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MULTIDIMENSIONAL ROBUST CONTROL, UNCERTAINTY AND FINANCE

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Multidimensional Robust Control, Uncertainty and Finance

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1. Aversão ao Risco (Risk Aversion)
2. Incerteza (Uncertainty)
3. Erros de Especificações de Modelos Econômicos (Model Misspecifications)
4. Controle Robusto (Robust Control)

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RESUMO

A teoria do controle ótimo tem sido uma fonte bastante útil de ferramentas para o estudo de problemas econômicos. Um campo mais recente do controle ótimo, denominado controle robusto, tem sido adotado por alguns economista de renome internacional no estudo de problemas econômicos onde há uma preocupação com erros de especificação dos modelos utilizados. Neste caso é possível construir modelagens onde é feita a separação entre elementos de aversão ao risco e aversão à incerteza (Knightiana) e, a partir deste arcabouço teórico, conseguir resolver alguns dos importantes enigmas empíricos de economia e finanças.

A maioria dos modelos atuais que levam em consideração esta análise de erros de especificação nos modelos considera uma representação unidimensional para o conceito de incerteza. Esta tese objetiva em construir uma modelagem para o apreçamento de ativos (e para outros problemas econômicos) que seja bidimensional no sentido de permitir a existência de dois parâmetros relacionados a intenção do modelador em ter seus resultados como sendo robustos a erros de especificação. Cada um dos parâmetros é relacionado a dois conceitos econômicos importantes (taxa de desconto e elasticidade de substituição intertemporal) não necessariamente em um correspondência biunívoca entre eles. Esta abordagem permite que sejam explicados dois enigmas: excesso de retorno para ativos arriscados (*equiti premium puzzle*) e enigma da alta taxa livre de risco (*risk-free rate puzzle*).

O tratamento bidimensional é um passo na tentativa de mostrar a necessidade de uma multidimensionalidade na representação da incerteza econômica. Como um produto adicional a tese define o conceito de Preço Multifatorial da Incerteza Knightiana (*Multifactor Price of Knightian Uncertainty* –MFPU) que estende um clássico conceito de Preço de Risco de Mercado (Market Price of Risk - MPR).

Estes resultados da tese mostram que um modelo com multidimensionalidade para modelos robustos a erros de especificação é um ferramental útil na explicação de anomalias no apreçamento de ativos e, em alguns casos, os resultados são melhores do que os obtidos com modelos clássicos de finanças.

Esta tese contribui com a enorme agenda de pesquisa relacionada a intenção de obter modelagem robustas a erros de especificações em problemas aplicados à economia e finanças.

ABSTRACT

Optimal control theory has being a source of useful tools (including Euler equations, z-transforms, lag operators, Bellman equations, Kalman filtering) to study dynamic economics problems. A more recent field of optimal control, namely robust control, has recently been adopted by some leading economists (Thomas Sargent, Lars Hansen and coauthors) to study important problems in

economics where there is a concern about model misspecification. With such new feature one can disentangle the concepts of risk aversion and (Knightian) uncertainty aversion and get some hope to explain famous empirical puzzles.

The majority of current models that take into consideration the fear of model misspecifications work with a single representation for uncertainty. This dissertation aims at building a bi—dimensional robust pricing model by allowing for two free parameters related to the fear of model misspecification. Both parameters may be linked to the discount rate and to the elasticity of intertemporal substitution not necessarily with a bijective mapping. This approach allows for a full explanation of the equity premium puzzle and the risk-free rate puzzle.

This bi-dimensional treatment is a step in trying to show the necessary multidimensionality of the representation of economic uncertainty. As a by-product we define a new concept, the Multifactor Price of Knightian Uncertainty (MFPU), that extends the classical Market Price of Risk (MPR) and Hansen and Sargent’s Market Price of Uncertainty (MPU).

The main results of the dissertation show that a model with multidimensional representation of concern for model misspecification is a valuable tool for explaining asset pricing anomalies and, in some cases, it outperforms standard neoclassical financial models.

This work is just part of the prolegomena of the research agenda of robustness concerns in economics and finance and shows some of its potential and weakness.

Contents of the Thesis

1 Introduction

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The majority of current models that take into consideration the fear of model misspecifications work with a single representation for uncertainty. This dissertation aims at building a bi–dimensional robust pricing model by allowing for two free parameters related to the fear of model misspecification. Both parameters may be linked to the discount rate and to the elasticity of intertemporal substitution not necessarily with a bijective mapping. This approach allows for a full explanation of the equity premium puzzle and the risk-free rate puzzle. This bi-dimensional treatment is a step in trying to show the necessary multidimensionality of the representation of economic uncertainty. As a by-product we define a new concept, the Multifactor Price of Knightian Uncertainty (MFPU), that extends the classical Market Price of Risk (MPR) and Hansen and Sargent's Market Price of Uncertainty (MPU).

The main contributions of this dissertation is twofold: (i) introduce a second parameter to model concern about model uncertainty and analyze what are the main implications in terms of power to explain puzzles like the risk-free rate puzzle in addition to the explanation of the equity premium puzzle; (ii) provide an explicit role for a second economic concept (the elasticity of intertemporal substitution, EIS) in the framework of uncertainty modelling with a quantification of its impact in the value of the fundamentals of asset pricing (mainly the first two moments of the stochastic discount factor).

One feature of the modelling treated here is that it considered uncertainty in a bi-dimensional fashion by looking at the very basic primitives of the economy. This is in contrast to the recent attempt by Hansen and Sargent (2006) which work with two parameters that model uncertainty but this is done in a two step procedure: the agent does not know his actual model (first parameter of uncertainty needed to treat this) and he does not observe part of the state vector which force him to filter. This filtering problem adds a second parameter related to uncertainty since they argue that estimation of the unobservable part is subject to misspecification too. We treat the bi-dimensionality of uncertainty up front: the malevolent nature acts in diverse ways and this affect the decision rule of the maximizing controller.[1]

As a side product the process of including a second parameter forced us to scrutinize the one-dimensional model and see possible way of improvement and also some problems with the modelling setup. It is worth stressing that both

[1] See section 4 on the boost of empirical power for a description for the failure of HSW model.

the multi prior approach and the robust control approach are very intellectual demanding areas of research and any eventual contribution of this dissertation is likely to be only incremental to the rich paths already set by their intellectual mentors.

1.1 Motivation for Modelling Uncertainty

Robust control seems to be a workable formalization of Knightian Uncertainty and this is a long searched issue by economists.

Before describing the aspects of robust control we briefly mention the three phases of control theory.

3 Generations of Control Theory

i) Classical Control: Euler equations, z-transforms, lag operators

ii) Modern Control: Bellman equations, Kalman filtering (Dual Problem)

iii) Robust Control: feedback control, Riccati Equations.

The phases (i) and (ii) were key for development of dynamic theory in economics (e.g. for studying rational expectations model). Phase (iii) has promising avenues in some economic areas.

1.1.1 Frank Knight, X-29, and NYSE: What Do They Have in Common?

Frank Knight leaded the idea of uncertainty in economic problems (in the University of Chicago and in the 1920's). He made a distinction of risk and uncertainty. The first is related to variable that are possible to attach a probability distribution.

By its turn the X-29 is a fine piece of engineering system control that was made possible by the use of the tools of robust control (it is one of the most successful aircraft design).

The New York security exchange (NYSE) is the most well known place in the world for trading stocks. It can be argued that price of asset trades in an stock exchange bears more than a compensation for risk since it has some components of uncertainty aversion. The stock pricing is a famous application of finance theory.

All those three issues are related to the idea of uncertainty: a very complex and fascinating topic in both engineering and social sciences.

Remark 1 *Frank Knight's (1921) distinction can be stated more formally: Risk refers to situations where an investor is able to calculate probabilities on the basis of an objective classification of instances. Uncertainty refers to cases where no objective classification is possible. This distinction is useless if one work with subjective probabilities in the sense of Savage (1954), i.e., probability is simply the degree of belief. Hence investors will be always in Knight's world of uncertainty. See Hirshleifer and Riley (1997) pages 9-11 and Mas-Collel et AL (1996), chapter 6, section 6.F for the definition of subjective probability and comments about the Ellsberg's paradox. See also Gilboa and Schmeidler (1989)'s atemporal Axiomatization approach.*

1.1.2 Representing Uncertainty in Engineering and Other Technical Fields

The modelling of uncertainty using robust control has been done for more than two decades in engineering and other technical fields. Much of the research in this area start by establishing the differences between the various types of uncertainty. One classification ranks uncertainties as unstructured, and structured (which includes the parametric uncertainties).

Definition 2 *Unstructured uncertainties refers to unmodelled and or nonlinear dynamics not captured by the model.*

Uncertainty as treated in the feedback control theory has two sources:
(i) discrepancy between the physical plant and the mathematical model used for controller design and
(ii) unmeasured noises and disturbances that impact the physical plant. Feedback is used to **desensitize** the control system from the effect of both types of uncertainty.

Searching ways of coping with uncertainty have been a key motivation for use of feedback rules in those technical fields. This was also a motivation for Hansen, Sargent and coauthors. Likewise this dissertatiton will focus on the robust control related with feedback rules (sometimes the literature interchange the names robust control and feedback control).

Robust control was not the only branch of optimal control that tries to model uncertainty. The adaptive control literature is another approach. The main difference between the two can be stated informally as: robust control models uncertainty without reducing it, just live with it while adaptive control explicitly tries to reduce it. That is why some models of uncertainty with learning (agents using the data to get better future decision rules) is likely to use adaptive control tools.

Robust control do not rely on a single parameter to represent uncertainty even though for many multivariable systems the literature works with what is called a singular value (or SSV which stands for single structured value). Many references explicitly deals with multidimensional versions of uncertainty. See for example Malakorn (2003) and Ball (2003).

The Idea of feedback (of the distortion to the relevant state variable) in modelling uncertainty is attractive because allow a comparison of the actual result with the desired result and allow control systems to take actions based on the difference of the two. The use of state feedbacking (feedback rules that force uncertainty to impact the state variable) allows new controller structures and efficient design algorithms

Since it is very difficult to capture uncertainty modelers used setup that permit calculations. One of them is the Linear Quadratic Gaussian framework (LQG for short). One drawback os this approach is that sometimes the state space approach delivers a nice design methods but also innocent design problems that give extremely non-robust systems.

The tools that are in common use in robust control are: algebraic Riccati equation, doubling algorithms, matrix sign algorithm, H_2 and H_∞ criteria, robustness bound, entropy (sometimes used in the indirect utility function of the multiplier problem, extremization and minmax solution, certainty equivalent issues. Many of these tools will be used or at least cited in this dissertation. For example:

i) The solution of a Riccati equation is key for getting the value of the worst-case distortion (v in our setup).

ii) The use of a doubling algorithm is essential for solving numerically the equations (involving Riccati-like equations). The matlab code in the Appendix, which is a copy with slight modification of the code done by Neng E. Wang, use a function called doublex by HST that is a doubling algorithm.

iii) H_2 and H_∞ can be interpreted as special cases of the value of η and w (see the model below).

iv) Entropy was also the metric used to deal with uncertainty by HST and HSW.

v) Extremization is the problem solved by HST and HSW (they have a $\min\max$ problem). This dissertation work with a slight modification by working with a $\min\min\max$ problem.

vi) The use of certainty equivalent is a source of much insight in economics (as it is in optimal control!). The work by HST and HSW also make strong use of this tool (with some modifications). This dissertation does not extend this issue on certainty equivalence beyond what they have done.

One of the main contribution of Hansen and Sargent (1995) to the robust control literature was to introduce discounting in the model. It turned out that this was key to the empirical strategy adopted by HST and HST.

1.2 Modelling Economic Uncertainty: Robustness Preference and Multi Prior Approaches

This dissertation will focus on two branches in economics that try model uncertainty. They can be understood as ways of studying non standard preferences in economics (and finance).

a) The Multi Prior Approach: in this field of knowledge model uncertainty is formalized in such a way that the decision maker is left with a set of models over which the decision maker does not put a unique prior.

b) The Robustness Preference Approach: in the words of Thomas Sargent we have that "...control theorists have altered their formulations to cope with their distrust of their models. We shall describe probabilistic formulations of robust control theory, link them to Epstein and Schneider's work on model ambiguity, and describe how they can be used to model situations in which decision makers distrust their models and want decision rules that will work for a set of models. Robust control theory can be thought of as a **practical way to generate a plausible set of priors**. We shall discuss the **kinds of behavior** that a concern for robustness leads to." The most famous current application of robustness are asset pricing (see HST and HSW) and monetary economics (see Onatsky (2001)).

There is a vivid debate between these two approaches. Eventough they both try to handle the difficult question of how to model uncertainty their motivations are quite different and some disagreement come at times. For example Epstein and Schneider (2003) argues that the type of preferences modeled with robust concern is not time consistent. Hansen, Sargent, Turmuhumbetova and Williams (2005) provides a defense of robust control approach regarding the time consistency issue. They also argue that the type of enlargement of the ordinary set of priors (considered to be admissible) proposed by Epstein and Schneider is too large to be useful for robust control modelling.[2] For the model used in this dissertation the introduction of a second uncertainty parameter is not likely to imply an implausible enlargement of the neighborhood (ball in topology jargon) represented by a circle. Indeed it may even imply a shrinkage of the admissible set of priors.

Note that both approaches tries to model preferences (instead of just modelling behavior). This option is more complex and intellectual demanding. The gains are that preferences provide an unchanging feature of a model such that agent model can be analyzed in different environment (say different institutions or polices). Modelling behavior directly will have the drawback of a need to adjust the model each time we change, say, policy.

Are those modelling just an excuse for using free parameters? No, since they put greater or novel demands on the data and raise empirical issues that are relevant.

[2] The enlargement may bee too much for robust approach purposes even if the duality between multi prior and robust control is still valid.

1.2.1 A Single-Parameter Robust Pricing of Uncertainty

In a survey paper Campbell (2000) cites a work by Hansen, Sargent and Tallarini (1999), henceforth **HST**, that signals a promising road to explaining empirical puzzles. HST provides a general equilibrium model to the treatment of investor behavior that is commonly presented in partial equilibrium models, i.e., it is, in a sense, an extension and improvement of what the bulk of models such as behavioral finance attempts (partial equilibrium analysis is indeed one of the main criticisms to behavioral models). HST deals with the usefulness of robust control by considering concerns about model misspecification and, by doing this, looks in a particular way of relaxing neoclassical finance assumptions. Exploring these concerns is fruitful because other model modifications such as Cambpell and Cochrane (1999)'s external habit, Constantinides and Duffie (1996)'s market frictions and Heaton and Lucas (1998)'s transaction costs are only partially successful in accounting fully for the market price of risk (or market price of knightian uncertainty to be defined later).

In a recent extension to HST, Hansen, Sargent and Wang (2002), henceforth **HSW**, study the contribution of a preference for robustness in the market price of risk in three different models: the basic HST model and two modified versions of HST in which agents do not (fully) observe the state vector and hence must filter. These two versions imply two robust filtering problems. The main conclusion is that regardless of the selected model the relationship between the detection error probability (DEP) and the contribution of robustness to the market price of risk is very strong (even though the value of the parameter measuring preference for robustness (σ) depends on the model). Moreover a preference for robustness corresponding to a plausible value of the DEP (small values) leads to a substantial increase in the market price of risk.

In order to understand DEP consider a simple problem of statistical discrimination: from historical data make a pairwise choice between two models: approximate model and worst-case model. This can be reformulated as a Bayesian decision problem. Accordingly to the selected model two types of errors are possible: the resulting DEP quantifies the statistical discrimination. For some values of σ the worst-case model is hard to detect statistically given the approximating model. In a parallel to rational expectations intuition, the decision maker want to be protected against some misspecified models: those that could not have been ascertained easily given historical data. This mean that factor risk prices brings model uncertainty premia in the model of robust decision making with these factor prices being largest precisely when, under the approximating model, investors are most unsure of the hidden state. Hence there is a strong relationship between ambiguity (encoded in state probabilities) and model uncertainty (reflected in local factor prices). (for further details see Cagetti et AL (2002)).

HSW conceals elements of the state from the planner and the agents, i.e., there is hidden state, forcing them to filter. Then the modeler needs to work jointly with robust filtering and control. In order to compute the appropriate market prices of Knightian uncertainty the DEP are used to discipline (constrain) the single free parameter that robust decision making adds to the standard rational expectations paradigm, i.e., that governs the taste for robustness.

HST reinterpret Epstein-Zin (1989) recursions as reflecting a preference for robustness instead of aversion to risk. Moreover it shows that in a class of stochastic growth models the consequences of either a preference for robustness (robust control problem) or a risk-sensitive adjustment to preferences (recursive risk-sensitive control problem) are difficult to detect in the behavior of quantity data they use: consumption (C) and investment (I). The reason is that altering a preference for robustness has effects on quantities (C, I) identical to changes in the discount factor (β), i.e., a robust decision maker with a lower discount factor would use the same optimal decision rules for (C,I). This is called the observational equivalence for quantities (C,I). One consequence is that alterations in the parameter measuring preference for robustness (σ, also called the robustness penalty parameter) can be offset by a change in the subjective discount factor (β). Hence, the vector (C, I) is unchanged. The main implication concerning asset pricing is that this observational equivalence result does not extend to equilibrium asset prices (and specially the market prices of risk). That is why HST and HSW are able to use this observational equivalence result to study the equity premium puzzle (robust decision-making induces a behavior similar to the one produced by risk aversion; preference for robustness can be interpreted as aversion to Knightian uncertainty). (The equity premium puzzle is the puzzle of the large gap between expected returns on stocks and government bonds. See Mehra and Prescott (1985). We will stress the interpretation of this puzzle given by Hansen and Jaganathan (1991)).

Hence they can calculate how much preference for robustness is necessary for market prices of risk to match the empirical findings, i.e., how much model misspecification the decision maker should fear given his historical data record.

In sum: by applying single-agent robust decision theory to representative agent pricing models, HST and HSW show that for a discrete-time linear quadratic permanent income model when robustness is taken into consideration the impact of precautionary savings should enter the macroeconomic calibration of parameters. Specifically decreasing the robustness parameter (σ) leads to an increase in the robust precautionary motive and increase the average level of capital. This could also be attained by lowering the discount factor (β), i.e., a more patient decision maker will hold more capital. The extra precautionary motive due to robustness is fully offset by increasing β.

Some branches of the modern finance literature has been prone to criticism due to lack of model discipline. For example a severe criticism about Behavioral Finance literature is that they are a bunch of psychological bias looking for a theory. Behavioral finance models assume specific types of psychological bias such as overreaction to new events, underreaction to other events, diverse types of heuristic, etc. Fama (1998) argues that there too much ad hoc assumption in the model.

The solution presented by robust control is to consider a particular type of model misspecification. So it is arguable that is not the case that any misspecification goes. One needs to look for intuitive and reasonable types of model uncertainties and ideally attach an economic intuition to such misspecification.

Anderson, Hansen and Sargent (2000) extends HST to include more general misspecification errors. They are justified based on specification test statistics. The econometrician can get at least implicitly a penumbra of alternative specifications that are statistically close to the preferred specification. Hence, it is argued that such specifications doubts can be infused into the economy agents.

The robust control approach imply a departure from standard expected-utility (Von Neuman-Mongestern) results:

(i) Rational expectations models assume that agents know the model and are not concerned about specification errors. It attributes a common model to the econometrician and the agents within the model: they can have different information sets but agree about the stochastic processes that drives the model. Also agents make unbiased forecast of the future.

(ii) Robust control assumes that the agent is concerned about some endogenous worst-case possibility that differs from the result of the reference model due to (preference) parameter misspecification. We look for reasonable values for this parameter which is a measure of the strength of the preference for robustness. Coherently the agent tries to hedge against this worst-case possibility.

The idea of uncertainty aversion as interpreted by robust control literature is a possible solution for the equity premium puzzle since robustness (or uncertainty aversion) strongly reduces the demand for risky assets. The way to explain the equity premium puzzle with the idea of robustness is by making agents concerned about model uncertainty (besides the ordinary concern about market risk) when making decision about optimal dynamic portfolio and consumption. Or in other words: uncertainty aversion adds to the traditionally considered risk aversion. Coherently the desirable decision rule should be twofold:

(i) provide a good result when the state variables that were modeled correspond exactly to reality (no model misspecification)

(ii) provide a good approximation when there are some particular types of model misspecification about the state variables.

Robustness can help explain some other asset pricing puzzles that have been reported by the financial literature by improving on the neoclassical treatment to better match theory and stylized facts. The main point here is to show that factor risk prices have components that can be interpreted as factor prices of model uncertainty. Hence the risk premium that is claimed to appear in security market data may be partially explained by model misspecification premium (ou uncertainty premium)

We can interpret robustness (to parameter uncertainty) as increasing risk-aversion without changing the preference for intertemporal substitution. In an environment where you can separate the coefficient of risk aversion from the elasticity of intertemporal substitution like those worked by Epstein and Zin (1989, 1990) and Tallarini (1996, 2000) it is possible to get some resolution of both the equity premium and risk-free rate puzzle. We argue that is done by arbitrarily set the elasticity of intertemporal substitution (EIS) equals to one. In this paper we try to see how a variable EIS will affect the resolutions of both puzzles.

1.2.2 Potential Improvements to HST/HSW

Although the research agenda set forth by HST and HSW is impressive there are some points that can be thought as *a priori* improvement to those models (Sargent, Hansen and coauthors have themselves addressed some of the topics listed below). The focus of this dissertation is on the treatment of an additional parameter that represents concern for robustness. This is not included in the topics below and is developed in the next section.

(i) The models are done for single-agent decision problems. If **multiple agents** want robustness then HST or HSW are not suitable models. In this case we need a new equilibrium concept to replace rational expectations (instead of using a natural extension of rational expectations as HST and HSW do) where agents have different preferences for robustness. Making **decision-makers heterogeneous** should provide new insights about the role of robustness for decision-making. Chapter 16 of Hansen and Sargent (2007) tackles this topic.

(ii) Allowing for robustness repair only some of the deficiencies that benchmark stochastic growth models have. One needs a **richer transient dynamics** and **multiple sectors and consumers** to produce models that deliver better empirical findings. A **rich learning dynamics** may result in quantitatively important asymmetries in uncertainty premia in expansions and recessions.

(iii) **Nonlinear-quadratic Gaussian** (NLQG) control problem-framework may be less restrictive and more suitable in some situations than linear-quadratic Gaussian (LQG) set up. See chapter 17 of Hansen and Sargent (2007). Robust decision rules and prices can be computed for economies with **more general return and transition functions** (with a representative agent who prefers a robust rule). It is also possible to generalize to a **continuos-time setting** with manageable generalizations of Bellman (and Ricatti) equations and the correspondent asset pricing equations associated with a rational expectations model. See AHS (2003).

(iv) HST consider a limited **array of specification errors**. It takes the form of **shifts in the conditional means of shocks** that would be i.i.d. and normally distributed under the approximating model. This can be generalized to **Markov diffusion problems.** See chapter 17 of Hansen and Sargent (2007). **Specification test statistics** can be used to substantiate the form of specification errors used.

(v) There is **no learning**. The robust decision maker fully accepts model misspecification as a permanent state of affairs and design robust controls. The data is not used to improve his model specification over time. One has to bear this in mind since learning is the next logical step n decision making modelling under uncertainty.

(vi) Misspecifications are **small in a statistical sense** and the way decision-makers select an approximating model is merely an extension of rational expectations models.

(vii) Linking of premia in asset prices from Knightian uncertainty to **detection error statistics** for **discriminating between models** may be a desired feature.

(viii) The effects of concern about robustness are likely to be particularly important in economies with large shocks that occur infrequently (**rare events**). Hence, it is promising to model robustness in the presence of **jump components**. See Liu, Uppal and Wang (2005).

1.3 Related Literature

Incomplete (please skip the reading of this section)

1.3.1 Bansal and Yaron (2004)

1.3.2 Pathak (2002)

1.3.3 Maenhout (2004)

1.3.4 Maenhout (2006)

1.3.5 Uppal and Wang (2003)

1.3.6 Liu, Uppal and Wang (2005)

1.3.7 Kogan, Leonid and Wang (2002)

1.3.8 Tornell (2001)

Robust-H_∞ Forecasting and Asset Pricing Anomalies. UCLA working paper. Mimeo.

1.3.9 Trojani and Vanini (2002)

1.3.10 Skiadas (2003)

1.3.11 Vardas and Xepapadeas (2004)

1.3.12 Abel (2002)

2 Robust Control (Hansen & Sargent) and Risk-Sensitive Control (Epstein & Zin) Methods for Modelling Uncertainty

In this section we describe two approaches for modelling economic uncertainty. Before we briefly describe the economy using state space methods.

2.1 State Space Description

The state transition equation is

$$x_{t+1} = Ax_t + Bi_t + Cw_{t+1}$$

Where x_t is the Markov state vector, i_t is the control vector and w_{t+1} is an i.i.d. Gaussian random vector with $Ew_{t+1} = 0$, and $Ew_{t+1}w'_{t+1} = I$.

The one-period (quadratic) return function is:

$$u(i, x) = -i\prime Qi - x\prime Rx$$

with Q being a positive definite matrix and R being a positive semidefinite matrix.

2.2 Risk-Sensitive Control Problem (Epstein, Zin, and Coauthors)

Use of a recursive non-expected utility similar to that used by Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990) and Tallarini (2000):

$$V_t = W(c_t, \mu(V_{t+1}))$$

$$\mu(V_{t+1}) = f^{-1}\{E_t[f(V_{t+1})]\}$$

$$f(z) = \begin{cases} z^{1-\gamma} \text{ , if } 0 < \gamma \neq 1 \\ \log z, \text{ if } \gamma = 1 \end{cases}$$

Where $V(.)$ is the value function, $W(.)$ represents an aggregator function, $\mu(.)$ represents a certainty equivalent (or mean value) function, $f(.)$ is a (power) utility function that describes the atemporal risk attitudes, and γ can be interpretted as a coefficient of relative risk aversion.[3]

The most common used aggregator is the CES aggregator, i.e.:

$$W(c, \mu) = [(1-\beta)c^{1-\rho} + \beta\mu^{1-\rho}]^{\frac{1}{1-\rho}}, \text{ for } 0 < \rho \neq 1 \text{ or}$$

[3] The coefficient γ will be compared to the standard Arrow-Pratt measure or risk aversion in section XXXX.

$$\lim_{\rho \longrightarrow 1} W(c,\mu) = c^{1-\beta}\mu^{\beta}$$

Where $\frac{1}{\rho} > 0$ is the Elasticity of intertemporal substitution (EIS) and β is the discount factor. Note that when we have $\rho = \gamma$ we get the canonical case of additive expected power utility.

Lemma 3 *When $\rho = 1$ we get log preferences under certainty, i.e., $W(c, W^{'}) = c^{1-\beta}\mu^{\beta}$ where $W^{'}$ means next period value for W.*
Proof. *Straightfoward.* ■

Definition 4 *A power certainty equivalent function is of the form $\mu(V) = [E(V^{\alpha})]^{\frac{1}{\alpha}}$.*

Using a power certainty equivalent function we get a recursive utility under uncertainty:

$$V_t = c_t^{1-\beta}\left\{E_t(V_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}}\right\}^{\beta}$$

Or

$$\log V_t = (1-\beta)\log c_t + \frac{\beta}{1-\gamma}\log E_t(V_{t+1}^{1-\gamma})$$

Or

$$\frac{\log V_t}{(1-\beta)} = \log c_t + \frac{\beta}{(1-\beta)(1-\gamma)}\log E_t(V_{t+1}^{1-\gamma})$$

Define $U_t \equiv \frac{\log V_t}{(1-\beta)}$ which implies that $V_t^{1-\gamma} = \exp[(1-\beta)(1-\gamma)U_t]$
Then we can rewrite the above equation as:

$$U_t = \log c_t + \frac{\beta}{(1-\beta)(1-\gamma)}\log E_t\left\{\exp[(1-\beta)(1-\gamma)U_{t+1}]\right\}$$

This is the formulation of risk sensitive recursion of Hansen and Sargent (1995). If we define, as Tallarini (2000) did, the coefficient of risk-sensitivity σ as $\sigma = 2(1-\beta)(1-\gamma)$ we will get the following formulation:

$$U_t = \log c_t + \beta\frac{2}{\sigma}\log E_t\left\{\exp[\frac{\sigma}{2}U_{t+1}]\right\}$$

Then we can represent the recursion to induce intertemporal preferences as:

$$U_t = u(i_t, x_t) + \beta\Re_t(U_{t+1})$$

where we define the operator $\Re_t(U_{t+1}) \equiv \frac{2}{\sigma}\log E_t\left\{\exp[\frac{\sigma}{2}U_{t+1}]\right\}$.
Since we are in a LQG setup, Hansen and Sargent (1995), Whittle (1989) and Jacobson (1973) allow us to write the above value function as:

$$U = x_t'\Omega x_t + \varkappa$$

with the decision rule given by:

$$i_t = -Fx_t$$

for a feedback rule F solved in the robust control literature (for getting the solution one need to solve a canonical Riccati equation).

Derived expression for the negative semidefinite matrix Ω is:

$$\Omega = A^{*\prime} \left[\Omega + \sigma \Omega C (I - \sigma C' \Omega C)^{-1} C' \Omega \right] A^*$$

where $A^* = A - BF$.

There is also a derived expression for the nonpositive real number $\varkappa$ but is not of our main concern in this work.

Remark 5 *The name risk sensitivity may sound inappropriate at a first reading since a concept with that role should treat the third derivative of the utility function (u''') as was done by Kimball (1990) which defined a new concept called prudence to measure this features. As a matter of fact the concept is famous in the optimal control literature (see Whittle (1990)) and goes back to the seventies.*

2.3 Robust Control Problem (Hansen, Sargent and Coauthors)

The distorted law of motion for the state vector can be written as

$$x_{t+1} = Ax_t + Bi_t + C(w_{t+1} + v_t)$$

where the term v_t is the conditional mean distortion to the innovation w_{t+1}. The way v_t enters the above law of motion is crucial for our work.

The minimizing player (nature) choose a state feedback rule for v_t to minimize utility subject to:

$$\begin{cases} \widehat{E}_t \sum \beta^j v_{t+j} \cdot v_{t+j} \leq \vartheta_t \\ \vartheta_{t+1} = \beta^{-1}(\vartheta_t - v_t \cdot v_t) \end{cases}$$

where $\widehat{E}_t$ is a conditional distorted expectation operator, ϑ_0 is given and ϑ_t indexes the pessimism, i.e., it is a continuation pessimism bound at date t. We associate a constant Lagrange multiplier $\theta \equiv -\sigma^{-1} > 0$ to the constraint above.

The game to be solved is an extremization:[4]

$$W(x) = \inf_v \sup_i \left\{ -i\prime Qi - x\prime Rx + \beta \left[-\frac{1}{\sigma} v'v + EW\left(Ax_t + Bi_t + C(w_{t+1} + v_t)\right) \right] \right\}$$

The solution for the minimization problem gives us:

$$v_t = \left[\sigma(I - \sigma C'\Omega C)^{-1} C'\Omega A^*\right] x_t = \kappa x_t$$

Given that the solution for the control variable i is the same with and without the concern for robustness HST and HSW called this a modified version of certainty equivalence. In other words $i = -Fx$ is independent of the covariance matrix C.

Remark 6 *By virtue of the Lagrange multiplier theorem, HST and Hansen and Sargent (2007) shows that there is a duality between the solution to the risk sensitive control problem and the solution to the robust control problem when we define $\theta \equiv -\sigma^{-1}$, a very simple relation between the parameters that represent uncertainty in each approach, and for mild conditions: σ needs to obey: $\underline{\sigma} < \sigma < 0$, for some breakdown point $\underline{\sigma}$, or equivalently $\underline{\theta} < \theta < \infty$. This condition is easily satisfied for the empirical relevant range of model parameter values used by HST and HSW).*

[4] There is also consideration of nature as benevolent agent such that the existence of this second controller will imply a $\max\max$ problem to be solved instead of a $\max\min$ setup as HST and HSW use. See Resende (2006) and Schneider and XXXX (1999).

Remark 7 *At this point is worth comments on some of the features of the matrix κ: i) it is not necessarily symmetric, ii) it can be sparse. If the eigenvalues of κ are not close to zero than the solution to the equation is stable. These issues are not a main concern in this dissertation (since our model is simple and well-posed when compared to the complex ones treated in the robust and optimal control literature). For more on stability and performance of feedback systems see Hansen and Sargent (2007) and chapters 4, 5, and 6 of Zhou, Glover and Doyle (1998).*

Remark 8 *For our purpose a weak concept of* ***absolute continuity*** *is used. Here it means that over finite intervals two models are difficult to distinguish given samples of finite length. In the standard notion we have that two stochastic processes are considered absolutely continuous with respect to each other if they agree about ,as the specialized literature points out, "tail events". The concept that is being used here requires that the two probability measures being compared both put positive probability on all of the same events, except tail events. See HSTW for a continuos-time treatment of this issue.*

2.4 Econometric Specification Analysis x Robust Decision Making

At this point it is worth mention the main difference between the standard econometric specification and the robust control approach. We use two figures of Hansen and Sargent (2007) to describe them.

For the econometric specification assume that the data generating process (DGP) is f, and that the econometrician fits a parametric class of models $f_\alpha \in A$ to the data with $f \notin A$. Estimation by maximum likelihood of parameter α may select the misspecified model f_{α_0} that is closest to f as measured by entropy $I(f)$.

For the robust control approach we assume that a decision maker with the approximating model f_{α_0} suspects that the DGP is actually generated by the unknown model f, such that $I(\alpha_0, f) \leq \eta$.

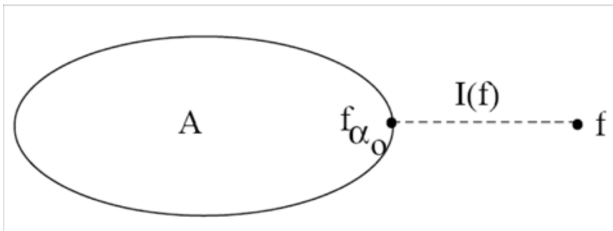


Figure XX: Source Hansen and Sargent (2007).

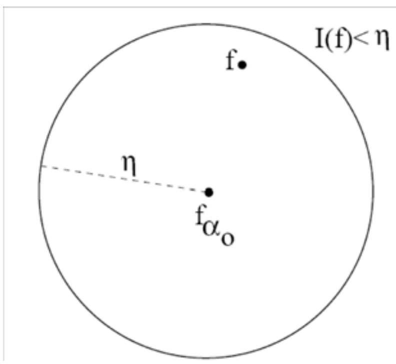


Figure XX: Source Hansen and Sargent (2007).

2.5 Permanent Income Growth Model

We need some structure on the data (C,I) for consumption and investment in order to proceed with the estimation of the parameter $\sigma \equiv -\theta^{-1}$. Let's assume that the utility index is defined by the following equations:

$$U_t = -(s_t - b_t)^2 + \beta \Re(U_{t+1})$$

$$s_t = (1+\lambda)c_t - \lambda h_{t-1} \qquad (2\beta^t \mu_{st})$$

$$h_t = \delta_h h_{t-1} + (1-\delta_h)c_t \qquad (2\beta^t \mu_{ht})$$

where s stands for service consumption and h for (internal) habit (since we are considering a representative agent model).

The production technology is given by

$$c_t + i_t = \varepsilon k_{t-1} + d_t$$

$$k_t = \delta_k k_{t-1} + i_t$$

By eliminating i we have

$$c_t + k_t = (\varepsilon + \delta_k)k_{t-1} + d_t = Rk_{t-1} + d_t \qquad (2\beta^t \mu_{ct})$$

Note that the risk free rate $R = \varepsilon + \delta_k$ was unambiguously pinned down by the production technology. Also note that in three of the above equations we listed in parenthesis the Lagrange multipliers μ_{st}, μ_{ht} and μ_{ct} for the appropriate equations (multiplied by 2β for mathematical convenience).

HST argued that the solution of the model is established by an observational equivalence for the quantity observations, the consumption (c) and investment (i). They proceed in two steps: (i) compute the solution for the permanent income economy without concern for robustness, using $\sigma = 0$ and $\beta = 1$. (ii) Use the allocation (c,i) for this benchmark economy to construct an equivalent class of alternative pairs (σ, β) that generate the same allocation, considering all other parameters of the economy to be fixed. This imply a non identification of pair (σ, β) from knowledge of (c,i).

From standard calculus of first-order conditions we get a martingale representation for the Lagrange multiplier for service consumption, i.e.,

$$\mu_{st} = \mu_{st-1} + \sqrt{}' w_t$$

It is straightforward to show that this martingale representation extends to μ_{ht} and μ_{ct}. Moreover there a possible way of writing as an additive function of key model elements ($b_{t+j}, d_{t+j}, h_{t-1}, k_{t-1}$). Without loss of generality we can assume $b_t = \mu_b$, i.e., we have a fixed bliss point.

2.6 Relation between β and σ

To compute the relation between β and σ we need to first note that $\mu_{st} = s_t - b_t$. We have that:

$$\Omega x^2 = -x^2 + \beta \min_v \left[-\frac{1}{\sigma} v^2 + \Omega(x + \alpha v)^2 \right]$$

and write $\mu_{st} = \mu_{st-1} + \alpha(v + w)$, with $\alpha^2 = \surd' \surd$

Solving the above problem let us with

$$\Omega = \Omega(\beta) = \frac{(\beta - 1 + \sigma\alpha^2) + \sqrt{(\beta - 1 + \sigma\alpha^2)^2 + 4\sigma\alpha^2}}{-2\sigma\alpha^2}$$

Where this come from a solution (Baskara formulation) of a quadratic equation with coefficients given by $\left(-\sigma\alpha^2, \beta - 1 + \sigma\alpha^2, 1\right)$. Now we are ready to compute the distorted expectation operator:

$$\widehat{E}_t \mu_{st+1} = \zeta \mu_{st}$$

with

$$\widehat{\zeta} = \widehat{\zeta}(\beta) = \frac{1}{1 - \sigma\alpha^2\Omega(\beta)} = 1 + \frac{\sigma\alpha^2\Omega(\beta)}{1 - \sigma\alpha^2\Omega(\beta)}$$

Note that for $\sigma \neq 0 \Rightarrow \widehat{\zeta}(\beta) \neq 1$.

Given the martingale representation for consumption we also have:

$$\widehat{E}_t \mu_{ct+1} = \zeta \mu_{ct}$$

This give us an Euler Equation for consumption. Now take $\widehat{\beta} R \widehat{\zeta}(\widehat{\beta}) = 1$. Then we have

$$\widehat{\beta} R \widehat{E}_t \mu_{ct+1} = \mu_{ct}$$

To find an explicit equation for $\widehat{\beta}$ work with the quadratic forms to obtain:

$$\widehat{\beta} = \frac{1}{R} + \frac{\alpha^2}{R - 1}\sigma$$

which shows that β and σ are related in a very simple mapping!

Remark 9 *The value for $\underline{\sigma}$ can be find by calculating the lowest value for σ such that the quadratic equation above has a real solution. Just work with the discriminant of the equation (Δ) with the triple $\left(-\sigma\alpha^2, \beta - 1 + \sigma\alpha^2, 1\right)$.*

Remark 10 *It is possible to interpret $\surd$ in another way: $\surd = M_s C$, where C is the volatility matrix of the state vector equation and M_s is the marginal utility of consumption services. The equivalence hinges on the fact that $\mu_{st} = M_s x_t$, i.e., we can take μ_{st} as the state vector and solve for it.*

2.7 Bivariate Stochastic Endowment Process and Parameter Values

In order to proceed HST and HSW took a very workable formulation for the endowment process.

$$\begin{cases} d_t = \mu_d + d^1_{t+1} + d^2_{t+1} \\ d^1_{t+1} = g_1 d^1_t + g_2 d^1_{t-1} + c_1 w^1_{t+1} = (\phi_1 + \phi_2) d^1_t - \phi_1 \phi_2 d^1_{t-1} + c_1 w^1_{t+1} \\ d^2_{t+1} = a_1 d^2_t + a_2 d^2_{t-1} + c_1 w^2_{t+1} = (\alpha_1 + \alpha_2) d^2_t + \alpha_1 \alpha_2 d^2_{t-1} + c_1 w^2_{t+1} \end{cases}$$

Hence the law of motion for the state variable can be written as:

$$x_{t+1} = \begin{bmatrix} h_t \\ k_t \\ d_t \\ 1 \\ d_{t+1} \\ d^1_{t+1} \\ d^1_t \end{bmatrix} = \begin{bmatrix} \delta_h & (1-\delta_h)\gamma & 0 & 0 & (1-\delta_h) & 0 & 0 \\ 0 & \delta_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & \mu_d(1-a_1-a_2) & a_1 & g_1 - a_1 & g_2 - a_2 \\ 0 & 0 & 0 & 0 & 0 & g_1 & g_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot$$

$$\begin{bmatrix} h_{t-1} \\ k_{t-1} \\ d_{t-1} \\ 1 \\ d_t \\ d^1_t \\ d^1_{t-1} \end{bmatrix} + \begin{bmatrix} -(1-\delta_h) \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} i_t + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \\ c_1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} w^1_{t+1} \\ w^2_{t+1} \end{bmatrix}$$

The parameters value that HST and HSW worked with were:

$$\begin{cases} \beta_0 = 0.9971 \Rightarrow R = 1{,}025 \text{ or an interest rate of 2,5\%} \\ \delta_h = 0.6817 \\ \lambda = 2.443 \\ \alpha_1 = 0.8131 \\ \alpha_2 = 0.1888 \\ \phi_1 = 0.9978 \\ \phi_2 = 0.7044 \\ \mu_d = 13.7099 \\ c_1 = 0.1084 \\ c_2 = 0.1551 \\ \text{Four values for } \mu_b = \{18, 24, 30, 36\} \\ \text{Three values for } \sigma = \{0.25, 0.50, 0.75\} \cdot 10^{-4} \end{cases}$$

2.8 Relevant Asset Pricing Relations

In this section we stressed the key relations for asset pricing implication of the model at hand.

If we denote by q_t the asset price of a unit vector of consumption good in period t than the relevant Euler equation will give us the fundamental relation about pricing:

$$q_t(x^t \mid x_o) = \beta^t \frac{u'(c_t(x^t))}{e_1 \cdot u'(c_0(x_0))} f^{(t)}(x^t \mid x_o)$$

where x^t represents the history, $c_t(x^t)$ represents a history-dependent state-contingent consumption process, e_1 is a selector vector that pulls off the first consumption good (we take the time-zero value of it as a numeraire) and $f(x_{t+1} \mid x_t)$ represents the transition density. Note that q_t is also known as the pricing kernel.

If we assume that consumption is not history-dependent than we will obtain the following t-step pricing kernel:

$$q_t(x_t \mid x_o) = \beta^t \frac{u'(c_t(x_t))}{e_1 \cdot u'(c_0(x_0))} f_t(x_t \mid x_o)$$

Let's represent by $\{y(x_t)\}_{t=0}^{\infty}$ the stream of payoffs that the owner of the asset has the right to receive.

Then the time-0 price of the asset is

$$a_0 = \sum_{t=0}^{\infty} \int_{x_t} q_t(x_t \mid x_o) \cdot y(x_t) dx_t$$

Which can be also written as

$$a_0 = E_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \cdot y(x_t) dx_t$$

Where $p_c(x_t) = \frac{u'(c_t(x_t))}{e_1 \cdot u'(c_0(x_0))}$ is the (scaled) Arrow-Debreu state price. The term $\beta^t p_c(x_t)$ has a special denomination and we definite below.

Definition 11 *The t-period stochastic discount factor (SDF) is defined as* $m_{0,t} = \beta^t p_c(x_t)$*, where* $p_c(x_t) = \frac{u'(c_t(x_t))}{e_1 \cdot u'(c_0(x_0))}$ *is the appropriate Arrow-Debreu state price.*

Hence we can write the price of the asset as:

$$a_0 = E_0 \sum_{t=0}^{\infty} m_{0,t} \cdot y(x_t) dx_t$$

For the setup that we are working with (linear-quadratic Gaussian general equilibrium models, i.e. LQG-GE) one can obtain explicit and closed-form solutions for pricing (as stated below).

2.8.1 Asset Pricing with Robustness Concerns

Given the above structure and suppose that the distorted model is governed by the distorted transition density $\widehat{f}_t(x_t \mid x_o)$.

In this case the time-0 price of the asset is represented by:

$$a_0 = \sum_{t=0}^{\infty} \int_{x_t} \beta^t p_c(x_t) \cdot y(x_t) \widehat{f}_t(x_t \mid x_o) dx_t$$

which can be written using the distorted expectation operator as:

$$a_0 = \widehat{E}_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \cdot y(x_t)$$

It is common to work with the above formulation and include the concept of likelihood ratio (or the Radon-Nykodin derivative) in the following way:

$$a_0 = \sum_{t=0}^{\infty} \int_{x_t} \beta^t p_c(x_t) \frac{\widehat{f}_t(x_t \mid x_o)}{f_t(x_t \mid x_o)} \cdot y(x_t) f_t(x_t \mid x_o) dx_t$$

or using the (non-distorted) expectation operator:

$$a_0 = E_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \frac{\widehat{f}_t(x_t \mid x_o)}{f_t(x_t \mid x_o)} \cdot y(x_t)$$

Hence we obtained a modified SDF of the form defined below:

Definition 12 *The modified stochastic discount factor is a multiplicative adjustment to the ordinary SDF using the likelihood ratio* $L_t = \frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)}$*, i.e., it is given by* $m_{0,t} \left[\frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} \right]$.

Note that this representation was made possible by virtue of the LQG framework. Given the properties of the normal-exponential integrals it is straightforward to show that:

Lemma 13 *The likelihood ratio is given by*

$$L_t = \frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} = \exp\left\{ \sum_{s=1}^{t} \left[w_s v_s - \tfrac{1}{2} v_s' v_s \right] \right\}$$

where s is the appropriate time index.

Proof. We have that $f(x) = (2\pi)^{-1/2} \exp\left\{ -\frac{x^2}{2} \right\}$. Then

$$\frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} = \frac{(2\pi C)^{-1/2} \exp\left\{ -\frac{v^2}{2} - \frac{w^2}{2} + vw \right\}}{(2\pi C)^{-1/2} \exp\left\{ -\frac{w^2}{2} \right\}} = \exp\left\{ -\frac{v^2}{2} - \frac{w^2}{2} + vw + \frac{w^2}{2} \right\}$$

$$= \exp\left\{ -\frac{v^2}{2} + vw \right\} . \blacksquare$$

Remark 14 *The appearance of the likelihood ratio above is the same as the one used to define entropy (for our purposes) and to describe the detection error probabilities (DEP) as stated below.*

2.9 Market Price of Risk (MPR)

Now we will specialize our laboratory economy (the permanent income model with habit) to have a scalar consumption process and a scalar random payoff y_t.

When there is no concern about robustness the asset price is given by

$$a_t = E_t m_{t,t+1} y_{t+1}$$

This can be reworked by applying the definition of conditional covariance and by a direct application of the Cauchy-Schwartz inequality delivering the following expression:

$$\left(\frac{a_t}{E_t m_{t,t+1}}\right) \geq E_t y_{t+1} - \left(\frac{std_t(m_{t,t+1})}{E_t m_{t,t+1}}\right) std_t(y_{t+1})$$

The left side shows the ratio of the price of a claim to payoff y_{t+1} to the price of a riskless claim to one unit of consumption next period. Along the efficient-frontier, the price of risk can be defined in the following way:

Definition 15 *The market price of risk (MPR) is represented by* $\left(\frac{std_t(m_{t,t+1})}{E_t m_{t,t+1}}\right)$ *and provides an estimate of the rate at which the price of an asset decreases with an increase in the conditional standard deviation of its payoff* ($std_t(y_{t+1})$)*. This ratio encodes information about the degree of aversion to risk the consumers display at the equilibrium consumption process.*

The bound above was studied by Hansen and Jaganhatan (1991) and are attained by payoffs on the efficient frontier (in the space $(E_t(m_{t,t+1}), std_t(m_{t,t+1}))$, i.e., the inequality become an equality for payoffs y_{t+1} on the conditional mean-standard deviation frontier. Another way to express this bound is: $\frac{|E[\tau]|}{std(\tau)} \leq \frac{std_t(m_{t,t+1})}{E_t m_{t,t+1}}$, for an excess return τ, where the left term can be easily related to the famous Sharpe ratio.

Remark 16 *Optimal Portfolio and Merton's Rule can be linked to the above theory. Usually the model can nests the special cases of the static CAPM and the intertemporal CAPM of Merton (1976). This is also true for the Consumption-CAPM. See Maenhout (2004 and 2006), Koogan and Wang (2003), Liu, Pan and Wang (2005), Uppal and Wang (2003) and Skiadas (2003).*

2.10 Empirical Puzzles of Interest and the Market Price of Uncertainty (MPU)

Some studies have estimated the MPR from data on the pair (a_t, y_{t+1}) . For post WWII quarterly data in the U.S. the result was something around 0.25. Barrilas, Hansen and Sargent (2006) found 0.2550 while Tallarini (2000) found 0.2525.

The characterization of the Equity Premium puzzle by Hansen and Jaganathan (1991) is that data on asset market prices and returns imply a MPR too high to be reconciled with many particular (specialized) models of the SDF $m_{t,t+1}$. The reason is that most of the theories make $std_t(m_{t,t+1})$ too small. Bellow we discuss some possible theories for the SDF.

a) Shiller (1981) considered $m_{t,t+1} = \beta =$ constant. This implies that $std_t(m_{t,t+1}) = 0$.

b) Lucas (1973) and Breeden (1979) considered $m_{t,t+1} = m^f_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ with u(.) been a CRRA one-period utility function with power γ for the assumed representative consumer. This implies $std_t(m_{t,t+1}) = \beta \left(\frac{c_t}{c_{t+1}}\right)^\gamma$ which is small given the empirically reasonable value for γ evaluated at aggregate U.S. consumption growth rates. This happens because aggregate consumption is a smooth series. Hence one needs γ to be huge for the MPR to get close to the empirical value of 0.25.

The work by HST and HST (and the diversity of papers related to pricing with robustness concern) deals with a multiplicative adjustment to SDF namely:

$$m_{t,t+1} = m^f_{t,t+1} m^u_{t,t+1}$$

where the superscripts f and u stand for familiar (ordinary) and unfamiliar (or uncertain) part of the SDF, respectively. This is the key expression for our concern with bi-dimensionality representation of uncertainty and with a plain role of the idea of elasticity of substitution (all of this worked in a friendly Linear Quadratic Gaussian-LQG framework).

Definition 17 *(due to HST) The one-period market price of Knightian uncertainty (MPU) is the standard deviation of the multiplicative adjustment, i.e.,* $std_t(m^u_{t,t+1})$.

When we have concern about robustness the asset price is given by

$$a_t = E_t \left(m^f_{t,t+1} m^u_{t,t+1} \right) y_{t+1}$$

where we omitted the summation because we are dealing with a single period and omitted the inner product because consumption and payoff are scalars.

Moreover

$$m^u_{t,t+1} = \exp\left\{ -\frac{v'v}{2} + w'v \right\}$$

by construction (see next section) we have that $E_t\left(m^u_{t,t+1}\right) = 1$. Than HST computed that

$$MPR = \frac{std_t(m_{t,t+1})}{E_t\left[m_{t,t+1}\right]} = \frac{\sqrt{E_t\left[\left(m^u_{t,t+1}\right)^2\right] - \left(E_t\left[m^u_{t,t+1}\right]\right)^2}}{E_t\left[m^u_{t,t+1}\right]} = \frac{\sqrt{\exp(v'v) - (1)^2}}{1} = \sqrt{\exp(v'v) - 1}$$

Note that in the above derivation it was needed to assume that $m^f_{t,t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$ is constant such that

$MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]} = MPR = \frac{std_t(m^f_{t,t+1}m^u_{t,t+1})}{E_t\left[m^f_{t,t+1}m^u_{t,t+1}\right]} = \frac{m^f_{t,t+1}std_t(m^f_{t,t+1}m^u_{t,t+1})}{m^f_{t,t+1}E_t\left[m^u_{t,t+1}\right]} = \frac{std_t(m^f_{t,t+1}m^u_{t,t+1})}{E_t\left[m^u_{t,t+1}\right]}$

Remark 18 *The result $MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]} = \sqrt{\exp(v'v) - 1}$ is an upper bound on the enhancement of the SDF due to consideration of robustness. Ideally the formula will explicit the term correlation$(m^f_{t,t+1}, m^u_{t,t+1})$ in the formula for MPR.*

This result within the robust control framework has a counterpart in the risk sensitive control problem with the SDF also being suitably decomposed in two factor: the standard (familiar) $m^f_{t,t+1}$ and the adjustment due to the risk sensitive parameter $m^r_{t,t+1}$.

$$m_{t,t+1} = m^f_{t,t+1}m^r_{t,t+1}$$

Note that although $m^r_{t,t+1} = m^u_{t,t+1}$ for the two frameworks to deliver the same asset pricing implications the economic intuition are strikingly different. The justification for $m^f_{t,t+1}$ is that agent (maximizing controller) fears misspecifications of the model he is working with. By its turn $m^u_{t,t+1}$ is justified in the ground that the preferences are adjusted to be recursive with a parameter σ driving the twist in preferences.

The equation for $m^r_{t,t+1}$ is

$$m^r_{t,t+1} = \frac{\exp(\frac{\sigma}{2}U_{t+1})}{E_t\left[\exp(\frac{\sigma}{2}U_{t+1})\right]}$$

Note that this is just an "exponential tilting" in the SDF.

Remark 19 *The effect of the multiplicative adjustment $m^u_{t,t+j}$ increases with time. HSW calculated the values for the MPR for up to four periods (i.e., for $j = 1, 2, 3, 4$). The equation for $m^u_{t,t+4}$, for example, is straightforward.*

A simple way of stating the two relevant puzzles for the purposes of this dissertation is to look at the space $(E(m), std(m))$. See Tallarini (1996 and 2000) and Barrilas, Hansen and Sargent (2006). Below the figure XXX gives a good

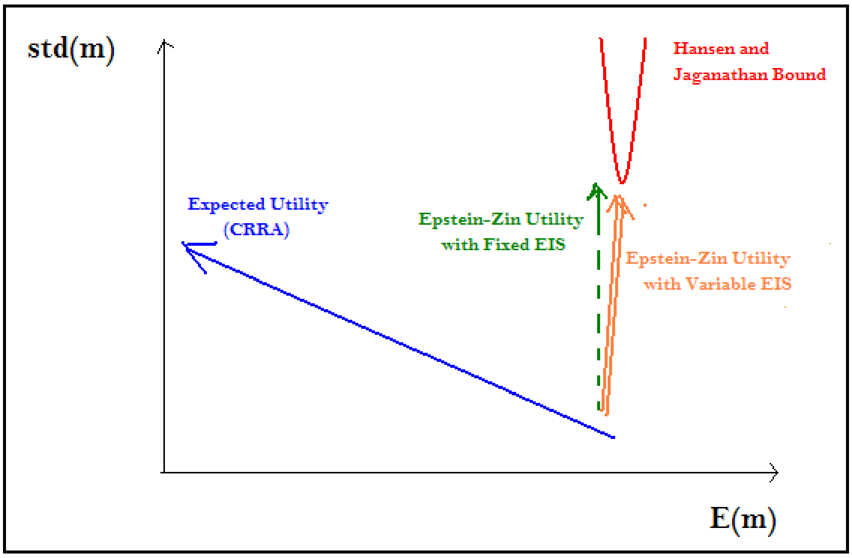


Figure 1:

illustration of what are the important features. The arrows for the three situations (expected utility, Epstein-Zin with fixed EIS and Epstein-Zin with variable EIS) points to the path obtained when the coefficient of risk aversion is increasing. Success in the solution of the puzzles is declared if the pair $(E(m), std(m))$ is set within the parabola representing the Hansen-Jaganathan bound.

a) For the solution of the Equity Premium puzzle (EPP) it is necessary to increase $MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]}$ which can be done by increasing the numerator $std_t(m_{t,t+1})$ and or decreasing the denominator $E_t\left[m_{t,t+1}\right]$. Note that the MPR is just the slope of a line drawn in the space $(E(m), std(m))$.

b) For the solution of the risk free rate puzzle (RFRP) it is necessary to increase (or at least to maintain constant) the value of $E_t\left[m_{t,t+1}\right]$ since $r^f = \frac{1}{E_t[m_{t,t+1}]}$.

The model using expected utility solves EPP but not the RFRP. The model using Epstein-Zin preferences with an ad hoc value for EIS (set at 1) solves the EPP but not the RFRP (though the value for r^f obtained was not too far from the reasonable value of 0.78%). Finally the model that we propose (Epstein-Zin with variable EIS) is able to both increase the MPR and consequently solving the EPP and decreasing the r^f by increasing the denominator $E_t\left[m_{t,t+1}\right]$. Note that the slope of the line is much bigger than one (for the MPR to really increase), i.e., we need a bigger increase in the denominator than the increase in the numerator.

2.11 The Use of Martingales to Represent Perturbed Models (in a Nutshell)

Martingales are useful tools for the characterization of distorted (alternative) probability models. For easiness of exposition we will work with the very basic concepts of martingale theory. See Duffie (2002) and Hansen (2005).

Definition 20 *For a probability space* $(\Omega, \Im, P)$ *with a finite number of events (finite set* Ω*) and for a nonnegative random variable Y with E(Y)=1, we can create a new probability measure Q by defining* $Q(B) = E(1_B Y)$ *for any event B and for* 1_B *representing the indicator function. Then we can write* $\frac{dQ}{dP} = Y$ *where Y is the Radon-Nykodyn derivative of Q with respect to P.*

For any random variable X we can assert that

$$E_Q[X] = E_P[YX]$$

where $E_Q[.]$ means the expectation under probability measure Q (likewise for $E_P[.]$)

Now let χ_t represents the date t information set. Let a nonnegative martingale $\{M_t : t \geq 0\}$ with $M_0 = 0$. In particular $E[M_t \mid \chi_0] = 1$. The distorted expectation $\widehat{E}$ can be represented as

$$\widehat{E}[x_t \mid \chi_0] = E[M_t x_t \mid \chi_0]$$

with M_t being a likelihood ratio or a Radon-Nykodin derivative.

Now define entropy

Definition 21 *Entropy is defined as* $I(\widehat{p}, p) = \widehat{E} \log\left(\frac{\widehat{p}}{p}\right) = EM \log M$*, where EM=1*

Remark 22 *For the static entropy penalization problem (similar to the robust control problem described above) we have that the problem* $\min_{M \geq 0, EM=1} \{E[M(U + \theta \log(M))]\}$ *has the solution given by an exponential tilting: the worst case is* $M^* = \frac{\exp(\frac{\sigma}{2} U_{t+1}(i,x))}{E\left[\exp(\frac{\sigma}{2} U_{t+1}(i,x))\right]}$*, with* $\theta = -\sigma^{-1}$*. Note that the minimized objective is* $-\theta \log(E\left[\exp(-\frac{1}{\theta} U_{t+1})\right])$ *and that* M^* *depends on the control i.*

The extremization is written as:

$$\sup_i \min_{M \geq 0, EM=1} \{E[M(U(i,x) + \theta \log(M))]\}$$

2.12 Results of HST and HSW Models

For the values used for the 10 free parameters HST and HSW found a significant boost in the MPR. For example: a one-period HST model with $\mu_b = 36$ and $\sigma = 0.75 \cdot 10^{-4}$ they get a MPR of 0.1186 and for the four-period with the same values for the pair (μ_b, σ) they get a MPR of 0.2405 which is a very good match to the empirical value. HSW managed to get a MPR of 0.2405 even with a one-period version of the model (for the four-period the MPR jumped to 0.3894).

By construction they kept the value for the interest rate constant at 2,5% which is high for the data set analyzed (the empirical value is around 0,78%). Hence both HST and HSW could managed to help solve the Equity Premium puzzle but both were powerless in facing the risk free rate puzzle. We argue that this was due to the implicit assumption that the elasticity of intertemporal substitution (EIS) was fixed at 1 (that was the way the risk sensitive control problem was solved with reflex for its dual solution in the robust control problem). Moreover if one come with a second parameter that captures the concern for robusteness it is possible to get an optimal mix of the two parameter values to solve both puzzles (since each parameter affects differently the key variables that explain the mentioned puzzles).

3 Functional Analysis Treatment

Below we try to draw a very synthetic map of the relevant approaches that is pursuing modelling economic uncertainty. It has the mere objective to serve as a guide since each one of the approaches are very complex and deserves long studies. This guide will help to see what is needed to work with when one wants to model uncertainty in a multidimensional fashion.

3.1 Eight Relevant Sets and Seven Respective Mappings

Below is a **functional analysis treatment** of the relevant mappings necessary to treat multidimensional robust control with applications to economic problems. For simplicity we assumed that the dimension of the eight listed vectors (Ə,Þ, Σ, Θ,Ł, Γ, E,Ω) are equal to N.

Ə $= (\zeta_1, \cdots, \zeta_?)$: vector or parameters representing the Gilboa and Schmeidler (1989)'s atemporal axiomatization treatment of preferences (including support of behavior of Ellsberg's types).

Þ$_{Nx1}$ $=$ (þ$_1, \cdots,$ þ$_N$) : vector of Multi-Prior treatment (Epstein and Schneider and coauthors). Each þ represents a prior (or belief).

$\Sigma_{Nx1} = (\sigma_1, \cdots, \sigma_N)$: vector of parameter in the risk-sensitive control problem. (constraint problem)

$\Theta = (\theta_1, \cdots, \theta_N)$: vector of technical (extra or free) parameters in the robust control problem that allow for representation of model uncertainty. (multiplier problem)

Ł $= (I_1, \cdots, I_N)$: vector of distance (or discrepancy or entropy measure).

$\Gamma = (\gamma_1, \cdots, \gamma_N)$: vector of economic parameters. In HST and HSW treatment we had that $\gamma_1 = \beta$

$E = (e_1, \cdots, e_N)$: vector of famous empirical puzzles in economics/finance. In HST and HSW treatment we had that $\mathrm{e}_1 =$ equity premium puzzle.

$\Omega = (v_1, \cdots, v_N)$: vector of distortions

Hence the set composed of one element of each vector is : $[(\sigma_1, \theta_1, I_1, \omega_1), (\zeta_1,$ þ$_1), (\gamma_1, e_1)]$.

The seven mappings related the above set are:

$m(\cdot)$ from Ə to Σ

$l(\cdot)$ from Ə to Þ

$g(\cdot)$ **from** Σ **to** Θ

$f(\cdot)$ **from** Θ **to** Γ

$h(\cdot)$ from Ł to Θ

$t(\cdot)$ from Γ to E

s$(\cdot)$ from Ω to Ł

Remark 23 *Lucas (1976) analyze the one-to-one mapping from transition laws (x_{t+1} in our case) to decision rules ($u_t = -Fx_{t+1}$). See Lucas (1976) and Chapter 1 Footnote 16 of Hansen and Sargent (2007)).*

The mapping $g(\cdot)$ from Σ to Θ is particularly important for our purposes.

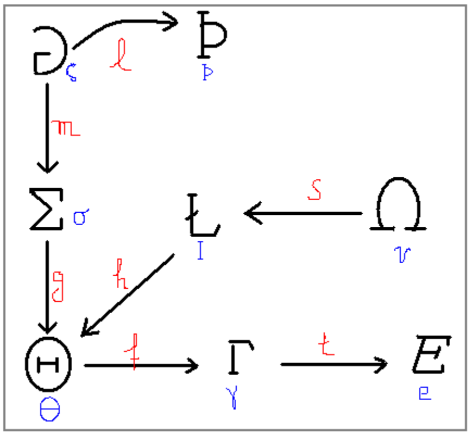


Figure 2:

When $\theta_1 = \theta$ in HST we had that the only cited economic parameter was $\gamma_1 = \beta$. In this case the model addressed the equity premium puzzle (e_1)

When considering a bi-dimensional robust control problem a new parameter θ_2 is introduced in the analysis and a candidate for γ_2 is the Elasticity of intertemporal substitution (EIS) represented by (ψ), and the additional puzzle to be addressed is the risk-free rate puzzle (e_2).

Similarly when one has the third free-parameter θ_3 it may be possible to treat a third puzzle, say a desire for less diversification in the optimal portfolio than advised by standard (Markovitz, Tobin, Merton) portfolio theory (one version of this behavior is the home-bias puzzle).

We can list the likely set of candidate elements (puzzles) to be explained by the use of robust control tools:

$$\left\{ \begin{array}{c} e_1 = \text{equity premium puzzle} \\ e_2 = \text{risk-free rate puzzle} \\ e_3 = \text{low diversification (home-bias) puzzle} \\ e_4 = \text{limited arbitrage feature (closed end fund puzzle)} \\ \cdots \\ e_{N-1} = \text{volatility/predictability puzzle} \end{array} \right\}$$

Although the task of formalizing a relationship between the eight sets discussed above is clearly a daunting task, the objective of this section is less ambitious and it is to set forth a first step in that direction by working with a bi-dimensional robust control regulator.

Remark 24 *Note that not all the elements presented in HST and HSW are necessarily represented in the sets above. For example there is no representation for the element* $\surd$ *(recall that* $\surd' \surd = \alpha^2$*).*

The fundamental question is what the relation between each one of the elements of vectors such as $[(\sigma_1, \theta_1, I_1, \omega_1), (\zeta_1, þ_1), (\gamma_1, e_1)]$. For example for HST and HSW it derived that:

i) $\sigma_1 = -\theta_1^{-1}$, i.e., $g(\cdot)$ is trivially derived as $g(\sigma_1) = \theta_1 = -\sigma_1^{-1} \Rightarrow g(x) = -x^{-1}$

ii) There is an upper bound on $I_1 = \overline{I}_1 = \frac{v_1^2}{2}$

Also note that we should work with $h^{-1}(\cdot)$ but this represents no problem since $h(\cdot)$ is assumed bijective.

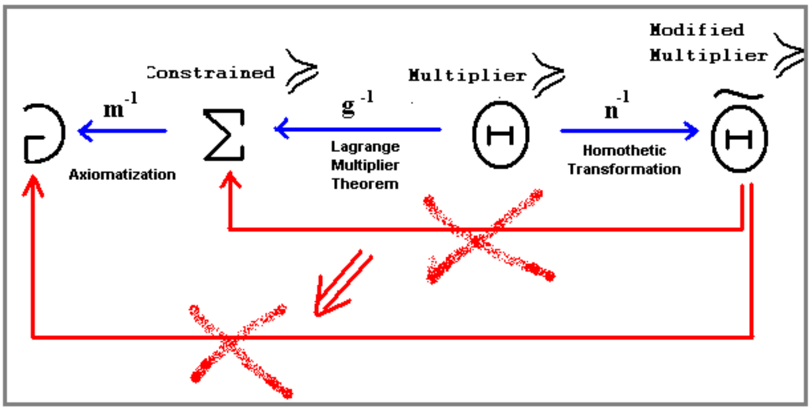


Figure 3:

3.2 Note on Phatak (2002)'s Critique on Maenhout (2004)'s Homothetic Transformation

Phatak (2002) argued that Maenhout (2004)'s homothetic transformation break the link with Gilboa and Schmeidler (1989) atemporal axiomatization. If such a single transformation broke the link, one can argue that the sort of transformations proposed by this dissertation will force us to find another axiomatization in substitution of the one provided by Gilboa and Schmeidler (1989). Fortunately this is not the case given the simple modification that we to the HST-HSW setup: a linear additive transformation.

3.3 Some Comments on the Functional Analysis Exposition and Related Issues

We are now in position to make some comments about the analysis made so far about modelling economic uncertainty:

1) The parameters of the sets Θ and Γ (like θ and ψ) may have asymmetric treatment, i.e., a bijection within two sets does not guarantee the others set will also be related by bijections.

2) We should avoid "anything goes" solution by placing a reasonable restriction on the type of the additional misspecification added to the model. Specifically we can have the same entropy constraint for the new v_2.

3) The structure of modelling may allow for N version of the Certainty Equivalent Principle (CEP), one for each free-parameter introduced in the analysis. Each version of CEP has an analytical translation: it is a sufficient condition for the mapping $f(\cdot)$ to be bijective. As it will be seen below this is not a necessary condition.

4) What is the limit of the dimension N for the eight listed vectors (Ә,Þ, Σ, Θ,Ł, Γ, E,Ω)? We need to work with a reasonable value for N. The answer may depend heavily on the set of equivalent economic parameters of interest (recall that even in HST and HSW - one dimensional treatment the application of robust control is environment specific).

5) The relevant mathematical technique for mastering each of the mapping between the eight relevant sets should be similar to the tools used by HST and HSW. The most important mappings for our purposes are $f(\cdot)$ **from** Θ **to** Γ and $g(\cdot)$ **from** Σ **to** Θ.

6) The literature has virtually overlooked the issue of multidimensionality in economic application of robust control. See Maenhout (2004) and Uppal and Wang (2003). In particular Uppal and Wang (2003) worked with different degrees of model uncertainty for each risky asset of the portfolio allocation problem. But they still used a single parameter. Ideally our work can be combined with theirs to provide a Matrix Θ of element $\theta_{ij}; 1 \leq i \leq N; 1 \leq j \leq J$, where $N$is the number of free parameters and J is the number of risky assets.

7) It is important to be aware of the various possibilities of properties for the seven mappings ($l(\cdot), g(\cdot),\ f(\cdot),\ h(\cdot), t(\cdot), m(\cdot)$ and $s(\cdot)$). The most comfortable situation would be to obtain all of these mappings to be bijective (i.e. they are both injective and surjective). Again, the results below will prove that this in not the case.

8) Each mapping is supported by well established theorems of functional analysis**.** For example the relationship $g(\cdot)$ from Σ to Θ is based on the Lagrange Multiplier Theorem. It is desirable to make explicit use of each one of these theorems.

9) HST and HSW name the control optimizations as Constraint Problem and Multiplier Problem. They can be cast in our terminology. See chapters 6 and 7 of the Robustness of Hansen and Sargent (2007).

10) Maenhout (2004) does an extension of HST treatment by adding a mod-

ified set $\widetilde{\Theta} = \left(\widetilde{\theta}_1, \cdots, \widetilde{\theta}_N\right)$ for the vector of technical (extra or free) parameters in the robust control problem. He worked in a continuos time framework. It is the case that in a **continuos time** setting it may be easier to see how multidimensionality will play a role in the model set up. This mean that instead of augmenting the term dZ_t in Maenhout (2004) to $(g_t dt + d\widehat{Z}_t)$ one should act directly in the dZ_t term. One would also need to model the Permanent Income Model with habit in a continuos time setting. It is also arguable that the misspecification term may enter the diffusion term (instead of the drift term). But Hansen, Sargent, Turmuhambetova and Williams (2004) proved that an absolute continuity property prevented the modelling of misspecification in the diffusion term.

Maenhout (2004) extends HST by allowing a homothetic transformation of the distortion and it has the following punch line: it provided a heuristic argument for the portfolio rule (α) to be homothetic, namely that the formula for the optimal portfolio rule does resemble the classical result of Merton's optimal portfolio if one let the distortion to be written as $\Psi_t = \frac{\theta}{(1-\gamma)V_t}$, i.e., the distortion is proportional to the inverse of the continuation value function V_t. That worked smoothingly in a power settings $(V_t(W) = kW^{1-\gamma})$. Hence his justification for a homothetic transformation comes for seeing what the model needs to present in order to obey neoclassical results. One can extend his pragmatic procedure and see the holes of current theory with respect to a list of empirical puzzles $E = (e_1, \cdots, e_N)$.

Note that a particular puzzle (the volatility puzzles as named in Campbell (2000) seems to be linked to a change in the volatility of the diffusion term dZ_t (in sharp contrast to comment 10 above).

11) The elaboration of the last argument is to allow for misspecification to enter not only the conditional mean of distortion (first moment) or the diffusion term (second moment) but **the whole probability distribution function** of the innovation. This is technically demanding but it has some useful insights. It will allow for distorting higher order moments. This can be done using Markov chains. Hansen and Sargent (2007) work in a way that provide some insights about that issue: they worked by choosing a Gaussian distribution with the same mean distortion.

4 Bidimensional Robust and Risk Sensitive Controllers

4.1 Motivation for a Second Free Parameter (θ_2)

The fundamental question to be addressed is: what is the adequate **representation of uncertainty** in economic problems?

The missing element on the relevant literature on model uncertainty is that modelling fear of misspecification should be **multidimensional**, i.e., the concern about model uncertainty should be represented by a **vector** of parameters. This dissertation takes a first step by designing a by-dimensional robust control problem and stress the claim that Knightian uncertainty is essentially multidimensional: it cannot be expressed (represented) in terms of a single parameter.

We argue that in order to get a higher degree of freedom for explaining more than a single empirical puzzle (like HST and HSW did for the equity premium puzzle) one needs to find a second free parameter in the framework of robust control and risk sensitive control problems. In other words we try to get an additional dimension and work with a vector (θ_1, θ_2) of uncertainty parameters. As stated in the functional analysis section the candidates are: the risk-free rate puzzle for (e_2) and EIS for (γ_2).

The EIS is a possible and reasonable candidate for γ_2 due to many reasons:

i) it is tightly related to the CRRA in power utility functions ($EIS = CRRA^{-1}$), such that breaking this link gives some hope to explain both e_1 and e_2

ii) It appeared at Epstein-Zin (1989)'s main equation.

Remark 25 *Indeed there is only two economic parameters (EIS and β) in Epstein-Zin (1989)'s setup which may suggest that we may not be able to come with another (third) parameter for modelling uncertainty, at least for the framework that we are working with. In other words, the number of economic parameters presented in the main equations may work as an upper bound for the dimensionality of uncertainty modelling as worked here. (recall that we need to obtain a mapping between elements of Γ and elements of Θ).*

Remark 26 *It is possible to model uncertainty by attaching a new distortion v_2 to a new (standard) restriction, say a liquidity constraint (like the one used in the literature of the equity premium puzzle). Yet this modelling can be interpreted in the framework that we are working with: if a measure of liquidity is a state variable than adding a new (liquidity) constraint is just like enhancing the vector of state variables.*

4.1.1 Fama and French's Multifactor Model and the Role of Multidimension Risk

The idea treated here has some similarity with the extension to the CAPM. This equilibrium pricing equation uses a single factor, namely the market (and associated with a parameter called beta of the asset), to quantify the return of a single asset (or portfolio of assets). Fama and French (1993 and 1996) extends this work for three factors, i.e., they added two factors to the regression model (this supposedly would improve the fitness with a better r-squared).

Two observations motivated the work by Fama and French: small caps and stocks with a high book-value-to-price ratio (called value stocks; as opposed to the so called growth stocks) have tended to do better than the market taken as a whole. So they add two extra factors in order to reflect the exposure of the portfolio's exposure to these two classes of assets. Their equation can be summarized as:

$$r - R_f = \alpha + beta_3(K_m - R_f) + b_s SMB + b_v HML$$

$$\text{Where: } \begin{cases} r = \text{ rate of return of the portfolio} \\ R_f = \text{risk-free rate of return} \\ K_m = \text{return of the whole stock market} \\ SMB = \text{small [cap] minus big; } 0 < b_s < 1 \\ HML = \text{high [book/price] minus low; } 0 < b_v < 1 \end{cases}$$

It is noticeable a weakness in explaining the intuition for dealing with 3 dimensions for the risk feature (instead of a single factor). Two factors are reasonably linked to economic issues but the third factor is hardly accepted as economic meaningful.

But one can argue that we have a essentially multifactor Arbitrage, i.e. the notion of arbitrage is intrinsically multidimensional.

Hence our defense of use of multidimensional uncertainty may be compared with the use of multidimensional risk. But the not consensual declared success of models that treat risk as multidimensional in explaining empirical puzzles encourage us to treat uncertainty as a multidimensional feature.

For the issue of why care about modelling uncertainty in a multidimensional way there is a reference that show the limited use and applications of multidimensionality of the attitude toward risk: Karni and Schmeidler (1981) analyze the use of multivariate risk aversion. Their main conclusion is that to put a meaning in comparison of multivariate risk aversions it is necessary to deal with the following problem: the individual is more risk averse the smaller is his certainty equivalence for a given risk. If u and v are two von Neumann-Mongestern utility functions on $\mathbb{R}^n_+$, it is possible to state that u displays greater multivariate risk aversion than v if $C^u(W) \geq C^v(W)$ for all $W \in W$, the set of random vectors with $E[u(W)]$ been finite, and for C^u representing the certainty equivalent for u, i.e., $u(C) = E[u(W)]$. But for this definition to make sense we need u and v to represent the same preference relation. If they do not have this feature then the certainty equivalent measures are non-comparable. Or in other words, if the utility functions u and v represent different ordinal preferences then the comparison of certainty equivalences is dependent on the direction in which they are measured. Kilhstrom and Mirmam (1981) proved that a necessary condition for comparability is for the preference relations to be homothetic.

4.2 Possible Classes of Functions and Nesting Conditions with Multidimensionality

The purpose of this section is to study the criteria for altering the modelling of economic uncertainty as proposed by HST and HSW. The main point is that the proposed model must obey some rules to be considered worth pursuing the effort. We propose to modify the distorted law of motion for the state vector x_t to be:

$$x_{t+1} = Ax_t + Bi_t + C(w_{t+1} + v_{1t} + v_{2t})$$

This seems to be the most simple and natural way for adding a new parameter to express agent's fear of model misspecification. It is done in an additive and linear fashion.

There are many alternative candidates to this distorted law of motion for x_t. Let's represent by $F(v_{1t}, v_{2t})$ the various candidate functions for relating the last part of the law of motion which is composed by the volatility matrix C, the noise w_{t+1}, and the two distortions v_{1t} and v_{2t}. We will stick to this procedure eventough it may be worth pursuing other modelling such as having a misspecification of the return function instead of the transition law. Also note that each different $F(.)$ will imply a different distorted transition probability density for x_t.

The core part of the law of motion that interest us is:

$$x_{t+1} - Ax_t - Bi_t - Cw_{t+1} = f(C, F(v_{1t}, v_{2t}))$$

or in a slightly different form:

$$C^{-1}(x_{t+1} - Ax_t - Bi_t - Cw_{t+1}) = F(v_{1t}, v_{2t})$$

Some examples of possible $F(.)$ are:
i) $C(w_{t+1} + v_{1t} - v_{2t})$
ii) $C(w_{t+1} + v_{1t} \cdot v_{2t})$
iii) $C(w_{t+1} + v_{1t}) \cdot v_{2t}$
iv) $C(w_{t+1} + v_{1t})/v_{2t}$
v) $C(w_{t+1} + \exp v_{1t} + \exp v_{2t})$
vi) $C(w_{t+1} + \exp(v_{1t} + v_{2t})$
vii) $C(w_{t+1} + 1/v_{1t} + 1/v_{2t})$
viii) $C(w_{t+1} + \ln v_{1t} + \ln v_{2t})$
ix) $C(w_{t+1} + \ln v_{1t} \cdot \ln v_{2t})$
x) $C(w_{t+1} + |v_{1t}| + |v_{2t}|)$
xi) $C(w_{t+1} + v_{1t}^2 + v_{2t}^2)$
xii) $C(w_{t+1} + v_{1t}^n + v_{2t}^n), n > 2$ and $n \in Z$
xiii) $C(w_{t+1} + \max(v_{1t}, v_{2t}))$
xiii) $C(w_{t+1} + \min(v_{1t}, v_{2t}))$
xiv) $C(w_{t+1} + v_{1t}^{v_{2t}})$

It is easy to come up with combination of the above alternatives (these are considered elementary functions). Indeed the number of possibilities is infinite.

Note that we are not modifying the simple additive relation between C and the distortions: we can proposed a format like $C(w_{t+1} \cdot F(v_{1t}, v_{2t}))$.This is a very modification but that brings about considerable mathematical difficulties to be handle in the model. If the distortion takes a multiplicative format we will need a new type of normalization, such that C'C will be the Covariance Matrix.

The **nine properties** that we propose to be satisfied by $F(v_{1t}, v_{2t})$ are:

1) Nesting for the law of motion for the state vector. This is easily checked in our proposed additive model (since as $v_{2t} = 0$ we get to HST-HSW model).

2) No destruction of the observation equivalence result (OER) of the elements of Θ and Γ.

2.1.) Nesting of the OER for the bidimensional case. This is obtained in our model since the discount rate β is just a linear function of $\sigma_1 + \sigma_2$.

3) Nesting formula for the optimal value for $v = (v_{1t}, v_{2t})$.

4) Multiplicative representation for the stochastic discount factor (SDF), i.e., $m_{t,t+1} = m^f_{t,t+1} m^u_{t,t+1}$. When evaluating asset prices under the approximating model we want to adjust the SDF in a simple manner. The most simple way in the LQG setup is this multiplicative adjustment.

5) Nesting for the SDF: $E_t m^u_{t,t+1} = 1$ when $v_{2t} = 0$.

6) The estimated value for (σ_1, σ_2) should not be beyond the breakdown point $\underline{\sigma}$ (a concern of the risk sensitive control theory). See HST footnote 11. This condition is likely to satisfy a less stringent one that the implied values for the measure of risk aversion be an empirically reasonable one.

7) The link between the market price of Knightian uncertainty (MPU) and the detection error probability (DEP) is maintained. This can be relaxed but we will need to come up with another statistical detection method to substitute DEP.

8) The modelling must preserve the fact that robustness parameters (θ_1, θ_2) will affect only the mean of the disturbance (but not the variance), i.e., $v = (v_{1t}, v_{2t})$ is still just a mean distortion. In a continuous time setting Anderson, Hansen and Sargent (2003) proved that the one-dimensional approach implies that the perturbation v can alter the drift but not the volatility of the diffusion (because it is infinitely cost in terms of entropy).

9) Ensure that "stochastic singularity" is avoided. The estimating strategy adopted by HST and HSW used two observed time series, namely (c, i), so their econometric specification needed at least two shock processes to avoid stochastic singularity. They accomplished that by specifying a bivariate or two-factor stochastic process for the endowment: $d_t = \mu_d + d^1_{t+1} + d^2_{t+1}$. (stochastic singularity is a spectral density of full rank for the observable vector of variables (c, i) for which it is constructed the (log) likelihood function used for estimating the free parameters of the model). Note that in our case we continue to use the same two observed series. Hence modelling the endowment with four shock processes $d_t = \mu_d + d^1_{t+1} + d^2_{t+1} + d^3_{t+1} + d^4_{t+1}$ will make even more easy satisfying this criteria.

Remark 27 *Some straightforward and boring algebra is able to rule out all the proposed candidates except the linear additive one. The rest of this dissertation* *shows that the additive solution obey all nine criteria. Failure for a candidate function is declared if it violates at least one of the nine criteria listed in this section.*

To illustrate how the possibilities of alternatives that obey the criteria is really constrained we state below a famous theorem (Cauchy's Equation) that show that even working we the simple additive function we may have a stringent restriction on its format.

Theorem 28 *The set of continuos functions defined in* $\mathbb{R} \to \mathbb{R}$ *such that* $f(x+y) = f(x) + f(y)$ *, for real* x *and* y*, is given by* $f(x) = x.f(1)$.
Proof. *Start by working with* $x = 0$ *to get that* $f(0) = 0$*. Then show that* $f(.)$ *is odd, i.e.,* $f(x) = -f(x)$*. For* $n \in N$ *show that* $f(n.x) = n.f(x)$*. The same applies for* $x \in \mathbb{Q}$*. For* $x \in \mathbb{Q}^c$ *use the the fact that it is continuos and take a sequence of rational numbers to obtain that* $f(x) = x.f(1)$*.* ■

Sometimes it is necessary to state that the effect of the distortion v_1 and v_2 in the state variable are independent. Since we are studying diverse classes of (v_1, v_2) and since orthogonality is implied by independence it is important to define what are orthogonal functions.

Definition 29 *Two functions* $f(v)$*and* $g(v)$ *are orthogonal over the interval* $a \leq v \leq b$ *with weighting function* $\varphi(v)$ *if* $\langle f \mid g \rangle = \int\limits_a^b f(v)g(v)\varphi(v)dv = 0$.

Remark 30 *The desirable result is to show what are the admissible classes of functions within all the existing functions that obey all criteria listed above. This is a very demanding question (to say a minimum about it). For our purposes it suffices to prove that the most common family functions fail at least one of the criteria and that the specification we work with is the only one that satisfies all stated criteria.*

4.3 Static Setting Case

This section will provide the basis insight of working with bidimensionality to model economic uncertainty. We choose a static setting because it is sufficient to provide the main results that a dynamic counterpart will do with a much more easy mathematics involved).

4.3.1 State Space and Preference Description

Let's assume that the law of motion for the state variable is given by

$$x = Ax_0 + Bi + C(w + v_1 + v_2)$$

and that the quadratic utility function is given as:

$$u(i, x) = -Qi^2 - Rx^2$$

This is a standard LQG setup where x is the state variable, i is the control variable, x_0 is a fixed initial value of x, w is the noise with w following a $N(O, 1)$ and v_1 and v_2 are the two additive mean distortion to w. The matrices used in the two equations above are assumed to obey: $A \neq 0, B \neq 0, C \neq 0, Q > 0, R > 0$.

If we have the standard optimal control setup (without robustness concern and without risk-sensitivity) it is straighforward to show that we can rewrite $Eu = -\left[Qi^2 + R(Ax + Bi)^2\right] - RC^2$. This will give the following optimal solution for the control variable:

$$i^* = -(Q + B^2R)^{-1}(ABR)x$$

Note that in this case we have standard preferences (a la vonNeumann-Mongestern) and distortions taken to be zero. Moreover note that even in this setup the control variable can be written as $i^* = -Fx$, with $F = (Q + B^2R)^{-1}(ABR)$. This gives some hint for the importance of feedbackin in modelling uncertainty.

4.3.2 Risk-Sensitive Control Problem

The setup of the problem is: we do not work with the standard preferences but instead maximize an exponential certainty equivalent of utility U as defined below. This change of preference representation is in the tradition of the recusive utility literature pioneered by Kreps and Porteus (1979), Chew (1983, 1989), Dekel (1986) and latter developed by Eptein-Zin (1989) and related papers. For the law of motion of the state vector we make no change (we work as if the distortions are all zero).

Definition 31 *Suppose that the state x is drawn with probability $p(x)$ from a finite set $X = \{1, 2, \cdots, X\}$. The certainty equivalent or mean value functional μ of a set of state-contingent consequences (for example consumption $c(x)$) is a certain consequence that give the same level of utility: $U(\mu, \cdots, \mu) = U(c(1), \cdots, c(X))$. The functional μ represents the same preferences as U.*

The agent maximizes the following exponential certainty equivalent of $U_t = V[u_t, \mu(U_{t+1})]$ which in this static case become $V[u, \mu(U)]$:

$$\mu(U) = -(\theta_1 + \theta_2)^{-1} \log E \{\exp[-(\theta_1 + \theta_2)U]\}$$

where (θ_1, θ_2) are parameters linked to the idea of risk aversion (not uncertainty). Note that when $\theta_2 = 0$ we get to the one-dimensional case.

Remark 32 *The way of modelling uncertainty stated as* $U_t = V[u_t, \mu(f(\theta_1, \theta_2), U_{t+1}), \mu_2(U_{t+1})]$ *above is superior (in all senses of the nine criteria stated above in this section) when compared with a modelling such as* $U_t = V[u_t, \mu_1(\theta_1, U_{t+1}), \mu_2(\theta_2, U_{t+1})]$ *that assume separability of* (θ_1, θ_2) *in two different mean-value functionals* (μ_1, μ_2)*. If we choose this second modelling we will face problems in defining two discount factors* β*. More on this below. This result reinforces the result to be stated in the next subsection that* $\beta = \beta(\sigma_1 + \sigma_2)$*.*

Remark 33 *Elaborating a little on the above remark: we cannot have two different* $\mu_1(\theta_1, U_{t+1})$ *and* $\mu_2(\theta_2, U_{t+1})$ *since by definition (see Epstein and Zin (1989), page 944) the mean value functional* μ *is a certainty equivalent for random future utility (and it does not make economic sense to have two different mean values for the same concept).*

The way we choose to model uncertainty allow us to use the nice proporteis of the exponential functions when dealing with normal-exponential integrals. In particular one can show that

$$\mu(U) = -\frac{1}{2} \log \left[1 - 2(\theta_1 + \theta_2)RC^2\right] - \left[Qi^2 + \frac{R}{1 - 2(\theta_1 + \theta_2)RC^2} (Ax_0 + Bi)^2\right]$$

Where we assum that $1 - 2(\theta_1 + \theta_2)RC^2 > 0$ or equivalently $\theta_1 + \theta_2 < \frac{1}{2RC^2}$ which is an upper bound on the value of the sum of the risk aversion parameters.

Note that $1 - 2(\theta_1 + \theta_2)RC^2 < 1$ since $(\theta_1 + \theta_2) \geq 0, R > 0$, and $C > 0$. This allow us to compare the risk sensitive control solution with the standard optimal control solution.

Solving $\max_i \{\mu(U)\}$ we find that:

$$\frac{\partial \mu(U)}{\partial i} = 0 \Rightarrow 2Qi + \frac{R}{1 - 2(\theta_1 + \theta_2)RC^2} 2 (Ax_0 + Bi) B = 0 \Rightarrow$$

$$i^* = -(Q + B^2R - 2(\theta_1 + \theta_2)QRC^2)^{-1}(ABR)x_0$$

Note that:

a) $i^* = f(\theta_1, \theta_2)$,

b) $i^* = -Fx$, and

c) The signal of AB is of high importance.

We have that for a fixed control variable i the relation between θ_1 and θ_2 is an affine function. This suggest that we can define a new concept that is

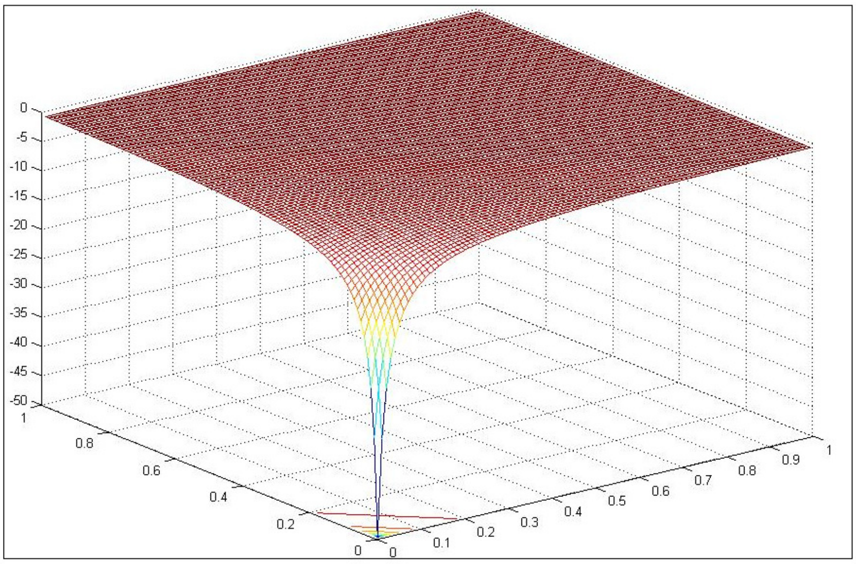


Figure 4:

valid only in the realm of multidimensional modelling of uncertainty: isocontrol curve, the locus where i takes a particular (constant) value regardless of the values of the risk sensitive parameters.

Definition 34 *The Isocontrol Curve is defined as a projection of the curve $i = i(\theta_1, \theta_2)$ for a particular constant value for the control variable i. This curve is in fact represented by a line in the (θ_1, θ_2) plane.* [5]

Figure XXX illustrates the aforementioned concept. Note that the shapes of the two curves $i = f(\theta_1)$ holding θ_2 fixed and $i = g(\theta_2)$ holding θ_1 fixed, for two different function $f(.)$ and $g(.)$, are similar to a hyperbole which makes the particular format of the isocontrol curve, i.e. a line, a surprising result. The figure was drawn using the Matlab command meshc which draws the level curves (in our case these are the projection of the isocontrol curves in the (θ_1, θ_2) plane.

It is worth mentioning that the isocontrol curves will also be present in the robust control framework but it will be stated as $i = i(v_1, v_2)$, i.e, the control variable as a funtion of the distortions (instead of the uncertainty parameters themselves).

Lemma 35 *The expression for the exponential certainty equivalent $\mu(U) = -(\theta)^{-1} \log E \{\exp[-\theta U]\}$ used here (and in Backus, Routledge and Zin (2004))*

[5] The name isocontrol was suggested by Paulo Cesar Coutinho.

is similar to the power exponential certainty equivalent $\mu(U) = [E(U^\theta)]^{\frac{1}{\theta}}$ *used in Epstein-Zin (1989) and related papers (like Tallarini (1996,2000).*

Proof. *Just take logs and recollect terms.* ■

How Modelling of Uncertainty will not Work Let's now formalize a comment made above.

Proposition 36 *Modelling uncertainty in the risk sensitive framework with nonseparable mean value funtional is superior ($\succ$) to modelling that assume separability, i.e.,*

$$U_t = V[u_t, \mu(f(\theta_1, \theta_2), U_{t+1}), \mu_2(U_{t+1})] \succ U_t = V[u_t, \mu_1(\theta_1, U_{t+1}), \mu_2(\theta_2, U_{t+1})]$$

The separability of (θ_1, θ_2) in two different mean-value functionals (μ_1, μ_2) of the second approach will deliver results that will fail to satisfy some of the nine criteria for modelling uncertainty proposed by this dissertation.

Proof. *The approach $U_t = V[u_t, \mu_1(\theta_1, U_{t+1}), \mu_2(\theta_2, U_{t+1})]$ will deliver $U_t = u_t + \beta_1\mu_1(\theta_1, U_{t+1}) + \beta_1\mu_2(\theta_2, U_{t+1})$. But this is problematic from the economic point of view (unless $\beta_1 = \beta_2$) since the discount factor does not have risk or uncertainty in its definition (it is not a concept to deal with risk preference or uncertainty preference but only with time preference). But even if we suppose that $\beta_1 = \beta_2 = \beta$ and proceed with the calculus (normal-exponential integrals) you get to:*

$$U_t = u_t + \beta \left\{ \begin{array}{c} -\frac{1}{2}\log\left[1 - 2\theta_1 RC^2\right] - \left[Qi^2 + \frac{R}{1-2\theta_1 RC^2}\left(Ax_0 + Bi\right)^2\right] + \\ -\frac{1}{2}\log\left[1 - 2\theta_2 RC^2\right] - \left[Qi^2 + \frac{R}{1-2\theta_1 RC^2}\left(Ax_0 + Bi\right)^2\right] \end{array} \right\}$$

The first order condition

Solving $\max_i \{\mu(U)\}$ we find after some tedious algebra that:

$$\frac{\partial\mu(U)}{\partial i} = 0 \Rightarrow i^* = -\left[\frac{(1-2\theta_1 RC^2)(1-2\theta_2 RC^2)}{1-\theta_1 RC^2 - \theta_2 RC^2}Q + B^2R\right]^{-1}(ABR)x_0$$

This resemble i=-Fx which is a good feature. But there also some drawbacks: the formula for i using this approach for modelling uncertainty does not became the formula for i in the one-dimensional case when we have that $\theta_2 = 0$, i.e., the nesting rule fails for the first order condition that gives the value for the optimum control variable. But this could be anticipated because by construction the parameter θ_2 cannot be equal to zero since it will make the mean value functional not well defined (in the first place). ■

4.3.3 Robust Control Problem

We have the following unconstrained problem to solve

$$\max_{i}\min_{v_1}\min_{v_2}\left\{Eu+\theta_1 v_1^2+\theta_2 v_2^2\right\}$$

where we took θ_1 and θ_2 as the parameters that governs the penalties added to the malevolent agent (nature) in its two versions. Those penalties discipline the way we introduce uncertainty in the model, i.e., they limit how much the "player" nature distorts the model. Small values of θ_1 and θ_2 means weaker limits on nature actions.[6]

The constrained version of the problem had the following equations:

$$\begin{cases} v_1^2 \leq \eta_1 & ,(\theta_1) \\ v_2^2 \leq \eta_2 & ,(\theta_2) \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are the Lagrange multipliers.

We can write that (recall that $Ew = 0$):

$$\begin{aligned} Eu &= E(-Qi^2 - Rx^2) = -Qi^2 - R\left[Ax_0 + Bi + Cv_1 + Cv_2\right)]^2 = \\ &= -Qi^2 - R\left(Ax_0 + Bi\right)^2 - 2R\left(Ax_0 + Bi\right)Cv_1 - 2R\left(Ax_0 + Bi\right)Cv_2 \\ &\quad -2RC^2 v_1 v_2 - RC^2 v_1^2 - RC^2 v_2^2 \end{aligned}$$

Note the presence of a cross-term $-2RC^2 v_1 v_2$.

The two first order conditions are (stars denote optima):

a) $\frac{\partial \pounds}{\partial v_1} = 0 \Rightarrow$

$$v_1^* = (\theta_1 - RC^2)^{-1}[R\left(Ax_0 + Bi\right)C - RC^2 v_2]$$

Note that when $v_2 = 0$ we can recover the one-dimensional solution (i.e., nesting is obeyed for this equation).

b) $\frac{\partial \pounds}{\partial v_2} = 0 \Rightarrow$

$$v_2^* = (\theta_2 - RC^2)^{-1}[R\left(Ax_0 + Bi\right)C - RC^2 v_1]$$

Note that when $v_1 = 0$ we can recover the one-dimensional solution (i.e., nesting is obeyed for this equation).

Remark 37 *The term $-(\theta_1 - RC^2)^{-1}RC^2 v_2$ that appears in the first F.O.C. can be written as $\frac{v_2}{1-\frac{\theta_1}{RC^2}}$ which is <0 if we assume that there exists a lower bound on θ_1, i.e., $\theta_1 > RC^2 > 0$. Similar results apply to the last term in the second F.O.C. assuming $\theta_2 > RC^2 > 0$.*

[6] We obtained a max min min solution which differs a little of the max min scheme of HST and HSW. There are versions of max max for related economic decision problems. See Resende (2006) and Schmeidler and XXX (1999XXX).

Note the affine relationship between the distortions v_1^* and v_2^* (at the optimum values). This will have impact on the relationship between θ_1 and θ_2 (compare to the relationship between σ_1 and σ_2 for the risk sensitive problem). It is also an affine relationship.

Consider a constant value for the control variable, say $i = i^*$.

Then we have:

$$v_2^* = (\theta_2 - RC^2)^{-1}[R(Ax_0 + Bi^*)C - RC^2 v_2^*] = \frac{R(Ax_0 + Bi^*)C}{(\theta_2 - RC^2)} - \frac{RC^2}{(\theta_2 - RC^2)} v_1^*$$

which can be synthetically be stated as:

Lemma 38 *In the two-dimensional static case the distortions are related by an affine function.*

Lemma 39 $v_2^* = i + jv_1^*$, *where* $i = \frac{R(Ax_0+Bi^*)C}{(\theta_2 - RC^2)} \gtreqless 0$ *and* $j = -\frac{RC^2}{(\theta_2 - RC^2)} < 0$. *Similar results apply to the other distortion* $v_1^* = f(v_2^*)$.

Proof. *Immediate from the stated above. The sign of i depends on the magnitude of A, B and* i^*. ■

Remark 40 *For the special case that* θ_1 *and* θ_2 *are equal we have that* $v_1^* + v_2^* = \frac{Ax_0 + Bi^*}{C}$, *implying that the decision rule will not be influenced by the distortions (a degenerate case for feedback control). The effect of one distortion is canceled out by the other one (except for a constant term).*

Given this affine relationship between the distortions v_1^* and v_2^* we can draw a figure to represent curves where the level of the control variable is the same (regardless of particular values of the distortions). The definition below follow the similar one stated for the risk sensitive control problem.

Definition 41 *The Isocontrol Curve is defined as a projection of the curve* $i = i(v_1, v_2)$ *for a particular constant value for the control variable i. This curve is in fact represented by a line in the* (v_1, v_2) *plane.*

Substitute out the values for v_1^* and v_2^* in the robust control objective function we obtain the following expression:

$$\max_i \left\{ \begin{array}{c} -Qi^2 - R(Ax_0 + Bi)^2 - 2R(Ax_0 + Bi)Cv_1^* - 2R(Ax_0 + Bi)Cv_2^* - 2RC^2 v_1^* v_2^* - RC^2 (v_1^*)^2 \\ -RC^2 (v_2^*)^2 + \theta_1 (v_1^*)^2 + \theta_2 (v_2^*)^2 \end{array} \right\} =$$

$$= \max_i \left\{ -\left[Qi^2 + \left(\frac{R}{1 - \frac{RC^2}{(\theta_1 + \theta_2)}}\right)(Ax_0 + Bi)^2 \right] \right\} - RC^2$$

Comparing the above equation with the expression for i in the risk sensitive control problem we note that we obtain the same result for the optimum control variable if we set $(\theta_1 + \theta_2) = -(\sigma_1 + \sigma_2)^{-1}$. This is a very strinklingy simple relation between (θ_1, θ_2) and (σ_1, σ_2) but note that the hypothesis of bijection between sets Θ and Σ is rejected.

Remark 42 *Since we obtained that $v_2^* = f(v_1^*)$ one may suggest that we substitute this at the original formulation and work only with one uncertainty parameter v_1^* which will make us dismiss the power or even the use of bi-dimensionality. But this is not appropriate since the value $v_2^* = f(v_1^*)$ are valid only at the optimum (it come from a first order condition).*

4.4 Relation between β and (σ_1, σ_2)

The choice followed by this dissertation of modelling uncertainty using two paramaters is fully compatible with the laboraty economy of the permanent income growth model that HST and HSW used for their one-dimensional case. As in their setup the quantities (c,i) will not be affected but there are important asset pricing modifications via-a-vis a model with no robustness concerns. Hence we will keep all the results of section 2.5 while deriving a new relation between the discount factor and the uncertainty parameters.

Recall that we have a martingale representation for the shadow price of consumption services

$$\mu_{st} = \mu_{st-1} + \surd' w_t$$

This imply a similar martingale representation for the other shadow prices: μ_{ht} and μ_{ct}.

To compute the relation between β and $(\sigma_{1,}\sigma_2)$ we first state the program to be solved (using μ_{st} as the state variable).

$$\Omega x^2 = -x^2 + \beta \min_v \left[-\frac{1}{\sigma_1} v_1^2 - \frac{1}{\sigma_2} v_2^2 + \Omega [x + \alpha(v_1 + v_2)]^2 \right]$$

where $\mu_{st} = \mu_{st-1} + \alpha(v + w)$, with $\alpha^2 = \surd' \surd$

The two first order conditions are:

i) $\frac{\partial(.)}{\partial v_1} = 0 \Rightarrow \frac{v_1}{\sigma_1} = \Omega[x + \alpha(v_1 + v_2)]\alpha \Rightarrow v_1(1 - \sigma_1 \alpha^2 \Omega) = \sigma_1 \alpha \Omega x + \sigma_1 \alpha^2 \Omega v_2$

ii) $\frac{\partial(.)}{\partial v_2} = 0 \Rightarrow \frac{v_2}{\sigma_2} = \Omega[x + \alpha(v_1 + v_2)]\alpha \Rightarrow v_2 = \frac{\sigma_2}{\sigma_1} v_1$

Remark 43 *The relation $v_2 = \frac{\sigma_2}{\sigma_1} v_1$ confirms our conjecture that we can draw an isocontrol curve for the space (σ_1, σ_2) in the same fashion that we do for the space (v_1, v_2).*

Then we have

$$v_1 = \frac{\sigma_1 \alpha \Omega}{1 - \sigma_1 \alpha^2 \Omega - \sigma_2 \alpha^2 \Omega} x$$

Note that the above expression nests the one-dimensional case (which was written as $v = \frac{\sigma \alpha \Omega}{1 - \sigma \alpha^2 \Omega} x$). This implies that

$$v_2 = \frac{\sigma_2 \alpha \Omega}{1 - \sigma_1 \alpha^2 \Omega - \sigma_2 \alpha^2 \Omega} x$$

Substituting out those values for v_1 and v_2 in the control program we have

$$\begin{aligned} \Omega x^2 &= -x^2 - \beta \frac{1}{\sigma_1} \left(\frac{\sigma_1 \alpha \Omega}{1 - \sigma_1 \alpha^2 \Omega - \sigma_2 \alpha^2 \Omega} x \right)^2 - \beta \frac{1}{\sigma_2} \left(\frac{\sigma_2 \alpha \Omega}{1 - \sigma_1 \alpha^2 \Omega - \sigma_2 \alpha^2 \Omega} x \right)^2 \\ &\quad + \beta \Omega \left[x + \alpha \left(\frac{\sigma_1 + \sigma_2}{\sigma_1} \right) \frac{\sigma_1 \alpha \Omega}{1 - \sigma_1 \alpha^2 \Omega - \sigma_2 \alpha^2 \Omega} x \right]^2 \end{aligned}$$

The above expression allows us to eliminate x^2 which is very convenient (since the solution will be free of the assumption made about what is appropriate state vector to consider).

Rewriting the above problem will deliver

$$(-\sigma_1\alpha^2 - \sigma_1\alpha^2)\Omega^2 + (\beta - 1 + \sigma_1\alpha^2 + \sigma_2\alpha^2)\Omega + 1 = 0$$

which allow us to write Ω as $\Omega(\beta)$.

Compute the distorted expectation operator used in the martingale representation of the shadow price μ_{st} : :

$$\widehat{E}_t\mu_{st+1} = \zeta\mu_{st}$$

with

$$\widehat{\zeta} = \widehat{\zeta}(\beta) = \frac{1}{1 - \sigma\alpha^2\Omega(\beta)} = 1 + \frac{\sigma\alpha^2\Omega(\beta)}{1 - \sigma\alpha^2\Omega(\beta)}$$

Given the martingale representation for consumption we also have:

$$\widehat{E}_t\mu_{ct+1} = \zeta\mu_{ct}$$

This give us an Euler Equation for consumption. Now take $\widehat{\beta}R\widehat{\zeta}(\widehat{\beta}) = 1$. Then we have

$$\widehat{\beta}R\widehat{E}_t\mu_{ct+1} = \mu_{ct}$$

To find an explicit equation for $\widehat{\beta}$ work with the quadratic forms to obtain:

$$\widehat{\beta} = \frac{1}{R} + \frac{\alpha^2}{R-1}(\sigma_1 + \sigma_1)$$

which shows that β has a non bijective relation with (σ_1, σ_2) altough it shows a nice simple mapping.

Remark 44 *for a fixed β we can see that we obtain an affine relationship between (σ_1, σ_2). We can write: $\sigma_2 = \left(\overline{\beta} - \frac{1}{R}\right)\left(\frac{R-1}{\alpha^2}\right) - \sigma_1$, showing an inverse relation between uncertainty parameters. Also note that the space $(\beta, \sigma_1, \sigma_2)$ has a representation a bit different when compared with space (i, v_1, v_2): both will show a line as a projection for fixed i or β, but the former space will display some linearities that the second will not.*

Remark 45 *The value for $\underline{\sigma_1 + \sigma_2}$ can be find by calculating the lowest value for $\sigma_1 + \sigma_1$ such that the quadratic equation above has a real solution. Just work with the discriminant of the equation (Δ) with the triple $\left(-\sigma_1\alpha^2 - \sigma_1\alpha^2, \beta - 1 + \sigma_1\alpha^2 + \sigma_2\alpha^2, 1\right)$.*

Remark 46 *HST and HSW argued that there is a observational equivalence for their one-dimensional case since for fixed the quantity data (c, i) we cannot simultaneously identify (β, σ). For our case holding fixed (c, i) will deliver an identification problem of higher magnitude since we will not be able to simultaneously identify $(\beta, \sigma_1, \sigma_2)$. By its turn the result of HST and HSW that there* *is a modified certaint equivalent result (in the sense that the decision rule i=-Fx will be the same with ou without robustness concern) will be kept. But it is a bit surprising that we do not come with an additional certainty equivalent result by adding a second uncertainty parameter.*

4.4.1 Stochastic Endowment Process and Potential Boost in Empirical Power

There is a potential boost in empirical power for the bi-dimensional uncertainty modelling case in terms of the fitting with data of the assumed stochastic process for the endowment d_t. In HST there was an explicit choice of not using the whole sample of post WWII data for the U.S. economy. Instead of using the available data at the time of their publication (from 1948 1996) they worked with the time span of 1970:1 to 1996:1. Their justification was that working with the whole sample they get worse result than working with only the second subsample due to earlier period of higher productivity. By using the whole sample HST would capture the productivity slowdown in their likelihood estimation by an initial slow decline in the b_t term of the utility function, that is followed by a slow increase. Hence they stated that "our illustrative permanent income model is not well suited to capture productivity slowdowns" (see HST footnote 16). By working with two uncertainty parameters we suggest that the endowment process can be assumed to follow a more rich process than just a sum of two AR(2) processes like HST did. Initially one can work with a sum of four AR(2) processes but there are other combinations to be analyzed to get a better match with data. The bottom line is that we get a higher degree of freedom to match data (by construction and not by a virtue of the bi-dimensional uncertainty modelling method).

Below we reproduce the HST-HSW workable assumption about d_t.

$$\left\{ \begin{array}{c} d_t = \mu_d + d^1_{t+1} + d^2_{t+1} \\ d^1_{t+1} = g_1 d^1_t + g_2 d^1_{t-1} + c_1 w^1_{t+1} = (\phi_1 + \phi_2) d^1_t - \phi_1 \phi_2 d^1_{t-1} + c_1 w^1_{t+1} \\ d^2_{t+1} = a_1 d^2_t + a_2 d^2_{t-1} + c_1 w^2_{t+1} = (\alpha_1 + \alpha_2) d^2_t + \alpha_1 \alpha_2 d^2_{t-1} + c_1 w^2_{t+1} \end{array} \right.$$

One possible modification is to assume $d_t = \mu_d + d^1_{t+1} + d^2_{t+1} + d^3_{t+1} + d^3_{t+1}$, i.e., endowment has four component: very persistent, mildly persistent, mildly transient and transient components.

Conjecture 47 *The use of two parameter modelling uncertainty allow for the machinery set forth by HST and HSW to use the whole WWWII sampla data for the U.S. economy.*

The law of motion for the state variable would be written in a very convenint way as the one-dimensional uncertainty modelling case. It is necessary however to change the state vector accordingly. One possible way is to set $x_t = (h_{t-1}, k_{t-1}, d_{t-1}, 1, d_t, d^1_t, d^1_{t-1}, d^2_t, d^2_{t-1}, d^3_t, d^3_{t-1})'$, i.e., we would get a 1x11 vector instead of the 1x7 vector of HST-HSW. Note that $\left(d^4_t, d^4_{t-1}\right)$ is implictly represented in x_t by displaying (d_t, d_{t+1}).

Remark 48 *Tallarini (1996,2000) and Barrilas et AL (2006) used the whole available data sample (from 1948:2 to 1993:4 and from 1948:2 to 2005:5, respectively) because they assumed different processes for the consumption stream (a random walk and a trend stationary processes).*

4.5 Market Price of Risk (MPR) and Multifactor Price of Uncertainty (MFPU)

Recall from the asset pricing section of chapter 2 that we have the following fundamental relations:

a) The asset pricing can be state as $a_0 = E_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} \cdot y(x_t)$, where there is a multiplicative adjustment to the ordinary SDF using the likelihood ratio $L_t = \frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} = \exp\left\{-\frac{v^2}{2} + vw\right\}$, i.e., the SDF it is given by $m_{0,t}\left[\frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)}\right]$

b) HST defined the one-period market price of Knightian uncertainty (MPU) as the standard deviation of the multiplicative adjustment, i.e., $std_t(m^u_{t,t+1})$.

c) Under the framework of robust control problem we have that $a_t = E_t\left(m^f_{t,t+1} m^u_{t,t+1}\right) y_{t+1}$, with $m^u_{t,t+1} = \exp\left\{-\frac{v'v}{2} + w'v\right\}$ and $MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]} = \sqrt{\exp(v'v) - 1}$

d) Under the framework of risk sensitive control problem we have that $a_t = E_t\left(m^f_{t,t+1} m^u_{t,t+1}\right) y_{t+1}$, with $m^r_{t,t+1} = \frac{\exp(\frac{\sigma}{2}U_{t+1})}{E_t\left[\exp(\frac{\sigma}{2}U_{t+1})\right]}$, which is just an "exponential tilting" in the SDF.

When considering two uncertainty parameters the properties of normal exponential integrals give us the following expression for the likelihood ratio:

Lemma 49 *The likelihood ratio is given by* $L_t = \frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} = \exp\left\{-\frac{(v_1+v_2)^2}{2} + (v_1+v_2)w\right\}$

Proof. Use the fact that $f(x) = (2\pi)^{-1/2}\exp\left\{-\frac{x^2}{2}\right\}$. Then

$$\frac{\widehat{f}_t(x_t|x_o)}{f_t(x_t|x_o)} = \frac{(2\pi C)^{-1/2}\exp\left\{-\frac{1}{2}(w_{t+1}-v_{1t}-v_{2t})'(w_{t+1}-v_{1t}-v_{2t})\right\}}{(2\pi C)^{-1/2}\exp\left\{-\frac{w^2}{2}\right\}} = \exp\left\{-\frac{(v_1+v_2)^2}{2} + (v_1+v_2)w\right\}.$$

■

Remark 50 *The formulation of the likelihood ratio continue to be the the same as the one used to define entropy (for our purposes) and to describe the detection error probabilities (DEP).*

Given the resulte above we can define a related concept to MPU.

Definition 51 *The one-period multifactor price of Knightian uncertainty (MFPU) is the standard deviation of the multiplicative adjustment, i.e.,* $std_t(m^u_{t,t+1})$*. for the bi-dimentional case. It is given by* $std_t(m_{t,t+1}) = \sqrt{\exp(v_1+v_2)^2 - 1}$

The MPR is therefore given by $MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]} = \sqrt{\exp(v_1+v_2)^2 - 1}$

Hence we get a nice expression for the MPR helping to more easily solve the equity premium puzzle since we do not need to rely in any specific distortion to increase the MPR but only in their sum (we can have, for example, a smaller value for μ_b than what was assumed by HST and HSW). But there is also the following problem: to solve the equity premium puzzle we need to boost the theoretical MPR by an increase of (v_1+v_2) but this will affect the value of the risk free rate (which is equal to $\frac{1}{E[m_{t,t+1}]}$).

At this point we can comment in a possible problem with the estimation strategy used by HST and HSW: not only the quantitative data (c,i) needed to stay fixed but also the value for the risk free rate r^f. But as long as the $E[m_{t,t+1}]$ was changed to solve the equity premium puzzle r^f could not stay fixed. We need another variable to be change such that r^f would stay fixed. Recall that in HST and HSW (c,i) stayed fixed because as σ was introduced (and changed its value) there was an exact cancelling out effect by changing β accordingly. We propose that (c,i,r^f) will stay fixed as the two uncertainty parameters (σ_1,σ_2) were introduced by changing (β,EIS) accordingly.[7] Recall that we used the same feature that the r^f is tied down by the technology. HST argued on page 879 that "adding knowledge of the risk-free rate, which is constant in this model, does not achieve identification".

Remark 52 *In Epstein and Zin (1989) r^f was not constant but a function of the risk aversion coefficient, the discount rate β and the elasticity of substitution. Obtaining a closed-form solution for r^f is very difficult (at a minimum we need to get $\frac{\partial r^f}{\partial EIS}$ to get hope of convincily solve the risk free rate puzzle).*

Remark 53 *Given the approximation $MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]} = \sqrt{\exp(v_1+v_2)^2 - 1} \approx |v_1+v_2|$ in order to get a sense of the effect of v_1+v_2 in solving both puzzles we need to compare the functions*

$$\begin{cases} \exp(v_1+v_2)^2 \\ \quad |v_1+v_2| \end{cases}$$

This allows us to write the following result:

Proposition 54 *In the bidimentional linear additive case for modelling economic uncertainty the effect of an increase in v_1+v_2 on the risk free rate is always bigger than the effect on the MPR. This increses the importance of using EIS to solve the risk-free rate puzzle.*
Proof. *Strightforward.* ■

Figure XXX gives an illustration of the effect of the two functions. Of particular interest is the value for $v_1+v_2 = 0.2550$ which is the empirical value for the MPR. Note that $\exp(v_1+v_2)^2$ is always above $|v_1+v_2|$ for all the domain function (the effect on the risk free rate of an increase in v_1+v_2 is always bigger than the effect on the MPR), that for v ranging from zero to approximately 0.75 the increase in $|v_1+v_2|$ is bigger than the increase in $\exp(v_1+v_2)^2$, i.e, this happens after the value 0.2550.

When $v_1+v_2 = 0$ we have the biggest difference between the two effect on MPR and on r^f. As long as $v_1+v_2 \neq 0$ the effect will decrease monotically and symmetrically up to the point where the difference is minimum (approximately 0.75) but this happen after the desired target for the MPR (for solving the

[7] Iddeally we find an implicit function related the two economic conceps: $g(EIS,\beta) = 0$. Note that an increase in EIS tend to lower the value of r^f.

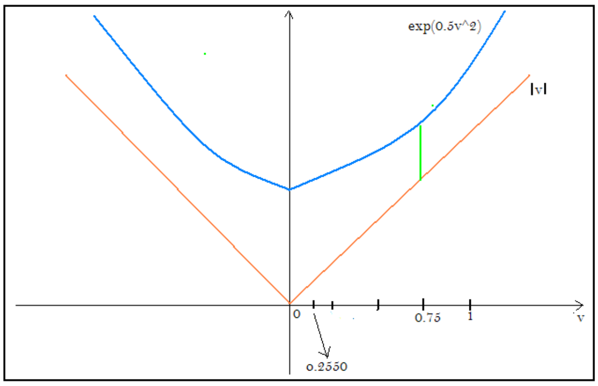


Figure 5:

equity premium puzzle) which hovers around 0.2550. Without changing the value of EIS solving the equity premium puzzle will have the side effect that r^f will increase by approximately 3.3% (and this exarcebates instead of solve the risk free rate puzzle).

4.6 Elasticity of Intertemporal Substitution (EIS) and Explanation of the Two Puzzles

Elasticity of substitution is an old and important concept. It was proposed by Hicks (1932) and has many uses in diverse branches of economics. The constant elasticity of substitution (CES) function was initially proposed by Arrow et aL (1961) and it is also a workhorse for many problems in dynamic economics.

In utility theory it is usual to define the elasticity of substitution between goods 1 and 2 (see Mas-Colell et aL (1996)) as:

$$\xi_{12}(p,w) \triangleq -\frac{\partial\left[\frac{x_1(p,w)}{x_2(p,w)}\right]}{\partial\left[\frac{p_1}{p_2}\right]}\frac{\frac{p_1}{p_2}}{\frac{x_1(p,w)}{x_2(p,w)}}$$

where x represents the Walrasian demand function, p is the price of the commodity and w is the wealth level. For the CES utility function we have that $\xi_{12}(p,w) = \frac{1}{1-\rho}$ which is indeed a constant. This is a reasonable measure of curvature of the indifference curve.

This section will deal with the concepts of EIS and CES within the framework of modelling robustness concern with two parameters.

We want to obtain both EIS and β as functions of (σ_1, σ_2) but with different magnitudes. The main point is that EIS should increase to help solve the risk-free rate puzzle.

When considering a variable EIS (instead of fixed at 1) some algebraic manupulations of Epstein and Zin (1989) and Hansen (2006) shows that the following expression is valid for the SDF:

Lemma 55 *For the case of variable EIS we have that*

$$E_t[m_{t+1,t}] = E_t\left\{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\frac{\exp\left[(1-\beta)(\rho-\gamma)U_{t+1}\right]}{\left\{E_t\left[\exp((1-\beta)(\rho-\gamma)U_{t+1})\right]\right\}^{\frac{\rho-\gamma}{1-\gamma}}}\right\}$$

where $0 \neq \rho < 1$ is the index of substitutability (ρ was also used in section 2.2). Note that $EIS = \frac{1}{\rho}$.

Proof. *Use the result of Hansen (2005) that $m_{t+1,t} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left(\frac{V_{t+1}}{\Re(V_{t+1})}\right)^{\rho-\gamma}$ and proceed with an adequate change of variable $U_t = \frac{\log V_t}{1-\beta}$.* ■

Lemma 56 *For the case of variable EIS we have that*

$$E_t[m_{t+1,t}] = E_t\left\{\beta\left(\frac{c_t}{c_{t+1}}\right)\left(\frac{c_t}{c_{t+1}}\right)^{\rho-1}\frac{\exp\left[\frac{\sigma}{2}U_{t+1}\right]}{\left\{E_t\left[\exp(\frac{\sigma}{2}U_{t+1})\right]\right\}^{\frac{\rho-\gamma}{1-\gamma}}}\right\},$$

i.e., $m^u_{t,t+1} = \left(\frac{c_t}{c_{t+1}}\right)^{\rho-1}\frac{\exp\left[\frac{\sigma}{2}U_{t+1}\right]}{\left\{E_t\left[\exp(\frac{\sigma}{2}U_{t+1})\right]\right\}^{\frac{\rho-\gamma}{1-\gamma}}}$.

Proof. *Straigthforward.* ■

Note that the nesting property is satisfied: the new $m^u_{t,t+1}$ nests the old $m^u_{t,t+1}$ when ρ is specialized bo te equal to one. Note also that we let $m^f_{t,t+1} = \beta\left(\frac{c_t}{c_{t+1}}\right)$ to be independent of ρ to conform to the standard literature on SDF (including HST and HSW).

4.6.1 Local Recovery of EIS

If we define the coefficient of risk-sensitivity as $\sigma \equiv 2(1-\beta)(\rho-\gamma)$ than we get the following expression:

$$E_t[m_{t+1,t}] = E_t\left\{\beta\left(\frac{c_t}{c_{t+1}}\right)^{\rho}\frac{\exp\left[\frac{\sigma}{2}U_{t+1}\right]}{\left\{E_t\left[\exp(\frac{\sigma}{2}U_{t+1})\right]\right\}^{\frac{\rho-\gamma}{1-\gamma}}}\right\}$$

Note that when $\begin{cases} \rho = 1 \Rightarrow \text{ get } \sigma \text{ used by Tallarini (1996,2000)} \\ \rho = \gamma \Rightarrow \text{ get } \sigma \ = 0 \text{ (expected utiliyt case)} \end{cases}$

Moreover we can work with the inverse map of the definition to get a (local) recovery[8] of EIS:

$$\rho = \frac{\sigma}{2(1-\beta)} + \gamma$$

hence $\frac{\partial\rho}{\partial\sigma} = \frac{1}{2(1-\beta)} > 0$, i.e., an increase in $|\sigma|$ imply an increase in EIS which would help solve the risk-free rate puzzle (since r^f will shrink).

[8] See Wang (1993) for a related issue on recoverability of EIS.

4.6.2 Disentangle the Relation of EIS with (σ_1, σ_2)

Now we have to discuss what is the effect of having a variable ρ in the two-dimensional case. Note that ρ has effect in r^f (through the effect on $E_t[m_{t+1,t}]$ as discussed above) and also on $std_t(m_{t+1,t})$ since this is a function of $\sigma = \sigma(\rho)$.

Note that the relation $\beta = \beta(\sigma_1, \sigma_2)$ is more difficult to disentangle as $\beta = \beta(f(\sigma_1), g(\sigma_2))$, i.e., get separability on the two parameters, when compared to separability of $EIS = EIS(\sigma_1, \sigma_2) = EIS(\overline{f}(\sigma_1), \overline{g}(\sigma_2))$. This happens because the relation $\sigma \equiv 2(1-\beta)(\rho-\gamma)$ was derived for a very specific agregator, the CES function (recall that $W(c,\mu) = [(1-\beta)c^{1-\rho} + \beta\mu^{1-\rho}]^{\frac{1}{1-\rho}}$, for $0 < \rho \neq 1$) and a special power mean value functional μ. For others types of aggregators the mathematics is much more complex and some issues of existence and non uniquiness of utility function V= $W(c, \mu(V))$ arises (see page 946 and footnote 7 of Epstein and Zin (1989)).

Remark 57 *There is a direct link between ρ and the risk sensitive parameter but there is no such a link between ρ and the robust parameter, i.e., EIS is a direct matter i the modelling a la Epstein-Zin risk sensitive approach but in only implicitly treated in Hansen-Sargent robust control methodology.*

For getting separability in EIS we need to define two elements of the substitutability issue (say ρ_1 and ρ_2). There are two channels for that: acting with the aggregator $W(c,\mu)$ or the mean value functional μ. By definition of μ we ruled out the existence of two different mean value funtionals for the same controller problem (recall that μ is a parameterized by one feature only (γ) while W is paramaterized by two features (β, ρ)). The next section shows our proposal to this problem.

4.6.3 Two Parameter Constant Elasticity of Substitution (2CES) Aggregator

Definition 58 *(2CES) Let the aggregator be of the following format*

$$\widetilde{W}(c,\mu) = [(1-\beta)c^{1-\rho_1} + \beta\mu^{1-\rho_2}]^{\frac{1}{1-\overline{\rho}}}$$

where $\overline{\rho} = \frac{\rho_1+\rho_2}{2}$ *and* $0 < \rho_1 \neq 1$ *and* $0 < \rho_2 \neq 1$.

Note the nesting feature being satisfied by this definition: when $\rho_1 = \rho_2$ we get back to the ordinary CES aggregator. Now we state an important result where we define two convenient risk sensitive parameters to conform to ρ_1 and ρ_2.

Proposition 59 *For $\rho_1 \neq \rho_2$ we have*

$$E_t[m_{t+1,t}] = E_t\left\{\beta\left(\frac{c_t}{c_{t+1}}\right)\left(\frac{c_t}{c_{t+1}}\right)^{\rho_1-1}\frac{\exp\left[\frac{\sigma_2}{2}U_{t+1}\right]}{\left\{E_t\left[\exp(\frac{\sigma_2}{2}U_{t+1})\right]\right\}^{\frac{\rho_2-\gamma}{1-\gamma}}}\right\}$$

with $\sigma_1 = 2(1-\beta)(\rho_1-\gamma) = \sigma_1(\rho_1)$ *and* $\sigma_2 = 2(1-\beta)(\rho_2-\gamma) = \sigma_2(\rho_2)$

From the above result we get that MPR (which can be susbtituted by the concept MFPU) is affected in different ways by ρ_1 and ρ_2 since: $MPR = \frac{std_t(m_{t,t+1})}{E_t[m_{t,t+1}]} = \sqrt{\exp(v_1+v_2)^2-1} = \sqrt{\exp(v_1(\sigma_1)+v_2(\sigma_2))^2-1} =$

$\sqrt{\exp(v_1(\rho_1)+v_2(\rho_2))^2-1}$.

Proposition 60 *It is possible to find a suitable combination of* (ρ_1, ρ_2)*, or* (σ_1, σ_2)*, such that* $std_t(m_{t,t+1})$ *will raise and* $E_t[m_{t,t+1}]{=}\frac{1}{r^f}$ *will (mildly) increase, allowing to help solve both the equity premium and the risk-free rate puzzle.*

Proof. *This is a direct result of the observation of the statis comparative results:* $\frac{\partial E_t[m_{t,t+1}]}{\partial \rho_1} > 0$, $\frac{\partial E_t[m_{t,t+1}]}{\partial \rho_1} \gtreqless 0$, $\frac{\partial std_t(m_{t,t+1})}{\partial \rho_1} \gtreqless 0$, $\frac{\partial std_t(m_{t,t+1})}{\partial \rho_1} \gtreqless 0$, $\frac{\partial |v|}{\partial \rho} > 0$, $\frac{\partial |v|}{\partial \rho_1} \gtreqless 0$, $\frac{\partial |v|}{\partial \rho_2} \gtreqless 0$*. Also use the fact that the only restrictions to substitutability parameters are* $0 < \rho_1 \neq 1$ *and* $0 < \rho_2 \neq 1$. ∎

The solution of the two puzzles can be reached by a suitable selection of (γ, ρ_1, ρ_2) instead of a selection of the risk sensitive parameter γ only (like Tallarini (1996, 2000) and Barrilas et AL (2006) do. This new channel may solve the critic to the value for γ (=50) that Tallarini (1996, 2000) calculated as necessary to put the pair $(E_t[m_{t,t+1}], std_t(m_{t,t+1}))$ inside the Hansen-Jaganathan bound. This is arguably very high and in part was a motivation for the critic stated by Lucas (2003) for the results obtained so far (including Tallarini (1996,2000)):

"No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high estimates of risk aversion to do it."

Barrilas et AL (2006) work on this critic but there is no consideration of EIS in the paper since it assumed to be equal to one.

Remark 61 *There is a well defined region of factability for the vector* (ρ_1, ρ_2) *due to economies that satisfies the proposition above. For a description of a similar problem with a direct result for the region of factability for* ρ *see Schroeder and Skiadas (1999) and specially page 964 of Maenhout (2004).*

Remark 62 *It is possible to prove that the proposed 2-CES has indeed a constant elasticity. Some hints are: we need to prove that* $\xi = \frac{d\ln(\frac{\mu}{c})}{d\ln(RS)}$ *is constant, where* $RS = \text{-}\frac{\frac{\partial \widetilde{W}(c,\mu)}{\partial c}}{\frac{\partial \widetilde{W}(c,\mu)}{\partial \mu}}$ *is the rate of substitution. After isolating* $\frac{\mu}{c}$ *take log and attach a suitable combination of factors* $(1-\rho_1)$ *and* $(1-\rho_2)$*. The results should nests the elasticity for the standard CES aggregator (which has an elasticity equal to* $\frac{1}{\rho}$*).*

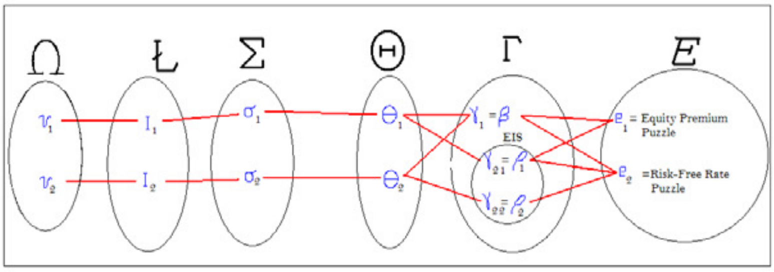


Figure 6:

Remark 63 *The 2-CES concept is only a slight modification of the standard CES aggregator such that the considerations made by Epstein and Zin (1989) for incorporating asset pricing results like the static CAPM and I-CAPM could be still valid.*

Remark 64 *For a comparison with other alternatives formulation of CES functions with more than one parameter see Rutheford (2002) and specially the non separable nested CES function of Perroni and Rutheford (1995).*

Remark 65 *The fact that β has an over powering role by beeing the economic concept (from set Γ) that captures both uncertainty parameters (recall the relation $\beta = \beta(\sigma_1 + \sigma_2)$ dispensing the role of the other economic concept (EIS) to act as the economic counterpart of the pure robust control modelling) arguably comes with no surprise since EIS was not present in the constraint set used in the robust control program:* $\begin{cases} \widehat{E}_t \sum \beta^j v_{t+j} \cdot v_{t+j} \leq \vartheta_t \\ \vartheta_{t+1} = \beta^{-1}(\vartheta_t - v_t \cdot v_t) \end{cases}$

4.7 Illustration of the Sets and Mappings for the Bi-Dimensinal Case

Given the use of two dimensions for modelling uncertainty the figure below illustrates the relevant sets and mappings. Note that some mappings are not bijective. This is a reduced version of the graph showed in section 3.

4.8 Geometry of the Set of Priors: Circle or Ellipse?

We start this section with a caveat: Epstein and Schneider (2003, JET) define the set of priors to be "rectangular" if its induced set of conditional and marginals probabilities shows a particular form of decomposition. They illustrated rectangularity in the probability simplex (see their Figure 1). HSTW argues that this particular (rectangular) set of beliefs advocated by Epstein and Schneider means an enlargement that is too immense to be useful for the robust control framework (because the enlargement reduces the content of the original set of prior beliefs).

What we intend to describe here with the figure XX above is *not* a formal topological treatment of the representation of the space of probability distributions but only a much vague idea of how the probability distributions are dispersed with respect to the approximating model representation. Specifically if the representation used is not a circle of ray η_1 but an ellipse of parameters (η_1, η_2) with, say, $\eta_1 > \eta_2$, we may have situations where a probability representation of a distorted model is included (in the set of robust decisions) in the first case and it is excluded in the second. We argue that the topological idea of a neighborhood of a point may be have a better geometric interpretation when using an ellipse instead of a circle.

The approach by Epstein, Zin, Schneider and coauthors (Recursive Multi-Priors Model = RMP) delivers rectangular sets of priors (with a prior-by-prior updating rule). The Robust Control (RC) approach by Hansen, Sargent and coauthors (mainly the papers HST and AHS) delivers sets of priors constrained by relative entropy (linked to one parameter η) that has a circle as a vague illustration. Our work with robust control treatment using two parameters delivers (under some mild conditions) a vague illustration of an ellipse. Since η is directly linked to the concept of entropy we will obtain a $\eta_1 - entropy$ and $\eta_2 - entropy$ "axis" of an ellipse instead of a $\eta - entropy$ "radius" of a circle.

5 Conclusions

This dissertation models the non uniquiness of beliefs from the primitives (or fundamentals) of the economy: it assumed up front that uncertainty is multidimensional. We advocate that adding more layers of specification doubt in that way seems a more natural manner for modelling uncertainty. The hunch for our model comes from the belief that structured uncertainty is inherently multidimensional.

Our work does not deliver a closed and definitive framework for a bidimensional treatment of uncertainty (in the sense of fear of model misspecification used by Hansen, Sargent and coauthors). It is a step in that direction and adds to the prolegomena stated by those researchers. Indeed both approaches of multi prior and robustness for modelling economic uncertainty are still under work and face a viviv debate between their advocates.[9] In order to get a sense of the various approaches that tackle the problem of modelling uncertainty we stated a simple functional analysis treatment of eight relevant sets of key concepts and seven respective mapping relating those sets. This gives a road map of what is in play.

Working with a two dimensions has a side benefit of checking at least the main results of the one-dimensional case which requires a serious study of this difficult literature. Some of the empirical results using bidimensionality may seem more plausible: for example the value for μ_b may be smaller than those worked by HST and HSW. The bottom line is that one gets more degree of freedom to attach value to parameters that have more sound economic intuition.

Most of Hansen, Sargent and coauthors attempt to model uncertainty with two parameters is done by assuming that the state vector has some unobservable part which forces the agent to filter. In Hansen and Sargent (2006) it is explicitly used the term "fragile" to describe the fact that the representative consumer is not confident about his posterior probabilities given that the model has a sort of Bayesian learning. For the first time they come with a paper that has a concept that leading researchers in robust control theory (like John C. Doyle) is using frequently: the fact that the design of some systems may be robust to the presence of structured uncertainties but still fragile for other types of uncertainties. [10]

One of the contributions of this dissertation is to show that there is not necessarily a bijection between the elements of the set of economic parameters

[9] Some of the dissagreement are: Epstein and Zin (2003) argues that the type of preferences advocated by robust control defenders is not time consistent. Hansen and Sargent (2005) maek a defense of their point in terms of time consistency and claim that the enlargement of admissible priors proposed by Epstein and co-authors are too big to be useful for robust control purposes.

[10] The way Bayesian learning from history of state vector is justified by Hansen and Sargent (2006)'s use of hidden markov states may be better modelled with the use of adaptive control (instead of robust control). Moreover there is no explanation for the way the second uncertainty parameter θ_2 is disciplined (for example by providing a detection error probability analysis to θ_2). They argued that introduction of θ_2 implies that representative agent's beliefs are fragile since he is not confident of his posterior probabilities. The horsework of the paper was the mixing of model selection and state estimation.

(Γ) and the elements of the set of parameters in the robust control problem (Θ), as one can initially assume without going through a formal treatment of a second parameter to model concern for robustness. By assuming a second uncertainty parameter we need to work with three controllers: an agent that maximizes and two versions of nature that minimizes. We proposed a list of nine criteria to be satisfied by possible functions $F(v_{1t}, v_{2t})$ of the two distortions to the model specification.

Another contribution of the dissertation is to work explicitly with a variable elasticity of intertemporal substitution (EIS). If you shut down the channel of EIS effect by assuming it to be equal to one (like explicitly did Tallarini (1996, 2000) and implicitly assumed HST, HSW and others) than you loose a considerable richness and power of modelling economic uncertainty. A suitable 2-CES function was proposed to handle that issue.

One possible extension of the work presented here is to consider a new concept (suitably defined) of covariabality between v_1 and v_2 different from the standard notion of covariance (this last concept is impossible to work in a robust control framework since we do not have knowledge of the probability distributions of neither v_1 or v_2. [11]

One open question is when we should stop adding new perturbations like (v_1, v_2). A possible rule of thumb is to consider how many important empirical puzzles are outstanding in economics and finance. The list has certainly less than seven really important and cataloged puzzles. This may be an upper bound to the number of extra parameters related to robustness concern. Still this number may be arguably too big. One defense for our case to multidimensionality is that it adds a new possible explanation but one may not focus all the explanations to puzzles as coming from the uncertainty world (at least the type of uncertainty treated in this dissertation, with a feedback on the vector of state variables). The enlargement of the admissible set of priors with more than two uncertainty parameters will likely to be a complex issue. For our work with (v_{1t}, v_{2t}) entering the law of motion for the state vector we heuristically defined an ellipse as the neighborhood of the probability density that governs the approximate model.[12]

Applications of our work to policy are possible but some rich ingredients will needed to be modelled. Since it is unavoidable that the models used to make policy evaluation are misspecified. One element that will need to be introduced are robust multi-agents.

[11] This observation was made by Rodrigo A. S. Penaloza.

[12] This "ball" was represented in the probability space.

6 Appendix A: Analysis of Risk Aversion Measures in Uncertainty Modelling

This section will discuss how the measures for risk aversion for risk sensitive and robust control frameworks are calculated and how they compare to each other. We stress that this comparison is not made in the literature and in particular is missing in HST and HSW. Through the use of the detection error probability (DEP) method HST and HST argues that they are able to select reasonable value for θ. From the definition $\theta \equiv -\sigma^{-1}$ this will imply a corresponding value for σ. It is expected that DEP as a statistical detection tool will deliver uncertainty parameters (θ, σ) linked to reasonable values for risk aversion.

6.1 Risk Aversion (γ) in the Risk Sensitive Control Framework

In the risk sensitive problem the values for σ that will imply, via the definition $\sigma \equiv 2(1-\beta)(1-\gamma)$, the respective value for γ can be found by reversing engineering the formula, i.e.,

$$\gamma = 1 - \frac{\sigma}{2(1-\beta)} = 1 + \frac{1}{2(1-\beta)\theta}$$

Hence for γ to assume a reasonable value, say between 1 and 10, and assuming (as HST and HSW did) that $\beta = 0.9971$, we have that:

$$1 < 1 + \frac{1}{2(1-\beta)\theta} < 10 \Rightarrow \theta > \frac{1}{18(1-\beta)} > 0 \Rightarrow$$

$$\theta > 19,1571$$

Or equivalently

$$522 \cdot 10^{-4} < \sigma < 0$$

For the four values for σ $\left((0, -0.25, -0.50, -0.75) \cdot 10^{-4}\right)$ that HST and HSW worked with the condition is satisfied. These four values (and two other important ones) are reported on the table below. Note that $\frac{\partial \gamma}{\partial |\sigma|} = \frac{1}{2(1-\beta)} > 0$.

TABLE 01:

$\sigma(10^{-4})$	θ	γ
0	∞	1
-0.25	4	1.00431
-0.50	2	1.00862
-0.75	1.33	1.01293
-1.00	1	1.017224
-522.00	0.001957	10

Remark 66 *Note that the formula implies that $\gamma > 1$ and, in particular, $\gamma = 0$ (i.e. the risk-neutrality case) is discharged here. Maenhout (2006) states that $\gamma > 1$ is arguably defensible since: (i) It is the empirically relevant part of the parameter space, (ii) Assures that the solution of most models, Maenhout (2004 and 2006) in particular, are well-behaved and (iii) it seems most plausible for relatively risk-averse investors given that robustness induces a relatively conservative behavior. There is also an explanation regarding time and risk preferences assumption: $\gamma > (<)\rho$ implies that the agent has a preference for early (later) resolution of uncertainty (see Backus, Routledge and Zin (2004)). In the special case that $\rho = 1$, $\gamma > 1$ is a sufficient condition for a risk preference for early resolution of uncertainty.*

Remark 67 *There is a much more stringent condition for σ to satisfy: the breakdown condition as posited by the risk sensitive control theory. The value $\sigma = -522x10^{-4}$ as the limit to get a "reasonable" value for $\gamma = 10$ is far beyond the breakdown point $\underline{\sigma}$. The breakdown analysis states that for $\sigma <\underline{\sigma}$ the positive definiteness of the expression $(I - \sigma C'\Omega C)$ ceases to hold and the risk-adjusted recursive utility is $-\infty$ whatever is the controller (maximizing agent) actions. An equivalent way to express this condition is to check if $\log\det(I - \sigma C'\Omega C) > -\infty$, i.e., if the eigenvalues of $(I - \sigma C'\Omega C)$ are all positive. This happens because the minimizing agent (nature) is sufficiently unconstrained that he can force the criterion function to be $-\infty$ regardless of the best response of the maximizing agent. One consequence is that it comes with no result to seek for more robustness beyond $\underline{\sigma}$. For the general equilibrium considerations of HST, HSW and this dissertation this breakdown point hovers around $-1.00 \cdot 10^{-4}$ (which implies a risk aversion coefficient $\gamma = 1.017224$, an empirical reasonable value). Hence the type of risk corrections (in the sense of the risk sensitive control problem) that we are working with is much smaller (in absolute value) than the limit imposed by the breakdown analysis. See also Glover and Doyle (1988) and chapter 7 of Hansen and Sargent (2007) for a detailed explanation of the breakdown point. Whittle (1990) calls $\underline{\sigma}$ the "utter psychotic despair".*

6.1.1 Model with Variable EIS

For the model a variable EIS that we propose in this dissertation we have that $\sigma \equiv 2(1-\beta)(\rho - \gamma)$ which implies

$$\gamma = \rho - \frac{\sigma}{2(1-\beta)} = \rho + \frac{1}{2(1-\beta)\theta}$$

Then for the range $1 < \gamma < 10$, assuming the discount factor is $\beta = 0.9971$ and for the value of the inverse of the EIS, that Bansal and Yaron (2004) and the references cited therein assumed for, i.e., $\rho = 1.5$, we have that:

$$1 < 1.5 + \frac{1}{2(1-\beta)\theta} < 10 \Rightarrow -0.5 < 0 < \theta < \frac{1}{17(1-\beta)} \Rightarrow$$

$$\theta > 20,284$$

Or equivalently

$$-493 \cdot 10^{-4} < \sigma < 0$$

Note that the restriction for both θ and σ are more restrictive for the case with $\rho = 1.5$, i.e., consideration of a variable EIS gives more discipline (less degree of freedom) to the way uncertainty is modelled. But again the consideration of the breakdown point $\underline{\sigma}$ is much more stringent. Lastly, the introduction of ρ will deliver agents that are even more far for the risk-neutral case.

6.2 Risk Aversion Measure ($CRRA$) in the Robust Control Framework

Below we show how the standard Arrow-Pratt measure of risk aversion is calculated in the quadratic utility setup of the robust control framework and compare it to the measure of risk aversion (γ) calculated above for the risk sensitive approach.

For the robust control problem the bliss point μ_b is a curvature parameter since $U_t = -(s_t - b_t)^2 + \beta \Re(U_{t+1})$, with $b_t = \mu_b$. From the equation for consumption service $s_t = (1+\lambda)c_t - \lambda h_{t-1}$ we get $u_c = \frac{\partial u}{\partial c} = -2(1+\lambda)(s_t - \mu_b)$. Note that no habit term h_t will appear in the marginal utility of consumption u_c since $h_t = \delta_h h_{t-1} + (1-\delta_h)c_t$. The second derivative is $u_{cc} = -2(1+\lambda)^2$. Hence the CRRA for consumption gambles is:

$$CRRA \equiv -\frac{c_t u_{cc}}{u_c} = -\frac{c_t\left[-2(1+\lambda)^2\right]}{-2(1+\lambda)(s_t - \mu_b)} = \frac{c_t(1+\lambda)}{\mu_b - s_t}, \text{ or}$$

$$CRRA = \frac{c_t(1+\lambda)}{\mu_b - (1+\lambda)c_t - \lambda h_{t-1}}$$

Note that this can be easily evaluated for the sample data used by HST and HSW since:

a) The (bi-dimensional case) state vector is

$$x_t = (h_{t-1}, k_{t-1}, d_{t-1}, 1, d_t, d_t^1, d_{t-1}^1, d_t^2, d_{t-1}^2, d_t^3, d_{t-1}^3)' \ ,$$

b) The consumption is given by $c_t = (\varepsilon + \delta_k)k_{t-1} + d_t - k_t$,

c) The (one-dimensional) endowment is modelled as

$$\begin{cases} d_t = \mu_d + d_{t+1}^1 + d_{t+1}^2 \\ d_{t+1}^1 = g_1 d_t^1 + g_2 d_{t-1}^1 + c_1 w_{t+1}^1 = (\phi_1 + \phi_2) d_t^1 - \phi_1 \phi_2 d_{t-1}^1 + c_1 w_{t+1}^1 \quad \text{and} \\ d_{t+1}^2 = a_1 d_t^2 + a_2 d_{t-1}^2 + c_1 w_{t+1}^2 = (\alpha_1 + \alpha_2) d_t^2 + \alpha_1 \alpha_2 d_{t-1}^2 + c_1 w_{t+1}^2 \end{cases}$$

d) The value for the parameters $[(\lambda, \varepsilon, \delta_k), (\mu_d, \phi_1, \phi_2, \alpha_1, \alpha_2, c_1, c_{2,})\,(\mu_b)]$ are known.

For example when $\mu_b = 36$, we get the reasonable value of $CRRA = 2.3$. Note that this value is not equal to the parameter of risk aversion γ used in the risk sensitive control problem which hovers a little bit above 1. Indeed $\gamma = \gamma(\beta, \sigma)$ is not a function of μ_b, for example. By its turn, $CRRA = CRRA\left((\lambda, \varepsilon, \delta_k), (\mu_d, \phi_1, \phi_2, \alpha_1, \alpha_2, c_1, c_{2,})\,(\mu_b)\right)$ is not a function of σ, for example.

Remark 68 *It is important to obtain reasonable values for both γ and CRRA since the solution to the robust control problem will provide a unique optimum θ which is related by construction to a unique σ. Note that σ is defined as $\sigma \equiv -\theta^{-1}$ such that the two approaches match, i.e., the objective functions will differ only from some constant terms that will not affect the optimal control variable. We do not set σ first but θ. For details of the Lagrange multiplier theorem, that guarantees the duality between the risk sensitive approach and the robust control method see Luenberger (1969), pages 216-221.*

Remark 69 *Getting a reasonable value for the measure of risk aversion cannot be listed as a criteria for the way uncertainty should be modelled simply because $CRRA$ is not a function of σ, and consequently it is not a function of distortions (v_1, v_2) (or the way these distortions enter the model). Anyway the already sixth listed criteria (i.e., the estimated value for (σ_1, σ_2) should not be beyond the breakdown point) is a very stringent condition.*

6.3 Unreasonable Risk Aversion of Tallarini (1996, 2000)

Tallarini (1996, 2000) worked with two parametric assumptions about the log consumption stream, namely random walk and trend stationary models, with the results for the random walk hypothesis getting better results.

For values of the discount factor of $\beta = 0.999$ and $\beta = 0.9999$ (i.e., very close to 1) the test statistics reported by Tallarini (based on the Hansen, Heaton and Luttmer (1995) method) under the assumption of non-expected utility and random walk consumption assert that the hypothesis that the mean-standard deviation pair $(E(m), std(m))$ is in the Hansen-Jaganathan region cannot be rejected at 10% for all values of risk aversion used ($\gamma = 1, 5, 10, 15, 20, 25$). But for $\beta = 0.995$ only the high values of $\gamma = 50$ for the random walk model and 250 for the trend stationary model puts the pair $(E(m), std(m))$ in the bounds of Hansen-Jaganathan region. Recall that the value used for HST and HSW was $\beta = 0.9971$. Hence it is likely that for the framework adopted by Tallarini (1996,2000) the value of the atemporal risk aversion coefficient γ is not empirically reasonable. This rests mainly in the assumption for the consumption stream adopted: in HST and HST we get that the shadow price of consumption μ_{ct} can be represented by a martingale but this is not the same as assuming that consumption is a random walk or a trend stationary process à la Tallini (1996,2000).

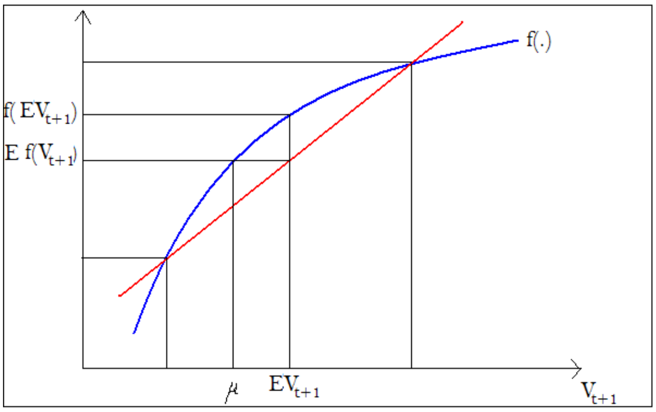
6.4 Certainty Equivalent Interpretation

One may justifies that the values for risk aversion in risk sensitive framework (γ) and in the robust control framework ($CRRA$) are not equal because the two setups use different notion of certainty equivalent. This does not proceed due to the following reasoning: in the risk sensitive control approach we use the certainty equivalent (or mean value) function $\mu(V_{t+1}) \equiv f^{-1}\{E_t[f(V_{t+1})]\}$, with the value function $V_t = W(c_t, \mu(V_{t+1}))$ and $W(.)$ representing an aggregator function. Most of the literature on recursive utility function use power utility specifications, i.e.,

$$f(z) = \begin{cases} z^{1-\gamma} \text{ , if } 0 < \gamma \neq 1 \\ \log z, \text{ if } \gamma = 1 \end{cases}$$

Hence this is exactly what is stated in standard microeconomics theory books (see Mas-Collel et AL (1996), for example) as the certainty equivalent $u^{-1}(E(u))$, just take $f(.) = u(.)$, where u(.) is the Bernoulli utility function. Usually the definition is stated as $u(\mu) \equiv E_t[f(V_{t+1})]$, which is a variant for the above definition of μ.

The restriction imposed by the risk sensitive approach taken here is for $U(.)$ to be of a power function format. In particular we are ruling out the linear utility function ($\gamma = 0$). The figure below gives an illustration of the certainty equivalent function μ.



Note that $\mu \leq E_t[V_{t+1}]$ which means that some expected return is exchanged for getting the certainty payoff μ. As a matter of fact we can state the following result.

Lemma 70 *The condition $\mu \leq E_t[V_{t+1}]$ is equivalent to the claim that the agent is risk averse.*

Proof. *From the nondecreasing feature of $f(.)$ and the definition of μ we have that $\mu \leq E_t[V_{t+1}] \Leftrightarrow f(\mu) \leq f(E_t[V_{t+1}]) \Leftrightarrow E_t[f(V_{t+1})] \leq f(E_t[V_{t+1}])$.* ■

7 Appendix B: Some Possible Extensions

7.1 Multidimensional Model Uncertainty and Equivalent Martingale Measure (EMM)

When modelling uncertainty in a multidimensional fashion some implications may occur for the existence and uniqueness of the equivalent martingale measure (EMM). Recall than in standard finance theory it is proved that there is an equivalence between: a) there is no arbitrage opportunities in the market and b) it is possible to price using a probability measure (the EMM itself) that is equivalent to the original probability density (f) (and this is linked to risk neutral pricing). This may result in a " no-arbitrage theorem with robustness concerns". Some promising road may be the use of the concept of approximate arbitrage property. Note that when making arbitrage considerations with the concept of risk there is no limitation for risk to be a one-dimensional variable. Arguably the same apply to uncertainty.

Risk neutral pricing relies on the existence of a probability (f_{RNP}) that describe the economy as if all investors are risk neutral. Would we have something similar to an "uncertainty neutral pricing"? One may use a measure to the notion of entropy for the distance between f and f_{RNP}. An important reference for starting this line of research is Duffie (2002) (specially pages 121 and 132 of chapter 6).

There are some implications for other issues in theoretical and applied finance when using n**onstandard specification of preferences (like the preferences with robustness concern that we are studying). Ther**e is some room for using the robustness artillery in issues such as value-at-risk and risk management. There are also some branches of finance for what we see no role for robusteness like the theory of capital structure.

7.2 Modelling Uncertainty in Emerging Economies

HST and HSW used U.S. dataset to check if the model could better explain the equity premium puzzle. Arguably emerging economies inherit a higher degree of uncertainty and may be a valuable task to compare the performance of HST model in rich and developing economies

First one should provide the intuition for an emerging economy to show agents that are more likely (*vis-a-vis* developed or rich economies) to fear model misspecifications. Another topic to be studied is the eventual maintenance of the observational equivalence when considering emerging economies. Finally: exam if it matters if the economy is rich or poor, have strong or weak institutions (etc) for the theory (robustness helping to explain empirical puzzle) to hold.

To the best of my knowledge there is no application of HST and HSW models to other country other the U.S. (accordingly to Econlit, ProQuest, Webscience, IDEAS, SSRN, JSTOR, and other search devices in use for economic research).

Some more general lessons (regarding the role of robustness in explaining asset pricing anomalies) may be extracted from extensions of HST-HSW-like models to emerging economy asset price data and maybe one can get new insights about the way financial markets operate in general.

To be more formal: it is arguable that it is more difficult to model the transition law for the vector of state variables of an emerging economy (B) compared to the developed economy case (A). In other words model misspecification of the transition law of B is likely to be more substantial (with this feature to be formally defined) than the distortion of the approximating model for case A. The discrepancy (as measured by relative entropy) between the approximate and distorted model is likely to be higher in case A (or for that matter the pessimism bound η need to be greater for economy B.

Some specific key points to be considered are:

i) What are the relevant economic variables to be included in the state variable vector x_t? One can initialy work with an important state variable, capital k_t: why there is more uncertainty in, say, capital accumulation optimal rule for B? This may have some appealing intuition and maybe it can be formally related to the level of investments in the economy.

ii) How is relative entropy defined in case B?

iii) In what sense will a multidimensional analysis of economic uncertainty help in matching the empirical results?

iv) Why feedbacking (of the uncertainty parameter θ_1 into the state variable x_{t+1}) will be more significant in case B?

v) Related the idiosyncrasies of economy B. In particular analyze the financial market data

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For ease of reading we divide the (many) references in sections of similar themes. Within each section we obeyed the required lexicographic ordering.

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