Annali di Matematica Pura ed Applicata (1923 -) (2021) 200:2213–2233
https://doi.org/10.1007/s10231-021-01077-7

Positive solutions of the prescribed mean curvature equation with exponential critical growth

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Received: 10 September 2019 / Accepted: 6 February 2021 / Published online: 1 March 2021

Abstract

In this paper, we are concerned with the problem

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=f(u) \ \text{ in } \ \Omega, \quad u=0 \ \text{ on } \ \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and $f:\mathbb{R}\to\mathbb{R}$ is a superlinear continuous function with critical exponential growth. We first make a truncation on the prescribed mean curvature operator and obtain an auxiliary problem. Next, we show the existence of positive solutions of this auxiliary problem by using the Nehari manifold method. Finally, we conclude that the solution of the auxiliary problem is a solution of the original problem by using the Moser iteration method and Stampacchia's estimates.

Keywords Prescribed mean curvature problem · Critical exponential growth · Nehari manifold method · Moser iterations · Stampacchia estimates

Mathematics Subject Classification Primary 35J60 · Secondary 35C20 · 35B33 · 49J45

1 Introduction

In this paper, we are concerned with the existence of positive solutions for the following prescribed mean curvature problem with Dirichlet boundary condition

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$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \qquad (P)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. The function f is given, and we seek a solution u satisfying (P).

Since the left-hand side is the mean curvature of the graph of u, problem (P) is a prescribed mean curvature equation whose prescription depends on the location of the graph. Problems of this type have been studied starting with the pioneering contributions of Gethardt [15] and Miranda [25] who constructed $H^{1,1}$ solutions, respectively BV solutions of the prescribed mean curvature equation with Dirichlet boundary condition. We also refer to the seminal paper by [18], where there are established necessary and sufficient conditions for the existence of solutions in a particular case but without boundary conditions. Moreover, prescribed mean curvature equation has been the object of extensive studies in the past due to arises from some problems associated with differential geometry and physics such as combustible gas dynamics [4–7, 10, 14, 19, 31] and also due to the close connection with the capillary problem. For example, radial solution of (P) in $\mathbb{R}^N$ when $f(u)$ is replaced by κu has been studied in the context of the analysis of capillary surfaces, as can be seen in [9, 13, 16, 20, 22, 28] and [32].

Recently, by using variational methods, Obersnel and Omari [29] have considered the existence and multiplicity of positive solutions to problem (P) with respect to the behavior of the nonlinearity near the origin and at infinity. In the references of [29], the reader will find different contributions to the study of the prescribed mean curvature equation.

To state our main result, we need some hypotheses. The hypotheses on the continuous function f are the following:

(f_1) There exists $\alpha_0 > 0$ such that the function $f(t)$ satisfies

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = 0 \quad \text{for} \quad \alpha > \alpha_0 \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = \infty \quad \text{for} \quad \alpha < \alpha_0.$$

(f_2) The following limit holds:

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 0.$$

Moreover, $f(t) = 0$ for all $t \le 0$.

(f_3) The function $\quad t \mapsto \dfrac{f(t)}{t} \quad$ is increasing in $(0, +\infty)$.

(f_4) There exist $r > \frac{32}{7}\sqrt{2}$ and $\tau > \tau^*$ such that

$$f(t) \ge \tau t^{r-1},$$

for all $t \ge 0$, where

$$\tau > \tau^* := \max\left\{ \left[\frac{16r\sqrt{2}}{7r - 32\sqrt{2}} \frac{c_r \alpha_0}{\pi}\right]^{r-2/2}, \frac{K_2}{\delta}\left[\frac{32r\sqrt{2}}{7r - 32\sqrt{2}} c_r\right]^{r-2/2}, 1 \right\}$$

and the constant c_r will appear in the Sect. 4, $K_2 > 0$ will appear in Lemma 5.1, and $\delta > 0$ will appear in (3.4).

(f_5) The following inequality holds:

$$0 < rF(t) \leq f(t)t,$$

for all $t > 0$, where $F(t) := \int_0^t f(s)\mathrm{d}s$.

The main result of this paper establishes the following existence and regularity property.

Theorem 1.1 *Assume that conditions* $(f_1) - (f_5)$ *hold. Then, problem* (P) *has a positive solution* $u \in C^1(\overline{\Omega})$.

Hypothesis (f_1) is closely related to the Trudinger–Moser inequality and establishes that the function f has an exponential critical growth in $\mathbb{R}^2$.

We would like to highlight that our theorem can be applied for the model nonlinearity

$$f(t) = \tau t^{r-1} \exp(\alpha_0 t^2) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad f(t) = 0, \quad \text{for all} \quad t \leq 0, \tag{1.1}$$

where τ and r are the constants in (f_4) and α_0 is the constant in (f_1).

Nonlinear problems with exponential growth have been considered recently by Alves and de Freitas [1], Alves and Santos [2], Ambrosio [3], Figueiredo and Severo [12], Li, Santos and Yang [23], Medeiros, Severo and Silva [24], etc.

There are some recent papers to prescribe mean curvature problem in two-dimensional case. In [27] the authors studied the prescribed mean curvature problem with nonhomogeneous boundary condition. More precisely, the authors investigate the boundary behavior of variational solutions of problem (P) at smooth boundary points where certain boundary curvature conditions are satisfied. In [11] the authors show a nonexistence result. To the best of our knowledge, the main result in this paper is the first work on the problem of medium curvature in dimension two and non-linearity with critical exponential growth.

The plan of the paper is as follows. We first associate to problem (P) a related nonhomogeneous auxiliary problem with Dirichlet boundary condition. In Sect. 3 we study the variational structure of this auxiliary nonlinear problem and we establish several qualitative properties of the associated energy functional. The key abstract tools in these arguments are the Trudinger–Moser inequality and the Nehari manifold method. Next, minimizing the energy function on the Nehari manifold, we prove the existence of solutions to the auxiliary problem. In the final section of this paper, we prove that the solution of the auxiliary problem is a solution of the original problem. This is essentially done by using the Moser iteration method and Stampacchia's estimates. We refer to the recent monograph by Papageorgiou, Rădulescu and Repovš [30] for some of the abstract methods used in this paper.

2 An auxiliary problem

Consider the following auxiliary problem

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^2)\nabla u\right) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{Aux}$$

where

$$a(s)=\begin{cases}\dfrac{1}{\sqrt{1+s}} & s\in[0,1],\\ \dfrac{(s-2)^2+7}{8\sqrt{2}} & s\in[1,2),\\ \dfrac{7}{8\sqrt{2}} & s\in[2,\infty).\end{cases}$$

Lemma 2.1 *The function $a:\mathbb{R}_+\to\mathbb{R}_+$ is decreasing and of C^1 class. Moreover, it satisfies the following conditions:*

(a_1) $\dfrac{7}{8\sqrt{2}}\le a(s^2)\le 1$, for all $s\ge 0$.
(a_2) $a'(s)s\le 0<a(s)$, for all $s\ge 0$.
(a_3) *The function*

$$s\mapsto A(s^2)\ \text{ is convex for }\ s\ge 0,$$

where $A(s)=\displaystyle\int_0^s a(t)\mathrm{d}t$.

Proof Since

$$a'(s)=\begin{cases}\dfrac{-1}{2\sqrt{(1+s)^3}} & s\in[0,1],\\ \dfrac{s-2}{4\sqrt{2}} & s\in[1,2),\\ 0 & s\in[2,\infty),\end{cases}$$

the items (a_1) and (a_2) follow by straightforward computation. For item (a_3) note that

$$(A(s^2))''=2[a(s^2)+2s^2a'(s^2)].$$

If we define $b(s):=a(s)+2sa'(s)$, we can prove that b is strictly decreasing in $[0,\frac{6}{5}]$, strictly increasing in $[\frac{6}{5},2]$ and constant in $[2,+\infty)$. Then,

$$(A(s^2))''\ge b\Big(\frac{6}{5}\Big)=\frac{19}{40\sqrt{2}}>0,$$

for all $s\ge 0$ and this completes the proof. □

In this section, we prove some auxiliary results which will be very useful throughout the paper.

Lemma 2.2 *If $(a_1)-(a_2)$ are true, then:*

(i) *The function $s\mapsto a(s^2)s$ is increasing.*

(*ii*) *For all* $x, y \in \mathbb{R}^2$, *we have*

$$\langle a(|x|^2)x - a(|y|^2)y, x - y\rangle \geq \frac{7}{8\sqrt{2}}|x-y|^2. \tag{2.1}$$

Proof In order to prove (*i*), note that using $(a_1)-(a_2)$, we get

$$(a(s^2)s)' = 2s^2a'(s^2) + a(s^2) = b(s^2) > 0.$$

Let us prove (*ii*). Firstly, note that for $z \in \mathbb{R}^2$, we have

$$\frac{\partial}{\partial z_i}(a(|z|^2)z_j) = a(|z|^2)\delta_{ij} + 2a'(|z|^2)z_iz_j,$$

where we have denoted δ_{ij} by the Kronecker delta. Hence, for all $z, \xi \in \mathbb{R}^2$ we get

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial z_i}(a(|z|^2)z_j)\xi_i\xi_j = a(|z|^2)|\xi|^2 + 2a'(|z|^2)\sum_{i,j=1}^{2} z_iz_j\xi_i\xi_j. \tag{2.2}$$

Since

$$\sum_{i,j=1}^{2} z_iz_j\xi_i\xi_j = \left(\sum_{j=1}^{2} z_j\xi_j\right)^2, \tag{2.3}$$

we have

$$\sum_{i,j=1}^{2} z_iz_j\xi_i\xi_j = |z|^2|\xi|^2\cos^2(\theta) \quad \text{for some } \theta \in [0, 2\pi).$$

Thus, using (2.2) we deduce that

$$\begin{aligned}\sum_{i,j=1}^{2} \frac{\partial}{\partial z_i}(a(|z|^2)z_j)\xi_i\xi_j &= |\xi|^2[a(|z|^2) + 2a'(|z|^2)|z|^2\cos^2(\theta)]\\ &= |\xi|^2[a(|z|^2) + 2a'(|z|^2)|z|^2 + 2a'(|z|^2)|z|^2(\cos^2(\theta)-1).\end{aligned}$$

From (a_2), we have that

$$2a'(|z|^2)|z|^2(\cos^2(\theta)-1) \geq 0.$$

Hence,

$$\begin{aligned}\sum_{i,j=1}^{2} \frac{\partial}{\partial z_i}(a(|z|^2)z_j)\xi_i\xi_j &\geq |\xi|^2[a(|z|^2) + 2a'(|z|^2)|z|^2]\\ &= b(|z|^2)|\xi|^2 \geq \frac{7}{8\sqrt{2}}|\xi|^2 \quad \text{for all} \quad z, \xi \in \mathbb{R}^2.\end{aligned} \tag{2.4}$$

Now for $z = y + t(x-y)$, $t \in [0,1]$ and $\xi = x - y$, we have

$$\langle a(|x|^2)x - a(|y|^2)y, x-y\rangle = \sum_{j=1}^{2}(a(|x|^2)x_j - a(|y|^2)y_j)(x_j-y_j)$$
$$= \int_0^1 \sum_{i,j=1}^{2} \frac{\partial}{\partial z_i}(a(|z|^2)z_j)\xi_i\xi_j \mathrm{d}t.$$

Finally, using (2.4) we get (2.1). □

The next result due to Stampacchia [33] will be useful in the arguments used in this paper.

Lemma 2.3 *Let $B(\eta)$ be a given C^1 vector field in $\mathbb{R}^2$ and $f(x, s)$ a bounded Carathéodory function in $\Omega\times\mathbb{R}$. Let $u\in H_0^1(\Omega)$ be a solution of*

$$-\mathrm{div}\,(B(\nabla u)) = f(x,u) \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

that is,

$$\int_\Omega B(\nabla u)\cdot\nabla\varphi = f(x,u)\varphi \qquad \forall\varphi\in H_0^1(\Omega).$$

Assume that there exists $0<\nu<K$ such that

$$\nu|\xi|^2 \le \sum_{i,j=1}^{2}\frac{\partial B_i}{\partial\eta_j}(\nabla u)\xi_i\xi_j \quad \text{and} \quad \left|\frac{\partial B_i}{\partial\eta_j}(\nabla u)\right| \le K, \tag{2.5}$$

for all $i,j=1,2$ and $\xi\in\mathbb{R}^2$. Then, $u\in W^{2,p}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$, for all $\alpha\in(0,1)$ and for all $p\in(1,\infty)$. Moreover,

$$\|u\|_{1,\alpha} \le O(\nu, K, \Omega, \|f(\cdot,u)\|_\infty). \tag{2.6}$$

In the following result, we show that the differential operator involved in (Aux) verifies conditions (2.5).

Lemma 2.4 *Assume that hypotheses $(a_1)-(a_2)$ are fulfilled. Then, for all $u\in H_0^1(\Omega)$, the second-order differential operator $B(\nabla u)=a(|\nabla u|^2)\nabla u$ satisfies (2.5) of Lemma 2.3.*

Proof Note that

$$\frac{\partial B_i}{\partial\eta_j}(\eta) = \frac{\partial}{\partial\eta_j}(a(|\eta|^2)\eta_i) = a(|\eta|^2)\delta_{ij} + 2a'(|\eta|^2)\eta_i\eta_j,$$

and then

$$\sum_{i,j=1}^{2}\frac{\partial B_i}{\partial\eta_j}(\nabla u)\xi_i\xi_j = a(|\nabla u|^2)|\xi|^2 + 2a'(|\nabla u|^2)\sum_{i,j=1}^{2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\xi_i\xi_j.$$

Now, observing (2.2), we can repeat the reasoning of the proof of Lemma 2.2, and using $(a_1)-(a_2)$, we conclude that

$$\sum_{i,j=1}^{2} \frac{\partial B_i}{\partial \eta_j}(\nabla u)\xi_i\xi_j \geq \frac{7}{8\sqrt{2}}|\xi|^2.$$

On the other hand, using $(a_1) - (a_2)$, we get

$$\left|\frac{\partial B_i}{\partial \eta_j}(\nabla u)\right| = \left|a(|\nabla u|^2)\delta_{ij} + 2a'(|\nabla u|^2)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right|$$
$$\leq a(|\nabla u|^2) + |a'(|\nabla u|^2)||\nabla u|^2 \leq K,$$

for some positive constant $K > 0$. The proof is now complete. □

3 The variational framework and some technical lemmas

Note that, by the hypothesis (a_1), we have that the functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x$$

is well defined, where $F(t) = \displaystyle\int_0^t f(s)\mathrm{d}s$.

Moreover, we have

$$I'(u)\phi = \int_\Omega a(|\nabla u|^2)|\nabla u\nabla\phi\ \mathrm{d}x - \int_\Omega f(u)\phi\ \mathrm{d}x,$$

for all $\phi \in H_0^1(\Omega)$. Thus, I is a $\mathcal{C}^1$ functional on $H_0^1(\Omega)$ and its critical points are weak solution of problem (Aux).

The Nehari manifold associated to the functional I is given by

$$\mathcal{N} = \{u \in H_0^1(\Omega)\backslash\{0\} : J(u) = 0\},$$

where $J(u) = I'(u)u$ for $u \in H_0^1(\Omega)$. Let us start with the following important result due to Trudinger [34] and Moser [26].

Theorem 3.1 *For every $u \in H_0^1(\Omega)$ and $\alpha > 0$*

$$\exp(\alpha u^2) \in L^1(\Omega) \tag{3.1}$$

and there is a constant $M > 0$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)}\leq 1} \int_\Omega \exp(\alpha u^2)\mathrm{d}x \leq M, \tag{3.2}$$

for every $\alpha \leq 4\pi$.

Moreover, if $\alpha > 4\pi$, then

$$\sup_{\|u\|_{H_0^1(\Omega)}\leq 1}\int_\Omega \exp(\alpha u^2)\mathrm{d}x=\infty. \tag{3.3}$$

Note that, from (f_2), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t)|\leq \varepsilon |t| \tag{3.4}$$

and

$$|F(t)|\leq \frac{1}{2}\varepsilon |t|^2, \tag{3.5}$$

for all $0 < t \leq \delta$.

Furthermore, from (f_1), given $\alpha > \alpha_0$, there exists $K > 0$ such that

$$|f(t)|\leq \varepsilon \exp(\alpha t^2),$$

for all $t \geq K$. In particular, we get

$$|f(t)|\leq \frac{\varepsilon}{K} t \exp(\alpha t^2), \tag{3.6}$$

with implies

$$|f(t)t|\leq \frac{\varepsilon}{K}t^2\exp(\alpha t^2)\leq \frac{\varepsilon}{K^{q-2}}|t|^q\exp(\alpha t^2)=C_\varepsilon |t|^q\exp(\alpha t^2), \tag{3.7}$$

where $C_\varepsilon=\frac{\varepsilon}{K^{q-2}}$. Moreover, from (3.6), we get

$$|F(t)|\leq \frac{\varepsilon}{\alpha K}\exp(\alpha t^2)\leq \frac{\varepsilon}{\alpha_0 K^{q+1}}|t|^q\exp(\alpha t^2)=\tilde{C}_\varepsilon |t|^q\exp(\alpha t^2), \tag{3.8}$$

for all $t \geq K$, for all $\alpha > \alpha_0$, for all $q \geq 0$ with $\tilde{C}_\varepsilon=\frac{\varepsilon}{\alpha_0 K^{q+1}}$.

Consequently, using (3.4), (3.5), (3.7) and (3.8), for all $\varepsilon > 0$ and for all $\alpha > \alpha_0$, there exists $C_\varepsilon > 0$ such that

$$\int_\Omega f(u)u\mathrm{d}x\leq \varepsilon\int_\Omega |u|^2\mathrm{d}x+C_\varepsilon\int_\Omega |u|^q\exp(\alpha|u|^2)\mathrm{d}x \tag{3.9}$$

and

$$\int_\Omega F(u)\mathrm{d}x\leq \frac{\varepsilon}{2}\int_\Omega |u|^2\mathrm{d}x+\tilde{C}_\varepsilon\int_\Omega |u|^q\exp(\alpha|u|^2)\mathrm{d}x, \tag{3.10}$$

for all $u \in H_0^1(\Omega)$. In particular, in this paper, we will use $q > 2$.

In the next result, we prove that $\mathcal{N}$ is not empty and that I restricted to $\mathcal{N}$ is bounded from below.

Lemma 3.1 *For each* $u \in H_0^1(\Omega)\setminus\{0\}$, *there exists a unique* $t > 0$ such that $tu \in \mathcal{N}$. *Moreover,* $I(u) > 0$ *for every* $u \in \mathcal{N}$.

Proof Given $u \in H_0^1(\Omega)\setminus\{0\}$, let $\mathcal{T}_u(t) = I(tu)$ for $t > 0$. Then, $tu \in \mathcal{N}$ if and only if $\mathcal{T}'_u(t) = 0$. Note that, taking $\varepsilon > 0$ sufficiently small in (3.9) and using (a_1) and Sobolev embedding, there exists $C > 0$ such that

$$\begin{aligned}\mathcal{T}_u(t) &= \frac{1}{2}\int_\Omega A(|t\nabla u|^2)\mathrm{d}x - \int_\Omega F(tu)\mathrm{d}x\\ &\geq \frac{7/8\sqrt{2}-C\varepsilon}{2}t^2\|u\|^2 - t^q\tilde{C}_\varepsilon\int_\Omega |u|^q\exp(\alpha|tu|^2)\mathrm{d}x.\end{aligned}$$

Using Hölder's inequality with $s', s > 1$, we get

$$\mathcal{T}_u(t) \geq \frac{7/8\sqrt{2}-C\varepsilon}{2}t^2\|u\|^2 - t^q\tilde{C}_\varepsilon\left(\int_\Omega |u|^{qs'}\mathrm{d}x\right)^{1/s'}\left(\int_\Omega \exp\left(\alpha s\|tu\|^2\left(\frac{u}{\|u\|}\right)^2\right)\mathrm{d}x\right)^{1/s}.$$

Choosing $\alpha > \alpha_0$ and $t_1 > 0$ such that $\alpha s\|t_1 u\|^2 < 4\pi$, using (3.2) we obtain

$$\mathcal{T}_u(t) \geq D_1 t_1^2 - D_2 t_1^q$$

for some $D_1, D_2 > 0$ and for all $0 \leq t \leq t_1$. Thus, since $2 < q$, there exists $0 < t^* \leq t_1$ such that $\mathcal{T}_u(t) > 0$ for all $0 < t < t^* \leq t_1$.

Now, from (a_1) and (f_4), we have

$$\frac{\mathcal{T}_u(t)}{t^2} \leq \frac{1}{2}\|u\|^2 - \frac{\tau}{r}t^{r-2}\int_{\Omega_u^+} u^r\mathrm{d}x,$$

where $\Omega_u^+ = \{x \in \Omega : u(x) > 0\}$. Therefore, since $r > 2$, we conclude $\lim_{t\to+\infty}\mathcal{T}_u(t) = -\infty$. Consequently, there exists at least one $t(u) > 0$ such that $\mathcal{T}_u'(t(u)) = 0$, that is, $t(u)u \in \mathcal{N}$.

Suppose, by contradiction, that there are $t > 0$ and $\widetilde{t} > 0$ such that

$$\int_\Omega a(|t\nabla u|^2)|\nabla u|^2\mathrm{d}x = \int_\Omega \frac{f(tu)}{t}u\mathrm{d}x$$

and

$$\int_\Omega a(|\widetilde{t}\nabla u|^2)|\nabla u|^2\mathrm{d}x = \int_\Omega \frac{f(\widetilde{t}u)}{\widetilde{t}}u\mathrm{d}x.$$

Then,

$$\int_\Omega \left[a(|t\nabla u|^2)|\nabla u|^2 - a(|\widetilde{t}\nabla u|^2)|\nabla u|^2\right]\mathrm{d}x = \int_\Omega\left[\frac{f(tu)}{t} - \frac{f(\widetilde{t}u)}{\widetilde{t}}\right]u\mathrm{d}x.$$

If $t > \widetilde{t}$, from Lemma 2.1 and (f_3), we have

$$0 > \int_\Omega \left[a(|t\nabla u|^2)|\nabla u|^2 - a(|\widetilde{t}\nabla u|^2)|\nabla u|^2\right]\mathrm{d}x = \int_\Omega\left[\frac{f(tu)}{t} - \frac{f(\widetilde{t}u)}{\widetilde{t}}\right]u\mathrm{d}x > 0,$$

which is a contradiction. In the same way, we obtain that we cannot have the case $t < \widetilde{t}$. We conclude that there is a unique parameter $t > 0$ such that $t(u)u \in \mathcal{N}$. Note, in particular, that $t(u)$ is a global maximum point of $\mathcal{T}_u$ and $\mathcal{T}_u(t(u)) > 0$, i.e. $I(t(u)u) > 0$. Since $t(u) = 1$ if $u \in \mathcal{N}$, we deduce that $I(u) > 0$ for every $u \in \mathcal{N}$. □

In the next result we prove that sequences in $\mathcal{N}$ cannot converge to 0.

Lemma 3.2 *There exists a constant* $C > 0$ *such that* $0 < C \leq \|u\|$, *for every* $u \in \mathcal{N}$.

Proof Suppose, by contradiction, that there is $(u_n) \subset \mathcal{N}$ such that

$$u_n \to 0 \quad \text{in} \quad H_0^1(\Omega). \tag{3.11}$$

Then, using (3.10), we have

$$\int_\Omega a(|\nabla u_n|^2)|\nabla u_n|^2 \mathrm{d}x = \int_\Omega f(u_n)u_n \mathrm{d}x \le \varepsilon \int_\Omega |u_n|^2 \mathrm{d}x + C_\varepsilon \int_\Omega |u_n|^q \exp(\alpha|u_n|^2)\mathrm{d}x.$$

Now, from (a_1) we get

$$\frac{7}{8\sqrt{2}} \int_\Omega |\nabla u_n|^2 \mathrm{d}x \le \varepsilon \int_\Omega |u_n|^2 \mathrm{d}x + C_\varepsilon \int_\Omega |u_n|^q \exp(\alpha|u_n|^2)\mathrm{d}x.$$

Using Sobolev embedding, there exists $C > 0$ such that

$$\left(\frac{7}{8\sqrt{2}} - C\varepsilon\right)\|u_n\|^2 \le C_\varepsilon \int_\Omega |u_n|^q \exp\left(\alpha\|u_n\|^2\left(\frac{|u_n|}{\|u_n\|}\right)^2\right)\mathrm{d}x.$$

Using Hölder's inequality with $s', s > 1$, we have

$$\left(\frac{7}{8\sqrt{2}} - C\varepsilon\right)\|u_n\|^2 \le C_\varepsilon\left(\int_\Omega |u_n|^{qs'}\mathrm{d}x\right)^{1/s'}\left(\int_\Omega \exp\left(\alpha s\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2\right)\mathrm{d}x\right)^{1/s}. \tag{3.12}$$

Note that by (3.11), there is $n_0 \in \mathbb{N}$ such that

$$\|u_n\|^2 < \frac{4\pi}{\alpha s}$$

for all $n \ge n_0$ and for some $\alpha > \alpha_0$. Then, from (3.2) and Sobolev embedding again, we have

$$\left(\frac{7}{8\sqrt{2}} - C\varepsilon\right)\|u_n\|^2 \le MC_\varepsilon\left(\int_\Omega |u_n|^{qs'}\mathrm{d}x\right)^{1/s'} \le MC_\varepsilon C\|u_n\|^q.$$

This inequality implies

$$\frac{\left(\frac{7}{8\sqrt{2}} - C\varepsilon\right)}{MC_\varepsilon C} \le \|u_n\|^{q-2}.$$

Since $q > 2$, the above inequality contradicts (3.11) and the lemma is proved. □

We set $c := \inf_{\mathcal{N}} I$, and in the next result we will prove that minimizing sequence for c are bounded.

Lemma 3.3 *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for c, then (u_n) is bounded.*

Proof Suppose, by contradiction, that up to a subsequence, $\|u_n\| \to \infty$ and consider $v_n = \frac{u_n}{\|u_n\|} \rightharpoonup v_0$. If $v_0 = 0$, then for all $t > 0$, from (a_1) we obtain

$$
\begin{aligned}
c + o_n(1) = I(u_n) = I(\|u_n\| v_n) \geq I(tv_n) \\
\geq \frac{7}{16\sqrt{2}} t^2 - \int_\Omega F(tv_n)\,\mathrm{d}x,
\end{aligned}
$$

where $o_n(1) \to 0$ as $n \to +\infty$. Since $v_n \to 0$ in $L^q(\Omega)$ and $\|v_n\| < 4\pi$, using (3.10), Hölder inequality as in (3.12) and Theorem 3.1, we get

$$
\int_\Omega F(tv_n)\,\mathrm{d}x \to 0.
$$

But this last convergence implies

$$
c \geq \frac{7}{16\sqrt{2}} t^2, \quad \text{for all} \quad t > 0,
$$

which is a contradiction.

Suppose now that $v_0 \neq 0$. Then,

$$
\frac{1}{\|u_n\|^2} I(u_n) = \frac{c}{\|u_n\|^2} = o_n(1),
$$

where $o_n(1) \to 0$ as $n \to +\infty$. Hence, using (a_1) and (f_4), we get

$$
\frac{\tau}{r} \|u_n\|^{r-2} \int_\Omega |v_n|^r \mathrm{d}x \leq \int_\Omega \frac{F(\|u_n\| v_n)}{\|u_n\|^2} = \frac{1}{2\|u_n\|^2} \int_\Omega A(|\nabla u_n|^2 \mathrm{d}x + o_n(1) \leq \frac{1}{2} + o_n(1).
$$

Since $v_n \to v$ in $L^r(\Omega)$ and $\|u_n\| \to +\infty$, we have a contradiction. □

To end up this section, let us prove that if the minimum of I over $\mathcal{N}$ is achieved in some $u_0 \in \mathcal{N}$, then in fact u_0 is a critical point of I. This follows from some arguments used in [8, Proof of Theorem 1.1 (completed)].

Lemma 3.4 *If $u_0 \in \mathcal{N}$ is such that*

$$
I(u_0) = \min_{\mathcal{N}} I = c,
$$

then $I'(u_0) = 0$.

Proof Suppose, by contradiction, that c is achieved by u_0 and this one is not a critical point of I. Then, there exists $\phi \in H_0^1(\Omega)$ such that

$$
I'(u_0)\phi < 0.
$$

By the continuity of I', we can find $\widetilde{\varepsilon}, \widetilde{\delta} > 0$ small such that

$$
I'(t(u_0 + \sigma\phi))\phi < 0, \quad \text{for} \quad t \in [1 - \widetilde{\varepsilon}, 1 + \widetilde{\varepsilon}] \quad \text{and} \quad \sigma \in [-\widetilde{\delta}, \widetilde{\delta}]. \tag{3.13}
$$

Moreover, since the map $t \mapsto I(tu_0)$ attains its maximum at $t = 1$ as shown in the proof of Lemma 3.1, we have

$$
I'((1 - \widetilde{\varepsilon})u_0)u_0 = \mathcal{T}_u{}'(1 - \widetilde{\varepsilon}) > 0 > \mathcal{T}_u{}'(1 + \widetilde{\varepsilon}) = I'((1 + \widetilde{\varepsilon})u_0)u_0.
$$

Then, again by the continuity of I', there exists $\overline{\sigma} \in (0, \widetilde{\delta})$ such that

$$I'((1-\widetilde{\varepsilon})(u_0+\overline{\sigma}\phi)(u_0+\overline{\sigma}\phi) > 0 > I'((1+\widetilde{\varepsilon})(u_0+\overline{\sigma}\phi)(u_0+\overline{\sigma}\phi),$$

i.e. $\mathcal{T}'_{u+\overline{\sigma}\phi}(1-\widetilde{\varepsilon}) > 0 > \mathcal{T}'_{u+\overline{\sigma}\phi}(1+\widetilde{\varepsilon})$. It follows that

$$\bar{t}(u_0+\overline{\sigma}\phi) \in \mathcal{N}, \quad \text{for some} \quad \bar{t} \in (1-\widetilde{\varepsilon}, 1+\widetilde{\varepsilon}). \tag{3.14}$$

From (3.13), we have

$$I(\bar{t}(u_0+\overline{\sigma}\phi)) - I(u_0) \le I(\bar{t}(u_0+\overline{\sigma}\phi)) - I(\bar{t}u_0) = \bar{t}\int_0^{\overline{\sigma}} I'(\bar{t}(u_0+\sigma\phi))\phi d\sigma < 0,$$

so that

$$I(\bar{t}(u_0+\overline{\sigma}\phi)) < I(u_0) = c,$$

which contradicts (3.14) [also, because $I(u_0) = \min_{\mathcal{N}} I$]. Therefore, $I'(u_0) = 0$ and the proof is complete. □

4 Existence of solution to the auxiliary problem

In this section, in order to prove the existence of result in the exponential critical case, we consider the auxiliary problem

$$\begin{cases} -\Delta u = |u|^{r-2}u \ \text{ in } \ \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{A}$$

where r is the constant that appear in the hypothesis (f_4).

The energy functional associated to problem (A) is defined by

$$I_r(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{r}\int_\Omega |u|^r \mathrm{d}x.$$

We also define the Nehari manifold

$$\mathcal{N}_r = \{u \in H_0^1(\Omega); u \neq 0 \,:\, I_r'(u)u = 0\}.$$

Since the embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ is compact, using the mountain pass theorem and the classical maximum principle, we can prove that there exists a positive solution to problems (A) given by $w_r \in H_0^1(\Omega)$ such that

$$I_r(w_r) = c_r, \;\; I_r'(w_r) = 0$$

and

$$c_r = \frac{r-2}{2r}\int_\Omega |w_r|^r \mathrm{d}x, \tag{4.1}$$

where $c_r = \inf\limits_{\mathcal{N}_r} I_r$. The next result is an estimate to $c = \inf\limits_{\mathcal{N}} I$.

Lemma 4.1 *The value $c = \inf_{\mathcal{N}} I$ satisfies*

$$c \le \frac{c_r}{\tau^{2/(r-2)}}.$$

Proof Note that, by the hypotheses (a_1) and (f_4), we have

$$\int_\Omega a(|\nabla w_r|^2)|\nabla w_r|^2 dx \le \int_\Omega |\nabla w_r|^2 dx = \int_\Omega |w_r|^r \le \int_\Omega f(w_r)w_r dx.$$

This inequality implies that $I'(w_r)w_r \le 0$. Then, from (f_3), there exists $\beta \in (0, 1]$ such that $\beta w_r \in \mathcal{N}$. Using (a_1) and (f_4) again, we obtain

$$c \le I(\beta w_r) \le \frac{\beta^2}{2}\int_\Omega |\nabla w_r|^2 dx - \frac{\tau}{r}\beta^r \int_\Omega |w_r|^r dx.$$

Since $I_r'(w_r) = 0$, we conclude that

$$c \le \left[\frac{\beta^2}{2} - \tau\frac{\beta^r}{r}\right]\int_\Omega |w_r|^r dx.$$

Using (4.1), we have

$$c \le \left[\frac{\beta^2}{2} - \tau\frac{\beta^r}{r}\right]\frac{2rc_r}{(r-2)} \le \max_{s\ge 0}\left[\frac{s^2}{2} - \tau\frac{s^r}{r}\right]\frac{2rc_r}{(r-2)} = \frac{(r-2)}{2r}\frac{1}{\tau^{2/(r-2)}}\frac{2rc_r}{(r-2)} = \frac{c_r}{\tau^{2/(r-2)}}.$$

The proof is now complete. □

Lemma 4.2 *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for c, then*

$$\limsup_{n\to\infty} \|u_n\|^2 \le \frac{2\pi}{\alpha_0}.$$

Proof Note that

$$c + o_n(1) = I(u_n) - \frac{1}{r}I'(u_n)u_n = \frac{1}{2}\int_\Omega A(|\nabla u_n|^2)dx - \frac{1}{r}\int_\Omega a(|\nabla u_n|^2)|\nabla u_n|^2 dx + \frac{1}{r}\int_\Omega f(u_n)u_n dx - \int_\Omega F(u_n)dx.$$

From (a_1) and (f_5), we get

$$c + o_n(1) \ge \left(\frac{7}{32\sqrt{2}} - \frac{1}{r}\right)\int_\Omega |\nabla u_n|^2 dx = \left(\frac{7}{32\sqrt{2}} - \frac{1}{r}\right)\|u_n\|^2.$$

Since $r > \frac{32\sqrt{2}}{7}$, we obtain

$$\|u_n\|^2 \le \frac{32\sqrt{2}r}{(7r - 32\sqrt{2})}c + o_n(1)$$

By the estimate on c in Lemma 4.1, we find

$$\|u_n\|^2 \leq \frac{32\sqrt{2}r}{(7r-32\sqrt{2})}\frac{c_r}{\tau^{2/(r-2)}}+o_n(1).$$

Since $\tau > \tau^*$ in (f_4), then

$$\|u_n\|^2 < \frac{2\pi}{\alpha_0}+o_n(1)$$

and the result follows. ◻

The next result establishes some compactness properties of minimizing sequences.

Lemma 4.3 *If* $(u_n) \subset \mathcal{N}$ *is a minimizing sequence for* c, *then there exists* $u \in H_0^1(\Omega)$ *such that*

$$\int_\Omega f(u_n)u_n\mathrm{d}x \to \int_\Omega f(u)u\mathrm{d}x$$

and

$$\int_\Omega F(u_n)\mathrm{d}x \to \int_\Omega F(u)\mathrm{d}x.$$

Proof It is enough to prove the first limit, since the second one is analogous. By Lemma 4.2, we have

$$\limsup_{n\to\infty}\|u_n\|^2 \leq \frac{2\pi}{\alpha_0} \tag{4.2}$$

and, up to a subsequence, then there exists $u \in H_0^1(\Omega)$ such that

$$u_n(x) \to u(x) \quad \text{a.e. in} \quad \Omega$$

and

$$f(u_n(x))u_n(x) \to f(u(x))u(x) \quad \text{a.e. in} \quad \Omega.$$

Now it is sufficient to prove that there is $g : \mathbb{R} \to \mathbb{R}$ such that $|f(u_n)u_n| \leq g(u_n)$ with $(g(u_n))$ convergent in $L^1(\Omega)$.

Note that by the inequality (3.9) we have

$$|f(u_n(x))u_n(x)| \leq \varepsilon|u_n(x)|^2 + C_\varepsilon|u_n(x)|^q\exp\left(\alpha|u_n(x)|^2\right) := g(u_n(x)).$$

We will prove that $(g(u_n))$ is convergent in $L^1(\Omega)$. First, note that

$$\int_\Omega |u_n|^2\mathrm{d}x \to \int_\Omega |u|^2\mathrm{d}x. \tag{4.3}$$

Considering $s, s' > 1$ such that $\frac{1}{s}+\frac{1}{s'}=1$ and s close to 1, we get

$$|u_n|^q \to |u|^q \quad \text{in} \quad L^{s'}(\Omega). \tag{4.4}$$

Now choosing $\alpha > \alpha_0$ but close to 1, we have that

$$\int_\Omega \exp\left(\alpha s|u_n(x)|^2\right)\mathrm{d}x = \int_\Omega \exp\left(\alpha s\|u_n\|^2\left(\frac{|u_n(x)|}{\|u_n\|}\right)^2\right)\mathrm{d}x.$$

Using Lemma 4.2, we can conclude that

$$\int_\Omega \exp\left(\alpha s|u_n(x)|^2\right)\mathrm{d}x \le \int_\Omega \exp\left(4\pi\left(\frac{|u_n(x)|}{\|u_n\|}\right)^2\right)\mathrm{d}x$$

Then, it follows by Theorem 3.1 that there is $M > 0$ such that

$$\int_\Omega \exp\left(\alpha s|u_n(x)|^2\right)\mathrm{d}x \le M.$$

Since

$$\exp\left(\alpha|u_n(x)|^2\right) \to \exp\left(\alpha|u(x)|^2\right) \quad \text{a.e. in} \quad \Omega,$$

we use [21, Lemma 4.8] and conclude that

$$\exp\left(\alpha|u_n|^2\right) \rightharpoonup \exp\left(\alpha|u|^2\right) \quad \text{in} \quad L^s(\Omega). \tag{4.5}$$

Now using (4.4), (4.5) and [21, Lemma 4.8] again, we conclude

$$\int_\Omega f(u_n)u_n\mathrm{d}x \to \int_\Omega f(u)u\mathrm{d}x.$$

The proof is complete. □

Theorem 4.1 *The auxiliary problem has a nonnegative solution $v_0 \in H_0^1(\Omega)$.*

Proof Consider $(u_n) \subset \mathcal{N}$ a minimizing sequence for c. Then, by Lemma 3.3, (u_n) is bounded in $H_0^1(\Omega)$ and, up to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in} \quad H_0^1(\Omega).$$

We claim that $u_0 \not\equiv 0$. Indeed, if $u_0 \equiv 0$, then, from (a_1) and Lemma 4.3, we get

$$\frac{7}{8\sqrt{2}}\|u_n\|^2 \le \int_\Omega a(|\nabla u_n|^2)|\nabla u_n|^2\mathrm{d}x = \int_\Omega f(u_n)u_n\mathrm{d}x \to 0,$$

which implies

$$\|u_n\| \to 0,$$

contradicting Lemma 3.2. Let $t_0 > 0$ such that $v_0 := t_0u_0 \in \mathcal{N}$. Since by (a_3) the function $s \mapsto A(s^2)$ is convex, we get $\int_\Omega A(|\nabla tu_0|^2)\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega A(|\nabla tu_n|^2)\mathrm{d}x$. From Lemma 4.3, we infer that $\int_\Omega F(tu_0)\mathrm{d}x = \lim_{n\to\infty} \int_\Omega F(tu_n)\mathrm{d}x$. Then,

$$c \le I(v_0) \le \liminf_{n\to\infty} I(t_0 u_n).$$

From Lemma 3.1, we conclude that

$$\liminf_{n\to\infty} I(t_0 u_n) \le \liminf_{n\to\infty} \max_{t\ge 0} I(tu_n) = \liminf_{n\to\infty} I(u_n) = c.$$

The equality $I'(v_0) = 0$ is a consequence of Lemma 3.4. Since $f(t) = 0$, for all $t \le 0$, we get $v_0 \ge 0$ in Ω. □

5 Proof of Theorem 1.1

We first establish some estimates on solutions of the auxiliary problem from which the existence of positive solution of problem (P) will be deduced. Let us point out that the classical elliptic regularity theory [17] cannot be applied immediately because the coefficients in the differential operator are not necessarily continuous. Throughout this section, we assume that $\gamma = 7/(8\sqrt{2})$ and $\Gamma = 1$.

Lemma 5.1 *If $v_0 \in H_0^1(\Omega)$ is a solution of the auxiliary problem, then $v_0 \in L^\infty(\Omega)$ and there exists $K_2 > 0$ not depending on v_0 such that*

$$\|v_0\|_\infty \le K_2 \left[\frac{4r}{(r\gamma - 4\Gamma)} \frac{c_r}{\tau^{2/(r-2)}} \right]^2.$$

Proof Since $I(v_0) = c$ and $I'(v_0) = 0$, arguing as Lemma 4.2, we have

$$\|v_0\|^2 \le \frac{4r}{(r\gamma - 4\Gamma)} \frac{c_r}{\tau^{2/(r-2)}}. \tag{5.1}$$

Considering $\tau > \tau^*$ in (f_4), then

$$\|v_0\|^2 < \frac{2\pi}{\alpha_0}. \tag{5.2}$$

In what follows, let $R > R_1 > 0$ with $R > 1$ and take a cutoff function $\eta_R \in C_0^\infty(\Omega)$ such that $0 \le \eta_R \le 1$, $\eta_R \equiv 0$ in B_R^c, $\eta_R \equiv 1$ in B_{R_1} and $|\nabla \eta_R| \le C/R$, where $B_R(0) \subset \Omega$ is a ball in $\mathbb{R}^2$ and $C > 0$ is a constant.

Define for $L > 0$,

$$v_{L,0}(x) = \begin{cases} v_0(x), & \text{if} \quad v_0(x) \le L \\ L, & \text{if} \quad v_0(x) \ge L, \end{cases}$$

$$z_{L,0} = \eta_R^2 v_{L,0}^{2(\sigma-1)} v_0 \quad \text{and} \quad w_{L,0} = \eta_R v_0 v_{L,0}^{\sigma-1}$$

with $\sigma > 1$ to be determined later. Taking $z_{L,0}$ as a test function we obtain

$$I'(v_0) z_{L,0} = 0.$$

In other words,

$$\int_\Omega a(|\nabla v_0|^2)\nabla v_0 \nabla z_{L,0}\mathrm{d}x = \int_\Omega f(v_0) z_{L,0}\mathrm{d}x.$$

By (3.9), (5.2) and Theorem 3.1, we obtain

$$\int_\Omega a(|\nabla v_0|^2)\nabla v_0 \nabla z_{L,0}\mathrm{d}x \le \epsilon \int_\Omega v_0 z_{L,0}\mathrm{d}x + MC_\epsilon \left(\int_\Omega |v_0^{q-1} z_{L,0}|^{s'}\mathrm{d}x\right)^{1/s'}.$$

Using $z_{L,0}$ and (a_1), we obtain

$$\begin{aligned}\gamma \int_\Omega \eta_R^2 v_{L,0}^{2(\sigma-1)}|\nabla v_0|^2\,\mathrm{d}x \le &- \int_\Omega a(|\nabla v_0|^2)\eta_R v_0 v_{L,0}^{2(\sigma-1)}\nabla \eta_R \nabla v_0\,\mathrm{d}x\\ &- 2(\sigma-1)\int_\Omega a(|\nabla v_0|^2) v_{L,0}^{(2\sigma-3)} v_0 \nabla v_0 \nabla v_{L,0}\mathrm{d}x\\ &+ \epsilon \int_\Omega |v_0|^2 \eta_R^2 v_{L,0}^{2(\sigma-1)}\mathrm{d}x + MC_\epsilon\left(\int_\Omega |v_0^q \eta_R^2 v_{L,0}^{2(\sigma-1)}|^{s'}\mathrm{d}x\right)^{1/s'}.\end{aligned}$$

The definition of $v_{L,0}$ implies

$$-2(\sigma-1)\int_\Omega a(|\nabla v_0|^2) v_{L,0}^{(2\sigma-3)} v_0 \nabla v_0 \nabla v_{L,0}\mathrm{d}x \le 0.$$

Thus, by (a_1) again

$$\begin{aligned}\gamma \int_\Omega \eta_R^2 v_{L,0}^{2(\sigma-1)}|\nabla v_0|^2\,\mathrm{d}x \le &\,\Gamma C_1 \int_\Omega \eta_R v_0 v_{L,0}^{2(\sigma-1)}|\nabla \eta_R||\nabla v_0|\,\mathrm{d}x\\ &+ \epsilon \int_\Omega |v_0|^2 \eta_R^2 v_{L,0}^{2(\sigma-1)}\mathrm{d}x + MC_\epsilon\left(\int_\Omega |v_0^q \eta_R^2 v_{L,0}^{2(\sigma-1)}|^{s'}\mathrm{d}x\right)^{1/s'}.\end{aligned}$$

Taking $\widetilde{\tau} > 0$ and using Young's inequality, we obtain

$$\begin{aligned}\gamma \int_\Omega \eta_R^2 v_{L,0}^{2(\sigma-1)}|\nabla v_0|^2\,\mathrm{d}x \le &\,\Gamma C_1 \int_\Omega \left(\widetilde{\tau}\eta_R^2|\nabla v_0|^2 + C_{\widetilde{\tau}} v_0^2 |\nabla \eta_R|^2\right) v_{L,0}^{2(\sigma-1)}\,\mathrm{d}x\\ &+ \epsilon \int_\Omega |v_0|^2 \eta_R^2 v_{L,0}^{2(\sigma-1)}\mathrm{d}x + MC_\epsilon\left(\int_\Omega |v_0^q \eta_R^2 v_{L,0}^{2(\sigma-1)}|^{s'}\mathrm{d}x\right)^{1/s'}.\end{aligned}$$

Choosing $\widetilde{\tau}$ sufficient small, it follows that

$$\begin{aligned}\int_\Omega \eta_R^2 v_{L,0}^{2(\sigma-1)}|\nabla v_0|^2\,\mathrm{d}x \le C_2\Bigg(&\int_\Omega v_0^2 v_{L,0}^{2(\sigma-1)}|\nabla \eta_R|^2\,\mathrm{d}x\\ &+ \epsilon \int_\Omega |v_0|^2 \eta_R^2 v_{L,0}^{2(\sigma-1)}\mathrm{d}x + MC_\epsilon\left(\int_\Omega |v_0^q \eta_R^2 v_{L,0}^{2(\sigma-1)}|^{s'}\mathrm{d}x\right)^{1/s'}\Bigg).\end{aligned} \tag{5.3}$$

On the other hand, we get

$$S_\Upsilon \|w_{L,0}\|^2_{L^\Upsilon(\Omega)} \le \int_\Omega \left|\nabla\left(\eta_R v_0 v_{L,0}^{\sigma-1}\right)\right|^2$$
$$\le \int_\Omega |v_0|^2 v_{L,0}^{2(\sigma-1)} |\nabla \eta_R|^2 + \int_\Omega \eta_R^2 \left|\nabla\left(v_0 v_{L,0}^{\sigma-1}\right)\right|^2,$$

where S_Υ is the best Sobolev constant of $H^1_0(\Omega)$ in $L^\Upsilon(\Omega)$ and $\Upsilon.1$ that will fix after. But

$$\int_\Omega \eta_R^2 \left|\nabla\left(v_0 v_{L,0}^{\sigma-1}\right)\right|^2 = \int_{\{|v_0|\le L\}} \eta_R^2 \left|\nabla\left(v_0 v_{L,0}^{\sigma-1}\right)\right|^2 + \int_{\{|v_0|>L\}} \eta_R^2 \left|\nabla\left(v_0 v_{L,0}^{\sigma-1}\right)\right|^2$$
$$= \int_{\{|v_0|\le L\}} \eta_R^2 |\nabla v_0^\sigma|^2 + \int_{\{|v_0|>L\}} \eta_R^2 L^{2(\sigma-1)} |\nabla v_0|^2$$
$$\le \sigma^2 \int_\Omega \eta_R^2 v_{L,0}^{2(\sigma-1)} |\nabla v_0|^2,$$

and therefore,

$$\|w_{L,0}\|^2_{L^\Upsilon(\Omega)} \le C_3 \sigma^2 \left(\int_\Omega |v_0|^2 v_{L,0}^{2(\sigma-1)} |\nabla \eta_R|^2 + \int_\Omega \eta_R^2 v_{L,0}^{2(\sigma-1)} |\nabla v_0|^2 \right).$$

From this estimate and (5.3),

$$\|w_{L,0}\|^2_{L^\Upsilon(\Omega)} \le C_4 \sigma^2 \int_\Omega |v_0|^2 v_{L,0}^{2(\sigma-1)} |\nabla \eta_R|^2$$
$$+ \sigma^2 C_4 \left(\epsilon \int_\Omega |v_0|^2 \eta_R^2 v_{L,0}^{2(\sigma-1)} \mathrm{d}x + M C_\epsilon \left(\int_\Omega |v_0^q \eta_R^2 v_{L,0}^{2(\sigma-1)}|^{s'} \mathrm{d}x \right)^{1/s'} \right). \quad (5.4)$$

for every $\sigma > 1$.

The above expression, the properties of η_R and $v_{L,0} \le v_0$ imply that

$$\|w_{L,0}\|^2_{L^r(\Omega)} \le C_4 \sigma^2 \int_\Omega |v_0|^{2\sigma} |\nabla \eta_R|^2$$
$$+ \sigma^2 C_4 \left(\epsilon \int_\Omega |v_0|^{2\sigma} \eta_R^2 \mathrm{d}x + M C_\epsilon \left(\int_\Omega \eta_R^2 ||v_0|^{q-2} |v_0|^{2\sigma}|^{s'} \mathrm{d}x \right)^{1/s'} \right). \quad (5.5)$$

Taking

$$t := \frac{qq}{2(q-2)} > 1, \quad \Upsilon := \frac{2t}{t-1}, \quad (5.6)$$

then we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (5.5) to get

$$\|w_{L,0}\|^2_{L^\Upsilon(\Omega)} \le C_4 \sigma^2 \|v_0\|^{2\sigma}_{L^{\sigma\Upsilon}(B_R)} \left(\int_{B_R} |\nabla \eta_R|^{2t} \right)^{1/t}$$
$$+ C_4 \sigma^2 \|v_0\|^{2\sigma}_{L^{\sigma\Upsilon}(B_R)} \left(\int_{B_R} |\eta_R|^{2t} \right)^{1/t} \quad (5.7)$$
$$+ M_1 C_\epsilon C_4 \sigma^2 \|v_0\|^{2\sigma}_{L^{\sigma\Upsilon}(B_R)} \left(\int_{B_R} |v_0|^{qq/2} \right)^{1/t}.$$

Since η_R is constant on $B_{R_1} \cup B_R^c$ and $|\nabla \eta_R| \leq C/R$, we conclude that

$$\int_{B_R} |\nabla \eta_R|^{2t} = \int_{B_R \setminus B_{R_1}} |\nabla \eta_R|^{2t} \leq \frac{C_5}{R^{2t-2}} \leq C_5. \tag{5.8}$$

We have used $R > 1$ and $2t > q > 2$ in the last inequality.

Considering

$$\int_\Omega |v_0|^{qq/2} \leq K,$$

we can use (5.7) and (5.8) to conclude that

$$\|w_{L,0}\|^2_{L^\Upsilon(\Omega)} \leq C_6 \sigma^2 \|v_0\|^{2\sigma}_{L^{\sigma\Upsilon}(B_R)}.$$

Since

$$\begin{aligned}
\|v_{L,0}\|^{2\sigma}_{L^{\sigma\Upsilon}(B_{R_1})} &= \left(\int_{B_{R_1}} v_{L,0}^{\sigma\Upsilon} \right)^{2/\Upsilon} \\
&\leq \left(\int_\Omega \eta_R^\Upsilon |v_0|^\Upsilon v_{L,0}^{\Upsilon(\sigma-1)} \right)^{2/\Upsilon} \\
&= \|w_{L,0}\|^2_{L^\Upsilon(\Omega)} \leq C_6 \sigma^2 \|v_0\|^{2\sigma}_{L^{\sigma\Upsilon}(\Omega)},
\end{aligned}$$

we can apply Fatou's lemma in the variable L and Sobolev embedding to obtain

$$\|v_0\|_{L^{\sigma\Upsilon}(B_{R_1})} \leq C_7^{1/\sigma} \sigma^{1/\sigma} \|v_0\|.$$

Here, C_7 is a positive constant independent on R. Iterating this process, for each $k \in \mathbb{N}$, it follows that

$$\|v_0\|_{L^{\sigma^k\Upsilon}(B_{R1})} \leq C_7^{\sum_{i=1}^k \sigma^{-i}} \sigma^{\sum_{i=1}^m i\sigma^{-i}} \|v_0\|.$$

Since Ω can be covered by a finite number of balls $B^j_{R_1}$, we have that

$$\|v_0\|_{L^{\sigma^k\Upsilon}(\Omega)} \leq \sum_j^{finite} \|v_0\|_{L^{\sigma^k\Upsilon}(B^j_{R_1})} \leq \sum_j^{finite} C_7^{\sum_{i=1}^k \sigma^{-i}} \sigma^{\sum_{i=1}^m i\sigma^{-i}} \|v_0\|.$$

Using (5.1) and since $\sigma > 1$, we let $k \to \infty$ to get $K_2 > 0$ such that

$$\|v_0\|_{L^\infty(\Omega)} \leq K_2 \|v_0\| \leq K_2 \left[\frac{4r}{(r\gamma - 4\Gamma)} \frac{c_r}{\tau^{2/(r-2)}} \right]^2.$$

The proof is now complete. □

5.1 Proof of Theorem 1.1 completed

By Lemma (5.1) and (f_4), we obtain

$$\|v_0\|_{L^\infty(\Omega)} \leq \delta.$$

Now from (3.4), given $\epsilon > 0$, we get

$$\|f(v_0)\|_{L^\infty(\Omega)} \leq \epsilon \|v_0\|_{L^\infty(\Omega)} \leq \epsilon\delta.$$

Using Lemma 2.3, then $v_0 \in C^{1,\alpha}(\Omega)$, and for $\epsilon > 0$ sufficient small, we have

$$\|\nabla v_0\|_{L^\infty(\Omega)} < 1.$$

The proof of Theorem 1.1 is now complete. □

Acknowledgements The authors thank a knowledgeable referee for very rigorous enlightening comments, which considerably improved the initial version of this paper. Giovany M. Figueiredo was partially supported by CNPq, Capes and Fap-DF - Brazil. The work of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Education and Research, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2020-0068, within PNCDI III.

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