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mathematics

A new characterization of simple K_3-groups using same-order type

Uma nova caracterização de k_3-grupos simples usando o mesmo tipo de ordem

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ABSTRACT

Let G be a group, define an equivalence relation ~ as below:

$$\forall\, g, h \in G,\ g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of G and denoted by $\alpha(G)$. And G is said a α_n-group if $|\alpha(G)| = n$. Let $\pi(G)$ be the set of prime divisors of the order of G. A simple group of G is called a simple K_n-group if $|\pi(G)| = n$. We give a new characterization of simple K_3-groups using same-order type. Indeed we prove that a nonabelian simple group G has same-order type {r, m, n, k, l} if and only if $G \cong PSL(2,q)$, with q = 7, 8 or 9. This result generalizes the main results in (4), (6) and (8). Moreover based on the main result in (8) we have the natural question: Let S be a nonabelian simple α_n-group and G a α_n-group such that $|S| = |G|$. Then $S \cong G$. In this paper with a counterexample we give a negative answer to this question.

Keywords: Element order; Same-order type; Characterization; Simple group; K_n-group simple

RESUMO

Seja G um grupo, definimos como uma relação de equivalência ~:

$$\forall\, g, h \in G,\ g \sim h \iff |g| = |h|$$

O tamanho do conjunto de classes de equivalência dado por essa relação é chamado de mesmo tipo de ordem de G e denotado por $\alpha(G)$. E G é chamado de um α_n-group se $|\alpha(G)| = n$. Seja $\pi(G)$ o conjunto dos divisores primos da ordem de G. Um grupo simples de ordem G é chamado de K_n- grupos simples se $|\pi(G)| = n$. Caracterizamos esses K_3- grupos simples usando outros de mesma ordem. Na verdade nós provamos que um grupo não abeliano G tem o mesmo tipo de ordem {r, m, n, k, l}, se e somente se, G

≅ PSL(2,q), com q = 7, 8 ou 9. Este é um resultado generalizado e os principais resultados em (4), (6) e (8). Além disso, com base no resultado principal em (8) nós temos uma questionamento natural: Seja S um grupo simples não abeliano α_n-grupo e G a α_n-grupo de tal modo que |S| = |G|. Então S ≅ G. Neste artigo, com um contra-exemplo, damos uma resposta negativa a essa pergunta.

Palavras-chave: Ordem dos elementos; Mesmo tipo de ordem; Caracterização; Grupo simples; K_n-grupos simples

1 INTRODUCTION

In this paper all the groups we consider are finite.

Let G a group and $\pi_e(G)$ be the set of element orders of G. Let $t \in \pi_e(G)$ and s_t be the number of elements of order t in G. Let $nse(G) = \{s_t | \, t \in \pi_e(G)\}$ the set of sizes of elements with the same order in G. Some authors have studied the influence of $nse(G)$ on the structure of G (see (1), (5), (8) and (9)). For instance ***R***. Shen in (6) proved that $A_4 \cong PSL(2, 3)$, $A_5 \cong PSL(2, 4) \cong PSL(2, 5)$ and $A_6 \cong PSL(2, 9)$ are uniquely determined by $nse(G)$. As a continuation in (4) was proved that if G is a group such that $nse(G) = nse(PSL(2, q))$, where $q \in \{7, 8, 11, 13\}$, then $G \cong PSL(2, q)$. In (7) and (8) new characterizations of A_5 were given using $nse(A_5)$. The authors in (7) proved that A_5 is the only group such that $nse(A_5) = \{1, 15,20,24\}$ and the authors in (8) generalized that a nonabelian simple group G has same-order type $\{r, m, n, k\}$ if and only if $G \cong A_5$ (see Th. 1.1 (8)).

Let G be a group, in (8) was defined an equivalence relation ~ as below:

$$\forall\, g, h \in G,\ g \sim h \Longleftrightarrow |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of G and denoted by $\alpha(G)$. And G is said a α_n-group if $|\alpha(G)| = n$. Note that $\alpha(G)$ is equal to the set of sizes of elements with the same order in G, hence $|nse(G)| = |\alpha(G)|$.

We give a new characterization of $PSL(2,7)$, $PSL(2,8)$ and $PSL(2,9)$ using same-order type.

THEOREM 1.1. Let G be a simple K_3-*group* with same-order type *{r, m, n, k, l}*. Then $G \cong PSL(2,7)$, $PSL(2,8)$ *or* $PSL(2,9)$.

This result generalizes the main results in (4), (6) and (8). Combination the main results in (4) and (6) with Theorem 1.1 we have the following result

COROLLARY 1.2. A simple K_3-*group* G has same-order type $\{r, m, n, k, l\}$ if and only if $G \cong PSL(2,7)$, $PSL(2,8)$ or $PSL(2,9)$.

We see easily that the only α_1-groups are 1 and a cyclic group of order 2. In (6) R. Shen characterized α_2-*group* as nilpotent groups and α_3-*group* as solvable groups. Moreover Taghvasani-Zarrin (see Th. 1.1 in (8)) showed that the only nonabelian simple α_4-*group* is the A_5. As noted in (4) and (8) finite groups G cannot be determined by $nse(G)$. Indeed in 1987 Thompson gave a first example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = PSL(3,4) \rtimes C_2$ be the maximal subgroups of Mathieu group M_{23}. Then $nse(G_1) = nse(G_2)$, but $G_1 \not\cong G_2$.

Motived by the main result in (8) about a new characterization of A_5 using same-order type, we have the natural question.

QUESTION 1.3. Let S be a nonabelian simple α_n-*group* and G *a* α_n-*group* such that $|S| = |G|$. *Then* $S \cong G$.

We give a negative answer to this question in the last section.

2 PROOF OF THEOREM 1.1

We need of one preliminary result to prove the main Theorem. The following result is a property very interesting of simple groups (see Lemma 2.7 in (8)).

LEMMA 2.1. Let G be a nonabelian simple group. Then there exist two odd prime divisors p and q of the order of G such that $s_p \neq s_q$.

In fact if G is a nonabelian simple group then there exist two odd prime divisors p and q of the order of G such that $\{1, s_2, s_p, s_q\} \subseteq \alpha(G)$ (see Corollary 2.8 in (8)).

We are now ready to conclude the proof of main Theorem.

Proof of Theorem 1.1: As G is a nonabelian simple group, it follows that $s_2 > 1$, w.l.g. $r = 1$ and $s_2 = m$. From Lemma there exist odd prime divisors p and q of the order of G such that $n = s_p \neq s_q = k$, hence $\pi(G) = \{2,p,q\}$ because G is a simple K_3-*group*. Therefore $\{1, s_2, s_p, s_q\} \subseteq \alpha(G) = \{r, m, n, k, l\}$. So there exist a divisor $t \notin \pi(G)$ of order of G such that $s_t = l$. It's well known that the only nonabelian simple groups of order divisible by exactly three primes are the following eight groups: PSL(2,q), where $q \in \{5, 7, 8, 9, 17\}$, PSL(3,3), PSU (3,3) and PSU (4,2), see Th. 1 and Th. 2 in (3). Now we arguing as in the proof of Th. 1.1 in (8) and a GAP check yields that $|\alpha(PSL(2,7))| = 5$, $|\alpha(PSL(2,8))| = 5$, $|\alpha(PSL(2,9))| = 5$ and all others groups are α_n-*group* with $n \geq 6$ (except A_5 since $|\alpha(A_5)| = 4$). The result is follows.

3 A COUNTEREXAMPLE TO A QUESTION 1.3

Now we give a counterexample to the Question 1.3. Firstly we observed that by the main Theorem in (6), we have that $\alpha(PSL(2,7))$ is uniquely determined and we have that $\alpha(PSL(2,7)) = \{1,21,56,42,48\}$ hence $PSL(2,7)$ is a α_5-*group*. Let $G = Q_8 \times (C_7 \rtimes C_3)$, where Q_8 is the quaternion group of order 8. As $|G| = 168$ and G is a soluble group then is sufficient to prove that $|\alpha(G)| = 5$. Indeed the only 2-Sylow subgroup Q_8 is a normal subgroup of G and using Sylow's Theorem it follows that $s_2 = 8$. Note that a 7-Sylow subgroup of G is isomorphic to C_7 and is a normal subgroup of $Q_8 \cdot C_7$ and $C_7 \rtimes C_3$, hence the normalizer N of C_7 has same order of G. Again from Sylow's Theorem we have that C_7 is a normal subgroup of G and $s_7 = 56$.

As the number of 3-Sylow subgroup of G is 7, then $s_3 = 14$. The number of elements of G of order 2, 4 are respectively 1 and 6, hence $s_2 = 1, s_4 = 6$ and consequently $s_6 = 14$, $s_{12} = 84$, $s_{14} = 6$ and $s_{28} = 36$ (because of direct product in the structure of G). Therefore $\alpha(G) = \{1,1,14,6,14,6,84,6,36\}$ and G is a α_5-group. Clearly $|PSL(2,7)| = 168 = |G|$ but $PSL(2,7) \ncong G$.

We can obtain others groups G with the computational group theory system GAP (2): $G = C_7 \times (Q_8 \rtimes C_3)$ or $G = C_2 \times ((C_{14} \times C_2) \rtimes C_3)$. These groups are also counterexamples to the Question 1.3.

4 CONCLUSION

We give the new characterization of some simple groups using the same-order type. Also we give a negative answer for a natural question. Our main result generalizes some known results. There is a natural interest in this theme. This result depends on classification of finite simple groups (CFSG).

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