Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

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por

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Aos meus pais,

”O que é, o que é?
Clara e salgada,
Cabe em um olho
E pesa uma tonelada

Tem sabor de mar,
Pode ser discreta
Inquilina da dor,
Morada predileta

Na calada ela vem,
Refém da vingança,
Irmã do desespero,
Rival da esperança

Pode ser causada por
Vermes e mundanas
E o espinho da flor,
Cruel que você ama

Amante do drama,
Vem pra minha cama, por querer
Sem me perguntar, me fez sofrer

E eu que me julguei forte,
E eu que me senti,
Serei um fraco quando outras delas vir

Se o barato é louco e o processo é lento,
No momento, deixa eu caminhar contra o vento

Do que adianta eu ser durão e o coração ser vulnerável?
O vento não, ele é suave, mas é frio e implacável...”

Jesus chorou . (Racionais MC’s)

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Resumo

Neste trabalho, estudamos a existência de soluções para uma classe de problemas envolvendo um operador de crescimento rápido com peso e diferentes tipos de não linearidade. Na primeira parte do trabalho, estudamos o problema

$$(P) \qquad \left\{ \; -\mathrm{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + K(x)f(u) \quad \text{in } \mathbb{R}^N, \right.$$

onde $N \geq 3$, $K(x) = \exp(\frac{|x|^2}{4})$, $1 < q < 2$ e f é uma função contínua com crescimento arbitrário no infinito. Assumindo algumas hipóteses sobre o potencial a, fazemos um truncamento sobre a não linearidade f de modo a nos permitir usar Teoria de Genêro no problema truncado e finalmente, usando Iteração de Moser, nós mostramos que toda solução do problema truncado é também solução do problema original. Obtendo assim uma infinidade de soluções para este problema.

Na segunda parte, consideramos o sistema

$$(S) \qquad \begin{cases} -\mathrm{div}(K(x)\nabla u) = K(x)Q_u(u,v) + \frac{1}{2^*}K(x)H_u(u,v) \text{ in } \mathbb{R}^N, \\ \\ -\mathrm{div}(K(x)\nabla v) = K(x)Q_v(u,v) + \frac{1}{2^*}K(x)H_v(u,v) \text{ in } \mathbb{R}^N, \end{cases}$$

onde $N \geq 3$, $K(x) = \exp\left(|x|^2/4\right)$, Q e H são funções homogeneas de classe C^1 com H tendo crescimento crítico. Usando métodos variacionais, obtemos a existência de uma solução ground state para este sistema. Além disso, também provamos um resultado de existência para uma versão com crescimento supercrítico deste sistema.

Por último, consideramos o seguinte problema com crescimento crítico e um salto de descontinuidade

$$(P_a) \qquad -\mathrm{div}(K(x)\nabla u) = K(x)\left(\lambda h(x) + \mathbf{H}(u-a)|u|^{2^*-2}u\right) \quad \text{in } \mathbb{R}^N.$$

onde, a e λ são parâmetros positivos, h é uma função não negativa e $\mathbf{H}$ é a função de Heaviside definida por

$$\mathbf{H}(s) := \begin{cases} 0 & \text{if } s \leq 0, \\ 1 & \text{if } s > 0. \end{cases}.$$

Obtemos para $a > 0$ suficientemente pequeno duas soluções não negativas $u_i, i = 1,2$ para esta equação. A primeira solução u_1 é obtida usando uma versão do Teorema do Passo da Montanha para funcionais não diferenciáveis. A segunda solução u_2 foi encontrada através de uma aplicação local do Princípio Variacional de Ekeland. Além disso, mostramos também que os conjuntos de pontos $x \in \mathbb{R}^N$ tais que $u_i(x) > a$ têm medida positiva e os conjuntos de pontos $x \in \mathbb{R}^N$ tais que $u_i(x) = a$ têm medida nula.

Abstract

In this work, we study the existence of solutions for a class of problems involving an operator with rapidly growing weights and differents types of nonlinearities. First of all we study the problem

$$(P) \qquad \left\{ \; -\mathrm{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + K(x)f(u) \quad \text{in } \mathbb{R}^N, \right.$$

where $N \geq 3$, $K(x) = \exp(\frac{|x|^2}{4})$, $1 < q < 2$ and f is a continuous function with arbitrary growth at infinity. Under some assumptions on the potential a, we make a suitable truncation on the nonlinearity f in such a way that we can apply Genus Theory with the truncated problem and finally, using Moser iteration we show that each solution of truncated problem is a solution of the original problem.

In the second part, we consider the system

$$(S) \qquad \begin{cases} -\mathrm{div}(K(x)\nabla u) = K(x)Q_u(u,v) + \frac{1}{2^*}K(x)H_u(u,v) \text{ in } \mathbb{R}^N, \\ \\ -\mathrm{div}(K(x)\nabla v) = K(x)Q_v(u,v) + \frac{1}{2^*}K(x)H_v(u,v) \text{ in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $K(x) = \exp\left(|x|^2/4\right)$, Q and H are homogeneous functions of class C^1 with H having critical growth. Using variational methods, we obtain the existence of a ground state solution for this system. Furthermore, we also proved an existing result for a supercritical growth version of this system.

Finally, we consider the following problem with critical growth and a jump of discontinuity

$$(P_H) \qquad -\mathrm{div}(K(x)\nabla u) = K(x)\left(\lambda h(x) + \mathbf{H}(u-a)|u|^{2^*-2}u\right) \text{ in } \mathbb{R}^N.$$

where, a e λ are positive parameters, h is a nonnegative function and $\mathbf{H}$ is a Heaviside function defined by

$$\mathbf{H}(s) := \begin{cases} 0 \text{ if } s \leq 0, \\ 1 \text{ if } s > 0. \end{cases}$$

We obtain for $a > 0$ sufficiently small two nonnegative solutions $u_i, i = 1, 2$ for this equation. The first solution u_1 is obtained using a version of the Mountain Pass Theorem for nonsmooth functionals. The second solution u_2 was obtained through a local application of the Ekeland Variational Principle. In addition, we also show that the set of points $x \in \mathbb{R}^N$ such that $u_i(x) > a$ has positive measure and the set of points $x \in \mathbb{R}^N$ such that $u_i(x) = a$ has zero measure.

Contents

Introduction

In this work we are going to study equations involving an operator with rapidly growing weights. That is, the operator that appears on the left-hand side of the problems we deal with in this work, appears naturally when we look for solutions on the form

$$v(x,t) = t^{(2-N)/(N+2)}u(xt^{-1/2})$$

for the following parabolic equation

$$(P) \qquad v_t - \Delta v = |v|^{\frac{4}{(N-2)}}v, \ \mathbb{R}^N \times (0,+\infty).$$

According to [54], a function like $v(t,x)$ is called a self-similiar solution for (P). So, v is a solution of (P) if, and only if, u is a solution of the problem

$$(PE) \qquad -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \frac{1}{2^*-1}u + |u|^{2^*-2}u \ \text{ in } \ \mathbb{R}^N,$$

where $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = +\infty$ for $N = 1$ or $N = 2$. Haraux and Weissler considered in [54] problem (PE) in order to prove some nonuniqueness results for the Cauchy problem associated to (P) in the case $N = 1$. Among other properties, the solutions of (PE) have decay to zero exponentially at infinity (e.g., see [36] and [75]) and give some information on the asymptotic behavior of (P) (e.g., see [54] and [74]).

As far as we now, the first variational approach for this class of problems was done by Escobedo and Kavian in [36]. The artifice to start this variational approach is due to the observation that the exponential-type weight $K(x)$ verifies $\nabla K(x) = \frac{1}{2}xK(x)$. In this way, it was possible to write the equation in (PE) in the divergent form

$$(PE') \qquad -\operatorname{div}(K(x)\nabla u) = K(x)\left(\frac{1}{2^*-1}u + |u|^{2^*-2}u\right) \ \text{ in } \ \mathbb{R}^N,$$

in which the authors proved that the existence of positive solutions is related to the interaction of the parameter λ with the first positive eigenvalue of the associated linear problem

$$-\operatorname{div}(K(x)\nabla u) = \lambda K(x)u \ \text{ in } \mathbb{R}^N,$$

which is $\lambda_1 = \frac{N}{2}$. To this end, they prove some compactness results for the embedding $H^1(K_\theta) \subset L^2(K_\theta)$ under suitable conditions on the weights $K(\theta(x))$, where $H^1(K_\theta)$ is a weighted Sobolev space with a weight of the form $K_\theta(x) = \exp(\theta(x))$ with $\theta \in C^2(\mathbb{R}^N, \mathbb{R}_+)$. Furthermore, due to the similarity with the problem treated by Brezis and Nirenberg in [22], an attempt would be to use the same method, but as no test function was known to approximate the constant $S(K)$, the idea was to compare S and $S(K)$, where

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}} \quad \text{and} \quad S(K_\theta) = \inf_{u \in H^1(K_\theta)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} K_\theta(x)|\nabla u|^2}{\left(\int_{\mathbb{R}^N} K_\theta(x)|u|^{2^*}\right)^{2/2^*}}.$$

Thus, they proved that, when $N \geq 4$ the equation (EK) has positive solution if, and only if, $\lambda \in (N/4, N/2)$. When $N = 3$ there exists a positive solution if, and only if, $\lambda \in (1, N/2)$, and there is no positive solution for $\lambda \leq N/4$ and $\lambda \geq N/2$.

This dichotomy observed by Escobedo and Kavian was later extended by Catrina, Furtado and Montenegro in [25], where the authors considered the problem

$$-\Delta u - \frac{1}{4}\alpha|x|^{\alpha-2}(x \cdot \nabla u) = \lambda|x|^{\alpha-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \tag{$\mathcal{P}$}$$

where $\alpha \geq 2$, and they observed an analogous dichotomy happens, dependind on the parameters α and λ. Moreover, they also showed a non-existence result similar to the one in [13] relative to radial solutions of $(\mathcal{P})$. Analogously to the results of Escobedo and Kavian, the results obtained by Catrina, Furtado and Montenegro in [25] were established in terms of the first eigenvalue

$$\lambda_1(\alpha) = \frac{1}{4}\alpha(N - 2 + \alpha),$$

of the problem

$$-\text{div}(K(x)\nabla u) = \lambda K(x)|x|^{\alpha-2}u \quad \text{in } \mathbb{R}^N,$$

where $K(x) = \exp(\frac{|x|^\alpha}{4})$. To calculate the first eigenvalue $\lambda_1(\alpha)$, they considered the eigenfunction $\varphi_1(x) = e^{|x|^\alpha/4}$. The dichotomy obtained was as follows: If $2 \leq \alpha \leq N-2$, then the problem $(\mathcal{P}_2)$ has a positive solution if, and only if, $\lambda \in (\lambda_1(\alpha)/2, \lambda_1(\alpha))$; If $\alpha > N-2$ and $\lambda \in (\alpha^2/4, \lambda_1(\alpha))$ then the problem $(\mathcal{P})$ has a positive solution; There is no positive solution if $\lambda \leq \lambda_1(\alpha)/2$ or $\lambda \geq \lambda_1(\alpha)$. So if $\alpha > 2$ then the critical dimension depends on α. Other papers with existence, multiplicity results of positive or nodal solution on this class of problem can be seen in [37], [38], [40], [42], [45], [47], [48], [49], [50], [60], [64], [65] and references therein.

In this thesis, we study a class of elliptic problems involving the same operator mentioned above. This thesis has three chapters and two appendices. In each of the first three chapters we study a different problem. To make it easier to read, each chapter has an introduction where we repeat all the hypotheses about the problem studied in that chapter. We first consider an equation involving a potential a and a nonlinearity with no growth restriction at infinity. For this problem we obtain a result of multiplicity solution making use of genus theory. In the second part of the work, we consider a system with critical or supercritical growth and obtain a result of the existence of a nontrivial solution. In the final chapter, we consider an equation with critical growth and a jump of discontinuity given by the presence of a Heaviside function in the equation. For this problem we obtain a result of existence and asymptotic behavior.

As mentioned earlier, we started our work by studying the existence of infinitely many solutions for the equation

$$(P_a) \qquad -\Delta u - \frac{1}{2}(x \cdot \nabla u) = a(x)|u|^{q-2}u + f(u),$$

where $1 < q < 2$, $N \geq 3$, a is a positive function and the function $f : \mathbb{R} \to \mathbb{R}$ has arbitrary growth at infinity, that is, the function f is superlinear with no growth restriction and can be allowed to be in the range subcritical, critical or supercritical. More precisely, the nonlinearity f is assumed to be a $C^0(\mathbb{R})$ function satisfying:

(f_1) There exists $\delta > 0$ such that f is odd for $|s| \leq \delta$.

(f_2) There exists $2 < p < 2^*$ such that

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{p-1}} = 0.$$

(f_3) The function $\frac{f(s)}{s}$ is decreasing in $[-\delta, 0]$ and increasing in $[0, \delta]$.

Since our approach is variational, we need to rewrite the problem (P_a) in its divergent form

$$(P_a') \qquad -\operatorname{div}(K(x)\nabla u) = K(x)a(x)|u|^{q-2}u + K(x)f(u),$$

where $K(x) = \exp(\frac{|x|^2}{4})$, and we also need to make one hypothesis on the function a so that the functional associated to problem will be well-defined:

(a_1) The function a is positive in $\mathbb{R}^N$ and there exists $\sigma_q \in \mathbb{R}$ such that $a \in L_K^{\sigma_q}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with

$$\frac{p}{p-q} < \sigma_q \leq \frac{2}{2-q}.$$

In this first part of the work, our main result is:

Theorem 1.1.1. *Suppose that the function f satisfies $(f_1)-(f_3)$ and the function a satisfies (a_1). Then there exists $\lambda^* > 0$ such that, if $\|a\|_{\sigma_q} \in (0, \lambda^*)$, problem (P) has infinitely many weak solutions.*

In the proof of this result, we apply variational methods. However, since we have no control on the behavior of f at infinity, the associated functional to (P_a')

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx - \int_{\mathbb{R}^N} K(x)F(u)dx,$$

is not well defined in the entire space X, for which we will look for solutions and is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2 \, dx\right)^{\frac{1}{2}}.$$

In order to overcome this, we consider an auxiliary function g defined from f such that $g(s) = f(s)$ if s is small enough, and we consider the functional I of C^1 class given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx - \int_{\mathbb{R}^N} K(x)G(u)dx,$$

where $G(s) = \displaystyle\int_0^s g(t)dt$. So, the idea is to get critical points u of I such that $\|u\|_{L^\infty}$ is small enough in such a way that each solution of the modified problem was a solution to the original problem. According to this space function X, the solutions found are forced to have a rapid decay at infinity.

There is a vast literature concerning nonlinearities with arbitrary growth, for example, see [24], [31], [53], [59] and [73].

Chapter 1 is devoted to the proof of Theorem 1.1.1 and is organized as follows. In order to use variational methods, in Section 2 we define the proper spaces to address the problem and we use an argument inspired by [33]. In Section 2 we show existence of infinitely many critical points of the functional associated to problem (P_a'). In order to use Genus theory, it was necessary to make another truncation. In Section 3 we prove the main result making use of Moser's iteration.

In Chapter 2 we study the existence of nontrivial solutions for some systems with critical or supercritical growth. More precisely, we consider the systems

$$(S)\qquad \begin{cases} -\Delta u - \frac{1}{2}x\cdot\nabla u = Q_u(u,v) + \frac{1}{2^*}H_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v - \frac{1}{2}x\cdot\nabla v = Q_v(u,v) + \frac{1}{2^*}H_v(u,v) \text{ in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, Q and H are functions of class C^1 and

$$(SC)\qquad \begin{cases} -\Delta u - \frac{1}{2}x\cdot\nabla u = Q_u(u,v) + |u|^{\Upsilon_1-2}u \text{ in } \mathbb{R}^N, \\ -\Delta v - \frac{1}{2}x\cdot\nabla v = Q_v(u,v) + |v|^{\Upsilon_2-2}v \text{ in } \mathbb{R}^N, \end{cases}$$

where $\Upsilon_i > 2^*,\ i = 1,2$.

Setting $\mathbb{R}^2_+ := [0,\infty)\times[0,\infty)$, for any given $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+,\mathbb{R})$ satisfying the following properties.

($\mathcal{H}^q_0$) F is q-homogeneous, that is,

$$F(\lambda s,\lambda t) = \lambda^q F(u,v), \quad \text{for each} \quad \lambda > 0 \ \text{ and } \ (s,t)\in\mathbb{R}^2_+.$$

($\mathcal{H}^q_1$) There exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t))| \leq c_1\left(s^{q-1} + t^{q-1}\right) \quad \text{for each } (s,t)\in\mathbb{R}^2_+.$$

($\mathcal{H}_2$) $F(s,t) > 0$ for each $s,t > 0$.

($\mathcal{H}_3$) $\nabla F(1,0) = \nabla F(0,1) = (0,0)$.

($\mathcal{H}_4$) $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t)\in\mathbb{R}^2_+$.

The hypotheses on the functions Q and H are the following:

(A_1) $H \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$.

(A_2) The 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := H(s,t)$ is concave.

(A_3) $Q(s,t) \geq \sigma s^\gamma t^\beta$ for all $(s,t)\in\mathbb{R}^2_+$, with $\gamma,\beta > 1$, $\gamma+\beta =: p_1 \in (2,2^*)$ and σ satisfies

(i) $\sigma > 0$ if either $N \geq 4$, or $N = 3$ and $2^*-2 < p_1 < 2^*$;

(ii) σ is sufficiently large if $N = 3$ and $2 < p_1 < 2^*-2$.

($\widetilde{A}_3$) There exists $\sigma^* > 0$ such that $Q(s,t) \geq \sigma s^\gamma t^\beta$ for all $(s,t)\in\mathbb{R}^2_+$, $\gamma,\beta > 1$, $\gamma+\beta =: p_1 \in (2,2^*)$, for all $\sigma > \sigma^*$ and σ^* to be fixed later.

In this second part of the work, our main results are:

Theorem 2.1.1. *Assume that conditions* $(A_1),(A_2),(A_3)$ *are hold. Then, system* (S) *has a weak positive solution.*

Theorem 2.1.2. *Assume that conditions* $(A_1),(A_2),(\widetilde{A}_3)$ *are hold. Then, system* (SC) *has a weak positive solution.*

The hypotheses $(A_1)-(A_3)$ have already appeared in [32]. In that paper, de Morais Filho and Souto [32] investigate the existence of solutions for the following system:

$$(DM) \qquad \begin{cases} -\Delta_p u = Q_u(u,v) + H_u(u,v) \text{ in } \Omega, \\ -\Delta_p v = Q_v(u,v) + H_v(u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where Δ_p denotes the $p-$Laplacian operator, $p > 1$ and Ω is a bounded domain in $\mathbb{R}^N$.

In [32] they showed that the number

$$\widetilde{S}_H := \inf_{u,v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \mathrm{d}x + \int_\Omega |\nabla v|^p \mathrm{d}x}{\left(\displaystyle\int_\Omega H(u,v)\mathrm{d}x\right)^{p/p^*}},$$

plays an important role in the study of the system (DM), where they obtained a relationship between S, the best Sobolev constant, and $\widetilde{S}_H$ using the hypothesis (A_2) and guaranteed that S_H does not depend on Ω.

As always, the biggest difficulty in dealing with nonlinearities with critical growth is the lack of compact immersion. To get around this problem in (DM), the authors proved a version of the Concentration and Compactness Lemma due to Lions [61, Lemma 1.2].

In the proof of our results about the systems (S) and (SC), we use variational methods. Again, we use the fact that $2\nabla K(x) = xK(x)$ to rewrite the systems in its divergence form, so we look at systems

$$(S') \qquad \begin{cases} -\operatorname{div}(K(x)\nabla u) = K(x)Q_u(u,v) + \frac{1}{2^*}K(x)H_u(u,v) \text{ in } \mathbb{R}^N, \\ -\operatorname{div}(K(x)\nabla v) = K(x)Q_v(u,v) + \frac{1}{2^*}K(x)H_v(u,v) \text{ in } \mathbb{R}^N, \end{cases}$$

and

$$(SC') \qquad \begin{cases} -\operatorname{div}(K(x)\nabla u) = K(x)Q_u(u,v) + K(x)|u|^{\Upsilon_1-2}u \text{ in } \mathbb{R}^N, \\ -\operatorname{div}(K(x)\nabla v) = K(x)Q_v(u,v) + K(x)|v|^{\Upsilon_2-2}v \text{ in } \mathbb{R}^N, \end{cases}$$

with the same assumptions stated above, where $K(x) = \exp(|x|^2/4)$.

Our arguments were strongly influenced by [25], [32] and [71]. It was necessary to adapt some estimates found in [32] for the $\mathbb{R}^N$ considering the weight function K, in order to obtain a type of Brézis-Lieb lemma, as can be seen in Lemma 2.2.1. Also inspired by [32], making strong use of hypothesis (A_2) we proved a kind of Hölder inequality involving the function H and the weight function K, this is the content of Lemma 2.2.4. This Hölder inequality together with the estimates found in [25] allowed us to obtain a relationship between the constants S, S_K, $\widetilde{S}_{K,H}$ and $\widetilde{S}_H$.

In order to get around the lack of compactness, in [25] estimates are used involving the Talenti's functions type and the weight function K. As can be seen in Lemma 2.2.5 and Lemma 2.2.6, the estimates to get around the lack of compactness with the systems in $\mathbb{R}^N$ are more delicate. We completed the study that was done in [71] in the sense that we are studying another class of systems considering the critical and supercritical cases. To circumvent the lack of variational structure of the supercritical system, we use a truncation argument. To recover the solution to the original problem, we used the study on a problem in a bounded domain.

The Chapter 2 is organized as follows. Section 2 is devoted to variational framework and some preliminary results for the critical case. In Section 3 we show the solution of the critical case proving the second main result. In the Section 4 we study the supercritical case and prove the third main result.

In the Chapter 3 we study the existence of nonegative solutions to a class of elliptic problems with fast increasing weights and critical growth in $\mathbb{R}^N$. More precisely, we consider the following problem

$$-\Delta u - \frac{1}{2}(x\cdot\nabla u)) = \lambda h(x) + \mathbf{H}(u-a)|u|^{2^*-2}u \ \text{ in } \ \mathbb{R}^N, \tag{3.1.1}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $\lambda > 0$, $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a positive function such that

$$h \in L^{\theta}_K(\mathbb{R}^N) \text{ with } \frac{1}{\theta} + \frac{1}{2^*} = 1 \tag{3.1.5}$$

and $\mathbf{H}$ is the Heaviside function, i.e,

$$\mathbf{H}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

As in the studies of the previous problems, we use the fact noted by Escobedo and Kavian that the exponential-type weight $K(x) = \exp(|x|^2/4)$ verifies $\nabla K(x) = \frac{1}{2}xK(x)$, to check that the problem ((3.1.1)) can be written as

$$-\operatorname{div}(K(x)\nabla u) = K(x)\left(\lambda h(x) + \mathbf{H}(u-a)|u|^{2^*-2}u\right) \ \text{ in } \ \mathbb{R}^N, \tag{3.1.2}$$

Thanks to the presence of the discontinuity caused by the Heaviside function in the equation, the associated functional

$$I_{\lambda,a}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F_H(u)\,dx - \lambda\int_{\mathbb{R}^N} K(x)h(x)u\,dx,$$

is not $C^1(X,\mathbb{R})$ class. So, the presence of a discontinuity avoids the immediate application of usual variational techniques. We also have the difficult of the lack of compactness due to critical exponent and also due to the problem is on whole $\mathbb{R}^N$. Many authors have treated problems with discontinuous non-linearity in different ways, in our work we will use the techniques and results of Convex Analisys since the associated functional is locally Lipschitz.

Our main result in Chapter 3 is:

Theorem 0.0.1. *Assume that* (3.1.5) *holds. Then, there exists* $\lambda_* > 0$ *and* $a_* > 0$ *such that for all* $\lambda \in (0,\lambda_*)$ *and* $a \in (0,a_*)$*, problem* (3.1.2) *has two nonnegative solutions* $u_i = u_i(a), i = 1,2,$ *with the following properties:*

(i) $-div(K(x)\nabla u_i) \in L^{\theta}_K(\mathbb{R}^N)$ *with* $\frac{1}{\theta} + \frac{1}{2^*} = 1$.

(ii) *meas*$(\{u_i = a\} := \{x \in \mathbb{R}^N : u_i(x) = a\}) = 0$.

(iii) *meas*$(\{u_i > a\} := \{x \in \mathbb{R}^N : u_i(x) > a\}) > 0$.

(iv) $I_{\lambda,a}(u_2) < 0 < I_{\lambda,a}(u_1)$,

where meas(.) denotes the Lebesgue measure. Moreover, if $a_n \to 0^+$ *there exist two functions* $v_i \in X, i = 1, 2$, *such that, up to a subsequence,* $u_i(a_n) \to v_i$ *in* X, $I_{\lambda,0}(v_2) < 0 < I_{\lambda,0}(v_1)$ *and* v_1, v_2 *are solutions of*

$$\begin{cases} -div(K(x)\nabla v) = K(x)\left(\lambda h(x) + |v|^{2^*-2}v\right) & a.e \ in \ \mathbb{R}^N, \\ v \in X, v \geq 0 & a.e \ in \ \mathbb{R}^N, \end{cases}$$

for all $\lambda \in (0, \lambda_*)$.

Chapter 3 first introduce some basic results about Convex Analysis. Since we are dealing with a problem involving critical growth in $\mathbb{R}^N$, we need to work around the lack of compactness problem. For this, we use the version of compactness and concentration principle due to Lions involving the weight function K found in [46]. The first solution was obtained using a version of the Mountain Pass Theorem for Locally Lipschitz functionals. The second soluton was found using a version of Ekeland Variational Principle.

We would like to emphasize that the results obtained in Chapter 1 were published in the journal Complex Variables and Elliptic Equations, and the results in Chapter 2 were published in the journal Nonlinear Analysis: Real World Applications, 64(2022), 103431. The results obtained in chapter three were also submitted for publication. Although the problems studied here are connected via the operator, all chapters are independent and can be read separately.

Chapter 1

Multiple solutions for an equation with weights and nonlinearity with arbitrary growth

1.1 Introduction

As already said, in this first chapter of the work we are interested in the existence of solutions for the following equation

$$(P'_a) \qquad \left\{ \; -\mathrm{div}(K(x)\nabla u) = a(x)K(x)|u|^{q-2}u + K(x)f(u) \quad \text{in } \mathbb{R}^N, \right.$$

where $K(x) = \exp\left(\frac{|x|^2}{4}\right)$, $N \geq 3$ and $1 < q < 2$. The nonlinearity f has arbitrary growth at infinity, that is, the function f is superlinear with no growth restriction and can be allowed to be in the range subcritical, critical or supercritical. More precisely, the nonlinearity f is a $C^1(\mathbb{R})$ function satisfying:

(f_1) There exists $\delta > 0$ such that f is odd for $|s| \leq \delta$.

(f_2) There exists $2 < p < 2^*$ such that

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{p-1}} = 0.$$

(f_3) The function $\frac{f(s)}{s}$ is decreasing in $[-\delta, 0]$ and increasing in $[0, \delta]$.

The function a is a positive function satisfying:

(a_1) There exists $\sigma_q \in \mathbb{R}$ such that $a \in L_K^{\sigma_q}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with

$$\frac{p}{p-q} < \sigma_q \leq \frac{2}{2-q}. \tag{1.1.1}$$

Below, we set out the main result of this chapter.

Theorem 1.1.1. *Suppose that the function f satisfies $(f_1)-(f_3)$ and the function a satisfies (a_1). Then there exists $\lambda^* > 0$ such that, if $\|a\|_{\sigma_q} \in (0, \lambda^*)$, problem (P'_a) has infinitely many weak solutions.*

We emphasize that the theorem holds independently of the growth of f at infinity. Some typical examples of functions satisfying the conditions $(f_1) - (f_3)$ are the following:

(1)
$$f(s) = |s|^{r-2} s \exp(s^t),$$

(2)
$$f(s) = |s|^{r-2} s,$$

(3)
$$f(s) = |s|^{r-2} s \exp(s^t) + |s|^{r-2} s$$

where $p < r$ such that p satisfies the condition (f_2) and $t \geq 1$.

In the proof of the main result, we apply variational methods. However, since we have no control on the behaviour of f at infinity, the associated functional to (P_a') is not well defined in the entire space X. In the literature, there are some papers with nonlinearities with arbitrary growth. For example, in [73] the existence of infinitely many solutions were obtained for some elliptic problems with Dirichlet boundary condition, Neumann boundary condition and for an Hamiltonian system considering nonlinearities with behaviour sublinear on the origin. The strategy consisted of modifying the nonlinearity, obtaining solutions with small L^∞ norms in such a way that each solution of the modified problem was a solution to the original problem. The version of [73] with the nonlinearity being able to change the sign was considered in [53]. In [44] was studied Kirchhoff problem considering nonlinearities with behaviour sublinear and linear on the origin using the strategy from [73].

Hamiltonian system also was studied with nonlinearities with arbitrary growth in [24], [31] and [59].

On the other hand, equations with this class of weights have been studied extensively in the literature. For example, the version of classical Brezis - Nirenberg problem was studied in [25]. The version with critical concave-convex nonlinearities was studied in [45]. The study in dimension two and nonlinearity with exponential growth was made in [39], [42] and [43].

The present work is strongly influenced by the articles above. Below we list what we believe that are the main contributions of our paper.

(1) Unlike [25], [39], [42], [43] and [45], we show existence of infinitely many solutions with the nonlinearity with arbitrary growth.

(2) The truncation used here is different of the truncation used in [44], [53] and [73]. We were influenced by arguments that can be found in [33].

(3) We complement the study that can be found in [44], because we consider the nonlinearity with behaviour superlinear on the origin. We were influenced by arguments that can be found in [14].

This chapter is organized as follows. In Section 1.2, in order to be able to deal variationally, we use introduce some definitions and we use an argument inspired by [33]. In Section 1.3 we show existence of infinitely many critical points of the functional associated to problem (P_a'). In order to use Genus theory, it was necessary to make another truncation. In Section 1.4 we prove the main result.

1.2 Variational framework and a modified problem

We define the space X as the completion of the smooth functions with compact support $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_K^2 = \int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx.$$

As quoted in [45, Proposition 2.1], X is a Banach space and the weighted Lebesgue spaces

$$L_K^s(\mathbb{R}^N) := \left\{ u \text{ measurable in } \mathbb{R}^N : \|u\|_s^s := \int_{\mathbb{R}^N} K(x)|u|^s dx < \infty \right\}$$

are such that the embedding $X \hookrightarrow L_K^s(\mathbb{R}^N)$ are continuous for $2 \le s \le 2^* := \frac{2N}{N-2}$ and compact for $2 \le s < 2^*$.

A weak solution of problem (P) is a function $u \in X$ such that

$$\int_{\mathbb{R}^N} K(x)\nabla u \nabla v dx - \int_{\mathbb{R}^N} K(x)a(x)|u|^{q-2}uv dx - \int_{\mathbb{R}^N} K(x)f(u)v dx = 0,$$

for all $v \in X$ and if $\int_{\mathbb{R}^N} K(x)f(u)v dx < \infty$.

From the variational point of view, the equation in (P_a') is the Euler - Lagrange equation of the energy functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx - \int_{\mathbb{R}^N} K(x)F(u)dx,$$

where $F(s) = \int_0^s f(t)dt$. Note that the term $\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx$ is finite because from (1.1.1), we have $2 \le q\beta_q \le p < 2^*$ and from (a_1) we obtain

$$\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx \le \|a\|_{\sigma_q}\|u\|_{q\beta_q}^q,$$

where β_q is the conjugated exponent of σ_q, that is, $\frac{1}{\sigma_q} + \frac{1}{\beta_q} = 1$.

However we have no control on the behaviour of f at infinity. Then the functional Φ is not well defined in the entire space X. In order to be able to deal variationally, we consider the following auxiliary function:

$$g(s) = \begin{cases} \dfrac{f(\delta)}{\delta^{p-1}}|s|^{p-2}s & \text{if} \quad s < -\delta, \\ f(s) \quad \text{if} & -\delta \le s \le \delta, \\ \dfrac{f(\delta)}{\delta^{p-1}}|s|^{p-2}s & \text{if} \quad s > \delta, \end{cases}$$

where p was given in (f_2).

Now we consider the functional I of C^1 class given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx - \int_{\mathbb{R}^N} K(x)G(u)dx,$$

where $G(s) = \int_0^s g(t)dt$. By some direct calculations, we get

$$I'(u)v = \int_{\mathbb{R}^N} K(x)\nabla u \nabla v dx - \int_{\mathbb{R}^N} K(x)a(x)|u|^{q-2}uv dx - \int_{\mathbb{R}^N} K(x)g(u)v dx,$$

for all $v \in X$. Note that if $u \in X \cap L^\infty(\mathbb{R}^N)$ is a critical point of I such that $\|u\|_\infty \leq \delta$, then $g(u) = f(u)$ and u is a weak solution of problem (P'_a).

1.3 Existence of infinitely many critical points of the functional I

The main result in this section is:

Theorem 1.3.1. *Suppose that the function f satisfies $(f_1)-(f_3)$ and the function a satisfies (a_1). Then there exists $\lambda^* > 0$ such that, if $\|a\|_{\sigma_q} \in (0, \lambda^*)$, the functional I has infinitely many critical points.*

In order to use variational methods, we first derive some results related to the Palais-Smale compactness condition.

A sequence $(u_n) \subset X$ is a $(PS)_c$ sequence for I if

$$I(u_n) \to c \text{ and } \|I'(u_n)\|_{X'} \to 0, \quad \text{as } n \to \infty \tag{1.3.1}$$

where

$$c := \inf_{\pi \in \Gamma} \max_{t \in [0,1]} I(\pi(t)) > 0$$

and

$$\Gamma := \{\pi \in C([0,1], X) : \pi(0) = 0,\ I(\pi(1)) < 0\}.$$

If (1.3.1) implies the existence of a subsequence $(u_{n_j}) \subset (u_n)$ which converges in X, we say that I satisfies the Palais-Smale condition on the level c.

1.3.1 Technical results

The genus theory requires that the functional I is bounded from below. Since this is not the case, it is necessary to define a new functional J which is bounded from below such that a critical point of J is a critical point of I. The definiton of such functional J follows by some ideas contained in [14].

From Sobolev embedding, we define the function h given by

$$h(\|u\|^2) := \frac{1}{2}\|u\|^2 - \frac{\|a\|_{\sigma_q}}{qS_{q\beta_q}^{q\beta_q/2}}\|u\|^{\beta_q q} - \frac{1}{pS_p^{p/2}}\|u\|^p \leq I(u), \tag{1.3.2}$$

where

$$S_p = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^2}{\|u\|_p^2} \quad \text{and} \quad S_{q\beta_q} = \inf_{u \in X \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{q\beta_q}^2}.$$

Hence, there exists $\tau_1 > 0$ such that, if $\|a\|_{\sigma_q} \in (0, \tau_1)$, then h attains its positive maximum. Let $0 < R_0 < R_1$ be the only roots of h. We have that $R_0 = R_0(\|a\|_{\sigma_q})$ and the following result holds:

Lemma 1.3.2. *Using the definition of h in (1.3.2), we have*

$$R_0(\|a\|_{\sigma_q}) \to 0 \quad \textit{as} \quad \|a\|_{\sigma_q} \to 0.$$

Proof. From $h(R_0(\|a\|_{\sigma_q})) = 0$ and $h'(R_0(\|a\|_{\sigma_q})) > 0$, we have

$$\frac{1}{2}R_0(\|a\|_{\sigma_q}) = \frac{\|a\|_{\sigma_q}}{qS_{q\beta_q}^{q\beta_q/2}}R_0(\|a\|_{\sigma_q})^{\beta_q q/2} + \frac{1}{pS_p^{p/2}}R_0(\|a\|_{\sigma_q})^{p/2} \tag{1.3.3}$$

and

$$\frac{1}{2} > \frac{\beta_q}{2}\frac{\|a\|_{\sigma_q}}{S_{q\beta_q}^{q\beta_q/2}}R_0(\|a\|_{\sigma_q})^{(\beta_q q-2)/2} + \frac{1}{2S_p^{p/2}}R_0(\|a\|_{\sigma_q})^{(p-2)/2}, \tag{1.3.4}$$

for all $\|a\|_{\sigma_q} \in (0, \tau_1)$. Since $p > 2$, from (1.3.3) we conclude that $R_0(\|a\|_{\sigma_q})$ is bounded.

Suppose, by contradiction, that up to a subsequence, we get $R_0(\|a\|_{\sigma_q}) \to \alpha > 0$ as $\|a\|_{\sigma_q} \to 0$. Then, passing to the limit as $\|a\|_{\sigma_q} \to 0$ in (1.3.3) and (1.3.4), we obtain

$$\frac{1}{2} = \frac{1}{pS^{p/2}}\alpha^{(p-2)/2} \tag{1.3.5}$$

and

$$\frac{1}{2} \geq \frac{1}{2S_p^{p/2}}\alpha^{(p-2)/2}. \tag{1.3.6}$$

Using (1.3.5) and (1.3.6) we derive a contradiction, because $2 < p$. Therefore $\alpha = 0$. □

We modify the functional I in the following way. Take $\phi \in C^\infty([0, +\infty))$, $0 \leq \phi \leq 1$ such that $\phi(t) = 1$ if $t \in [0, R_0]$ and $\phi(t) = 0$ if $t \in [R_1, +\infty)$. Now, we consider the truncated functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2\, dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q\, dx - \phi(\|u\|^2)\int_{\mathbb{R}^N} K(x)G(u)\, dx.$$

Note that $J \in C^1(X, \mathbb{R})$ and, as in (1.3.2), $J(u) \geq \overline{h}(\|u\|^2)$, where

$$\overline{h}(t^2) := \frac{1}{2}t^2 - \frac{\|a\|_{\sigma_q}}{qS_{q\beta_q}^{q\beta_q/2}}t^{\beta_q q} - \phi(t^2)\frac{1}{pS_p^{p/2}}t^p.$$

Let us remark that if $\|u\|^2 \leq R_0$, then $J(u) = I(u)$. In the light of Proposition 3.4.5, it seems to be useful proving that the set of critical points of J has genus greater than 2, in order to obtain infinitely many critical points of J.

Note that if $\|u\|^2 \geq R_1$, then $J(u) = \frac{1}{2}\int_{\mathbb{R}^N} K(x)|\nabla u|^2\, dx - \frac{1}{q}\int_{\mathbb{R}^N} K(x)a(x)|u|^q\, dx$, which implies that J is coercive and hence bounded from below.

In the next lemma we show that J satisfy the Palais-Smale condition for any level c.

Lemma 1.3.3. *For $c \in \mathbb{R}$, let $(u_n) \subset X$ be a bounded sequence such that*

$$I(u_n) \to c \quad \textit{and} \quad I'(u_n) \to 0.$$

Then, up to a subsequence, (u_n) is strongly convergent in X.

Proof. Note that, up to a subsequence, using the compactness result that can be found in [45, Proposition 2.1], we have,

$$u_n \rightharpoonup u \ \text{ in } \ X,$$
$$u_n \to u \ \text{ in } \ L^s_K(\mathbb{R}^N),$$
$$u_n \to u \ \text{ a.e in } \ \mathbb{R}^N$$

and

$$|u_n| \le \psi \ \text{ a.e in } \ \mathbb{R}^N,$$

for some $\psi \in L^s_K(\mathbb{R}^N)$ and $2 \le s < 2^*$. Then, from Lebesgue's Theorem, we obtain

$$\int_{\mathbb{R}^N} K(x)a(x)|u_n|^{q-2}(u_n - u)dx \to 0$$

and

$$\int_{\mathbb{R}^N} K(x)g(u_n)(u_n - u)dx \to 0.$$

Then,

$$o_n(1) = \int_{\mathbb{R}^N} K(x)\nabla u_n \nabla(u_n - u)dx = \|u_n\|^2 - \|u\|^2$$

and the proof is over. □

Lemma 1.3.4. *If $J(u) < 0$, then $\|u\|^2 < R_0$ and $J(v) = I(v)$, for all v in a small neighborhood of u. Moreover, J verifies a local Palais-Smale condition for $c < 0$.*

Proof. Since $\overline{h}(\|u\|^2) \le J(u) < 0$, then $\|u\|^2 < R_0$. By the definition of J, we have that $J(u) = I(u)$. Moreover, since J is continuous, we conclude that $J(v) = I(v)$, for all $v \in B_{R_0/2}(0)$. Besides, if (u_n) is a sequence such that $J(u_n) \to c < 0$ and $J'(u_n) \to 0$ as $n \to \infty$, then for n sufficiently large $I(u_n) = J(u_n) \to c < 0$ and $I'(u_n) = J'(u_n) \to 0$ as $n \to \infty$. Since J is coercive, we get that (u_n) is bounded in X. From Lemma 1.3.3, up to a subsequence (u_n) is strongly convergent in X. □

Now, we construct an appropriate minimax sequence of negative critical values.

Lemma 1.3.5. *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that*

$$\gamma(J^{-\epsilon}) \ge k,$$

where $J^{-\epsilon} = \{u \in X : J(u) \le -\epsilon\}$.

Proof. Consider $k \in \mathbb{N}$ and let X_k be a k-dimensional subspace of X and note that

$$\left(\int_{\mathbb{R}^N} K(x)a(x)|u|^q \, dx\right)^{\frac{1}{q}}$$

is a norm in X_k. Since in X_k all norms are equivalent, there exists $C(k) > 0$ such that

$$-C(k)\|u\|^q \ge -\int_{\mathbb{R}^N} K(x)a(x)|u|^q dx,$$

for all $u \in X_k$.

We now use the inequality above to conclude that

$$J(u) \le \frac{1}{2}\|u\|^2 - \|a\|_{\sigma_q}\frac{C(k)}{q}\|u\|^q = \|u\|^q\left(\frac{1}{2}\|u\|^{2-q} - \frac{C(k)\|a\|_{\sigma_q}}{q}\right).$$

Considering $R > 0$ sufficiently small, there exists $\epsilon = \epsilon(R) > 0$ such that

$$J(u) < -\epsilon < 0,$$

for all $u \in \mathcal{S}_R = \{u \in X_k; \|u\| = R\}$. Since X_k and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}_R$ and S^{k-1} are homeomorphic, we conclude from Corollary 3.4.2 that $\gamma(\mathcal{S}_R) = \gamma(S^{k-1}) = k$. Moreover, once that $\mathcal{S}_R \subset J^{-\epsilon}$ and $J^{-\epsilon}$ is symmetric and closed, we have

$$k = \gamma(\mathcal{S}_R) \leq \gamma(J^{-\epsilon}).$$

□

We define now, for each $k \in \mathbb{N}$, the sets

$$\Gamma_k = \{C \subset X : C \ \text{ is closed}, C = -C \ \text{ and } \ \gamma(C) \geq k\},$$

$$K_c = \{u \in X : J'(u) = 0 \ \text{ and } \ J(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J(u).$$

Lemma 1.3.6. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 1.3.5, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J^{-\epsilon}) \geq k$. Moreover, $0 \notin J^{-\epsilon}$ and $J^{-\epsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J^{-\epsilon}} J(u) \leq -\epsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J(u) \leq \sup_{u \in J^{-\epsilon}} J(u) \leq -\epsilon < 0.$$

□

The next Lemma allows us to prove the existence of critical points of J.

Lemma 1.3.7. *If $c = c_k = c_{k+1} = ... = c_{k+r}$ for some $r \in \mathbb{N}$, then*

$$\gamma(K_c) \geq r + 1.$$

Proof. Since $c = c_k = c_{k+1} = ... = c_{k+r} < 0$, from Lemma 1.3.3 and Lemma 1.3.6, we get that K_c is compact. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J^0$ because $c < 0$. By the deformation lemma [17] we have an odd homeomorphism $\eta : X \to X$ such that $\eta(J^{c+\delta} - U) \subset J^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J^{c+\delta} \subset J^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup\limits_{u \in A} J < c + \delta$, that is, $A \subset J^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J^{c+\delta} - U) \subset J^{c-\delta}. \tag{1.3.7}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$. Then $\eta(\overline{A - U}) \in \Gamma_k$. Then, by (1.3.7)

$$c = c_k \leq \sup_{u \in \eta(A-U)} J(u) \leq \sup_{u \in J^{c-\delta}} J(u) < c - \delta,$$

which is a contradiction. □

1.4 Proof of Theorem 1.1

Proof. If $-\infty < c_1 < c_2 < ... < c_k < ... < 0$ with $c_i \neq c_j$, once each c_k is a critical value of J, we obtain infinitely many critical points of J.

On the other hand, if $c_k = c_{k+r}$ for some k and r, then $c = c_k = c_{k+1} = ... = c_{k+r}$ and from Lemma 1.3.7, we have

$$\gamma(K_c) \geq r + 1 \geq 2.$$

From Proposition 3.4.5, we conclude that K_c has infinitely many points.

If $u_0 \in X$ is a critical point of J in level c_k, from Lemma 1.3.6 we conclude that $c_k < 0$. Using Lemma 1.3.4, $u_0 \in X$ is a critical point of I with $\|u_0\|^2 \leq R_0$.

Now it is sufficient to prove that there exists a positive constant C, independent on $\|a\|_{\sigma_q}$ such that

$$\|u_0\|_\infty \leq C\|u_0\|. \tag{1.4.1}$$

In this case, we can use Lemma 1.3.2 to conclude that, there exists a $\lambda^* > 0$ such that $\|a\|_{\sigma_q} \in (0, \lambda^*)$, which implies

$$\|u_0\|_\infty \leq \delta,$$

where we conclude that $u_0 \in X$ is a weak solution of problem (P). Now we use the Moser iteration technique in order to prove (1.4.1). In order to save the notation, from now on we denote u_0 by u.

For each $L > 0$, we define $u_L \in H^1(\mathbb{R}^N)$ by setting

$$u_L(x) := \min\{u(x), L\}, \qquad \Upsilon_L := u_L^{2(\beta-1)}u$$

with $\beta > 1$ to be determined later.

We would like to emphasize that we will consider in these calculations that u is non-negative, because if u change the sign, we can consider the negative part of u in Υ_L and get the estimates for the positive part. After that, we consider the positive part of u in Υ_L and obtain the estimates for the negative part.

Now we define the function $\mathcal{H}(x,t) := a(x)t^{q-1} + t^{p-1}$, for all $t \geq 0$. Since $a \in L^\infty(\mathbb{R}^N)$, we have that

$$\lim_{t\to 0} \mathcal{H}(x,t) = 0 \text{ and } \lim_{t\to+\infty} \frac{\mathcal{H}(x,t)}{t^p - 1} = 1, \text{ uniformly in } x \in \mathbb{R}^N.$$

Then, there exists a positive contant C such that

$$\mathcal{H}(x,t) \leq Ct^{p-1}. \tag{1.4.2}$$

Note that $I'(u)\Upsilon_L = 0$. Considering that

$$2(\beta-1)\int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)-1}\nabla u \nabla u_L dx \geq 0,$$

we obtain

$$\int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)}|\nabla u|^2 dx \leq \int_{\mathbb{R}^N} K(x)a(x)|u|^{q-2}u^2 u_L^{2(\beta-1)}\, dx + \int_{\mathbb{R}^N} K(x)g(u)uu_L^{2(\beta-1)}dx. \tag{1.4.3}$$

By definition of g, we derive

$$\int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)}|\nabla u|^2 dx \leq \int_{\mathbb{R}^N} K(x)a(x)|u|^q u_L^{2(\beta-1)}\, dx + \int_{\mathbb{R}^N} K(x)u^p u_L^{2(\beta-1)} dx$$
$$= \int_{\mathbb{R}^N} K(x)\mathcal{H}(x,u)uu_L^{2(\beta-1)} dx$$

Using the inequality (1.4.2) we conclude

$$\int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)}|\nabla u|^2 dx \leq C \int_{\mathbb{R}^N} K(x)u^p u_L^{2(\beta-1)} dx. \tag{1.4.4}$$

Let S be the best constant of the embedding $X \hookrightarrow L_K^{2^*}(\mathbb{R}^N)$ and define $\widehat{u}_L := uu_L^{\beta-1}$. Since $u_L \leq u$, we have that

$$S\|\widehat{u}_L\|_{2^*}^2 \leq \int_{\mathbb{R}^N} \left|K(x)\nabla\left(uu_L^{\beta-1}\right)\right|^2 dx \leq \beta^2 \int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)}|\nabla u|^2 dx.$$

The last inequality and (1.4.4) provide

$$S\|\widehat{u}_L\|_{2^*}^2 \leq C_4\beta^2 \int_{\mathbb{R}^N} K(x)|u|^p u_L^{2(\beta-1)} dx, \tag{1.4.5}$$

for all $\beta > 1$.

The definition of $\widehat{u}_L$ imply that

$$S\|\widehat{u}_L\|_{2^*}^2 \leq C_4\beta^2 \int_{\mathbb{R}^N} K(x)|u|^{p-2}|\widehat{u}_L|^2 dx. \tag{1.4.6}$$

Using Hölder's inequality with $\frac{2^*}{p-2}$ and $\frac{2^*}{2^*-(p-2)}$, we get

$$S\|\widehat{u}_L\|_{2^*}^2 \leq C_4\beta^2 \left(\int_{\mathbb{R}^N} K(x)|u|^{2^*} dx\right)^{(p-2)/2^*} \left(\int_{\mathbb{R}^N} K(x)|\widehat{u}_L|^{\frac{22^*}{(2^*-(p-2))}} dx\right)^{\frac{(2^*-(p-2))}{2^*}}, \tag{1.4.7}$$

where $2 < \frac{22^*}{(2^*-(p-2))} < 2^*$.

Note that $\left(\int_{\mathbb{R}^N} K(x)|u|^{2^*} dx\right)^{(p-2)/2^*} = \|u\|_{2^*}^{p-2} \leq S^{\frac{2-p}{2}}\|u\|^{p-2} \leq S^{\frac{2-p}{2}} R_0^{p-2} \leq 1$, for $\|a\|_{\sigma_q}$ sufficient small.

Then,

$$S\|\widehat{u}_L\|_{2^*}^2 \leq C_4\beta^2 \left(\int_{\mathbb{R}^N} K(x)|\widehat{u}_L|^{\frac{22^*}{(2^*-(p-2))}} dx\right)^{\frac{(2^*-(p-2))}{2^*}},$$

where we conclude that

$$S\|\widehat{u}_L\|_{2^*}^2 \leq C_5\beta^2\|\widehat{u}_L\|_{\zeta}^2,$$

where $\zeta = \frac{22^*}{(2^*-(p-2))}$.

Using $u_L \leq |u|$, we get

$$S\|\widehat{u}_L\|_{L^{2^*}}^2 \leq C_5\beta^2 \left(\int_{\mathbb{R}^N} |u|^{\beta\zeta} dx\right)^{2/\zeta}.$$

From Fatou's lemma in the variable L we obtain

$$\|u\|_{\beta 2^*} \leq C_6^{1/\beta} \beta^{1/\beta} \|u\|_{\beta\zeta},$$

whenever $u^{\beta\zeta} \in L^1_K(\mathbb{R}^N)$.

We now set $\beta := 2^*/\zeta > 1$ and note that, since $u \in L^{2^*}_K(\mathbb{R}^N)$, the above inequality holds for this choice of β. Moreover, since $\beta^2\zeta = \beta 2^*$, it follows that the inequality also holds with β replaced by β^2.

Hence,

$$\|u\|_{\beta^2 2^*} \leq C_7^{1/\beta^2} \beta^{1/\beta^2} \|u\|_{\beta^2\zeta}.$$

By iterating this process and recalling that $\beta\zeta = 2^*$, we obtain, for $k \in \mathbb{N}$,

$$\|u\|_{\beta^k 2^*} \leq C_7^{\sum_{i=1}^{k} \beta^{-i}} \beta^{\sum_{i=1}^{k} i\beta^{-i}} \|u\|_{2^*}.$$

Since $\beta > 1$ we can take the limit as $k \to \infty$ to get

$$\|u\|_\infty \leq C_8 \|u\|_{2^*} \leq C_8 S^{-1/2} \|u\| \leq C_8 S^{-1/2} R_0^{1/2} \leq \delta,$$

for $\|a\|_{\sigma_q}$ sufficient small, which prove the main result. □

Chapter 2

On a critical and a supercritical system with fast increasing weights

2.1 Introduction

In this chapter we prove a result as in [25] for some systems with critical or supercritical growth. More precisely we show the existence of nontrivial solutions to the systems

$$(S') \qquad \begin{cases} -\operatorname{div}(K(x)\nabla u) = K(x)Q_u(u,v) + \frac{1}{2^*}K(x)H_u(u,v) \text{ in } \mathbb{R}^N, \\ -\operatorname{div}(K(x)\nabla v) = K(x)Q_v(u,v) + \frac{1}{2^*}K(x)H_v(u,v) \text{ in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $K(x) = \exp\left(|x|^2/4\right)$, Q and H are functions of class C^1 and

$$(SC') \qquad \begin{cases} -\operatorname{div}(K(x)\nabla u) = K(x)Q_u(u,v) + K(x)|u|^{\Upsilon_1-2}u \text{ in } \mathbb{R}^N, \\ -\operatorname{div}(K(x)\nabla v) = K(x)Q_v(u,v) + K(x)|v|^{\Upsilon_2-2}v \text{ in } \mathbb{R}^N, \end{cases}$$

where $\Upsilon_i > 2^*,\ i = 1, 2$.

Setting $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$, for any given $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+, \mathbb{R})$ satisfying the following properties.

$(\mathcal{H}^q_0)$ F is q-homogeneous, that is,

$$F(\lambda s, \lambda t) = \lambda^q F(s,t), \quad \text{for each} \quad \lambda > 0 \quad \text{and} \quad (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{H}^q_1)$ There exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t))| \leq c_1\left(s^{q-1} + t^{q-1}\right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{H}_2)$ $F(s,t) > 0$ for each $s, t > 0$.

$(\mathcal{H}_3)$ $\nabla F(1,0) = \nabla F(0,1) = (0,0)$.

$(\mathcal{H}_4)$ $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

The hypotheses on the functions Q and H are the following:

(A_1) $H \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$.

(A_2) The 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := H(s,t)$ is concave.

(A_3) $Q(s,t) \geq \sigma s^{\gamma} t^{\beta}$ for all $(s,t) \in \mathbb{R}^2_+$, with $\gamma, \beta > 1$, $\gamma + \beta =: p_1 \in (2, 2^*)$ and σ satisfies

(i) $\sigma > 0$ if either $N \geq 4$, or $N = 3$ and $2^* - 2 < p_1 < 2^*$;

(ii) σ is sufficiently large if $N = 3$ and $2 < p_1 < 2^* - 2$.

$(\widetilde{A}_3)$ There exists $\sigma^* > 0$ such that $Q(s,t) \geq \sigma s^{\gamma} t^{\beta}$ for all $(s,t) \in \mathbb{R}^2_+$, $\gamma, \beta > 1$, $\gamma + \beta =: p_1 \in (2, 2^*)$, for all $\sigma > \sigma^*$ and σ^* to be fixed later.

The main results are:

Theorem 2.1.1. *Assume that conditions* $(A_1), (A_2), (A_3)$ *are hold. Then, system* (S') *has a weak positive solution.*

Theorem 2.1.2. *Assume that conditions* $(A_1), (A_2), (\widetilde{A}_3)$ *are hold. Then, system* (SC') *has a weak positive solution.*

The search for self-similar solutions for systems appeared in the first time in [71]. In that paper, Yuan-wei Qi shows the existence of both slowly and fast decaying positive solutions for the system

$$(Q) \qquad \begin{cases} -\Delta u - \frac{1}{2}(x.\nabla u) = \frac{k_1}{2} u + v^p \text{ in } \mathbb{R}^N, \\ \\ -\Delta v - \frac{1}{2}(x.\nabla v) = \frac{k_2}{2} v + v^q \text{ in } \mathbb{R}^N, \end{cases}$$

where $k_1, k_2 > 0$ and $p, q > 1$. The asymptotic behaviour of positive solutions such system also was studied by Yuan-wei Qi in [71].

The hypotheses $(A_1) - (A_3)$ had already appeared in [32]. In that paper, de Morais Filho and Souto [32] investigate the existence of solutions for the following system:

$$(DM) \qquad \begin{cases} -\Delta_p u = Q_u(u,v) + H_u(u,v) \text{ in } \Omega, \\ \\ -\Delta_p v = Q_v(u,v) + H_v(u,v) \text{ in } \Omega, \\ \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where Δ_p denotes the $p-$Laplacian operator, $p > 1$ and Ω is a bounded domain in $\mathbb{R}^N$.

Our arguments were strongly influenced by [25], [32] and [71]. Below we list what we believe to be the main contributions of our paper.

i) In order to get around the lack of compactness, in [25] estimates are used involving the Talenti's functions type and the weight function K. As can be seen in Lemma 2.2.5 and Lemma 2.2.6, the estimates to get around the lack of compactness with the systems in $\mathbb{R}^N$ are more delicate.

ii) We completed the study that was done in [71] in the sense that we are studying another class of systems considering the critical and supercritical cases.

iii) It was necessary to adapt some estimates found in [32] for the $\mathbb{R}^N$ considering the weight function K, as can be seen in Lemma 2.2.1 and Lemma 2.2.4.

iv) To circumvent the lack of variational structure of the supercritical system, we use a truncation argument. To recover the solution to the original problem, we used the study on a problem in a bounded domain.

Concerning the class of nonlinearities we are dealing, we have the following examples from [32]. Let $q \geq 1$ and

$$P_q(s,t) = \sum_{\alpha_i+\beta_i=q} a_i s^{\alpha_i} t^{\beta_i},$$

where $i \in \{1,\ldots,k\}$, $\alpha_i, \beta_i \geq 1$ and $a_i \in \mathbb{R}$. The following functions and its possible combinations, with appropriated choices of the coefficients a_i, satisfy our hypothesis on Q

$$Q_1(s,t) = P_p(s,t), \quad Q_2(s,t) = \sqrt[r]{P_l(s,t)} \quad \text{and} \quad Q_3(s,t) = \frac{P_{l_1}(s,t)}{P_{l_2}(s,t)},$$

with $r = pl$ and $l_1 - l_2 = p$. Condition (A_2) restricts the expression of the critical function H. However, it can have the polynomial form $H(s,t) = P_{2^*}(s,t)$.

The chapter is organized as follows. In Section 2 is devoted to variational framework and some preliminary results for the critical case. In Section 3 we show the solution of the critical case proving the first main result. In the Section 4 we study the supercritical case and prove the second main result.

2.2 Variational framework and some preliminary results for the critical case

We define the space X as the completion of the smooth functions with compact support $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_K^2 = \int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx.$$

We are looking for solution on the space $X \times X$ with respect to the norm

$$\|(u,v)\|^2 = \|u\|_K^2 + \|v\|_K^2.$$

As quoted in [45, Proposition 2.1], X is a Banach space and the weighted Lebesgue spaces

$$L_K^s(\mathbb{R}^N) := \left\{ u \text{ mensurable in } \mathbb{R}^N : \|u\|_{s,K}^s := \int_{\mathbb{R}^N} K(x)|u|^s dx < \infty \right\}$$

are such that the embedding $X \hookrightarrow L_K^s(\mathbb{R}^N)$ are continuous for $2 \leq s \leq 2^*$ and compact for $2 \leq s < 2^*$.

A pair $(u,v) \in X \times X$ is a weak positive solution of system (S') if $u > 0$ and $v > 0$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)\nabla u \nabla\phi dx + \int_{\mathbb{R}^N} K(x)\nabla v \nabla\psi dx &= \int_{\mathbb{R}^N} K(x)[\phi Q_u(u,v) + \psi Q_v(u,v)]dx \\ + \quad \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)[\phi H_u(u,v) + \psi H_v(u,v)]dx,\end{aligned}$$

for all $(\phi,\psi) \in X \times X$. With the same reasoning, we defined a weak positive solution for the system with supercritical growth (SC').

Since we are interested in positive solutions we extend the functions Q and H to the whole $\mathbb{R}^2$ by setting $Q(u,v) = H(u,v) = 0$ if $u \leq 0$ or $v \leq 0$. We also note that for any function $F \in \mathcal{H}^q$, we can use the homogeneity condition $(\mathcal{H}_0^q)$ to conclude that

$$qF(s,t) = sF_s(s,t) + tF_t(s,t). \tag{2.2.1}$$

for any $(s,t) \in \mathbb{R}^2$. Moreover, since $\{(s,t) \in \mathbb{R}^2 : |s|^q + |t|^q = 1\}$ is a compact set and F is a continuous function on it, then there exists $M_F > 0$ such that

$$|F(s,t)| \leq M_F(|s|^q + |t|^q), \tag{2.2.2}$$

where $M_F = \max\{F(s,t) : s,t \in \mathbb{R},\ |s|^q + |t|^q = 1\}$ and the maximum M_F is attained for some $(s_0, t_0) \in \mathbb{R}^2$.

The associated functional to system (S') is given by

$$I(u,v) := \frac{1}{2}\|(u,v)\|^2 - \int_{\mathbb{R}^N} K(x)Q(u,v)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)H(u,v)dx,$$

is well defined for $(u,v) \in X \times X$. Thus,

$$\begin{aligned} I'(u,v)(\phi,\psi) : &= \int_{\mathbb{R}^N} K(x)\nabla u \nabla \phi dx + \int_{\mathbb{R}^N} K(x)\nabla v \nabla \psi dx \\ &- \int_{\mathbb{R}^N} K(x)Q_u(u,v)\phi dx - \int_{\mathbb{R}^N} K(x)Q_v(u,v)\psi dx \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)H_u(u,v)\phi dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)H_v(u,v)\psi dx. \end{aligned}$$

Hence, critical points of I are weak solutions of (S').

Now we prove a version of Brezis-Lieb lemma for class of the system that we are studying.

Lemma 2.2.1. *Let (u_n, v_n) be bounded sequence in $X \times X$ and $u, v \in X$ such that $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ a.e in $\mathbb{R}^N$. Then,*

$$\int_{\mathbb{R}^N} K(x)Q(u_n,v_n)dx - \int_{\mathbb{R}^N} K(x)Q(u_n - u, v_n - v)dx = \int_{\mathbb{R}^N} K(x)Q(u,v)dx + o_n(1).$$

and

$$\int_{\mathbb{R}^N} K(x)H(u_n,v_n)dx - \int_{\mathbb{R}^N} K(x)H(u_n - u, v_n - v)dx = \int_{\mathbb{R}^N} K(x)H(u,v)dx + o_n(1).$$

Proof. Since Q is a q-homogeneous function, from $(\mathcal{H}_1^q)$ and arguing as [32, Lemma 7], given $\epsilon > 0$ there exists C_ϵ such that

$$\begin{aligned} &K(x)|Q(u_n,v_n) - Q(u_n - u, v_n - v)| \\ \leq\ &\epsilon K(x)\left(|u_n - u|^q + |v_n - v|^q\right) + C_\epsilon K(x)\left(|u|^q + |v|^q\right). \end{aligned} \tag{2.2.3}$$

Now, for each $x \in \mathbb{R}^N$, let us define the function

$$\begin{aligned} &h_{\epsilon,n}(x) := \max\big[K(x)|Q(u_n(x),v_n(x)) - Q(u(x),v(x)) - Q(u_n(x)-u(x), v_n(x)-v(x))| \\ -\ &\epsilon K(x)\left(|u_n(x)-u(x)|^q + |v_n(x)-v(x)|^q\right), 0\big]. \end{aligned} \tag{2.2.4}$$

Note that by hypothesis $h_{\epsilon,n}(x) \to 0$, a.e in $\mathbb{R}^N$. Now from (2.2.3) and (2.2.4), we obtain

$$|h_{\epsilon,n}(x)| \leq K(x)Q(u,v) + C_\epsilon K(x)\left(|u|^q + |v|^q\right).$$

Then, from Dominated Convergence Theorem we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_{\epsilon,n}dx = 0.$$

From definition of $h_{\epsilon,n}$, there exists $C > 0$ such that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} K(x)Q(u_n,v_n)dx - \int_{\mathbb{R}^N} K(x)Q(u_n-u,v_n-v)dx - \int_{\mathbb{R}^N} K(x)Q(u,v)dx\right|$$
$$\le \limsup_{n\to\infty}\int_{\mathbb{R}^N} \epsilon K(x)\left(|u_n(x)-u(x)|^q+|v_n(x)-v(x)|^q\right)dx \le \epsilon C$$

and the result follows taking $\epsilon \to 0^+$. The second convergence follows by using the same argument and we omit it. □

In order to use variational methods, we first derive some results related to the Palais-Smale compactness condition.

We say that a sequence (u_n, v_n) is a Palais-Smale sequence for the functional I at the level $d \in \mathbb{R}$ if

$$I(u_n, v_n) \to d \text{ and } \|I'(u_n, v_n)\| \to 0 \text{ in } X \times X.$$

If every Palais-Smale sequence of I has a strong convergent subsequence, then one says that I satisfies the Palais-Smale condition ((PS) for short).

In the sequel, we prove that the functional I has the Mountain Pass Geometry. This fact is proved in the next lemmas:

Lemma 2.2.2. *Assume that condition (A_1) is hold. Then, there exist positive numbers ρ and α such that,*

$$I(u,v) \ge \alpha > 0, \forall (u,v) \in \ X \times X : \|(u,v)\| = \rho.$$

Proof. It follows from (A_1) that there exists a positive constant $c_1 > 0$ such that

$$I(u,v) \ge \frac{1}{2}\|u\|_K^2 + \frac{1}{2}\|v\|_K^2 - c_1(\|u\|_{p,K}^p + \|v\|_{p,K}^p) - \frac{c_1}{2^*}(\|u\|_{2^*,K}^{2^*} + \|v\|_{2^*,K}^{2^*}).$$

Now, from Sobolev embedding, there is a positive constant $C > 0$ such that

$$I(u,v) \ge \frac{1}{2}\|(u,v)\|^2 - C\|(u,v)\|^p - C\|(u,v)\|^{2^*}.$$

Since $2 < p < 2^*$, the result follows by choosing $\rho > 0$ small enough. □

Lemma 2.2.3. *Assume that condition (A_1) is hold. Then, there exists $(e_1, e_2) \in \ X \times X$ with $I(e_1, e_2) < 0$ and $\|(e_1, e_2)\| > \rho$.*

Proof. Fix $(u_0, v_0) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N) \setminus \{(0,0)\}$ with $u_0, v_0 \ge 0$ in $\mathbb{R}^N$. From (A_1), we get

$$I(t(u_0,v_0)) = \frac{t^2}{2}\|(u_0,v_0)\|^2 - t^p\int_{\mathbb{R}^N} K(x)Q(u_0,v_0)\,dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} K(x)H(u_0,v_0)\,dx.$$

Since $2 < p < 2^*$, the result follows by considering $(e_1 = t_*u_0, e_2 = t_*v_0)$ for some $t_* > 0$ large enough. □

Using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [10], without (PS) condition (see [76, Theorem p.12]), there exists a sequence $(u_n, v_n) \subset X \times X$ satisfying

$$I(u_n, v_n) \to c_* \quad \text{and} \quad I'(u_n, v_n) \to 0,$$

where

$$c_* = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0, \tag{2.2.5}$$

and

$$\Gamma := \{\gamma \in C([0,1], X \times X) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

As in [32, Lemma 3], we need a Hölder type inequality involving the weight function K.

Lemma 2.2.4. *Let H be a function in $\mathcal{H}^q$ such that, the 1-homogeneous function G, defined by*

$$G(s^q, t^q) = H(s,t) \text{ for all } s,t \geq 0, \text{ is concave.} \tag{2.2.6}$$

Then the Hölder type inequality holds:

$$\int_{\mathbb{R}^N} K(x)H(u,v)dx \leq H(\|u\|_{q,K}, \|v\|_{q,K}) \tag{2.2.7}$$

for all $u, v \in L^q_K(\mathbb{R}^N)$, $u, v \geq 0$.

Proof. Firstly we will treat the case $q = 1$. Let $u, v \in L^q_K(\mathbb{R}^N)$, with $u, v \geq 0$. Thus, $Ku, Kv \in L^1(\mathbb{R}^N)$. Since H is 1-homogeneous, from [32, Proposition 4] we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)H(u,v)dx &= \int_{\mathbb{R}^N} H(K(x)u, K(x)v)dx \\
&\leq H(|Ku|_1, |Kv|_1) \\
&= H(\|u\|_{1,K}, \|v\|_{1,K}).
\end{aligned}$$

To prove the general case, we use the case $q = 1$ and the fact that the function G is 1-homogeneous. That is,

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)H(u,v)dx &= \int_{\mathbb{R}^N} K(x)G(u^q, v^q)dx \\
&= \int_{\mathbb{R}^N} G(K(x)u^q, K(x)v^q)dx \\
&\leq G(|Ku^q|_1, |Kv^q|_1) \\
&= G(\|u\|^q_{q,K}, \|v\|^q_{q,K}) \\
&= H(\|u\|_{q,K}, \|v\|_{q,K}).
\end{aligned}$$

□

In the following, we will use the number $\widetilde{S}_{K,H}$, $\widetilde{S}_K$, S_H and S_0 given by

$$\widetilde{S}_{K,H} := \inf_{u,v \in X\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} K(x)|\nabla v|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)H(u,v)\mathrm{d}x\right)^{2/2^*}},$$

$$S_K := \inf_{u \in X\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)|u|^{2^*} dx\right)^{2/2^*}},$$

$$\widetilde{S}_H := \inf_{u,v \in X\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} H(u,v)\mathrm{d}x\right)^{2/2^*}},$$

$$S := \inf_{u\in D^{1,2}(\mathbb{R}^N)\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}.$$

From now on, we consider the function $\Phi_{\delta,y} \in D^{1,2}(\mathbb{R}^N)$ given by

$$\Phi_{\delta,y}(x) = \frac{(\delta N(N-2))^{(N-2)/4}}{(\delta + |x-y|^2)^{(N-2)/2}}, \quad x, y \in \mathbb{R}^N \text{ and } \delta > 0. \tag{2.2.8}$$

In [70] we can see that every positive solution of

$$(P_\infty) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), & N \geq 3. \end{cases}$$

is as (2.2.8). Moreover, it satisfies

$$\int_{\mathbb{R}^N} |\nabla \Phi_{\delta,y}|^2 dx = S \quad \text{and} \quad \int_{\mathbb{R}^N} |\Phi_{\delta,y}|^{2^*} dx = 1. \tag{2.2.9}$$

By [32, Lemma 3], there exist $s_o, t_o > 0$ such that $\widetilde{S}_H$ is attained by $(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y})$. Moreover,

$$M_H \widetilde{S}_H = S, \tag{2.2.10}$$

where $M_H = \max_{s^2+t^2=1} H(s,t)^{2/2^*} = H(s_o, t_o)^{2/2^*}$.

In the next result let us prove a relation between $\widetilde{S}_{K,H}$, $\widetilde{S}_K$ and S.

Lemma 2.2.5. *Assume that condition (A_2) is hold. Then,*

$$M_H \widetilde{S}_{K,H} = S_K = S.$$

Proof. From the definition of M_H, we get

$$H\Big(\frac{s}{(|s|^2+|t|^2)^{1/2^*}}, \frac{t}{(|s|^2+|t|^2)^{1/2^*}}\Big)^{2/2^*} \leq M_H$$

and homogeneity of H, we have

$$\frac{1}{M_H} H(s,t)^{2/2^*} \leq |s|^2 + |t|^2, \text{ for all } s, t \in \mathbb{R}. \tag{2.2.11}$$

Moreover,

$$\frac{1}{M_H} H(s_0, t_0)^{2/2^*} = s_0^2 + t_0^2. \tag{2.2.12}$$

Consider $(\omega_n) \subset X$ a minimizing sequence for S_K and the sequence $(s_0\omega_n, t_0\omega_n) \subset X \times X$.

Thus, using (2.2.12) we have that

$$
\begin{aligned}
&\frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla(s_0\omega_n)|^2 \mathrm{d}x + \int_{\mathbb{R}^N} K(x)|\nabla(t_0\omega_n)|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)H(s_0\omega_n, t_0\omega_n)\mathrm{d}x\right)^{2/2^*}} \\
= \;& \frac{(s_0^2+t_0^2)\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla\omega_n|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)\omega_n^{2^*}H(s_0,t_0)\mathrm{d}x\right)^{2/2^*}} = \frac{(s_0^2+t_0^2)\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla\omega_n|^2 \mathrm{d}x}{H(s_0,t_0)^{2/2^*}\left(\displaystyle\int_{\mathbb{R}^N} K(x)\omega_n^{2^*}\mathrm{d}x\right)^{2/2^*}} \\
= \;& \frac{1}{M_H}\frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla\omega_n|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)\omega_n^{2^*}\mathrm{d}x\right)^{2/2^*}}.
\end{aligned}
$$

Taking the limit on n in the last equality, we obtain

$$
\tilde{S}_{K,H} \le \frac{1}{M_H} S_K.
$$

To prove the reverse inequality, we use Lemma 2.2.4. Let $(u_n, v_n) \subset X \times X$, be a minimizing sequence for $\tilde{S}_{K,H}$. By definition of S_K we have

$$
S_K\|u_n\|_{2^*,K}^2 \le \int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \mathrm{d}x
$$

and

$$
S_K\|v_n\|_{2^*,K}^2 \le \int_{\mathbb{R}^N} K(x)|\nabla v_n|^2 \mathrm{d}x
$$

which gives us

$$
S_K\left(\|u_n\|_{2^*,K}^2 + \|v_n\|_{2^*,K}^2\right) \le \int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} K(x)|\nabla v_n|^2 \mathrm{d}x.
$$

Using Lemma 2.2.4 and the inequality (2.2.11), we obtain

$$
\begin{aligned}
&\frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} K(x)|\nabla v_n)|^2 \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)H(u_n, v_n)\mathrm{d}x\right)^{2/2^*}} \\
\ge \;& S_K \frac{\left(\|u_n\|_{2^*,K}^2 + \|v_n\|_{2^*,K}^2\right)}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)H(u_n, v_n)\mathrm{d}x\right)^{2/2^*}} \\
\ge \;& S_K \frac{\left(\|u_n\|_{2^*,K}^2 + \|v_n\|_{2^*,K}^2\right)}{H(\|u_n\|_{2^*,K}, \|v_n\|_{2^*K})^{2/2^*}} \\
\ge \;& \frac{1}{M_H} S_K.
\end{aligned}
$$

Taking the limit on n we have

$$
\tilde{S}_{K,H} \ge \frac{1}{M_H} S_K.
$$

The equality $S_K = S$ can be seen in [25, section 4].

□

Lemma 2.2.6. *If the conditions $(A_1)-(A_3)$ are hold, then $0 < c_* < \frac{1}{N}\widetilde{S}_{K,H}^{N/2}$, where c_* was defined in (2.4.3).*

Proof. We adapt the arguments and some calculations perfomed in [50, Proposition 3.2]. We taking a smooth function $\varphi \in C_c^\infty(\mathbb{R}^N, [0,1])$ satisfying $\varphi \equiv 1$ in $B_R(0)$ and $\varphi \equiv 0$ outside $B_{2R}(0)$. We consider the function

$$u_\epsilon(x) := K(x)^{-1/2}\varphi(x)\Phi_{\epsilon,0}(x).$$

Setting

$$v_\epsilon(x) := \frac{u_\epsilon(x)}{\|u_\epsilon\|_{2^*,K}}$$

we can use the definition of v_ϵ and (A_3) to get

$$\begin{aligned} I(ts_o v_\epsilon, tt_o v_\epsilon) &= \frac{1}{2}\|(ts_0 v_\epsilon, tt_0 v_\epsilon)\|^2 - \int_{\mathbb{R}^N} K(x)Q(ts_0 v_\epsilon, tt_0 v_\epsilon)dx \\ - & \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)H(ts_0 v_\epsilon, tt_0 v_\epsilon)dx \le \frac{t^2}{2}(s_o^2 + t_o^2)\|v_\epsilon\|_K^2 \\ - & \sigma t^{p_1} s_o^\gamma t_o^\beta \int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} K(x)|v_\epsilon|^{2^*}H(s_o,t_o)dx \\ = & \frac{t^2}{2}(s_o^2+t_o^2)\|v_\epsilon\|_K^2 - \sigma t^{p_1} s_o^\gamma t_o^\beta \int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx - \frac{t^{2^*}}{2^*}H(s_o,t_o), \end{aligned}$$

where (s_o, t_o) appeared in (2.2.10) and $p_1 \in (2, 2^*)$. Denoting the right size of the above equality by $h_\epsilon(t)$, as in [18, Lemma 3.5] we conclude that $h_\epsilon(t)$ has a unique critical point $t_\epsilon > 0$ such that

$$h_\epsilon(t_\epsilon) = \max_{t\ge 0} h_\epsilon(t). \tag{2.2.13}$$

Define

$$g_\epsilon(t) := \frac{t^2}{2}(s_o^2+t_o^2)\|v_\epsilon\|_K^2 - \frac{t^{2^*}}{2^*}H(s_o,t_o),\ t \ge 0, \tag{2.2.14}$$

and notice that the maximum of $g_\epsilon(t)$ is attained at

$$\tilde{t}_\epsilon = \left\{\frac{s_o^2+t_o^2}{H(s_o,t_o)}\|v_\epsilon\|_K^2\right\}^{1/(2^*-2)} = \left\{\frac{1}{M_H}\|v_\epsilon\|_K^2\right\}^{1/(2^*-2)} \ge t_\epsilon. \tag{2.2.15}$$

Since the function g_ϵ is increasing in $(0, \tilde{t}_\epsilon)$, we can use the definition of h_ϵ to get

$$h_\epsilon(t_\epsilon) \le \frac{1}{N}\left(\frac{s_o^2+t_o^2}{H(s_o,t_o)}\|v_\epsilon\|_K^2\right)^{N/2} - \sigma t_\epsilon^{p_1} s_o^\gamma t_o^\beta \int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx.$$

From [50, pp.1043–1046], we have

$$\|v_\epsilon\|_K^2 = S + O(\epsilon), \tag{2.2.16}$$

which implies

$$h_\epsilon(t_\epsilon) \;\le\; \frac{1}{N}\left(\frac{s_o^2+t_o^2}{H(s_o,t_o)}S + O(\epsilon)\right)^{N/2} - \sigma t_\epsilon^{p_1} s_o^\gamma t_o^\beta \int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx. \tag{2.2.17}$$

From Lemma 2.2.5, we obtain

$$h_\epsilon(t_\epsilon) \le \frac{1}{N}\left(\tilde{S}_{K,H}+O(\epsilon)\right)^{N/2} - \sigma t_\epsilon^{p_1} s_o^{\gamma} t_o^{\beta}\int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx.$$

If $a, b \ge 0$ and $m \ge 1$, then $(a+b)^m \le a^m + m(a+b)^{m-1}b$. Therefore,

$$h_\epsilon(t_\epsilon) \le \frac{1}{N}\widetilde{S}_{K,H}^{N/2} + O(\epsilon^{(N-2)/2}) - \sigma t_\epsilon^{p_1} s_o^{p_1} t_o^{p_1}\int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx.$$

Moreover, we can obtain $\rho > 0$ such that $t_\epsilon > \rho$ for ϵ small. Hence, it follows from the above inequality that

$$h_\epsilon(t_\epsilon) \le \frac{1}{N}\widetilde{S}_{K,H}^{N/2} + \epsilon^{(N-2)/2}\left(C - \frac{\sigma\rho^{p_1} s_o^{p_1} t_o^{p_1}}{\epsilon^{(N-2)/2}}\int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx\right).$$

From [50, Proposition 3.2], we have that $\lim\limits_{\epsilon\to 0^+}\frac{1}{\epsilon^{(N-2)/2}}\int_{\mathbb{R}^N} K(x)|v_\epsilon|^{p_1}dx = +\infty$, which we concluded that, for ϵ small enough

$$c_* \le \sup_{t\ge 0} I(ts_o v_\epsilon, tt_o v_\epsilon) \le h_\epsilon(t_\epsilon) < \frac{1}{N}\widetilde{S}_{K,H}^{N/2}.$$

□

2.3 Proof of Theorem 2.1.1

Proof. From Lemmas 2.2.2, 2.2.3 and 2.2.6, there exists a sequence $(u_n, v_n) \subset X \times X$ verifying $I(u_n, v_n) \to c_*$ and $I'(u_n, v_n) \to 0$ with

$$0 < c_* < \frac{1}{N}\widetilde{S}_{K,H}^{N/2}.$$

From (A_1) we have

$$\begin{aligned} c_* + o_n(1) &= I(u_n,v_n) - \frac{1}{p}I'(u_n,v_n)(u_n,v_n) \\ &= (\frac{1}{2}-\frac{1}{p})\|(u_n,v_n)\|^2 + \left(\frac{1}{p}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N} K(x)H(u_n,v_n)dx \\ &\ge (\frac{1}{2}-\frac{1}{p})\|(u_n,v_n)\|^2, \end{aligned}$$

which implies that the sequence (u_n, v_n) is bounded in $X \times X$. Then there exists $(u, v) \in X \times X$ such that, and, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ in $X \times X$. From [45, Proposition 2.1], up to a subsequence, we have that $(u_n, v_n) \to (u, v)$ in $L^p_K(\mathbb{R}^N) \times L^p_K(\mathbb{R}^N)$.

Considering $(w_n, z_n) := (u_n - u, v_n - v)$ and using Lemma 2.2.1 we get (w_n, z_n) is a $(PS)_d$ for I and $I(w_n, z_n) = I(u_n, v_n) - I(u, v) + o_n(1)$.

Moreover, from weak convergence and [45, Proposition 2.1] again, we obtain $I'(u, v) = 0$ and $I'(w_n, z_n) = I'(u_n, v_n) = o_n(1)$. Since

$$\int_{\mathbb{R}^N} K(x)Q(w_n,z_n)dx = o_n(1), \tag{2.3.1}$$

where conclude that

$$\|(w_n,z_n)\|^2 = \int_{\mathbb{R}^N} K(x)H(w_n,z_n)dx + o_n(1).$$

Then, there exists $l \geq 0$ such that

$$\|(w_n, z_n)\|^2 \to l \quad \text{and} \quad \int_{\mathbb{R}^N} K(x)H(w_n, z_n)dx \to l.$$

Suppose, by contradiction, that $l > 0$ and note that by definition of $\widetilde{S}_{K,H}$, we have that $l \geq \widetilde{S}_{K,H}^{N/2}$. Then,

$$\begin{aligned} c_* + o_n(1) &= I(u_n, v_n) - \frac{1}{2}I'(u_n, v_n)(u_n, v_n) \\ &= I(w_n, z_n) - \frac{1}{2}I'(w_n, z_n)(w_n, z_n) + o_n(1) \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N} K(x)H(w_n, z_n)dx + o_n(1) = \frac{1}{N}l + o_n(1) \geq \frac{1}{N}\widetilde{S}_{K,H}^{N/2}, \end{aligned}$$

which is absurd. Hence, $l = 0$, which implies that $(u_n, v_n) \to (u, v)$ in $X \times X$. □

2.4 Supercritical case

To solve system (SC'), we first consider a truncated problem which involves only a subcritical Sobolev exponent. As in the case of critical growth, we will use the Mountain Pass Theorem to show the existence of a positive solution for the truncated system. After that we are going to show that any positive solution of truncated system is a positive solution of system (SC').

2.4.1 Truncated problem

First of all, note that since $\Upsilon_i > 2^*$ we cannot use directly variational techniques to study system (SC'). Hence we construct a suitable truncation of on the nonlinearity in order to use variational methods. This truncation was used in [26] in the escalar case and in [66] in the system case. Consider the functions $\mathfrak{l}_i : \mathbb{R} \to \mathbb{R}$ given by

$$\mathfrak{l}_i(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t^{\Upsilon_i - 1} & \text{if } 0 < t \leq 1, \\ t^{p-1} & \text{if } t \geq 1, \end{cases}$$

where $2 < p < 2^*$ and $i = 1, 2$. Considering $\mathfrak{L}_i(t) = \int_0^t \mathfrak{l}_i(s)ds$, we have

$$\mathfrak{l}_i(t) \leq t^{p-1} \text{ and } \mathfrak{L}_i(t) \leq \frac{1}{p}t^p \tag{2.4.1}$$

and the truncated system

$$(SCT) \qquad \begin{cases} -\text{div}(K(x)\nabla u) = K(x)Q_u(u, v) + K(x)\mathfrak{l}_1(u) \text{ in } \mathbb{R}^N, \\ -\text{div}(K(x)\nabla v) = K(x)Q_v(u, v) + K(x)\mathfrak{l}_2(v) \text{ in } \mathbb{R}^N. \end{cases}$$

We recall that the weak solutions of (SCT) are the critical points of the functional

$$J_\sigma(u, v) = \frac{1}{2}\|(u, v)\|^2 - \int_{\mathbb{R}^N} K(x)Q(u, v)dx - \int_{\mathbb{R}^N} K(x)\mathfrak{L}_1(u)dx - \int_{\mathbb{R}^N} K(x)\mathfrak{L}_2(v)dx$$

and

$$\begin{aligned} J_\sigma{}'(u,v)(\varphi,\psi) &= \int_{\mathbb{R}^N} K(x)\left[\nabla u \nabla \varphi + \nabla v \nabla \psi\right] dx \\ &- \int_{\mathbb{R}^N} K(x)\left[Q_u(u,v)\varphi + Q_v(u,v)\psi\right] dx \\ &- \int_{\mathbb{R}^N} K(x)\left[\mathfrak{l}_1(u)\varphi + \mathfrak{l}_2(v)\psi\right] dx. \end{aligned}$$

Moreover

$$J_\sigma(u,v) - \frac{1}{p} J_\sigma{}'(u,v)(u,v) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|(u,v)\|^2 \tag{2.4.2}$$

Note that the functional associated to system (SCT) depends on σ due to the hypothesis $(\widetilde{A}_3)$.

2.4.2 The existence result for the truncated system

Theorem 2.4.1. *Assume that conditions* $(A_1), (A_2), (\widetilde{A}_3)$ *are hold. Then, system* (SCT) *has a weak positive solution.*

Proof. Arguing as Lemma 2.2.2, there exist positive numbers ρ and α such that,

$$J_\sigma(u,v) \geq \alpha > 0, \forall (u,v) \in \ X \times X : \|(u,v)\| = \rho.$$

Now since $2 < p < 2$, arguing as Lemma 2.2.3, there exists $(e_1, e_2) \in \ X \times X$ with $J_\sigma(e_1,e_2) < 0$ and $\|(e_1,e_2)\| > \rho$.

Using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [10], without (PS) condition (see [76, Theorem p.12]), there exists a sequence $(u_n, v_n) \subset X \times X$ satisfying

$$J_\sigma(u_n,v_n) \to c_\sigma \quad \text{and} \quad J_\sigma{}'(u_n,v_n) \to 0,$$

where

$$c_\sigma = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\sigma(\gamma(t)) > 0, \tag{2.4.3}$$

and

$$\Gamma := \{\gamma \in C([0,1], X \times X) : \gamma(0) = 0, \ J_\sigma(\gamma(1)) < 0\}.$$

Since the embedding $X \hookrightarrow L^s_K(\mathbb{R}^N)$ are continuous for $2 \leq s \leq 2^*$ and compact for $2 \leq s < 2^*$ [45, Proposition 2.1], we can use well-known arguments to prove that there is $(u,v) \in X \times X$ such that $J_\sigma(u,v) = c_\sigma$ and $J_\sigma{}'(u,v) = 0$. □

Let us now consider the following problem

$$\begin{cases} -\Delta \omega = |\omega|^{p_1 - 2}\omega \ \text{ in } \Omega, \\ \omega \in H^1_0(\Omega), \end{cases} \tag{P_Ω}$$

where p_1 is the constant which appears in the hypothesis $(\widetilde{A}_3)$. It is well-known that using the Mountain Pass Theorem [10], problem (P_Ω) has a nontrivial solution $\omega \in H^1_0(\Omega)$ satisfying

$$\|\omega\|^2_{H^1_0(\Omega)} = \int_\Omega |\omega|^{p_1} dx. \tag{2.4.4}$$

This information will be used to obtain an estimate for c_σ and it will be crucial to show the existence of a solution for the supercritical system.

Lemma 2.4.2. $\lim_{\sigma\to+\infty} c_\sigma = 0$.

Proof. Note that, by definition of c_σ, (2.4.4) and $(\widetilde{A}_3)$, we have

$$\begin{aligned} c_\sigma &\leq \max_{t\geq 0} J_\sigma(t\omega) \leq \max_{t\geq 0}\left[t^2\int_\Omega K(x)|\nabla\omega|^2dx - \int_\Omega K(x)Q(t\omega,t\omega)dx\right] \\ &\leq \max_{t\geq 0}\left[t^2K_M - K_m\sigma t^{p_1}\right]\|\omega\|^2_{H^1_0(\Omega)} \\ &= \frac{K_M^{p_1/(p_1-2)}}{(k_m\sigma)^{2/(p_1-2)}}\left[(\frac{2}{p_1})^{2/(p_1-2)} - (\frac{2}{p_1})^{p_1/(p_1-2)}\right], \end{aligned}$$

where $K_M = \max_{x\in\Omega} K(x)$ and $K_m = \min_{x\in\Omega} K(x)$. Since $2 < p_1$, we have that $\left[(\frac{2}{p_1})^{2/(p_1-2)} - (\frac{2}{p_1})^{p_1/(p_1-2)}\right] > 0$ and the prove is over. □

2.4.3 Proof of Theorem 2.1.2

Let (u,v) be a solution of (SCT), by definition of Q and $\mathfrak{L}_i$ we can assume without loss of generality, that $u,v\geq 0$. It is sufficient to show that $|u|_\infty \leq \mathfrak{M}$ and $|v|_\infty \leq \mathfrak{M}$.

For each $L\geq 1$ we can define

$$u_L(x) = \begin{cases} u(x), & \text{if } u(x)\leq L \\ L, & \text{if } u(x) > L \end{cases}$$

and

$$v_L(x) = \begin{cases} u(x), & \text{if } u(x)\leq L \\ L, & \text{if } u(x) > L \end{cases}.$$

Consider $w_1 = uu_L^{2(\beta-1)}$ and $w_2 = vv_L^{2(\beta-1)}$, where $\beta > 1$ is a constant to be determined later. Taking $(w_1, 0)$ as a test function, we obtain

$$\int_{\mathbb{R}^N} K(x)\nabla u\nabla w_1 dx = \int_{\mathbb{R}^N} K(x)\left[Q_u(u,v)w_1dx + \mathfrak{l}_1(u)w_1\right]dx.$$

Since

$$\int_{\mathbb{R}^N} K(x)uu_L^{2(\beta-1)-1}\nabla u_L\nabla u = \int_{u\leq L} K(x)u^{2(\beta-1)}|\nabla u|^2 \geq 0,$$

it follows that

$$\int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)}|\nabla u|^2 \leq \int_{\mathbb{R}^N} K(x)\left[Q_u(u,v)uu_L^{2(\beta-1)}dx + \mathfrak{l}_1(u)uu_L^{2(\beta-1)}\right]dx. \quad (2.4.5)$$

Note that

$$\lim_{s\to\infty}\frac{Q_s(s,t)}{s^{p-1}} = c_1 \quad \lim_{s\to 0} Q_s(s,t) = 0 \quad (2.4.6)$$

where c_1 is the constant that appeared in $(\mathcal{H}_1^q)$. Combining (2.4.6), the definition of u_L and (2.4.1) with (2.4.5), we conclude that there exists $M_1 > 0$ suchat

$$\int_{\mathbb{R}^N} K(x)u^{2(\beta-1)}|\nabla u|^2dx \leq (M_1+1)\int_{\mathbb{R}^N} K(x)u^pu_L^{2(\beta-1)}dx. \quad (2.4.7)$$

On the other hand, consider $\widehat{u}_L = uu_L^{\beta-1}$. Since $u_L\leq u$, we have

$$S_K\|\widehat{u}_L\|^2_{2^*,K} \leq \int_{\mathbb{R}^N} K(x)|\nabla\left(uu_L^{\beta-1}\right)|^2dx \leq \beta^2\int_{\mathbb{R}^N} K(x)u_L^{2(\beta-1)}|\nabla u|^2dx.$$

Then, using (2.4.7) we obtain

$$\begin{aligned} S_K \|\widehat{u}_L\|^2_{2^*,K} &\leq \beta^2 (M_1+1) \int_{\mathbb{R}^N} K(x) u^p u_L^{2(\beta-1)} dx \\ &= \beta^2 (M_1+1) \int_{\mathbb{R}^N} K(x) u^{p-2} |\widehat{u}_L|^2 dx. \end{aligned}$$

Now, applying the Hölder inequality with exponents $\frac{2^*}{p-2}$ and $\frac{2^*}{2^*-(p-2)}$, we get

$$S_K \|\widehat{u}_L\|^2_{2^*,K} \leq \beta^2 (M_1+1) \|u\|^{\frac{p-2}{2^*}}_{2^*,K} \left(\int_{\mathbb{R}^N} K(x) |\widehat{u}_L|^{\frac{22^*}{2^*-(p-2)}} \right)^{\frac{2^*-(p-2)}{2^*}}.$$

From continuous embedding from $X \times X$ into $L^{2^*}_K(\mathbb{R}^N) \times L^{2^*}_K(\mathbb{R}^N)$ and (2.4.2) we have

$$S_K \|\widehat{u}_L\|^2_{2^*,K} \leq \beta^2 (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \left(\int_{\mathbb{R}^N} K(x) |\widehat{u}_L|^{\frac{22^*}{2^*-(p-2)}} \right)^{\frac{2^*-(p-2)}{2^*}},$$

where we conclude that

$$S_K \|\widehat{u}_L\|^2_{2^*,K} \leq \beta^2 (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \|\widehat{u}_L\|^2_{\zeta,K},$$

with $\zeta = \frac{22^*}{2^*-(p-2)}$. Using $u_L \leq |u|$, we have

$$S_K \|\widehat{u}_L\|^2_{2^*,K} \leq \beta^2 (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \left(\int_{\mathbb{R}^N} K(x) |u|^{\beta\zeta} dx \right)^{2/\zeta}.$$

By Fatou's Lemma in the variable L we obtain

$$S_K \|u\|^{2\beta}_{2^*,K} \leq \beta^2 (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \|u\|^{2\beta}_{\beta\zeta,K},$$

which implies

$$\|u\|_{\beta 2^*,K} \leq \beta^{1/\beta} \left[S_K^{-1} (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \right]^{\frac{1}{2\beta}} \|u\|_{\beta\zeta,K}.$$

We now taking $\beta = 2^*/\zeta > 1$, and note that, since $u, v \in L^{2^*}_K(\mathbb{R}^N)$, the above inequality is holds for this choice of β. Moreover, since $\beta^2\zeta = \beta 2^*$, it follows that the inequality also holds with β replaced by β^2.

Hence,

$$\|u\|_{\beta^2 2^*,K} \leq (\beta^2)^{1/\beta^2} \left[S_K^{-1} (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \right]^{\frac{1}{2\beta^2}} \|u\|_{\beta^2\zeta,K}.$$

By iterating this process and recalling that $\beta\zeta = 2^*$, we obtain, for $k \in \mathbb{N}$,

$$\|u\|_{\beta^k 2^*,K} \leq \beta^{\sum_{i=1}^k i\beta^{-i}} \left[S_K^{-1} (M_1+1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \right]^{\sum_{i=1}^k \beta^{-i}} \|u\|_{2^*,K}.$$

Note that since $K(x) \geq 1$ for all $x \in \mathbb{R}^N$, $\beta > 1$, from continuous embedding from $X \times X$ into $L_K^{2*}(\mathbb{R}^N) \times L_K^{2*}(\mathbb{R}^N)$ and (2.4.2), we have we can take the limit as $k \to \infty$ to get

$$\|u\|_\infty \leq \beta^{\sigma_1} \left[S_K^{-1} (M_1 + 1) \left(\frac{2pc_\sigma}{p-2} \right)^{\frac{p-2}{22^*}} \right]^{\sigma_2} \left(\frac{2pc_\sigma}{p-2} \right)^{1/2},$$

where

$$\sigma_1 = \sum_{i=1}^{\infty} i\beta^{-i} \quad \sigma_2 = \sum_{i=1}^{\infty} \beta^{-i}.$$

Choosing σ^* sufficient large and fixing $\sigma^* \leq \sigma$, we have $\|u\|_\infty \leq 1$. Repeating the same reasoning with the test function $(0, w_2)$, we have $\|v\|_\infty \leq 1$ and, in this case, if $(u, v) \in X \times X$ is a positive solution of system (SCT), then it is a solution of system (SC').

Chapter 3

Existence of positive solutions for a class of elliptic problems with fast increasing and critical exponent and discontinuous nonlinearity

3.1 Introduction

In this chapter we are looking for positive solutions to a problem with nonlinearity discontinuous. To be specific, we are looking positive solutions for the following class of problems

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda h(x) + \mathbf{H}(u-a)|u|^{2^*-2}u \ \ in \ \ \mathbb{R}^N, \tag{3.1.1}$$

where $a > 0, N \geq 3$, 2^* is the critical Sobolev exponent (i.e. $2^* := 2N/(N-2)$), $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a nonnegative function and $\mathbf{H}$ is the Heaviside function, defined as

$$\mathbf{H}(s) := \begin{cases} 0 & \text{if } \ s \leq 0, \\ 1 & \text{if } \ s > 0. \end{cases}$$

Since many obstacle problems and free boundary problems (that appear in certain physical situations) may be reduced to partial differential equations with discontinuous nonlinearities, in recent decades, the study of existence, nonexistence and multiplicity of solutions for problems with discontinuous nonlinearity has attracted the interest of several researchers, see [8, 9, 11, 15, 16, 18, 19, 27–30, 38, 51, 58, 67, 70] and the references therein. For more recent papers, see [5–7, 34, 68, 72]. Moreover, some physical problems are related to discontinuous surface

$$\Gamma_a(u) = \{x \in \mathbb{R}^N; u(x) = a\}$$

which causes difficulties in analyzing this kind of problems, as can be seen in [8] and [12].

On the operator, we would like to emphasize that, as observed by Escobedo and Kavian in [36], problem

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{p-2}u \ \ in \ \ \mathbb{R}^N,$$

with $2 < p \leq 2^*$ naturally appears when we deal with the nonlinear heat equation

$$u_t - \Delta u = |u|^{p-2}u \ \ in \ \ (0, \infty) \times \mathbb{R}^N,$$

and look for solutions with the special form $u_\lambda(t,x) = t^\lambda u(t^{-1/2}x)$, for $\lambda = 1/(p-1)$. We quote the works [13], [21], [25], [46] , [54], [62], [63] and references therein for information about existence, nonexistence, decay rate and many other aspects concerning this subject.

Since the exponential-type weight $K(x) = \exp(|x|^2/4)$ verifies $\nabla K(x) = \frac{1}{2}xK(x)$, problem (3.1.1) can be written as

$$-\operatorname{div}(K(x)\nabla u) = K(x)\left(\lambda h(x) + \mathbf{H}(u-a)|u|^{2^*-2}u\right) \quad in \quad \mathbb{R}^N. \tag{3.1.2}$$

We are looking for solution of (3.1.2) on the space X defined as the completion of the smooth functions with compact support $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx;\ u \in X. \tag{3.1.3}$$

As quoted in [45, Proposition 2.1], X is a Banach space and the weighted Lebesgue spaces

$$L^s_K(\mathbb{R}^N) := \left\{u \text{ measurable in } \mathbb{R}^N : \|u\|^s_{K,s} := \int_{\mathbb{R}^N} K(x)|u|^s dx < \infty\right\}$$

are such that the embedding $X \hookrightarrow L^s_K(\mathbb{R}^N)$ are continuous for $2 \le s \le 2^* := \frac{2N}{N-2}$ and compact for $2 \le s < 2^*$.

Since the nonlinearity in (3.1.2) is discontinuous, we will consider the following notion of solution to (3.1.2) inspired by Chang in [27] and [28]. A function $u \in X$ is a solution of (3.1.2) if

$$-\operatorname{div}(K(x)\nabla u) - \lambda K(x)h(x) \in K(x)\widehat{f}_{\mathbf{H}}(u) \ \text{ a.e in } \mathbb{R}^N, \tag{3.1.4}$$

where $\widehat{f}_{\mathbf{H}}$ is a multi-valued function

$$\widehat{f}_{\mathbf{H}}(s) = \begin{cases} \{0\} & \text{if } s < a, \\ \{s^{2^*-1}\} & \text{if } s > a, \\ [0, a^{2^*-1}] & \text{if } s = a. \end{cases}$$

We emphasize that the solutions of (3.1.2) are critical points of locally Lipschitz functional $I_{\lambda,a} : X \to \mathbb{R}$ given by

$$I_{\lambda,a}(u) = \int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \int_{\mathbb{R}^N} K(x)F_{\mathbf{H}}(u)dx - \lambda\int_{\mathbb{R}^N} K(x)h(x)u dx.$$

Throughout the paper, we will assume h is a nonnegative function which satisfies

$$h \in L^\theta_K(\mathbb{R}^N) \ \text{ with } \frac{1}{\theta} + \frac{1}{2^*} = 1 \ \text{ and } \ h \neq 0. \tag{3.1.5}$$

The main result is:

Theorem 3.1.1. *Assume that (3.1.5) holds. Then, there exists $\lambda_* > 0$ and $a_* > 0$ such that for all $\lambda \in (0, \lambda_*)$ and $a \in (0, a_*)$, problem (3.1.2) has two nonnegative solutions $u_i = u_i(a), i = 1, 2$, with the following properties:*

(i) $-div(K(x)\nabla u_i) \in L^\theta_K(\mathbb{R}^N)$.

(ii) $meas(\{u_i = a\} := \{x \in \mathbb{R}^N : u_i(x) = a\}) = 0$.

(iii) $meas(\{u_i > a\} := \{x \in \mathbb{R}^N : u_i(x) > a\}) > 0$.

(iv) $I_{\lambda,a}(u_2) < 0 < I_{\lambda,a}(u_1)$,

where meas$(\cdot)$ denote the Lebesgue measure. Moreover, if $a_n \to 0^+$ there exist two functions $v_i \in X, i = 1, 2$, such that, up to a subsequence, $u_i(a_n) \to v_i$ in X, $I_{\lambda,0}(v_2) < 0 < I_{\lambda,0}(v_1)$ and v_1, v_2 are solutions of

$$\begin{cases} -div(K(x)\nabla v) = K(x)\left(\lambda h(x) + |v|^{2^*-2}v\right) & a.e \text{ in } \mathbb{R}^N, \\ v \in X, v \geq 0 & a.e \text{ in } \mathbb{R}^N, \end{cases} \tag{3.1.6}$$

for all $\lambda \in (0, \lambda_*)$.

Remark 1. *The item (ii) of Theorem 3.1.1 is very important because it ensures that the following equality is true*

$$-div(K(x)\nabla u_i) = K(x)\left(\lambda h(x) + \boldsymbol{H}(u_i - a)|u_i|^{2^*-2}u_i\right) \quad a.e \text{ in } \mathbb{R}^N.$$

Item (iii) implies that each u_i is different from of solution of the equation $-div(K(x)\nabla u) = \lambda K(x)h(x)$ in $\mathbb{R}^N$.

Our arguments were strongly influenced by [2], [3], [4] and [34]. In [2], using convex analysis, the authors establish the existence of at least two nonnegative solutions for the quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda h(x) + \mathbf{H}(u - a)|u|^{p^*-2}u & \text{a.e in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), u \geq 0 & \text{a.e in } \mathbb{R}^N, \end{cases} \tag{3.1.7}$$

where Δ_p is the p-Laplacian operator and h is a positive function.

The authors in [3] study the existence and multiplicity of positive solutions for a class of semilinear elliptic problems of second order, posed in all of $\mathbb{R}^N$, where the nonlinearity is discontinuous and of the form $\lambda h(x)\mathbf{H}(u - a)u^q + |u|^{2^*-2}u$.

Assuming that f is a discontinuous function with exponential critical growth, in [4], the authors have applied variational methods for locally Lipschitz functional to get two solutions for

$$\begin{cases} -\Delta u = \epsilon h(x) + \mathbf{H}(u - a)f(u) & \text{a.e in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^N), u \geq 0 & \text{a.e in } \mathbb{R}^2, \end{cases}$$

where ϵ is a positive small parameter. The version of problem (3.1.7) with fractional Laplacian was studied in [34].

Below we list what we believe that are the main contributions of our paper.

1) The arguments involved in the study of the problem (3.1.2) are not standard ones: First of all because we are working with the exponential-type weight $K(x) = \exp(|x|^2/4)$, which causes some difficulties, as can be seen in Lemma 3.1, Lemma 3.3 and Lemma 3.4.

2) In [2], [3] and [4] the asymptotic behavior of the solutions found was not studied. Furthermore, in order to carry out this study of asymptotic behavior, some independent estimates of the parameter a were necessary.

3) We study the asymptotic behavior of the solutions $u_i = u_i(a)$, $i = 1, 2$, when the parameter a goes to 0, which, in general, it is not studied in problems of this nature. This study requires delicate uniform estimates of the parameter a to prove that $(u_{i,a})$ is bounded and that the weak limit is nontrivial.

4) Since we have the presence of the function $H(u-a)$, we are not able to use Variational Methods for C^1 functionals. For this reason, we use Variational Methods for nondifferentiable functionals, motivated by the works of Chang [27], [28], [29], [30] and [52], see Section 2.

This chapter is organized as follows. In Section 2 we remember some results from Convex Analysis. Some estimates on the minimax level is given in Section 3. The proof of the main result is in Section 4.

3.2 Basic results from convex analysis

In this section, for the reader's convenience, we recall some definitions and basic results on the critical point theory of locally Lipschitz continuous functionals as developed by Chang [28], Clarke [29, 30] and Grossinho and Tersian [52].

Let E be a real Banach space. A functional $I: E \to \mathbb{R}$ is locally Lipschitz continuous, $I \in Lip_{loc}(E, \mathbb{R})$ for short, if given $u \in E$ there is an open neighborhood $V := V_u \subset E$ and some constant $M = M_V > 0$ such that

$$\mid I(v_2) - I(v_1) \mid \leq M\|v_2 - v_1\|,\ v_i \in V,\ i = 1, 2.$$

The directional derivative of I at u in the direction of $v \in E$ is defined by

$$I^0(u; v) = \limsup_{h \to 0,\ \sigma \downarrow 0} \frac{I(u + h + \sigma v) - I(u + h)}{\sigma}.$$

Hence $I^0(u; .)$ is continuous, convex and its subdifferential at $z \in X$ is given by

$$I^0(u; z) = \{\mu \in E^*; I^0(u; v) \geq I^0(u; z) + \langle \mu, v - z \rangle,\ v \in X\},$$

where $\langle ., . \rangle$ is the duality pairing between E^* and X. The generalized gradient of I at u is the set

$$\partial I(u) = \{\mu \in E^*; \langle \mu, v \rangle \leq I^0(u; v),\ v \in E\}.$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is the subdifferential of $I^0(u; 0)$. A few definitions and properties will be recalled below.

$$\partial I(u) \subset E^* \text{ is convex, non-empty and weak*-compact,}$$

$$m_I(u) = \min\{\parallel \mu \parallel_{E^*}; \mu \in \partial I(u)\}, \tag{3.2.1}$$

and

$$\partial I(u) = \{I'(u)\}, \quad \text{if } I \in C^1(E, \mathbb{R}).$$

A critical point of I is an element $u_0 \in E$ such that $0 \in \partial I(u_0)$ and a critical value of I is a real number c such that $I(u_0) = c$ for some critical point $u_0 \in E$.

A sequence $(u_n) \subset E$ is called Palais-Smale sequence at level c $(PS)_c$ if

$$I(u_n) \to c \ \text{ and } m_I(u_n) \to 0.$$

A functional I satisfies the $(PS)_c$ condition if any Palais-Smale sequence at nivel c has a convergent subsequence.

Proposition 3.2.1. *(See [29, 30, 52]) Let $I_1, I_2 : E \to R$ be locally Lipschitz functions, then:*

(i) $I_1 + I_2 \in Lip_{loc}(E, \mathbb{R})$ *and* $\partial(I_1 + I_2)(u) \subseteq \partial I_1(u) + \partial I_2(u)$*, for all* $u \in E$.

(ii) $\partial(\lambda I_1)(u) = \lambda \partial I_1(u)$ *for each* $\lambda \in \mathbb{R}, u \in E$.

(iii) Suppose that for each point v in a neighborhood of u, I_1 admits a Gateaux derivative $I_1'(v)$ and that $I_1' : E \to E^*$ is continuous, then $\partial I_1(u) = \{I_1'(u)\}$.

Theorem 3.2.2. *(See [29, 30, 52]) Let E be a Banach space and let $I \in Lip_{loc}(E, \mathbb{R})$ with $I(0) = 0$. Suppose there are numbers $\alpha, r > 0$ and $e \in E$, such that*

(i) $I(u) \geq \alpha, \quad$ *for all* $\quad u \in E; \|u\| = r$,

(ii) $I(e) < 0$ *and* $\|e\| > r$.

Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}. \tag{3.2.2}$$

Then $c \geq \alpha$ and there is a sequence $(u_n) \subset X$ satisfying

$$I(u_n) \to c \quad \text{and} \quad m_I(u_n) \to 0.$$

If, in addition, I satisfies the $(PS)_c$-condition, then c is a critical value of I.

Theorem 3.2.3. *(Riesz representation theorem) Let Φ be a bounded linear functional on $L^r_K(\mathbb{R}^N), 1 < r < \infty$. Then, there is a unique function $u \in L^{r'}_K(\mathbb{R}^N)$ with $\frac{1}{r'} + \frac{1}{r} = 1$, such that*

$$\langle \Phi, \varphi \rangle = \int_{\mathbb{R}^N} K(x) u \varphi dx, \quad \text{for all} \quad \varphi \in L^r_K(\mathbb{R}^N).$$

Moreover, $\|\Phi\|_{(L^r_K(\mathbb{R}^N))^*} = \|u\|_{K,r}$, where $(L^r_K(\mathbb{R}^N))^*$ is the dual space of $L^r_K(\mathbb{R}^N)$ and $\|.\|_{K,r}$ is given in (3.1.3).

Proof. Consider $T : L^{r'}_K(\mathbb{R}^N) \to (L^r_K(\mathbb{R}^N))^*$ be the operator given by

$$\langle Tu, \varphi \rangle = \int_{\mathbb{R}^N} K(x) u \varphi dx,$$

by the Hölder inequality, it follows that

$$\begin{aligned} |\langle Tu, \varphi \rangle| &= \left| \int_{\mathbb{R}^N} K(x)^{\frac{1}{r'}} u K(x)^{\frac{1}{r}} \varphi dx \right| \\ &\leq \|u\|_{K,r'} \|\varphi\|_{K,r}, \quad \text{for all} \quad u \in L^{r'}_K(\mathbb{R}^N), \varphi \in L^r_K(\mathbb{R}^N). \end{aligned}$$

Hence, arguing as in the proof of [20, Theorem 4.11], for the case $K(x) \equiv 1$, we conclude that $\|Tu\|_{(L^{r'}_K(\mathbb{R}^N))^*} = \|u\|_{K,r}$ for all $u \in L^r_K(\mathbb{R}^N)$ and T is surjective.

□

For the next lemma we need of the following definitions: Let $\underline{f}_{\mathbf{H}}, \overline{f}_{\mathbf{H}} : \mathbb{R} \to \mathbb{R}$ be the functions N-mensurable, see [28], defined by

$$\underline{f}_{\mathbf{H}}(t) := \lim_{\delta \downarrow 0} \text{ess inf}_{|t-s|<\delta} f_{\mathbf{H}}(s) \quad \text{and} \quad \overline{f}_{\mathbf{H}}(t) := \lim_{\delta \downarrow 0} \text{ess sup}_{|t-s|<\delta} f_{\mathbf{H}}(s).$$

Using the Theorem 3.2.3 and arguing as in Chang [28], we obtain the following version of the [28, Theorems 2.1 and 2.2] and your proof will be omitted.

Lemma 3.2.4. *The functional $\Phi_{\boldsymbol{H}}: L^{2^*}_K(\mathbb{R}^N) \to \mathbb{R}$ given by*

$$\Phi_{\boldsymbol{H}}(u) = \int_{\mathbb{R}^N} K(x) F_{\boldsymbol{H}}(u)dx, \tag{3.2.3}$$

where $F_{\boldsymbol{H}}(t) = \displaystyle\int_0^t f_{\boldsymbol{H}}(s)ds$ and $f_{\boldsymbol{H}}(s) = \boldsymbol{H}(s-a)s^{2^*-1}$, satisfies:

(i) $\Phi_{\boldsymbol{H}} \in Lip_{loc}(L^{2^*}_K(\mathbb{R}^N), \mathbb{R})$ and for every $\rho^* \in \partial\Phi_{\boldsymbol{H}}(u)$ there exists $\rho \in L_K^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$ such that $\rho \in [\underline{f}_{\boldsymbol{H}}(u), \overline{f}_{\boldsymbol{H}}(u)]$ a.e in $\mathbb{R}^N$ and

$$\langle \rho^*, \varphi \rangle = \int_{\mathbb{R}^N} K(x)\rho\varphi dx, \quad \textit{for all } \varphi \in L^{2^*}_K(\mathbb{R}^N).$$

(ii) If $\Phi_{\boldsymbol{H}}\big|_X$ is the restriction to X of $\Phi_{\boldsymbol{H}}$, then $\partial\big(\Phi_{\boldsymbol{H}}\big|_X\big)(u) = \partial\Phi_{\boldsymbol{H}}(u)$, for all $u \in X$.

Remark 2. *By definition of $f_{\boldsymbol{H}}$, it is clear that $\underline{f}_{\boldsymbol{H}}(s) = \overline{f}_{\boldsymbol{H}}(s) = 0$ for all $s < a$, $\underline{f}_{\boldsymbol{H}}(s) = \overline{f}_{\boldsymbol{H}}(s) = f(s)$ for all $s > a$ and $\underline{f}_{\boldsymbol{H}}(s) = 0, \overline{f}_{\boldsymbol{H}}(s) = f(a)$ for $s = a$. Then, defining $\widehat{f}_{\boldsymbol{H}}(s) = [\underline{f}_{\boldsymbol{H}}(s), \overline{f}_{\boldsymbol{H}}(s)]$, we have*

$$\widehat{f}_{\boldsymbol{H}}(s) = \begin{cases} \{0\} & \textit{if } s < a, \\ \{s^{2^*-1}\} & \textit{if } s > a, \\ [0, a^{2^*-1}] & \textit{if } s = a. \end{cases}$$

Now, let us consider the energy functional $I_{\lambda,a}: X \to \mathbb{R}$ defined as follows:

$$I_{\lambda,a}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F_{\mathbf{H}}(u)dx - \lambda\int_{\mathbb{R}^N} K(x)h(x)udx, \tag{3.2.4}$$

for $a \geq 0$. We observe that for the case $a = 0$, the functional $I_{\lambda,0}$ is given by

$$I_{\lambda,0}(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)(u_+)^{2^*}dx - \lambda\int_{\mathbb{R}^N} K(x)h(x)udx,$$

where $u^+ = \max\{u, 0\}$. It is clear that, different of the case $a > 0$, the functional $I_{\lambda,0}$ is of class $C^1(X, \mathbb{R})$.

Lemma 3.2.5. *For each $a > 0$, the functional $I_{\lambda,a} \in Lip_{loc}(X, \mathbb{R})$ and critical points of $I_{\lambda,a}$ are solutions of (3.1.2) in the sense of (3.1.4).*

Proof. In fact, note that by (ii) of Lemma 3.2.4 we can write $I_{\lambda,a}(u) = \Psi(u) - \Phi_H(u)$ with $\Psi \in C^1(X, \mathbb{R})$ and $\Phi_H \in Lip_{loc}(X, \mathbb{R})$, where

$$Q(u) = \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N} K(x)h(x)udx$$

and Φ_H is given in (3.2.3). Hence, by Proposition 3.2.1, we have $I_{\lambda,a} \in Lip_{loc}(X, \mathbb{R})$ and $\partial I_{\lambda,a}(u) \subseteq \{Q'(u)\} - \partial\Phi_H(u)$, for all $u \in X$.

Therefore, if $u \in X$ is a critical point of $I_{\lambda,a}$ there exists $\rho \in L^{2^*}_K(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} K(x)\nabla u\nabla\varphi dx = \lambda\int_{\mathbb{R}^N} K(x)h(x)\varphi dx + \int_{\mathbb{R}^N} K(x)\rho\varphi dx, \quad \text{for all } \varphi \in X, \tag{3.2.5}$$

where $\rho \in \partial\Phi(u)$. From Remark 2, we have $\rho \in \widehat{f}_H(u)$ with

$$\widehat{f}_H(u) = \begin{cases} \{0\} & \text{if } u < a, \\ \{u^{2^*-1}\} & \text{if } u > a, \\ [0, a^{2^*-1}] & \text{if } u = a. \end{cases} \tag{3.2.6}$$

Taking $\varphi = u^- := \min\{u, 0\}$ as a test function in (3.2.5) and using (3.2.6), we get $\|u^-\| \leq 0$, then $u = u^+ \geq 0$ a.e in $\mathbb{R}^N$. From (3.2.5), we have

$$-\text{div}(K(x)\nabla u) = L_1 + L_2 \quad \text{in} \quad X^*,$$

where $L_1, L_2 : X \to \mathbb{R}$ are linear functionals given by

$$L_1(v) = \lambda \int_{\mathbb{R}^N} K(x)h(x)v \, dx \text{ and } L_2(v) = \int_{\mathbb{R}^N} K(x)\rho v \, dx.$$

Since $L_1, L_2 \in \left(L^\theta_K(\mathbb{R}^N)\right)^* \subset X^*$, by Riesz's Theorem, see Theorem 3.2.3, we have $L_1, L_2 \in L^\theta_K(\mathbb{R}^N)$ and so

$$-\,\text{div}(K(x)\nabla u) \in L^\theta_K(\mathbb{R}^N). \tag{3.2.7}$$

Since (3.2.5) holds, then

$$-\text{div}(K(x)\nabla u) = \lambda K(x)h + K(x)\rho \quad \text{a.e. in } \mathbb{R}^N$$

which implies that

$$-\text{div}(K(x)\nabla u) - \lambda K(x)h(x) = K(x)\rho \in K(x)\widehat{f}(u) \quad \text{a.e. in } \mathbb{R}^N.$$

This conclude that u is a solution of (3.1.2) in the sense of (3.1.4).

□

3.3 Preliminary results

The next result says that the functional $I_{\lambda,a}$ satisfies the Palais-Smale condition at any level c smaller than a certain threshold related to the best critical Sobolev constant S_K of the injection $X \hookrightarrow L^{2^*}_K(\mathbb{R}^N)$. More precisely, S_K is defined by

$$S_K := \inf_{u \in X \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(x)|u|^{2^*} dx\right)^{2/2^*}} > 0, \tag{3.3.1}$$

see [25, Section 4] for more details.

Lemma 3.3.1. *$I_{\lambda,a}$ satisfies the $(PS)_c$ condition for each $\lambda, a > 0$ and*

$$c < \frac{1}{N} S_K^{N/2} - c_K \lambda^{\frac{2N}{N+2}}, \tag{3.3.2}$$

where $c_K = c_K(N, \theta, \|h\|_{K,\theta})$ is a positive constant that will be fixed later.

Proof. Let $(u_n) \subset X$ be a $(PS)_c$ sequence for $I_{\lambda,a}$, that is,

$$I_{\lambda,a}(u_n) \to c \quad \text{and} \quad m_{I_{\lambda,a}}(u_n) \to 0.$$

From (3.2.1) and Lemma 3.2.4, there exists $(w_n) \subset \partial I_{\lambda,a}(u_n)$ such that

$$\|w_n\|_* = m_{I_{\lambda,a}}(u_n) = o_n(1) \quad \text{and} \quad w_n = \Psi'(u_n) - \rho_n,$$

where $\rho_n \in \partial\Phi_H(u_n)$. Then,

$$\begin{aligned} c + 1 + \|u_n\| &\geq I_{\lambda,a}(u_n) - \frac{1}{2^*}\langle w_n, u_n\rangle + o_n(1) \\ &= I_{\lambda,a}(u_n) - \frac{1}{2^*}\langle \Psi'(u_n) - \rho_n, u_n\rangle + o_n(1) \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|u_n\|^2 + \left(\frac{1}{2^*} - 1\right)\lambda\int_{\mathbb{R}^N} K(x)h(x)u_n dx \\ &+ \int_{\mathbb{R}^N} K(x)\left(\frac{1}{2^*}\rho_n u_n - F_{\mathbf{H}}(u_n)\right)dx + o_n(1). \end{aligned} \tag{3.3.3}$$

Note that by Lemma 3.2.4 and Remark 2,

$$\int_{\mathbb{R}^N} K(x)\left(\frac{1}{2^*}\rho_n u_n - F_H(u_n)\right)dx = \frac{a}{2^*}\int_{\{u_n=a\}} K(x)\rho_n dx \geq 0. \tag{3.3.4}$$

Moreover, by (3.1.5), Hölder inequality and the embedding properties of the space X, we have

$$\begin{aligned} \lambda\int_{\mathbb{R}^N} K(x)h(x)udx &\leq \lambda\left(\int_{\mathbb{R}^N} K(x)h(x)^\theta dx\right)^{\frac{1}{\theta}}\left(\int_{\mathbb{R}^N} K(x)|u|^{2^*}dx\right)^{\frac{1}{2^*}} \\ &\leq \lambda S_K^{-\frac{1}{2}}\|h\|_{K,\theta}\|u\|, \quad \text{for all } u \in X, \end{aligned} \tag{3.3.5}$$

where S_K is given in (3.3.1).

Hence, by (3.3.3), (3.3.4) and (3.3.5), we obtain a constant $C_1 > 0$ such that

$$\frac{\lambda C_1}{\theta}\|h\|_{K,\theta}\|u_n\| + c + 1 + \|u_n\| \geq \frac{1}{N}\|u_n\|^2 + o_n(1),$$

which implies that the sequence (u_n) is bounded in X. Using [45, Proposition 2.1], passing to a subsequence if necessary, we obtain

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \; u_n \to u \text{ in } L^s_K(\mathbb{R}^N) \\ K(x)u_n(x) \to K(x)u(x) \text{ a.e in } \mathbb{R}^N \\ |u_n(x)| \leq \varphi(x) \text{ for some } \varphi \in L^s_K(\mathbb{R}^N), s \in [2, 2^*). \end{cases} \tag{3.3.6}$$

Since $X \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow D^{1,2}(\mathbb{R}^N)$ we can argue along the same lines of the proof the classical concentration-compactness principle due to Lions [61, Lemma 1.1], to obtain J an at most countable index set, sequences $(\mu_j), (\nu_j) \subset [0, +\infty)$ and $(x_j) \in \mathbb{R}^N$ such that

$$\mu_n := K(x)|\nabla u_n|^2 \rightharpoonup \mu \quad \text{and} \quad \nu_n := K(x)|u_n|^{2^*} \rightharpoonup \nu \tag{3.3.7}$$

in weak*-sense of measure, with

$$\mu \geq K(x)|\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j}, \quad \nu = K(x)|u|^{2^*} + \sum_{j\in J}\nu_j\delta_{x_j}, \tag{3.3.8}$$

and $S_K \nu_j^{2/2^*} \leq \mu_j$ for all $j \in J$, where δ_{x_j} is the Dirac mass at x_j.

We claim that $J = \emptyset$. Arguing by contradiction that $J \neq \emptyset$, we fixe $i \in J$. Considering $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that

$$\phi(x) = \begin{cases} 1, & \text{if } \quad x \in B_1(0), \\ 0, & \text{if } \quad x \in \mathbb{R}^N \setminus B_2(0), \\ |\nabla \phi|_\infty \leq 2 \end{cases}$$

and we define $\phi_r(x) = \phi\left(\frac{x-x_i}{r}\right)$, where $r > 0$. Hence, $\nabla \phi_r(x) = \frac{1}{r}\nabla \phi\left(\frac{x-x_i}{r}\right)$, which implies that the sequence $(\phi_r u_n)$ is bounded in X, hence $o_n(1) = \langle w_n, \phi_r u_n \rangle$, that is,

$$\begin{aligned} o_n(1) &= \int_{\mathbb{R}^N} K(x)\nabla u_n \nabla(\phi_r u_n)dx - \lambda \int_{\mathbb{R}^N} K(x)h(x)\phi_r u_n dx - \int_{\mathbb{R}^N} K(x)\rho_n \phi_r u_n dx \\ &= \int_{\mathbb{R}^N} K(x)u_n \nabla u_n \nabla \phi_r dx + \int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \phi_r dx \\ &- \lambda \int_{\mathbb{R}^N} K(x)h(x)\phi_r u_n dx - \int_{\mathbb{R}^N} K(x)\rho_n \phi_r u_n dx. \end{aligned} \tag{3.3.9}$$

Since $\text{supp}(\phi_r)$ is contained in $B_{2r}(x_i)$, by Hölder's inequality and the boundedness of (u_n), we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} K(x)u_n \nabla u_n \nabla \phi_r \, dx \right| &\leq \left(\int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^N} K(x)|u_n|^2 |\nabla \phi_r|^2 \, dx \right)^{1/2} \\ &\leq M \left(\int_{\mathbb{R}^N} K(x)|u|^2 |\nabla \phi_r|^2 \, dx \right)^{1/2} + o_n(1), \end{aligned}$$

and taking the change variable $x = ry + x_i$, we obtain

$$\begin{aligned} M \left(\int_{\mathbb{R}^N} K(x)|u|^2 |\nabla \phi_r|^2 \, dx \right)^{1/2} &= \frac{M}{r} \left(\int_{r<|x-x_i|<2r} K(ry+x_i)u(ry+x_i)^2 |\nabla \phi(y)|^2 r^N \, dy \right)^{1/2} \\ &\leq M r^{\frac{N}{2}-1} |\nabla \phi|_\infty \left(\int_{r<|x-x_i|<2r} K(ry+x_i)u(ry+x_i)^2 \, dy \right)^{1/2} \\ &\leq 2M r^{\frac{N}{2}-1} \left(\int_{\mathbb{R}^N} K(x)u(x)^2 \, dx \right)^{1/2} = 2M r^{\frac{N}{2}-1} \|u\|_{K,2}. \end{aligned}$$

Thus,

$$\lim_{r\to 0} \left[\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)u_n \nabla u_n \nabla \phi_r \, dx \right] = 0. \tag{3.3.10}$$

Moreover, using the same argument we also obtain

$$\lim_{r\to 0} \left[\lim_{n\to\infty} \lambda \int_{\mathbb{R}^N} K(x)h(x)\phi_r u_n \, dx \right] = 0. \tag{3.3.11}$$

Since $0 \leq \rho_n \leq |u_n|^{2^*-1}$ a.e. in $\mathbb{R}^N$ from (3.3.9), (3.3.10) and (3.3.11), it follows that

$$o_n(1) \geq o_{n,r}(1) + \int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \phi_r dx - \int_{\mathbb{R}^N} K(x)|u_n|^{2^*} \phi_r dx, \tag{3.3.12}$$

where $o_{n,r}(1)$ is a quantity that satisfies $\lim_{r\to 0} \lim_{n\to\infty} o_{r,n} = 0$. From (3.3.7),

$$\int_{\mathbb{R}^N} K(x)|\nabla u_n|^2 \phi_r dx \geq \int_{\mathbb{R}^N} \phi_r d\mu \quad \text{and} \quad \int_{\mathbb{R}^N} K(x)|u_n|^{2*} \phi_r dx \to \int_{\mathbb{R}^N} \phi_r d\nu,$$

then, by (3.3.12) we obtain

$$\lim_{r\to 0}\int_{\mathbb{R}^N}\phi_r d\nu \geq \lim_{r\to 0}\int_{\mathbb{R}^N}\phi_r d\mu. \tag{3.3.13}$$

From Concentration-Compactness Principle, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\phi_r d\nu = \int_{\mathbb{R}^N}\phi_r \nu dx &= \int_{\mathbb{R}^N}\phi_r K(x)|u|^{2^*}dx + \sum_{j\in J}\nu_j\delta_{x_j}(\phi_r)\\ &= \int_{\mathbb{R}^N}\phi_r K(x)|u|^{2^*}dx + \sum_{j\in J}\nu_j\phi_r(x_j)\\ &= \int_{\mathbb{R}^N}\phi_r K(x)|u|^{2^*}dx + \sum_{j\in J}\nu_j\phi\left(\frac{x_j-x_i}{r}\right).\end{aligned}$$

which give us

$$\lim_{r\to 0}\int_{\mathbb{R}^N}\phi_r d\nu = \nu_i.$$

In the same way,

$$\int_{\mathbb{R}^N}\phi_r d\mu \geq \int_{\mathbb{R}^N}\phi_r K(x)|\nabla u|^2 dx + \sum_{j\in J}\mu_j\phi\left(\frac{x_j-x_i}{r}\right),$$

thus, we have

$$\lim_{r\to 0}\int_{\Omega}\phi_r d\mu \geq \mu_i.$$

Hence, from (3.3.13), we conclude that $\nu_i \geq \mu_i$, wich implies $S_K\nu_i^{2/2^*} \leq \nu_i$, so we obtain $\nu_i^{2/N} \geq S_K$.

Now, we shall prove that the above inequality cannot occur, and therefore the set J is empty. Indeed, arguing by contradiction, let us suppose that $\nu_i^{2/N} \geq S_K$ for some $i \in J$. Then,

$$\begin{aligned} c &= I_{\lambda,a}(u_n) - \frac{1}{2}\left\langle \Psi'(u_n) - \rho_n, u_n\right\rangle + o_n(1)\\ &= \int_{\mathbb{R}^N}K(x)\left(\frac{1}{2}\rho_n u_n - F_{\mathbf{H}}(u_n)\right)dx - \frac{\lambda}{2}\int_{\mathbb{R}^N}K(x)h(x)u_n dx + o_n(1)\\ &\geq \frac{1}{N}\int_{\{u_n>a\}}K(x)|u_n|^{2^*}\phi_r dx + \frac{1}{N}\int_{\{u_n=a\}}K(x)\rho_n\phi_r dx - \frac{\lambda}{2}\int_{\mathbb{R}^N}K(x)h(x)u_n dx + o_n(1)\\ &= \frac{1}{N}\int_{\mathbb{R}^N}K(x)|u_n|^{2^*}\phi_r dx - \frac{\lambda}{2}\int_{\mathbb{R}^N}K(x)h(x)u_n dx + o_{r,n} + o_n(1),\end{aligned}$$

in the last equality we use the fact that

$$\int_{\{u_n\leq a\}}K(x)|u_n|^{2^*}\phi_r dx = o_{r,n} \text{ and } \int_{\{u_n=a\}}K(x)\rho_n\phi_r dx = o_{r,n},$$

where $\lim_{r\to 0}\lim_{n\to\infty} o_{r,n} = 0$.

Since (u_n) is bounded in $L_K^{2^*}(\mathbb{R}^N)$, going if necessary to a subsequence, we may assume $u_n \rightharpoonup u$ in $L_K^{2^*}(\mathbb{R}^N)$, or equivalently (by Riesz representation theorem),

$$\int_{\mathbb{R}^N}K(x)u_n\varphi dx \to \int_{\mathbb{R}^N}K(x)u\varphi dx, \text{ for all } \varphi \in L_K^{\theta}(\mathbb{R}^N),$$

where $\theta = 2^*/(2^*-1)$. This fact and (3.1.5), imply that

$$c \geq \frac{1}{N}\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\phi_r dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} K(x)h(x)udx + o_{r,n} + o_n(1),$$

by Hölder inequality and by passing to the limit, we get

$$c \geq \frac{1}{N}\left(\int_{\mathbb{R}^N} K(x)|u|^{2^*}\phi_r dx + \sum_{j\in J}\nu_j\delta_{x_j}(\phi_r)\right) - \frac{\lambda}{2}\|h\|_{K,\theta}\|u\|_{K,2*}.$$

Hence, using $\nu_i \geq S_K^{N/2}$ we conclude that

$$\begin{aligned} c &\geq \frac{1}{N}S_K^{\frac{N}{2}} + \frac{1}{N}\|u\|_{K,2^*}^{2^*} - \frac{\lambda}{2}\|h\|_{K,\theta}\|u\|_{K,2^*} \\ &= \frac{1}{N}S_K^{\frac{N}{2}} + g(\|u\|_{K,2^*}^{2^*}), \end{aligned}$$

where g is the function given by $g(t) = \dfrac{1}{N}t - \dfrac{\lambda}{2}\|h\|_{K,\theta}t^{\frac{1}{2^*}}$. Hence, if we define the constant $c_K = c_K(N,\theta,\|h\|_{K,\theta}) > 0$ by $\min\limits_{t\in(0,\infty)} g(t) = -c_K\lambda^{\frac{2N}{N+2}}$, we have $c \geq \frac{1}{N}S_K^{\frac{N}{2}} - c_K\lambda^{\frac{2N}{N+2}}$, which contradicts (3.3.2).

□

The next lemma shows the functional $I_{\lambda,a}$ verifies the mountain pass geometry.

Lemma 3.3.2. *There is $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0)$ and $a > 0$ the functional $I_{\lambda,a}$ satisfies:*
(i) There exist $r,\alpha > 0$, which are independent on a, such that $I_{\lambda,a}(u) \geq \alpha$ for all $u \in X; \|u\| = r$.
(ii) There exists $e = e(a) \in C_0^\infty(\mathbb{R}^N)$ such that $I_{\lambda,a}(e) < 0$ and $\|e\| > r$.

Proof. Using $f_H(s) \leq |s|^{2^*}$ for all $s \in \mathbb{R}$ and (3.3.5), we have

$$\begin{aligned} I_{\lambda,a}(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\|u\|_{K,2^*}^{2^*} - \lambda\|h\|_{K,\theta}\|u\|_{K,2^*} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{S_K^{-2^*/2}}{2^*}\|u\|^{2^*} - \lambda S_K^{-1/2}\|h\|_{K,\theta}\|u\| \\ &= \|u\|^2\left(\frac{1}{2} - \frac{S_K^{-2^*/2}}{2^*}\|u\|^{2^*-2} - \lambda S_K^{-1/2}\|h\|_{K,\theta}\|u\|^{-1}\right), \end{aligned}$$

where S_K is given in (3.3.1). Making $P(t) = \frac{1}{2} - \frac{S_K^{-2^*/2}}{2^*}t^{2^*-2}$ we have

$$P(t) > \frac{1}{4} \text{ if } t \leq r = \left(\frac{2^*}{4S_K^{-2^*/2}}\right)^{\frac{1}{2^*-2}}.$$

Thus,

$$P(t) - \lambda\|h\|_{K,\theta}t^{-1} > \frac{1}{8} \text{ if } \lambda < \lambda_0 = \frac{r}{8\|h\|_{K,\theta}},$$

and so there exists $\alpha > 0$, independent on a, such that $I_{\lambda,a}(u) \geq \alpha$ whenever $\|u\| = r$, for all $\lambda \in (0,\lambda_0)$ and $a > 0$.

Taking $\varphi \in C_0^\infty(\mathbb{R}^N)$, such that $\text{meas}(\{\varphi > a\}) > 0$. We recall that $\text{meas}(A)$ is the Lebesgue measure of the measurable set $A \subset \mathbb{R}^N$ and $\{\varphi > a\} := \{x \in \mathbb{R}^N : \varphi(x) > a\}$. We get for each $t \geq 1$,

$$\begin{aligned} I_{\lambda,a}(t\varphi) &= \frac{t^2}{2}\|\varphi\|^2 - \int_{\mathbb{R}^N} K(x)F_H(t\varphi)dx - t\lambda \int_{\mathbb{R}^N} K(x)h(x)\varphi\, dx \\ &\leq \frac{t^2}{2}\|\varphi\|^2 - \frac{t^{2*}}{2^*}\int_{\{\varphi>a\}} K(x)\varphi^{2^*} dx + \frac{a^{2^*}}{2^*}\text{meas}(supp\varphi) \\ &- t\lambda \int_{\mathbb{R}^N} K(x)h(x)\varphi dx, \end{aligned}$$

which implies in the existence of e satisfying (ii). □

Lemma 3.3.3. *There exist $\lambda_*, a_* > 0$ and $\beta \in (1, 3/2)$, independent of a, such that for all $\lambda \in (0, \lambda_*)$ and $a \in (0, a_*)$, we have*

$$0 < \alpha \leq c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I_{\lambda,a}(\gamma(t)) < \frac{1}{N}S_K^{N/2} - \beta c_K \lambda^{\frac{2N}{N+2}}, \tag{3.3.14}$$

with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$, where α, e and c_K are as in Lemma 3.3.2 and Lemma 3.3.1, respectively.

Proof. Consider the family of functions $w_\epsilon : \mathbb{R}^N \to \mathbb{R}$ given by

$$w_\epsilon(x) = \frac{[N\epsilon(N-2)]^{\frac{N-2}{4}}}{(\epsilon + |x|^2)^{\frac{N-2}{2}}}; \quad \epsilon > 0.$$

We taking a smooth function $\varphi \in C_c^\infty(\mathbb{R}^N, [0,1])$ satisfying $\varphi \equiv 1$ in $B_1(0)$ and $\varphi \equiv 0$ outside $B_2(0)$. We consider the function

$$u_\epsilon(x) := K(x)^{-1/2}\varphi(x)w_\epsilon(x), x \in \mathbb{R}^N.$$

Now, let

$$v_\epsilon(x) := \frac{u_\epsilon(x)}{\|u_\epsilon\|_{2^*,K}},$$

Note that there exists $t_a = t(a) > 0$ such that $I_{\lambda,a}(t_a v_\epsilon) = \max_{t\geq 0} I_{\lambda,a}(tv_\epsilon)$. We claim that $(t_a)_{a\in(0,a_*)}$ is bounded in $\mathbb{R}$, for some $a_* > 0$ fixed. In fact, let $\Upsilon(t) = I_{\lambda,a}(tv_\epsilon)$, then,

$$\Upsilon(t) = \frac{t^2}{2}\|v_\epsilon\|^2 - \int_{\{tv_\epsilon>a\}} K(x)F_H(tv_\epsilon)dx - \lambda t \int_{\mathbb{R}^N} K(x)h(x)v_\epsilon dx.$$

Since $\Upsilon(t_a) > 0$ we have that $t_a \geq \underline{t} := \frac{2\lambda}{\|v_\epsilon\|^2}\int_{\mathbb{R}^N} K(x)h(x)v_\epsilon dx$. If $\text{meas}(\{t_a v_\epsilon > a\}) = 0$, from $\Upsilon'(t_a) = 0$ and $\lambda > 0$, we get $t_a = \frac{2\lambda}{\|v_\epsilon\|^2}\int_{\mathbb{R}^N} K(x)h(x)v_\epsilon dx$. If $\text{meas}(\{t_a v_\epsilon > a\}) > 0$, using once more that $\lambda > 0$ and $\Upsilon(t_a) > 0$, we have

$$\frac{t_a^2}{2}\|v_\epsilon\|^2 \geq \frac{t_a^{2^*}}{2^*}\int_{\{t_a v_\epsilon > a\}} K(x)v_\epsilon^{2^*} dx - \text{meas}(\{t_a v_\epsilon > a\})\frac{a^{2^*}}{2^*},$$

then, as $supp\ v_\epsilon \subset B_2(0), t_a \geq \underline{t}$ and $\{\underline{t}v_\epsilon > a^*\} \subset \{\underline{t}v_\epsilon > a\}$, it follows that

$$\text{meas}(B_2(0))\frac{a^{2^*}}{2^*} + \frac{t_a^2}{2}\|v_\epsilon\|^2 \geq \frac{t_a^{2^*}}{2^*}\int_{\{\underline{t}v_\epsilon > a_*\}} K(x)v_\epsilon^{2^*} dx,$$

hence, $(t_a)_{a\in(0,a_*)}$ is bounded in $\mathbb{R}$, for some $a_*>0$ fixed.

Now, observe that, $\Omega_a:=\left\{x\in\mathbb{R}^N : \underline{t}v_\epsilon(x)>a\right\}\subset\left\{x\in\mathbb{R}^N : tv_\epsilon(x)>a\right\}$, thus,

$$\begin{aligned}I_{\lambda,a}(tv_\epsilon) &\le \frac{t^2}{2}\|v_\epsilon\|^2-\int_{\Omega_a}K(x)F_H(tv_\epsilon)dx-\lambda t\int_{\mathbb{R}^N}K(x)h(x)v_\epsilon dx\\ &= \frac{t^2}{2}\|v_\epsilon\|^2-\lambda t\int_{\mathbb{R}^N}K(x)h(x)v_\epsilon dx-\frac{t^{2^*}}{2^*}\int_{\Omega_a}K(x)v_\epsilon^{2^*}dx+\frac{a^{2^*}}{2^*}\int_{\Omega_a}K(x)dx,\end{aligned}$$

using that $\|v_\epsilon\|_{2^*,K}=1$ and $K(x)\ge 1$ in $\mathbb{R}^N$, for all $t\ge 0$ we get

$$I_{\lambda,a}(tv_\epsilon)\le g(t)-\lambda t\int_{\mathbb{R}^N}K(x)h(x)v_\epsilon dx+\frac{t^{2^*}}{2^*}\int_{\Omega_a^c}K(x)v_\epsilon^{2^*}dx+\frac{a^{2^*}}{2^*}\int_{\Omega_a}K(x)dx,$$

where $g(t):=\frac{t^2}{2}\|v_\epsilon\|^2-\frac{t^{2^*}}{2^*};t\ge 0$ and $\Omega_a^c:=\{x\in\ \mathbb{R}^N : t_0u_\epsilon(x)\le a\}$. The function g has a maximum at $\bar{t}=\|v_\epsilon\|^{\frac{2}{(2^*-2)}}$ which satisfy

$$g(\bar{t})=\frac{1}{N}(\|v_\epsilon\|^2)^{\frac{N}{2}}=\frac{1}{N}(S_K+O(\epsilon))^{\frac{N}{2}},\quad \text{see [50, Proof of Proposition 3.2].}$$

Hence,

$$\begin{aligned}I_{\lambda,a}(t_av_\epsilon) &\le \frac{1}{N}\left(S_K+O(\epsilon)\right)^{N/2}-\lambda t_a\int_{\mathbb{R}^N}K(x)h(x)v_\epsilon dx\\ &+ \frac{t_a^{2^*}}{2^*}\int_{\Omega_a^c}K(x)v_\epsilon^{2^*}dx+\frac{a^{2^*}}{2^*}\int_{\Omega_a}K(x)dx.\end{aligned}$$

Moreover, note that $\Omega_a\subset B_2(0)$ because if $|x|\ge 2$ we have $v_\epsilon(x)=0<a$. So, $\int_{\Omega_a}K(x)\,dx\le\int_{B_2(0)}K(x)\,dx<\infty$, for all $a>0$. This and the fact that $(t_a)_{a\in(0,a_*)}$ is bounded imply that

$$\frac{a^{2^*}}{2^*}\int_{\Omega_a}K(x)\,dx\to 0\quad\text{and}\quad\frac{t_a^{2^*}}{2^*}\int_{\Omega_a^c}K(x)v_\epsilon^{2^*}dx\to 0\quad\text{as}\quad a\to 0. \tag{3.3.15}$$

Since $1<2N/(N+2)$ we can find $\lambda_1>0$ small enough, such that

$$t_0\lambda\int_{\mathbb{R}^N}K(x)h(x)v_\epsilon\,dx>\frac{3c_K}{2}\lambda^{\frac{2N}{N+2}},\quad\text{for all }\lambda\in(0,\lambda_1). \tag{3.3.16}$$

Hence, by (3.3.15) and (3.3.16), we choose $a_*=a(\lambda_1)$ satisfying

$$-\lambda t\int_{\mathbb{R}^N}K(x)h(x)v_\epsilon\,dx+\frac{a^{2^*}}{2^*}\int_{\Omega_a}K(x)\,dx+\frac{t_a^{2^*}}{2^*}\int_{\Omega_a^c}K(x)v_\epsilon^{2^*}dx<-\frac{3c_K}{2}\lambda^{\frac{2N}{N+2}},$$

for all $a\in(0,a_*)$ and $\lambda\in(0,\lambda_1)$.

Thus, it follows that for all $a\in(0,a_*)$ and $\lambda\in(0,\lambda_1)$,

$$\begin{aligned}I_{\lambda,a}(t_av_\epsilon) &\le \frac{1}{N}\left(S_K+O(\epsilon)\right)^{N/2}-\frac{3c_K}{2}\lambda^{\frac{2N}{N+2}}\\ &\le \frac{1}{N}S_K^{N/2}+O(\epsilon^{(N-2)/2})-\frac{3c_K}{2}\lambda^{\frac{2N}{N+2}}\\ &= \frac{1}{N}S_K^{N/2}+\epsilon^{(N-2)/2}\left(O(1)-\frac{1}{\epsilon^{(N-2)/2}}\frac{3c_K}{2}\lambda^{\frac{2N}{N+2}}\right).\end{aligned}$$

Since

$$\lim_{\epsilon\to 0^+}\frac{c_1}{\epsilon^{(N-2)/2}}=+\infty,$$

so, by choosing $\varphi = v_\epsilon$ in Lemma 3.3.2, we conclude that, for $\epsilon > 0$ small enough, such that $\text{meas}(\{w_\epsilon > a^*\}) > 0$, there exists $\beta \in (1, 3/2)$ independent of a, such that

$$c \leq \sup_{t \geq 0} I_{\lambda,a}(tv_\epsilon) < \frac{1}{N} S_K^{N/2} - c_K \beta \lambda^{\frac{2N}{N+2}}, \tag{3.3.17}$$

for all $a \in (0, a_*)$ and $\lambda \in (0, \lambda_*)$, where $\lambda_* = \min\{\lambda_0, \lambda_1\}$ with λ_0 and λ_1 give in Lemma 3.3.2 and (3.3.16), respectively.

□

Lemma 3.3.4. *Let $\overline{B}_r$ be the ball of radius r and a_*, λ_* given in Lemma 3.3.2 and 3.3.3, respectively. There exist a sequence $(u_n) \subset \overline{B}_r$ and a constant $\overline{M} < 0$, independent on a, such that for all $a \in (0, a_*)$ and $\lambda \in (0, \lambda_*)$, we have*

$$I_{\lambda,a}(u_n) \to \tilde{c} \quad \text{and} \quad m_{I_{\lambda,a}}(u_n) \to 0, \tag{3.3.18}$$

where

$$\tilde{c} := \inf_{\overline{B}_r} I_{\lambda,a} \leq \overline{M} < 0. \tag{3.3.19}$$

Proof. Fixed a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, we have

$$I_{\lambda,a}(t\varphi) \leq M(t) := \frac{t^2}{2}\|\varphi\|^2 - t\lambda \int_{\mathbb{R}^N} K(x)h(x)\varphi \, dx.$$

We can choose $\overline{M} := M(\bar{t})$ with $\bar{t} > 0$ small enough, independent on a, such that for all $a \in (0, a_*)$ and $\lambda \in (0, \lambda_*)$, we have

$$\tilde{c} := \inf_{\overline{B}_r} I_{\lambda,a} \leq \overline{M} < 0. \tag{3.3.20}$$

Now, considering $I_{\lambda,a}$ restricted to $\overline{B}_r$, we can apply the Ekeland variational principle, see [35], to obtain $u_\epsilon \in \overline{B}_r$ such that

$$I_{\lambda,a}(u_\epsilon) < \inf_{\overline{B}_r} I_{\lambda,a} + \epsilon \quad \text{and} \quad I_{\lambda,a}(u_\epsilon) < I_{\lambda,a}(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon. \tag{3.3.21}$$

Since

$$\inf_{\overline{B}_r} I_{\lambda,a} < 0 < \alpha \leq \inf_{\partial B_r} I_{\lambda,a},$$

we can consider $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{\partial B_r} I_{\lambda,a} - \inf_{\overline{B}_r} I_{\lambda,a}.$$

For this choice of ϵ, one has

$$I_{\lambda,a}(u_\epsilon) \leq \inf_{\overline{B}_r} I_{\lambda,a} + \epsilon < \inf_{\partial B_r} I_{\lambda,a},$$

which implies that $u_\epsilon \in B_r$. Let $v \in X$ and take $\delta > 0$ small enough such that $u_\delta = u_\epsilon + \delta v \in B_r$. From (3.3.21) we get

$$I_{\lambda,a}(u_\epsilon + \delta v) - I_{\lambda,a}(u_\epsilon) + \delta\|v\| \geq 0.$$

Thus we have

$$-\epsilon\|v\| \leq \limsup_{\delta \downarrow 0} \frac{I_{\lambda,a}(u_\epsilon + \delta v) - I_{\lambda,a}(u_\epsilon)}{\delta} \leq I^0_{\lambda,a}(u_\epsilon; v).$$

From

$$I^0_{\lambda,a}(u;v) = \max_{\mu\in\partial I_{\lambda,a}(u)} \langle \mu, v\rangle\,, \quad u,v\in X,$$

we get

$$-\epsilon\|v\| \le I^0_{\lambda,a}(u_\epsilon;v) = \max_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle\,, \quad \text{for all } v\in X$$

and interchanging v with $-v$, we obtain

$$-\epsilon\|v\| \le \max_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, -v\rangle = -\min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle\,, \quad \text{for all } v\in X.$$

Hence,

$$\min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle \le \epsilon\|v\|, \quad v\in X,$$

which gives

$$\sup_{\|v\|=1} \min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle \le \epsilon.$$

Finally, by Ky Fan's Min-max theorem [23, Proposition 1.8] , we get

$$\min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \sup_{\|v\|=1} \langle \omega, v\rangle \le \epsilon.$$

This together with (3.3.21) gives that there exists $(u_n)\subset B_r$ such that

$$I_{\lambda,a}(u_n)\to \tilde{c} \;\text{ and } m_{I_{\lambda,a}}(u_n) := \min_{\omega\in\partial I_{\lambda,a}(u_n)} \|\omega\|_{X^*}\to 0 \quad \text{as } n\to\infty.$$

□

3.4 Proof of Theorem 3.1.1

In this section we will use the previous results to prove Theorem 3.1.1.

Proof of Theorem 3.1.1: First solution (Mountain Pass):

Let λ_*, a_* and $c = c(a)$ be as in Lemma 3.3.3. For each $a\in(0,a_*)$ and $\lambda\in(0,\lambda_*)$, combining Lemma 3.3.1 and Lemma 3.3.3 with the Mountain pass theorem, see Theorem 3.2.2, we obtain $u_1 = u_1(a)\in X$ with $I_{\lambda,a}(u_1) = c > 0$ and $0\in\partial I_{\lambda,a}(u_1)$. Hence, there exists $\rho_1\in L^{2^*}_K(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} K(x)\nabla u_1\nabla\varphi dx = \lambda\int_{\mathbb{R}^N} K(x)h(x)\varphi dx + \int_{\mathbb{R}^N} K(x)\rho_1\varphi dx, \text{ for all } \varphi\in X, \qquad (3.4.1)$$

where $\rho_1\in \widehat{f}_H(u_1)$ with $\widehat{f}_H(s)$ given in (3.2.6). Therefore, by Lemma 3.2.5 we conclude that u_1 is a solution of (3.1.2) in the sense of (3.1.4). Note that the proof of i) of Theorem 3.1.1 was given (3.2.7). Now we will show ii) and iii).

Assume by contradiction that meas$(\{u_1 = a\}) > 0$. By using the Morrey-Stampacchia's Theorem, we have that $-\text{div}(K(x)\nabla u_1(x)) = 0$ a.e. in $\{u_1 = a\}$. Then, by (3.4.1) and Remark 2,

$$-\lambda K(x)h(x)\in K(x)\widehat{f}(a) = [0, K(x)a^{2^*-1}] \text{ a.e in } \{u_1 = a\},$$

which is a contradiction. Therefore, meas$(\{u_1 = a\}) = 0$, which proves ii).

Now we shall prove that meas$(\{u_1 > a\}) > 0$ for all $a\in(0,a_*)$. Suppose, by contradiction, that $u_1(x)\le a$ a.e in $\mathbb{R}^N$. Then, using u_1 as a test function in (3.4.1), we obtain

$$\|u_1\|^2 = \int_{\{u_1=a\}} K(x)\rho_1 u_1\; dx + \lambda\int_{\mathbb{R}^N} K(x)h(x)u_1\; dx,$$

and as a consequence, from ii) and $c = I_{\lambda,a}(u_1)$ we have

$$0 < c = \frac{1}{2}\|u_1\|^2 - \lambda \int_{\mathbb{R}^N} K(x)h(x)u_1\,dx = -\frac{\lambda}{2}\int_{\mathbb{R}^N} K(x)h(x)u_1 dx < 0,$$

which is a contradiction.

Second solution(Local Minimization):

To prove the existence of the second solution, we observe that by Lemma 3.3.1 and Lemma 3.3.4, we can conclude that there is a function $u_2 = u_2(a) \in \overline{B}_r$ such that, for all $a \in (0, a_*)$ and $\lambda \in (0, \lambda_*)$,

$$I_{\lambda,a}(u_2) = \tilde{c} \leq \overline{M} < 0 \ \text{ and } 0 \in \partial I_{\lambda,a}(u_2). \tag{3.4.2}$$

Thus, u_2 is nontrivial critical point de $I_{\lambda,a}$. To verify that u_2 also satisfies $i), ii)$ and $u_2 \geq 0$ a.e in $\mathbb{R}^N$ we use the same arguments used for u_1. Moreover, we claim that, reducing a_* if necessary, we have meas$(\{u_2 > a\}) > 0$, for all $a \in (0, a_*)$. In fact, otherwise for each $a_* = 1/n$ there exists $u_2(a_n) =: u_n \in X$ with $0 \leq u_n(x) \leq 1/n$, for all $n \in \mathbb{N}$. This and ii) imply that u_n satisfies

$$-\operatorname{div}(K(x)\nabla u_n) = \lambda K(x)h(x) \ \text{ in } \ \mathbb{R}^N. \tag{3.4.3}$$

Hence,

$$\int_{\mathbb{R}^N} K(x)\nabla u_{n+1}\nabla\varphi = \int_{\mathbb{R}^N} K(x)\nabla u_n \nabla\varphi, \ \text{ for all } \varphi \in X.$$

Setting $\varphi = (u_{n+1} - u_n)$ we obtain

$$\|(u_{n+1} - u_n)\|^2 = \int_{\mathbb{R}^N} K(x)|\nabla(u_{n+1} - u_n)|^2 = 0,$$

then $u_{n+1} = u_n$ for all $n \in \mathbb{N}$. Moreover, by (3.4.3), Hölder inequality and the embedding properties of X, we have

$$\|u_n\|^2 \leq \lambda\|h\|_{K,\theta}\|u_n\|_{K,2^*} \leq \lambda C\|h\|_{K,\theta}\|u_n\|,$$

hence, $\|u_n\| \leq \lambda C\|h\|_{K,\theta}$ for all $n \in \mathbb{N}$. In particular, by this and (3.4.3) we get

$$\sup_{n\in\mathbb{N}} \lambda \int_{\mathbb{R}^N} h(x)K(x)u_n \leq (\lambda C\|h\|_{K,\theta})^2.$$

Since $u_n(x) \to 0$ a.e in $\mathbb{R}^N$, then the Monotone convergence theorem and once more (3.4.3), imply that

$$\|u_n\|^2 = \lambda \int_{\mathbb{R}^N} K(x)h(x)u_n \to 0.$$

Thus, we have $\widetilde{c}_n = I_{\lambda,a}(u_n) \to 0$ as $n \to \infty$, which is impossible by (3.4.2). Hence, the claim is proved and $iii)$ holds for u_2.

Finally, we will show the last part of the theorem. Let $u_n^i := u_i(a_n), i = 1, 2$, be the solutions of (3.1.2), where $= u_1(a_n)$ and $u_2(a_n)$ was obtained by Mountain pass theorem and Ekeland variational principle, respectively. Since $\mathbf{H}(s - a^*)s^{2^*-1} \leq \mathbf{H}(s - a)s^{2^*-1} \leq (s_+)^{2^*-1}$, for all $s \in \mathbb{R}$, we get

$$I_{\lambda,0}(u) \leq I_{\lambda,a}(u) \leq I_{\lambda,a_*}(u) \ \text{ for all } u \in X, a \in [0, a^*), \tag{3.4.4}$$

so, we conclude that $c(0) \leq c(a_n) \leq c(a_*)$, where $c(a)$ is the minimax level associated with $I_{\lambda,a}$.

Arguing as in the proof of Lemma 3.3.1, we obtain a constant $C_1 > 0$ such that

$$\frac{\lambda C_1}{\theta}\|h\|_{K,\theta}\|u_n^1\| + c(a_n) + 1 + \|u_n^1\| \geq \frac{1}{N}\|u_n^1\|^2 + o_n(1),$$

then, (u_n^1) is bounded in X. Hence, passing to a subsequence if necessary, we obtain

$$\begin{cases} u_n^1 \rightharpoonup v_1 \ \text{ in } X,\ u_n^1 \to v_1 \ \text{ in } \ L_K^s(\mathbb{R}^N) \\ u_n^1(x) \to v_1(x) \ \text{ a.e in } \mathbb{R}^N \\ |u_n^1(x)| \leq \varphi_s(x) \ \text{ for some } \ \varphi_s \in L_K^s(\mathbb{R}^N), s \in [2, 2^*). \end{cases} \tag{3.4.5}$$

Passing to the limit as $n \to \infty$ in (3.4.1), by using ii), (3.4.5), $v_1(x) \geq 0$ and $H(u_n^1(x) - a_n)(u_n^1(x))^{2^*-1} \to (v_1(x))_+^{2^*-1}$ a.e. in $\mathbb{R}^N$, we conclude that v_1 satisfies

$$\int_{\mathbb{R}^N} K(x)\nabla v\nabla\varphi dx = \lambda\int_{\mathbb{R}^N} K(x)h(x)\varphi dx + \int_{\mathbb{R}^N} K(x)v^{2^*-1}\varphi dx, \ \text{ for all } \ \varphi \in X. \tag{3.4.6}$$

We claim that $v_1 \neq 0$. In fact, note that as $u_n^1 \geq 0$ we have

$$\begin{aligned} 0 \leq I_{\lambda,a_n}(u_n^1) - I_{\lambda,0}(u_n^1) &= \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)(u_n^1)^{2^*}dx - \int_{\mathbb{R}^N} K(x)F_H(u_n^1)dx \\ &= \int_{\mathbb{R}^N} K(x)\int_0^{u_n^1}\left[(s_+)^{2^*-1} - H(s-a_n)s^{2^*-1}\right]dsdx \\ &= \frac{1}{2^*}\int_{\{u_n^1 \leq a_n\}} K(x)(u_n^1)^{2^*}dx \\ &\leq \frac{a_*}{2^*}\int_{\mathbb{R}^N} K(x)(u_n^1)^{2^*-1}\chi_{\{u_n^1 \leq a_n\}}dx \to 0 \ \text{ as } n \to \infty, \end{aligned}$$

where $\chi_{\{u_n^1 \leq a_n\}}$ is the characteristic function of set $\{u_n^1 \leq a_n\}$.

Hence, $I_{\lambda,a_n}(u_n^1) = I_{\lambda,0}(u_n^1) + o_n(1)$. Similarly, $I_{\lambda,a_n}(u) = I_{\lambda,0}(u) + o_n(1)$ for all $u \in X$. Then, it follows that

$$c(a_n) = c(0) + o_n(1) \ \text{ and } \ I_{\lambda,0}(u_n^1) = c(0) + o_n(1). \tag{3.4.7}$$

Now, note that by (3.3.17) and (3.4.7), for all $\lambda \in (0, \lambda_*)$, we have

$$c(0) \leq \frac{1}{N}S_K^{N/2} - c_K\beta\lambda^{\frac{2N}{N+2}}, \tag{3.4.8}$$

where $\beta \in (1, 3/2)$. So, using the fact that u_n^1 is the solution of (3.1.2), (3.4.8) and same argument of the proof of Lemma 3.3.1, we conclude that $u_n^1 \to v_1$ in X. Since $I_{\lambda,a}(u_n^1) = c(a_n) \geq \alpha$ we have $2\alpha \leq \|u_n^1\|^2$ and so $2\alpha \leq \|v_1\|^2$ because α is independent of a, see i) of Lemma 3.3.2. Hence, the claim is proved.

Now, since $(u_n^2) \subset \overline{B_r}$ and r is is independent of a, we have, up to a subsequence, $u_n^2 \rightharpoonup v_2$ in X. It is clear that v_2 is a solution of (3.4.6).

To see that $v_2 \neq 0$ it is sufficient to use a similar argument as above combined with the fact that by (3.3.20) and (3.4.4),

$$\tilde{c}(0) \leq \tilde{c}(a_n) \leq \overline{M} < 0, \ \text{ for all } n \in \mathbb{N}, \lambda \in (0, \lambda_*),$$

where $\tilde{c}(a) := \inf\limits_{\overline{B_r}} I_{\lambda,a}$ with $a \in [0, a^*)$. This conclude the proof of Theorem 3.1.1.

Appendix A: Genus Theory

This section is dedicated to recalling some basic facts on Krasnoselskii genus theory as well as its demonstrations, which we use in the proof of Theorem 1.3.1. More informations on this subject may be found in [57].

Let E be a real Banach space. Let us denote by $\mathfrak{A}$ the class off all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $f \in C^1(\overline{\Omega}, \mathbb{R}^N)$. Recall that $f'(x) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ and hence $f'(x)$ can be represented by an $N \times N$ matrix. Let $\mathcal{S}$ be the set of critical points of f. In order to make this section clearer, we recall the definition of topological degree.

Definition 3.4.1. *Let $f : \Omega \to \mathbb{R}^N$ be a function in $C^1(\overline{\Omega}, \mathbb{R}^N)$ and $b \notin f(\mathcal{S}) \cup f(\partial\Omega)$. Then we define the degree of f in Ω with respect to b as*

$$\deg(f, \Omega, b) = \begin{cases} 0, \text{ if } f^{-1}(b) = \emptyset, \\ \sum_{x \in f^{-1}(b)} \operatorname{sgn}(\det f'(x)), \text{ otherwise} \end{cases}. \tag{3.4.9}$$

The function sgn denotes the sing, i.e., $+1$ if positive and -1 if negative. In [56] the reader can find many properties about deg as well as their demonstrations.

Definition 3.4.2. *Let $A \in \mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. When such number does not exist we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

Proposition 3.4.1. *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ such that $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Proof. Trivially, $\gamma(\partial\Omega) \leq N$. (Choose $h = id$.) Let $\gamma(\partial\Omega) = k$ and let $h \in C^0\left(\mathbb{R}^N; \mathbb{R}^k\right)$ be an odd map such that $h(\partial\Omega) \not\supset 0$. We may consider $\mathbb{R}^k \subset \mathbb{R}^N$. But then the topological degree of $h : \mathbb{R}^N \to \mathbb{R}^k \subset \mathbb{R}^N$ on Ω with respect to 0 is well-defined (see [55, Definition 1.2.3]). In fact, since h is odd, by the Borsuk-Ulam theorem (see [55, Theorem 1.4.1]) we have

$$\deg(h, \Omega, 0) = 1$$

Hence by continuity of the degree also

$$\deg(h, \Omega, y) = 1 \neq 0$$

for $y \in \mathbb{R}^N$ close to 0 and thus, by the solution property of the degree, h covers a neighborhood of the origin in $\mathbb{R}^N$; see Deimling [1; Theorem 1.3.1]. But then $k = N$, as claimed.

□

Corollary 3.4.2. $\gamma(S^{N-1}) = N$.

The genus has the following properties:

Proposition 3.4.3. *Let $A, A_1, A_2 \in \mathfrak{A}, h \in C^0(E; E)$ an odd map. Then the following hold:*

(i) $\gamma(A) \geq 0, \quad \gamma(A) = 0 \quad \Leftrightarrow \quad A = \emptyset$

(ii) $A_1 \subset A_2 \Rightarrow \gamma(A_1) \leq \gamma(A_2)$.

(iii) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$

(iv) $\gamma(A) \leq \gamma(\overline{h(A)})$.

(v) If $A \in \mathcal{A}$ is compact and $0 \notin A$, then $\gamma(A) < \infty$ and there is a neighborhood V of A in E such that $\bar{V} \in \mathfrak{A}$ and $\gamma(A) = \gamma(\bar{V})$.

That is, γ is a definite, monotone, sub-additive, supervariant and "continuous" map $\gamma : \mathfrak{A} \to \mathbb{N}_0 \cup \{\infty\}$

Remark 3. *It is easy to see that if A is a finite collection of antipodal pairs $u_i, -u_i$ $(u_i \neq 0)$, then $\gamma(A) = 1$*

Let I be a functional of class C^1 on a closed symmetric $C^{1,1}$-submanifold M of a Banach space E and satisfies the $(P.S)$ condition. Moreover, suppose that I is even, that is, $I(u) = I(-u)$ for all u. Also let $\mathfrak{A}$ be as above. Then for any $k \leq \gamma(M) \leq \infty$ by (iv) in the previous proposition, the family

$$\mathcal{F}_k = \{A \in \mathfrak{A}; A \subset M, \gamma(A) \geq k\}$$

is invariant under any odd and continuous map and non-empty. Hence, for any $k \leq \gamma(M)$, if

$$\beta_k = \inf_{A \in \mathcal{F}_k} \sup_{u \in A} I(u)$$

is finite, then β_k is a critical value of I.

Proposition 3.4.4. *Suppose for some k, l there holds*

$$-\infty < \beta_k = \beta_{k+1} = \ldots = \beta_{k+l-1} = \beta < \infty$$

Then $\gamma(K_\beta) \geq l$. By observation 3, in particular, if $l > 1, K_\beta$ is infinite.

In consequence, we have

Proposition 3.4.5. *If $B \in \mathfrak{A}$, $0 \notin B$ and $\gamma(B) \geq 2$, then B has infinitely many points.*

The reader interested in the demonstrations of these propositions can consult, for example, [56, 69]

Appendix B: Some Classical Results

This section is devoted to recall some classical results that were used throughout this work. As this section is just for viewing the results, we will not give any demonstrations.

Theorem 3.4.6. *(Dominated convergence theorem, Lebesgue). Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) $f_n(x) \to f(x)$ a.e on Ω,

(b) there is a function $g \in L^1$ such that for all $n, |f_n(x)| \leq g(x)$ a.e on Ω.

Then $f \in L^1$ and $\|f_n - f\| \to 0$.

Proof. See [41, p.54].

Theorem 3.4.7. *(Fatou's lemma). Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) for all $n, f_n \geq 0$ a.e,

(b) $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq +\infty$. Then $f \in L^1$ and

$$\int f dx \leq \liminf_{n\to\infty} \int f_n dx.$$

Proof. See [41, p. 52].

Theorem 3.4.8. *(Hölder's inequality). Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int |fg| dx \leq \left(\int |f|^p dx\right)^{1/p} \left(\int |g|^{p'} dx\right)^{1/p'}.$$

Proof. See [41, p. 182].

Theorem 3.4.9. *Let (f_n) be a sequence in L^p and let $f \in L^p$ be such that $\|f_n - f\|_p \to 0$. Then, there exist a subsequence (f_{n_k}) and a function $h \in L^p$ such that*

(a) $f_{n_k}(x) \to f(x)$ a.e on Ω,

(b) $|f_{n_k}(x)| \leq h(x)\ \forall\ k \in \mathbb{N}$, a.e on Ω.

Proof. See [41].

The following theorem gives a useful embedding result for L^p spaces over domains with finite measure. This result was used when we applied Moser's iteration.

Theorem 3.4.10. *Suppose* $\operatorname{vol}\Omega = \int_\Omega 1 \; dx < \infty$ *and* $1 \leq p \leq q \leq \infty$. *If* $u \in L^q(\Omega)$, *then* $u \in L^p(\Omega)$ *and*

$$\|u\|_p \leq (\operatorname{vol}\Omega)^{(1/p-1/q)} \, \|u\|_q.$$

Hence $L^q(\Omega) \hookrightarrow L^p(\Omega)$. *If* $u \in L^\infty(\Omega)$, *then*

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty.$$

Finally, if $u \in L^p(\Omega)$ *for* $1 \leq p < \infty$ *and if there is a constant* C *such that for all* p

$$\|u\|_p \leq C,$$

then

$$u \in L^\infty(\Omega) \text{ and } \|u\| \leq C.$$

Proof. See [1, Theorem 2.8].

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