OPINION DYNAMICS CONTROL AS A CONSENSUS PROBLEM OF MULTI-AGENT SYSTEMS: AN LMI APPROACH

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DISSERTAÇÃO DE MESTRADO SUBMETIDA AO DEPARTAMENTO DE ENGENHARIA ELÉTRICA DA FACULDADE DE TECNOLOGIA DA UNIVERSIDADE DE BRASÍLIA COMO PARTE DOS REQUISITOS NECESSÁRIOS PARA A OBTENÇÃO DO GRAU DE MESTRE.

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O thou in whom hopes are placed, thy lofty aspiration is drawing this (poem) God knows wither. - Rumi

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RESUMO ESTENDIDO

Título: CONTROLE DE DINÂMICA DE OPINIÃO COMO UM PROBLEMA DE CONSENSO DE SISTEMAS MULTI-AGENTES: UMA ABORDAGEM LMI
Autor: Daniel Rostami Alkhorshid
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Brasília, 27 de maio de 2022

Enquanto que um acordo em sistemas multi-agentes (MAS) pode ser assegurado impondo algumas propriedades de conectividade entre os agentes, o consenso resultante depende das condições iniciais e da topologia da rede. Nesse contexto, nosso principal objetivo nesta dissertação é influenciar o valor do consenso de sistemas multi-agentes em direção a um valor desejado. A estabilidade assintótica e a maximização do domínio de atração para o modelo bilinear representando a dinâmica de opinião na presença de limitação na amplitude e na energia da ação de controle para uma rede fixa e conectada está sendo investigada. Usando a teoria dos grafos algébricos e desigualdades matriciais lineares (LMI), fornecemos condições suficientes para garantir a convergência dos agentes em direção ao consenso desejado. Além disso, exemplos numéricos mostram a eficiência do método proposto.

Palavras-chave: Sistemas multi-agentes, Dinâmica de opinião, Desigualdades matricias lineares, Sistemas bilineares.

ABSTRACT

Title: Opinion dynamics control as a consensus problem of multi-agent systems: an LMI approach
Author: Daniel Rostami Alkhorshid
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While reaching an agreement in multi-agent systems (MAS) can be ensured by enforcing some connectivity properties between agents, the resulted consensus depends on their initial conditions and the network topology. In this context, our main objective in this manuscript is to sway the consensus value of multi-agent systems towards a desired value. The asymptotic stability and maximization of the domain of attraction for the bilinear model representing the opinion dynamics in the presence of limited control action for a fixed and connected network is being investigated. By using algebraic graph theory and linear matrix inequality (LMI), we provide sufficient conditions guaranteeing the convergence of agents toward the desired consensus. Furthermore, examples are being driven to display the effectiveness of the proposed method.

Keywords: Multi-agent systems, Opinion dynamics, Linear matrix inequality, Bilinear systems.

SUMMARY

LIST OF FIGURES

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LIST OF SYMBOLS AND NOTATIONS

SOME NOTATIONS

$X_{(i)}$	Denotes the i-th row of matrix X
$\mathbf{diag}(v)$	A diagonal matrix composed of the elements of the vector v
$He\{X\}$	Notation that stands for $X + X^T$
I_n	The identity matrix of order n
X^{-1}	Inverse of a square matrix X
$\mathcal{L}_2^n[0,\infty)$	Denotes the space of square-integrable vectors of n functions over $[0,\infty)$
$X_{(ij)}$	Denotes the i-th row and the j-th column of matrix X
$\mathbb{R}^{n\times m}$	The space of real matrices with dimension $n \times m$
$X \ll 0$	Indicates that all entries of matrix X are negative
$\mathbb{1}_m$	Stands for a vector of ones of dimension m
$\otimes$	Denotes the Kronecker product
$X_\perp$	Matrix whose columns form a basis for the null space of X
$X \gg 0$	Indicates that all entries of matrix X are positive
$X < 0$	Indicates that matrix X is negative definite
$\star$	Denotes symmetric blocks in partitioned matrices
$X > 0$	Indicates that matrix X is positive definite
X^T	Transpose of matrix X
$\mathbb{R}_+^n$	The space of vectors of dimension n with real positive entries
$0_{m\times n}$	The null matrix of $m \times n$ order

LIST OF ACRONYMS AND ABBREVIATIONS

LMI Linear Matrix Inequalities. 1

MAS Multi-Agent System. 1

1 INTRODUCTION

Multi-agent systems (MASs) are composed of multiple autonomous agents that work toward reaching a specific goal. Due to their applicability in many research fields such as biology, robotic teams, power grids, opinion dynamics etc., they have recently received tremendous attention. The flexibility that MASs offer has made them instrumental in several engineering fields like artificial intelligence and control engineering.

Since the main focus of this manuscript is on the opinion dynamics, it is essential to introduce the basic parts of a MAS for such application. To address the workspace in which a MAS operate, it is vital to mention the following definitions [1].

- Agent: An entity on the network capable of collecting information and making decisions toward a specific goal.
- Network: The environment in which the agents communicate and take action. The network could have continuous or discrete characteristics as well as fixed or time-varying topology.
- Parameters: The presented information in the network that could be collected by the agents.
- Action: an operation performed by an agent or an external entity appearing as changes in some of the network and agent characteristics.

It is noteworthy that in MASs, every agent has a limited view of the network characterized by its connections with neighboring agents and local collected parameters. From this point of view, one can decompose a complex task (that can not be addressed by a single system) into divided simpler ones, and use agents to fulfill each divided task in order to reach a desired solution.

The applications of MASs is not limited to engineering problems. The agent-based models and their local rules are widely used to produce the collective behavior in different species as they could be easily implemented by numerical simulations [2]. To name a few, in herds of mammals, individuals use the information of their neighbors to remain in the herd for its certain advantages [3], small birds like pigeons fly in coordinated flocks with impressive synchronization in directional movements and landing [4], and fish schools organize themselves in large groups and manage their movement with respect to the nearest neighbors [5]. The real world behavior of the mentioned species is displayed in Fig. 1.1.


(a) Swarm of birds.


(b) School of fish.


(c) Mammal herds.

Figure 1.1 – Different species with collective behavior in nature. (Google Images)

Thus, it could be concluded that cooperative behaviors exist in nature due to the certain advantages that it creates in the process of achieving a common and certain goal. In this manuscript, we are interested in the concept of opinion dynamics. Opinion dynamics is the study of how opinions progress and the evolution during the interactions between agents in a social group [6]. The collective behavior of different species could be interpreted in terms of opinion dynamics. Moreover, this concept appears in different fields such as finance and business and multiple engineering fields. Since in MASs, we usually aim to reach a particular outcome or a common goal (consensus), it is reasonable to employ the concept of opinion dynamics in MASs as it creates a framework in which, opinions evolution of a group of agents belonging to a MAS could be studied. Hence, this combination of characteristics and dynamics could study and manipulate a particular consensus in MASs. In the view of opinion dynamics, consensus is defined as an accepted opinion that a group of agents tend to agree on. Such concept in practice appears when a particular social group of agents tend to reach a common ground through their interactions. For instance, imagine a group of travelers that want to decide on the duration of their trip. As every individual in the group may have a different and unique opinion about it, they would discuss it through their interaction to finally agree on a unanimous decision (consensus) at the end. Such concept has a great appeal in some of the engineering fields and marketing business. For instance, a producer might be interested in swaying a group of potential customers into buying a certain product. Opinion dynamics could be useful in this regard, as it can create a framework to study how a considered budget for such operation should be spent and in what way, to be able to persuade the average opinions of potential customers into buying the product of interest in an optimal way. In other words, opinion dynamics could also be used to manipulate the final common ground or goal of a network of agents and in this regard, it could be useful in some of control problems of the fields related to MASs. As we would observe in later sections, an external control input (for instance, a campaign) through opinion dynamics framework has the ability to modify the average opinion of a group of agents through its influence defined by the dynamics of agents' opinions evolution.

1.1 LITERATURE REVIEW

1.1.1 Consensus in multi-agent systems

The concept of consensus in MASs could be interpreted as an agreement by autonomous agents on a distinct value [7, 8, 9, 10], through their operations and interactions with a number of nearest neighbors present in the network. A great review exist on the consensus algorithms in networks of MASs which gives an extensive view of consensus in a variety of networks with different characteristics [11]. Such problems often arise in distributed systems subject to control or coordination challenges. The consensus problem mainly divides to two categories, the leader-follower consensus in which the follower agents aim to reach to the leader coordinates and the the leaderless consensus in which all the autonomous agents should agree on a common state [12]. These types of problems could be linked to their similar projection in the nature. As depicted in Fig 1.1b and 1.1a, their network is quite similar to the leaderless consensus while all the individuals converge toward a certain consensus (like flying or swimming in a certain direction), while in Fig 1.1c, the herd usually follows a leader toward reaching certain coordinates.

There are multiple literature concerned by the design of consensus algorithms for agents with linear dynamics. With respect to this topic, there are works on fixed and time-variant network properties [13, 14], directed and undirected graphs [11, 15], synchronous and asynchronous interactions [16, 17], transmission of information with and without delay [8, 9], etc. Moreover, there are many works related to the consensus problem in the presence of input saturation [18, 19] and the control effort constraints [18, 20, 21].

By using the great review of [22], we are able to discuss the literature of consensus problem in MASs. the starting point of consensus problem goes back to "Parimutuel" linear method for agreement on a consensus of individual distributions for subjective probability distribution [23]. Likewise, DeGroot in [24] presented a solution to address the linear consensus problem using stochastic matrix by the DeGroot model. Afterwards, the linear adaptive control of Markov chain was presented in [25]. Also in 1981, Berger enhanced the DeGroot model in [26], and proved that reaching to a consensus depends on the initial values of opinions in DeGroot model. Some years later, in [27] and [28] an agreement-based protocol by the form of a linear algorithm for asynchronous problem in parallel computing was proposed. Moreover, Lynch in [29] proposed a generalization concept of distributed linear algorithm for the networks with fixed topology. Later, Olfati-Saber and Murray presented nonlinear and linear consensus protocol for distributed systems using undirected graph theory [30].

Saturation in control signal is a well-known problem associated with consensus problem in MASs. The saturation effect makes the closed-loop system behavior nonlinear. It may lead to instability or prevention in reaching the consensus. Generally, there are two main techniques to deal with saturation in linear systems [31]. The first approach is to take into account the saturation effect in obtaining the performance related conditions which creates a trade-off performance [32, 33]. The second approach is to obtain the performance related conditions without taking saturation into account and then, introduce some mechanisms or modifications to minimize the saturation impact on the system [34].

Another concept that is related to consensus in MASs is state constraint. Some existing researches on consensus consider the assumption that no state constraint exist, and it can simplify the design conditions and convergence analysis. However, in engineering problems state constraints exist as an interpretation of physical constraint and power limitations [35]. Consensus was studied in [36] while agents should stay in their own conex sets. An extension of [36] for unballanced graphs with communication delays was considered in [37]. Moreover, for a continuous-time system case in [38], a gradient based consensus algorithm was proposed for undirected graphs and additional input based on logarithmic barrier function were employed to prevent violation in the constraints [35]. Additionally, in [39] the authors consider consensus control of MAS where the states of the agents lie within individually-defined constraints.

Consensus of positive MASs is another important topic. A system is considered positive if for non-negative initial conditions, all of its states remain in non-negative orthant [40]. To name a few practical examples for positive MASs, we can mention the coordination of multiple vehicle system [41], and wireless sensor network for physical variables monitoring [42, 43]. To achieve sufficient conditions for guaranteeing positivity, some researches have been done to solve consensus problem of positive MASs [44, 45, 46, 47]. In [44], using a static output feedback, a positive consensus problem is solved for positive agents. Moreover, in [45, 46, 47], the authors solved the consensus problem of positive nature by state feedback control law. Also, in [48], the positive consensus problem is solved using state feedback and observer based framework.

In power engineering field, as it is discussed in [49], MASs are applied for some large scale applications such as Simulation of power system and modeling [50], smart grid operation [51], management od power network [52], planning of power systems [53], substation automation [54] and electricity market [55], diagnostic and monitoring [56], protection [57] and control of power system [58]. To name some works about MASs in the other fields of engineering, in [59] the author has applied MASs alongside artificial neural network. There are also some work on Fuzzy logic using MASs [60, 61]. Moreover, there is a work on particle swarm optimization (a type of evolutionary algorithm) based MAS by [62]. Additionally, in [63] a new technique called MAPSO is designed based on MASs using particle swarm optimization. To name an application of MAS in another fields, a Markov decision process based MAS is applied to overcome obstacles of conventional maintenance policy optimization of aeroengines in [64].

1.1.2 Opinion Dynamics

As the social studies revealed, opinions tend to converge toward each other when the interaction exist. Thus, it is the reason why the consensus problem received attention in opinion dynamics related works such as [65, 66]. It is notable that while some models naturally converge toward consensus [24, 67], others might lead to clustering [10, 68, 69]. There are some recent studies about using consensus in social networks by controlling a few agents [70, 71]. Additionally, in [18], the authors have considered a discrete time implementation of the external control effort to alter the consensus value toward a desired one. In [72], the control of network opinion dynamics by a single selfish agent is discussed and in [73], the modeling and control of opinion dynamics over a time evolving network is presented.

As social phenomena became more and more popular in different fields of science, opinion dynamics as the process of opinions evolution became appealing to many researchers [74]. This manuscript is in accordance with opinion dynamics in viral marketing that is about sale strategies using word-of-mouth advertisement targeting potential buyers. The effectiveness of this method is well established by economists and social scientists [75, 76]. The work [77] presents the basic model of opinion dynamics at first. Years later, different models of opinion dynamics were presented with different opinion development rules that can be divided into continuous and discrete opinion form. For the continuous form, we can mention the FJ model [67], DeGroot model [24], HK model [10] and the CODA model [78] in which we have continuous opinions and discrete actions. On another hand, for the discrete form of opinions we can mention the voter model [79], majority rule model [80] and sznajd model [81].

Opinion dynamics is a powerful tool that could be used not only for the administration of opinions, but also for influence and guidance toward them in order to form a specific consensus point. In [82], the authors presented a management method of consensus through a number of interactions based on leadership. Moreover, [83] describes and compares opinion control algorithm for reaching a consensus in MASs subject to bounded opinion updates. The work [84] studies opinion consensus in a network with stubborn individuals and [85] establishes some efficient criteria for finite-time consensus of opinion dynamics for distributed optimization over digraph. Additionally, in [18] authors address the problem of opinion dynamics in a social network where opinions are influenced by the agents neighbors and an external influential entity.

1.2 OBJECTIVES AND CONTRIBUTIONS

Through this manuscript we aim to solve the consensus problem in multi-agent systems. Our main objectives in this study is to apply different control laws and obtain an estimation of the domain of attraction of the system to provide asymptotic stability of agents' trajectories toward the origin. To express it in details, our general objective is to use different control laws such as state feedback and take advantage of the Lyapunov stability concept to drive the equations. Furthermore, we aim to replace the bilinear product present in the system dynamics with different approaches and try to obtain a feedback gain such that all the trajectories converge to the desired consensus (the origin). The remainder of this manuscript is structured as follows.

- **Chapter 2:** This chapter define and discuss the instrumental preliminary concepts that play a crucial role in the development of the next chapter. From the characteristics provided by the definition of Laplacian matrix [30] to different saturation models in [86, 87] and positive system characteristics and the properties dictated by it in [40, 88] are proposed in this chapter in order to provide a general perspective to the reader.

- **Chapter 3:** This chapter is dedicated to the main contributions of our work that primarily focus on consensus control of networks of MASs. The first part is about defining the main problem we aim to solve and its formulation. Furthermore, several approaches are expressed in the form of multiple theorems to sufficiently solve the mentioned problem. To be precise, after expressing the main problem, we aim to exploit the characteristics of positive systems defined in [40] and replace the bilinear products of the system dynamics by different approaches alongside using the saturation models from [86] and [87]. The presented approaches are closely related to [18]. However, in contrast with the discrete-time case presented in [18], our work is focused on continuous-time case.

- **Chapter 4:** This chapter is dedicated to the numerical examples displaying the effectiveness of the proposed results. It consists of case studies represented by directed and undirected graphs in different orders that exploit the results of the mentioned approaches in solving similar problems in different networks.

The main contributions of this manuscript are presented in the following.

- To propose new models to represent bilinear dynamics in multi-agent systems. The main advantage of the proposed approach, based on norm-bounded uncertainties, is the applicability to large networks.

- To propose new approaches for the estimation of the domain of attraction of the closed-loop system subject to state constraints, bilinear product and amplitude saturation of the control signal. The first approach is based on invariant ellipsoids taking advantage of properties of positive systems. The second approach is based on invariant polyhedrals.

- To propose new methods to take into account the energy limitation of the control law. The first approach is based on state-feedback control law and the estimation of the region where the initial conditions can reach the origin. The second technique is based on constant control signal that linearizes the bilinear dynamics and minimize a cost function related to the performance of the states' trajectories.

2 PRELIMINARIES

In this chapter, we are going to explore the concepts of graph theory which are related to our work. This benefits us by creating a transparent image of the properties we would discuss further. Afterwards, we formulate the main problem that we aim to solve and discuss the saturation model and bilinear terms in the system dynamics. We conclude this chapter by expressing some important lemmas that are essential to further developments in the next chapters.

2.1 GRAPH THEORY

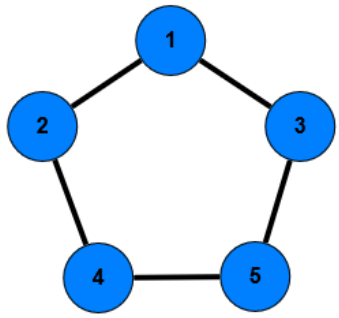
As we know, MASs are defined as the systems including a number of agents and their unique internal interaction links with each other that work cooperatively in achieving a certain common goal. To process such networks mathematically, it is vital to properly model their behavior using mathematical concepts. To this end, the MASs could be interpreted as graphs with every node representing each agent and every edge embodying a communication line between them.

There are two types of communication in literature concerning MASs represented with graphs. In the first case, the information exchanged between the agents can flow in both directions, that is, each agent can send and receive information with its neighbors. The graph with this characteristic is denoted undirected. The second case is the situation in which, each agent only send or receive information to-or-from other agents. This property is manifested using directed graphs.

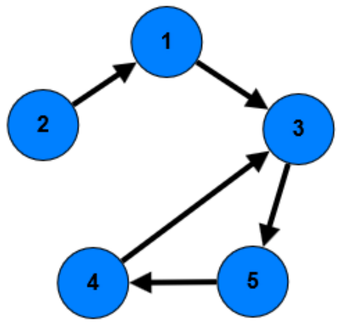
In Figure 2.1, different types of graphs are displayed. The part (a) displays an example for a network of agents presented with an undirected graph, and the part (b) displays the same agents with different connections. Each node in these manifestations represent an agent, and every edge of the respected graphs represent a connection between each particular pair of agents. In part (b) of Figure 2.1, the point of each arrow indicate the flow of information between their respective pair of agents. The next subsection is going to talk about the mathematical justifications and properties of these graphs.

2.1.1 Algebraic Interpretation of Graph Theory

A directed graph could be represented by $\mathcal{G}(\mathcal{V}, \xi)$, where $\mathcal{V} = \{v_1, \cdots, v_N\}$ is the set of graph nodes and $\xi \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges associated with this graph. If a pair (v_i, v_j)



(a) Undirected Graph.



(b) Directed Graph.

Figure 2.1 – Graphs representing different network characteristics.

belongs to the the set ξ, It manifests a directed communication line from agent i to the agent j. The adjacency matrix of such graph is defined through matrix $\mathcal{A} = [a_{ij}]$, for $a_{ij} > 0$ a weighted gain for its respective edge (v_i, v_j) and $a_{ij} = 0$ for the case where there is no connection. The Laplacian matrix associated with graph $\mathcal{G}(\mathcal{V}, \xi)$ is defined by

$$L = [l_{ij}] = \begin{cases} l_{ii} = \sum_{j=1}^{N} a_{ij}, & \forall i = 1, \ldots, N \\ l_{ij} = -a_{ij}, & i \neq j \end{cases} \tag{2.1}$$

It is evident that the diagonal entries of Laplacian matrix reflects the degree of each node that is equal to the sum of outgoing edges. The neighboring agent of every agent v_i is described by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \xi\}$, that is, v_j is a neighbor of v_i if $a_{ij} > 0$. In a directed graph case, since the Laplacian matrix L is not symmetric, imaginary eigenvalues are always expected. Although, in an undirected topology the Laplacian matrix becomes a symmetric matrix with real entries and thus, all of its eigenvalues are real.

As it is shown in Figure 2.2, every connection of the graph could be customized using gains associated with the Adjacency matrix. In this network topology, agent v_1 only sends information to agent v_3 and do not receive any information from it (directed connection), while agents v_2 and v_5 send and receive information toward each other (undirected connection). Under assumption that all the non-zero gains of the Adjacency matrix are equal to 1, we could obtain the following Adjacency and Laplacian matrix for the graph represented in

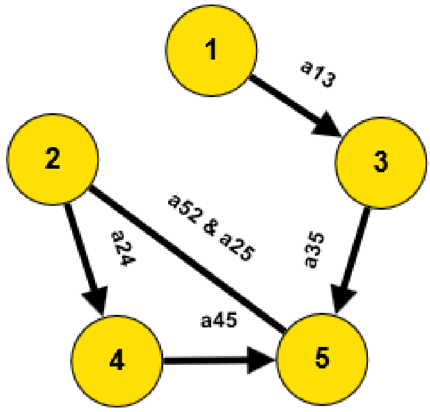


Figure 2.2 – A graph with weighted undirected and directed connections.

Figure 2.2 as

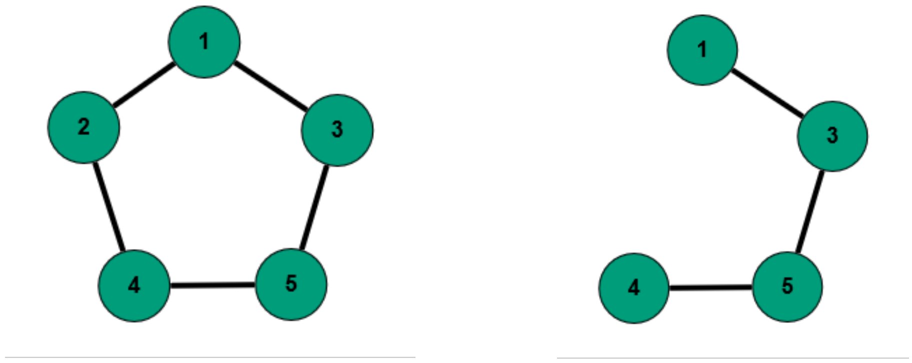
$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

2.1.2 Subgraphs

In this part we talk about a useful concept associated with graph theory. As it is shown, a graph $\mathcal{G}(\mathcal{V}, \xi)$ contains a number of vertices along with some defined edges involving them. The graph $\mathcal{G}_s(\mathcal{V}_s, \xi_s)$ is a subgraph of $\mathcal{G}(\mathcal{V}, \xi)$ if the following conditions are met

$$\mathcal{V}_s \subseteq \mathcal{V}$$
$$\xi_s \subseteq \xi$$

That is, it is possible to obtain a subgraph by eliminating some of the nodes and edges of the original graph. In figure 2.3 it is obvious that the sets of nodes and edges of the graph in (b) are a subset of the nodes and edges of graph in (a). The following subsection talks about the connectivity of a graph and its related concepts.



(a) Original Graph. (b) Subgraph.

Figure 2.3 – An undirected graph and its subgraph.

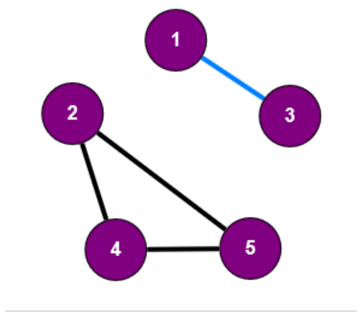


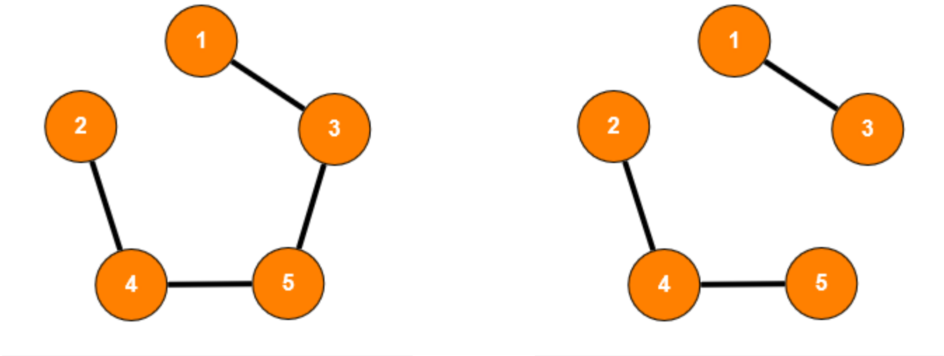
Figure 2.4 – Illustration of path and cycle in a graph.

2.1.3 Connectivity

There are some essential concept to define before discussing the connectivity of a graph. In a graph, a path is defined as the trajectory that connect a number of distinct vertices using a sequence of edges. A cycle is defined as a path where it return to its initial vertex by crossing some other vertices. Consider the illustration in Figure 2.4.

In the illustration, a path of the graph is shown in blue color. Additionally, a cycle of the graph is displayed with black color. As it is shown, one can start from v_5 and go through v_4 and then v_2 and finally return to v_5. It is obvious that a cycle contains multiple paths.

A connected graph is defined as a graph that all its vertices are linked with at least one path (for instance the graph in Figure 2.3a is connected). Moreover, a tree is a graph which every vertex of it is connected by only one path, and a forest is a graph with every vertex connected with a maximum of one path, that is, a set of unconnected uniting trees as illustrated in Figure 2.5.



(a) A tree. (b) A forest.

Figure 2.5 – An illustration of a tree and a forest.

Now we can introduce a useful definition of graph theory that is essential in studying consensus problem in MASs. A spanning tree of a graph is a tree that contains all of its vertices. It is an essential concept to be considered as a network of agents should be connected to achieve consensus [13].

2.1.4 Consensus Problem and linear consensus protocol

Consensus by its original meaning is the agreement of a set of agents on a common goal. A certain rule that governs the convergence process is called the consensus protocol. the consensus protocol is normally based on the interaction of an agent with its neighbors. Let us introduce some important concepts in this regard. Let x_i denote a scalar real value assigned to the node $v_i \in \mathcal{V}$. Then, $x = (x_1, \ldots, x_N)^T$ denotes the state of the graph $\mathcal{G}$.

LEMMA 2.1 ([30]) The Laplacian potential of a graph $\mathcal{G}$, defined as $\Psi_{\mathcal{G}}(x) = 0.5\, x^T L x$, is positive semi-definite such that

$$x^T L x = \sum_{i,j \in \xi} (x_i - x_j)^2$$

Additionally, by considering graph $\mathcal{G}$ as connected, $\Psi_{\mathcal{G}}(x) = 0$ if and only if $x_i = x_j,\ \forall i, j$.

proof.

We have $L = CC^T$ where C is the incidence matrix. Considering

$$x^T L x = x^T C C^T x = ||C^T x||^2$$

$x^T L x$ is positive semi-definite. Additionally, $\sum_{i,j \in \xi}(x_i - x_j)^2 = 0$ means that $x_j - x_i = 0$ for all the edges of graph $\mathcal{G}$. If the graph $\mathcal{G}$ is connected, then all the nodes values must be equal. The opposite statement is trivial, that is, if $x_j - x_i = 0$, $\forall (i,j) \in \xi$, then $\Psi_{\mathcal{G}}(x) = 0$ [30]. ■

Definition 2.1 (Consensus) Let x_i denote the value of node v_i for all $i = 1, \ldots, N$. We say all the nodes of the graph have reached consensus if and only if

$$\lim_{t \to \infty} ||x_i(t) - x_j(t)|| = 0, \quad i,j = 1, \ldots, N.$$

A simple consensus algorithm to reach an agreement regarding the state of n integrator agents with dynamics $\dot{x}_i(t) = u_i(t)$ is given by the following result.

Theorem 2.1 [30]

Consider $\mathcal{G}$ as a connected graph and let the following linear protocol be applied to each of nodes of the graph $\mathcal{G}$

$$u_i(t) = \sum_{j \in \xi}(x_j(t) - x_i(t))$$

then, the vector of nodes x value is the solution to a gradient system related to the Laplacian potential $\Psi_{\mathcal{G}}(x)$, that is

$$\dot{x} = -Lx = -\nabla \Psi_{\mathcal{G}}(x), \quad x(0) \in \mathbb{R}$$

Moreover, all the graph nodes globally asymptotically reach to an average consensus, that is, if $x^* = \lim_{t \to \infty} x(t)$, then $x_j^* = x_i^* = \frac{1}{N}\sum_{j \in \xi} x_j(0)$, $\forall i,j,\ i \neq j$.

proof.

Consider x^* as an equilibrium point of the system $\dot{x} = -Lx$. Hence, $Lx^* = 0$,that is, x^* is the eigenvector of L associated with zero eigenvalue of L ($\lambda_1 = 0$). Additionally, we have $\Psi_{\mathcal{G}}(x) = 0$ and from $\mathcal{G}$ connectivity, we have $x_j^* - x_i^* = a$, $\forall i,j$, that is, $x^* = (a, \cdots, a)^T, a \in \mathbb{R}$. Notice also that $\sum_{i=1}^N u_i = 0$. Therefore, $\bar{x} = Average(x)$, where the operator $Average(x)$ is defined as the average value of the elements of x, is an invariant quantity, that is, $\bar{x} = 0$. This means $Average(x^*) = Average(x(0))$. Since $Average(x^*) = a$, we have $x_i^* = Average(x(0))$, $\forall i \in 1, \cdots, N$. Note that all eigenvalues of L are negative but $\lambda_1 = 0$. Thus, any system solution asymptotically converge to a point x^* in the eigenspace associated with $\lambda_1 = 0$. This means that an average consensus is globally asymptotically achieved by all nodes [30]. ■

REMARK 2.1 In other words, the properties of the Laplacian matrix dictated by considering graph $\mathcal{G}$ as connected ensure global asymptotic convergence of network consensus toward the average of initial states.

Now consider the collective dynamics of the group of agents

$$\dot{x}(t) = -Lx(t).$$

To demonstrate the meaning of consensus in a network of agents, we provided an example. Consider the following network of 3 individuals illustrated in Fig 2.6.

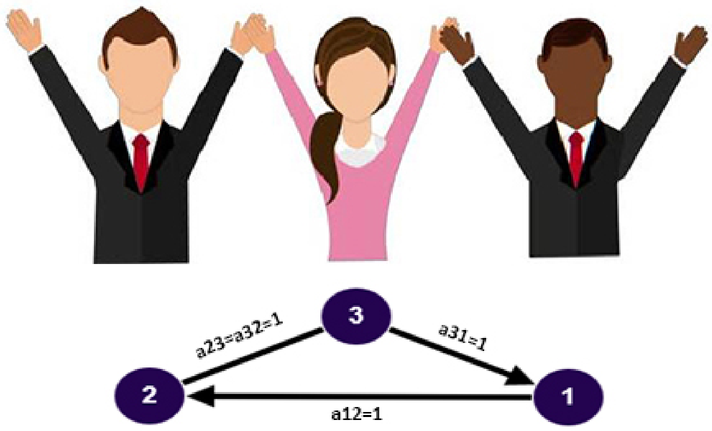


Figure 2.6 – The illustration of a network of 3 individuals planning for a short trip.

The set of 3 individuals displayed in Fig 2.6 want to decide on the amount of time they would spent in nature, during a short trip at the weekend. The individual on the right side (agent 1) opinion is to spent 24 hours in the nature, while the individual on the left side (agent 2) thinks it should be 11 hours and the one on the center (agent 3) believes that 5 hours would suffice. Consequently, we can express the initial states of this network by $x(0) = (24, 11, 5)$. Another properties that is of importance is the interactions between these individuals. It is observable that all the connections in this network carry the same gain. The agents 2 and 3 clearly have a mutual interaction as $a_{23} = a_{32} = 1$, meaning that these agents can talk and listen to each other. Moreover, the agent 1 can only listen to agent 3 as $a_{31} = 1$ and $a_{13} = 0$, and it can only speak to agent 2 as $a_{12} = 1$ and $a_{21} = 0$. By considering these properties of the defined network and the consensus protocol defined earlier, a simulation is taken place to demonstrate the agents' opinions evolution in reaching a common consensus.

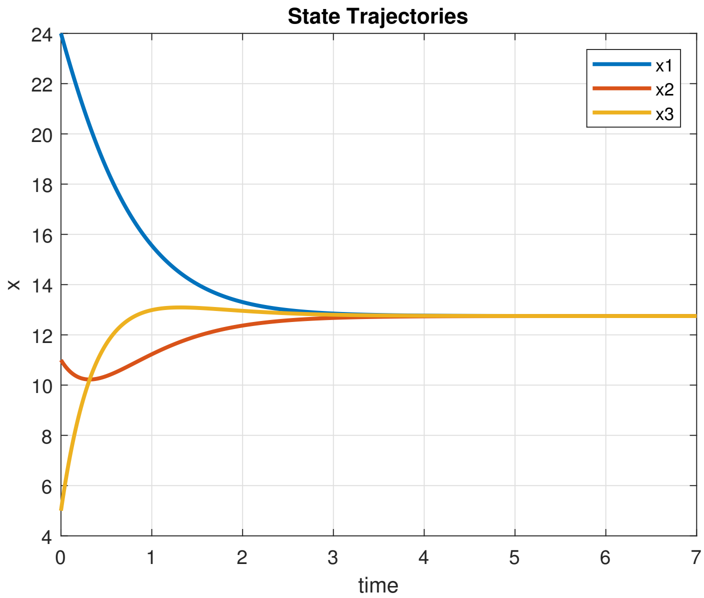


Figure 2.7 – The evolution of agents' opinion through their interactions to reach consensus.

From Fig. 2.7, reaching a common value (consensus) by the agents is evident. Moreover, the value of the consensus could be extracted from it and it is equal to 12.75 which represents a unanimous decision on spending 12 hours and 45 minutes in the nature during their short trip, that is, $\lim_{t\to\infty} x_i(t) \approx 12.75$. The concept of consensus in a network of agents appealed many works in different fields of study as it is able to model different kind of problems such as communication networks, power grids and opinion dynamics.

2.2 INPUT SATURATION

The saturation effect in systems comes from the concept of saturation effect in physical real-world systems and it happens when a feedback control input reaches its physical limits. These limits could occur in many form such as a spring that can not be compressed anymore as it is compressed to its maximum level or an operational amplifier that its output voltage is physically limited by its supply voltage. The problem of this effect is elements that reach saturation can not be linearized and thus, the operational points of any system subject to feedback control should be placed in regions that the saturation would not occur [89]. In other words, even when a system has a linear open-loop system, when the saturation occurs the closed-loop system becomes nonlinear and might have an impact on the system stability or performance. Hence, a bound could be defined for the control input. When this bound exist, it affects the set of admissible initial states as when this bound exist, one can not guarantee that all the valid initial conditions belonging to the invariant region of the closed-loop system would converge successfully to the origin. Moreover, as it may occur to the reader as a simple setting, it can make the treatment of inequalities extracted from Lyapunov function difficult. Consequently, it is recommended to follow certain models representing this effect and take them into account in the stabilization conditions. To express this effect in terms of algebraic conditions, we can consider the following general model

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.2}$$

where $x(t) \in \mathbb{R}^N$ and $u(t) \in \mathbb{R}^N$ are the state vector and control input[1], respectively. We assume that the pair (A, B) is stabilizable in (2.2), then one may define the following condition to express a region where the saturation conditions are met

$$\mathcal{S}(\bar{u}) = \{u \in \mathbb{R}^N : -\bar{u} \leq u_i \leq \bar{u}, \quad \forall i = 1, \cdots, N\}$$

where $0 < \bar{u} \in \mathbb{R}$ is the component defining the bounds on the control input [87]. In the following, we express the saturation in two different forms that are practical in developing adequate conditions for stability. For the first case, let us incorporate $u(t) = \text{sat}(v(t))$ using the standard decentralized saturation function $\text{sat}(v_i) = \text{sign}(v_i)\min(|v_i|, \bar{u}), i = 1, \cdots, N$ where v is the unbounded control signal. Then, we can express (2.2) with

$$\dot{x}(t) = Ax(t) + B\text{sat}(v(t)) \rightarrow \dot{x}(t) = Ax(t) + Bv(t) - B\Psi(v(t)) \tag{2.3}$$

where the elements of $\Psi(v) = [\Psi(v_1), \cdots, \Psi(v_N)]^T$ are defined by

$$\psi(v_i) = \begin{cases} v_i - \bar{u}, & if \ v_i > \bar{u} \\ 0 & if \ -\bar{u} \leq v_i \leq \bar{u} \\ v_i + \bar{u} & if \ v_i < -\bar{u} \end{cases}$$

The following lemma provide sufficient conditions to apply this model of saturation.

LEMMA 2.2 ([86]) By considering a matrix $G \in \mathbb{R}^{N \times N}$ related to the region

$$\Pi = \left\{x \in \mathbb{R}^N : |v_i - G_i x| \preccurlyeq \bar{u}, \ i = 1, \ldots, N\right\}. \tag{2.4}$$

[1]For ease of reference for the next chapter, we consider the input signal of the same dimension as the state vector but without loss of generality.

If $x \subseteq \Pi$, the following expression holds

$$\psi(v)^T T(\psi(v) - Gx) \leq 0 \tag{2.5}$$

for a positive definite and diagonal matrix $T \in \mathbb{R}^{N\times N}$.

Now let us consider the system (2.3) such that $v(t) = Kx(t)$, we have

$$\dot{x}(t) = \underbrace{(A + BK)}_{A_c} x(t) - B\Psi(Kx(t)) \tag{2.6}$$

If we consider the candidate Lyapunov matrix as $V(x) = x^T Px$ for a P matrix symmetric and positive definite, we could express the following theorem.

Theorem 2.2 [86]

If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{N\times N}$, a diagonal positive definite matrix $S \in \mathbb{R}^{N\times N}$ and a matrix $Y \in \mathbb{R}^{N\times N}$ such that the following inequalities hold

$$\begin{bmatrix} WA_c^T + A_cW & \star \\ SB^T - Y & -2S \end{bmatrix} < 0$$

$$\begin{bmatrix} W & \star \\ K_{(i)}W - Y_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0, \quad i = 1, \cdots, N$$

Then the ellipsoid $\mathcal{S} = \{x \in \mathbb{R}^N : x^T Px \leq 1\}$ with $P = W^{-1}$ is an asymptotic stability domain of (2.6).

Another approach is to express the saturation as a polytopic model following [87]:

$$\text{sat}(Kx) = \Gamma(\gamma)Kx, \;\; \Gamma(\gamma) = \sum_{i=1}^{2^N} \gamma_i(x)\Gamma_i(\nu), \quad \forall x \in \mathcal{H}, \;\; \gamma(x) \in \mathcal{U}, \tag{2.7}$$

where $\mathcal{H} = \{x \in \mathbb{R}^N : \; -\bar{u}^\nu \preccurlyeq Kx \preccurlyeq \bar{u}^\nu, \bar{u}^\nu = [\bar{u}_1^\nu, \ldots, \bar{u}_N^\nu]^T, \; \bar{u}_i^\nu = \bar{u}/\nu_i, \; i = 1, \ldots, N\}$, $0 < \nu_i \leq 1$ scalars to be find, and $\Gamma_i(\nu) \in \mathbb{R}^{N\times N}$ diagonal matrices constituted from all the combinations formed with 1, ν_i, $i = 1, \ldots, N$ and $\gamma(x)$ belonging to the unit simplex defined by

$$\mathcal{U} = \{\gamma \in \mathbb{R}^{2^N} : \sum_{i=1}^{2^N} \gamma_i = 1, \; \gamma_i \geq 0, \; i = 1, \ldots, 2^N\}$$

By this definition, one can extract 2^N diagonal matrices to ensure $x \in \mathcal{H}$. As an example, for the case $N = 2$, we would have

$$\Gamma_1(\nu) = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}, \quad \Gamma_2(\nu) = \begin{bmatrix} \nu_1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Gamma_3(\nu) = \begin{bmatrix} 1 & 0 \\ 0 & \nu_2 \end{bmatrix}, \quad \Gamma_4(\nu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $\forall x \in \mathcal{H}$ we can obtain the value of $\text{sat}(Kx)$ computed as the convex combination of $\Gamma_i(\gamma)Kx$, $i = 1, \ldots, N$.

2.3 POSITIVE SYSTEMS

In the last years, positive systems attracted a great deal of attention from researchers around the world (see for instance [90] and [91]). The following descriptions are in accordance with this concept.

Definition 2.2 ([92]) A linear system is positive if for every non-negative initial state, its state remain non-negative. Moreover, a matrix $A \in \mathbb{R}^{N \times N}$ is called Metzler if and only if all of its diagonal elements are non-negative (i.e., $\forall i \neq j : A_{(ij)} \geq 0$).

In literature, the term "positive system" applies to the class of systems where the states carry only non-negative values if the initial states would be non-negative as well. Consider a system $\dot{x}(t) = Ax(t)$, $x \in \mathbb{R}^N$, then, based on Lemma 2.3, this system is positive if the matrix A is a Metzler matrix. If we consider A as a Metzler matrix, it means that for some positive initial states, the evolution of the system states would never include negative values. Such property has its own advantages in control theory. In our own case study as we would observe in later sections, this property could help through its specification to achieve a larger domain of attraction in some cases. Moreover, In [88] authors have shown that the feedback gain of a positive system reveals a diagonal structure of quadratic function for a feasible answer. Moreover, for the asymptotical stability of a positive system, we can express the following theorem from (Theorem 15,[40]) as Lyapunov theorem for positive systems.

Theorem 2.3 [40]

A continuous positive system is asymptotically stable if and only if for a strictly positive **diagonal** matrix $P \in \mathbb{R}^{N \times N}$ the following inequality holds

$$A^T P + PA < 0.$$

It could be concluded that the advantages offered by the properties of positive systems bring out impressive benefits in control theory [93]. In the following, we express an essential lemma that provide the sufficient conditions for the realization of positive systems.

LEMMA 2.3 ([40]) The positivity of the system $\dot{x}(t) = Ax(t) + Bu(t)$ is verified if and only if $B \geq 0$ and A appears as a Metzler matrix (i.e., $\forall i \neq j : A_{(ij)} \geq 0$).

2.4 INVARIANT POLYHEDRAL SETS

A locally stable time invariant dynamical system have a domain in the state-space such that the states included in it can not escape from it. This domain is denoted as invariant set of the system. For a controllable system, one can try to stabilize it such that the states belong and remain in an established invariant region. Through this definition, it is also possible to convert the constraint on the control input to an equivalent projected region in the state-space. Consider the following continuous system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^N \tag{2.8}$$

We express the following definition with respect to (2.8).

DEFINITION 2.3 ([94]) A non-empty set Φ is an invariant set of (2.8) if and only if for any $x_0 \in \Phi$, the complete trajectory of the state vector remains in Φ, that is, $e^{At}\Phi \in \Phi, \forall t$.

The invariant set may be of ellipsoidal, polyhedral or cone subspace. In this part we are interested to the polyhedral invariant sets. Let us define the general structure of convex polyhedral sets.

DEFINITION 2.4 ([94]) A non-empty convex polyhedron of $\mathbb{R}^N$ could be established by a matrix $Q \in \mathbb{R}^{g \times N}$ and a vector $\rho \in \mathbb{R}^g$ such that

$$R[Q, \rho] = \{x \in \mathbb{R}^N : Qx \preccurlyeq \rho\} \tag{2.9}$$

With respect to Definitions 2.3 and 2.4 and by considering the characteristics of positive systems, we propose the positive invariant polyhedron sets.

DEFINITION 2.5 Considering Definition 2.2, a positive invariant polyhedron could be characterized by the following definition

$$\mathcal{X} = \left\{ x \in \mathbb{R}_+^N : \Omega x \preccurlyeq \mathbb{1} \right\}, \tag{2.10}$$

where $\Omega \in \mathbb{R}^{g \times N}$.

In the following, with respect to Definition 2.5, we propose sufficient conditions to establish polyhedral invariant region for the system (2.8).

LEMMA 2.4 ([95]) With respect to the system (2.8), the set in (2.10) is considered positively invariant in regard of this system if and only if there exist a Metzler matrix $H \in \mathbb{R}^{g \times g}$ such that

$$\Omega A = H \Omega$$
$$H \mathbb{1} \preccurlyeq 0.$$

2.5 AUXILIARY RESULTS

Here we express some Auxiliary lemmas and definitions that play an important role in further discussions.

LEMMA 2.5 (Petersen's Lemma [96]) Let $G = G^T \in \mathbb{R}^{n \times n}$, $M \in \mathbb{R}^{n \times p}$, and $N \in \mathbb{R}^{q \times n}$ be pre-defined matrices. For all $\Delta(t) \in \mathbb{R}^{p \times q}$ confirming $\Delta(t)^T \Delta(t) \leq I$, the inequality

$$G + M\Delta(t)N + N^T \Delta(t)^T M^T \leq 0$$

holds if and only if there exists a scalar $\lambda > 0$ such that

$$G + \lambda M M^T + \frac{1}{\lambda} N^T N \leq 0.$$

LEMMA 2.6 ([97]) Consider $w \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}$ with $rank(B) < n$ and $B_\perp$ a basis of B nullspace ($BB_\perp = 0$). The following statements are equivalent:
1. $w^T M w < 0, \forall w \neq 0, Bw = 0.$

2. $B_{\perp}^{T} M B_{\perp} < 0$.
3. $\exists N \in \mathbb{R}^{n \times m} : M + NB + B^T N^T < 0$.

3 MAIN CONTRIBUTIONS

In this chapter, we discuss the main problem we are dealing with alongside the proposed approaches to solve the problem effectively.

3.1 PROBLEM FORMULATION

Consider a network represented by a directed graph $\mathcal{G}$ in which each agent corresponds to a vertex belonging the set $\mathcal{V} = \{1, \ldots, N\}$ associated with N agents. For every agent $i \in \mathcal{V}$ we assign a time varying opinion $x_i(t)$ described by a scalar value normalized between 0 and 1, i.e., $x_i(t) \in [0, 1]$. There are two factors that affect the agent's opinion, the internal influence of neighbors opinions and the action of an external control effort (temptation/persuasion), that sway the agents' opinions toward a desired opinion. The progression of the agents' states (opinions) is characterized by the following dynamic model:

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + (x_i(t) - d)u_i(t), \quad \forall i \in \{1, \ldots, N\}, \tag{3.1}$$

where $x_i(t) \in [0,\ 1]$ are the states, $u_i(t) \in [-\bar{u}, \bar{u}]$, $\bar{u} \in (0, 1)$, is a bounded external control action with limited amount of energy, and $d \in \{0, 1\}$ is the desired opinion.

Remark 3.1 We can observe from (3.1) that in the presence of interactions between agents ($a_{ij} \neq 0$), the changes in an agent's opinion rely on the opinion difference with its neighbors. Moreover, we also note the effect of external influence that affects the agent's opinion regarding the desired opinion d. Consequently, the agents converge to the desired opinion and toward each other, which explains the convergence of agents toward the desired opinion in a network.

In this work, we mainly focus on binary values of d representing a two-fold decision making process (agree/disagree, do/not to do, etc.) in which most problems could be categorized in. On the other hand, the case $d \in (0, 1)$ embodies the median decision making procedure (medium consumption, for instance) presented in the end of this chapter. The afforded budget is represented by a energy constraint in the external control action. In practice, budget limitation may avoid the agents to converge to the desired opinion [18].

Consider $x(t) = (x_1(t), \ldots, x_N(t))^T$ and $u(t) = (u_1(t), \ldots, u_N(t))^T$ the vectors collecting the individual states and controllers, respectively. The collective dynamics of the system (3.1) is given by

$$\dot{x}(t) = -Lx(t) + B(x(t) - \mathbb{1}d)u(t) \tag{3.2}$$

where $B(\cdot) : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is a function described by $B(x) = \text{diag}(x_1, x_2, \ldots, x_N)$. As it will become clear later, let us define the unit hypercube in $\mathbb{R}^N$ ($[0,1]^N$) as

$$\mathcal{X} = \left\{ x \in \mathbb{R}^N_+ : \Omega x \preccurlyeq \mathbb{1} \right\}, \tag{3.3}$$

where $\mathbb{1} \in \mathbb{R}^N$ and $\Omega = I_N \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^{2N\times N}$.

Consider the definition $x_{d_i}(t) = x_i(t) - d$ ($x_d(t) = x(t) - \mathbb{1}d$). With respect to Lemma 3.1, the system described by (3.2) is rewritten as

$$\dot{x}_d(t) = -Lx_d(t) + B(x_d(t))u(t), \tag{3.4}$$

where $x_{d_i}(t) \in [-d,\ 1-d]$, $i = 1, \ldots, N$. Hereafter, we present a solution for $d = 0$ since $d = 1$ is a symmetric case. Thus, without losing the generality, we adopt $d = 0$ and consequently, $x_d = x$ unless stated otherwise. The following problem is tackled in this paper.

Problem 1. *To design the control component u_i subjected to the following optimization problem:*

$$\min_{u(t)} J_x = \int_0^\infty x(t)^T R x(t)\, dt \tag{3.5}$$

subjected to

$$x(t) \in \mathcal{X} \tag{3.6}$$

$$|u_i(t)| \le \bar{u} \tag{3.7}$$

$$J_u = \int_0^\infty u(t)^T Q u(t)\, dt < \mu \tag{3.8}$$

where $\bar{u}$ and μ are given positive scalar from the problem specification; J_x is the global cost used as performance criterion; J_u is the total budget; and R and Q are positive definite matrices that balance the agents' convergence and the budget required for synchronization, respectively. The conditions (3.7)–(3.8) *embody the bounded amplitude and the limited energy of external control action, respectively.*

REMARK 3.2 To put it simply, a campaign (a marketing campaign, for instance) that aims to persuade a social network (customers) toward a desired opinion (buying a certain product) tries to sway the original consensus of the network as close as possible to the desired opinion minimizing (3.5) through a limited budget in the form of a bounded external control effort (3.7)–(3.8). This can be explained as follows. If an entity is aiming at convincing or persuading a network of interest into a certain choice or habit, it tries to alter the asymptotic consensus value such that it gets as close as possible to the desired value and consequently minimize J_x. For a different case like an election that a campaign tries to get the opinions as close as possible to the desired one (d) through a limited amount of time (T), we aim to minimize $J_x(T) = \int_0^T (x(t) - d)^T R(x(t) - d) dt$. Thus, without additional campaigns the opinions asymptotically converge to a certain local or global agreement through their interactions. It is noteworthy that in the absence of campaigns, the system dynamics become $\dot{x}(t) = -Lx(t)$ which has a global exponential stable attractor. Also, we know that the limitation in (3.8) might avoid the asymptotic consensus (resulting to a consensus different than d). Thus, in the absence of campaigns or when the external control action is active, for an adequately large time (T), $x_i(T)$ is a good approximation of $\lim\limits_{t\to\infty} x_i(t)$.

The afforded budget μ in (3.8) embodies how much we are willing to invest in swaying the average consensus of our network of interest toward the desired opinion. It is noteworthy that the functions J_x and J_u could be expressed as $\int_0^\infty z(t)^T z(t)\, dt$ and $\int_0^\infty y(t)^T y(t)\, dt$, respectively, with

$$\begin{aligned} z(t) &= R^{\frac{1}{2}} x(t) \\ y(t) &= Q^{\frac{1}{2}} u(t). \end{aligned} \tag{3.9}$$

Consider the following assumption for the network.

ASSUMPTION 3.1 We assume that the graph $\mathcal{G}$ representing the social network is weakly connected, *i.e.*, it contains at least one directed spanning tree.

LEMMA 3.1 Considering Assumption 3.1, the Laplacian matrix L has a simple eigenvalue equal to 0 associated with the right eigenvector $\mathbb{1} \in \mathbb{R}^N$, meaning $L\mathbb{1} = 0$. The other $N - 1$ eigenvalues of matrix L have positive real parts.

3.2 REPRESENTATION OF THE BILINEAR TERMS AND SATURATION MODEL

The restriction in (3.7) could be expressed in the system dynamics by $u(t) = \text{sat}(v(t))$ considering the decentralized saturation function $\text{sat}(v_{(\ell)}) = \text{sign}(v_{(\ell)}) \min(|v_{(\ell)}|, \bar{u})$, $\ell = 1, \ldots, N$, where v is the unlimited control action that we aim to establish. Consequently, we have

$$\dot{x}(t) = -Lx(t) + B(x(t))\text{sat}(v(t)). \tag{3.10}$$

The system described in (3.10) could be interpreted by involving the decentralized dead-zone nonlinearity $\psi(v) = v - \text{sat}(v)$

$$\dot{x}(t) = -Lx(t) + B(x(t))v(t) - B(x(t))\psi(v(t)). \tag{3.11}$$

The bilinear product in (3.11) for $x \in \mathcal{X}$ is handled by the norm-bounded uncertainty representation:

$$B(x) = B_0 + B_1\Delta(t), \tag{3.12}$$

where $B_0 = B(0.5\mathbb{1})$, $B_1 = 0.5I$, and $\Delta(t) = \text{diag}(\delta_1(t), \ldots, \delta_N(t)) \in \mathbb{R}^{N\times N}$, where $\delta_i(t)$ is a bounded Lebesgue measurable uncertainty associated to the set $\mathcal{D} = \{\delta \in \mathbb{R} : \ \delta^T\delta \leq 1\}$.

In this part, we propose the state feedback control law

$$v(t) = Kx(t), \qquad K = \text{diag}(k_1, \ldots, k_N) \in \mathbb{R}^{N\times N}, \tag{3.13}$$

where k_i, $i = 1, \ldots, N$, are gains to be designed. The diagonal characteristic of the gain matrix K yields a control action v_i depending only on its respected opinion x_i. The closed-loop system with respect to (3.9), (3.11) and (3.13) is given by

$$\dot{x}(t) = (-L + (B_0 + B_1\Delta(t))K)x(t) - (B_0 + B_1\Delta(t))\psi(Kx(t)) \tag{3.14a}$$
$$z(t) = R^{\frac{1}{2}}x(t) \tag{3.14b}$$
$$y(t) = Q^{\frac{1}{2}}Kx(t) + Q^{\frac{1}{2}}\psi(Kx(t)). \tag{3.14c}$$

REMARK 3.3 Observe that the controllability matrix of the pair $(-L, B(x))$ loses rank for $x = 0$. Therefore, the closed-loop system (3.14) is not controllable for all $\Delta(t) \in \mathcal{D}$ with $B_0 = B(0.5\mathbb{1})$, $B_1 = 0.5I$. To circumvent this issue, matrix B_0 is redefined as $B_0 = B((0.5 + \varepsilon)\mathbb{1})$ such that (3.12) models $B(x)$ for the interval $x_i \in [\varepsilon, 1]$, where $\varepsilon > 0$ is arbitrarily small.

3.3 CONSENSUS CONDITIONS BASED ON NORM-BOUNDED APPROACH

In this section, we aim to propose conditions to establish a state feedback control law to solve Problem 1. These conditions should guarantee that all the trajectories remain in $\mathcal{X} \cap \Pi$ with respect to the constraint (2.5). In other words, an invariant region $\mathcal{S}$ should be established such that if for all $x(0) \in \mathcal{S}$, then $x(t) \in \mathcal{X} \cap \Pi$ for all $t \geq 0$. The candidate set $\mathcal{S}$ is defined as

$$\mathcal{S} := \left\{ x \in \mathbb{R}_+^N : x^T W^{-1} x \leq 1, \quad W = W^T > 0 \right\}, \tag{3.15}$$

where W is a positive definite matrix to be determined.

Generally, conditions verifying $\mathcal{S} \subset \mathcal{X}$ are not easily obtainable and in general, the set $\mathcal{S}$ is not an invariant region with $V(t) = x^T W^{-1} x$ a candidate Lyapunov function. Thus, we define the following level curve

$$\mathcal{S}_a := \left\{ x \in \mathbb{R}^N : x^T W^{-1} x \leq 1, \quad W = W^T > 0 \right\} \tag{3.16}$$

associated to the quadratic Lyapunov function $V(x) = x(t)^T W^{-1} x(t)$. Let us also define the augmented polyhedral region

$$\mathcal{X}_a = \left\{ x \in \mathbb{R}^N : \Omega x \preccurlyeq \mathbb{1} \right\}, \tag{3.17}$$

with Ω as in (3.3), representing the region $x_i \in [-1,\ 1]$.

The approach adopted in this work to guarantee $\mathcal{S} \subset \mathcal{X} \cap \Pi$, with $\mathcal{S}$ an estimate of the domain of attraction of the origin of (3.14), is first to propose conditions for $\mathcal{S}_a \subset \mathcal{X}_a \cap \Pi$, and then to show that $\mathcal{S} = \mathcal{S}_a \cap \mathbb{R}_+^N$ is an invariant region. Observe that $-L$ is a Metzler matrix and $B(x) \geq 0$ for all $x \in \mathcal{X}$, then the proposed approach relies on the property that the closed-loop system (3.14) remains positive (see Definition 2.2) for the state feedback control law (3.13). This establishes the set $\mathcal{S}$ as an invariant region. Figure 3.1 depicts the sets $\mathcal{X}$, $\mathcal{X}_a$, $\mathcal{S}$, $\mathcal{S}_a$ and a trajectory of $x(t)$.

The control energy limitation (3.8) may prevent the agents' trajectories belonging to $\mathcal{S}$ from reaching the origin. In practice, the limitation in the budget investment leads the opinions to a neighbor value of d, that is, the convergence of opinions to the exact desired consensus could be expensive and overpriced. As a result, we present a method to establish an invariant region $\mathcal{S}_u \subseteq \mathcal{S}$ to ensure the convergence of opinions to the origin for $x(0) \in \mathcal{S}_u$ by guaranteeing the availability of enough budget $J_u < \mu$ for such procedure. Moreover, we also propose a method to limit budget investment ($u = 0$) for initial conditions in $\mathcal{S} \setminus \mathcal{S}_u$ when the investment constraint μ is reached.

One can rewrite Problem 1 as:

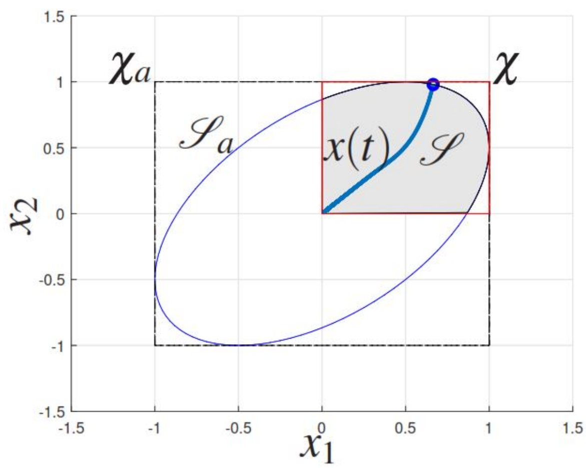


Figure 3.1 – Sets $\mathcal{X}_a$ (black dashed box), $\mathcal{X}$ (red box), $\mathcal{S}$ (gray region), $\mathcal{S}_a$ (blue curve), and trajectory $x(t)$ (bold blue line).

Problem 2. *To design the state feedback gain K such that the closed-loop system* (3.14) *is asymptotically stable and to find*

(I) an estimation for the domain of attraction $\mathcal{S} \subset (\mathcal{X} \cap \Pi)$ such that for all initial states $x(0) \in \mathcal{S}$, the trajectories of (3.14) *converge asymptotically to the origin with respect to guaranteed cost function J_x for all $\delta_i(t) \in \mathcal{D}$ and $|u_i(t)| \leq \bar{u}$;*

(II) an estimation of an invariant region $\mathcal{S}_u$ such that for all initial states $x(0) \in \mathcal{S}_u \cap \mathcal{S}$, the trajectories of (3.14) *converge asymptotically toward the origin with guaranteed cost function $J_u < \mu$ for all $\delta_i(t) \in \mathcal{D}$ and $|u_i(t)| \leq \bar{u}$;*

(III) a mechanism to restrict the investment ($u = 0$) for all of initial states belonging to $\mathcal{S} \setminus \mathcal{S}_u$ when the investment limitation $J_u = \mu$ is reached.

Methodologically, inspiring from ideas proposed in [18, Proposition 4.1], we first present a solution for Problem 2 (I) using the investment as soon as possible to minimize J_x subjected to constraints (3.6)-(3.7). In the sequence, we present conditions to estimate the set $\mathcal{S}_u$ solving Problems 2 (II) and (III).

Remark 3.4 Observe that for K diagonal the closed-loop system is positive. If we express the input saturation as (2.7), the closed-loop system is given by $\dot{x}(t) = (-L + B(x)\Gamma(\gamma)Kx)x(t)$ and matrix $-L + B(x)\Gamma(\gamma)Kx$ is Metzler since $-L$ is Metzler and $B(x)\Gamma(\gamma)K$ is diagonal.

Theorem 3.1

If there exist diagonal positive definite matrices $W \in \mathbb{R}^{N\times N}$ and $S \in \mathbb{R}^{N\times N}$, a matrix $Y \in \mathbb{R}^{N\times N}$, a diagonal matrix $Z \in \mathbb{R}^{N\times N}$, and a scalar $\lambda > 0$, such that the further inequalities are satisfied

$$\begin{bmatrix} \mathrm{He}\{-LW + B_0 Z\} + \lambda I & \star & \star & \star \\ SB_0^T + Y & -2S & \star & \star \\ W & 0 & -R^{-1} & \star \\ Z & S & 0 & -4\lambda I \end{bmatrix} < 0 \tag{3.18}$$

$$\begin{bmatrix} W & \star \\ \Omega_{(i)} W & 1 \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, 2N \tag{3.19}$$

$$\begin{bmatrix} W & \star \\ Z_{(i)} - Y_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, N, \tag{3.20}$$

then the state feedback gain $K = ZW^{-1}$ ensures exponential stability of the closed-loop system (3.14a) with respect to $\mathcal{S} \subseteq \mathcal{X} \cap \Pi$ as estimation of the domain of attraction of origin considering the guaranteed cost $J_x \leq x(0)^T W^{-1} x(0)$.

proof.

Consider the Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$ and the closed-loop system (3.14). It is evident that the integral from 0 to ∞ of

$$\dot{V}(t) + z(t)^T z(t) < 0, \quad \forall x \in \mathcal{X}, \tag{3.21}$$

confirms $J_x < V(0)$. The inequality $\dot{V}(t) < -cx(t)^T x(t)$ is the equivalent expression of (3.21), where c denotes the maximum eigenvalue of matrix R. Consequently, the exponential stability of the origin is verified.

Using Lemma 2.2, if $\dot{V}(t) + z(t)^T z(t) - 2\psi(v(t))^T T \psi(v(t)) + 2\psi(v(t))^T T G x(t) < 0$, the inequality (3.21) is met, and by taking (3.12) and (3.14) into account, the last inequality could be expressed as

$$\begin{bmatrix} x \\ \psi(v) \end{bmatrix}^T \begin{bmatrix} \mathrm{He}\{-W^{-1}L + W^{-1}B(x)K\} + R & \star \\ B(x)^T W^{-1} + TG & -2T \end{bmatrix} \begin{bmatrix} x \\ \psi(v) \end{bmatrix} < 0,$$

then, with pre- and post-multiplying it by diag(W, T^{-1}), we have

$$\begin{bmatrix} \mathrm{He}\{-LW + B(x)Z\} + W^T R W & \star \\ SB(x)^T + Y & -2S \end{bmatrix} < 0,$$

where $Y = GW$, $S = T^{-1}$, and $Z = KW$. The utilization of Schur complement lemma leads to

$$\begin{bmatrix} \text{He}\{-LW + B(x)Z\} & \star & \star \\ SB(x)^T + Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0,$$

replacing $B(x)$ in the previous inequality by (3.12) for all $x \in \mathcal{X}$ yields

$$\begin{bmatrix} \text{He}\{-LW + B_0 Z\} & \star & \star \\ SB_0^T + Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} + \text{He}\{ \begin{bmatrix} \Delta \\ 0 \\ 0 \end{bmatrix} 0.5 \begin{bmatrix} Z & S & 0 \end{bmatrix} \} < 0.$$

exploiting Lemma 2.5 for $\Delta(t) = \text{diag}(\delta_1(t), \ldots, \delta_N(t))$, $\delta_i(t) \in \mathcal{D}$, and the Schur Complement lemma, we obtain (3.18).

By pre- and post-multiplying (3.19) with diag(W^{-1}, I), we have

$$\begin{bmatrix} W^{-1} & \star \\ \Omega_{(i)} & 1 \end{bmatrix} \geq 0$$

that verifies $\mathcal{S}_a \subseteq \mathcal{X}_a$ [97].

As the last part, by pre-and-post multiplying inequality (3.20) by diag(W^{-1}, I), one has

$$\begin{bmatrix} W^{-1} & \star \\ K_{(i)} - G_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0$$

considering the set Π in Lemma 2.2 and $\mathcal{S}_a$ in (3.16), the above inequality guarantees $\mathcal{S}_a \subseteq \Pi$ [86] and, consequently, $\mathcal{S} \subseteq \Pi$. Considering the closed-loop system (3.14) positive (Remark 3.4) and $\mathcal{S}_a$ an invariant region, one has that $\mathcal{S}$ is invariant and contractive [98]. ∎

Theorem 3.1 provides adequate criteria to ensure exponential stability of the closed-loop system (3.14) while the energy constraint of (3.8) for the control action is considered. It is noteworthy that the obtained W by Theorem 3.1 for the Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$ would not provide the conditions of guaranteeing $J_u < \mu$ for all $x \in \mathcal{S}$ and the invariant region $\mathcal{S}_u$. Thus, a method is presented in the following to estimate the level curves related to the control energy cost function J_u and deal with Problems 2 (II) and (III).

Theorem 3.2

Suppose that there exist diagonal positive definite matrices $P \in \mathbb{R}^{N \times N}$ and $S \in$

$\mathbb{R}^{N\times N}$, a matrix $Y \in \mathbb{R}^{N\times N}$, and a scalar $\lambda > 0$, such that the further inequality holds

$$\begin{bmatrix} \text{He}\{-LP + B_0KP\} + \lambda I & \star & \star & \star \\ SB_0^T + Y & -2S & \star & \star \\ KP & S & -Q^{-1} & \star \\ KP & S & 0 & -4\lambda I \end{bmatrix} < 0, \qquad (3.22)$$

Henceforth, the closed-loop system (3.14a) achieves the guaranteed cost $J_u \leq x(0)^T P^{-1} x(0)$.

proof.

The proof follows the same line of Theorem 3.1 considering the Lyapunov function $V(t) = x(t)^T P^{-1} x(t)$. ■

Remark 3.5 Theorem 3.1 assures that the opinions (x_i, $i = 1, \dots, N$) subjected to a convenient external bounded influence (u) converge to the desired value ($d = 0$) with exponential rate if the initial opinion belongs to $\mathcal{S}$. Theorem 3.2 provides a way to take into account the fact that the energy of the external control is limited, which avoids, in general, the opinions to reach exactly the desired opinion, as is usually the case in practice.

The following section presents a systematic way to optimize the domain of attraction of the origin and a mechanism to respect the bound on the control input energy.

3.4 ESTIMATION OF DOMAIN OF ATTRACTION

In this subsection, first we state a number of remarks that aim to enhance the perspective that Theorems 3.1 and 3.2 offer. Afterwards, the polyhedral approach is discussed to maximize the domain of attraction in order to verify all possible initial states as valid (i.e., to recognize a maximal admissible set of all possible initial conditions).

Remark 3.6 The assessment of the region of all initial states with respect to $J_u \leq \mu$ is expressed by $\mathcal{S}_u = \{x \in \mathbb{R}_+^N : x^T P^{-1} x \leq \mu\}$. For all $x(0) \in \mathcal{S} \setminus \mathcal{S}_u$, the following control mechanism guarantees $J_u < \mu$

$$u(t) = \begin{cases} sat(Kx), & x(t)^T P^{-1} x(t) > x(0)^T P^{-1} x(0) - \mu \\ 0, & \text{otherwise}. \end{cases} \tag{3.23}$$

The control mechanism (3.23) drives the trajectories of $x(0) \in \mathcal{S} \setminus \mathcal{S}_u$ to approach the boundary of the region $\{x \in \mathcal{S} : x(t)^T P^{-1} x(t) \leq x(0)^T P^{-1} x(0) - \mu\}$. Note that cutting off the control action ($u = 0$) alters the dynamics of system to become $\dot{x}(t) = -Lx(t)$ that has a uniform globally exponentially stable attractor specified by consensus manifold. Observe that since $\mathcal{S}$ and $\mathcal{S}_u$ are invariant sets, $\mathcal{S} \setminus \mathcal{S}_u$ is also invariant.

Remark 3.7 To indirectly minimize J_x and maximize the set $\mathcal{S}$ in (3.15) simultaneously, one can maximize the trace of W (see [97]). Accordingly, the maximization of the estimation of the domain of attraction alongside (3.5) and (3.7) are achievable through solving the optimization problem

$$\max \ \text{Trace(W)} \tag{3.24}$$

subjected to (3.18)–(3.20).

to achieve a more accurate estimation of upper bound for cost function J_u in Theorem 3.2, one can solve the optimization problem

$$\max \ \text{Trace(P)} \tag{3.25}$$

subjected to (3.22).

Remark 3.8 If the initial conditions $x(0)$ are known, one can minimize J_x solving the optimization problem

$$\min \ \tau \tag{3.26}$$

subjected to (3.18)–(3.20) and

$$\begin{bmatrix} \tau & \star \\ x(0) & W \end{bmatrix} > 0.$$

REMARK 3.9 In order to directly maximize the set $\mathcal{S}$ in (3.15), one can use matrix decomposition such that $W_0^{-1} = VDV'$ when $W = W_0 + W_1$ and W_1 is a variable matrix. By considering $D = \text{diag}(\lambda_1, \cdots, \lambda_N)$ and $V = [e_1, \cdots, e_N]$, where λ_i entries represent the eigenvalues of W_0^{-1} and columns of V (e_i) are their respective eigenvectors, it is possible to maximize the set $\mathcal{S}$ by customizing the entries of matrices D and V. Consequently, a maximized estimation of the domain of attraction could be obtained by solving the following optimization problem:

$$\max \ \text{Trace(W)} \tag{3.27}$$

subjected to (3.18)–(3.20)

$$\begin{gathered} W_0 > 0 \\ \min_{\rho} : \begin{bmatrix} \rho I & \star \\ W_1 & I \end{bmatrix} \geq 0 \end{gathered} \tag{3.28}$$

REMARK 3.10 With respect to $\varepsilon > 0$ in $B_0 = B((0.5+\varepsilon)\mathbb{1})$, (3.21) holds for $x \in \mathcal{X} \backslash \mathcal{B}_\varepsilon$, where $\mathcal{B}_\varepsilon = \{x \in \mathcal{S} : x^T x \leq \varepsilon\}$. It clarifies that only the trajectories convergence to the $\mathcal{B}_\varepsilon$ is assured. Observe that for an adequately small ε any practical consequence is avoided as the limitation (3.8) generally prohibits $x_i(t)$ from reaching d.

3.4.1 Estimation of Domain of Attraction using polyhedral invariant sets

In this section, we consider an approach to verify if $\mathcal{X}$ is a domain of attraction for the closed-loop system instead of considering the level curves of the Lyapunov function inside $\mathcal{X}$ as an estimation of domain of attraction.

In the following, we discuss the problem of designing a state feedback gain K for the polytopic approach. We use the concept presented in Lemma 2.3 to ensure the internal positivity of the closed-loop system (3.29) that is defined in the following. Moreover, we define a maximized domain of attraction over the polytope set $\mathcal{X}_a$ to ensure exponential stability for initial conditions belonging to the invariant ellipsoid such that the trajectories do not leave the boundaries of $\mathcal{X}_a$.

We present a solution to the Problem 1 using polyhedral invariant set along with a quadratic Lyapunov function. For the polytopic approach, we describe the bilinear product in (3.11) using a polytopic interpretation.

The set (3.17) can also be represented in terms of its 2^n vertices obtained from Ω by

$$\mathcal{X} = \text{co}\left\{h_1, h_2, \ldots, h_{2^N}\right\}$$

Hence, the polyhedral set $\mathcal{X}_a$ induces the following matrix valued polytope

$$\mathcal{P} = \text{co}\{B(h_i),\ i = 1, \ldots, 2^N\}$$

and for any $x \in \mathcal{X}_a$, one has

$$B(x) = \sum_{i=1}^{2^N} \alpha_i(x) h_i =: \mathbb{B}(\alpha)$$

with $\alpha(x)$ belonging to the unit simplex

$$\mathcal{U} = \{\alpha \in \mathbb{R}^{2^N} : \sum_{i=1}^{2^N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, 2^N\}$$

for all $t \geq 0$.

With respect to (2.7), the closed-loop system in this case is obtained as

$$\dot{x}(t) = (-L + \mathbb{B}(\alpha)\Gamma(\gamma)K)x(t) \tag{3.29}$$

Based on these properties, we propose a theorem based on polytopic approach to extend the domain of attraction in a way that all the possible states are covered by it, that is, to achieve a maximal admissible set of initial states for the agents.

Theorem 3.3

Let $K \in \mathbb{R}^{N \times N}$ be a given matrix such that the closed-loop system (3.10)–(3.13) is asymptotically stable, if there exist scalars $\nu_i,\ \forall i = 1, \cdots, N$ and a Metzler matrix $H \in \mathbb{R}^{g \times g}$ such that the following inequalities hold

$$\Theta(-L + B(h_i)\Gamma_j(\nu)K) - H\Theta \ll 0, \quad i = 1, \ldots, 2^N,\ \ j = 1, \ldots, 2^N, \tag{3.30}$$

$$H\mathbb{1} \preccurlyeq 0 \tag{3.31}$$

$$0 < \nu_i \leq 1, \quad i = 1, \ldots, N, \tag{3.32}$$

$$\begin{bmatrix} \Theta_{(\ell)}^T \Theta_{(\ell)} & \star \\ \nu_\ell K_{(\ell)} & \bar{u}^2 \end{bmatrix} \geq 0, \quad \ell = 1, \ldots, N, \tag{3.33}$$

where $\Theta = I_N \otimes \begin{bmatrix} 1 \\ -\epsilon^{-1} \end{bmatrix} \in \mathbb{R}^{g \times N}$, $0 < \epsilon \ll 1$, then $\mathcal{X}$ is an estimation of the domain of attraction of (3.29).

proof.

For the (3.30)–(3.31), with respect to the set $\mathcal{K} = \left\{ x \in \mathbb{R}^N : -Ix \preccurlyeq 0 \right\}$ and $\mathcal{X}_\epsilon = \left\{ x \in \mathbb{R}^N : \Theta x \preccurlyeq \mathbb{1} \right\}$, we have

$$\mathcal{X}^+ = \mathcal{X}_\epsilon \cap \mathcal{K} = \left\{ x \in \mathbb{R}^N : \begin{bmatrix} \Theta \\ -I \end{bmatrix} x \preccurlyeq \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \right\}$$

With respect to Lemma 2.4, the set $\mathcal{X}^+$ is a positive polyhedron invariant set for the closed-loop system of the form $\dot{x}(t) = A_{cl}x(t)$, with $A_{cl} = -L + \mathbb{B}(\alpha)\Gamma(\gamma)K$, from (3.29) if and only if there exist two Metzler matrices $H_1 \in \mathbb{R}^{g \times g}$ and $H_4 \in \mathbb{R}^{N \times N}$, and two non-negative matrices $H_2 \in \mathbb{R}^{g \times N}$ and $H_3 \in \mathbb{R}^{N \times g}$, such that

$$\begin{bmatrix} \Theta \\ -I \end{bmatrix} A_{cl} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} \Theta \\ -I \end{bmatrix}, \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \preccurlyeq 0$$

without loss of generality, it is possible to replace $H_3 = 0$ and $H_4 = A_{cl}$, resulting in

$$\begin{cases} \Theta A_{cl} = H_1\Theta - H_2 \\ H_1\mathbb{1} \preccurlyeq 0 \end{cases} \Rightarrow \begin{cases} \Theta A_{cl} - H_1\Theta \ll 0 \\ H_1\mathbb{1} \preccurlyeq 0 \end{cases} \tag{3.34}$$

from (3.34), we recover (3.30) and (3.31) for H_1 being Metzler [99].

Moreover, the (3.33) ensures $\mathcal{X}_\epsilon \subseteq \mathcal{H}$ where $\mathcal{H}$ is the set associated with saturation definition in (2.7), that is, to ensure $\mathcal{X}_\epsilon \subseteq \mathcal{H}$ and with respect to the definition of each set and $\ell = 1, \cdots, N$,we have

$$\Theta_{(\ell)}^T \Theta_{(\ell)} \geq (\nu_\ell K_{(\ell)})^T \bar{u}^{-2} (\nu_\ell K_{(\ell)})$$

by using Schur complement lemma, one can recover (3.33). ■

The advantage of this method is the independence of estimation of the domain of attraction $\mathcal{X}$ from the obtained gain K which results in larger estimation of domain of attraction. Moreover, a maximal admissible set is established covering all the possible system initial values resulting in guaranteed asymptotic stability for $\forall x(0) \in \mathcal{X}$. In opposite, the main disadvantage of this method is its incapability to deal with large networks due to the limits caused by polytopic representation in bigger networks. Thus, to address a large network, one may prefer a norm-bounded uncertainty model. Since Theorem 3.3 demands the polytopic representation of $B(x)$, one can adapt the conditions of Theorem 3.1 to obtain a feedback gain K to be used in Theorem 3.3 as described in the following theorem.

Theorem 3.4

If there exist diagonal positive definite matrices $W \in \mathbb{R}^{N\times N}$ and $S \in \mathbb{R}^{N\times N}$, a matrix $Y \in \mathbb{R}^{N\times N}$, a diagonal matrix $Z \in \mathbb{R}^{N\times N}$ such that inequality (3.20) and

$$\begin{bmatrix} \mathrm{He}\{-LW + B(h_i)Z\} & \star & \star \\ SB(h_i)^T + Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0, \quad i = 1, \dots, 2^N, \tag{3.35}$$

are satisfied, then the designed state feedback gain $K = ZW^{-1}$ ensures exponential stability of the closed-loop system (3.10)–(3.13) considering the guaranteed cost $J_x \leq x(0)^T W^{-1} x(0)$.

Moreover, in this case, to obtain the level curves associated with the cost J_u we consider the following theorem.

Theorem 3.5

Suppose that there exist diagonal positive definite matrices $P \in \mathbb{R}^{N\times N}$ and $S \in \mathbb{R}^{N\times N}$, a matrix $Y \in \mathbb{R}^{N\times N}$ such that the further inequality holds

$$\begin{bmatrix} \mathrm{He}\{-LP + B(h_i)KP\} & \star & \star \\ SB(h_i)^T + Y & -2S & \star \\ KP & S & -Q^{-1} \end{bmatrix} < 0, \quad i = 1, \cdots, 2^N \tag{3.36}$$

Henceforth, the closed-loop system (3.10)–(3.13) achieves the guaranteed cost $J_u \leq x(0)^T P^{-1} x(0)$.

3.5 CONSENSUS CONDITIONS BASED ON ON-OFF CONTROL LAW

We propose the following on-off control approach based on on-off control law

$$u_i(t) = \begin{cases} k_i, & t \leq T \\ 0, & t > T \end{cases} \tag{3.37}$$

where $|k_i| < \bar{u}, i = 1, ..., N$, are gains to be determined, and T is given by the energy constraint (3.8) as following

$$J_u = \int_0^T \mathbb{1}^T KQK\mathbb{1} dt = \mathbb{1}^T KQK\mathbb{1} T < \mu \tag{3.38}$$

with $K = \text{diag}(k_1, ..., k_N) \in \mathbb{R}^{N\times N}$. Thus, $T = \mu/\mathbb{1}^T KQK\mathbb{1}$ yields the maximum use of the budget. The closed loop system with respect to (3.37) is obtained as

$$\dot{x}(t) = \begin{cases} -Lx(t) + B(x(t))K\mathbb{1} = (-L+K)x(t), & t \leq T \\ -Lx(t), & t > T \end{cases} \tag{3.39}$$

Observe that, in the simple control law (3.37), there is no bilinear term and the saturation in the control input can be treated straightforward independent of initial conditions. The following conditions design (3.37) and solves Problem 1.

Theorem 3.6

If there exist diagonal positive definite matrix $W \in \mathbb{R}^{N\times N}$ and a diagonal matrix $Z \in \mathbb{R}^{N\times N}$ such that the following inequalities are satisfied

$$\begin{bmatrix} He\{-LW + Z\} & \star \\ W & -R^{-1} \end{bmatrix} < 0 \tag{3.40}$$

$$\begin{bmatrix} W & \star \\ \Omega_{(i)} W & 1 \end{bmatrix} \geq 0, \quad \forall i = 1, \cdots, N \tag{3.41}$$

$$\begin{bmatrix} W_{(ii)} & \star \\ Z_{(ii)} & \bar{u}^2 W_{(ii)} \end{bmatrix} \geq 0, \quad \forall i = 1, \cdots, N, \tag{3.42}$$

then the control law (3.37) with gain $K = ZW^{-1}$ ensures $\lim_{t\to\infty} x_i(t) = \frac{1}{N}\mathbb{1}^T x(T)$, with $x(T) = exp((-L+K)T)x(0)$, $T = \mu/(\mathbb{1}^T KQK\mathbb{1})$, for the closed-loop system (3.39), for all $x(0) \in \mathcal{S} \subseteq \mathcal{X}$, with guaranteed cost $J_u \leq \mu$.

proof.

From $\dot{V} + x'Rx < 0$, one has

$$He\{-LW + KW\} + WRW < 0$$

Using Schur complement lemma and by considering $Z = KW$,we obtain (3.40) that assures asymptotic stability of (3.39).

The proof of inequalities (3.41) and (3.42) follows the same reasoning as (3.19)-

(3.20), that is, if we pre-and-post multiply (3.41) with diag(W^{-1}, I), we could obtain an inequality that assures $\mathcal{S}_a \subseteq \mathcal{X}_a$. Moreover, by pre-and-post multiplying (3.42) with diag$(W_{(ii)}^{-1}, I)$, we obtain

$$\begin{bmatrix} W_{(ii)}^{-1} & \star \\ k_i & \bar{u}^2 W_{(ii)} \end{bmatrix} \geq 0$$

where $K = ZW^{-1}$. This is equivalent to $||k_i|| \leq \bar{u}$.

Finally, observe that the system is under control action until $t = T$. After that, the MAS is governed by the open-loop dynamics $\dot{x}(t) = -Lx(t)$, as given by (3.39), and with initial condition $x(T)$, where $x(T) = exp((-L + K)T)x(0)$. By Theorem 2.1, one has $\lim_{t\to\infty} x_i(t) = \frac{1}{N}\mathbb{1}^T x(T)$. From (3.38), $T = \mu/(\mathbb{1}^T KQK\mathbb{1})$ assures $J_u \leq \mu$. ■

The Theorem 3.6 establishes the domain of attraction using matrix W, that is, it obtains an ellipsoidal invariant region which in larger networks carries its disadvantage (the limits in valid initial conditions for the agents). Hence, we propose the next result to address this issue by establishing a polyhedral invariant region such that all the possible initial values of states are valid. In other words, the Theorem 3.7 address the same problem as Theorem 3.6 with an extended domain of attraction of the origin.

Theorem 3.7

If there exist diagonal positive definite matrix $W \in \mathbb{R}^{N\times N}$, a Metzler matrix $H \in \mathbb{R}^{g\times g}$, a positive scalar ξ and a diagonal matrix $K \in \mathbb{R}^{N\times N}$ such that the following inequalities hold

$$\begin{bmatrix} He\{-LW - W\} & \star & \star \\ \xi W + K + I & -2\xi I & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0 \tag{3.43}$$

$$\Theta(-L + K) - H\Theta \ll 0 \tag{3.44a}$$

$$H\mathbb{1} \preccurlyeq 0 \tag{3.44b}$$

$$\begin{bmatrix} 1 & \star \\ k_i & \bar{u}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \cdots, N \tag{3.45}$$

The designed state feedback gain K ensures $\lim_{t\to\infty} x_i(t) = \frac{1}{N}\mathbb{1}^T x(T)$, with $x(T) = exp((-L+K)T)x(0)$, $T = \mu/(\mathbb{1}^T KQK\mathbb{1})$, for the closed-loop system (3.39), for all $x(0) \in \mathcal{X}$, with guaranteed cost $J_u \leq \mu$.

proof.

For the proof of (3.43), from (3.40) we have

$$\begin{bmatrix} He\{-LW+Z\} & \star \\ W & -R^{-1} \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ W & 0 \\ 0 & I \end{bmatrix}^T}_{B_\perp^T} \underbrace{\begin{bmatrix} He{-}LW & \star & \star \\ K^T & 0 & \star \\ W & 0 & -R^{-1} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} I & 0 \\ W & 0 \\ 0 & I \end{bmatrix}}_{B_\perp} < 0$$

Now, with respect to the equivalence of case 2 and case 3 in Lemma 2.6, for the following choice of B and M we have $BB_\perp = 0$

$$B = \begin{bmatrix} W & -I & 0 \end{bmatrix}, \quad N = \begin{bmatrix} -I \\ \xi I \\ 0 \end{bmatrix}$$

Thus we have

$$M + NB + B^T N^T = \begin{bmatrix} He\{-LW - W\} & \star & \star \\ \xi W + K + I & -2\xi I & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0$$

The proof of (3.44) follows the same line as the proof of (3.30)–(3.31).

Moreover, the (3.45) is equivalent to $||k_i|| \leq \bar{u}$.

■

The Theorem 3.7 provides a clear advantage. It provides perfect conditions for dealing with large networks by avoiding the conservatism, limitations and computational burdens that the other approaches carry. To optimize this method, we propose the next remark with emphasis on minimizing the cost J_x.

REMARK 3.11 If the initial conditions $x(0)$ are known, one can minimize J_x solving the optimization problem

$$\min \quad \alpha\tau_1 + (1-\alpha)\tau_2 \tag{3.46}$$

for a given parameter $\alpha \in [0,1]$ subjected to (3.43)-(3.45) and

$$\begin{bmatrix} \tau_1 & \star \\ x(0) & W \end{bmatrix} > 0 \tag{3.47a}$$

$$\begin{bmatrix} \tau_2 & \star \\ K\mathbb{1} & Q^{-1} \end{bmatrix} > 0 \tag{3.47b}$$

Observe that

$$J_x = \int_0^\infty x(t)^T Rx(t)\, dt = \underbrace{\int_0^T x(t)^T Rx(t)\, dt}_{J_{x_1}} + \underbrace{\lim_{T_f \to \infty} (T_f - \mu/(\mathbb{1}^T KQK\mathbb{1}))\bar{x}^T R\bar{x}}_{J_{x_2}}$$

With $\bar{x} = x(T) = \exp\left(\dfrac{\mu(-L+K)}{\mathbb{1}^T KQK\mathbb{1}}\right) x(0)$. The minimization of τ_2 yields the minimization of J_{x_2}, that is, $V(T) \approx 0$ with $T = \mu/\mathbb{1}^T KQK\mathbb{1}$. The minimization of τ_1 yields the minimization of J_{x_1}, that means $J_{x_1} \leq x(0)^T W^{-1} x(0)$ according to (3.43).

3.6 POLYTOPIC APPROACH FOR INTERMEDIATE CONSENSUS

In this section that should be considered as an extension separate from the other approaches, we deal with the problem of intermediate consensus in MASs. This approach deals with cases where $d \in (0,1)$ using polytopic approach along with quadratic Lyapunov function for obtaining ellipsoidal level curves that estimate the invariant region in this case. Let us define $x_{d_i}(t) = x_i(t) - d$ ($x_d(t) = x(t) - \mathbb{1}d$) and the closed-loop system (3.2) is rewritten as

$$\dot{x}_d(t) = -Lx_d(t) + B(x_d(t))u(t) \tag{3.48}$$

Where $x_{d_i} \in [-d,\ 1-d]$, $i = 1, \ldots, N$. Observe that the functionals J_x and J_u can be rewritten as $\int_0^\infty z(t)^T z(t)\, dt$ and $\int_0^\infty y(t)^T y(t)\, dt$, respectively, where

$$\begin{aligned} z(t) &= R^{\frac{1}{2}} x_d(t) \\ y(t) &= Q^{\frac{1}{2}} u(t). \end{aligned} \tag{3.49}$$

The constraint (3.7) can be incorporated in the dynamics as $u(t) = sat(v(t))$ using the standard decentralized saturation function sat$(v_{(\ell)}) =$ sign$(v_{(\ell)}) \min(|v_{(\ell)}|, \bar{u})$, $\ell = 1, \ldots, N$, where v is an unbounded control signal to be designed. The, one has

$$\dot{x}_d(t) = -Lx_d(t) + B(x_d(t))\text{sat}(v(t)). \tag{3.50}$$

The system (3.50) can be rewritten using the decentralized deadzone nonlinearity $\psi(v) = v - sat(v)$

$$\dot{x}_d(t) = -Lx_d(t) + B(x_d(t))v(t) - B(x_d(t))\psi(v(t)). \tag{3.51}$$

We describe the bilinear product in (3.51) using a polytopic approach. In this approach, we define a polyhedral set to which all the initial conditions belong. The polyhedral set of interest in this work representing $x_{d_i} \in [-d,\ 1-d]$, $i = 1, \ldots, N$, is given by

$$\chi_d = \left\{ x_d \in \mathbb{R}^N : \Omega x_d \preccurlyeq \mathbb{1} \right\} \tag{3.52}$$

where $\Omega = I_N \otimes \begin{bmatrix} -1/d \\ 1/(1-d) \end{bmatrix} \in \mathbb{R}^{2N \times N}$, $\mathbb{1} \in \mathbb{R}^N$, and $0 \in \chi_d$.

The set (3.52) can also be represented in terms of its 2^n vertices obtained from Ω by

$$\chi_d = \text{co}\left\{ h_1, h_2, \ldots, h_{2^N} \right\}$$

Hence, the polyhedral set χ_d induces the following matrix valued polytope

$$\mathcal{P} = \text{co}\{B(h_i),\ i = 1, \ldots, 2^N\}$$

and for any $x_d \in \chi_d$, one has

$$B(x_d) = \sum_{i=1}^{2^N} \alpha_i(t) h_i =: \mathbb{B}(\alpha(t))$$

with $\alpha(t)$ belonging to the unit simplex

$$\mathcal{U} = \{\alpha \in \mathbb{R}^{2^N} : \sum_{i=1}^{2^N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, 2^N\}$$

for all $t \geq 0$.

For the case $d \in (0,1)$, observe that if we consider the control law $v(t) = Kx_d(t)$, $K \in \mathbb{R}^{N \times N}$, the closed-loop system $\dot{x}_d(t) = (-L + \mathbb{B}(\alpha(t))K)x_d(t) - \mathbb{B}(\alpha(t))\psi(Kx_d(t))$ loses controllability for $0 \in \chi_d$ and it is not possible to exclude it for the case $d \in (0,1)$. To circumvent the lack the controllability in $\mathcal{P}$, we propose the following control law

$$v(t) = B(x_d)Kx_d(t), \quad K = diag(k_1, \ldots, k_N) \in \mathbb{R}^{N \times N} \tag{3.53}$$

where $k_i \in \mathbb{R}$ are state feedback gains to be designed. Note that, $B(x_d)B(x_d) = B(\hat{x}_d)$, $\hat{x}_d = \text{diag}(x_{d_1}^2, \ldots, x_{d_N}^2)$. Let us consider the following interval for $x_{d_i}^2$

$$x_{d_i}^2 \in [\varepsilon, \max(d^2, (1-d)^2)] \tag{3.54}$$

where ε a positive scalar arbitrarily small. This means that we are excluding 0 from the interval of x_d. Therefore, one can write $B(\hat{x}_d)$ as

$$B(\hat{x}_d) = \sum_{i=1}^{2^N} \gamma(t)\vartheta_i = \mathcal{B}(\gamma(t)) \tag{3.55}$$

with $\gamma(t) \in \mathcal{U}$ and ϑ_i vertices obtained from (3.54).

The closed-loop system is given by

$$\dot{x_d}(t) = (-L + \mathcal{B}(\gamma(t))K)x_d(t) - \mathbb{B}(\alpha(t))\psi(\mathbb{B}(\alpha(t))Kx_d(t)) \tag{3.56a}$$

$$z(t) = R^{\frac{1}{2}}x_d(t) \tag{3.56b}$$

$$y(t) = Q^{\frac{1}{2}}\mathbb{B}(\alpha(t))Kx_d(t) + Q^{\frac{1}{2}}\psi(\mathbb{B}(\alpha(t))Kx_d(t)). \tag{3.56c}$$

Note that the price paid to obtain a controllable linear part in (3.56a) is to introduce an extra polytope in the closed-loop system.

The objective is To design the state feedback gain K and to determine a region $\mathcal{S}_d \subseteq \chi_d \cap \Pi$, as large as possible, such that the trajectories of the closed-loop system (3.56a) starting from any initial condition $x_d(0) \in \mathcal{S}_d$ converge exponentially toward the origin of (3.56a) for $d \in (0, 1)$. To deal with the state constraints we adopted an ellipsoid invariant method with the following level curve of the Lyapunov function $V(t) = x_d(t)^T W^{-1} x_d(t)$

$$\mathcal{S}_d := \left\{ x_d \in \mathbb{R}^N : x_d^T W^{-1} x_d \leq 1 \right\}. \tag{3.57}$$

The solution of this part is stated in the form of a theorem that is expressed in the following.

Theorem 3.8

If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{N\times N}$, a diagonal positive definite matrix $S \in \mathbb{R}^{N\times N}$, diagonal matrices $F \in \mathbb{R}^{N\times N}$ and $Z \in \mathbb{R}^{N\times N}$, a matrix $Y \in \mathbb{R}^{N\times N}$, and a given scalar $\varepsilon > 0$ such that the following inequalities hold for all $(\alpha(t), \gamma(t)) \in \mathcal{U} \times \mathcal{U}$

$$\begin{bmatrix} \mathrm{He}\{-LF + \mathcal{B}(\gamma(t))Z\} & \star & \star & \star & \star \\ W - F + \varepsilon(-LF + \mathcal{B}(\gamma(t))Z)^T & -\varepsilon(F + F^T) & \star & \star & \star \\ S\mathbb{B}(\alpha(t))^T + Y & 0 & -2S & \star & \star \\ \mathbb{B}(\alpha(t))Z & 0 & S & -Q^{-1} & \star \\ W & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0 \tag{3.58}$$

$$\begin{bmatrix} -W + F + F^T & \star \\ \Omega_{(i)}F & 1 \end{bmatrix} \geq 0, \quad \forall i = 1, \ldots, 2N \tag{3.59}$$

$$\begin{bmatrix} -W + F + F^T & \star \\ (\mathbb{B}(\alpha(t))Z)_{(i)} - Y_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, N \tag{3.60}$$

Then, the closed-loop system (3.56a) with $K = ZF^{-1}$ is asymptotically stable with $\mathcal{S}_d \subseteq \chi_d \cap \Pi$ as an estimation of the domain of attraction of the origin, and $J_x \leq x_d(0)^T W^{-1} x_d(0)$. Moreover, if the initial condition $x(0)$ satisfies

$$\begin{bmatrix} W & x(0) - d\mathbb{1} \\ \star & \mu \end{bmatrix} > 0 \tag{3.61}$$

then $J_u < \mu$.

proof.

Proof of (3.58) and (3.61):

First, consider the closed-loop system (3.56a) and the Lyapunov function $V(t) = x_d(t)^T W^{-1} x_d(t)$. The integral from 0 to ∞ of

$$\dot{V}(t) + y(t)^T y(t) + z(t)^T z(t) < 0 \tag{3.62}$$

implies $J_x < V(0)$ and $J_u < V(0)$. Therefore, if we apply the Schur complement in (3.61), we obtain $x_d(0)^T W^{-1} x_d(0 < \mu$ and assure (3.8). The conservatism of $J_u < V(0)$ can be relaxed by imposing $R < \gamma Q$ with $0 < \gamma \ll 1$.

Using Lemma 2.2, the inequality (3.62) holds if $\dot{V} + y^T y + z^T z - 2\psi(v)^T T \psi(v) + 2\psi(v)^T T G x_d < 0$, and, considering (3.56), the last inequality is rewritten as

$$\begin{bmatrix} x_d \\ \psi(v) \end{bmatrix}^T \begin{bmatrix} \text{He}\{-W^{-1}L + B(x_d)(B(x_d)K)\} + (B(x_d)K)^T Q (B(x_d)K) + R & * \\ B(x_d)^T W^{-1} + Q(B(x_d)K) + TG & Q - 2T \end{bmatrix} \begin{bmatrix} x_d \\ \psi(v) \end{bmatrix} < 0 \tag{3.63}$$

By pre- and post-multiplying the above inequality by diag(W, T^{-1}), one has

$$\begin{bmatrix} \text{He}\{-LW + B(x_d)(B(x_d)Z)\} + (B(x_d)Z)^T Q (B(x_d)Z) + W^T R W & \star \\ SB(x_d)^T + SQ(B(x_d)Z) + Y & SQS - 2S \end{bmatrix} < 0,$$

where $Z = KW$, $S = T^{-1}$, and $Y = GW$. Using the Schur complement lemma, one has

$$\begin{bmatrix} \text{He}\{-LW + B(x_d)B(x_d)Z\} & \star & \star & \star \\ SB(x_d)^T + Y & -2S & \star & \star \\ B(x_d)Z & S & -Q^{-1} & \star \\ W & 0 & 0 & -R^{-1} \end{bmatrix} < 0$$

In this LMI, there is a multiplication of Lyapunov matrix W and $B(x_d)K$ that limits the choice of a proper W to a class of diagonal matrices. In order to overcome this limit, we use Lemma 2.6 and derive the LMI such that

$$\begin{bmatrix} \text{He}\{-LF + B(\hat{x_d})Z\} & \star & \star & \star & \star \\ W - F + \varepsilon(-LF + B(\hat{x_d})Z)^T & -\varepsilon(F + F^T) & \star & \star & \star \\ SB(x_d)^T + Y & 0 & -2S & \star & \star \\ B(x_d)Z & 0 & S & -Q^{-1} & \star \\ W & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0.$$

Finally, the above condition holds for all $x_d \in \chi_d - \{0\}$ if (3.58) is verified.

Proof of (3.59):

Using the inequality

$$(F - W)^T W^{-1}(F - W) \geq 0 \ \Rightarrow \ F^T W^{-1} F \geq -W + F + F^T, \tag{3.64}$$

one has that (3.59) implies

$$\begin{bmatrix} F^T W^{-1} F & \star \\ \Omega_{(i)} F & 1 \end{bmatrix} \geq 0$$

and, by pre-and-post multiplying the above inequality by $\text{diag}(F^{-1}, I)$, one has

$$\begin{bmatrix} W^{-1} & \star \\ \Omega_{(i)} & 1 \end{bmatrix} \geq 0$$

that verifies $\mathcal{S}_d \subseteq \chi_d$ [97].

Proof of (3.60):

Consider the set Π in Lemma 2.2 and $\mathcal{S}$ in (3.57). By using (3.64), (3.60) implies

$$\begin{bmatrix} F^T W^{-1} F & \star \\ (\mathbb{B}(\alpha(t))Z)_{(i)} - Y_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0$$

Then, by pre-and-post multiplying the above inequality by diag(F^{-1}, I), we have

$$\begin{bmatrix} W^{-1} & \star \\ (\mathbb{B}(\alpha(t))K)_{(i)} - G_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0$$

Where $K = \mathbb{B}(\alpha(t))ZF^{-1}$ and $G = YF^{-1}$. This verifies $\mathcal{S}_d \subseteq \Pi$ [86].

■

This approach makes it possible to achieve an intermediate consensus in a network of MASs. However, it carries a certain disadvantage. In this method, the estimation of the domain of attraction is made using quadratic Lyapunov function that results in ellipsoidal invariant region. The quadratic Lyapunov function is of symmetric nature and consequently, the value of the desired consensus d could limit the size of domain of attraction. To phrase it clearly, as the value of the desired intermediate consensus moves toward its edges (0 and 1), the obtainable domain of attraction gets smaller. With respect to this fact, this approach works adequately for consensus values away from its edges.

3.7 NUMERICAL COMPLEXITY

In this part, we express the numerical complexity of each theorem with respect to the network size. The goal is to clarify the computational burden of each theorem and verify the ability of the proposed methods in dealing with networks with a large number of agents.

The algorithms are implemented employing YALMIP [100] and SeDuMi [101]. The comparison of the numerical complexity of the proposed LMIs is verified by the number of LMI rows and scalar variables presented in the following table.

	Scalar variables	LMI rows
Theorem 3.1	$N^2 + 3N + 1$	$3N^2 + 7N$
Theorem 3.2	$N^2 + 2N + 1$	$4N$
Theorem 3.3	$4N^2 + N$	$N^2 + N(4 + 2^{2N+1})$
Theorem 3.4	$N^2 + 3N$	$3N \times 2^N$
Theorem 3.5	$N^2 + 2N$	$3N \times 2^N$
Theorem 3.6	$2N$	$N^2 + 5N$
Theorem 3.7	$4N^2 + 2N$	$9N$
Theorem 3.8	$2N^2 + 3N + 1$	$3N^2 + N(4 + (5 \times 2^{2N})) + 1$

Table 3.1 – Numerical complexities of different approaches.

In the following table, we provide numerical values for the computational complexity for different network sizes N.

		$N=3$	$N=5$	$N=20$	$N=100$
Theorem 3.1	Scalar variables	19	41	461	10301
	LMI rows	48	110	1340	30700
Theorem 3.2	Scalar variables	16	36	441	10201
	LMI rows	12	20	80	400
Theorem 3.3	Scalar variables	39	105	1620	40100
	LMI rows	405	10285	$480 + (20 \times 2^{41})$	$10400 + (100 \times 2^{201})$
Theorem 3.4	Scalar variables	18	40	460	10300
	LMI rows	72	480	60×2^{20}	300×2^{100}
Theorem 3.5	Scalar variables	15	35	440	10200
	LMI rows	72	480	60×2^{20}	300×2^{100}
Theorem 3.6	Scalar variables	6	10	40	200
	LMI rows	24	50	500	10500
Theorem 3.7	Scalar variables	42	110	1640	40200
	LMI rows	27	45	180	900
Theorem 3.8	Scalar variables	28	66	861	20301
	LMI rows	1000	25696	$1281 + (100 \times 2^{40})$	$30401 + (500 \times 2^{200})$

Table 3.2 – Numerical complexity of different approaches for several network sizes N.

As we can observe in Tables 3.1 and 3.2, Theorems 3.3–3.5 and Theorem 3.8 are not suitable for higher order networks. On the other hand, Theorems 3.2 and 3.6 carry the lowest computational burden. Moreover, Theorems 3.1 and 3.7 maintain an intermediate computational cost in comparison with other approaches. Therefore, Theorems 3.1–3.2 and Theorems 3.6–3.7 are the most suitable approaches for dealing with large networks, although the computational effort is high for very large networks.

4 NUMERICAL EXAMPLES

This section is dedicated to presenting a number of numerical examples to indicate the effectiveness of the presented results.

4.1 EXAMPLE 1

For this example, we use a network from [20] in the form of a connected undirected graph $\mathcal{G}$ with $N = 3$ agents where the Laplacian matrix is expressed by

$$L = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

Case $d = 0$ (state feedback control law)

In this case we consider $Q = 10^{-1}I$, $R = 10^{-1}I$, $\mu = 0.675$, and $\bar{u} = 0.9$ for Problem 1. The state-feedback control law is designed using Theorems 3.1 and 3.2, where $B_0 = B((0.5 + \varepsilon)\mathbb{1})$, with $\varepsilon = 0.1$. Theorem 3.1 yields $K = \text{diag}(-1.4143, -1.4143, -1.3886)$ and the control mechanism (3.23) is implemented using P obtained from Theorem 3.2. From Theorems 3.1 and 3.2 and by taking advantage of Remark 3.7, we have $J_x \leq 1$ and $J_u \leq 0.7019$ as the guaranteed costs. The real costs using (3.5) and (3.8), obtained from the trajectories of Figure 4.3, are $J_x = 0.0988$ and $J_u = 0.1777$. One can see that the real costs are upper bounded by the values obtained by the theorems, although the presence of some conservatism causes the control signal to be cut off before J_u reaches μ.

Figure 4.1 confirms that the estimation of the domain of attraction $\mathcal{S}$ encloses most of the region χ. In this figure, the domain of attraction for a positive system is obtained as the intersection of the ellipsoidal region $\mathcal{S}_a$ and the positive polyhedral set χ. Different illustrations in this figure aim to give an adequate perspective in the three dimensional space about the domain of attraction.

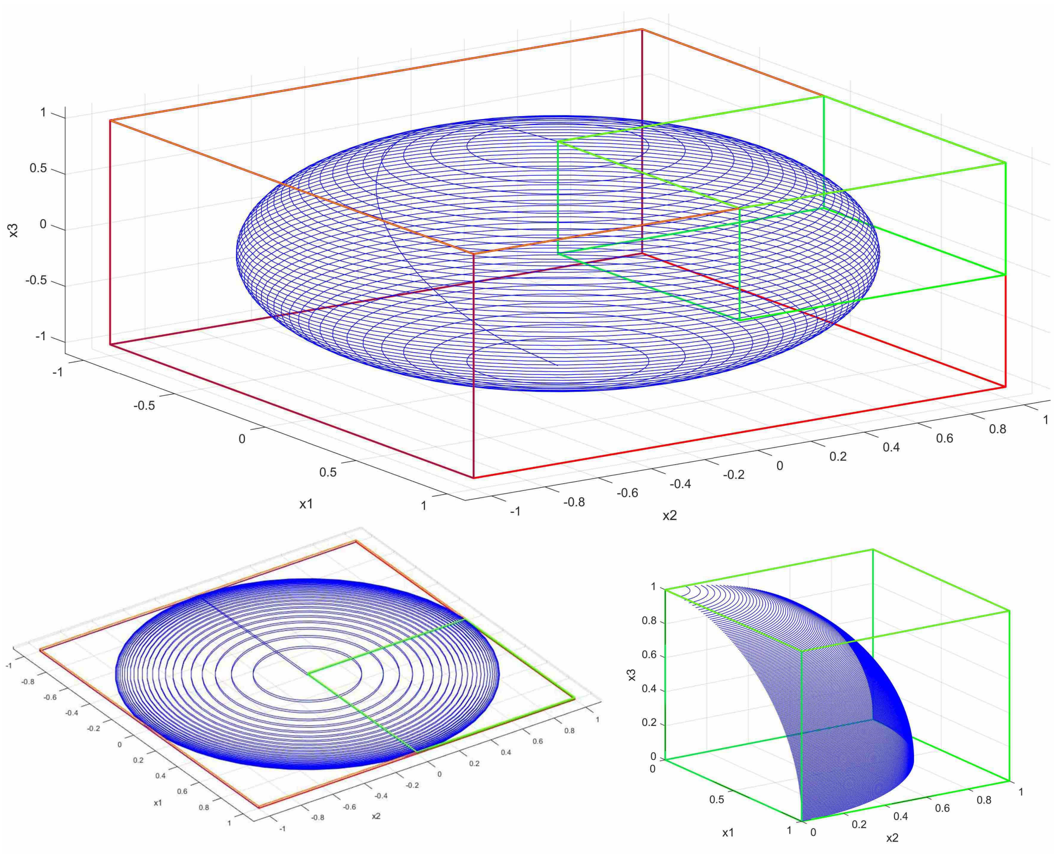


Figure 4.1 – Top: invariant region $\mathcal{S}_a$ (blue) contained in χ_a (red); bottom left: illustration of $\mathcal{S}_a$ (blue) and χ_a (red) in the 2-D plane ; bottom right: estimation of domain of attraction $\mathcal{S}$ (blue) enclosed in χ (green).

Figure 4.2 demonstrates the inclusion of $\mathcal{S}_u$ in $\mathcal{S}$, confirming the limitation caused by (3.8) in preventing agents' opinions from reaching the origin. In other words, not all agents in the domain of attraction have the adequate amount of energy to reach the origin with respect to defined budget limitations. Note that only the initial conditions inside the green region can reach the origin.

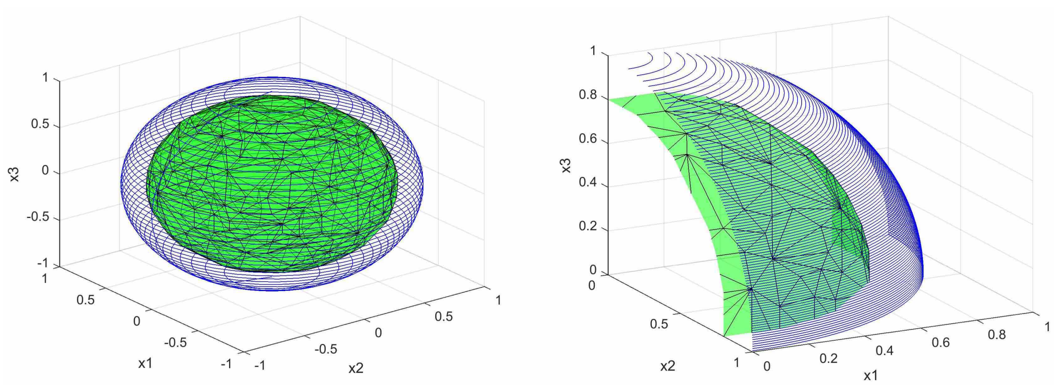


Figure 4.2 – Ellipsoidal regions $\mathcal{S}_a$ (blue, left figure) and $\mathcal{S}$ (blue, right figure) covering $\mathcal{S}_u$ (green).

Figure 4.3 shows the trajectories of agents' opinions (states) for the initial states

$$x(0) = (0.2673, 0.5345, 0.8018) \in \partial\mathcal{S}.$$

It can be seen that all trajectories converge towards the origin but without reaching it due to the energy limitation (3.8). One can observe that u_3 is saturated in the initial instant and the control signal is zero around $t = 5$s when afforded budget is finished according Remark 3.4. It should be noted that the final consensus value of agents' states is (0.1130) in this case.

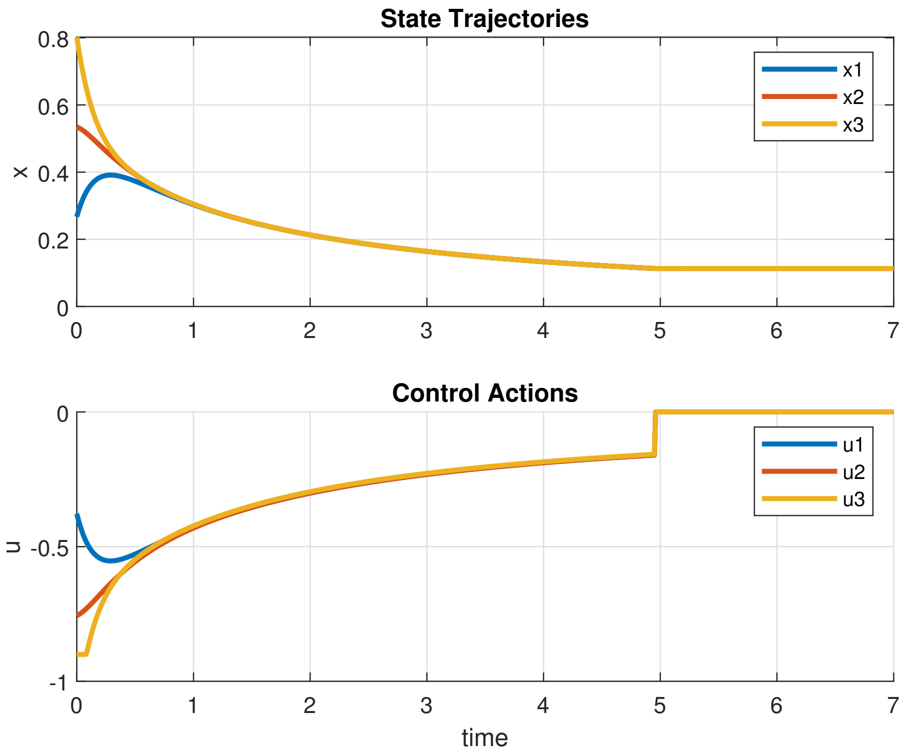


Figure 4.3 – Trajectories of agents' opinions for $x(0) = (0.2673, 0.5345, 0.8018)$ and their respective control actions for norm-bounded approach.

Remark 3.9 is employed to maximize the domain of attraction, as shown in Figure 4.4. The volume of the ellipsoid representing the estimation of the domain of attraction with Remark 3.7 (red ellipsoid) is approximately equal to 0.5175 while the one with Remark 3.9 (green ellipsoid) achieves a volume of approximately 0.7048, showing the efficiency of technique proposed in Remark 3.9. To be more explicit, while Remark 3.7 provide the necessary condition for maximizing the domain of attraction in a general way, Remark 3.9 provide specifications to achieve the maximization for a selected part of state-space (positive orthant, for instance).

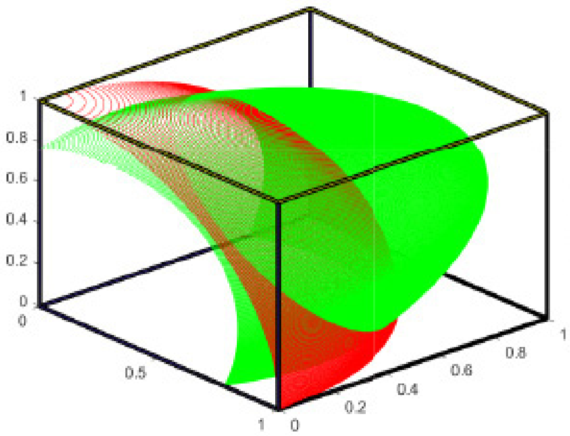


Figure 4.4 – The figure shows the original domain of attraction of Example 4.1 (red) and the maximized domain of attraction of the same example with Remark 3.9 (green).

Case $d = 0$ (on-off control law)

We keep the same parameters ($Q = 10^{-1}I$, $\mu = 0.675$ and $\bar{u} = 0.9$), also we adapt $R = I$ in Theorem 3.6 to design the control law (3.37). The difference in the parameter Q compared to the previous example is to augment the result for this particular case, that is, we modify parameter Q in each example to obtain a better performance. In this case The objective is to compare this approach with the state feedback control law designed by Theorem 3.1. Theorem 3.6 yields $K = \text{diag}(-0.6017, -0.6039, -0.6212)$ and $T = 6.05$s by exploiting the manimization criteria presented in Remark 3.11.

Figure 4.5 shows the domain of attraction and the agents's trajectories, with the same initial conditions as before. The agents start their trajectories at the edge of the domain of attraction and converge to the origin. Figure 4.6 illustrates the time simulation of the agents' trajectories. At the time $t = T$ the control input becomes zero as the switching mechanism gets activated. As we can see, the final value of the agents (0.01337) are closer to the origin compared with the state feedback control law illustrated in Figure 4.3.

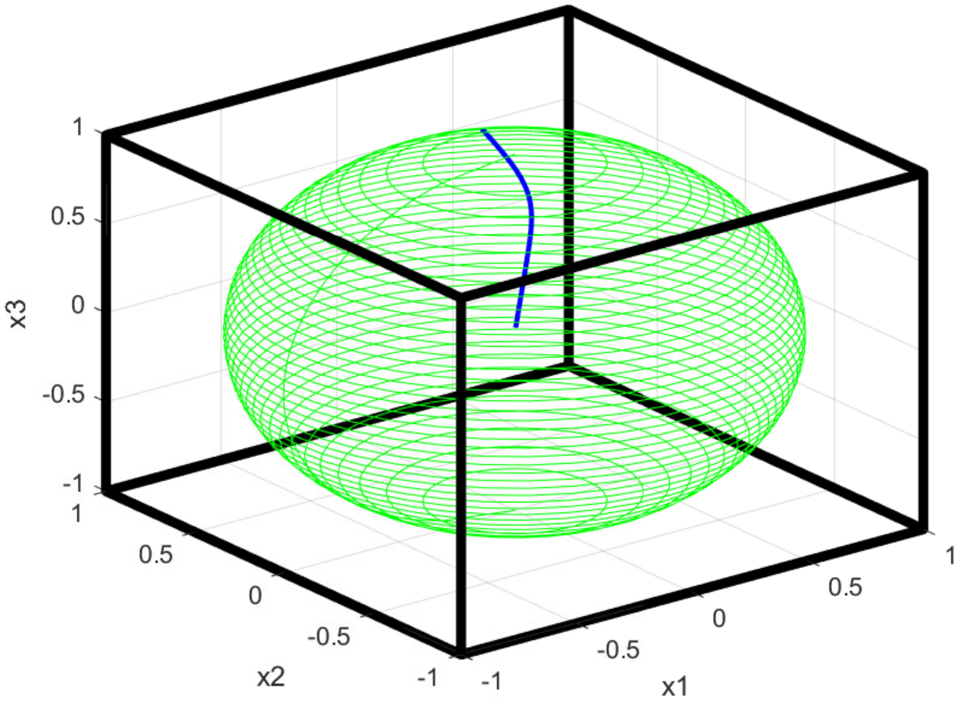


Figure 4.5 – Invariant region $\mathcal{S}_a$ obtained by Theorem 3.6 (green ellipsoid) and agents' trajectory (blue line) for the initial condindition $x(0) = (0.2673, 0.5345, 0.8018)$.

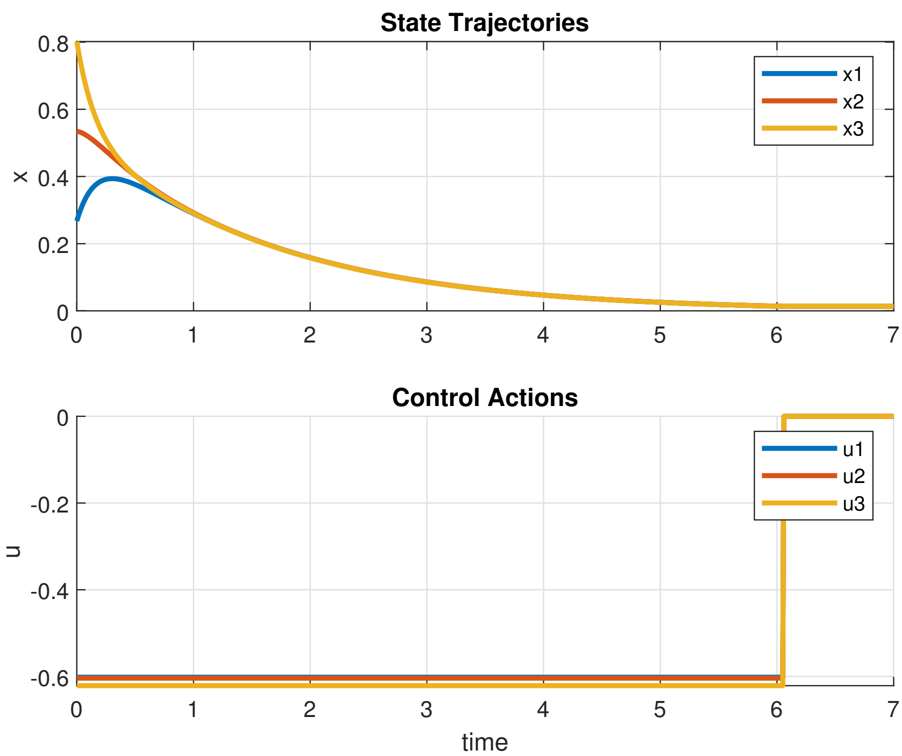


Figure 4.6 – Time simulation of agents' states and their control input using Theorem 3.6.

Additionally, we apply the same conditions ($R = I$,$Q = 10^{-1}I$, $\mu = 0.675$ and $\bar{u} = 0.9$) to Theorem 3.7 for making further comparison between different approaches in this case. By employing Remark 3.11 as the cost minimization criteria, Theorem 3.7 yields $K = \text{diag}(-0.7216, -0.7277, -0.7339)$ with $T = 4.24$. The time simulation of agents' trajectories is illustrated in Figure 4.7.

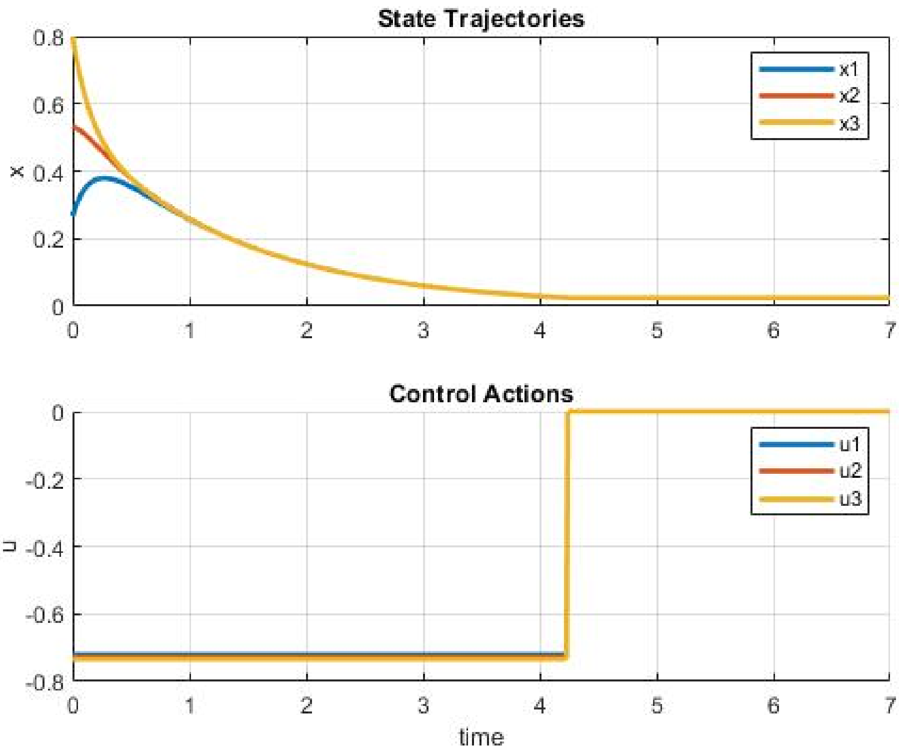


Figure 4.7 – Time simulation of agents states and their respective control input using Theorem 3.7.

From Theorem 3.7 and by using Remark 3.11, we have $J_x \leq 0.8449$ and $J_u \leq 0.675$ as guaranteed costs for the closed-loop system. Considering the time simulation of agents' trajectories in Figure 4.7, one has $J_x = 0.6090$ and $J_u = 0.6737$. We can observe that Theorem 3.7 with Remark 3.11 provides tight bounds on the costs and a precise mechanism to cut off the control signal to assure $J_u \leq \mu$. It is evident that the agents are successfully swayed to the desired consensus and the control action is deactivated at $t = T$, resulting in a final consensus value of (0.02445). Moreover, we have chosen different initial conditions on the border of the polyhedral estimation of domain of attraction to illustrate that all the points belonging to it would converge asymptotically toward the origin. We have depicted this illustration in Figure 4.8.

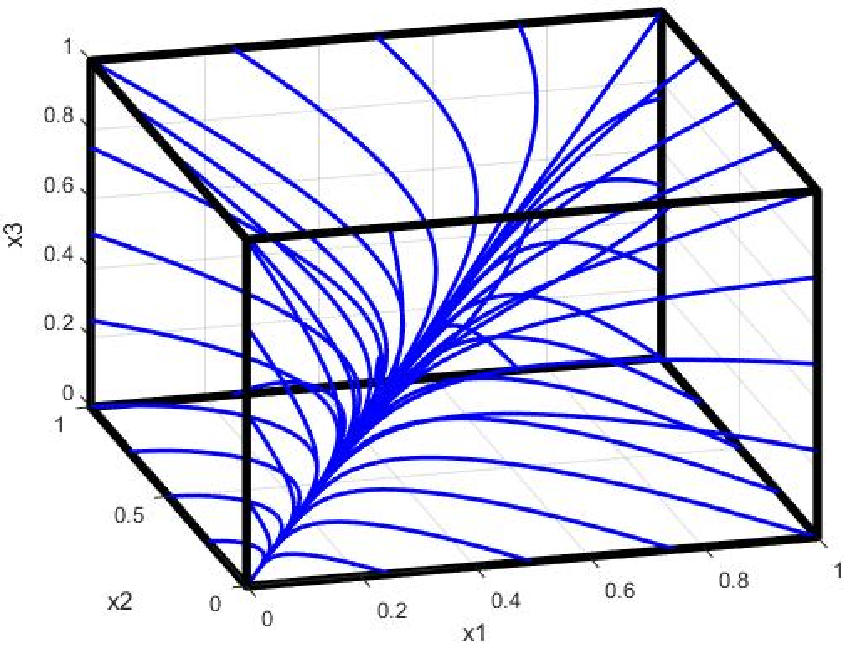


Figure 4.8 – Evolution of agents' trajectories considering different initial conditions on the border of the polyhedral estimation of domain of attraction.

As it could be observed, all the trajectories converge asymtotically to the origin. To provide a comparison between the performance of Theorem 3.6 and Theorem 3.7, it is evident that the final consensus value is slightly smaller by using Theorem 3.6. However, Theorem 3.7 provides a faster convergence instead. Thus, each of these two approaches carry their own advantages with respect to this particular example.

Finally, by comparing the results of the same case with different properties (with state-feedback and on-off control law), one can conclude that Theorems 3.6 and 3.7 have solved the same problem more effectively than Theorem 3.1 as the final consensus value is considerably closer to the origin alongside a higher convergence rate. Moreover, by the on-off control law, we avoid creating bilinear products in the system dynamics which makes it easier to apply. It is noteworthy that the estimation of domain of attraction in Theorem 3.6 and Theorem 3.1 reveal identical results.

Case $d = 0.5$

In this case, Theorem 3.8 is implemented with $Q = 10^{-6}I$, $R = 10^{-6}I$, $\mu = 2$, $\bar{u} = 0.99$ and $\varepsilon = 10^{-6}$. The estimation of the domain of attraction is depicted in Figures 4.9 and 4.10. As it could be observed in Figure 4.9, the estimation of the domain of attraction $\mathcal{S}_d$ is contained in χ_d. For the case $d = 0.5$, we observe that the estimation of the domain of attraction is the optimal one that can be obtained by an ellipsoidal region inside the polyhedral set. Figure 4.10 shows that all points of the domain of attractions are covered by the region where the energy constraint $J_u < \mu$ is satisfied. In other words, due to the specification of μ, all points belonging to the domain of attraction have enough energy to reach the origin.

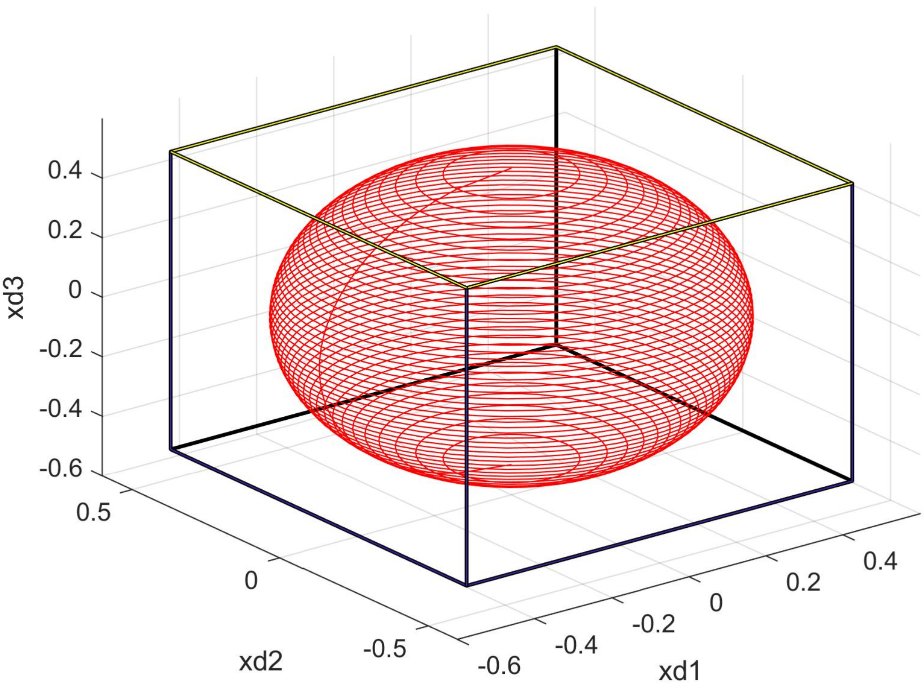


Figure 4.9 – Estimation of the domain of attraction $\mathcal{S}_d$ for the case $d = 0.5$ using Theorem 3.8 (red ellipsoid) contained in the polyhedral set χ_d (black box) with respect to the value of d.

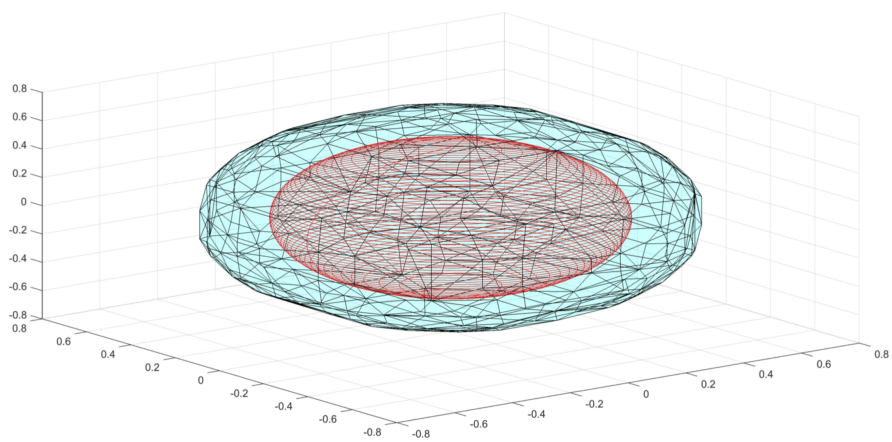


Figure 4.10 – Domain of attraction (red region) inside the space of initial conditions satisfing the energy constraint $J_u < \mu$ (cyan space).

The time simulation of agents' trajectories is illustrated in Figure 4.11 for the initial condition $x(0) = (0.05, 0.95, 0.3)$. We can observe that the trajectories converge to the desired consensus over time.

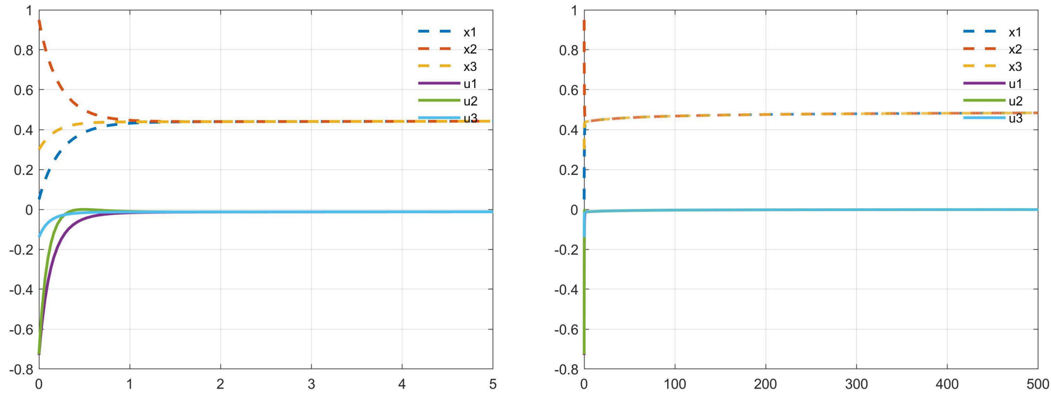


Figure 4.11 – The time simulations of agents' trajectories. Simulation placed on the left side considers $t \in [0, 5]$ and the simulations placed on the right side considers $t \in [0, 500]$ as their time-span.

4.2 EXAMPLE 2

This example deals with a social network of $N = 20$ agents represented by a connected directed graph displayed in Figure 4.12. Using Theorem 3.1 with Remark 3.8, the state feedback gain K is obtained to solve Problem 1 with $Q = 10^{-1}I$, $R = 10^{-1}I$, $\mu = 21$, $\bar{u} = 0.9$, and $\varepsilon = 0.02$. As it is visualized in Figure 4.13, all the trajectories converge successfully toward the origin. Implementing (3.23) using P designed by Theorem 3.2, at $t = 2.3$ the available investment is spent ($J_u = \mu$) and, from this point on, the trajectories converge towards the consensus (open-loop) manifold. We note that the control signal related with the agent 1 with higher initial condition ($x_1(0) = 0.8$) saturates in the initial instant ($u_1 = -0.9$), showing that the algorithm is able to deliver the highest control action to the to the agent furthest from the desired consensus value. Finally, due to the budget constraint, agents are prevented from reaching the exact consensus value at $d = 0$.

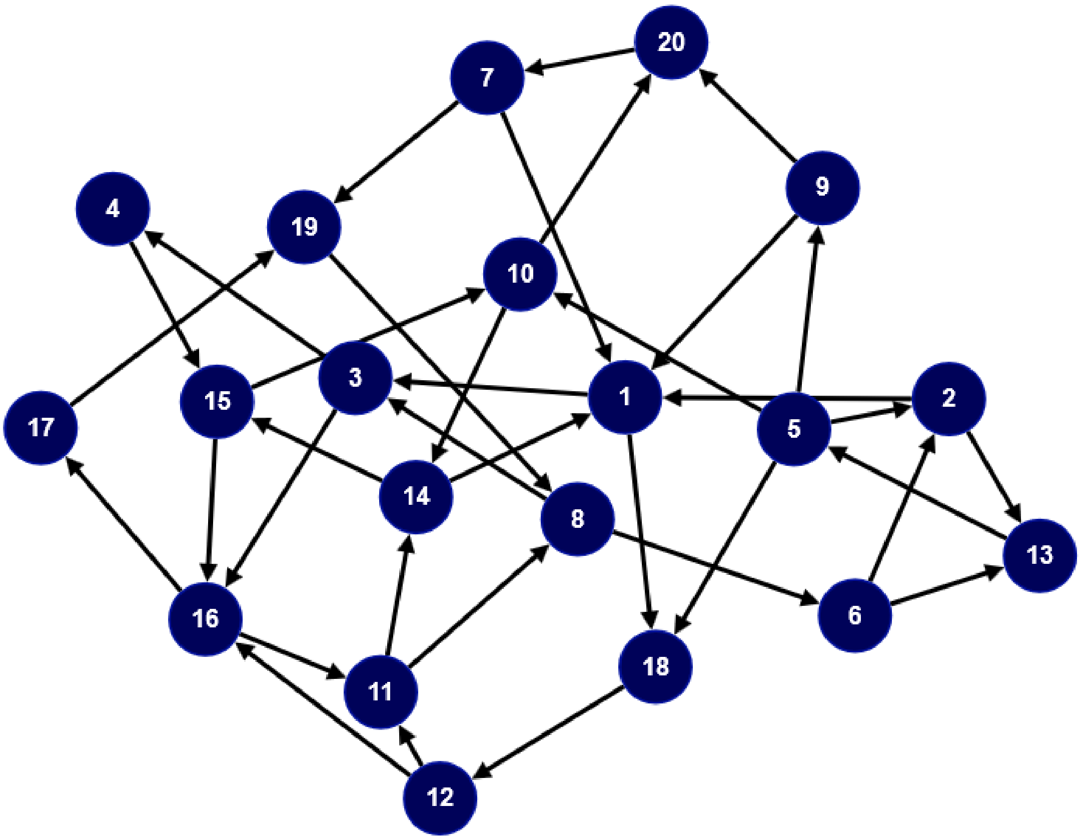


Figure 4.12 – The visualization of targeted network in the form of a connected directed graph with 20 agents.

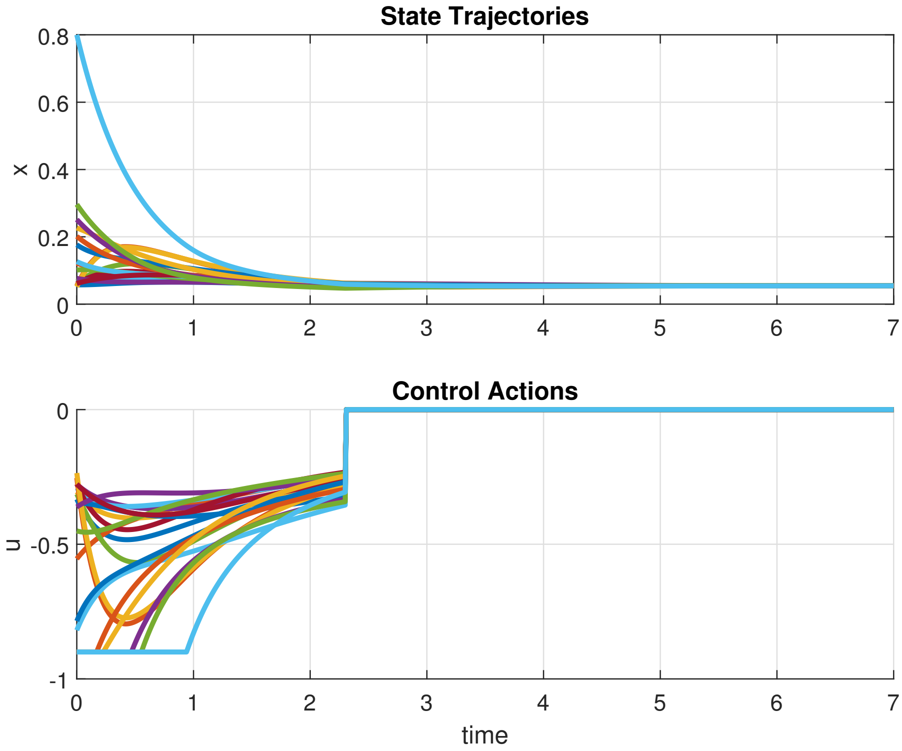


Figure 4.13 – Trajectories of the agents for some initial condition $x(0) \in \partial\mathcal{S}$ and control signal for Example 4.2.

4.3 EXAMPLE 3

This example deals with a network of $N = 5$ agents formed by the graph depicted in Figure 4.14. Theorem 3.3 is applied to achieve maximization of domain of attraction for the feedback gain K obtained with respect to Theorem 3.4 and $Q = R = 10^{-1}I$, $\bar{u} = 0.9$, $\epsilon = 0.01$ and $\mu = 32$. Figure 4.15 shows the trajectories of the agents. By using Theorem 3.5, at $t = 4.69$, when the guaranteed budget μ is reached, the external control action is deactivated and the exact desired consensus $d = 0$ is not achieved.

It could be concluded that by realizing a polyhedral invariant region instead of an ellipsoidal equivalent, we have extended the domain of attraction in a way that we are authorized to select any possible valid state as an initial state. Thus, despite the computional cost of this method, in smaller networks the polyhedral invariant region provides better results in terms of maximization of domain of attraction in comparison with ellipsoidal invariant regions.

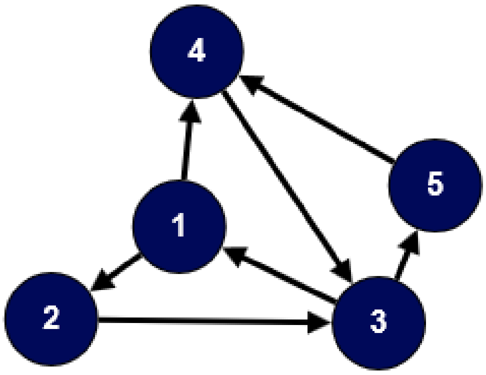


Figure 4.14 – Topology of the connected directed graph with 5 agents.

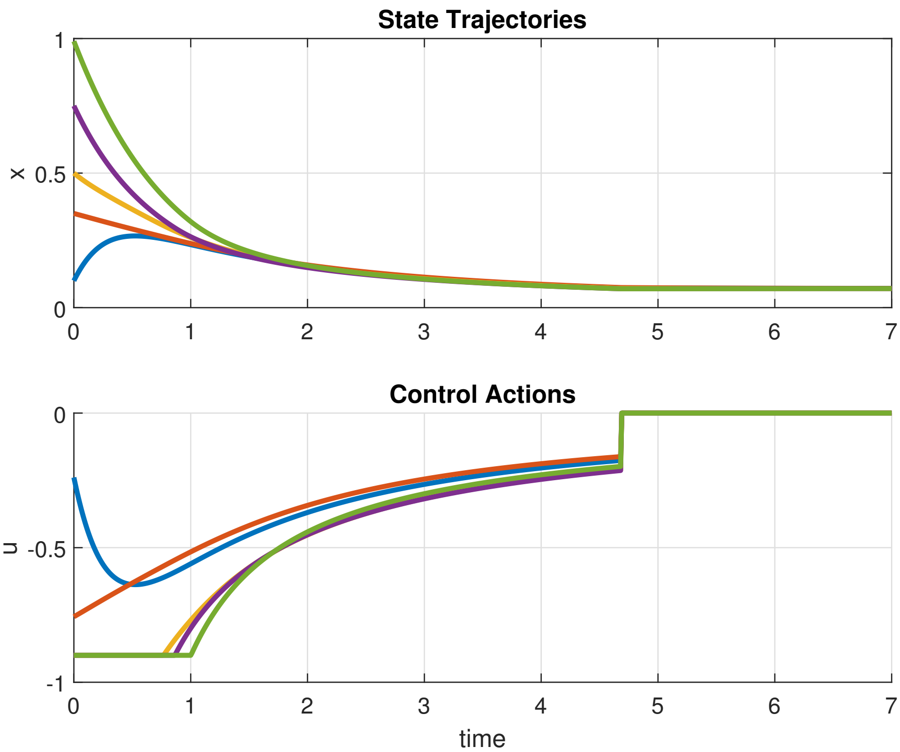


Figure 4.15 – Trajectories of the agents for initial values belonging to $\mathcal{X}$ and the control actions for Example 3.

4.4 EXAMPLE 4

In this example, we ilustrate the advantage of Theorem 3.7 on a large network of 100 agents. The goal of considering such large network is to assess the capability of Theorem 3.7

in dealing with large networks. The network topology is depicted in Figure 4.16.

Considering $R = I$, $Q = 10^{-1}I$, $\mu = 18$, $\bar{u} = 0.9$, $\xi = 1$ and $\epsilon = 10^{-3}$, the control law (3.37) is designed by Theorem 3.7. The time simulation of agents opinions is shown in Figure 4.17. The upper-bound of the agents' opinion value at the end of the simulation is equal to 0.0273. The initial conditions of the states are distributed in a uniform manner from 0.01 to 1. The switching mechanism becomes activated around $t = 4$ imposing no control action from this period. It is evident that all the agents' opinions converge successfully toward the desired consensus $d = 0$.

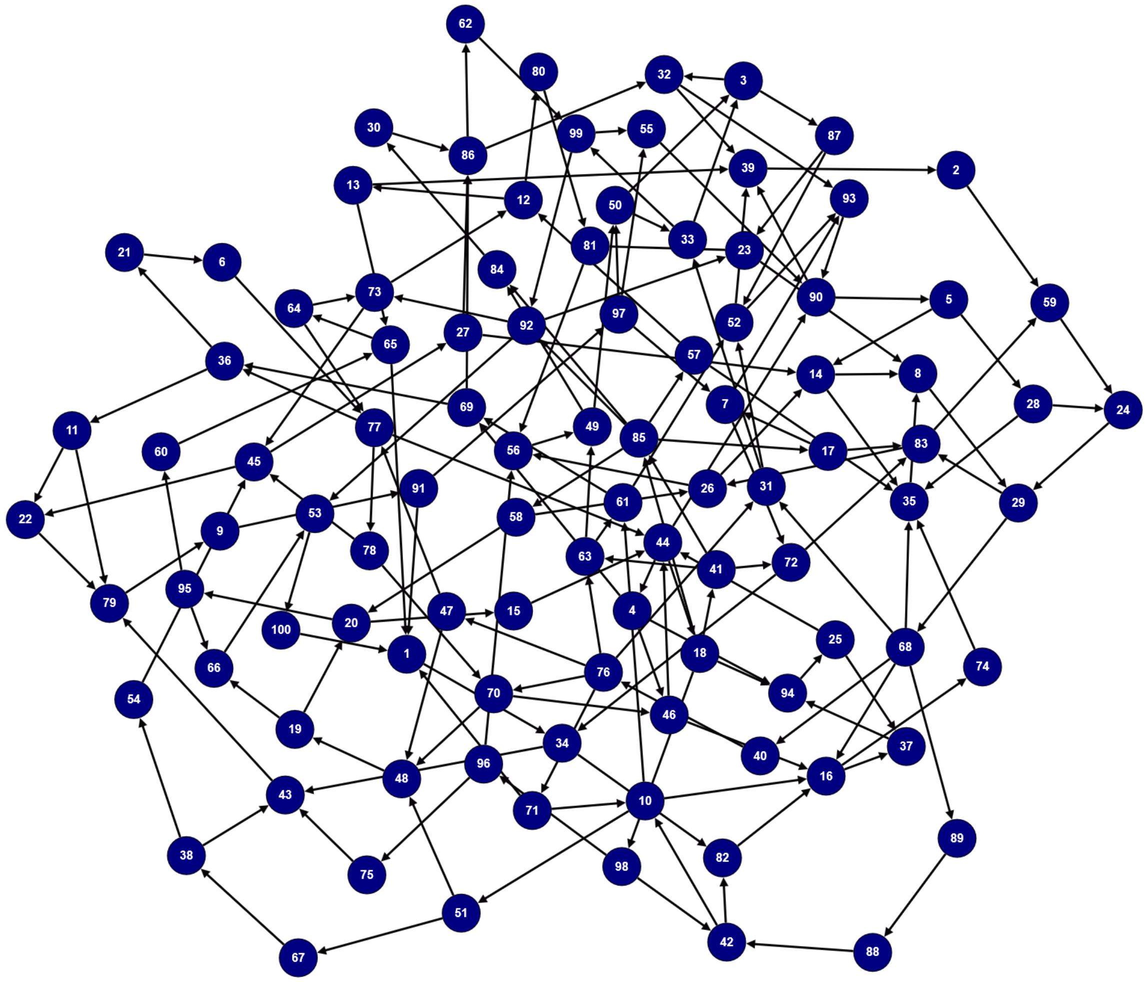


Figure 4.16 – Topology of the directed graph with 100 agents.

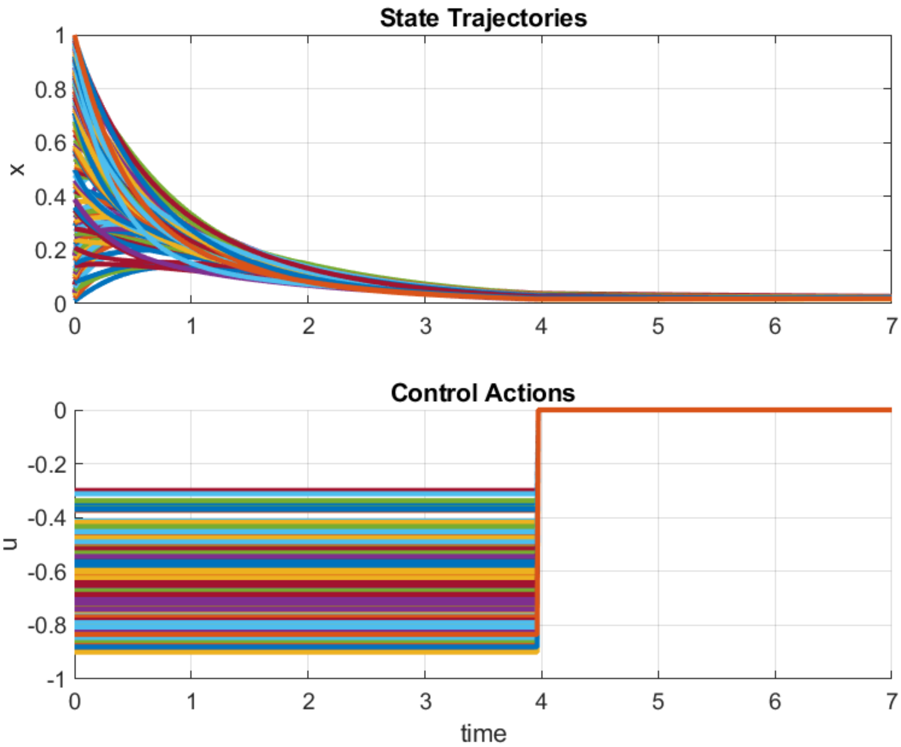


Figure 4.17 – The time simulation of 100 agents case using Theorem 3.7. It is evident that all the trajectories successfully converge to the desired consensus $d = 0$.

A very interesting capability of this method is its ability to deal with large networks, while the previous methods, based on the state feedback or on-off control law, are not able to provide a solution. Moreover, the performance of this method is quite adequate as illustrated in Figure 4.17. Therefore, this example shows the efficiency of the on-off control law (as it eliminate the bilinear product) for large networks.

5 CONCLUSION

This manuscript provided multiple LMI-based conditions to address consensus problem in MASs represented by connected graphs. Multiple Theorems are developed to deal with different conditions with respect to the desired consensus. Afterwards, the numerical examples took place to demonstrate the performance of each method and to make comparisons between them. The approaches are expressed considering practical conditions like budget constraint and limited influence. The bilinear product for the case of state feedback control law caused extra challenges in obtaining a stabilizing feedback gain.

Through this work we have proposed new sufficient conditions for the consensus control of multi-agent systems via state feedback and on-off control laws and by replacing bilinear product of systems dynamic with norm-bounded uncertainty and polyhedral model. Moreover, a number of remarks has been stated to complete and justify different aspects of the proposed method. It is evident that there exist a trade-off between the obtained gain and the size of the domain of attraction for the norm-bounded method which creates some levels of conservativeness. Hence, it could be a limit toward network convergence rate as higher gains lead to faster convergence of agents' trajectories. Moreover, in norm-bounded method as the number of agents grow, it becomes harder to evaluate valid initial condition away from the origin that are a part of the established and estimated domain of attraction. In other words, for larger networks the valid initial states become closer to the origin. The mentioned issues associated with the norm-bound method are addressed using polyhedral approach and on-off control law to augment the results. While the polyhedral approach provide the maximization of domain of attraction, it still carry a high computational cost that makes it suitable only for smaller networks. On the other hand, the on-off control law is capable of maximizing the domain of attraction while it has a considerably lower computational cost which makes it suitable for larger networks. Moreover, it provides faster convergence speed toward the origin in comparison with the state feedback control law. Additionally, we have explored the case of intermediate decision making using polyhedral approach. While it provide the consensus control through time, it still carries major drawbacks such as slow convergence speed created by the trade-off between the size of domain of attraction and the limits in establishing the ellipsoidal domain of attraction for different cases. In other words, because of the symmetric nature of ellipsoidal invariant region, the domain of attraction could get very small, which is not adequate as an answer to our problem. A possibility to overcome these challenges might be in exploring different choices for establishing the domain of attraction or using a different control law. Hence, this approach is subject to further developments in the future.

5.1 FUTURE WORKS

In this part, we propose some topics that could be the subject of our research in the future.

- To extend the design procedure for the intermediate decision ($d \in (0, 1)$) to large networks and to improve the estimation of the domain of attraction;
- To include the convergence rate of the trajectories as a performance criterion and to consider the attenuation of exogenous (disturbances) inputs using the indices H_2 or H_∞.
- To deal with MASs subject to uncertainties (robust consensus) or with time-varying topologies;
- To consider the consensus problem of MASs in the presence of communication delay in the flow of information between the agents;
- To use hybrid models that employ discrete control actions in continuous-time dynamics.

5.2 LIST OF PUBLICATIONS

Publication in process or published during the elaboration of this thesis:

- Alkhorshid, D. R., Tognetti, E. S., Morarescu, I. C. (2022). A bilinear systems approach with input saturation to control the agreement value of multi-agent systems. In European Control Conference (ECC22). (Accepted for publication.)

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