditions and estimation of the domain of attraction of the origin. Finally, numerical examples show the methodology's effectiveness and applicability.

Keywords: Fuzzy Takagi-Sugeno; Dynamic Output Feedback, LMIs, Imperfect premise matching.

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1 Introduction

1.1 Takagi-Sugeno fuzzy systems

The study of nonlinear systems is important in many areas, including control theory. The main reason is that most physical phenomena are nonlinear, and the numerical and analytical analysis becomes complex. The most common methodology to study nonlinear systems is the Lyapunov theory, which is based on an energy function associated with the nonlinear system. However, obtaining the energy function is challenging because there is no general procedure. An alternative methodology is to linearize the nonlinear system in specific operation points, then analyze these linear systems. This technique is called Lyapunov indirect method (Khalil, 2002). The main advantage is to use established techniques developed for linear systems. However, the analysis is only valid around the point of linearization.

In the past few decades, Takagi-Sugeno (T-S) fuzzy systems (Takagi and Sugeno, 1985) have attracted great interest due to their ability to describe nonlinear systems as a compact set of linear time-invariant models (Tanaka and Wang, 2001). Hence, the motivation to study T-S fuzzy systems is to adapt methodologies of analysis and robust control developed for linear time-varying (LTV) systems to study nonlinear systems. In this way, there exist many methods to design control laws based on the Lyapunov direct method and linear matrix inequalities (LMIs) (see Feng (2006) and references therein).

The so-called *sector nonlinearity approach* can obtain exact representations of nonlinear systems (Tanaka and Wang, 2001). In this approach, the premise variables, which represent the nonlinear terms of the dynamical system, are used to generate the membership functions as a convex combination of the vertex models (Sala, 2009). However, in this case, if the the nonlinear terms of the model are not precisely known, then the premise variables will be inexact. Moreover, approximate fuzzy models obtained from identification methods may also provide imprecise representations for the membership functions (Babuška and Verbruggen, 2003).

In general, there are two kinds of T-S fuzzy control schemes: the parallel distribution compensation (PDC) (Tanaka et al., 1998) and non-PDC control. The PDC is the most employed controller structure in T-S systems, where the controller shares the same premise membership functions and the same number of rules from the T-S fuzzy system. The PDC approach requires the measurement of all premise variables and the perfect knowledge of the membership functions (Tanaka and Wang, 2001). In practice, these assumptions are rarely met, and realistic implementations should consider that the premise variables are usually immeasurable or measured with a certain degree of uncertainty. For example, we can cite sensors with offsets, low resolutions, imprecision due to calibration, weather changes, noise, and instrument quality, among other sources of uncertainties (Lacerda et al., 2016).

Many works present sufficient LMI conditions for the analysis and synthesis of continuous and discrete-time T-S fuzzy models. For example, in Tanaka and Wang (2001), the system stability is given by a Lyapunov function common to all linear models. Less conservative results can be obtained using fuzzy Lyapunov functions, which consist of the fuzzy combination of quadratic Lyapunov functions. Fuzzy Lyapunov functions are more widely used in discrete-time T-S systems. In continuous-time models, dealing with the time derivative of the membership functions in stability conditions is difficult. One way to circumvent this difficulty is to use upper bounds for the time derivatives at the price of obtaining conservative approximations (Mozelli et al., 2009).

1.2 Output feedback

The design of state feedback controllers for T-S systems is largely developed in the literature (Feng, 2006). State feedback methodologies assume that all states are measured and available for controller implementation. However, this is true only in a few practical cases. Therefore, output stabilization techniques through state observers (Tanaka et al., 1998; Mansouri et al., 2009), static (Huang and Nguang, 2007; Lee and Kim, 2009; Bouarar et al., 2009) and dynamic output feedback (DOF) controllers (Nguang and Shi, 2006; Dong and Yang, 2008; Razavi-Panah and Majd, 2008; Yoneyama, 2009; Yang and Dong, 2010; Guelton et al., 2009; Liu et al., 2017) have been considered in the literature of T-S systems. Quadratic (Nguang and Shi, 2006; Dong and Yang, 2008; Razavi-Panah and Majd, 2008; Yoneyama, 2009; Yang and Dong, 2010) or fuzzy (Guelton et al., 2009; Tognetti et al., 2012; Liu et al., 2017) Lyapunov functions have been considered to assess the stability of the closed-loop system.

For DOF control design, a descriptor redundancy approach is used to obtain convex conditions in Guelton et al. (2009) and, in Yang and Dong (2010), a switching strategy is applied, but the output matrix must be the same for all the local models. Two-step design procedures are also applied for the DOF problem, as in Tognetti et al. (2012), where a state feedback gain is first designed. However, there is no methodology to guess the ideal state feedback gain used in the design of the DOF controller. The work Razavi-Panah and Majd (2008) addresses the problem of robust pole placement with H_∞ performance criteria via DOF control for a class of uncertain fuzzy systems. More recently, Liu et al. (2017) adopted a particular structure for the Lyapunov matrix, as proposed in de Oliveira et al. (2000), by employing a linear fractional transformation (LFT) mechanism requiring, however, the measurement of all premise variables.

1.3 Premise variables availability

In T-S models, the premise variables usually depend on the plant states. For this reason, some or all of them may not be available for measurement, also known as the imperfect premise matching design problem (Lam and Narimani, 2009). Therefore, the main convenience when dealing with state feedback design is the possibility of using all premise variables in the control law. However, this assumption does not hold for the output feedback design problem when, for instance, the unmeasurable states are part of the membership functions. Additionally, a drawback of control systems under perfect premise matching is that the design flexibility is restricted, and the implementation complexity of the fuzzy controller increases when the fuzzy model has many fuzzy rules with complex membership functions (Lam, 2016).

The problem of nonmeasurement of premise variables naturally arises in fuzzy observers when the premise variables depend on the estimated state variables. Therefore, many solutions have been proposed to design observer-based controllers (Guerra et al., 2006; Asemani and Majd, 2013; Dong and Wang, 2017; Maalej et al., 2017; Guerra et al., 2018; Ichalal et al., 2018). In this context, some approaches require the premise variables to be estimated (Guerra et al., 2006; Asemani and Majd, 2013; Guerra et al., 2018), the use of Lipschitz constants (Dong and Wang, 2017), Input-to-State Stability (ISS) framework (Maalej et al., 2017) and immersion techniques (Ichalal et al., 2018).

The problem of control synthesis with partially or completely unmeasurable premise variables becomes more involved in the design of dynamic output feedback (DOF) controllers. For this reason, very few works have considered the design of DOF controllers that do not share the same membership functions and the number of rules with T-S fuzzy systems, as Nguang and Shi (2006); Tognetti et al. (2012); Zhao and Dian (2017) for continuous and Ueno et al. (2011) for discrete-time systems. The works Nguang and Shi (2006); Zhao and Dian (2017) consider different membership functions for the controller and the plant. In Nguang and Shi (2006), an upper bound is used to deal with the difference between the membership functions of the plant and the controller, yielding conservative results. The work Zhao and Dian (2017) claims to be the first work that designs fuzzy DOF controllers under imperfect premise matching where the membership functions can be chosen freely. However, there is no intuitive procedure to find the membership functions that are subjected to many restrictions and depend on several scalar variables found as a nonlinear optimization problem. Moreover, no uncertainties are allowed in the T-S model. A linear fractional transformation approach is adopted in Liu et al. (2017). The solution presented in Tognetti et al. (2012), based on the approach developed in Tognetti et al. (2011), selects the available premise variables for the control law by choosing appropriate degrees for the polynomial of the slack variables that synthesize the controller. However, as a drawback, the design problem is solved in two stages. For discrete-time systems, Ueno et al. (2011) proposes an output feedback controller whose premise variables are their estimates. Therefore, it is also necessary to include an observer for this purpose.

Notation. $\mathbb{R}^n$, $\mathbb{R}_{>0}$ and $\mathbb{R}^{n\times m}$ respectively denote sets of n-dimensional real vectors, positive real numbers and $n \times m$-dimensional real matrices. The identity matrix of order n is denoted by

I_n and the null $m \times n$ matrix is denoted by $0_{m,n}$ (or simply I and 0 if no confusion arises); $\mathbb{1}_{m,n}$ denotes an m-by-n matrix with ones. For a matrix X, X' denotes its transpose and $He\{X\}$ is a short notation for $X + X'$. The block-diagonal matrix is denoted by diag$(\cdots)$. The symbol $\star$ denotes symmetric blocks in partitioned matrices, $\blacksquare$ stands for an element that has no influence on the development and $\otimes$ denotes the Kronecker product.

Thesis structure

- **Chapter 2** presents a general description of T-S fuzzy models used in this thesis and a strategy to obtain these models, which is an exact representation of a nonlinear system. Moreover, auxiliary lemmas, classic results for stability analysis and state feedback control design for T-S fuzzy systems using Lyapunov functions and LMI conditions are presented.
- **Chapter 3** presents the synthesis of DOF controllers for continuous-time fuzzy systems with imperfect premise matching subjected to norm-bounded uncertainties.
- **Chapter 4** presents local synthesis conditions and estimation of the domain of attraction for DOF controllers for discrete-time fuzzy systems with imperfect premise matching subjected to saturation in the input signal.
- **Chapter 5** presents conditions for the design of DOF controllers for continuous-time fuzzy systems with inexact measurement of premise variables.
- **Chapter 6** presents the final conclusion and future works.

Publications

E. S. Tognetti and T. M. Linhares, "Dynamic output feedback controller design for uncertain Takagi-Sugeno fuzzy systems: a premise variable selection approach," in IEEE Transactions on Fuzzy Systems, 26(6):1590–1600, 2021.

T. M. Linhares and E. S. Tognetti, "Realimentação dinâmica de saída de sistemas fuzzy T-S discretos no tempo com medição parcial das variáveis premissas," in XXIV Congresso Brasileiro de Automática 2022 (CBA 2022), Fortaleza-CE, Brazil, October 2022.

2 Definitions and Preliminaries

This chapter presents a brief description of Takagi-Sugeno fuzzy model, some stability conditions presented in literature and the definitions and notations used in the text.

2.1 Takagi-Sugeno fuzzy models

T-S fuzzy models (Takagi and Sugeno, 1985) are described by *IF-THEN* fuzzy rules that represent, exactly or approximated, a wide class of nonlinear systems. The main feature of a T-S fuzzy model is to express the local dynamic of each rule by a linear system. The ith rule of T-S fuzzy model (Tanaka and Wang, 2001) is given by

$\mathscr{R}_i$: IF $z_1(t)$ is $\mathscr{M}_1^i$ and ... and $z_p(t)$ is $\mathscr{M}_p^i$ THEN

$$\begin{cases} \delta[x](t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t), \qquad i = 1, \dots, N, \end{cases} \tag{2.1}$$

where $\mathscr{R}_i$ denotes the ith fuzzy rule, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector. $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n_u}$ and $C_i \in \mathbb{R}^{n_y \times n}$ are the system matrices. The fuzzy set based at $z_j(t)$ for the ith rule is denoted by $\mathscr{M}_j^i$ and N is the total number of fuzzy rules. The vector $z(t) = [z_1(t), \dots, z_p(t)]\mathbb{R}^p$ gives the premise variables, which can be functions of state variables, external disturbance or time. $\delta[x](t)$ denotes a time-derivative for continuous-time systems ($\delta[x](t) = \dot{x}(t)$) and displacement operator for discrete-time systems ($\delta[x](t) = x(t+1)$). $M_1^i(z_j(t))$ is the membership degree of $z_j(t)$ in $\mathscr{M}_1^i$. The normalized membership function for each ith fuzzy rule is

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum\limits_{i=1}^{N} w_i(z(t))}, \quad i = 1, \dots, N,$$

with

$$w_i(z(t)) = \prod_{j=1}^{p} M_1^i(z_j(t)).$$

The membership function $h(z(t)) = (h_1(z(t)), \dots, h_N(z(t)))$ assumes values from the unit simplex $\mathscr{U}_N$, defined as

$$\mathscr{U}_N \triangleq \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \geq 0, i = 1, \dots, N \right\}.$$

The global fuzzy model is the fuzzy combination of the local linear models and can be described in the following polytopic form

$$\begin{cases} \delta[x](t) = A(h)x(t) + B(h)u(t) \\ y(t) = C(h)x(t) \end{cases} \tag{2.2}$$

where

$$(A, B, C)(h) = \sum_{i=1}^{N} h_i(z(t))(A_i, B_i, C_i) \tag{2.3}$$

As presented in Rhee and Won (2006), a more suited notation for fuzzy sets is used to specify which fuzzy set based on $z_j(t)$ is used at the ith rule in (2.1). Therefore, a lth rule of (2.1) is rewritten as

$\mathscr{R}_l$: IF $z_1(t)$ is $\mathscr{M}_1^{\alpha_{l1}}$ and ... $z_j(t)$ is $\mathscr{M}_j^{\alpha_{lj}}$ ... and $z_p(t)$ is $\mathscr{M}_p^{\alpha_{lp}}$ THEN

$$\begin{cases} \delta[x](t) = A_{\alpha_{l1}\dots\alpha_{lp}}x(t) + B_{\alpha_{l1}\dots\alpha_{lp}}u(t) \\ y(t) = C_{\alpha_{l1}\dots\alpha_{lp}}x(t), \qquad l = 1, \dots, N \end{cases} \tag{2.4}$$

with $A_{\alpha_{\ell 1}\dots\alpha_{\ell p}} \in \mathbb{R}^{n \times n}$, $B_{\alpha_{\ell 1}\dots\alpha_{\ell p}} \in \mathbb{R}^{n \times n_u}$ and $C_{\alpha_{\ell 1}\dots\alpha_{\ell p}} \in \mathbb{R}^{n_y \times n}$, and α_{lj} specifies which fuzzy set based at z_j is used in the ith fuzzy rule. For example, $\alpha_{11} = \alpha_{21} = k$ means that the same fuzzy set $\mathscr{M}_1^k$ based in the premise variable $z_1(t)$ is used in the rules 1 and 2.

Let r_j be the number of fuzzy sets based in $z_j(t)$. Then,

$$N = \prod_{j=1}^{p} r_j, \tag{2.5}$$

and $r = (r_1, \dots, r_p)$. Consider that $M_j^{\alpha_{lj}}(z_j(t))$ is the membership degree of $z_j(t)$ in $\mathscr{M}_j^{\alpha_{lj}}$ and, therefore, the normalized membership function for each $\alpha_{lj} = 1, \dots, r_j = i$ is given by

$$\mu_{ji}(z_j(t)) = \frac{M_j^i(z_j(t))}{\sum_{i=1}^{r_j} M_j^i(z_j(t))}, \qquad j = 1, \dots, p, \quad i = 1, \dots, r_j, \tag{2.6}$$

and

$$0 \leq \mu_{ji}(z_j(t)) \leq 1, \quad \sum_{i=1}^{r_j} \mu_{ji}(z_j(t)) = 1. \tag{2.7}$$

It is important to observe that each premise variable z_j is uniquely associated with a membership function $\mu_j(z_j(t)) = \mu_{j1}(z_j(t)), \mu_{j2}(z_j(t)), \dots, \mu_{jr_j}(z_j(t))$, $j = 1, \dots, p$ belonging to a unit simplex $\mathscr{U}_{r_j}$ and, consequently, $\mu(z(t)) = \mu_1(z_1(t)), \mu_2(z_2(t)), \dots, \mu_p(z_p(t))$ belongs to the Cartesian product of simplexes $\mathscr{U}_r$, also called multi-simplex, defined as follows (Oliveira et al., 2008).

Definition 2.1 *(Multi-simplex) A multi-simplex $\mathcal{U}_r$ is the Cartesian product of a finite number of finite simplexes $\mathcal{U}_{r_1}, \ldots, \mathcal{U}_{r_p}$, i.e.,*

$$\mathcal{U}_r \triangleq \mathcal{U}_{r_1} \times \mathcal{U}_{r_2} \times \ldots \mathcal{U}_{r_p}. \tag{2.8}$$

The dimension of $\mathcal{U}_r$ is defined as the index $r = (r_1, \ldots, r_p)$. To simplify the notation, $\mathbb{R}^r$ represents the space $\mathbb{R}^{r_1+\ldots+r_p}$. A given element μ of $\mathcal{U}_r$ is the composition of $(\mu_1, \mu_2, \ldots, \mu_p)$, subsequently, each μ_j (belonging to $\mathcal{U}_{r_j}$) is the composition of $(\mu_{j1}, \mu_{j2}, \ldots, \mu_{jr_j})$.

The main advantage of the (2.4) representation is to provide a polytopic representation of the fuzzy T-S system in the multi-simplex structure, which preserves the dependence in each premise variable, as follows

$$\delta[x](t) = A_z x(t) + B_z u(t) \tag{2.9a}$$

$$y(t) = C_z x(t) \tag{2.9b}$$

where

$$(A, B, C)_z = \sum_{i_1=1}^{r_1} \cdots \sum_{i_p=1}^{r_p} \mu_{1i_1}(z_1(t)) \ldots \mu_{pi_p}(z_p(t)) \times (A_{i_1 \ldots i_p}, B_{i_1 \ldots i_p}, C_{i_1 \ldots i_p}), \tag{2.10}$$

$$\mu(z(t)) \in \mathcal{U}_r, \quad \mu_j(z_j(t)) \in \mathcal{U}_{r_j}, j = 1, \ldots, p.$$

The following compact notation is adopted for (2.10)

$$(A, B, C)_z = \sum_{\mathbf{i} \in \mathcal{I}_p} \mu_{\mathbf{i}}(z)(A_{\mathbf{i}}, B_{\mathbf{i}}, C_{\mathbf{i}}), \qquad \forall \mu \in \mathcal{U}_r. \tag{2.11}$$

where $\mathcal{I}_p := \{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p : i_j \in \{1, \ldots, r_j\},\ j = 1, \ldots, p\}$ and $\mu_{\mathbf{i}}(z) = \mu_{1i_1}(z_1(t)) \cdots \mu_{pi_p}(z_p(t))$ for $\mathbf{i} \in \mathcal{I}_p$.

Observe that the systems matrices representation in (2.3) and (2.11) are equivalent. It can be done by checking that $h_i = \mu_{1i_1}\mu_{2i_2} \ldots \mu_{pi_p}$, for $i \in \{1, \ldots, N\}$ and $(i_1, \ldots, i_p) \in \{1, \ldots, r_1\} \times \ldots \times \{1, \ldots, r_p\}$. For example, for a continuous-time T-S fuzzy system case with two premise variables ($p = 2$) and four rules ($N = 4, r = (2, 2)$),

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(z(t)) A_i x(t) = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \mu_{1i_1}(z_1(t)) \mu_{2i_2}(z_2(t)) \mathcal{A}_{i_1 i_2}$$

with

$$h_1(z(t)) = \mu_{11}(z_1(t))\mu_{21}(z_2(t)) \quad h_2(z(t)) = \mu_{11}(z_1(t))\mu_{22}(z_2(t))$$

$$h_3(z(t)) = \mu_{12}(z_1(t))\mu_{21}(z_2(t)) \quad h_4(z(t)) = \mu_{12}(z_1(t))\mu_{22}(z_2(t))$$

and $A_1 = \mathcal{A}_{11}$, $A_2 = \mathcal{A}_{12}$, $A_3 = \mathcal{A}_{21}$ and $A_4 = \mathcal{A}_{22}$.

2.1.1 Nonlinear systems aproximated by T-S fuzzy systems

One of the most common techniques to obtain a T-S fuzzy model from a nonlinear model is the sector nonlinearity approach (Tanaka and Wang, 2001), which allows to exact represent a nonlinear system by a convex combination of linear time varying models in a compact sector of the state space.

Let a nonlinear system $\delta[x](t) = f(x(t), u(t))$, with $f(0,0) = 0$, where $x(t) \in \mathbb{R}^n$ is the state variable and $u(t) \in \mathbb{R}^{n_u}$ the control signal, which can be expressed as

$$\delta[x] = \xi(x,u)x + \gamma(x,u)u. \tag{2.12}$$

The goal is to obtain an exact representation of the system (2.12) in a compact set $\mathfrak{D}$ of state space (including the equilibrium point $x = 0$) for T-S fuzzy models. The strategy consists to look for a global sector for the nonlinear system $f(x) \in [\bar{z}\ \underline{z}]x$, $\bar{z}$, $\underline{z} \in \mathbb{R}$, or a local sector where the sector condition is valid for $x_i \in [-d\ d]$. Therefore, the nonlinear terms $z_j(x,u)$, $j = 1, \ldots, p$, where p is the number of nonlinearities in $f(x(t), u(t))$, can be expressed by sector nonlinearity as

$$\begin{cases} z_j(x,u) = \underline{z_j}\mu_{j1}(x,u) + \overline{z_j}\mu_{j2}(x,u) \\ 1 = \mu_{j1}(x,u) + \mu_{j2}(x,u) \end{cases} \tag{2.13}$$

where $\mu_{j1}(x,u)$ and $\mu_{j2}(x,u)$ are normalized membership functions and

$$\overline{z_j} = \max z_j(x,u) \qquad \underline{z_j} = \min z_j(x,u).$$

Solving (2.13)

$$\mu_{j1}(x,u) = \frac{\overline{z_j} - z_j(x,u)}{\overline{z_j} - \underline{z_j}}, \quad \mu_{j2}(x,u) = 1 - \mu_{j1}(x,u), \quad j = 1, \ldots, p. \tag{2.14}$$

By doing this process for the terms $\xi(x,u)$ and $\gamma(x,u)$ of (2.12), the following T-S fuzzy model can be written as (2.9a), where the linear subsystem matrices come from the functions $\xi(x,u)$ and $\gamma(x,u)$ calculated in the extreme points of the nonlinearities $z_j(x,u)$ in the $\mathfrak{D}$ set, i.e., $z_j(x,u)$ is written as (2.13) to build (2.9a). It is important to note that the representation (2.9a) is not unique, because there are different ways to rewrite $f(x(t), u(t))$ using (2.13).

Example 2.1 Consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -sin^2(x_1)x_1 + u \tag{2.15}$$

in the sector $\mathfrak{D} = \{(x_1, x_2) \in \mathbb{R}^n : |x_1| \leq 0.5\}$*. For all* $x \in \mathfrak{D}$*, the nonlinear term* $z_1(x_1) = sin^2(x_1)$

$\in [0\ sin^2(0.5)]$, *therefore, the system* (2.15) *can be exactly described in* $\mathfrak{D}$ *by* (2.9a) *with*

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ sin^2(0.5) & 0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T,$$

$$\mu_{11}(z_1) = 1 - sin^2(x_1)/sin^2(0.5), \quad \mu_{12}(z_1) = 1 - \mu_{11}(z_1).$$

2.1.2 Fuzzy Controllers

One of the most common controllers design for T-S fuzzy models is the Parallel Distributed Compensation (PDC) where each controller rule is designed from the respective T-S fuzzy model rule (Tanaka and Wang, 2001). Therefore, the designed fuzzy controller shares the same fuzzy sets with the system model in the premise parts. From the fuzzy model rule (2.1), the following rule for a state feedback fuzzy controller can be construct:

$\mathscr{R}_l$: IF $z_1(t)$ is $\mathscr{M}_1^{\alpha_{l1}}$ and ... $z_j(t)$ is $\mathscr{M}_j^{\alpha_{lj}}$ ... and $z_p(t)$ is $\mathscr{M}_p^{\alpha_{lp}}$ THEN

$$u(t) = K_{\alpha_{l1}\dots\alpha_{lp}} x(t), \qquad l = 1, \dots, N \tag{2.16}$$

The global fuzzy controller, which is nonlinear in general, is a combination of linear local controllers and can be represented by

$$u(t) = K_z x(t), \quad K_z = \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) K_{\mathbf{i}}, \quad \forall \mu \in \mathscr{U}_r. \tag{2.17}$$

A non-PDC control law means that the fuzzy controller does not share the same fuzzy rules with the T-S fuzzy model. For example, the following state feedback control law

$$u(t) = \left(\sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) F_{\mathbf{i}}\right) \left(\sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) G_{\mathbf{i}}\right)^{-1} x(t) = F_z G_z^{-1} x(t)$$

where $F_{\mathbf{i}}$ and $G_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{I}_p$, are matrices with appropriate dimensions to be determined by design conditions.

2.2 Auxiliary lemmas

The following lemma is commonly used in the analysis of norm-bounded uncertain systems

Lemma 2.1 (see Petersen (1987)) *Let* $\Psi_0 = \Psi_0'$, *M*, *N*, *be matrices of appropriate dimensions. For all* $\beta > 0$,

$$\Psi_0 + MN + N'M' \leq \Psi_0 + \beta MM' + \beta^{-1} N'N.$$

2.2.1 Linear Matrix Inequalities

A linear matrix inequality (LMI) can be represented by $F(x) > 0$ with $F(x) : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ symmetric positive-definite and affine in the search variables x. Therefore, a LMI can be generically represented as

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \tag{2.18}$$

where $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $i = 0, \ldots, m$ are given matrices and x_i, $i = 1, \ldots, m$ are scalar variables to be found by solving the LMI. If there is a solution x for $F(x) > 0$, the LMI is feasible. The LMI (2.18) is equivalent to a set of n polynomial inequalities, because $F(x) > 0$ implies that all leading principal minors of $F(x)$ are positive.

LMIs conditions are convex and can be solved in polynomial time by interior point algorithms. One of the advantages of using LMIs is the availability of softwares to solve these conditions, for example, LMI Control Toolbox, SeDuMi and Mosek.

In general, stability analysis and controller design conditions are not originally LMIs. Some tools are very useful to write the problem as a LMI, for example, the Schur complement is a common property used in the manipulation of matrix inequalities.

Lemma 2.2 (Boyd et al. (1994)) *Let $x \in \mathbb{R}$ be the vector of decision variables and $M_1(x)$, $M_2(x)$ and $M_3(x)$ be affine functions in x, where $M_1(x)$ and $M_2(x)$ are symmetric. Then, the following are equivalent*

$$\begin{aligned} &a) \quad M_1(x) - M_3(x)' M_2(x)^{-1} M_3(x) > 0 \text{ with } M_2(x) > 0, \\ &b) \quad \begin{bmatrix} M_1(x) & M_3(x)' \\ M_3(x) & M_2(x) \end{bmatrix} > 0. \end{aligned} \tag{2.19}$$

Note that $a)$ is not a LMI, because $M(x) = M_1(x) - M_3(x)' M_2(x)^{-1} M_3(x)$ is not a affine function in x. However, inequality $b)$ is a LMI and both are equivalent. Also, note that in $b)$ $M_1(x) > 0$ and $M_2(x) > 0$ are necessary, but not sufficient conditions.

Next, the Finsler's Lemma allows to eliminate or introduce slack variables in matrix positivity conditions.

Lemma 2.3 (de Oliveira and Skelton (2001)) *Let $\xi \in \mathbb{R}^a$, $\mathcal{D} = \mathcal{D}' \in \mathbb{R}^{a\times a}$, $\mathcal{B} \in \mathbb{R}^{b\times a}$ with rank $(\mathcal{B}) < a$, and $\mathcal{B}^\perp$ a base for the null space of $\mathcal{B}$ (i.e. $\mathcal{B}\mathcal{B}^\perp = 0$). The following conditions are equivalent:*

1. $\xi' \mathcal{D} \xi < 0, \ \forall \ \mathcal{B}\xi = 0, \ \xi \neq 0$;
2. $\mathcal{B}^{\perp\prime} \mathcal{D} \mathcal{B}^\perp < 0$;
3. $\exists \mu \in \mathbb{R}$: $\mathcal{D} - \mu \mathcal{B}' \mathcal{B} < 0$;
4. $\exists \mathcal{X} \in \mathbb{R}^{a\times b}$: $\mathcal{D} + \mathcal{X}\mathcal{B} + \mathcal{B}'\mathcal{X}' < 0$.

2.3 Stability and design conditions

In general, stability and controllers design conditions in T-S fuzzy systems come from Lyapunov stability theory. Next, some LMI-based conditions for polytopic T-S fuzzy systems are presented. Polytopic T-S fuzzy systems are described as

$$\delta[x](t) = \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) A_{\mathbf{i}} x(t) + \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) B_{\mathbf{i}} u(t) = A_z x(t) + B_z u(t), \tag{2.20}$$

with matrices A_z and B_z given as in (2.3).

Consider the state feedback controller (2.17), which gives the closed-loop system

$$\begin{aligned}
\delta[x](t) &= \sum_{\mathbf{i}\in\mathcal{I}_p} \sum_{\mathbf{j}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)\mu_{\mathbf{j}}(z) \left(A_{\mathbf{i}} + B_{\mathbf{i}} K_{\mathbf{j}}\right) x(t) \\
&= \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)^2 \left(A_{\mathbf{i}} + B_{\mathbf{i}} K_{\mathbf{i}}\right) x(t) + \sum_{\mathbf{i}\in\mathcal{I}_p} \sum_{\mathbf{i}<\mathbf{j}} \mu_{\mathbf{i}}(z)\mu_{\mathbf{j}}(z) \left(A_{\mathbf{i}} + B_{\mathbf{i}} K_{\mathbf{j}} + A_{\mathbf{j}} + B_{\mathbf{j}} K_{\mathbf{i}}\right) x(t) \\
&= \left(A_z + B_z K_z\right) x(t),
\end{aligned} \tag{2.21}$$

where the relation $\mathbf{i} < \mathbf{j}$ can be performed by associating $\mathbf{i}$ and $\mathbf{j}$ to scalar values in the following form $\mathbf{i} = (i_1, \ldots, i_p) \mapsto 1 + \sum_{j=1}^{p} (i_j - 1) {r_M}^{p-j}$, with $r_M = \max\{r_1, \ldots, r_p\}$.

2.3.1 Continuous-time systems

Consider a quadratic Lyapunov function $V(x) = x(t)' P x(t)$. A sufficient stability condition for continuous-time closed-loop (2.21), with $\Omega_{\mathbf{ij}} = A_{\mathbf{i}} + B_{\mathbf{i}} K_{\mathbf{j}}$, is given by

$$\begin{aligned}
\dot{V}(x) &= \left(\sum_{\mathbf{i}\in\mathcal{I}_p} \sum_{\mathbf{j}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)\mu_{\mathbf{j}}(z) \Omega_{\mathbf{ij}} x(t)\right)' P x(t) + x(t)' P \left(\sum_{\mathbf{i}\in\mathcal{I}_p} \sum_{\mathbf{j}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)\mu_{\mathbf{j}}(z) \Omega_{\mathbf{ij}} x(t)\right) \\
&= \sum_{\mathbf{i}\in\mathcal{I}_p} \sum_{\mathbf{j}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)\mu_{\mathbf{j}}(z) x(t)' \left(\Omega_{\mathbf{ij}}' P + P \Omega_{\mathbf{ij}}\right) x(t) \\
&= \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)^2 x(t)' \left(\Omega_{\mathbf{ii}}' P + P \Omega_{\mathbf{ii}}\right) x(t) \\
&\qquad + \sum_{\mathbf{i}\in\mathcal{I}_p} \sum_{\mathbf{i}<\mathbf{j}} \mu_{\mathbf{i}}(z)\mu_{\mathbf{j}}(z) x(t)' \left(\Omega_{\mathbf{ij}}' P + P \Omega_{\mathbf{ij}} + \Omega_{\mathbf{ji}}' P + P \Omega_{\mathbf{ji}}\right) x(t) < 0,
\end{aligned}$$

which can be verified by the following LMIs with the change of variables $W = P^{-1}$ and $Z_{\mathbf{i}} = K_{\mathbf{i}} W$,

$$\begin{aligned}
\Gamma_{\mathbf{ii}} &< 0, \quad \mathbf{i} \in \mathcal{I}_p \\
\Gamma_{\mathbf{ij}} + \Gamma_{\mathbf{ji}} &< 0, \quad \mathbf{i}, \mathbf{j} \in \mathcal{I}_p \quad \mathbf{i} < \mathbf{j}.
\end{aligned} \tag{2.22}$$

with

$$\Gamma_{\mathbf{ij}} \triangleq A_{\mathbf{i}} W + W A_{\mathbf{i}}' + B_{\mathbf{i}} Z_{\mathbf{j}} + Z_{\mathbf{j}}' B_{\mathbf{i}}' \tag{2.23}$$

where the controller gain is recovered by $K_{\mathbf{i}} = Z_{\mathbf{i}} W^{-1}$, $\mathbf{i} \in \mathcal{I}_p$. Note that LMIs (2.22) assure, for all $\mu \in \mathcal{U}_r$,

$$\Gamma_z \triangleq A_z W + W A_z' + B_z Z_z + Z_z' B_z' < 0. \tag{2.24}$$

Less conservative conditions in literature assure $\dot{V}(x) < 0$, for example, adding slack variables in right-hand side LMIs (2.22).

Lemma 2.4 (Xiaodong and Qingling (2003)) *If there are matrices $Z_{\mathbf{i}} \in \mathbb{R}^{m \times n}$, $W = W' > 0 \in \mathbb{R}^{n \times n}$, $Y_{\mathbf{j},\mathbf{i}} = Y_{\mathbf{i},\mathbf{j}}' \in \mathbb{R}^{n \times n}$, $\mathbf{i}, \mathbf{j} \in \mathcal{I}_p$, $\mathbf{i} \neq \mathbf{j}$, satisfying the following LMIs*

$$\Gamma_{\mathbf{ii}} < Y_{\mathbf{i},\mathbf{i}}, \quad \mathbf{i} \in \mathcal{I}_p$$
$$\Gamma_{\mathbf{ij}} + \Gamma_{\mathbf{ji}} < Y_{\mathbf{i},\mathbf{j}} + Y_{\mathbf{i},\mathbf{j}}', \quad \mathbf{i}, \mathbf{j} \in \mathcal{I}_p,\ \mathbf{i} < \mathbf{j}$$
$$\begin{bmatrix} Y_{(1\cdots1),(1\cdots1)} & \cdots & Y_{(1\cdots1),(r_1\cdots r_p)} \\ \vdots & \ddots & \vdots \\ Y_{(r_1\cdots r_p),(1\cdots1)} & \cdots & Y_{(r_1\cdots r_p),(r_1\cdots r_p)} \end{bmatrix} < 0,$$

where $\Gamma_{\mathbf{ij}}$ is given by (2.23)*, then the state feedback* (2.17) *stabilizes a continuous-time T-S fuzzy system given by* (2.21) *with $K_{\mathbf{i}} = Z_{\mathbf{i}} W^{-1}$.*

Using Lemma 2.3 (Finsler's Lemma), other relaxation can be expressed.

Lemma 2.5 (Montagner et al. (2009)) *If there exist matrices $Z_z \in \mathbb{R}^{n_u \times n}$, $\mathcal{X}_z \in \mathbb{R}^{2n \times n}$ and $W = W' > 0 \in \mathbb{R}^{n \times n}$, where for all $h(z) \in \mathcal{U}_N$,*

$$\mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}'\mathcal{X}' < 0, \tag{2.25}$$

with

$$\mathcal{Q} = \begin{bmatrix} B_z Z_z + Z_z' B_z' & W \\ W & 0 \end{bmatrix} \quad and \quad \mathcal{B} = \begin{bmatrix} A_z' & -I \end{bmatrix},$$

then the state feedback controller $K_z = Z_z W^{-1}$ stabilizes the continuous time T-S fuzzy system given by (2.21)*.*

Proof First, note that $\mathcal{B}^{\perp} = \begin{bmatrix} I & A_z \end{bmatrix}'$. By the equivalence between *ii*) and *iv*) of Lemma 2.3, we have that (2.25) is equivalent to

$$0 > \mathcal{B}_z^{\perp\prime} \mathcal{Q} \mathcal{B}_z^{\perp} = \begin{bmatrix} I \\ A_z' \end{bmatrix}' \begin{bmatrix} B_z Z_z + Z_z' B_z' & W \\ W & 0 \end{bmatrix} \begin{bmatrix} I \\ A_z' \end{bmatrix} = \Gamma_z,$$

with Γ_z given by (2.24). ∎

2.3.2 Discrete-time systems

Fuzzy Lyapunov functions have been extensively used in T-S discrete-time systems due to the facility in dealing with the advanced time instant $P_{z+} = \sum_{\mathbf{i} \in \mathcal{I}_p} \mu_{\mathbf{i}}(z(t+1)) P_{\mathbf{i}}$, $\mu \in \mathcal{U}_r$.

Consider the fuzzy Lyapunov function

$$V(x,z) = x(t)'P_z x(t), \tag{2.26}$$

we have that a sufficient condition for the stability of a discrete-time T-S fuzzy system given by (2.21) is

$$\Delta V(x,z) = x(t+1)'P_{z+}x(t+1) - x(t)'P_z x(t) < 0.$$

It is presented a condition using fuzzy Lyapunov function (2.26).

Theorem 2.1 *The discrete-time T-S fuzzy system given by* (2.20) *with* $u = 0$ *is asymptotically stable if exist* $P_z = P_z' > 0$ *and if one of the following equivalent conditions is satisfied*

$$\begin{aligned} i) \quad & A_z'P_{z+}A_z - P_z < 0, \\ ii) \quad & \begin{bmatrix} P_z & A_z'P_{z+} \\ \star & P_{z+} \end{bmatrix} > 0, \end{aligned} \tag{2.27}$$

for all $\mu \in \mathcal{U}_r$.

Theorem 2.2 *If there exist* $P_{\mathbf{i}} = P_{\mathbf{i}}' > 0 \in \mathbb{R}^{n\times n}$, $\mathbf{i} \in \mathcal{I}_p$, *where the following LMIs are verified*

$$\begin{bmatrix} P_{\mathbf{i}} & A_{\mathbf{i}}'P_{\mathbf{j}} \\ \star & P_{\mathbf{j}} \end{bmatrix} > 0, \quad \mathbf{i},\mathbf{j} \in \mathcal{I}_p,$$

then conditions (2.27) *hold.*

Theorem 2.3 (Euntai Kim and Heejin Lee (2000)) *If there exist matrices* $W = W' > 0$, $Q_{\mathbf{ii}} > 0$, $Q_{\mathbf{ij}} = Q_{\mathbf{ij}}'$, $S_{\mathbf{i}}$, *where*

$$\Gamma_{\mathbf{ii}} > Q_{\mathbf{ii}}, \quad \mathbf{i} \in \mathcal{I}_p$$

$$\Gamma_{\mathbf{ij}} + \Gamma_{\mathbf{ji}} > Q_{\mathbf{ij}}, \quad \mathbf{i},\mathbf{j} \in \mathcal{I}_p,\ \mathbf{i} < \mathbf{j}$$

$$\begin{bmatrix} 2Q_{(1\cdots1),(1\cdots1)} & \cdots & Q_{(1\cdots1),(r_1\cdots r_p)} \\ \vdots & \ddots & \vdots \\ Q_{(r_1\cdots r_p),(1\cdots1)} & \cdots & 2Q_{(r_1\cdots r_p),(r_1\cdots r_p)} \end{bmatrix} > 0$$

with

$$\Gamma_{\mathbf{ij}} = \begin{bmatrix} W & \star \\ A_{\mathbf{i}} + B_{\mathbf{i}}S_{\mathbf{j}} & W \end{bmatrix}, \quad \mathbf{i},\mathbf{j} \in \mathcal{I}_p,$$

then, the discrete-time T-S fuzzy system (2.20) *can be stabilized by PDC control law*

$$u(t) = \left(\sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)K_{\mathbf{i}} \right) x(t), \qquad K_{\mathbf{i}} = S_{\mathbf{i}}W^{-1}.$$

2.4 Dynamic Output Feedback

This section presents the design of dynamic output feedback controllers (Scherer et al., 1997) for T-S fuzzy system (2.9). Given the dynamic output feedback controller

$$\begin{cases} \delta[x_c] = A_{cz}x_c(t) + B_{cz}y(t) \\ u(t) = C_{cz}x_c(t) + D_{cz}y(t) \end{cases} \tag{2.28}$$

where $x_c \in \mathbb{R}^n$ is the controller states. The closed loop system is

$$\begin{bmatrix} \delta[x] \\ \delta[x_c] \end{bmatrix} = \underbrace{\begin{bmatrix} A_z + B_z D_{cz} C_z & B_z C_{cz} \\ B_{cz} C_z & A_{cz} \end{bmatrix}}_{A_{cl_z}} \begin{bmatrix} x \\ x_c \end{bmatrix}.$$

The controller matrices $A_{cz}, B_{cz}, C_{cz}, D_{cz}$ can be determined by applying on the closed-loop system analyses conditions and performance criteria.

2.4.1 Continuous-time Dynamic Output Feedback

A stabilizing controller for continuous time systems can be determined by the following lemma

Lemma 2.6 *The matrix A_{cl_z} is Hurwitz if and only if there exists matrix $W' = W > 0 \in \mathbb{R}^{2n\times 2n}$ such that*

$$A_{cl_z}W + WA'_{cl_z} < 0.$$

The resulting inequality is non linear due to the product between closed loop system's dynamic matrix and Lyapunov matrix W. In order to turn the inequality into a LMI, congruence transformation and change of variables are used, allowing to solve the product of variables between the Lyapunov matrix and controller's matrices. Consider the congruence transformation

$$T'\left(A_{cl_z}W + WA'_{cl_z}\right)T < 0.$$

LMIs conditions can be obtained if it is possible to find a matrix T that writes term $T'A_{cl_z}WT$ as a affine expression of decision variables. For this purpose, the following matrix partitions are defined

$$W = \begin{bmatrix} X & U' \\ U & \blacksquare \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} Y & V \\ V' & \blacksquare \end{bmatrix}, \quad T = \begin{bmatrix} I & Y \\ 0 & V' \end{bmatrix}, \tag{2.29}$$

where $\blacksquare$ means that these partitions are not important for the congruence transformation and the change of variables.

Therefore,

$$WT = \begin{bmatrix} X & I \\ U & 0 \end{bmatrix}, \quad T'WT = \begin{bmatrix} X & I \\ I & Y \end{bmatrix},$$

$$T'A_zWT = \begin{bmatrix} A_zX + B_z\left(C_{cz}U + D_{cz}C_zX\right) & A_z + B_zD_{cz}C_z \\ VA_{cz}U + YA_zX + VB_{cz}C_zX + YB_zC_{cz}U + YB_zD_{cz}C_zX & YA_z + \left(VB_{cz} + YB_zD_{cz}\right)C_z \end{bmatrix}.$$

Consider the change of variables

$$\begin{cases} \mathscr{A}_z = VA_{cz}U + YA_zX + VB_{cz}C_zX + YB_zC_{cz}U + YB_zD_{cz}C_zX \\ \mathscr{B}_z = VB_{cz} + YB_zD_{cz} \\ \mathscr{C}_z = C_{cz}U + D_{cz}C_zX \\ \mathscr{D}_z = D_{cz} \end{cases} \tag{2.30}$$

yielding

$$T'A_zWT = \begin{bmatrix} A_zX + B_z\mathscr{C}_z & A_z + B_z\mathscr{D}_zC_z \\ \mathscr{A}_z & YA_z + \mathscr{B}_zC_z \end{bmatrix},$$

which is affine in the decision variables $\mathscr{A}_z$, $\mathscr{B}_z$, $\mathscr{C}_z$, $\mathscr{D}_z$.

The change of variables of (2.30) can be written as

$$\begin{bmatrix} \mathscr{A}_z & \mathscr{B}_z \\ \mathscr{C}_z & \mathscr{D}_z \end{bmatrix} = \begin{bmatrix} V & YB_z \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cz} & B_{cz} \\ C_{cz} & D_{cz} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_zX & I \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} A_z \begin{bmatrix} X & 0 \end{bmatrix}. \tag{2.31}$$

If U and V are square ($n_c = n$), the transformation is invertible if U and V are non-singular, giving

$$\begin{bmatrix} A_{cz} & B_{cz} \\ C_{cz} & D_{cz} \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB_z \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_z - YA_zX & \mathscr{B}_z \\ \mathscr{C}_z & \mathscr{D}_z \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -C_zXU^{-1} & I \end{bmatrix}. \tag{2.32}$$

Theorem 2.4 (Gahinet and Apkarian (1994)) *Let $n_c = n$. The matrix A_{cl_z} is Hurwitz if and only if there exist matrices $X = X' \in \mathbb{R}^{n\times n}$, $Y = Y' \in \mathbb{R}^{n\times n}$, $\mathscr{A}_z \in \mathbb{R}^{n\times n}$, $\mathscr{B}_z \in \mathbb{R}^{n\times n_y}$, $\mathscr{C}_z \in \mathbb{R}^{n_u\times n}$ and $\mathscr{D}_z \in \mathbb{R}^{n_u\times n_y}$ such that*

$$\begin{bmatrix} A_zX + XA_z' + B_z\mathscr{C}_z + \mathscr{C}_z'B_z' & A_z + B_z\mathscr{D}_zC_z + \mathscr{A}_z' \\ \star & A_z'Y + YA_z + \mathscr{B}_zC_z + C_z'\mathscr{B}_z' \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0.$$

In this case,

$$\begin{bmatrix} A_{cz} & B_{cz} \\ C_{cz} & D_{cz} \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB \\ 0_{n_u\times n} & I_{n_u} \end{bmatrix} \begin{bmatrix} \mathscr{A}_z - YA_zX & \mathscr{B}_z \\ \mathscr{C}_z & \mathscr{D}_z \end{bmatrix} \begin{bmatrix} U^{-1} & 0_{n\times n_y} \\ -C_zX^{-1} & I_{n_y} \end{bmatrix},$$

where U and V are such that $YX + VU = I$.

2.4.2 Discrete-time Dynamic Output Feedback

A stabilizing controller for discrete-time systems can be determined by the following lemma.

Lemma 2.7 (de Oliveira et al. (2002)) *The matrix A_{cl_z} is Schur stable if and only if there exist matrices $W' = W \in \mathbb{R}^{2n\times 2n}$ and $G \in \mathbb{R}^{2n\times 2n}$ such that*

$$\begin{bmatrix} W & A_{cl_z}G \\ G'A'_{cl_z} & G+G'-W \end{bmatrix} > 0.$$

For discrete-time, note that the nonlinear terms are formed with the slack variable G and no more with Lyapunov matrix W. Differently from continuous-time where the parametrized matrix is the Lyapunov matrix, here the transformation matrix is defined by the partitions of the slack variable and its inverse

$$G = \begin{bmatrix} X & \blacksquare \\ U & \blacksquare \end{bmatrix}, \quad G^{-1} = \begin{bmatrix} Y' & \blacksquare \\ V' & \blacksquare \end{bmatrix}, \quad T = \begin{bmatrix} I & Y' \\ 0 & V' \end{bmatrix}, \tag{2.33}$$

where it is necessary that G^{-1} exists. The following steps are similar to the continuous-time case. First, the congruence transformations

$$T'(G+G'-W)T = \begin{bmatrix} X+X' & I+S' \\ I+S & Y+Y' \end{bmatrix} - \begin{bmatrix} P & J \\ J' & H \end{bmatrix}, \quad T'WT = \begin{bmatrix} P & J \\ J' & H \end{bmatrix},$$
$$T'AGT = \begin{bmatrix} A_zX + B_z\mathscr{C}_z & A_z + B_z\mathscr{D}_zC_z \\ \mathscr{A}_z & YA_z + \mathscr{B}_zC_z \end{bmatrix},$$

where $S = YX + VU$ and $\mathscr{A}_z$, $\mathscr{B}_z$, $\mathscr{C}_z$ and $\mathscr{D}_z$ are given by (2.30). The change of variables (2.30) is the same of the continuous-time case, therefore (2.31) and (2.32) also apply.

Theorem 2.5 (de Oliveira et al. (2002)) *Let $n_c = n$. The matrix A_{cl_z} is Schur if and only if there exist matrices $P = P' \in \mathbb{R}^{n\times n}$, $H = H' \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $\mathscr{A}_z \in \mathbb{R}^{n\times n}$, $\mathscr{B}_z \in \mathbb{R}^{n\times n_y}$, $\mathscr{C}_z \in \mathbb{R}^{n_u\times n}$, $\mathscr{D}_z \in \mathbb{R}^{n_u\times n_y}$, $S \in \mathbb{R}^{n\times n}$ and $J \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} P & J & A_zX + B_z\mathscr{C}_z & A_z + B_z\mathscr{D}_zC_z \\ \star & H & \mathscr{A}_z & YA_z + \mathscr{B}_zC_z \\ \star & \star & X+X'-P & I+S'-J \\ \star & \star & \star & Y+Y'-H \end{bmatrix} > 0.$$

In this case, the controllers' matrices can be determined by (2.32) *and U and V are given by $S = YX + VU$.*

3 Continuous-time DOF with imperfect premise matching

This chapter presents new design conditions of full-order dynamic output feedback controllers for continuous-time Takagi-Sugeno (T-S) fuzzy systems allowing the selection of premise variables to be used in the control law. The fuzzy output controller is allowed to have a different number of fuzzy rules and a different set of membership functions from the T-S model. This includes the cases of complete or partial immeasurable premise variables. The main aspect of the proposed methodology is to present conditions where the control gains are independent of the premise variables that cannot be measured allowing flexibility for the designer in a realistic output feedback context. For this purpose, the design conditions are expressed as linear matrix inequality relaxations combined with scalar parameters that provide extra degrees of freedom. The proposed control methodology also deals with model uncertainties and the use of fuzzy Lyapunov functions. The effectiveness and applicability of the methodology are shown through numerical examples.

3.1 Introduction

It is well known that almost all models describing physical systems present uncertainties. Unfortunately, most LMI-based techniques for designing DOF controllers are infeasible for uncertain T-S systems because the standard technique of change of variables presented in Chilali and Gahinet (1996); Scherer et al. (1997) results in crossing terms between the system's and the controller's matrices in the closed-loop systems, with an intricate membership interconnection. Therefore, Li et al. (2000); Dong and Yang (2009) are only suitable for a restrictive class of fuzzy systems without uncertainties, with matched membership functions between the system and the controller and PDC controller structure. The work Chang et al. (2016) proposes a solution for designing DOF controllers for discrete-time T-S systems subjected to uncertainties and non-measurable premise variables. However, the controller's gains are fixed, taking no advantage if some premise variables are available.

In this chapter, a LMI-based procedure is proposed to design non-PDC DOF controllers that do not share the same set of premise variables and the number of rules of the uncertain continuous-time T-S fuzzy system. Takagi-Sugeno models represented with a multi-simplex structure can be written as the product of one-variable membership functions. Therefore, thanks to the modeling of the membership functions in a space that is defined by the Cartesian product of simplexes, also known as multi-simplex (Oliveira et al., 2008), the proposed technique provides an intuitive approach that allows the selection of the membership functions for the control law that depends only on the measured premise variables. The designer can also discard the membership functions that are too complex for numerical implementation or contain uncertain terms. This feature allows flexibility for the designer in a realistic output feedback scenario.

The main contribution is to propose less conservative convex conditions for the design of DOF controllers such that the gains of the controller do not depend on the unavailable premise variables. For this, one proposes a systematic procedure to rewrite the dynamical matrices of the plant in terms of the desirable membership functions. It is demonstrated that the best scenario is to select a subset of membership functions of the plant. This procedure preserves the information of the membership functions depending on the measured premise variables. It also minimizes the impact of discarding the membership functions depending on the unmeasured premise variables. The technique also works for the case where the premise variables or the membership functions of the controller are entirely free, representing an opportunity for future developments. For this purpose, the presented conditions require fewer restrictions for the membership functions of the controller than Zhao and Dian (2017), no use of upper bounds as in Nguang and Shi (2006); Apkarian and Noll (2006), and no need for multiple steps for the control design, as in Tognetti et al. (2012). For simplicity, the conditions employ quadratic Lyapunov functions, but the results can be easily extended, as shown in the chapter, by using fuzzy Lyapunov functions (Tanaka et al., 2003) and requiring the treatment of the time derivatives of the membership functions. Finally, we also consider norm-bounded uncertainties in the plant to design robust controllers.

The main contributions of the chapter are:

- Systematic procedure to select the premise variables and membership functions for the control law without the use of bounds to deal with the difference between the membership functions of the plant and the controller;
- New LMI conditions with scalar parameters such that the controller gains do not depend on the matrices of the plant;
- Extensions using fuzzy Lyapunov functions, polynomial relaxations, and models subject to polytopic-type uncertainties are straightforward.

Numerical examples demonstrate the effectiveness of the proposed approach.

3.2 Preliminaries

3.2.1 Description of the system

Consider a class of uncertain T-S fuzzy system described by the following IF-THEN rules adapted from (2.4) to consider the presence of uncertainties:

Rule l**:** IF $z_1(t)$ is $\mathscr{M}_1^{\alpha_{l1}}$ and ... and $z_p(t)$ is $\mathscr{M}_p^{\alpha_{lp}}$ THEN

$$\begin{cases} \dot{x}(t) = (A_{\alpha_{l1}\dots\alpha_{lp}} + \Delta A_{\alpha_{l1}\dots\alpha_{lp}})x(t) + (B_{\alpha_{l1}\dots\alpha_{lp}} + \Delta B_{\alpha_{l1}\dots\alpha_{lp}})u(t) \\ y(t) = C_{\alpha_{l1}\dots\alpha_{lp}}x(t), \qquad \ell = 1,\dots,N. \end{cases} \tag{3.1}$$

The terms $\Delta A_{\alpha_{l1}\dots\alpha_{lp}} \in \mathbb{R}^{n\times n}$ and $\Delta B_{\alpha_{l1}\dots\alpha_{lp}} \in \mathbb{R}^{n\times n_u}$ are Lebesgue measurable uncertainties defined by Zhou and Khargonekar (1988)

$$\begin{aligned} \Delta A_{\alpha_{l1}\dots\alpha_{lp}} &= H^a_{\alpha_{l1}\dots\alpha_{lp}} f^a(t) N^a_{\alpha_{l1}\dots\alpha_{lp}}, \\ \Delta B_{\alpha_{l1}\dots\alpha_{lp}} &= H^b_{\alpha_{l1}\dots\alpha_{lp}} f^b(t) N^b_{\alpha_{l1}\dots\alpha_{lp}},. \end{aligned} \tag{3.2}$$

where $H^s_{\alpha_{l1}\dots\alpha_{lp}}$ and $N^s_{\alpha_{l1}\dots\alpha_{lp}}$, for $s = a, b$, are known real constant matrices and $f^s(t)$ are uncertain matrices bounded satisfying $f^s(t)^T f^s(t) \le I \ \forall t$.

It is important to the development of the proposed approach dealing with non-measured premise variables to observe that each premise variable z_j is uniquely associated with a membership function[1] $\mu_j(z_j)$. More specifically, the use of the multi-simplex $\mathscr{U}_r$ has an important role in the proposed technique by allowing a specific treatment of the membership functions that depend on the premise variables that are not measured.

3.2.2 Notation and definitions

Here, we detail an appropriate notation to deal with fuzzy summations with non-measured premise variables. The following two cases are considered: fuzzy summations depending on all premise variables or depending on a subset of the premise variables of the plant. To handle the multi-dimensional fuzzy summations of matrices, the following notation is adopted based on Sala and Ariño (2007): the set $\mathcal{Q}_s := \{(q_1,\dots,q_s) : q_j \in \mathbb{N},\ j = 1,\dots,s\}$ contains the index of the s premise variables $z_{\mathbf{q}} := (z_{q_1}, z_{q_2},\dots,z_{q_s})$ represented in the summation; $\mathcal{I}_s := \{(i_1,i_2,\dots,i_s) \in \mathbb{N}^s : i_j \in \{1,\dots,r_{q_j}\},\ j = 1,\dots,s\}$; $\mu_{\mathbf{i}}(z_{\mathbf{q}}) := \mu_{q_1 i_1}(z_{q_1})\mu_{q_2 i_2}(z_{q_2})\cdots\mu_{q_s i_s}(z_{q_s})$, $\mathbf{q} \in \mathcal{Q}_s$, $\mathbf{i} \in \mathcal{I}_s$, with $\mathbf{q} := (q_1,\dots,q_s)$ and $\mathbf{i} := (i_1,i_2,\dots,i_s)$. Given vertices $Q_{i_1 i_2\dots i_s}$, the following fuzzy summation is defined:

$$Q_{z_{\mathbf{q}}} := \sum_{\mathbf{i}\in\mathcal{I}_s} \mu_{\mathbf{i}}(z_{\mathbf{q}}) Q_{\mathbf{i}}$$

with $Q_{\mathbf{i}} := Q_{i_1 i_2\dots i_s}$ and

$$\sum_{\mathbf{i}\in\mathcal{I}_s} \mu_{\mathbf{i}}(z_{\mathbf{q}}) Q_{\mathbf{i}} = \sum_{i_1=1}^{r_{q_1}} \cdots \sum_{i_s=1}^{r_{q_s}} \mu_{q_1 i_1}(z_{q_1}) \cdots \mu_{q_s i_s}(z_{q_s}) Q_{i_1\dots i_s}.$$

For example, the fuzzy summation

$$\sum_{i_1=1}^{r_1} \cdots \sum_{i_m=1}^{r_m} \mu_{1i_1}(z_1)\dots\mu_{mi_m}(z_m) Q_{i_1\dots i_m}$$

[1]For simplicity of notation, the time-dependence in $\mu(z(t))$ and $z(t)$ is omitted.

is represented by

$$Q_{z_{\mathbf{q}}} = \sum_{\mathbf{i}\in\mathcal{I}_m} \mu_{\mathbf{i}}(z_{\mathbf{q}})Q_{\mathbf{i}} \tag{3.3}$$

where $\mathbf{q} = (1,\ldots,m) \in \mathcal{Q}_m$, $z_{\mathbf{q}} = (z_1,\ldots,z_m)$, $\mathbf{i} = (i_1,\ldots,i_m) \in \mathcal{I}_m$, $\mu_{\mathbf{i}}(z_{\mathbf{q}}) = \mu_{1i_1}(z_1)\mu_{2i_2}(z_2)\cdots \mu_{mi_m}(z_m)$, $Q_{\mathbf{i}} = Q_{i_1\ldots i_m}$. If $m = p$, that is, the summation contains all premise variables $z = (z_1,\ldots,z_p)$ of the fuzzy model, we will omit the index $\mathbf{q}$, since $z_{\mathbf{q}} = z$, and denote it simply by $Q_z \coloneqq \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)Q_{\mathbf{i}}$.

For instance, suppose that we are interested to represent a fuzzy summation with the premise variables z_2, z_4 and z_5. Then, one has $P_{z_{\mathbf{q}}} = \sum_{\mathbf{i}\in\mathcal{I}_3} \mu_{\mathbf{i}}(z_{\mathbf{q}})P_{\mathbf{i}}$ with: $\mathbf{q} = (2,4,5) \in \mathcal{Q}_3$, $s = 3$, $z_{\mathbf{q}} = (z_2,z_4,z_5)$, $\mathbf{i} = (i_1,i_2,i_3) \in \mathcal{I}_3$, $\mu_{\mathbf{i}}(z_{\mathbf{q}}) = \mu_{2i_1}(z_2)\mu_{4i_2}(z_4)\mu_{5i_3}(z_5)$, $P_{\mathbf{i}} = P_{i_1i_2i_3}$, that is,

$$P_{z_{\mathbf{q}}} = \sum_{i_1=1}^{r_2}\sum_{i_2=1}^{r_4}\sum_{i_3=1}^{r_5} \mu_{2i_1}(z_2)\mu_{4i_2}(z_4)\mu_{5i_3}(z_5)P_{i_1i_2i_3}.$$

We may also rewrite a fuzzy summation with less premise variables by imposing constant values for the premise functions that are not represented. For instance, consider the fuzzy summation (3.3) with $m = 2$ and $r_j = 2$, $j = 1,2$. The summation (3.3) represented only by the premise variable z_2 is obtained by imposing a constant value $\bar{z}_1$ in $\mu_1(z_1)$ and replacing it by $\mu_1(\bar{z}_1)$. Then the fuzzy summation (3.3) becomes

$$\hat{Q}_{z_{\mathbf{q}}} = \sum_{\mathbf{i}\in\mathcal{I}_1} \mu_{\mathbf{i}}(z_{\mathbf{q}})\hat{Q}_{\mathbf{i}}, \quad \mathbf{q} = 2,\ \mathbf{i} = i_1 \tag{3.4}$$

with vertices given by $\hat{Q}_{i_1} = \mu_{11}(\bar{z}_1)Q_{1i_1} + \mu_{12}(\bar{z}_1)Q_{2i_1}$, $i_1 = 1,2$.

Handling the difference and the product of fuzzy summations will be of particular interest to this work. For instance, consider again the fuzzy summation (3.3) with $m = 2$ and $r_j = 2$, $j = 1,2$, involving all premise variables $z = (z_1,z_2)$ and denoted by Q_z. Consider also a summation $\hat{Q}_{z_{\mathbf{q}}}$, as given in (3.4), obtained by assigning $z_1 = \bar{z}_1$, where $\bar{z}_1$ is an arbitrary constant value. Observe that $\hat{Q}_{z_{\mathbf{q}}}$ is represented only by the subset $z_{\mathbf{q}} = z_2$ ($\mathbf{q} = 2$) of the premise variables. The difference $\Gamma_Q \coloneqq Q_z - \hat{Q}_{z_{\mathbf{q}}}$ is given by

$$\Gamma_Q = \sum_{\mathbf{j}\in\mathcal{I}_2} \mu_{\mathbf{j}}(z)Q_{\mathbf{j}} - \sum_{\mathbf{i}\in\mathcal{I}_1} \mu_{\mathbf{i}}(z_{\mathbf{q}})\hat{Q}_{\mathbf{i}} = \Delta\mu_{11}(z_1)\sum_{\mathbf{i}\in\mathcal{I}_1} \mu_{\mathbf{i}}(z_{\mathbf{q}})Q_{1\mathbf{i}} + \Delta\mu_{12}(z_1)\sum_{\mathbf{i}\in\mathcal{I}_1} \mu_{\mathbf{i}}(z_{\mathbf{q}})Q_{2\mathbf{i}} \tag{3.5}$$

where $\mathbf{j} = (j_1,j_2) \in \mathcal{I}_2$, $\mathbf{i} = i_1 \in \mathcal{I}_1$, $\Delta\mu_{1\ell}(z_1) \coloneqq \mu_{1\ell}(z_1) - \mu_{1\ell}(\bar{z}_1)$, $\ell = 1,2$. Observe that, if we consider (3.4), we can rewrite (3.5) as

$$\begin{aligned}\Gamma_Q &= \sum_{\mathbf{j}\in\mathcal{I}_2} \mu_{\mathbf{j}}(z)Q_{\mathbf{j}} - \left(\sum_{i_1\in\mathcal{I}_1} \mu_{i_1}(z_1)\right) \times \sum_{i_2\in\mathcal{I}_1} \mu_{i_2}(z_2)(\mu_{11}(\bar{z}_1)Q_{1i_2} + \mu_{12}(\bar{z}_1)Q_{2i_2})\\ &= \sum_{(i_1,i_2)\in\mathcal{I}_2} \mu_{i_1}(z_1)\mu_{i_2}(z_2)\left(Q_{i_1i_2} - (\mu_{11}(\bar{z}_1)Q_{1i_2} - \mu_{12}(\bar{z}_1)Q_{2i_2}\right).\end{aligned}$$

Furthermore, if we consider $\bar{z}_1$ such that $\mu_{11}(\bar{z}_1) = \mu_{12}(\bar{z}_1) = 0.5$, then the vertices of Γ_Q are given by $\pm 0.5(Q_{1i_2} - Q_{2i_2})$, $i_2 \in \mathcal{I}_1$, and the norm $||Q_{\mathbf{j}} - \hat{Q}_{\mathbf{i}}||$ has a minimum upper bound for all $\mathbf{j} \in \mathcal{I}_2$ and $\mathbf{i} \in \mathcal{I}_1$.

For the sake of conciseness, we will adopt the index z for the product or sum of fuzzy summations involving z and $z_{\mathbf{q}}$, and the index $z_{\mathbf{q}}$ when only fuzzy summations that depend on $z_{\mathbf{q}}$ are involved. Finally, the homogenization of product of summations and the sum of matrices depending on different membership functions can be handled by using $\sum_{\mathbf{i}\in\mathcal{I}_s} \mu_{\mathbf{i}}(z_{\mathbf{q}}) = 1$. For more details, please refer to Sala and Ariño (2007) or to Tognetti et al. (2012), in the context of multi-simplex manipulations.

3.2.3 Problem statement

The global T-S fuzzy system obtained from (3.1) can be rewritten as

$$\begin{cases} \dot{x}(t) = (A_z + \Delta A_z)x(t) + (B_z + \Delta B_z)u(t) \\ y(t) = C_z x(t) \end{cases} \tag{3.6}$$

with

$$(A, B, C)_z = \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z)(A_{\mathbf{i}}, B_{\mathbf{i}}, C_{\mathbf{i}}) \tag{3.7}$$

and

$$\Delta A_z = H_z^a f^a(t) N_z^a, \quad \Delta B_z = H_z^b f^b(t) N_z^b, \qquad \forall \mu \in \mathscr{U}_r. \tag{3.8}$$

We will consider that only a subset $z_{\mathbf{q}}$, $\mathbf{q} \in \mathcal{Q}_s$, of the premise variables z are measurable and available for the control law. Hereafter, $\mathcal{Q}_s$ denotes the set of the index of the s premise variables used by the controller and $\mathcal{Q}_v$ the set of the index of the v premise variables that are not used by the controller, that is, $v = p - s$. The aim is to design a full-order dynamic output feedback (2.28), which can be written as

$$\begin{cases} \dot{x}_c(t) = A_{cz_{\mathbf{q}}} x_c(t) + B_{cz_{\mathbf{q}}} y(t) \\ u(t) = C_{cz_{\mathbf{q}}} x_c(t) + D_{cz_{\mathbf{q}}} y(t), \end{cases} \tag{3.9}$$

with

$$(A_c, B_c, C_c, D_c)_{z_{\mathbf{q}}} = \sum_{\mathbf{i}\in\mathcal{I}_s} \mu_{\mathbf{i}}(z_{\mathbf{q}})(A_{c\mathbf{i}}, B_{c\mathbf{i}}, C_{c\mathbf{i}}, D_{c\mathbf{i}}) \tag{3.10}$$

and $x_c(t) \in \mathbb{R}^n$ is the controller state. As a matter of fact, the control law depends on the output $y(t)$ and on some of the premise variables available from the measured outputs.

The closed-loop system can be represented as

$$\dot{x}_a(t) = \mathbb{A}_z x_a(t) \tag{3.11}$$

where $x_a(t) = \begin{bmatrix} x(t)' & x_c(t)' \end{bmatrix}'$ denotes the augmented state and

$$\mathbb{A}_z = \begin{bmatrix} (A_z + \Delta A_z) + (B_z + \Delta B_z)D_{cz_{\mathbf{q}}}C_z & (B_z + \Delta B_z)C_{cz_{\mathbf{q}}} \\ B_{cz_{\mathbf{q}}}C_z & A_{cz_{\mathbf{q}}} \end{bmatrix}.$$

The following problem is addressed in this work.

Problem 3.1 *Let the T-S fuzzy system* (3.6) *subject to norm-bounded uncertainties given by* (3.8) *and a subset* $z_{\mathbf{q}}$, $\mathbf{q} \in \mathcal{Q}_s$, *of the premise variables* z *that are measurable and available for the control law. Determine a fuzzy dynamic output feedback controller as* (3.9) *such that the closed-loop system* (3.11) *is asymptotically stable.*

Remark 3.1 *Problem 3.1 assumes that the premise variables are partially measurable. The proposed technique can also be applied when all premise variables are available, motivated by the scenario where the fuzzy model has a large number of fuzzy rules with complex membership functions and the control law under perfect premise matching has high implementation complexity. When none of the premise variables are measured, a constant gain controller (instead of fuzzy combinations) may be obtained. It is straightforward to consider in the proposed approach controllers depending on different membership functions or variables other than the premise variables. For instance,* $z_{\mathbf{q}}$ *could be the controller states or the estimated states of the plant. Although no advantages have been observed compared to constant gain controllers, in the case of completely immeasurable premise variables, the investigations of these possibilities are left for future works.*

The following lemma will be useful for the main results.

Lemma 3.1 *If there exists a continuously differentiable parameter-dependent symmetric matrix* $W_z \in \mathbb{R}^{2n \times 2n}$ *such that the following LMIs are satisfied for all* $\mu \in \mathcal{U}_r$

$$W_z > 0$$

$$He\{\mathbb{A}_z W_z\} - \dot{W}_z < 0, \tag{3.12}$$

where $\dot{W}_z$ *denotes the time-derivative of* W_z*, then the closed-loop system* (3.11) *is asymptotically stable.*

Proof The proof is based on the stability of (3.11) with the Lyapunov function $V(x_a) = x_a(t)'W_z^{-1}x_a(t)$ and from $\dot{W}_z^{-1} = -W_z^{-1}\dot{W}_z W_z^{-1}$. ∎

The results presented in the next section provide sufficient conditions for the design of full order dynamic output feedback controllers depending only on the available premise variables $z_{\mathbf{q}}$.

3.3 Main Results

Firstly, we will consider quadratic stability with $W_z = W$ and the parametrization adopted in Chilali and Gahinet (1996); Scherer et al. (1997). Using (2.29) with $U = U' = Y$, $Y = Z$ and

$V = V' = -Z$, we define the following matrices:

$$W = \begin{bmatrix} X & Y \\ Y & \blacksquare \end{bmatrix}, \; W^{-1} = \begin{bmatrix} Z & -Z \\ -Z & \blacksquare \end{bmatrix}, \; \mathscr{X} = \begin{bmatrix} I_n & Z \\ 0 & -Z \end{bmatrix} \tag{3.13}$$

where $Y = X - Z^{-1}$ and $X, Z \in \mathbb{R}^{n\times n}$ are symmetric. From $WW^{-1} = I$ and using the change of variables presented in (2.30), the condition (3.12) is equivalent to

$$\mathscr{X}' \left(He \left\{ \mathbb{A}_z W \right\} \right) \mathscr{X} = He \left\{ \begin{bmatrix} (A_z + \Delta A_z)X + (B_z + \Delta B_z)\bar{\mathscr{C}}_z & (A_z + \Delta A_z) + (B_z + \Delta B_z)\bar{\mathscr{D}}_{z_\mathbf{q}} C_z \\ \bar{\mathscr{A}}_z & Z(A_z + \Delta A_z) + \bar{\mathscr{B}}_z C_z \end{bmatrix} \right\} < 0 \tag{3.14}$$

with

$$\begin{aligned} \bar{\mathscr{A}}_z &= Z(A_z + \Delta A_z)X - ZA_{cz_\mathbf{q}}Y - ZB_{cz_\mathbf{q}}C_zX + \\ &\quad Z(B_z + \Delta B_z)C_{cz_\mathbf{q}}Y + Z(B_z + \Delta B_z)D_{cz_\mathbf{q}}C_zX \\ \bar{\mathscr{B}}_z &= Z(B_z + \Delta B_z)D_{cz_\mathbf{q}} - ZB_{cz_\mathbf{q}}, \\ \bar{\mathscr{C}}_z &= C_{cz_\mathbf{q}}Y + D_{cz_\mathbf{q}}C_zX, \;\; \bar{\mathscr{D}}_{z_\mathbf{q}} = D_{cz_\mathbf{q}}. \end{aligned} \tag{3.15}$$

Observe that the controller gains cannot be recovered from (3.15) due to the presence of the uncertain terms and the system matrices depending on the premise variables z that are not available for feedback.

To allow the design of a controller depending only on the available premise variables $z_\mathbf{q}$, the following approach is proposed. First, let us define

$$\Gamma_A := A_z - \hat{A}_{z_\mathbf{q}}, \quad \Gamma_B := B_z - \hat{B}_{z_\mathbf{q}}, \quad \Gamma_C := C_z - \hat{C}_{z_\mathbf{q}} \tag{3.16}$$

where $\hat{A}_{z_\mathbf{q}}$, $\hat{B}_{z_\mathbf{q}}$ and $\hat{C}_{z_\mathbf{q}}$ are the matrices obtained from A_z, B_z and C_z, respectively, by representing the fuzzy summation (3.7) exclusively by the premise variables $z_\mathbf{q}$, used in (3.10), following the same steps used to rewrite (3.3) as (3.4).

Remark 3.2 *As pointed out in Remark 3.1, several scenarios can be taken into account in this chapter. It is evident that, when all premise variables are available for feedback,* $\Gamma_A = \Gamma_B = \Gamma_C = 0$*. When one or more premise variables are not available, the matrices* $\hat{A}_{z_\mathbf{q}}$*,* $\hat{B}_{z_\mathbf{q}}$ *and* $\hat{C}_{z_\mathbf{q}}$ *can be constructed by assigning in* A_z*,* B_z *and* C_z *fixed values for the premise variables that are not used by the controller, denoted here by* $z_\mathbf{v}$*, where* $\mathbf{v} \in \mathcal{Q}_v$ *is the index of the* v *premise variables not available for the control law. Any value from the universe of discourse*[2] *for* $z_\mathbf{v}$ *can be used. However, since the conservativeness of the design conditions are related with the magnitude of the terms* Γ_A*,* Γ_B *and* Γ_C*, the best choice for a fixed value of* $z_\mathbf{v}$ *is such that the norm of* $\Delta\mu_{v_j\ell}(z_{v_j})$*,* $j \in \mathcal{Q}_v$*,* $\ell = 1, \ldots, r_\ell$*, as defined in* (3.5)*, is minimum. Thanks to the multi-simplex modeling, this can be obtained by choosing* $z_\mathbf{v}$ *such that* $\mu_{v_j}(z_{v_j}) = (0.5, \ldots, 0.5)$ *which yields* $|\Delta\mu_{v_j\ell}(z_{v_j})| \leq 0.5$*. Observe that, if one considers the controller's premise variables different from the plant's premise variables,*

[2]Space of all possible values of the premise variables.

as in Nguang and Shi (2006), one has $|\Delta\mu_{v_j\ell}(z_{v_j})| \leq 1$ and the conservativeness of the conditions increases. Finally, if the premise variables are estimated, as in Guerra et al. (2006); Asemani and Majd (2013); Guerra et al. (2018); Dong and Wang (2017) in the context of observer-based state-feedback controllers, Lipschitz conditions can improve the results, however it is not straightforward to use Lipschitz conditions in the dynamic output feedback control problem.

Remark 3.3 *It is important to observe that a systematic procedure is proposed to construct $\hat{A}_{z_\mathbf{q}}$, $\hat{B}_{z_\mathbf{q}}$ and $\hat{C}_{z_\mathbf{q}}$, as presented in Section 3.2.2, and, for that reason, no upper bounds are necessary to evaluate Γ_A, Γ_B and Γ_C, as occur in some approaches (Nguang and Shi, 2006; Apkarian and Noll, 2006).*

Then, one can replace A_z, B_z and C_z in (3.15) by $\hat{A}_{z_\mathbf{q}}+\Gamma_A$, $\hat{B}_{z_\mathbf{q}}+\Gamma_B$ and $\hat{C}_{z_\mathbf{q}}+\Gamma_C$, respectively, and rewrite (3.14)-(3.15) as

$$He\left\{\begin{bmatrix} A_zX + B_z(\mathscr{C}_{z_\mathbf{q}} + \mathscr{D}_{z_\mathbf{q}}\Gamma_C X) & A_z + B_z\mathscr{D}_{z_\mathbf{q}}C_z \\ \mathscr{A}_{z_\mathbf{q}} + \Phi & ZA_z + (\mathscr{B}_{z_\mathbf{q}} + Z\Gamma_B\mathscr{D}_{z_\mathbf{q}})C_z \end{bmatrix} + \begin{bmatrix} \Delta A_zX + \Delta B_z(\mathscr{C}_{z_\mathbf{q}} + \mathscr{D}_{z_\mathbf{q}}\Gamma_C X) & \Delta A_z + \Delta B_z\mathscr{D}_{z_\mathbf{q}}C_z \\ Z\Delta A_zX + Z\Delta B_z(\mathscr{C}_{z_\mathbf{q}} + \mathscr{D}_{z_\mathbf{q}}\Gamma_C X) & Z\Delta A_z + Z\Delta B_z\mathscr{D}_{z_\mathbf{q}}C_z \end{bmatrix}\right\} < 0 \quad (3.17)$$

with

$$\begin{aligned} \mathscr{A}_{z_\mathbf{q}} &= Z\hat{A}_{z_\mathbf{q}}X - ZA_{cz_\mathbf{q}}Y - ZB_{cz_\mathbf{q}}\hat{C}_{z_\mathbf{q}}X + Z\hat{B}_{z_\mathbf{q}}C_{cz_\mathbf{q}}Y + Z\hat{B}_{z_\mathbf{q}}D_{cz_\mathbf{q}}\hat{C}_{z_\mathbf{q}}X \\ \mathscr{B}_{z_\mathbf{q}} &= Z\hat{B}_{z_\mathbf{q}}D_{cz_\mathbf{q}} - ZB_{cz_\mathbf{q}}, \\ \mathscr{C}_{z_\mathbf{q}} &= C_{cz_\mathbf{q}}Y + D_{cz_\mathbf{q}}\hat{C}_{z_\mathbf{q}}X, \quad \mathscr{D}_{z_\mathbf{q}} = D_{cz_\mathbf{q}}, \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \Phi := {} & Z\Gamma_AX + (\underbrace{Z\hat{B}_{z_\mathbf{q}}\mathscr{D}_{z_\mathbf{q}} - ZB_{cz_\mathbf{q}}}_{\mathscr{B}_{z_\mathbf{q}}})\Gamma_CX + \\ & Z\Gamma_B(\underbrace{C_{cz_\mathbf{q}}Y + \mathscr{D}_{z_\mathbf{q}}\hat{C}_{z_\mathbf{q}}X}_{\mathscr{C}_{z_\mathbf{q}}}) + Z\Gamma_B\mathscr{D}_{z_\mathbf{q}}\Gamma_CX \\ = {} & Z\Gamma_AX + \mathscr{B}_{z_\mathbf{q}}\Gamma_CX + Z\Gamma_B\mathscr{C}_{z_\mathbf{q}} + Z\Gamma_B\mathscr{D}_{z_\mathbf{q}}\Gamma_CX. \end{aligned} \quad (3.19)$$

As can be observed, the change of variables (3.18) allows to recover the controller gains depending only on $z_\mathbf{q}$:

$$\begin{cases} A_{cz_\mathbf{q}} = Z^{-1}\left\{Z\hat{A}_{z_\mathbf{q}}X + Z\hat{B}_{z_\mathbf{q}}\mathscr{C}_{z_\mathbf{q}} - \mathscr{A}_{z_\mathbf{q}} \right. \\ \qquad\qquad \left. - \left[Z\hat{B}_{z_\mathbf{q}}\mathscr{D}_{z_\mathbf{q}} - \mathscr{B}_{z_\mathbf{q}}\right]\hat{C}_{z_\mathbf{q}}X\right\}Y^{-1} \\ B_{cz_\mathbf{q}} = Z^{-1}\left(Z\hat{B}_{z_\mathbf{q}}\mathscr{D}_{z_\mathbf{q}} - \mathscr{B}_{z_\mathbf{q}}\right) \\ C_{cz_\mathbf{q}} = \left(\mathscr{C}_{z_\mathbf{q}} - \mathscr{D}_{z_\mathbf{q}}\hat{C}_{z_\mathbf{q}}X\right)Y^{-1} \\ D_{cz_\mathbf{q}} = \mathscr{D}_{z_\mathbf{q}}, \end{cases} \quad (3.20)$$

Observe also that inequality (3.17) is nonlinear on the decision variables X, Z, $\mathscr{A}_{z_\mathbf{q}}$, $\mathscr{B}_{z_\mathbf{q}}$, $\mathscr{C}_{z_\mathbf{q}}$ and $\mathscr{D}_{z_\mathbf{q}}$. The case with no uncertainty will be discussed first and then we will take into account the uncertainties.

3.3.1 System without uncertainties

Consider the T-S system (3.6) without uncertainties, that is, $\Delta A_z = \Delta B_z = 0$. Then, (3.17) can be rewritten as

$$He\left\{\Upsilon + \begin{bmatrix} B_z\mathcal{D}_{z_\mathbf{q}}\Gamma_C X & 0 \\ \Phi & Z\Gamma_B\mathcal{D}_{z_\mathbf{q}}C_z \end{bmatrix}\right\} < 0 \tag{3.21}$$

with

$$\Upsilon := \begin{bmatrix} A_zX + B_z\mathcal{C}_{z_\mathbf{q}} & A_z + B_z\mathcal{D}_{z_\mathbf{q}}C_z \\ \mathcal{A}_{z_\mathbf{q}} & ZA_z + \mathcal{B}_{z_\mathbf{q}}C_z \end{bmatrix} \tag{3.22}$$

and Φ given by (3.19).

Observe that the second part of the above inequality, that contains the nonlinear terms, is rewritten as

$$He\left\{\underbrace{\begin{bmatrix} 0 & B_z\mathcal{D}_{z_\mathbf{q}}\Gamma_C \\ Z & \mathcal{B}_{z_\mathbf{q}}\Gamma_C \end{bmatrix}}_{M_1'}\underbrace{\begin{bmatrix} \Gamma_AX + \Gamma_B\mathcal{C}_{z_\mathbf{q}} & \Gamma_B\mathcal{D}_{z_\mathbf{q}}C_z \\ X & 0 \end{bmatrix}}_{M_2}\right\} + \underbrace{\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix}}_{M_3}\underbrace{\begin{bmatrix} 0 & \star \\ \Gamma_B\mathcal{D}_{z_\mathbf{q}}\Gamma_C & 0 \end{bmatrix}}_{R}\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix}. \tag{3.23}$$

The following theorem proposes convex conditions for the design of (3.9).

Theorem 3.1 *For given scalars $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}_{>0}$, if there exist symmetric positive definite matrices X , $Z \in \mathbb{R}^{n\times n}$, and parametrically affine matrices $\mathcal{A}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n}$, $\mathcal{B}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n_y}$, $\mathcal{C}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n}$ and $\mathcal{D}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n_y}$, such that, for all $\mu \in \mathcal{U}_r$, the following LMIs are satisfied:*

$$\begin{bmatrix} X & I_n \\ I_n & Z \end{bmatrix} > 0 \tag{3.24}$$

$$\Psi := \begin{bmatrix} \Psi_{11} & \star \\ \Psi_{21} & \Psi_{22} \end{bmatrix} < 0 \tag{3.25}$$

where

$$\Psi_{11} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\Psi_{111}} & \star \\ \hline -\lambda_1 I_n & \lambda_2 Z & \\ \lambda_2\Gamma_C'\mathcal{D}_{z_\mathbf{q}}'B_z' & -\lambda_1 I + \lambda_2\Gamma_C'\mathcal{B}_{z_\mathbf{q}}' & -2\lambda_2 I \end{array}\right]$$

$$\Psi_{111} = He\left\{\Upsilon + \lambda_1\begin{bmatrix} \Gamma_AX + \Gamma_B\mathcal{C}_{z_\mathbf{q}} + X & \Gamma_B\mathcal{D}_{z_\mathbf{q}}C_z + Z \\ \Gamma_C'\mathcal{D}_{z_\mathbf{q}}'B_z' + X & \Gamma_C'\mathcal{B}_{z_\mathbf{q}}' + Z \end{bmatrix}\right\}$$

$$\Psi_{21} = \left[\begin{array}{cc|c} -\lambda_1 I + \lambda_2\Gamma_AX + \lambda_2\Gamma_B\mathcal{C}_{z_\mathbf{q}} & \lambda_2\Gamma_B\mathcal{D}_{z_\mathbf{q}}C_z & I \\ \lambda_2 X & -\lambda_1 I & \\ \hline -\lambda_1 I + \lambda_2 X & 0 & 0 \\ 0 & -\lambda_1 I + \lambda_2 Z & \end{array}\right]$$

$$\Psi_{22} = \left[\begin{array}{c|cc} -2\lambda_2 I & \multicolumn{2}{c}{\star} \\ \hline 0 & -2\lambda_2 I & \star \\ & \Gamma_B \mathscr{D}_{z_\mathbf{q}} \Gamma_C & -2\lambda_2 I \end{array}\right],$$

Υ given by (3.22)*, and Γ_A, Γ_B and Γ_C as is* (3.16)*, then the controller* (3.9) *whose state-space matrices are given by* (3.20)*, with $Y = X - Z^{-1}$, makes the closed-loop system* (3.11) *asymptotically stable with $\Delta A_z = \Delta B_z = 0$.*

Proof First, from (3.24), it follows that $X > 0$, $Z > 0$ and $Y > 0$, since by definition $Y = X - Z^{-1}$. Then, the gains (3.20) are well defined.

Thus, pre- and post-multiplying (3.25) by $\begin{bmatrix} I_{2n} & M_1' & M_2' & M_3' \end{bmatrix}$ and its transpose, respectively, we obtain

$$\Pi := He\{\Upsilon\} + \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}' \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} < 0 \tag{3.26}$$

where M_1, M_2, M_3 are given in (3.23). Consider the change of variables (3.18) and (3.16), then the following inequality is obtained

$$He\left\{\begin{bmatrix} \Omega X + B_z C_{cz_\mathbf{q}} Y & \Omega \\ Z\left(\Omega - B_{cz_\mathbf{q}} C_z\right) X + Z\left(B_z C_{cz_\mathbf{q}} - A_{cz_\mathbf{q}}\right) Y & Z\left(\Omega - B_{cz_\mathbf{q}} C_z\right) \end{bmatrix}\right\} < 0$$

with $\Omega = A_z + B_z D_{cz_\mathbf{q}} C_z$.

Therefore, the above inequality can be written as $\mathscr{X}'\left(He\left\{\mathbb{A}_z W\right\}\right)\mathscr{X} < 0$ as in (3.14) with the definitions (3.13). One also observe that (3.24) is equivalent to $\mathscr{X}^T W \mathscr{X}$, then $W > 0$. Thus, from Lemma 5.1, the closed-loop system (3.11) is asymptotically stable.

■

Consider the particular case $\Gamma_A \neq 0$, $\Gamma_B = 0$ and $\Gamma_C = 0$. Then, one has the following corollary.

Corollary 3.1 *For given scalars $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}_{>0}$, if there exist symmetric positive definite matrices X , $Z \in \mathbb{R}^{n\times n}$, and parametrically affine matrices $\mathscr{A}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n}$, $\mathscr{B}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n_y}$, $\mathscr{C}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n}$ and $\mathscr{D}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n_y}$, such that* (3.24) *and the following LMI are satisfied for all $\mu \in \mathscr{U}_r$:*

$$diag(\Upsilon + \Upsilon', 0) + \begin{bmatrix} 0 & \star & \star \\ \lambda_1 Z \Gamma_A & 0 & \star \\ X - \lambda_1 I_n & \lambda_2 \Gamma_A' Z & -2\lambda_2 I_n \end{bmatrix} < 0 \tag{3.27}$$

where Υ and Γ_A are given by (3.22) *and* (3.16)*, respectively, then the controller* (3.9) *whose state-space matrices are given by* (3.20)*, with $Y = X - Z^{-1}$, asymptotically stabilize the closed-loop system (3.11) with $\Delta A_z = \Delta B_z = 0$.*

Proof By pre- and post-multiplying (3.27) by $\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & Z\Gamma_A \end{bmatrix}$ and its transpose, respectively, one has

$$He\left\{\begin{bmatrix} A_zX + B_z\mathscr{C}_{z_\mathbf{q}} & A_z + B_z\mathscr{D}_{z_\mathbf{q}}C_z \\ \mathscr{A}_{z_\mathbf{q}} + Z\Gamma_A X & ZA_z + \mathscr{B}_{z_\mathbf{q}}C_z \end{bmatrix}\right\} < 0.$$

The above inequality is equivalent to $\mathscr{X}'\left(He\left\{\mathbb{A}_z W\right\}\right)\mathscr{X} < 0$ with the definitions (3.13). From (3.24), one has $W > 0$, then the closed-loop system (3.11) is asymptotically stable. ■

3.3.2 System with Uncertainties

In this section, matrices A_z and B_z are subjected to uncertainties as in (3.6).

Observe that the second term of (3.17) that contains the uncertain terms can be rewritten as

$$He\left\{\begin{bmatrix} I \\ Z \end{bmatrix}\Delta H_z^a f^a(t) N_z^a \begin{bmatrix} X & I \end{bmatrix} + \begin{bmatrix} I \\ Z \end{bmatrix}\Delta H_z^b f^b(t) N_z^b \begin{bmatrix} \mathscr{C}_{z_\mathbf{q}} + \mathscr{D}_{z_\mathbf{q}}\Gamma_C X & \mathscr{D}_{z_\mathbf{q}}C_z \end{bmatrix}\right\},$$

where ΔA_z and ΔB_z were replaced using (3.8).

Therefore, defining $\Lambda \coloneqq \mathscr{C}_{z_\mathbf{q}} + \mathscr{D}_{z_\mathbf{q}}\Gamma_C X$, using Lemma 2.1 and the manipulations done in (3.21)–(3.26), (3.17) is rewritten as

$$\begin{aligned} \Pi + He\left\{\begin{bmatrix} I \\ Z \end{bmatrix} H_z^a f^a(t) N_z^a \begin{bmatrix} X & I \end{bmatrix} + \begin{bmatrix} I \\ Z \end{bmatrix} H_z^b f^b(t) N_z^b \begin{bmatrix} \Lambda & \mathscr{D}_{z_\mathbf{q}}C_z \end{bmatrix}\right\} < \\ \Pi + \beta_1 \begin{bmatrix} I \\ Z \end{bmatrix} H_z^a H_z^{a'} \begin{bmatrix} I & Z \end{bmatrix} + \beta_1^{-1}\begin{bmatrix} X \\ I \end{bmatrix} N_z^{a'} f^a(t)' f^a(t) N_z^a \begin{bmatrix} X & I \end{bmatrix} + \\ \beta_2 \begin{bmatrix} I \\ Z \end{bmatrix} H_z^b H_z^{b'} \begin{bmatrix} I & Z \end{bmatrix} + \beta_2^{-1}\begin{bmatrix} \Lambda' \\ C_z'\mathscr{D}_{z_\mathbf{q}}' \end{bmatrix} N_z^{b'} f^b(t)' f^b(t) N_z^b \begin{bmatrix} \Lambda & \mathscr{D}_{z_\mathbf{q}}C_z \end{bmatrix} < \\ \Pi + \beta_1 \begin{bmatrix} H_z^a \\ ZH_z^a \end{bmatrix}\begin{bmatrix} H_z^{a'} & H_z^{a'}Z \end{bmatrix} + \beta_1^{-1}\begin{bmatrix} XN_z^{a'} \\ N_z^{a'} \end{bmatrix}\begin{bmatrix} N_z^a X & N_z^a \end{bmatrix} + \\ \beta_2 \begin{bmatrix} H_z^b \\ ZH_z^b \end{bmatrix}\begin{bmatrix} H_z^{b'} & H_z^{b'}Z \end{bmatrix} + \beta_2^{-1}\begin{bmatrix} \Lambda' N_z^{b'} \\ C_z'\mathscr{D}_{z_\mathbf{q}}' N_z^{b'} \end{bmatrix}\begin{bmatrix} N_z^b\Lambda & N_z^b\mathscr{D}_{z_\mathbf{q}}C_z \end{bmatrix} < 0, \end{aligned} \tag{3.28}$$

where Π is defined in (3.26). The following theorem proposes convex conditions to synthesize (3.9).

Theorem 3.2 *For given scalars $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}_{>0}$, $\lambda_3 \in \mathbb{R}$, $\lambda_4 \in \mathbb{R}_{>0}$, $\beta_1 \in \mathbb{R}_{>0}$ and $\beta_2 \in \mathbb{R}_{>0}$, if there exist symmetric positive definite matrices X , $Z \in \mathbb{R}^{n\times n}$, and parametrically affine matrices $\mathscr{A}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n}$, $\mathscr{B}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n_y}$, $\mathscr{C}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n}$ and $\mathscr{D}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n_y}$, such that, for all $\mu \in \mathscr{U}_r$, (3.24) and the following LMI are satisfied:*

$$\begin{bmatrix} \Upsilon_{11} & \star \\ \Upsilon_{21} & \Upsilon_{22} \end{bmatrix} < 0 \tag{3.29}$$

where

$$\Upsilon_{11} = \left[\begin{array}{ccc|c} & \Psi & & \star \\ \hline H_z^{a'} & H_z^{a'}Z & 0 & -\beta_1^{-1}I \end{array}\right]$$

$$\Upsilon_{21} = \left[\begin{array}{ccc|c} N_z^a X & N_z^a & 0 & 0 \\ H_z^{b'} & H_z^{b'}Z & 0 & 0 \\ N_z^b(\mathscr{C}_{z_{\mathbf{q}}} + \Theta) & \Upsilon_{211} & \mathbb{1}_{n,6n} \otimes \Theta & \Theta \\ \hline (X - \lambda_3 I) & -\lambda_3 I & -\mathbb{1}_{n,6n} \otimes \lambda_3 I & -\lambda_3 I \end{array}\right]$$

$$\Upsilon_{211} = N_z^b \mathscr{D}_{z_{\mathbf{q}}}(C_z + \lambda_3 \Gamma_C), \quad \Theta = \lambda_3 N_z^b \mathscr{D}_{z_{\mathbf{q}}} \Gamma_C$$

$$\Upsilon_{22} = \left[\begin{array}{cc|c} diag(-\beta_1 I, -\beta_2^{-1} I) & \star & \star \\ \begin{bmatrix}\Theta & \Theta\end{bmatrix} & (\Theta - \beta_2 I) & \star \\ & & \star \\ \hline -\lambda_3 I \quad -\lambda_3 I & (\lambda_4 \Gamma_C' \mathscr{D}_{z_{\mathbf{q}}}' N_z^{b'} - \lambda_3 I) & -2\lambda_4 I \end{array}\right]$$

where Ψ *is given by* (3.25)*, then the controller* (3.9) *whose state-space matrices are given by* (3.20)*, with* $Y = X - Z^{-1}$*, robustly stabilizes the closed-loop system (3.11).*

Proof Pre- and post-multiplying (3.29) by $\text{diag}(\varphi, I_{5n})$, with $\varphi = \begin{bmatrix} I_{2n} & M_1' & M_2' & M_3' \end{bmatrix}$, and its transpose, respectively, one has

$$\begin{bmatrix} \Pi & \star & \star & \star & \star \\ \begin{bmatrix} H_z^{a'} & H_z^{a'}Z \end{bmatrix} & -\beta_1^{-1}I & \star & \star & \star \\ \begin{bmatrix} N_z^a X & N_z^a \end{bmatrix} & 0 & \beta_1 I & \star & \star \\ \begin{bmatrix} H_z^{b'} & H_z^{b'}Z \end{bmatrix} & 0 & 0 & -\beta_2^{-1}I & \star \\ \begin{bmatrix} N_z^b \Lambda & N_z^b \mathscr{D}_{z_{\mathbf{q}}} C_z \end{bmatrix} & 0 & 0 & 0 & -\beta_2^{-1}I \end{bmatrix} < 0$$

with Π defined in (3.26) and $\Lambda = \mathscr{C}_{z_{\mathbf{q}}} + \mathscr{D}_{z_{\mathbf{q}}} \Gamma_C X$. Then, applying Schur successive times in the above inequality and using (3.28), one has

$$\Pi + He\left\{ \begin{bmatrix} I \\ Z \end{bmatrix} H_z^a f^a(t) N_z^a \begin{bmatrix} X & I \end{bmatrix} + \begin{bmatrix} I \\ Z \end{bmatrix} H_z^b f^b(t) N_z^b \begin{bmatrix} \Lambda & \mathscr{D}_{z_{\mathbf{q}}} C_z \end{bmatrix} \right\} < 0.$$

Following the same lines of the proof of Theorem 3.1, we prove the asymptotically stability of the closed-loop system (3.11). ∎

Remark 3.4 *Theorem 3.2 depends on the choice of six scalar parameters what can be a high computational cost task. For the sake of computational time, some parameters can be set equal, for instance, one may impose* $\lambda_i = \lambda$*,* $i = 1, \ldots, 4$ *and* $\beta_j = \beta$*,* $j = 1, 2$*, at a price of more conservative results. In Section 3.4, the numerical examples found solution with* $\lambda_1 = \lambda_3 = 0$*,* $\lambda_2 = \lambda_4 = 1$*, and* $\beta_1 = \beta_2 = 1$*, and these values are suggested for an initial guess. We verify that only in particular cases, where the stability of the closed-loop systems is a difficult task, there is a need to search all scalar parameters, which can be seem as an extra degree of freedom for challenging cases.*

Remark 3.5 *It is worthy to note that quadratic stability can be relaxed by the use of fuzzy Lyapunov functions (Tanaka et al., 2003). For this, consider the Lyapunov function* $V(x_a) = x_a(t)'W_{z_{\mathbf{q}}}^{-1}x_a(t)$ *with*

$$W_{z_{\mathbf{q}}} = \begin{bmatrix} X_{z_{\mathbf{q}}} & Y_{z_{\mathbf{q}}} \\ Y_{z_{\mathbf{q}}} & Y_{z_{\mathbf{q}}} \end{bmatrix}, \quad W_{z_{\mathbf{q}}}^{-1} = \begin{bmatrix} Z_{z_{\mathbf{q}}} & -Z_{z_{\mathbf{q}}} \\ -Z_{z_{\mathbf{q}}} & V_{z_{\mathbf{q}}} \end{bmatrix}$$

where $X_{z_{\mathbf{q}}}$ *and* $Z_{z_{\mathbf{q}}}$ *are symmetric positive definite matrices,* $Y_{z_{\mathbf{q}}} = X_{z_{\mathbf{q}}} - Z_{z_{\mathbf{q}}}^{-1}$ *and* $V_{z_{\mathbf{q}}} = Y_{z_{\mathbf{q}}}^{-1}X_{z_{\mathbf{q}}}Z_{z_{\mathbf{q}}}$*.One can verify that* $W_{z_{\mathbf{q}}} = W_{z_{\mathbf{q}}}' > 0$ *and* $W_{z_{\mathbf{q}}}W_{z_{\mathbf{q}}}^{-1} = I$*. Then, the condition* (3.12) *is equivalent to*

$$\mathscr{X}'\left(He\left\{\mathbb{A}_z W\right\}\right)\mathscr{X} + \mathscr{X}'\dot{W}_{z_{\mathbf{q}}}\mathscr{X} < 0$$

where, for $\mathscr{X}$ *defined as* (3.13),

$$\mathscr{X}'\dot{W}_{z_{\mathbf{q}}}\mathscr{X} = \begin{bmatrix} \dot{X}_{z_{\mathbf{q}}} & \star \\ -\dot{Z}_{z_{\mathbf{q}}}Z_{z_{\mathbf{q}}}^{-1} & \dot{Z}_{z_{\mathbf{q}}} \end{bmatrix}.$$

Therefore, the conditions of Theorem 3.1, Corollary 3.1 and Theorem 3.2 are relaxed by replacing the term Υ *in* (3.22) *by*

$$\begin{bmatrix} A_z X_{z_{\mathbf{q}}} + B_z \mathscr{C}_{z_{\mathbf{q}}} + 0.5\dot{X}_{z_{\mathbf{q}}} & A_z + B_z \mathscr{D}_{z_{\mathbf{q}}} C_z \\ \tilde{\mathscr{A}}_z & Z_{z_{\mathbf{q}}} A_z + \mathscr{B}_{z_{\mathbf{q}}} C_z + 0.5\dot{Z}_{z_{\mathbf{q}}} \end{bmatrix}$$

with $\tilde{\mathscr{A}}_z = \mathscr{A}_{z_{\mathbf{q}}} - \dot{Z}_{z_{\mathbf{q}}}Z_{z_{\mathbf{q}}}^{-1}$*. The controller gains are given by* (3.20) *with* X*,* Z *and* $\mathscr{A}_{z_{\mathbf{q}}}$ *replaced by* $X_{z_{\mathbf{q}}}$*,* $Z_{z_{\mathbf{q}}}$ *and* $\tilde{\mathscr{A}}_z$*, respectively.*

Observe that matrices $X_{z_{\mathbf{q}}}$ *and* $Z_{z_{\mathbf{q}}}$ *must depend on the available premise variables* $z_{\mathbf{q}}$ *since they appear in the expressions that recover the gains of the controller. If the upper-bounds of the time-derivatives of the membership functions are known, then several techniques may be used to deal with the time-derivative of* $X_{z_{\mathbf{q}}}$ *and* $Z_{z_{\mathbf{q}}}$ *in the design conditions, as shown in Mozelli et al. (2009) and Tognetti et al. (2011). The main challenge for the implementation of the control law is the necessity of computing in real-time the term* $\dot{Z}_{z_{\mathbf{q}}}$*. Since the premise variables* $z_{\mathbf{q}}$ *are supposed to be measured,* $\dot{z}_{\mathbf{q}}$ *could be evaluated. However, to circumvent numerically issues, one may impose* $Z_{z_{\mathbf{q}}} = Z$*, yielding*

$$\mathscr{X}'\dot{W}_{z_{\mathbf{q}}}\mathscr{X} = \begin{bmatrix} \dot{X}_{z_{\mathbf{q}}} & \star \\ 0 & 0 \end{bmatrix}.$$

Remark 3.6 *In the ideal case, where all premise variables are available, one gets* $\Upsilon + \Upsilon' < 0$ *in Theorem 3.1 recovering the standard condition for the DOF design as presented in de Oliveira et al. (2000). Finally, one may observe that is straightforward to incorporate the* H_∞ *criteria in the design conditions.*

Remark 3.7 *The measurement of the premise variables with no accuracy (Agulhari et al., 2013; Sato and Peaucelle, 2013; Lacerda et al., 2016), in the case of noise or bias in the measurement, can also be taken into account using the presented technique. See, for instance, Agulhari et al. (2013); Lacerda et al. (2016) for further details of how to construct matrices* $\hat{A}_{z_{\mathbf{q}}}$*,* $\hat{B}_{z_{\mathbf{q}}}$ *and* $\hat{C}_{z_{\mathbf{q}}}$*, in this case.*

Remark 3.8 *The proposed technique can be also applied to T-S models subject to polytopic-type uncertainties. In this case, the procedure to construct the matrices* $\hat{A}_{z_{\mathbf{q}}}$*,* $\hat{B}_{z_{\mathbf{q}}}$ *and* $\hat{C}_{z_{\mathbf{q}}}$*, used to recover the controller gains, can be adapted such that these matrices are independent of the polytope that describes the uncertain space.*

Remark 3.9 *The design conditions can be adapted to incorporate a decay rate specification given by* $\gamma > 0$*. Condition* (3.12) *becomes*

$$He\left\{\mathbb{A}_z W\right\} + 2\gamma W < 0 \tag{3.30}$$

yielding

$$\Upsilon = \begin{bmatrix} A_z X + B_z \mathscr{C}_{z_{\mathbf{q}}} + 2\gamma X & A_z + B_z \mathscr{D}_{z_{\mathbf{q}}} C_z + 2\gamma I \\ \mathscr{A}_{z_{\mathbf{q}}} + 2\gamma I & Z A_z + \mathscr{B}_{z_{\mathbf{q}}} C_z + 2\gamma Z \end{bmatrix} \tag{3.31}$$

in (3.22)*. Therefore, using* (3.31) *in Theorems 3.1 and 3.2, and Corollary 3.1, one gets* $||x(t)|| \leq \beta||x(0)||e^{-\gamma t}$ *for a given* $\beta > 0$*.*

3.4 Numerical Examples

The computational routines were programmed in Matlab 7.10 (R2010a), using Yalmip (Löfberg, 2004), SeDuMi (Sturm, 1999) and Mosek (ApS, 2019) in a personal computer equipped with an Intel Core i7 CPU (3.40GHz), 8GB RAM. To handle the infinite dimensional problem described by the parameter-dependent conditions, the optimization variables are fixed as parameter-dependent matrices and the negativity of the inequalities is verified by testing a finite set of LMIs that are directly obtained by ROLMIP (Robust LMI Parser) toolbox (Agulhari et al., 2012).

The applicability of the proposed method is illustrated by numerical examples. The scalar parameters that must be provided in Theorems 3.1–3.2 and Corollary 3.1 have been selected following the suggestions presented in Remark 3.4. A line search algorithm could be used as well, probably yielding improved results, at the expense of increasing the computational burden.

Example 3.1 In this first example, the control design problem of the full-order dynamic output feedback controller (3.9) *for the Furuta Pendulum*[3] *(Mori et al., 1976) subject to imperfect premise matching in matrix* A_z *is solved using Corollary 3.1.*

[3] For the sake of computational burden, it is adopted the approximation $\sin(x_2) \approx x_2$.

The nonlinear system dynamics is represented by[4]

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = \left(gm_p^2 l_p^2 r\right) z_1 z_4 x_2 + \left(m_p^2 l_p^3 r\right) z_1 z_2 z_4 x_3 \\ -m_p l_p \left(m_p l_p^2 + J_p\right) (r z_3 + 2 l_p z_2) z_4 x_4 + \left(m_p l_p^2 + J_p\right) z_4 \tau \\ \dot{x}_4 = m_p l_p g \left(m_p l_p^2 z_5 + m_p r^2 + J_{arm}\right) z_4 x_2 + m_p l_p^2 \left(m_p l_p^2 z_5 + m_p r^2 + J_{arm}\right) z_2 z_4 x_3 \\ -m_p^2 l_p^2 r \left(2 l_p z_2 + r z_3\right) z_1 z_4 x_4 + m_p l_p r z_1 z_4 \tau \end{cases} \tag{3.32}$$

where

$$z_1 = cos(x_2), \quad z_2 = x_3 \sin x_2 cos x_2, \quad z_3 = x_4 \sin x_2, \quad z_4 = \frac{1}{\epsilon_1 - \epsilon_2 cos^2 x_2}, \quad z_5 = \sin^2 x_2,$$
$$\epsilon_1 = J_{arm} J_p + J_p m_p r^2 + J_{arm} l_p^2 m_p + \epsilon_2, \quad \epsilon_2 = \left[l_p^4 m_p^2 + J_p l_p^2 m_p + l_p^2 m_p^2 r^2\right]$$

are used as premise variables in the T-S model yielding $z = (z_1, z_2, z_3, z_4, z_5)$, $\mu = (\mu_1(z_1), \mu_2(z_2), \mu_3(z_3), \mu_4(z_4),\ \mu_5(z_5)) \in \mathcal{U}_{2,2,2,2,2}$. *The nonlinear system is represented by the T-S model* (3.6) *using the sector nonlinearity approach (Tanaka and Wang, 2001) in the domain* $x_2 \in [-5°,\ 5°]$, $x_3 \in [-20,\ 20]$ *and* $x_4 \in [-20,\ 20]$, *where* x_1 *is horizontal arm angle,* x_2 *is the pendulum angle,* x_3 *is the horizontal arm angular velocity and* x_4 *is the pendulum angular velocity. The system's parameters adopted in the simulation are presented in Table 3.1.*

Table 3.1: System's parameters.

Symbol	**Definition**	**Value**
l_o	horizontal arm length	0.216 m
l_1	distance between pendulum mass center and arm	0.156 m
m_p	pendulum mass	0.127kg
J_a	Inertia moment horizontal arm	0.002 $kg.m^2$
J_p	Inertia momentum pendulum	0.0012 $kg.m^2$
B_a	Damping coefficient	0.0024 $N.m.s.rad^{-1}$
B_p	Damping coefficient	0.0024 $N.m.s.rad^{-1}$
g	Gravity acceleration	9.71 $m.s^{-2}$

It is considered that only states x_1 *and* x_2 *are measured, that is,*

$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

therefore the control design can only utilize the premises variables $z_{\mathbf{q}} = (z_1, z_4, z_5)$.

For the initial conditions $x(0) = \begin{bmatrix} 0 & 0.1\pi & 0 & 0 \end{bmatrix}'$ *and* $x_c(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}'$, *the time-response*

[4] For conciseness, the dependence of $x(t)$ on t is omitted hereafter.

of the states of the closed-loop system with the controller designed with Corollary 3.1 with $\lambda_1 = 0$ *and* $\lambda_2 = 1$ *is shown in Figure 3.1.*

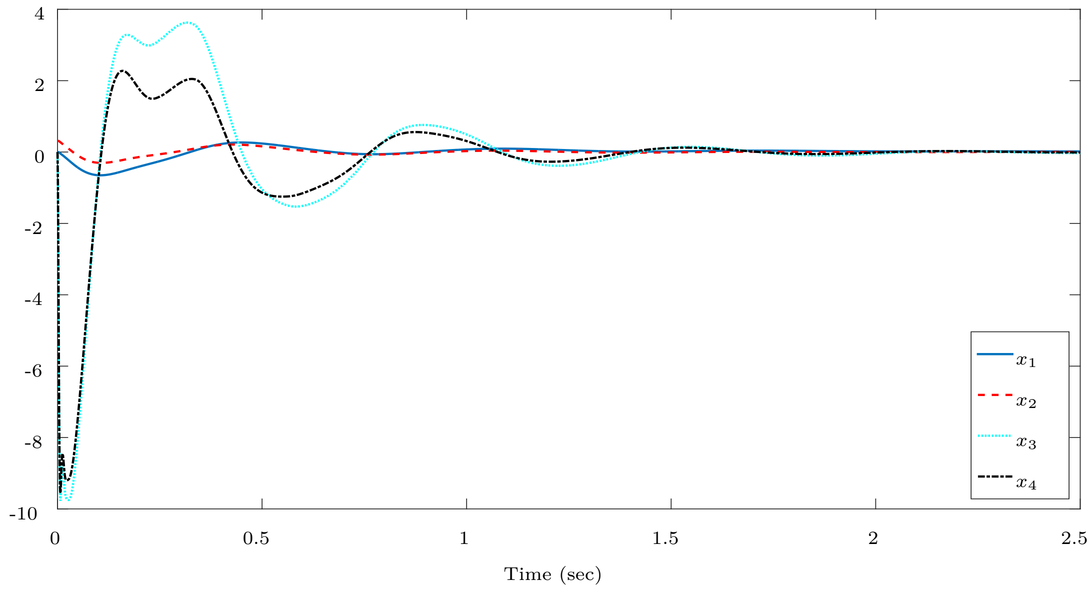

Figure 3.1: Trajectory of the states of the closed-loop system for the initial conditions $x(0) = [0 \;\; 0.1\pi \;\; 0 \;\; 0]'$ and $x_c(0) = [0 \;\; 0 \;\; 0 \;\; 0]'$ using Corollary 3.1.

Figure 3.2 shows the trajectories for the controller designed by Corollary 3.1 using Remark 3.9 with decay rate $\gamma = 0.4$. *We observe a faster convergence compared to the trajectories in Figure 3.1.*

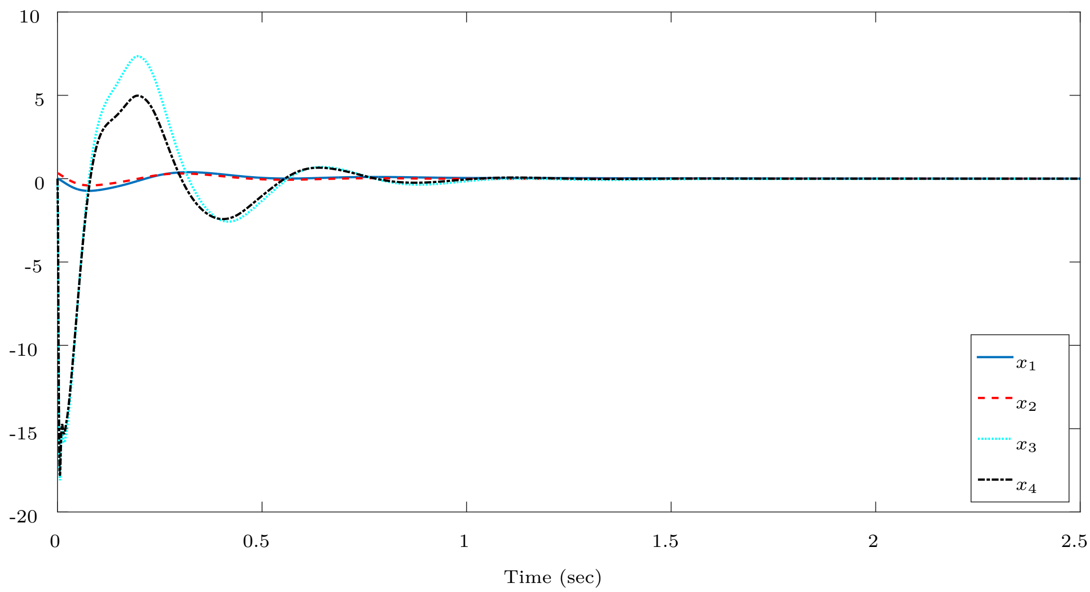

Figure 3.2: Trajectory of the states of the closed-loop system for the initial conditions $x(0) = [0 \;\; 0.1\pi \;\; 0 \;\; 0]'$ and $x_c(0) = [0 \;\; 0 \;\; 0 \;\; 0]'$ using Corollary 3.1 with $\gamma = 0.4$ from Remark 3.9.

Example 3.2 This example is used to evaluate the cases of imperfect premise matching presented *in Section 3.3.1. The nonlinear system is adapted from Tognetti et al. (2012) and given by*

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + \sin x_3 - 0.1x_4 + (x_1^2 + 1)u \\ \dot{x}_2 = x_1 - 2x_2 + \varphi_1 u \\ \dot{x}_3 = x_1 + x_1^2 x_2 - 0.3x_3 \\ \dot{x}_4 = \sin x_3 - x_4 \\ y_1 = x_2 + (x_1^2 + 1)x_4 + \varphi_2 \\ y_2 = x_1 \end{cases} \tag{3.33}$$

where

$$\varphi_1 = \frac{\sin x_3}{x_3}, \quad \varphi_2 = \sin x_3. \tag{3.34}$$

The system (3.33) is rewritten as $\dot{x} = A(z)x + B(z)u$, $y = C(z)x$, with

$$A(z) = \begin{bmatrix} 1 & 1 & z_2 & -0,1 \\ 1 & -2 & 0 & 0 \\ 1 & z_1 & -0,3 & 0 \\ 0 & 0 & z_2 & -1 \end{bmatrix}, \; B(z) = \begin{bmatrix} z_1 + 1 \\ z_2 \\ 0 \\ 0 \end{bmatrix} \tag{3.35}$$
$$C(z) = \begin{bmatrix} 0 & 1 & z_2 & z_1 + 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The premise variables are given by

$$z_1 = x_1^2, \qquad z_2 = \frac{\sin x_3}{x_3}.$$

The matrices A_z, B_z and C_z of the T-S fuzzy model (3.6) *are obtained from* (3.35) *evaluated in the extreme values of z_1 and z_2 for the domain $x_1 \in [-1.4,\ 1.4]$ and $x_3 \in [-0.7,\ 0.7]$ using the sector nonlinearity approach (Tanaka and Wang, 2001).*

Supposing that only premise z_1 is available for the controller by the measurement $y_2 = x_1$. Therefore, $z_{\mathbf{q}} = z_1$ and the vertices of $\hat{A}_{z_{\mathbf{q}}}$, $\hat{B}_{z_{\mathbf{q}}}$ and $\hat{C}_{z_{\mathbf{q}}}$ are obtained by considering $z_2 = \bar{z}_2$, where $\bar{z}_2$ is the mean value of z_2. Theorem 3.1 could find solution with $\lambda_1 = 0$ and $\lambda_2 = 1$.

The states trajectories of the closed-loop system for the initial conditions $x(0) = \begin{bmatrix} 1 & 2 & -0.6 & -4 \end{bmatrix}'$ and $x_c(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}'$ are illustrated by Figure 3.3.

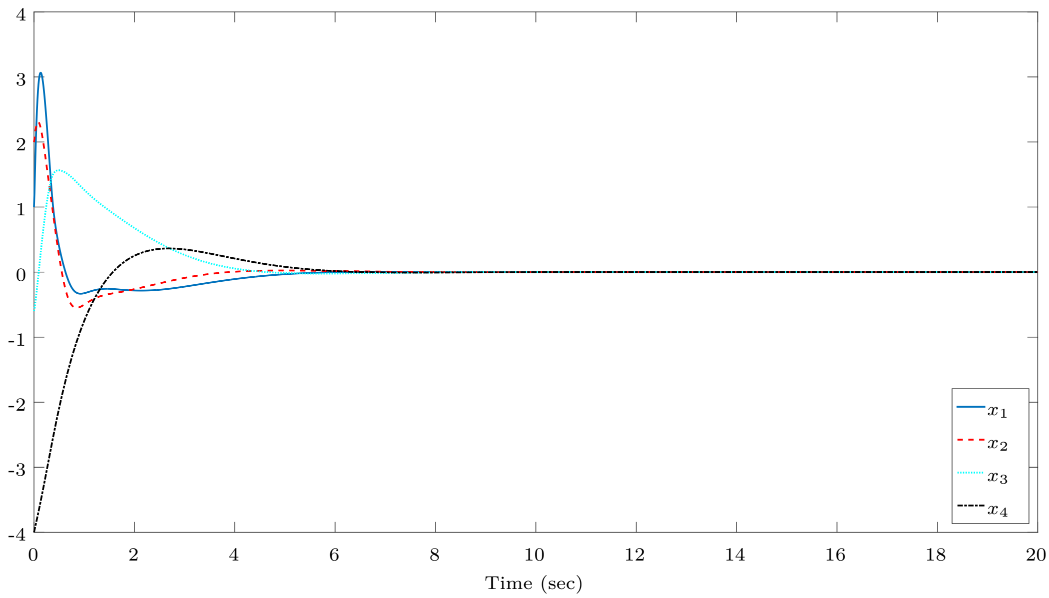

Figure 3.3: Trajectory of the states of the closed-loop system using Theorem 3.1.

Now, we would like to evaluate how the choice of $\hat{A}_{z_{\mathbf{q}}}$, $\hat{B}_{z_{\mathbf{q}}}$ and $\hat{C}_{z_{\mathbf{q}}}$ influence the stabilizability of the proposed conditions. For this, the maximum value of b in $x_1 \in [-b,\ b]$ is evaluated such that Theorem 3.1 is feasible for several choices of $\bar{z}_2$. As showed in Table 3.2, the maximum value of b is obtained with $\bar{z}_2 = 0.96$ as expected since this value represents the mean value of $z_2 \in [0.92,\ 1]$ considering $x_3 \in [-0.7,\ 0.7]$. We can also observe that the conservatism is related with the magnitude of Γ_A, Γ_B and Γ_C. To illustrate this, Table 3.2 shows the lower bounds of ρ and γ in $\|A_{ij} - \hat{A}_i\| < \rho$ and $\|Z(A_{ij} - \hat{A}_i)X\| < \gamma$, $i, j = 1, 2$, respectively, for Z and X obtained from Theorem 3.1. The smallest values of ρ and γ occur for $\bar{z}_2 = 0.96$ where the difference of $A(\mu) - \hat{A}(\hat{\mu})$ is expected to be minimum.

Table 3.2: Maximum values of b in $x_1 \in [-b\ \ b]$ such that Theorem 3.1 is feasible and the lower bounds of ρ and γ in $\|A_{ij} - \hat{A}_i\| < \rho$ and $\|Z(A_{ij} - \hat{A}_i)X\| < \gamma$, $i, j = 1, 2$, respectively.

$\bar{z}_2$	0.5	0.6	0.7	0.8	0.9	0.96	1
b	0.91	1.2	1.35	1.43	1.48	**1.49**	1.49
ρ	0.707	0.565	0.424	0.282	0.414	**0.056**	0.112
γ	0.961	0.766	0.615	0.419	0.212	**0.087**	0.174

To illustrate the flexibility of the proposed technique, we suppose now that only z_2 is measured, that is, the second row of matrix $C(z)$ is modified to $[0\ 0\ 1\ 0]$. It is verified that Theorem 3.1 finds a stabilizing controller for $x_1 \in [-0.6\ \ 0.6]$ and $x_3 \in [-1.65\ \ 1.65]$ considering $\bar{z}_1 = 0.18$ (middle value of z_1), that is, a smaller set for x_1 and a larger set for x_3.

Example 3.3 This example is used to evaluate the cases of imperfect premise matching with uncertainties presented in Section 3.3.2. The nonlinear system is adapted from Guelton et al. (2009)

and the vertices of the T-S fuzzy model (3.6) *are given by*

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 2 & -10 \\ 5 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} -3 & 20 \\ -7 & -2 \end{bmatrix}. H_1^a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H_1^a = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$N_1^a = \begin{bmatrix} 1 & 1 \end{bmatrix}, N_2^a = \begin{bmatrix} -1 & 1 \end{bmatrix}, H_1^b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, H_2^b = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$N_1^b = 1, N_2^b = -0,75,$$

where only $z_1 \in [-1,\ 1]$ *is the premise variable.*

Theorem 3.2 provided a controller (3.9) *with* $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = \lambda_4 = 1$, $\beta_1 = \beta_2 = 1$. *The closed-loop trajectories of the closed-loop system for the initial values* $x_1(0) = 1$, $x_2(0) = 1$, $x_{1c}(0) = 0$, $x_{2c}(0) = 0$ *are shown in Figure 3.4.*

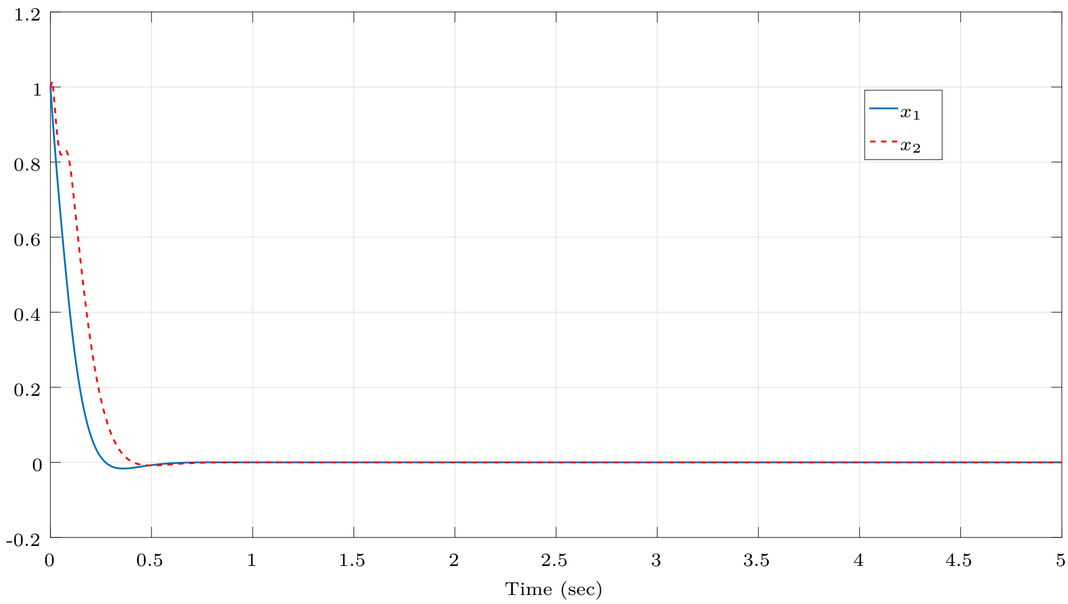

Figure 3.4: Trajectory of the states of the closed-loop system using Theorem 3.2.

To compare Theorem 3.2 with (Guelton et al., 2009, Theorem 2) the feasibility is verified in terms of z_1. *Theorem 3.2 was feasible for* $z_1 \in [-3.1,\ 3.1]$, *adopting* $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = 1$, $\lambda_4 = 1$, $\beta_1 = \beta_2 = 1$, *while (Guelton et al., 2009, Theorem 2) was feasible for a smaller interval,* $z_1 \in [-3.0,\ 3.0]$, *considering sufficiently large variation rate* ($\phi = -200$)*, as a matter of comparison. To compare the computational burden, it is verified that Theorem 3.2 uses 30 variables and 92 rows in the LMI conditions while (Guelton et al., 2009, Theorem 2) uses 58 variables and 40 LMI rows.*

3.5 Conclusion

This chapter contributes with new LMI conditions for the design of DOF controllers with the selection of the membership functions implemented in the control law. Partial or total immeasurable premise variables can be taken into account. The proposed conditions solve a common problem in the DOF design, the dependence of the controller gains on the membership functions of the plant. Allowing also, as an indirect result, robust controllers for systems with parameter uncertainty. The use of fuzzy Lyapunov functions can be easily encompassed and systems with norm-bounded uncertainties are also considered. The effectiveness and validity of the proposed approach are illustrated through numerical examples and time simulations. For future works, the choice of optimal membership functions for the controller in the case of no measurements of the premise variables and efficient methods to deal with the local stability may be addressed.

4 Discrete-time DOF with imperfect premise matching

This chapter aims to investigate the problem of designing locally stabilizing dynamic output feedback controllers and estimate the domain of attraction for discrete-time T-S fuzzy systems. A realistic scenario is assumed where the control signal is subject to saturation, and the premise variables are partially or completely unmeasured, that is, not available for the control law. As a result, the fuzzy output controller can have a different number of fuzzy rules and a different set of membership functions from the T-S model. To obtain locally stabilizable conditions, we propose modeling the variation rate of the membership functions without using upper bounds, a new contribution in the context of output control of discrete-time T-S systems. The design conditions are expressed as linear matrix inequality relaxations based on fuzzy Lyapunov functions using slack variables introduced by Finsler's lemma. The effectiveness and applicability of the methodology are shown through numerical examples.

4.1 Introduction

T-S models usually describe nonlinear dynamics in a compact region in the state space containing the origin. Then, the stability is only assured for trajectories remaining in this region. The simplest way to guarantee the convergence of all trajectories to the origin is to compute as an estimate for the domain of attraction the largest Lyapunov level surface inside the region of validity of the T-S model. However, this estimate may be conservative if quadratic Lyapunov functions are considered. For continuous-time systems, some works proposed conditions using the bounds (Pan et al., 2012) or manipulating (Gomes et al., 2020) the time derivatives of the membership functions. For discrete-time systems, the nonquadratic stabilization is facilitated (Guerra and Vermeiren, 2004), but few works consider models for the advanced instant of the Lyapunov matrix. Thus, the local nature of the evolution of the membership functions is disregarded.

An attempt to develop local conditions for the design of state-feedback laws for discrete-time T-S systems is presented in Lee and Joo (2014). However, an upper bound for the variation of the membership functions must be specified, impacting the estimation of the stability domain. Recent techniques such as Tognetti et al. (2015) and Lendek and Lauber (2022), based on Lyapunov functions depending on past samples, are still unable to take full advantage of the variation of the membership functions inside the domain of validity of the T-S model. Moreover, there is no method to choose the matrix structure that defines the set that contains the estimation of the domain of attraction (Lendek and Lauber, 2022). If saturation is also considered, the domain of attraction is affected (Gomes da Silva Jr. and Tarbouriech, 2006). For instance, the design of full-order DOF controllers assuming all premise variables are available for feedback is investigated in Klug et al. (2015) but considering arbitrary variations of the membership functions.

Motivated by the lack of results for designing DOF controllers for discrete-time systems with partially or completely unmeasurable premise variables, this chapter contributes to this problem by proposing local design conditions. Very few papers have addressed this problem outside the observer-based approach. The solution uses the mean value theorem in several variables (Buck, 1994), as presented in Lee and Joo (2014), but without using bounds on the variation of the membership functions. We also consider saturation in the control signal, making modeling the variation of the membership functions more challenging and intricate. Fuzzy Lyapunov functions are applied to estimate the domain of attraction, and an LMI-based procedure is proposed to design DOF controllers that do not share the same set of premise variables of the T-S system. Thanks to the multi-simplex modeling (Cartesian product of simplexes) (Oliveira et al., 2008), the dynamical matrices of the plant are rewritten in terms of the desirable membership functions. We extend the approach presented in Chapter 3 to the discrete-time case, the main differences are the development of local conditions due to the membership variation modeling and the presence of saturation in the control signal. The use of the membership variation model in the output-feedback design is a major innovation for discrete-time T-S fuzzy systems. Numerical examples demonstrate the effectiveness of the proposed approach.

4.2 Preliminaries

4.2.1 System description

Let us consider a discrete-time nonlinear system

$$\begin{aligned} x(t+1) &= f(x(t))x(t) + g(x(t))\text{sat}(u(t)) \\ y(t) &= h(x(t))x(t) \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, and $u(t) \in \mathbb{R}^{n_u}$ is the control input. The nonlinear functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n \times n_u}$, $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n_y \times n}$ are assumed to be bounded and smooth in a compact set $\mathcal{X}_0$ of the state-space containing the origin. The standard decentralized saturation function $sat(u_{(l)}) = sign(u_{(l)})min(|u_{(l)}|, \rho_{(l)})$, $l = 1, \dots, n_u$, describes the amplitude-bounded control input. We assume the origin is an equilibrium point of (4.1) for $u = 0$.

Using the sector nonlinearity approach (Tanaka and Wang, 2001), the nonlinear system (4.1) can be represented precisely in $\mathcal{X}_0$ through the T–S fuzzy system by a set of rules adapted from (2.4) to consider the presence of the saturation in the control input, given by

Rule ℓ: If $z_1(k)$ is $\mathscr{M}_1^{\alpha_{\ell 1}}$ and ... and $z_p(k)$ is $\mathscr{M}_p^{\alpha_{\ell p}}$, then

$$\begin{cases} x(t+1) = A_{\alpha_{\ell 1} \dots \alpha_{\ell p}} x(t) + B_{\alpha_{\ell 1} \dots \alpha_{\ell p}} \text{sat}(u(t)) \\ y(t) = C_{\alpha_{\ell 1} \dots \alpha_{\ell p}} x(t), \qquad \ell = 1, \dots, N. \end{cases} \tag{4.2}$$

The variation of the membership function is defined as[1]

$$\Delta\mu_j(z_j) \coloneqq \mu_j(z_j + 1) - \mu_j(z_j), \qquad j = 1, \dots, p$$

and

$$\Delta\mu(z) \coloneqq \begin{bmatrix} \Delta\mu_1(z_1) \\ \vdots \\ \Delta\mu_N(z_N) \end{bmatrix} = \mu(z+1) - \mu(z). \tag{4.3}$$

To handle the multi-dimensional fuzzy summations of matrices, the same notation presented in subsection 3.2.2 is adopted.

4.2.2 Problem statement

The global T-S fuzzy system obtained from (4.2) can be rewritten as

$$\begin{cases} x(t+1) = A_z x(t) + B_z \text{sat}(u(t)) \\ y(t) = C_z x(t) \end{cases} \tag{4.4}$$

with $(A, B, C)_z$ given by (2.11).

The domain of validity of the T-S fuzzy model (4.4) is given by the following polyhedral set

$$\mathcal{X}_0 = \{x \in \mathbb{R}^n : b_i' x \leq 1, \quad i = 1, \dots, 2q\} \subset \mathbb{R}^n \tag{4.5}$$

where $b_i \in \mathbb{R}^{n \times 1}$ are given vectors and q is the number of states that the premise variables depend on. Note that the linear constraints defining $\mathcal{X}_0$ guarantee $0 \in \mathcal{X}_0$. The set $\mathcal{X}_0$ can also be written as:

$$\mathcal{X}_0 = co\{h^1, h^2, \dots, h^{n_h}\}, \tag{4.6}$$

where the vectors h^i, $i = 1, \dots, n_h$, can be systematically obtained through the linear constraints in (4.5).

Consider that a subset $z_{\mathbf{q}} \in \mathbb{R}^s$, $s \leq p$, of the premise variables $z \in \mathbb{R}^p$ are measurable and available for the control law. The aim is to design a full-order dynamic output feedback controller given by

$$\begin{cases} x_c(t+1) = A_{cz_{\mathbf{q}}} x_c(t) + B_{cz_{\mathbf{q}}} y(t) \\ u(t) = C_{cz_{\mathbf{q}}} x_c(t) + D_{cz_{\mathbf{q}}} y(t), \end{cases} \tag{4.7}$$

[1] The time dependency in $\mu_j(z_j)$ is omitted for brevity, and then we adopt the notation $\mu_j(z_j + 1)$ to denote $\mu_j(z_j)$ in the instant $k + 1$.

with

$$(A_c, B_c, C_c, D_c)_{z_\mathbf{q}} = \sum_{\mathbf{i}\in\mathcal{I}_s} \mu_\mathbf{i}(z_\mathbf{q})(A_{c\mathbf{i}}, B_{c\mathbf{i}}, C_{c\mathbf{i}}, D_{c\mathbf{i}})$$

and $x_c(t) \in \mathbb{R}^n$ are the controller states. As matter of fact, it is required the measurement of the premise variables $z_\mathbf{q}$ for the implementation of the control law.

Let us define the decentralized deadzone nonlinearity $\psi(u) = u - sat(u)$. The closed-loop system can be represented as

$$x_a(t+1) = \mathbb{A}_z x_a(t) - \mathbb{B}_z \psi(u) \tag{4.8}$$

where $x_a(t) = \begin{bmatrix} x(t)' & x_c(t)' \end{bmatrix}'$ denotes the augmented state and

$$\mathbb{A}_z = \begin{bmatrix} A_z + B_z D_{cz_\mathbf{q}} C_z & B_z C_{cz_\mathbf{q}} \\ B_{cz_\mathbf{q}} C_z & A_{cz_\mathbf{q}} \end{bmatrix}, \qquad \mathbb{B}_z = \begin{bmatrix} B_z \\ 0 \end{bmatrix}.$$

The amplitude-limited control signal can be rewritten as $u(t) = K_z x_a(t)$, with

$$K_z = \begin{bmatrix} D_{cz_\mathbf{q}} C_z & C_{cz_\mathbf{q}} \end{bmatrix}. \tag{4.9}$$

The domain of validity of closed-loop system described by (4.8) is the subset

$$\mathcal{X} = \left\{ x_a \in \mathbb{R}^{2n} : \bar{b}_i' x_a \leq 1, \quad i = 1, \ldots, 2q \right\} \subset \mathbb{R}^{2n}, \tag{4.10}$$

with $\bar{b}_i' = \begin{bmatrix} b_i' & 0 \end{bmatrix} \in \mathbb{R}^{2n}$.

The following problem is addressed.

Problem 4.1 *Consider the T-S fuzzy system* (4.4) *with a subset* $z_\mathbf{q}$ *of the premise variables* z *that are measurable and available for the control law. Determine a dynamic output controller as* (4.7) *and a region* $\Omega_0 \subseteq \mathcal{X}_0$*, as large as possible, such that the trajectories of closed-loop system* (4.8) *starting from any initial condition* $(x(0), x_c(0)) \in \Omega_0 \times \{0\}$ *converge exponentially towards the origin.*

Remark 4.1 *Problem 4.1 assumes that the premise variables are partially measurable. The proposed technique can also be applied when all premise variables are available, recovering the standard case, or when none of them can be used, yielding a robust controller.*

Remark 4.2 *As a matter of simplicity, we will consider the dynamic controller* (4.7) *with zero initial conditions. Then, our interest is to maximize the estimation of the region of initial conditions* $x(0)$*, denoted by* $\Omega_0 \subseteq \mathbb{R}^n$*, such that the trajectories* $x_a(t)$ *converge to the origin. As we demonstrate later, we do not impose the estimate of the domain of attraction* Ω_0 *to be invariant.*

The following lemmas will be useful for the main results.

Lemma 4.1 (Gomes da Silva Jr. and Tarbouriech (2006)) *Consider a matrix* $\Upsilon_z \in \mathbb{R}^{n_u \times 2n}$ *and the region defined by*

$$\Pi = \left\{ x_a \in \mathbb{R}^{2n} : |(K_{z(\ell)} - \Upsilon_{z(\ell)})x_a| \leq \rho_{(\ell)},\ \ell = 1, \ldots, n_u \right\}, \tag{4.11}$$

where $\rho_{(\ell)}$ *determines the limit of the control effort, that is* $-\rho_{(\ell)} \leq u_{(\ell)} \leq \rho_{(\ell)}$. *If* $x \subseteq \Pi$, *then the following relation holds:*

$$\psi(u)'\Sigma^{-1}(\psi(u) - \Upsilon_z x_a) \leq 0 \tag{4.12}$$

for any matrix $\Sigma \in \mathbb{R}^{n_u \times n_u}$ *diagonal and positive definite.*

Lemma 4.2 (Mean value theorem (Buck, 1994)) *Let* $\mathcal{U} \in \mathbb{R}^p$ *be a convex set, and suppose* $f : \mathcal{U} \to \mathbb{R}$ *is continuously differentiable. Then, for any* $x,\ y \in \mathcal{U}$, *there is a real number* $c \in [0,\ 1]$ *such that*

$$f(y) - f(x) = \frac{\partial f((1-c)x + cy)}{\partial x}(y - x). \tag{4.13}$$

Lemma 4.3 *If there exists a symmetric positive definite matrix* $W_z \in \mathbb{R}^{2n \times 2n}$, *matrices* $G \in \mathbb{R}^{2n \times 2n}$ *and* $M_z \in \mathbb{R}^{2n \times n_u}$, *and a diagonal positive definite matrix* $\Sigma \in \mathbb{R}^{n_u \times n_u}$ *satisfying for all* $\mu \in \mathscr{U}_r$

$$\left[\begin{array}{cc|c} W_{z+} & \mathbb{A}_z G & \mathbb{B}_z \Sigma \\ \star & G + G' - W_z & -M_z \\ \hline \star & \star & 2\Sigma \end{array}\right] > 0, \tag{4.14}$$

$$\begin{bmatrix} G + G' - W_z & G'\bar{b}_i \\ \star & I \end{bmatrix} \geq 0, \quad i = 1, \ldots 2q, \tag{4.15}$$

$$\begin{bmatrix} G + G' - W_z & G'K'_{z(\ell)} - M_{z(\ell)} \\ \star & \rho^2_{(\ell)} \end{bmatrix} \geq 0, \quad \ell = 1, \ldots n_u, \tag{4.16}$$

then the origin of closed-loop system (4.8) *is locally exponentially stable and the trajectories for any initial conditions* $x_a(0)$ *belonging to the ellipsoid*

$$\Omega = \left\{ x_a \in \mathbb{R}^{2n} : x_a' W_z^{-1} x_a \leq 1 \right\}. \tag{4.17}$$

exponentially converge toward the origin.

Proof The difference of the Lyapunov function $V(x_a) = x_a(t)'W_z^{-1}x_a(t)$ along the solution of (4.4) is negative definite if $\Delta V(x_a) - 2\psi(u)'\Sigma^{-1}(\psi(u) - \Upsilon_z x_a) < 0$ due to (4.12) or, equivalently,

$$\begin{bmatrix} x_a \\ \psi(u) \end{bmatrix}' \left(\begin{bmatrix} W_z^{-1} & -\Upsilon_z'\Sigma^{-1} \\ \star & 2\Sigma^{-1} \end{bmatrix} - \begin{bmatrix} \mathbb{A}_z' \\ \mathbb{B}_z' \end{bmatrix} W_{z+}^{-1} \begin{bmatrix} \mathbb{A}_z & \mathbb{B}_z \end{bmatrix} \right) \begin{bmatrix} x_a \\ \psi(u) \end{bmatrix} > 0.$$

Applying the Schur complement in the above condition and pre- and post-multiplying the result by $\begin{bmatrix} 0 & 0 & I \\ G' & 0 & 0 \\ 0 & \Sigma & 0 \end{bmatrix}$, one has

$$\begin{bmatrix} W_{z+} & \mathbb{A}_z G & \mathbb{B}_z \Sigma \\ \star & G' W_z^{-1} G & -G' \Upsilon_z' \\ \star & \star & 2\Sigma \end{bmatrix} > 0.$$

Observe that

$$G' W_z^{-1} G > G + G' - W_z, \tag{4.18}$$

since

$$(G - W_z)' W_z^{-1} (G - W_z) \geq 0.$$

Considering (4.18) and defining $M_z = G' \Upsilon_z'$, one has that if (4.14) holds then there exists a positive scalar ϵ such that $\Delta V(x_a) \leq \epsilon V(x_a)$ and (4.8) is exponentially stable.

If (4.15) and (4.16) hold, then the same inequalities are verified replacing $G + G' - W_z$ by $G' W_z^{-1} G$. Post- and pre-multiplying the resulting conditions by $\text{diag}(G^{-1}, I)$, one obtains, respectively,

$$\begin{bmatrix} W_z^{-1} & \bar{b}_i \\ \star & I \end{bmatrix} \geq 0, \quad \begin{bmatrix} W_z^{-1} & K'_{z(\ell)} - \Upsilon_z' \\ \star & \rho_{(\ell)}^2 \end{bmatrix} \geq 0.$$

The first inequality assures $\Omega \subset \mathcal{X}$ (Boyd et al., 1994) and the second $\Omega \subset \Pi$ (Gomes da Silva Jr. and Tarbouriech, 2006). ■

The results presented in the next section provide sufficient conditions for the design of full order dynamic output feedback controllers depending only on the available premise variables $z_{\mathbf{q}}$.

4.3 Main Results

4.3.1 Stabilizability of the T-S system without saturation

Before presenting stability conditions for the closed-loop (4.8), we first consider the design of the DOF controller (4.7) for system (4.4) without the saturation in the control signal, that is, $x(t+1) = A_z x(t) + B_z u(t)$, $y(t) = C_z x(t)$. We focus on the design of controllers that do not depend on all premise variables.

Using (2.33) with $U = Y$, $Y' = Z$ and $V' = -Z$, we can define the following matrices inspired by the parametrization adopted in de Oliveira et al. (2002):

$$G = \begin{bmatrix} X & \blacksquare \\ Y & \blacksquare \end{bmatrix}, \; G^{-1} = \begin{bmatrix} Z & \blacksquare \\ -Z & \blacksquare \end{bmatrix}, \; \mathscr{X} = \begin{bmatrix} I_n & Z \\ 0 & -Z \end{bmatrix} \tag{4.19}$$

where $X, Z \in \mathbb{R}^{n \times n}$ and the elements represented by $\blacksquare$ are such that $GG^{-1} = I$, which yields

$G\mathscr{X} = \begin{bmatrix} X & I \\ Y & 0 \end{bmatrix}$. Then, applying a congruence transformation in the inequality formed by the first two block rows and columns in (4.14), one has

$$\begin{bmatrix} \mathscr{X} & 0 \\ 0 & \mathscr{X} \end{bmatrix}' \left[\begin{array}{c|c} W_{z+} & \mathbb{A}_z G \\ \hline G'\mathbb{A}_z' & G + G' - W_z \end{array}\right] \begin{bmatrix} \mathscr{X} & 0 \\ 0 & \mathscr{X} \end{bmatrix} =$$
$$\left[\begin{array}{cc|cc} P_{z+} & J_{z+} & A_z X + B_z \bar{\mathscr{C}}_z & A_z + B_z D_{cz_\mathbf{q}} C_z \\ \star & H_{z+} & \bar{\mathscr{A}}_z & Z' A_z + \bar{\mathscr{B}}_z C_z \\ \hline \star & \star & X + X' - P_z & I_n + S' - J_z \\ \star & \star & \star & Z + Z' - H_z \end{array}\right] > 0 \quad (4.20)$$

where matrices P_z, J_z, H_z and S are defined by

$$\begin{bmatrix} P_z & J_z \\ \star & H_z \end{bmatrix} := \mathscr{X}' W_z \mathscr{X}, \qquad S := Z'(X - Y) \quad (4.21)$$

and

$$\begin{aligned} \bar{\mathscr{A}}_z &= Z' A_z X - Z' A_{cz_\mathbf{q}} Y - Z' B_{cz_\mathbf{q}} C_z X + Z' B_z C_{cz_\mathbf{q}} Y + Z' B_z D_{cz_\mathbf{q}} C_z X \\ \bar{\mathscr{B}}_z &= Z' B_z D_{cz_\mathbf{q}} - Z' B_{cz_\mathbf{q}}, \\ \bar{\mathscr{C}}_z &= C_{cz_\mathbf{q}} Y + D_{cz_\mathbf{q}} C_z X. \end{aligned} \quad (4.22)$$

Observe that expressions to recover $A_{cz_\mathbf{q}}$, $B_{cz_\mathbf{q}}$ and $C_{cz_\mathbf{q}}$ from (4.22) will depend on A_z, B_z and C_z and, therefore, the controller gains cannot be implemented since z is not available for feedback. As in Chapter 3, (3.16) is used to allow the design of a controller depending only on the available premise variables $z_\mathbf{q}$.

Then, one can replace A_z, B_z and C_z in (4.22) by $\hat{A}_{z_\mathbf{q}} + \Gamma_A$, $\hat{B}_{z_\mathbf{q}} + \Gamma_B$ and $\hat{C}_{z_\mathbf{q}} + \Gamma_C$, respectively, and rewrite (4.20) as

$$\Psi_0 + \Psi_\Gamma > 0 \quad (4.23)$$

where

$$\Psi_0 := \begin{bmatrix} P_{z+} & J_{z+} & A_z X + B_z \mathscr{C}_{z_\mathbf{q}} & A_z + B_z D_{cz_\mathbf{q}} C_z \\ \star & H_{z+} & \mathscr{A}_{z_\mathbf{q}} & Z' A_z + \mathscr{B}_{z_\mathbf{q}} C_z \\ \star & \star & X + X' - P_z & I_n + S' - J_z \\ \star & \star & \star & Z + Z' - H_z \end{bmatrix}, \quad \Psi_\Gamma := \begin{bmatrix} 0 & \Phi_\Gamma + \Delta \\ \star & 0 \end{bmatrix}, \quad (4.24)$$

with

$$\begin{aligned} \mathscr{A}_{z_\mathbf{q}} &= Z' \hat{A}_{z_\mathbf{q}} X - Z' A_{cz_\mathbf{q}} Y - Z' B_{cz_\mathbf{q}} \hat{C}_{z_\mathbf{q}} X + Z' \hat{B}_{z_\mathbf{q}} C_{cz_\mathbf{q}} Y + Z' \hat{B}_{z_\mathbf{q}} D_{cz_\mathbf{q}} \hat{C}_{z_\mathbf{q}} X, \\ \mathscr{B}_{z_\mathbf{q}} &= Z' \hat{B}_{z_\mathbf{q}} D_{cz_\mathbf{q}} - Z' B_{cz_\mathbf{q}}, \\ \mathscr{C}_{z_\mathbf{q}} &= C_{cz_\mathbf{q}} Y + D_{cz_\mathbf{q}} \hat{C}_{z_\mathbf{q}} X, \end{aligned} \quad (4.25)$$

and

$$\Phi_\Gamma := \begin{bmatrix} 0 & B_z D_{cz_\mathbf{q}} \\ Z' & \mathscr{B}_{z_\mathbf{q}} \end{bmatrix} \Gamma \begin{bmatrix} X & 0 \\ \mathscr{C}_{z_\mathbf{q}} & D_{cz_\mathbf{q}} C_z \end{bmatrix}, \quad \Gamma := \begin{bmatrix} \Gamma_A & \Gamma_B \\ \Gamma_C & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & 0 \\ Z\Gamma_B D_{cz_\mathbf{q}} \Gamma_C X & 0 \end{bmatrix}.$$

Assumption 4.1 *We assume that either Γ_B or Γ_C are zero or, in the case $\Gamma_B \neq 0$ and $\Gamma_C \neq 0$, we impose $D_{cz_\mathbf{q}} = 0$. As consequence, one has $\Delta = 0$.*

Observe that the change of variables (4.25) allows to recover the controller gains depending only on $z_\mathbf{q}$:

$$\begin{aligned} A_{cz_\mathbf{q}} &= (Z')^{-1} \left\{ Z'\hat{A}_{z_\mathbf{q}} X + Z'\hat{B}_{z_\mathbf{q}} \mathscr{C}_{z_\mathbf{q}} - \mathscr{A}_{z_\mathbf{q}} + \mathscr{B}_{z_\mathbf{q}} \hat{C}_{z_\mathbf{q}} X \right\} Y^{-1}, \\ B_{cz_\mathbf{q}} &= -(Z')^{-1} \mathscr{B}_{z_\mathbf{q}} + \hat{B}_{z_\mathbf{q}} D_{cz_\mathbf{q}}, \\ C_{cz_\mathbf{q}} &= (\mathscr{C}_{z_\mathbf{q}} - D_{cz_\mathbf{q}} \hat{C}_{z_\mathbf{q}} X) Y^{-1}. \end{aligned} \tag{4.26}$$

Moreover, note that the inequality (4.23) is nonlinear due to the product of variables in term Φ_Γ. The following theorem proposes sufficient LMI conditions for (4.23).

Theorem 4.1 *For a given scalar $\lambda \in \mathbb{R}_{>0}$, if there exist matrices X, Z, S, $J_z \in \mathbb{R}^{n\times n}$, $\mathscr{A}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n}$, $\mathscr{B}_{z_\mathbf{q}} \in \mathbb{R}^{n\times n_y}$, $\mathscr{C}_{z_\mathbf{q}} \in \mathbb{R}^{n_u\times n}$ and $D_{cz_\mathbf{q}} \in \mathbb{R}^{n_u\times n_y}$, symmetric positive definite matrices P_z and $H_z \in \mathbb{R}^{n\times n}$, such that, for all $\mu \in \mathscr{U}_r$, the following LMI is satisfied:*

$$\begin{bmatrix} \Psi_0 & \Psi_1\Gamma - \lambda\Psi_2 \\ \star & 2\lambda I \end{bmatrix} > 0 \tag{4.27}$$

where

$$\Psi_1 = \begin{bmatrix} 0 & B_z D_{cz_\mathbf{q}} \\ Z' & \mathscr{B}_{z_\mathbf{q}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Psi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ X' & \mathscr{C}'_{z_\mathbf{q}} \\ 0 & C'_z D'_{cz_\mathbf{q}} \end{bmatrix} \tag{4.28}$$

and Ψ_0 given by (4.24). Then, the controller (4.7), whose state-space matrices are given by (4.26) with $Y = X - (Z')^{-1}S$, makes closed-loop system $x_a(t+1) = \mathbb{A}_z x_a(t) - \mathbb{B}_z u(t)$ asymptotically stable.

Proof

First, note that inequality (4.27) can be written as condition (4) of Lemma 2.3 with $\mathcal{X} = \lambda[0 \;\; I]'$,

$$\mathcal{D} = \begin{bmatrix} \Psi_0 & \Psi_1\Gamma \\ \star & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} -\Psi'_2 & I \end{bmatrix}.$$

Then, if (4.27) holds, condition (2) of Lemma 2.3 is verified with $\mathcal{B}'_\perp = \begin{bmatrix} I & \Psi'_2 \end{bmatrix}$ and it can be written as (4.23). If $\mathscr{Z}$ is non-singular, then (4.23) and the left-hand side of (4.20) are equivalent. By Lemma 4.3, considering only the inequality formed by the first two block rows and columns of (4.14), the closed-loop system $x_a(t+1) = \mathbb{A}_z x_a(t) - \mathbb{B}_z u(t)$ is asymptotically stable.

To assure $\mathscr{X}$ full rank and (4.26) well defined, one requires Z and Y non-singular. If (4.27) holds,

$$\begin{bmatrix} X+X' & I_n+S' \\ \star & Z+Z' \end{bmatrix} > \begin{bmatrix} P_z & J_z \\ \star & H_z \end{bmatrix} > 0 \tag{4.29}$$

which implies that X and Z are non-singular and, from (4.21), $W_z > 0$. If we multiply (4.29) by $T' = [I \quad -(Z')^{-1}]$ on the left and by T on the right one obtains $He\{(Z'X - S)'Z^{-1}\} > 0$. From the definition of S in (4.21), one has $Z'X - S = Z'Y$ thus $Y + Y' > 0$, implying Y non-singular. ■

Consider the particular case $\Gamma_A \neq 0$, $\Gamma_B = 0$ and $\Gamma_C = 0$. Then, one has the following corollary.

Corollary 4.1 *For a given scalar $\lambda \in \mathbb{R}_{>0}$, if there exist matrices X, Z, S, $J_z \in \mathbb{R}^{n\times n}$, $\mathscr{A}_{z_{\mathbf{q}}} \in \mathbb{R}^{n\times n}$, $\mathscr{B}_{z_{\mathbf{q}}} \in \mathbb{R}^{n\times n_y}$, $\mathscr{C}_{z_{\mathbf{q}}} \in \mathbb{R}^{n_u\times n}$ and $D_{cz_{\mathbf{q}}} \in \mathbb{R}^{n_u\times n_y}$, symmetric positive definite matrices P_z and $H_z \in \mathbb{R}^{n\times n}$, such that, for all $\mu \in \mathscr{U}_r$ (4.27) holds with*

$$\Psi_1 = \begin{bmatrix} 0 & \Gamma_A' Z & 0 & 0 \end{bmatrix}', \qquad \Psi_2 = \begin{bmatrix} 0 & 0 & X & 0 \end{bmatrix}',$$

then, the controller (4.7), whose state-space matrices are given by (4.26) with $Y = X - (Z')^{-1}S$, asymptotically stabilizes the closed-loop system $x_a(t+1) = \mathbb{A}_z x_a(t) - \mathbb{B}_z u(t)$

4.3.2 Saturation and Local stabilization

In this section, we present a solution for Problem 4.1. We consider the following assumption.

Assumption 4.2 *The output matrix C_z of the T-S fuzzy model (4.4) depends on measurable premise variables, that is, $\hat{C}_{z_{\mathbf{q}}} = C_z$ ($\Gamma_C = 0$).*

First, we take into account in the design conditions some property of the variation of the membership function $\Delta\mu_{\mathbf{i}}(z)$. From the definition of $\mu_{\mathbf{i}}(z)$, one has $\sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) = 1$. Then, one has the following hyperplane where $\Delta\mu_{\mathbf{i}}(z)$ is contained:

$$\sum_{\mathbf{i}\in\mathcal{I}_p} \Delta\mu_{\mathbf{i}}(z) = \mathbb{1}'\Delta\mu(z) = 0.$$

Any summation Q_z in the instant $k+1$, denoted by Q_{z+}, can be described by

$$Q_{z+} = \sum_{\mathbf{i}\in\mathcal{I}_p} (\mu_{\mathbf{i}}(z) + \Delta\mu_{\mathbf{i}}(z))Q_{\mathbf{i}} = Q_z + Q_{\Delta z},$$

where

$$Q_{\Delta z} = \sum_{\mathbf{i}\in\mathcal{I}_p} \Delta\mu_{\mathbf{i}}(z)Q_{\mathbf{i}}.$$

The term $Q_{\Delta z}$ can be rewritten as

$$Q_{\Delta z} = \tilde{Q}(\Delta\mu(z) \otimes I_n), \quad \tilde{Q} = \begin{bmatrix} Q_1 & \cdots & Q_N \end{bmatrix}. \tag{4.30}$$

If Q_z is a symmetric matrix, we can also write $Q_{\Delta z} = 0.5\text{He}\left\{\tilde{Q}(\Delta\mu(z) \otimes I_n)\right\}$.

Applying decomposition (4.30) in the first two block rows and columns of Ψ_0, defined in (4.24), one has

$$\begin{bmatrix} P_{z+} & J_{z+} \\ \star & H_{z+} \end{bmatrix} = \begin{bmatrix} P_z & J_z \\ \star & H_z \end{bmatrix} + \begin{bmatrix} P_{\Delta z} & J_{\Delta z} \\ \star & H_{\Delta z} \end{bmatrix}, \tag{4.31}$$

where

$$\begin{bmatrix} P_{\Delta z} & J_{\Delta z} \\ \star & H_{\Delta z} \end{bmatrix} = \text{He}\left\{\begin{bmatrix} 0.5\tilde{P} & \tilde{J} \\ \star & 0.5\tilde{H} \end{bmatrix} \begin{bmatrix} \Delta\mu(z) \otimes I_n & 0 \\ 0 & \Delta\mu(z) \otimes I_n \end{bmatrix}\right\}$$

with $\tilde{P}$, $\tilde{J}$, and $\tilde{H}$ described as in (4.30).

Note that, since the premise variables depend on the states, by Lemma 4.2, there exist real numbers $c_i \in [0,\ 1]$, $i \in \mathcal{I}_p$, such that, for each instant of time k, the variation of the membership function can be describe as:

$$\Delta\mu_{\mathbf{i}}(z) = \mu_{\mathbf{i}}(z+1) - \mu_{\mathbf{i}}(z) = \frac{\partial\mu_{\mathbf{i}}((1-c_{\mathbf{i}})z(t+1) + c_{\mathbf{i}}z(t))}{\partial x(t)}(x(t+1) - x(t)), \tag{4.32}$$

$\forall (x(t), x(t+1), \mathbf{i}) \in \mathcal{X}_0 \times \mathcal{X}_0 \times \mathcal{I}_p$.

The condition $x(t+1) \in \mathcal{X}_0$ can be reformulated as $x(t) \in \mathcal{R} \subset \mathbb{R}^n$, where

$$\mathcal{R} = \left\{x \in \mathbb{R}^n : (A_z + B_z D_{cz_{\mathbf{q}}} C_z)x + B_z C_{cz_{\mathbf{q}}} x_c - B_z \psi(u) \in \mathcal{X}_0, \ \ \forall x_c \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^{n_u}\right\}. \tag{4.33}$$

The set of all initial solutions $x_a(0)$ such that the trajectories $x_a(t)$ remain in the set $(\mathcal{X}_0 \cap \mathcal{R}) \times \mathbb{R}^n$ can be estimated as the level set of the Lyapunov function Ω given in (4.17).

The term $\partial\mu_{\mathbf{i}}(z(t))/\partial x(t)$ can be described in a polytope:

$$\nabla_x \mu(z) = J_\theta \coloneqq \sum_{i=1}^{\vartheta} \theta_i(x) J_i \tag{4.34}$$

where $\theta(x) \in \mathcal{U}_\vartheta$, and J_i are vertices obtained using the sector nonlinearity approach (Tanaka and Wang, 2001) over

$$\left[\nabla_x \mu(z)\right]_{ij} = \frac{\partial\mu_{\mathbf{i}}}{\partial x_j(t)}$$

for $\mu_{\mathbf{i}}(z)$ defined in set $\mathcal{X}_0$.

Then, using (4.4) and (4.32) in (4.34), $\Delta\mu(z)$ can be rewritten as

$$\Delta\mu(z) = \nabla_x \mu(z)(A_z x(t) + B_z \text{sat}(u(t)) - x(t)) = J_\theta((A_z - I)x(t) + B_z \text{sat}(u(t))). \tag{4.35}$$

Let $v = \text{sat}(u)$, then, from the definition of the saturation function, one has $v \in \mathcal{V}$ where

$\mathcal{V} = \{v \in \mathbb{R}^{n_u} : |v_{(\ell)}| \leq \rho_{(\ell)}, \ell = 1, \ldots, n_u\}$. Applying (4.35) in the representation of $P_{\Delta z}$, $J_{\Delta z}$, and $H_{\Delta z}$, for all $x \in \mathcal{X}_0$ and $v \in \mathcal{U}$, in (4.31), one has

$$\begin{bmatrix} P_{z+} & J_{z+} \\ \star & H_{z+} \end{bmatrix} = \begin{bmatrix} P_z & J_z \\ \star & H_z \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} 0.5\tilde{P} & \tilde{J} \\ 0 & 0.5\tilde{H} \end{bmatrix} (I_2 \otimes ((J_\theta((A_z - I)x_\gamma + B_z v_\delta)) \otimes I_n)) \right\} \tag{4.36}$$

where

$$x_\gamma = \sum_{i=1}^{n_h} \gamma_i h^i, \quad \gamma \in \mathcal{U}_{n_h}, \qquad v_\delta = \sum_{i=1}^{2^{n_u}} \delta_i \nu^i, \quad \delta \in \mathcal{U}_{2^{n_u}} \tag{4.37}$$

with vertices h^i given in (4.6) and

$$\nu^i = D_i \rho, \quad i = 1, \ldots, 2^{n_u}, \quad \rho = \begin{bmatrix} \rho_{(1)} & \cdots & \rho_{(n_u)} \end{bmatrix}'$$

where D_i, $i = 1, \ldots, 2^{n_u}$, are diagonal matrices in $\mathbb{R}^{n_u \times n_u}$ constituted from all the combinations formed with 1 and -1.

Theorem 4.2 *For a given scalar $\lambda \in \mathbb{R}_{>0}$, if there exist symmetric positive definite matrices P_z and $H_z \in \mathbb{R}^{n \times n}$, a diagonal positive definite matrix $\Sigma \in \mathbb{R}^{n_u \times n_u}$, and matrices X, Z, S, $J_z \in \mathbb{R}^{n \times n}$, $M_{1z} \in \mathbb{R}^{n \times n_u}$, $M_{2z} \in \mathbb{R}^{n \times n_u}$, $\mathcal{A}_{z_\mathbf{q}} \in \mathbb{R}^{n \times n}$, $\mathcal{B}_{z_\mathbf{q}} \in \mathbb{R}^{n \times n_y}$, $\mathcal{C}_{z_\mathbf{q}} \in \mathbb{R}^{n_u \times n}$, and $D_{cz_\mathbf{q}} \in \mathbb{R}^{n_u \times n_y}$, such that, for all $\mu \in \mathcal{U}_r$ the following LMIs are satisfied:*

$$\begin{bmatrix} \hat{\Psi}_{k\kappa j} & \hat{\Psi}_1 \Gamma - \lambda \hat{\Psi}_2 \\ \star & 2\lambda I \end{bmatrix} > 0, \quad k = 1, \ldots, n_h, \ \kappa = 1, \ldots, 2^{n_u}, \ j = 1, \ldots, \vartheta, \tag{4.38}$$

$$\begin{bmatrix} X + X' - P_z & I_n + S' - J_z & X' b_i \\ \star & Z + Z' - H_z & b_i \\ \star & \star & 1 \end{bmatrix} \geq 0, \quad i = 1, \ldots, 2q, \tag{4.39}$$

$$\begin{bmatrix} X + X' - P_z & I_n + S' - J_z & -0.5M_{1z(\ell)} & (A_z X + B_z \mathcal{C}_{z_\mathbf{q}})' b_i \\ \star & Z + Z' - H_z & -0.5M_{2z(\ell)} & (A_z + B_z D_{cz_\mathbf{q}} C_z)' b_i \\ \star & \star & \Sigma & -\Sigma B_z' b_i \\ \star & \star & \star & 1 \end{bmatrix} \geq 0, \quad i = 1, \ldots, 2q, \tag{4.40}$$

$$\begin{bmatrix} X + X' - P_z & I_n + S' - J_z & \mathcal{C}'_{z_\mathbf{q}(\ell)} - M_{1z(\ell)} \\ \star & Z + Z' - H_z & C_z' D'_{cz_\mathbf{q}(\ell)} - M_{2z(\ell)} \\ \star & \star & \rho^2_{(\ell)} \end{bmatrix} \geq 0, \quad \ell = 1, \ldots, n_u, \tag{4.41}$$

where

$$\hat{\Psi}_{k\kappa j} = \overline{\Psi}_0 + He\left\{\hat{\Psi}_3\overline{\Psi}_{k\kappa j}\right\}, \quad \overline{\Psi}_{k\kappa j} = \left[\begin{array}{c|c} I_2 \otimes \Big((J_j((A_z - I)h^k + B_z\nu^\kappa)) \otimes I_n\Big) & 0 \end{array}\right],$$

$$\overline{\Psi}_0 = \begin{bmatrix} P_z & J_z & A_zX + B_z\mathscr{C}_{z_\mathbf{q}} & A_z + B_zD_{cz_\mathbf{q}}C_z & B_z\Sigma \\ \star & H_z & \mathscr{A}_{z_\mathbf{q}} & Z'A_z + \mathscr{B}_{z_\mathbf{q}}C_z & 0 \\ \star & \star & X + X' - P_z & I_n + S' - J_z & M_{1z} \\ \star & \star & \star & Z + Z' - H_z & M_{2z} \\ \star & \star & \star & \star & 2\Sigma \end{bmatrix},$$

$$\hat{\Psi}_j = \left[\begin{array}{c} \Psi_j \\ \hline 0 \end{array}\right], \quad j = 1, 2, 3, \quad \Psi_3 = \frac{1}{2}\begin{bmatrix} \tilde{P} & 0 & 0 & 0 & 0 \\ 2\tilde{J}' & \tilde{H} & 0 & 0 & 0 \end{bmatrix}',$$

Ψ_1 and Ψ_2 defined in (4.28)*. Then the controller* (4.7)*, whose state-space matrices are given by* (4.26) *with $Y = X-(Z')^{-1}S$, makes the origin of the closed-loop system (4.8) locally asymptotically stable and the trajectories for any initial conditions $x_a(0) \in \Omega$ exponentially converge toward the origin. Moreover, when imposing a zero initial condition to the DOF controller* (4.7)*, the set $\Omega_0 = \{x \in \mathbb{R}^n \ : \ (x, 0) \in \Omega\} \subset \mathbb{R}^n$, that can be rewritten as*

$$\Omega_0 = \{x \in \mathbb{R}^n : \ x' \begin{bmatrix} I_n & 0_n \end{bmatrix} W_z^{-1} \begin{bmatrix} I_n & 0_n \end{bmatrix}' x \leq 1\}, \tag{4.42}$$

is an estimate of the basin of attraction of the origin related to the system (4.4)*, that is, $\forall x_a(0) \in \Omega_0 \times \{0\}$, $\lim_{k\to\infty} x_a(t) = 0$.*

Proof

First, observe that according to (4.36), one has

$$\mathcal{M} := \sum_{k=1}^{n_h}\sum_{\kappa=1}^{2^{nu}}\sum_{j=1}^{\vartheta} \gamma_k\delta_\kappa\theta_j\hat{\Psi}_{k\kappa j} = \overline{\Psi}_0 +$$

$$\text{He}\left\{\left[\begin{array}{cc} 0.5\tilde{P} & \tilde{J} \\ 0 & 0.5\tilde{H} \\ \hline \multicolumn{2}{c}{0} \end{array}\right]\left[\begin{array}{c|c} (I_2 \otimes ((J_\theta((A_z - I)x_\gamma + B_z v_\delta)) \otimes I_n)) & 0 \end{array}\right]\right\} = \left[\begin{array}{c|c} \Psi_0 & \begin{array}{c} \mathbb{B}_z\Sigma \\ -M_z \end{array} \\ \hline \star & 2\Sigma \end{array}\right],$$

with Ψ_0 given by (4.24). Therefore, inequality (4.38) is equivalent to condition (4) of Lemma 2.3 with

$$\mathcal{D} = \left[\begin{array}{c|c} \mathcal{M} & \begin{array}{c} \Psi_1\Gamma \\ 0 \end{array} \\ \hline \star & 0 \end{array}\right], \quad \mathcal{B} = \left[\begin{array}{cc|c} -\Psi_2 & 0 & I \end{array}\right], \quad \mathcal{X} = \lambda\left[\begin{array}{c} 0 \\ 0 \\ \hline I \end{array}\right].$$

Thus, we can write condition (2) of Lemma 2.3 with

$$\mathcal{B}^\perp = \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \Psi_2 & 0 \end{array}\right]$$

yielding

$$\left[\begin{array}{c|c} \Psi_0 + \Psi_\Gamma & \begin{matrix}\mathbb{B}_z\Sigma \\ -M_z\end{matrix} \\ \hline \star & 2\Sigma \end{array}\right] > 0, \quad \Psi_\Gamma = \mathrm{He}\left\{\Psi_1\Gamma\Psi_2'\right\}.$$

By observing the equivalence between (4.23) and (4.20), one obtains (4.14).

Applying congruence transformation as in (4.20), condition (4.39) is equivalent to

$$\begin{bmatrix} \mathscr{X}'(G+G'-W_z)\mathscr{X} & \mathscr{X}'G'\bar{b}_i \\ \star & 1 \end{bmatrix} \geq 0, \tag{4.43}$$

with $\mathscr{X}$ and G defined in (4.19). Then, pre- and post-multiplying (4.43) by $\mathrm{diag}((\mathscr{X}')^{-1}, I)$ and its transpose, respectively, and considering the relation (4.18), we get that (4.43) implies

$$\begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} W_z^{-1} & \bar{b}_i \\ \star & I \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \geq 0.$$

or, equivalently, $\bar{b}_i\bar{b}_i' \leq W_z^{-1}$. Thus, (4.39) implies $\Omega \subset \mathcal{X}$ and, consequently, $\Omega_0 \subset \mathcal{X}_0$.

Similarly, using (4.18), (4.40) implies

$$\begin{bmatrix} \mathscr{X}'(G'W_z^{-1}G)\mathscr{X} & -0.5\mathscr{X}'G'\Upsilon_z' & \mathscr{X}'G'(\hat{A}_z+B_zK_z)'b_i \\ \star & \Sigma & -\Sigma B_z' b_i \\ \star & \star & I \end{bmatrix} \geq 0, \tag{4.44}$$

with K_z given by (4.9) and the following definitions adopted

$$\hat{A}_z = \begin{bmatrix} A_z & 0 \end{bmatrix}, \quad \begin{bmatrix} M_{1z} \\ M_{2z} \end{bmatrix} = \mathscr{X}'G'\Upsilon_z'. \tag{4.45}$$

To obtain (4.44), observe that we used the definition of the product $G\mathscr{X}$ from (4.19), $\mathscr{C}_{z_\mathbf{q}}$ in (4.25) and Assumption 4.2 ($\hat{C}_{z_\mathbf{q}} = C_z$) yielding the relations

$$\hat{A}_z G\mathscr{X} = \begin{bmatrix} A_z X & A_z \end{bmatrix}, \quad K_z G\mathscr{X} = \begin{bmatrix} \mathscr{C}_{z_\mathbf{q}} & D_{cz_\mathbf{q}} C_z \end{bmatrix}. \tag{4.46}$$

By pre- and post-multiplying (4.44) by $\mathrm{diag}((\mathscr{X}'G')^{-1}, \Sigma^{-1})$ and its transpose, respectively, one has

$$\begin{bmatrix} W_z^{-1} & -0.5\Upsilon_z'\Sigma^{-1} & (\hat{A}_z+B_zK_z)'b_i \\ \star & \Sigma^{-1} & -B_z'b_i \\ \star & \star & I \end{bmatrix} \geq 0. \tag{4.47}$$

Now, using Schur's complement and relation (4.12), (4.47) can be written as

$$\begin{bmatrix} x_a \\ \psi(u) \end{bmatrix}' \begin{bmatrix} \hat{A}_z + B_z K_z & -B_z \end{bmatrix}' b_i b_i' \begin{bmatrix} \hat{A}_z + B_z K_z & -B_z \end{bmatrix} \begin{bmatrix} x_a \\ \psi(u) \end{bmatrix}$$
$$\leq \begin{bmatrix} x_a \\ \psi(u) \end{bmatrix}' \begin{bmatrix} W_z^{-1} & -0.5\Upsilon_z'\Sigma^{-1} \\ \star & \Sigma^{-1} \end{bmatrix} \begin{bmatrix} x_a \\ \psi(u) \end{bmatrix} = x_a' W_z^{-1} x_a + \psi(u)'\Sigma^{-1}(\psi(u) - \Upsilon_z x_a) < 1,$$

that is, $b_i' \begin{bmatrix} \hat{A}_z + B_z K_z & -B_z \end{bmatrix} \begin{bmatrix} x_a' & \psi(u)' \end{bmatrix}' \leq 1$, $i = 1, \ldots, 2q$, which guarantees $\Omega \subset \mathcal{R} \times \mathbb{R}^n$. We can also verify this relation by observing that the set $\mathcal{R}$ in (4.33) can be rewritten as

$$\mathcal{R} = \left\{ x(t) : b_i' x(k+1) \leq 1, \quad i = 1, \ldots, 2q \right\},$$

with

$$x(k+1) = A_z x(t) + B_z \text{sat}(u(t)) = (\hat{A}_z + B_z K_z) x_a(t) - B_z \psi(u).$$

Therefore, $b_i' x(k+1) \leq 1$ is equivalent to

$$b_i' \begin{bmatrix} \hat{A}_z + B_z K_z & -B_z \end{bmatrix} \begin{bmatrix} x_a \\ \psi(u) \end{bmatrix} \leq 1, \quad i = 1, \ldots, 2q.$$

From (4.18), (4.45) and (4.46), if the LMI (4.41) holds, then

$$\begin{bmatrix} \mathscr{X}'(G' W_z^{-1} G)\mathscr{X} & \mathscr{X}' G' (K_{z(\ell)}' - \Upsilon_z') \\ \star & \rho_{(\ell)}^2 \end{bmatrix} \geq 0.$$

If we pre- and post-multiply the above inequality by $\text{diag}((\mathscr{X}'G')^{-1}, I)$ and its transpose, respectively, one has

$$\begin{bmatrix} W_z^{-1} & K_{z(\ell)}' - \Upsilon_z' \\ \star & \rho_{(\ell)}^2 \end{bmatrix} \geq 0,$$

which guarantees $\Omega \subset \Pi$ (Gomes da Silva Jr. and Tarbouriech, 2006). ∎

Remark 4.3 *The maximization of the region of attraction Ω is obtained by solving the following optimization problem:*

$$\min \quad \mathit{Trace}(\Xi) \tag{4.48}$$

subjected to relations of Theorem 4.2 and

$$\left[\begin{array}{c|cc} \Xi & \multicolumn{2}{c}{\mathscr{X}} \\ \hline \star & P_z & J_z \\ & \star & H_z \end{array}\right] \geq 0, \quad \Xi = \Xi' \in \mathbb{R}^{2n \times 2n}. \tag{4.49}$$

Observe that (4.49) *is equivalent to*

$$\begin{bmatrix} I & 0 \\ 0 & \mathscr{X} \end{bmatrix}' \begin{bmatrix} \Xi & I \\ \star & W_z \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathscr{X} \end{bmatrix} \geq 0$$

that is, $\Xi \geq W_z^{-1}$*, thus* Ξ *is an upper bound for* W_z^{-1}*.*

Remark 4.4 *The Lyapunov matrix* W_z *can be recovered from Theorem 4.2 by the relation presented in* (4.21)*, that is,* $W_z = \sum_{\mathbf{i}\in\mathcal{I}_p} \mu_{\mathbf{i}}(z) W_{\mathbf{i}}$*, with*

$$W_{\mathbf{i}} = \mathscr{X}^{-T} \begin{bmatrix} P_{\mathbf{i}} & J_{\mathbf{i}} \\ \star & H_{\mathbf{i}} \end{bmatrix} \mathscr{X}^{-1} = \begin{bmatrix} P_{\mathbf{i}} & P_{\mathbf{i}} - J_{\mathbf{i}} Z \\ \star & P_{\mathbf{i}} + Z^{-T} H_{\mathbf{i}} Z^{-1} - He\{J_{\mathbf{i}} Z^{-1}\} \end{bmatrix}.$$

Remark 4.5 *It should be emphasized in particular that* Ω_0 *is not positively invariant. It is nevertheless interesting to notice that* Ω_0 *and* Ω *are both obtained thanks to the same Lyapunov function* V*.*

Remark 4.6 *The set* Ω_0 *contains the intersection of ellipsoidal sets*[2]*,*

$$\Omega_0 \supseteq \bigcap_{\mathbf{i}\in\mathcal{I}_p} \Omega_{0\mathbf{i}}, \quad \Omega_{0\mathbf{i}} = \{x \in \mathbb{R}^n : \; x' \begin{bmatrix} I_n & 0_n \end{bmatrix} W_{\mathbf{i}}^{-1} \begin{bmatrix} I_n & 0_n \end{bmatrix}' x \leq 1\},$$

however, since $\mu_{\mathbf{i}}(z)$ *depend on the states, this approach may yield a conservative region. As alternative, a fine grid on* $x \in \mathcal{X} \cap \mathcal{R}$ *can be performed to verify all points that satisfy* $x \in \Omega_0$*.*

Remark 4.7 *Condition* (4.39) *can be incorporated in Theorem 4.1 to impose* $\Omega \subset \mathcal{X}$*.*

Remark 4.8 *Dynamic robust controllers (constant gains) can be obtained quite straightforwardly by constructing* $\hat{A}_{z_\mathbf{q}}$*,* $\hat{B}_{z_\mathbf{q}}$ *and* $\hat{C}_{z_\mathbf{q}}$ *as constant matrices. This can be done, for instance, using the average of the vertices given by* $[\hat{A}_{z_\mathbf{q}} \; \hat{B}_{z_\mathbf{q}} \; \hat{C}_{z_\mathbf{q}}] = 1/N \sum_{\mathbf{i}\in\mathcal{I}_p} [A_{\mathbf{i}} \; B_{\mathbf{i}} \; C_{\mathbf{i}}]$*.*

Remark 4.9 *The set* Ω *is* $\gamma-$*contractive for* $\gamma \in (0, 1]$ *with respect to the trajectories of system* (4.8) *if (Blanchini and Miani, 2007)*

$$\Delta V(x_a(t)) = V(x_a(t)) - \gamma V(x_a(t)) < 0 \quad \forall x_a(t) \in \Omega. \tag{4.50}$$

Therefore, Theorems 4.1 and 4.2, and Corollary 4.1 can be adapted to guarantee the set Ω *to be* $\gamma-$*contractive by replacing* Ψ_0 *and* $\overline{\Psi}_0$*, respectively by*

$$\Psi_0 := \begin{bmatrix} P_{z+} & J_{z+} & A_z X + B_z \mathscr{C}_{z_\mathbf{q}} & A_z + B_z D_{cz_\mathbf{q}} C_z \\ \star & H_{z+} & \mathscr{A}_{z_\mathbf{q}} & Z' A_z + \mathscr{B}_{z_\mathbf{q}} C_z \\ \star & \star & X + X' - \gamma^{-1} P_z & I_n + S' - \gamma^{-1} J_z \\ \star & \star & \star & Z + Z' - \gamma^{-1} H_z \end{bmatrix} \tag{4.51}$$

[2]The equality holds when z is a time-varying parameter independent of the states (Tingshu Hu and Zongli Lin, 2003; Jungers and Castelan, 2011).

and

$$\overline{\Psi}_0 = \begin{bmatrix} P_z & J_z & A_zX + B_z\mathscr{C}_{z_\mathbf{q}} & A_z + B_zD_{cz_\mathbf{q}}C_z & B_z\Sigma \\ \star & H_z & \mathscr{A}_{z_\mathbf{q}} & Z'A_z + \mathscr{B}_{z_\mathbf{q}}C_z & 0 \\ \star & \star & X + X' - \gamma^{-1}P_z & I_n + S' - \gamma^{-1}J_z & M_{1z} \\ \star & \star & \star & Z + Z' - \gamma^{-1}H_z & M_{2z} \\ \star & \star & \star & \star & 2\Sigma \end{bmatrix}.$$

4.4 Examples

In this section, the applicability of the proposed technique is illustrated by means of numerical examples concerning the design of DOF controllers when some of the premise variables are not available for the control law. The conditions proposed are presented in terms of parameter-dependent LMIs. Finite dimension conditions can be obtained in terms of the vertices of the fuzzy summations or by using the ROLMIP (Robust LMI Parser) (Agulhari et al., 2019).

Example 4.1 In this example, Theorem 4.1 is compared with Algorithm 1 of Dong and Yang (2017) considering one premise variable unavailable for measurement. The discrete-time nonlinear system is expressed as

$$\begin{aligned} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} &= \left(h \begin{bmatrix} 0 & 1 \\ -0.01 - 0.1x_1^2(t) & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + h \begin{bmatrix} 0 \\ 1 + 0.13x_2^3(t) \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \tag{4.52}$$

where $x_1 \in [-a,\ a]$, $x_2 \in [-\beta,\ \beta]$, $a > 0$, $\beta > 0$ *and* $h = 0.001$ *by Euler's discretization method with a fixed step. The normalized membership functions are*

$$\begin{aligned} \mu_{11}(z_1) &= a^2 - z_1/a^2 \quad \mu_{12}(z_1) = z_1/a^2 \\ \mu_{21}(z_2) &= \beta^3 - z_2/2\beta^3 \quad \mu_{22}(z_2) = \beta^3 + z_2/2\beta^3 \end{aligned}$$

with $z_1 = x_1^2$, $z_2 = x_2^3$ *and* z_2 *is unmeasurable. The matrices coefficients are*

$$\begin{aligned} &A_{11} = A_{12} = \begin{bmatrix} 1 & h \\ -0.001h & 1-h \end{bmatrix} \quad A_{21} = A_{22} = \begin{bmatrix} 1 & h \\ -0.001h - 0.1a^2h & 1-h \end{bmatrix} \\ &B_{11} = B_{21} = \begin{bmatrix} 0 \\ h - 0.13\beta^3h \end{bmatrix} \quad B_{12} = B_{22} = \begin{bmatrix} 0 \\ h + 0.13\beta^3h \end{bmatrix} \\ &C_{11} = C_{12} = C_{21} = C_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

Table 4.1 shows the maximum values of β *in function of parameter* a *such that Theorem 4.1 with* $\lambda = 1$ *and (Dong and Yang, 2017, Algorithm 1) are feasible. One can observe in Table 4.1* *that Theorem 4.1 presents, except for one case, a wider feasible area* (a, β) *than the Algorithm 1 of Dong and Yang (2017).*

Table 4.1: Maximum values of β obtained in Example 4.1.

a	1	2	3	4	5	6	7	8	9	10
Theorem 4.1	2.8	3.2	3.2	1.9	1.9	1.9	1.9	1.9	1.9	1.9
(Dong and Yang, 2017, Algorithm 1)	2.7	2.7	2.6	2.6	1.9	1.7	1.5	1.5	1.5	1.4

Example 4.2 This example is used to evaluate the case of imperfect premise matching presented in Section 4.3. The following nonlinear system is adapted from Estrada-Manzo et al. (2019) to include the saturation in the control signal with $\rho = 0.2$*:* $x(t+1) = A(x)x(t) + B(x)sat(u(t))$, $y(t) = C(x)x(t)$*, with*

$$A(x) = \begin{bmatrix} 0.2 + 0.12\cos x_1 & 1.6 \\ -0.8 & 2.1 + 0.1\sin x_2 \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0.1 + 0.05\cos x_1 \\ -2 - 0.4\sin x_2 \end{bmatrix},$$
$$C(x) = \begin{bmatrix} 0.2 + 0.1\cos x_1 & 0.2 \end{bmatrix}, \quad |x_1| \leq \pi/2, \quad |x_2| \leq 1.$$

Defining the premise variables as $z_1 = \cos x_1 \in [0,\ 1]$ *and* $z_2 = \sin x_2 \in [-0.84,\ 0.84]$*, the membership functions* $\mu(z) = (\mu_1(z_1), \mu_2(z_2))$ *are given by* $\mu_{11}(z_1) = 0.5(\cos x_1 + 1)$, $\mu_{12}(z_1) = 1 - \mu_{11}(z_1)$, $\mu_{21}(z_2) = 0.5(\sin x_2 + 1)$ *and* $\mu_{22}(z_2) = 1 - \mu_{21}(z_2)$*. The matrices of the T-S fuzzy model* (4.4) *are:*

$$A_{11} = \begin{bmatrix} 0.32 & 1.60 \\ -0.80 & 0.78 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0.20 & 1.60 \\ -0.80 & 0.78 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} 0.32 & 1.60 \\ -0.80 & 0.78 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0.20 & 1.60 \\ -0.80 & 0.78 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0.15 \\ -2.34 \end{bmatrix} \quad B_{12} = \begin{bmatrix} 0.10 \\ -2.34 \end{bmatrix}$$
$$B_{21} = \begin{bmatrix} 0.15 \\ -1.66 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 0.10 \\ -1.66 \end{bmatrix}$$
$$C_{11} = C_{12} = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix} \quad C_{21} = C_{22} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}.$$

Considering the saturation in the control signal, the controller gains are obtained by Theorem 4.2 with $\lambda = 1$ *and given by:*

$$A_{cz_{\mathbf{q}}} = \sum_{i=1}^{2} \mu_{1i}(z_1)^2 A_{cii} + \mu_{11}(z_1)\mu_{12}(z_1)A_{ci2},$$
$$(B_{cz_{\mathbf{q}}}, C_{cz_{\mathbf{q}}}, D_{cz_{\mathbf{q}}}) = \sum_{i=1}^{2} \mu_{1i}(z_1)(B_{ci}, C_{ci}, D_{ci})$$

with

$$
\begin{aligned}
A_{c11} &= \begin{bmatrix} -0.66 & 0.48 \\ 0.14 & -0.14 \end{bmatrix}, \quad A_{c12} = \begin{bmatrix} -1.29 & 0.81 \\ 0.16 & -0.25 \end{bmatrix}, \quad A_{c22} = \begin{bmatrix} -0.55 & 0.37 \\ 0.06 & -0.10 \end{bmatrix}, \\
B_{c1} &= \begin{bmatrix} 3.20 \\ -0.44 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 3.80 \\ -0.17 \end{bmatrix}, \quad C_{c1} = \begin{bmatrix} -0.38 & 0.28 \end{bmatrix}, \quad C_{c2} = \begin{bmatrix} -0.37 & 0.28 \end{bmatrix}, \\
& D_{c1} = 0.05, \quad D_{c2} = -0.03.
\end{aligned} \tag{4.53}
$$

For the closed-loop system with controller gains (4.53), the trajectories and the estimate of the domain of attraction $\Omega_0 \subseteq \mathbb{R}^2$ *using Remark 4.4 are illustrated in Figures 4.1 and 4.2, respectively. To obtain the largest* Ω_0*, we employ the optimization problem described in Remark 4.3.*

Following Remark 4.5, we observe in Figure 4.2 that the set Ω_0 *is not invariant but for all* $x_a(0) \in \Omega_0 \times \{0\}$, $\lim_{k\to\infty} x_a(t) = 0$. *This is illustrated in Figure 4.3 that shows the evolution of the Lyapunov function* $V(x_a) = x_a(t)'W_z^{-1}x_a(t)$ *and* $V(x,0) = [x(t)' \quad 0]W_z^{-1}[x(t)' \quad 0]'$*, that characterizes the set* Ω_0*, for the initial condition* $x_a(0) = (0.37, \ 0.33, \ 0, \ 0)$. *We observe that* $V(x_a)$ *behaves as expected, whereas* $V(x,0)$ *increases its value before converging to zero according to the trajectories presented in Figure 4.2.*

Using Remark 4.9 to evaluate the $\gamma-$*contractiveness of* Ω *set, Figure 4.4 shows system and controller states time-response and control signal. Comparing with Figure 4.1, we note a faster state convergence in Figure 4.4, but as a drawback the control signal reaches a higher level before stabilization.*

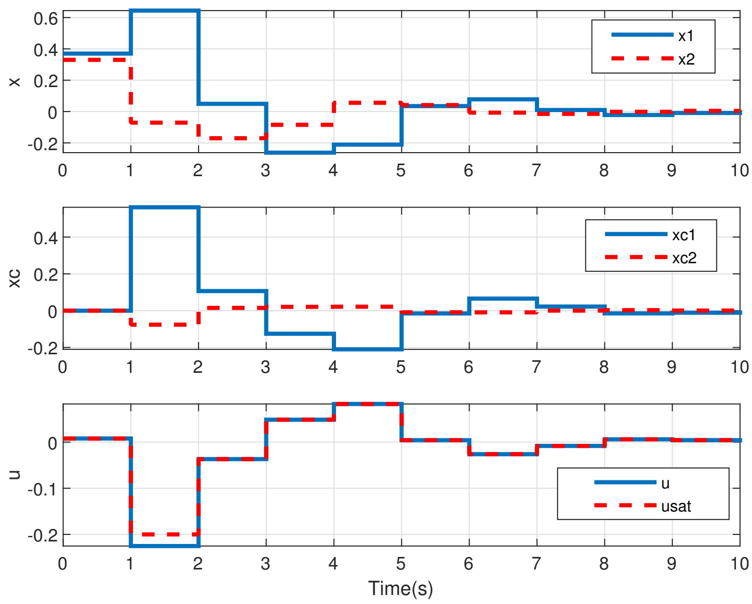

Figure 4.1: Trajectories and control signal for initial condition $x(0) = (0.37, \ 0.33)$ and $x_c(0) = (0, \ 0)$.

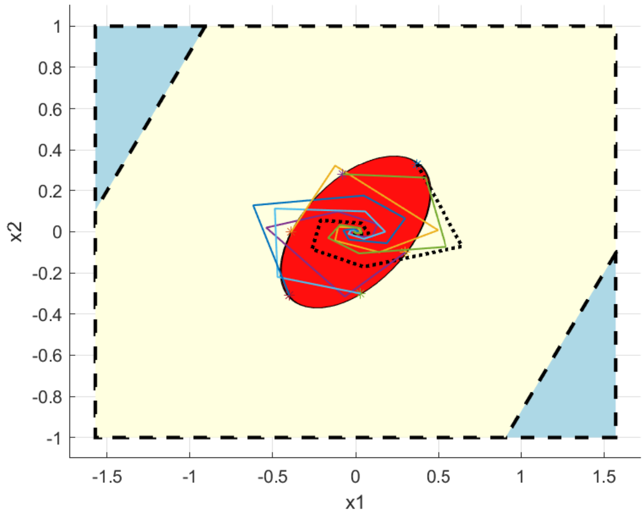

Figure 4.2: Regions $\mathcal{X}_0$ (blue and yellow region/ external dashed black line), Π defined in (4.11) (yellow region/ internal dashed black line), the estimation of the region of attraction Ω_0 (red ellipsoid), and trajectories of the closed-loop system. The trajectory illustrated in Figure 4.1 is the black dotted line.

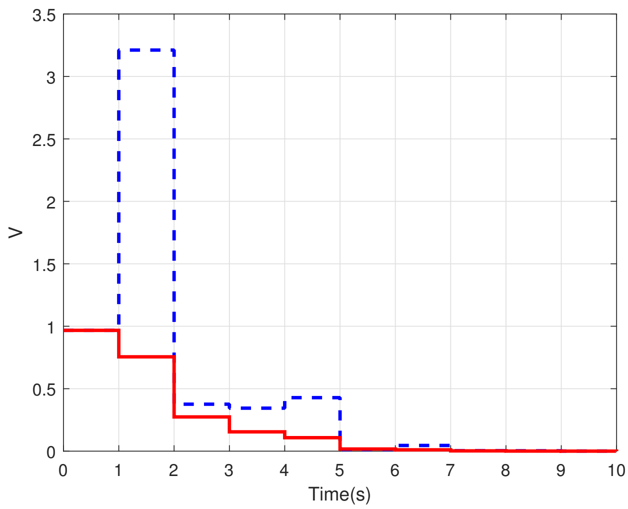

Figure 4.3: Lyapunov function $V(x_a) = x_a' W_z^{-1} x_a$ (solid red line) and $V(x,0) = [x' \;\; 0] W_z^{-1} [x' \;\; 0]'$ (dashed blue line) associated to the trajectory illustrated in Figure 4.1 (initial condition $x_a(0) = (0.37, \; 0.33, \; 0, \; 0)$).

4.5 Conclusion

This chapter proposed new LMI synthesis conditions for dynamic output feedback controllers for discrete-time Takagi-Sugeno fuzzy systems when the premise variables are partially or completely unavailable for the control law. The main idea is to rewrite the dynamics in terms of the available premise variables and use them to recover the controller's gains. The use of fuzzy Lyapunov functions is facilitated by additional instrumental variables making the controller matrices not functions of the Lyapunov matrix or the original system matrices, in opposition to the

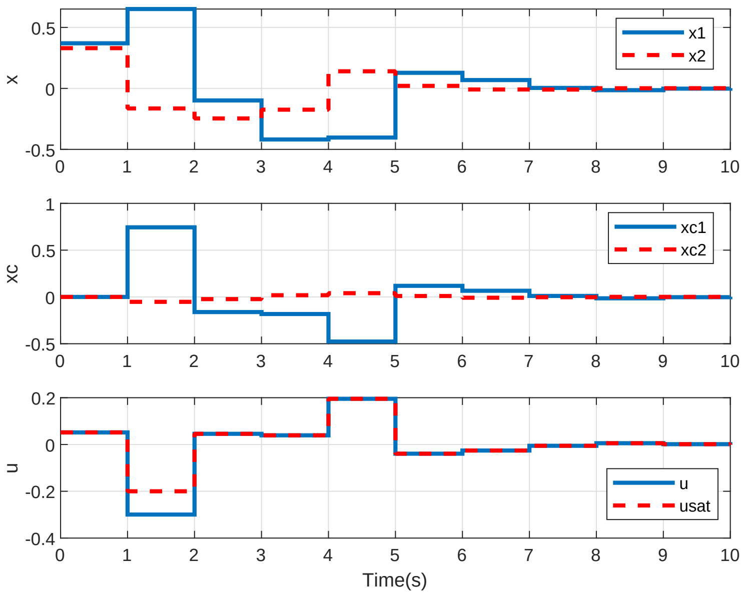

Figure 4.4: Trajectories and control signal for initial condition $x(0) = (0.37,\ 0.33)$, $x_c(0) = (0,\ 0)$ and $\gamma = 0.85$.

classic approach in the design of full-order dynamic output feedback controllers. As a novelty, we describe the variation rate of the membership functions using the system dynamics yielding local design conditions. The control signal saturation is also taken into account to estimate the domain of attraction of the origin. We show that for zero initial conditions of the controller, the domain of attraction in the systems' state space does not need to be invariant.

5 Inexact measurements of premise variables

This chapter addresses the design of dynamic output feedback controllers for continuous-time Takagi-Sugeno (T-S) fuzzy systems subjected to inexact measurements of premise variables. In contrast to the common assumption that the premise variables are precisely available for the controller, this work provides convex conditions as linear matrix inequalities (LMIs) to design controller gains depending on premise variables with additive uncertainties. This uncertainty models inexact measurements of the premise variables due to sensor devices or an approximate representation in the T-S modeling, which can assume the form of absolute uncertainties in the membership functions. Numerical examples validate the effectiveness of the approach.

5.1 Introduction

In the T-S literature, many works are devoted to the problem of immeasurable or partial measurements of premise variables, especially when the states are not available for feedback. However, even in the imperfect premise-matching design problem, one usually considers that the available premise variables are measured with high precision (Gómez-Peñate et al., 2020). Thus, this chapter focuses on handling different sets of membership functions of the T-S fuzzy model and the fuzzy controller due to imprecise measurement of premise variables. Among the few works that explicitly handle the inaccurate measurement of the premise variables, the work Gómez-Peñate et al. (2020) designs a sliding mode controller and an unknown input observer for T-S fuzzy systems.

The problem of control synthesis with inexact premise variables becomes intricate for designing dynamic output feedback (DOF) controllers. Although the existence of some works considering the design of DOF controllers that do not share the same membership functions and the number of rules with T-S fuzzy systems (Nguang and Shi, 2006; Tognetti et al., 2012; Tognetti and Linhares, 2021), taking into account uncertainties in the measurement is practically unexplored in designing fuzzy DOF controllers.

The problem of inaccurate measurement of premise variables is similar to the problem of inexact scheduling parameters for linear parameter varying (LPV) systems (Daafouz et al., 2008; Sato and Peaucelle, 2013; Lacerda et al., 2016). In Sato and Peaucelle (2013); Lacerda et al. (2016), the scheduling parameters may be affected by uncertainties that are proportional to the values of the actual parameters (proportional uncertainty), or they are supposed to lie within *a priori* defined intervals that are independent of the actual values of scheduling parameters (absolute uncertainty). Note, however, that techniques borrowed from the LPV literature (scheduling variables are free variables) can be conservative when applied in T-S systems by disregarding the nature of the fuzzy modeling (premise variables model nonlinear terms or heuristic knowledge).

Motivated by the lack of controller design techniques for T-S systems considering inexact measurements of premise variables, this chapter presents new results for designing DOF controllers depending on premise variables subject to absolute uncertainties with known upper bounds. We first propose a model to characterize the influence of the premise variables' uncertainties in the membership functions without introducing conservativeness. Then, we develop a fuzzy summation transformation that allows the recovery of the controller gains from the measured variables that do not precisely fit the real ones - the main difficulty in the DOF controller design. By establishing a relation between the real and the uncertain parameters in the fuzzy variables, this approach also allows building design LMI conditions depending only on one parameter set. The proposed approach also models uncertainties in the membership functions, an important feature in implementations. Numerical examples demonstrate the effectiveness of the proposed approach.

5.2 Preliminaries and Problem definition

Consider a class of T-S fuzzy systems described by (2.9) with $\delta[x](t) = \dot{x}$ and the membership functions $\mu(z)$ obtained from the sector-nonlinearity modeling (Tanaka and Wang, 2001) yielding $r_i = 2,\ i = 1, \dots, p$. The functions $\mu_j(z_j) = (\mu_{j1}(z_j), \mu_{j2}(z_j)),\ j = 1, ..., p,$ belong to the unit simplex $\mathcal{U}_2$ and $\mu(z) = (\mu_1(z_1), \mu_2(z_2), \dots, \mu_p(z_p)) \in \mathcal{U} \coloneqq \mathcal{U}_2 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_2$.

The sector-nonlinearity approach is used to model nonlinear terms of a dynamic system, represented as the premise variables z_j, as a convex combination of their upper and lower bounds, $\overline{z}_j$ and $\underline{z}_j$, respectively, in a given domain:

$$z_j = \mu_{j1}(z_j)\overline{z}_j + \mu_{j2}(z_j)\underline{z}_j, \quad \mu_{j1}(z_j) = \frac{z_j - \underline{z}_j}{\overline{z}_j - \underline{z}_j}, \quad \mu_{j2}(z_j) = 1 - \mu_{j1}.$$

One important aspect of this approach is that each premise variable z_j is uniquely associated with a membership function $\mu_j(z_j)$.

The aim is to design a full-order dynamic output feedback controller given by

$$\begin{cases} \dot{x}_c(t) & = A_{c\hat{z}} x_c(t) + B_{c\hat{z}} y(t) \\ u(t) & = C_{c\hat{z}} x_c(t) + D_{c\hat{z}} y(t), \end{cases} \tag{5.1}$$

with

$$(A, B, C, D)_{c\hat{z}} = \sum_{\mathbf{i} \in \mathcal{I}_p} \mu_{\mathbf{i}}(\hat{z})(A_{c\mathbf{i}}, B_{c\mathbf{i}}, C_{c\mathbf{i}}, D_{c\mathbf{i}}),$$

where $x_c(t) \in \mathbb{R}^n$ is the controller state, and $\hat{z}$ are the inexactly measured premise variables modeled as

$$\hat{z}_i = z_i + \theta_i, \tag{5.2}$$

where θ_i is the additive uncertainty due to measurement errors satisfying $|\theta_i| \leq \delta_i$, with δ_i a known upper bound. In practice, the measured data contains measurement errors, noise, etc, yielding different values from the actual value of the premise variable. In the case of T-S models obtained from the sector-nonlinearity approach, (5.2) can also model uncertainties in the representation of the nonlinear terms of the original system.

The closed-loop system is represented as

$$\dot{x}_a(t) = \mathbb{A}_{z,\hat{z}} x_a(t) \tag{5.3}$$

where $x_a(t) = \begin{bmatrix} x(t)' & x_c(t)' \end{bmatrix}'$ denotes the augmented state and

$$\mathbb{A}_{z,\hat{z}} = \begin{bmatrix} A_z + B_z D_{c\hat{z}} C_z & B_z C_{c\hat{z}} \\ B_{c\hat{z}} C_z & A_{c\hat{z}} \end{bmatrix}. \tag{5.4}$$

The following problem is addressed.

Problem 5.1 *Consider the T-S fuzzy system* (2.9) *and premise variables available for the control law with uncertainty modeled by* (5.2)*. Determine a fuzzy dynamic output feedback controller as* (5.1) *such that the closed-loop system* (5.3) *is asymptotically stable.*

The following lemma is standard in the literature and it will be useful for the main results.

Lemma 5.1 *If there exists a symmetric positive definite matrix $W \in \mathbb{R}^{2n\times 2n}$ such that the following LMIs are satisfied for all $\mu \in \mathcal{U}$*

$$He\{\mathbb{A}_{z,\hat{z}} W\} < 0, \tag{5.5}$$

then the closed-loop system (5.3) *is asymptotically stable.*

5.3 Main results

5.3.1 Uncertainty modeling

The membership functions depending on the inexact premise variables $\hat{z}_i$ are rewritten as

$$\mu_{j1}(\hat{z}_j) = \frac{\hat{z}_j - \underline{z}_j}{\overline{z}_j - \underline{z}_j} = \mu_{j1}(z_j) + \tilde{\theta}_j, \quad \mu_{j2}(\hat{z}_j) = \mu_{j2}(z_j) - \tilde{\theta}_j, \quad \tilde{\theta}_j = \frac{\theta_j}{\overline{z}_j - \underline{z}_j}. \tag{5.6}$$

Observe that $\mu_{j1}(\hat{z}_j) + \mu_{j2}(\hat{z}_j) = 1$ but $\mu_j(\hat{z}_j) \notin \mathcal{U}_2$. Note also that the proposed approach for modeling inexact premise variables can also be used to represent uncertainties in the membership functions. This is justified in approximation-based fuzzy models, where membership functions are estimated, or simply due to uncertainties in the nonlinear terms of the model.

To construct numerically implementable conditions to design (5.1) one needs parameter-dependent conditions such that the parameters belong to $\mathcal{U}$. Therefore, one needs to represent the controllers gains $A_{c\hat{z}}$, $B_{c\hat{z}}$, $C_{c\hat{z}}$ and $D_{c\hat{z}}$ as functions of $\mu(z)$ instead of $\mu(\hat{z})$. For this purpose, we can rewrite the controller gains as

$$\begin{aligned} A_{c\hat{z}} &= \sum_{i_1=1}^{2} \cdots \sum_{i_p=1}^{2} (\mu_{1i_1}(z_1) + (-1)^{i_1-1}\tilde{\theta}_1) \cdots (\mu_{pi_p}(z_p) + (-1)^{i_p-1}\tilde{\theta}_p) A_{c_{i_1 \ldots i_p}} \\ &= A_{cz} + \Gamma_{A_{cz}}, \end{aligned} \tag{5.7}$$

where $\Gamma_{A_{cz}}$ can be construct by a systematic procedure, and the same for $B_{c\hat{z}}$, $C_{c\hat{z}}$ and $D_{c\hat{z}}$.

For instance, if $p = 2$ one has

$$\begin{aligned} \Gamma_{A_{cz}} = \underbrace{\tilde{\theta}_2 \sum_{i_1=1}^{2} \mu_{1i_1}(z_1)(A_{c_{i_1 1}} - A_{c_{i_1 2}})}_{\Gamma^{(1)}_{A_c}(z_1)} + \underbrace{\tilde{\theta}_1 \sum_{i_2=1}^{2} \mu_{2i_2}(z_2)(A_{c_{1i_2}} - A_{c_{2i_2}})}_{\Gamma^{(2)}_{A_c}(z_2)} \\ + \tilde{\theta}_1\tilde{\theta}_2 \underbrace{(A_{c_{11}} - A_{c_{12}} - A_{c_{21}} + A_{c_{22}})}_{\Gamma^{(3)}_{A_c}}. \end{aligned} \tag{5.8}$$

In the design, the controller gains must be recovered by functions that depend on $\mu(\hat{z})$ instead of $\mu(z)$. For this reason, we rewrite the system matrices as

$$\begin{aligned} A_z &= A_{\hat{z}} + \Gamma_{A_z}, \\ B_z &= B_{\hat{z}} + \Gamma_{B_z}, \\ C_z &= C_{\hat{z}} + \Gamma_{C_z}, \end{aligned} \tag{5.9}$$

where Γ_{A_z}, Γ_{B_z} and Γ_{C_z} follow the same structure as (5.8).

5.3.2 Controller design

As defined in Section 3.3, we consider quadratic stability with $W_z = W$ and the parametrization adopted in Chilali and Gahinet (1996); Scherer et al. (1997). Using the matrices defined in (3.13), where $Y = X - Z^{-1}$ and $X, Z \in \mathbb{R}^{n \times n}$ are symmetric. From $WW^{-1} = I$ and using the change of variables presented in Scherer et al. (1997), the condition (5.5) is equivalent to

$$\mathscr{X}' \left(He \left\{ \mathbb{A}_{z,\hat{z}} W \right\} \right) \mathscr{X} = \text{He} \left\{ \begin{bmatrix} A_z X + B_z \bar{H}_{\hat{z}} & A_z + B_z D_{c\hat{z}} C_z \\ \bar{F}_{\hat{z}} & Z A_z + \bar{G}_{\hat{z}} C_z \end{bmatrix} \right\} < 0 \tag{5.10}$$

with

$$\begin{aligned} \bar{F}_{\hat{z}} &= Z A_z X - Z A_{c\hat{z}} Y - Z B_{c\hat{z}} C_z X + Z B_z C_{c\hat{z}} Y + Z B_z D_{c\hat{z}} C_z X, \\ \bar{G}_{\hat{z}} &= Z B_z D_{c\hat{z}} - Z B_{c\hat{z}}, \\ \bar{H}_{\hat{z}} &= C_{c\hat{z}} Y + D_{c\hat{z}} C_z X. \end{aligned} \tag{5.11}$$

Observe that the controller gains cannot be recovered from (5.11) due to the presence of the premise variable z, without uncertainty, that is not available for feedback.

To allow the design of a controller depending only on the measured premise variables $\hat{z}$, the following approach is proposed. First, we replace A_z, B_z and C_z in (5.11) by the relations in (5.9) yielding

$$\text{He}\left\{\begin{bmatrix} A_zX + B_z(H_{\hat{z}} + D_{c\hat{z}}\Gamma_{C_z}X) & A_z + B_zD_{c\hat{z}}C_z \\ F_{\hat{z}} + \Phi_{z,\hat{z}} & ZA_z + (G_{\hat{z}} + Z\Gamma_{B_z}D_{c\hat{z}})C_z \end{bmatrix}\right\} < 0 \tag{5.12}$$

with

$$\begin{aligned} F_{\hat{z}} &= ZA_{\hat{z}}X - ZA_{c\hat{z}}Y - ZB_{c\hat{z}}C_{\hat{z}}X + ZB_{\hat{z}}C_{c\hat{z}}Y + ZB_{\hat{z}}D_{c\hat{z}}C_{\hat{z}}X \\ G_{\hat{z}} &= ZB_{\hat{z}}D_{c\hat{z}} - ZB_{c\hat{z}}, \\ H_{\hat{z}} &= C_{c\hat{z}}Y + D_{c\hat{z}}C_{\hat{z}}X, \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \Phi_{z,\hat{z}} := Z\Gamma_{A_z}X + (\underbrace{ZB_{\hat{z}}D_{c\hat{z}} - ZB_{c\hat{z}}}_{G_{\hat{z}}})\Gamma_{C_z}X + \\ Z\Gamma_{B_z}(\underbrace{C_{c\hat{z}}Y + D_{c\hat{z}}C_{\hat{z}}X}_{H_{\hat{z}}}) + Z\Gamma_{B_z}D_{c\hat{z}}\Gamma_{C_z}X \\ = Z\Gamma_{A_z}X + G_{\hat{z}}\Gamma_{C_z}X + Z\Gamma_{B_z}H_{\hat{z}} + Z\Gamma_{B_z}D_{c\hat{z}}\Gamma_{C_z}X. \end{aligned}$$

As can be observed, the change of variables (5.13) allows to recover the controller gains depending only on $\hat{z}$:

$$\begin{cases} A_{c\hat{z}} = Z^{-1}\left\{ZA_{\hat{z}}X + ZB_{\hat{z}}H_{\hat{z}} - F_{\hat{z}} - \left[ZB_{\hat{z}}D_{c\hat{z}} - G_{\hat{z}}\right]C_{\hat{z}}X\right\}Y^{-1} \\ B_{c\hat{z}} = Z^{-1}\left(ZB_{\hat{z}}D_{c\hat{z}} - G_{\hat{z}}\right) \\ C_{c\hat{z}} = \left(H_{\hat{z}} - D_{c\hat{z}}C_{\hat{z}}X\right)Y^{-1} \end{cases} \tag{5.14}$$

Observe also that inequality (5.12) depends on the parameters z, $\hat{z}$ and $\tilde{\theta}$. The dependence on $\hat{z}$ can be removed by applying the equivalence (5.7), that is, replacing $F_{\hat{z}}$, $G_{\hat{z}}$, $H_{\hat{z}}$ and $D_{c\hat{z}}$ in (5.12) by $F_z + \Gamma_{F_z}$, $G_z + \Gamma_{G_z}$, $H_z + \Gamma_{H_z}$ and $D_{cz} + \Gamma_{D_{cz}}$, respectively, yielding

$$He\left\{\begin{bmatrix} A_zX + B_z(H_z + \Gamma_{H_z} + (D_{cz} + \Gamma_{D_{cz}})\Gamma_{C_z}X) & A_z + B_z(D_{cz} + \Gamma_{D_z})C_z \\ F_z + \Gamma_{F_z} + \tilde{\Phi}_z & ZA_z + (G_z + \Gamma_{G_z} + Z\Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}}))C_z \end{bmatrix}\right\} < 0 \tag{5.15}$$

where

$$\tilde{\Phi}_z = Z(\Gamma_{A_z} + \Gamma_{B_z}(D_{cz} + \Gamma_{D_z})\Gamma_{C_z})X + (G_z + \Gamma_{G_z})\Gamma_{C_z}X + Z\Gamma_{B_z}(H_z + \Gamma_{H_z}). \tag{5.16}$$

Observe also that inequality (5.15) is nonlinear on the decision variables X, Z, F_z, G_z, H_z and D_{cz}. We rewrite (5.15) separating the linear and nonlinear terms in Υ and Λ, respectively, yielding

$$\text{He}\left\{\Upsilon + \Lambda\right\} < 0 \tag{5.17}$$

with

$$\Upsilon = \begin{bmatrix} A_zX + B_z(H_z + \Gamma_{H_z}) & A_z + B_z(D_{cz} + \Gamma_{D_z})C_z \\ F_z + \Gamma_{F_z} & ZA_z + (G_z + \Gamma_{G_z})C_z \end{bmatrix}, \tag{5.18}$$

$$\Lambda = \begin{bmatrix} B_z(D_{cz} + \Gamma_{D_{cz}})\Gamma_{C_z}X & 0 \\ \tilde{\Phi}_z & Z\Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}})C_z \end{bmatrix} \tag{5.19}$$

and $\tilde{\Phi}_z$ given by (5.16).

To deal with the product of variables in (5.19), one has

$$\Lambda = \text{He}\left\{ \underbrace{\begin{bmatrix} 0 & B_z(D_{cz} + \Gamma_{D_{cz}})\Gamma_{C_z} \\ Z & (G_z + \Gamma_{G_z})\Gamma_{C_z} \end{bmatrix}}_{M_1'} \underbrace{\begin{bmatrix} \Gamma_{B_z}(H_z + \Gamma_{H_z}) & \Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}})C_z \\ X & 0 \end{bmatrix}}_{M_2} \right\} + \underbrace{\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix}}_{M_3} \underbrace{\begin{bmatrix} 0 & \star \\ \Gamma_{A_z} + \Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}})\Gamma_{C_z} & 0 \end{bmatrix}}_{R} \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix}. \tag{5.20}$$

Now, we are ready to propose convex conditions to solve (5.17) and design the controller (5.1).

Theorem 5.1 *For given scalars $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}_{>0}$, if there exist symmetric positive definite matrices X , $Z \in \mathbb{R}^{n\times n}$, and affine matrices $F_z \in \mathbb{R}^{n\times n}$, $G_z \in \mathbb{R}^{n\times n_y}$, $H_z \in \mathbb{R}^{n_u\times n}$ and $D_{cz} \in \mathbb{R}^{n_u\times n_y}$, such that, for all $\mu \in \mathcal{U}$, the following LMIs are satisfied:*

$$\begin{bmatrix} X & I_n \\ I_n & Z \end{bmatrix} > 0 \tag{5.21}$$

and

$$\Psi := \begin{bmatrix} \Psi_{11} & \star \\ \Psi_{21} & \Psi_{22} \end{bmatrix} < 0 \tag{5.22}$$

where

$$\Psi_{11} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\Psi_{111}} & \star \\ \hline -\lambda_1 I_n & \lambda_2 Z & \\ \lambda_2\Gamma_{C_z}'(D_{cz} + \Gamma_{D_{cz}})'B_z' & -\lambda_1 I + \lambda_2\Gamma_{C_z}'(G_z + \Gamma_{G_z})' & -2\lambda_2 I \end{array}\right],$$

$$\Psi_{111} = He\left\{ \Upsilon + \lambda_1 \begin{bmatrix} \Gamma_{B_z}(H_z + \Gamma_{H_z}) + X & \Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}})C_z + Z \\ \Gamma_{C_z}'(D_{cz} + \Gamma_{D_{cz}})'B_z' + X & \Gamma_{C_z}'(G_z + \Gamma_{G_z})' + Z \end{bmatrix} \right\},$$

$$\Psi_{21} = \left[\begin{array}{cc|c} -\lambda_1 I + \lambda_2 \Gamma_{B_z}(H_z + \Gamma_{H_z}) & \lambda_2 \Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}})C_z & \multirow{2}{*}{I} \\ \lambda_2 X & -\lambda_1 I & \\ \hline -\lambda_1 I + \lambda_2 X & 0 & \multirow{2}{*}{0} \\ 0 & -\lambda_1 I + \lambda_2 Z & \end{array}\right],$$

and

$$\Psi_{22} = \left[\begin{array}{c|cc} -2\lambda_2 I & \multicolumn{2}{c}{\star} \\ \hline \multirow{2}{*}{0} & -2\lambda_2 I & \star \\ & \Gamma_{A_z} + \Gamma_{B_z}(D_{cz} + \Gamma_{D_{cz}})\Gamma_{C_z} & -2\lambda_2 I \end{array}\right],$$

with Υ *given by* (5.18)*, then the controller* (5.1) *whose state-space matrices are given by* (5.14)*, with* $Y = X - Z^{-1}$*, and makes the closed-loop system* (5.4) *asymptotically stable.*

Proof First, from (5.21), it follows that $X > 0$, $Z > 0$ and $Y > 0$, since by definition $Y = X - Z^{-1}$. Then, the gains (5.14) are well defined.

Thus, pre- and post-multiplying (5.22) by $\begin{bmatrix} I_{2n} & M_1' & M_2' & M_3' \end{bmatrix}$ and its transpose, respectively, we obtain

$$\Pi := He\{\Upsilon\} + \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}' \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} < 0 \tag{5.23}$$

where M_1, M_2, M_3 are given in (5.20). Consider the change of variables (5.9) and (5.13), then the following inequality is obtained

$$He\left\{\begin{bmatrix} \Omega X + B_z C_{c\hat{z}} Y & \Omega \\ Z(\Omega - B_{c\hat{z}} C_z) X + Z(B_z C_{c\hat{z}} - A_{c\hat{z}}) Y & Z(\Omega - B_{c\hat{z}} C_z) \end{bmatrix}\right\} < 0$$

with $\Omega = A_z + B_z D_{c\hat{z}} C_z$.

Therefore, the above inequality can be written as $\mathscr{X}' (He\{\mathbb{A}_{z,\hat{z}} W\}) \mathscr{X} < 0$ as in (5.10) with the definitions (3.13). One also observe that (5.21) is equivalent to $\mathscr{X}^T W \mathscr{X}$, then $W > 0$. Thus, from Lemma 5.1, the closed-loop system (5.3) is asymptotically stable. ∎

Consider the particular case $B_z = B$ and $C_z = C$ implying $\Gamma_B = 0$ and $\Gamma_C = 0$, respectively. Then, (5.19) becomes

$$\Lambda = \begin{bmatrix} 0 & 0 \\ Z\Gamma_{A_z} X & 0 \end{bmatrix}$$

and one has the following corollary.

Corollary 5.1 *For given scalars* $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}_{>0}$*, if there exist symmetric positive definite matrices* X , $Z \in \mathbb{R}^{n \times n}$*, and affine matrices* $F_z \in \mathbb{R}^{n \times n}$, $G_z \in \mathbb{R}^{n \times n_y}$, $H_z \in \mathbb{R}^{n_u \times n}$ *and* $D_{cz} \in \mathbb{R}^{n_u \times n_y}$*, such that* (5.21) *and the following LMI are satisfied for all* $\mu \in \mathscr{U}$*:*

$$diag(\Upsilon + \Upsilon', 0) + \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \star \\ \hline \lambda_1 Z \Gamma_{A_z} & 0 & \multirow{2}{*}{$-2\lambda_2 I_{2n}$} \\ X - \lambda_1 I_n & \lambda_2 \Gamma_{A_z}' Z & \end{array}\right] < 0 \tag{5.24}$$

with Υ *given by* (5.18)*, then the controller* (5.1) *whose state-space matrices are given by* (5.14)*, with* $Y = X - Z^{-1}$*, makes the closed-loop system* (5.4) *asymptotically stable.*

Proof By pre- and post-multiplying (5.24) by

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & Z\Gamma_{A_z} \end{bmatrix}$$

and its transpose, respectively, one has

$$He\left\{\Upsilon + \begin{bmatrix} 0 \\ Z\Gamma_{A_z} \end{bmatrix} \begin{bmatrix} X & 0 \end{bmatrix}\right\} < 0.$$

The above inequality is equivalent to

$$\mathscr{X}' \left(He\left\{\mathbb{A}_{z,\hat{z}} W\right\}\right) \mathscr{X} < 0$$

with the definitions (3.13). From (5.21), one has $W > 0$, then the closed-loop system (5.4) is asymptotically stable.

■

Remark 5.1 *Note that the terms* $\Gamma_M(z)$*,* $M \in \{A, B, C, F, G, H, D_c\}$*, in the conditions of Theorem 5.1 depend on* $\tilde{\theta}$*, that are assumed to be unknown. To obtain finite dimension conditions, we can represent* $\tilde{\theta}_i$ *as convex combinations of their extreme values:*

$$\theta_i(\gamma_i) = \gamma_{i1}(-\tilde{\delta}_i) + \gamma_{i2}\tilde{\delta}_i,$$

where

$$\gamma = (\gamma_1, \gamma_2) \in \mathscr{U}_2 \times \mathscr{U}_2, \quad \gamma_i = (\gamma_{i1}, \gamma_{i2}).$$

The parameter $\tilde{\delta}_i$ *represents an upper bound for* $\tilde{\theta}$ *obtained from* (5.6) *as:*

$$|\tilde{\theta}_i| = \frac{|\theta_i|}{|\overline{z}_j - \underline{z}_j|} \leq \frac{\delta_i}{|\overline{z}_j - \underline{z}_j|} := \tilde{\delta}_i. \tag{5.25}$$

For instance, for $p = 2$*, one can rewrite* (5.8) *as*

$$\begin{aligned} \Gamma_M(z) &= \tilde{\theta}_1(\gamma_1)\Gamma_M^{(2)}(z_2) + \tilde{\theta}_2(\gamma_2)\Gamma_M^{(1)}(z_1) + \tilde{\theta}_1(\gamma_1)\tilde{\theta}_2(\gamma_2)\Gamma_M^{(3)} \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} \gamma_{1i}\gamma_{2j}\left((-1)^i\tilde{\delta}_1\Gamma_M^{(2)}(z_2) + (-1)^j\tilde{\delta}_2\Gamma_M^{(1)}(z_1) + (-1)^{i+j}\tilde{\delta}_1\tilde{\delta}_2\Gamma_M^{(3)}\right) \end{aligned}$$

with $\Gamma_M^{(1)}(z_1)$*,* $\Gamma_M^{(2)}(z_2)$ *and* $\Gamma_M^{(3)}$ *defined in* (5.8)*.*

Remark 5.2 *The terms* Γ_{M_z}*,* $M \in \{F, G, H, D_c\}$ *have coefficients containing the vertices of the variables* F_z*,* G_z*,* H_z *and* D_{cz} *and, therefore, they do not introduce new variables in the LMIs.*

Remark 5.3 *In Section 5.4, the numerical examples found solution with* $\lambda_1 = 0$ *and* $\lambda_2 = 1$*, and these values are suggested for an initial guess. We verify that only in particular cases, where the* *stability of the closed-loop systems is a difficult task, there is a need to perform a bisection search on these parameters, which can be seen as an extra degree of freedom for challenging cases.*

Remark 5.4 *A decay rate specification given by $\gamma > 0$ for the trajectories of the closed-loop system is obtained by replacing Υ defined in* (5.18) *with*

$$\Upsilon = \begin{bmatrix} A_z X + B_z(H_z + \Gamma_{H_z}) + 2\gamma X & A_z + B_z(D_{cz} + \Gamma_{D_z})C_z + 2\gamma I \\ F_z + \Gamma_{F_z} + 2\gamma I & Z A_z + (G_z + \Gamma_{G_z})C_z + 2\gamma Z \end{bmatrix}$$

in the conditions of Theorem 5.1 and Corollary 5.1.

5.4 Numerical Examples

The applicability of the proposed method is illustrated by numerical examples. To handle the infinite dimensional problem described by the parameter-dependent conditions, the optimization variables are fixed as parameter-dependent matrices and the negativity of the inequalities is verified by testing a finite set of LMIs that are directly obtained by ROLMIP (Robust LMI Parser) toolbox (Agulhari et al., 2012). The scalar parameters that must be provided in Theorem 5.1 have been selected following the suggestions presented in Remark 5.3. A line search algorithm could be used as well, probably yielding improved results, at the expense of increasing the computational burden.

Example 5.1 The nonlinear system is adapted from Tognetti et al. (2012) and given by

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + \sin x_3 - 0.1x_4 + (x_1^2 + 1)u \\ \dot{x}_2 = x_1 - 2x_2 + \varphi_1 u \\ \dot{x}_3 = x_1 + x_1^2 x_2 - 0.3x_3 \\ \dot{x}_4 = \sin x_3 - x_4 \\ y_1 = x_2 + (x_1^2 + 1)x_4 + \varphi_2 \\ y_2 = x_1 \end{cases} \tag{5.26}$$

where

$$\varphi_1 = \frac{\sin x_3}{x_3}, \quad \varphi_2 = \sin x_3. \tag{5.27}$$

The system (5.26) is rewritten as $\dot{x} = A(z)x + B(z)u$, $y = C(z)x$, with

$$A(z) = \begin{bmatrix} 1 & 1 & z_2 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & z_1 & -0.3 & 0 \\ 0 & 0 & z_2 & -1 \end{bmatrix}, \; B(z) = \begin{bmatrix} z_1 + 1 \\ z_2 \\ 0 \\ 0 \end{bmatrix},$$
$$C(z) = \begin{bmatrix} 0 & 1 & z_2 & z_1 + 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{5.28}$$

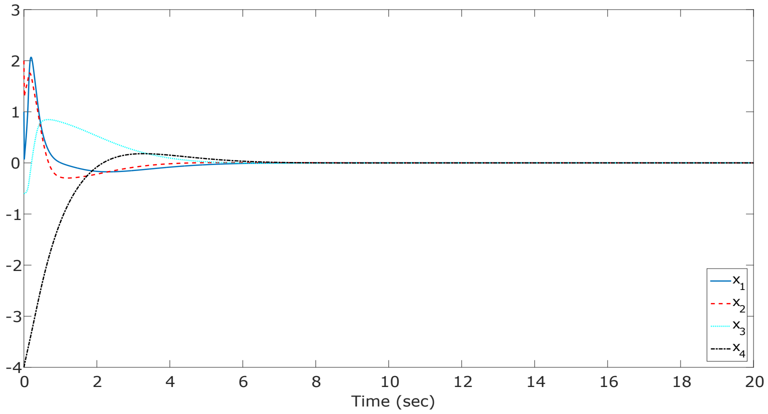

Figure 5.1: Time response of system states.

The premise variables are given by

$$z_1 = x_1^2, \qquad z_2 = \frac{\sin x_3}{x_3}.$$

The matrices A_z*,* B_z *and* C_z *of the T-S fuzzy model* (2.9) *are obtained from* (5.28) *evaluated in the extreme values of* z_1 *and* z_2 *for the domain* $x_1 \in [-1.4,\ 1.4]$ *and* $x_3 \in [-0.7,\ 0.7]$ *using the sector nonlinearity approach Tanaka and Wang (2001). Hence,* $z_1 \in [0,\ 1.96]$ *and* $z_2 \in [0.92,\ 1]$*.*

Considering the upper bound of additive uncertainties as $\delta_1 = 0.1$ *and* $\delta_2 = 0.09$*, we have* $\theta_1 \in [-0.1,\ 0.1]$ *and* $\theta_2 \in [-0.09,\ 0.09]$*. From* (5.25) *one has* $\tilde{\delta}_1 = 0.51$ *and* $\tilde{\delta}_2 = 0.12$*, therefore* $\tilde{\theta}_1 \in [-0.51,\ 0.51]$ *and* $\tilde{\theta}_2 \in [-0.12,\ 0.12]$*.*

Using Theorem 5.1, the closed loop system is stable and the time states response is shown in Figure 5.1.

Example 5.2 Consider the following nonlinear system from Ichalal et al. (2011)

$$\begin{cases} \dot{x}_1(t) = -\frac{F}{J}x_1(t) + K_m\frac{L}{J}x_2(t)^2 - \frac{C_r(t)}{J} \\ \dot{x}_2(t) = -K_m\frac{L}{L_t}x_2(t)x_1(t) - \frac{R_t}{L_t}x_2(t) - \frac{U(t)}{L_t} \\ y_1 = x_2 \end{cases} \tag{5.29}$$

where $F = 0.1N/(rad.s)$*,* $J = 30.1N/(rad.s)$*,* $K_m = 0.04329$*,* $L = 0.06H$*,* $C_r(t)$ *is the resisting torque,* $U(t)$ *is the motor voltage,* $L_t = L + l$*, but as* $L >> l$*, then* $L_t = L$*.* $R_t = R + r$*, where* $R = 0.01485\Omega$ *and* $r = 0.00989\Omega$*.*

System (5.29) is rewritten as $\dot{x} = A(z)x(t) + Bu(t)$*,* $y = Cx(t)$*, with*

$$\begin{aligned} A(z) &= \begin{bmatrix} -0.003 & 0.000086z_1 \\ -0.043z_1 & -0.412 \end{bmatrix}, \\ B &= \begin{bmatrix} -0.033 & 0 \\ 0 & 16.667 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \end{aligned} \tag{5.30}$$

where premise variable $z_1 = x_2$.

The matrix A_z *of the T-S fuzzy model* (2.9) *is obtained from* (5.30) *evaluated in the extreme values of* z_1 *for the domain* $x_2 \in [-100,\ 300]$ *using the sector nonlinearity approach (Tanaka and Wang, 2001). Hence,* $z_1 \in [-100,\ 300]$.

Therefore, the vertices of A_z *are given by*

$$A_1 = \begin{bmatrix} -0.003 & 0 \\ 0 & -0.412 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.003 & 0.035 \\ -17.317 & -0.412 \end{bmatrix},$$

and the membership functions are

$$\begin{cases} \mu_1(z_1) = \dfrac{\overline{z_1} - z_1}{\overline{z_1} - \underline{z_1}} = \dfrac{400 - z_1}{400 - 0} \\ \mu_2(z_1) = \dfrac{z_1 - \underline{z_1}}{\overline{z_1} - \underline{z_1}} = \dfrac{z_1 - 0}{400 - 0} \end{cases}$$

Consider the upper bound of additive uncertainty as $\delta_1 = 100$*, then* $\theta_1 = [-100,\ 100]$*. From* (5.25) *one has* $\tilde{\delta}_1 = 0.25$*, therefore* $\tilde{\theta}_1 = [-0.25,\ 0.25]$.

Figures 5.2 and 5.3 show the state trajectories and control signals, respectively, for the extreme scenario $\tilde{\theta}_1 = \tilde{\delta}_1$.

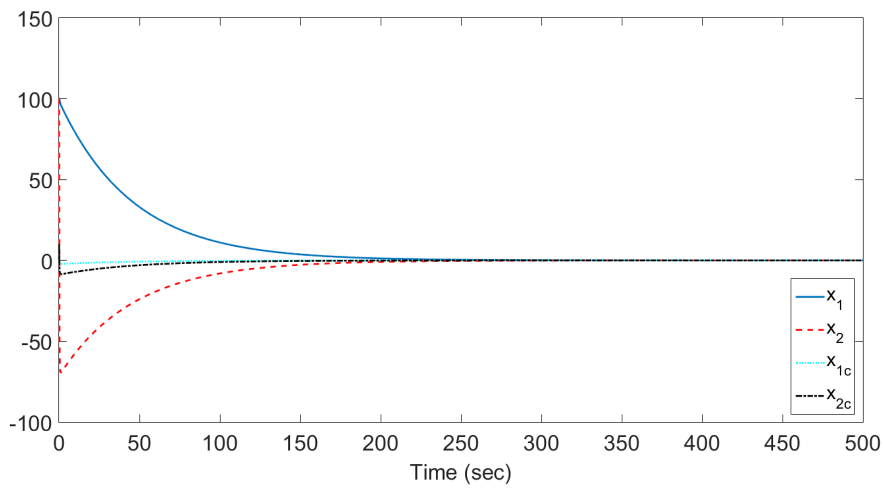

Figure 5.2: Trajectories of system and controller states of Example 5.2 .

5.5 Conclusion

This chapter provides an answer to the problem of imprecise measurement of the premise variables for the control law. As a contribution, new LMI conditions for the design of DOF controllers when there are uncertainties on the premise variables and, consequently, absolute uncertainties in the membership functions. The main appeal of the technique is to manipulate the fuzzy summations to obtain design conditions that can take into account the relation between

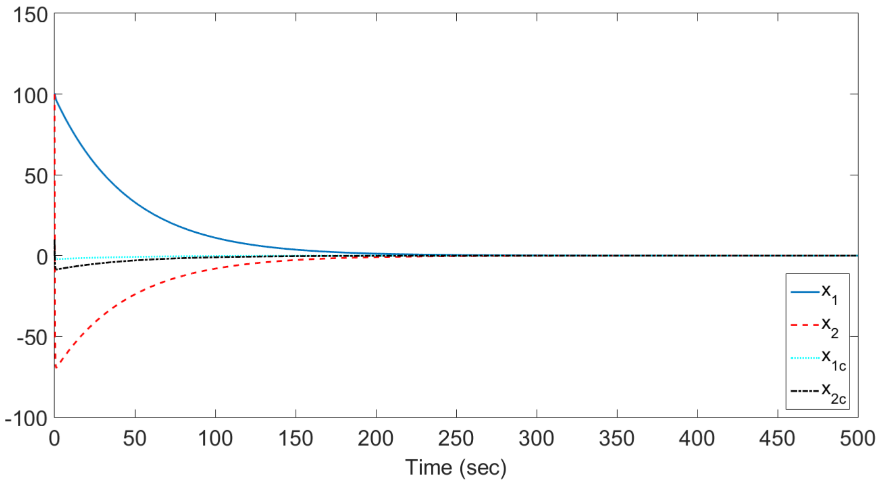

Figure 5.3: Control signals of Example 5.2 .

membership functions depending on the measured and the real premise variable, yielding less conservative results than the ones that consider independent variables. The effectiveness and validity of the proposed approach are illustrated through numerical examples and time simulations.

6 Conclusions

This work presents new LMI conditions for designing full-order DOF controllers for continuous or discrete-time T-S systems by selecting the membership functions used in the control law. Systems with partial or inaccurate measurement of premise variables are considered in the design. The proposed conditions solve a common problem in the standard technique of DOF design using a change of variables, the dependence of the controller gains on the membership functions of the plant on the dynamic output feedback design problem. With this approach, we avoid using alternative techniques, such as the two-stage approach, that result in the open problem of selecting an optimal state feedback controller before finding the DOF gains.

As an indirect result, we can also design robust controllers for systems with norm-bounded uncertainties or when no premise variable is available for the control law. The use of fuzzy Lyapunov functions can be easily encompassed in continuous-time systems using upper bounds on the derivative of the membership functions.

For discrete-time systems, we consider saturation on the control input. The novelty in the context of output feedback control design is the model of the variation rate of the membership functions that appears due to the use of fuzzy Lyapunov functions, thanks to the mean value theorem. Then, an estimate of the domain of attraction of the origin is obtained that does not need to be invariant if the controller has zero initial condition.

Finally, the problem of inaccurate measurement of the premise variables with addictive uncertainty is considered using the tools developed for the partial measurement of premise variables. The technique focuses on manipulating the fuzzy summations using the relation between membership functions depending on the measured and the real premise variable, yielding less conservative results than those considering independent variables. The proposed approach can also represent uncertainties in the membership functions that arrive in approximation-based fuzzy models or simply due to uncertainties in the nonlinear terms of the model.

6.1 Future works

The following ideas can be explored in future works:

- To consider T-S systems subjected to persistent and finite energy exogenous disturbances.
- The presence of multiplicative uncertainties in the measurement of the premise variables.

- To explore less conservative models for uncertainties in the nonlinear terms described by the premise variables in the sector nonliterary approach.
- The design of switched DOF controllers to deal with immeasurable premise variables.

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