Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

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Brasília

2020

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Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos necessários para obtenção do título de Doutor em Matemática.

†O autor contou com apoio financeiro CAPES e CNPq durante a realização deste trabalho.

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DOUTOR EM MATEMÁTICA

Brasília, 21 de outubro de 2020.

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Acknowlegments

This page is dedicated to pay homage to a select group of individuals and governmental Offices that influenced positively and decisively with the development of such complex work over the course of more than four long years.

I open with my sincere thanks to my lovely and understanding wife, Evelyn. Her patience and companionship, both remarkable and ever so rare to find, were fundamental to the birth of this piece of work. Such predicates also provided the raw materials needed for my emotional balance, much essential for the overcoming of the hardships inherent to the university's doctorate's program.

I also salute my professor doctors Lucio Centrone, Sérgio Antônio Tozoni, and Marcelo Santos for allowing, through their nominations, my admittance into this stage of the mathematics' post-graduate.

My gratitude goes toward the professor doctor Jaqueline Godoy Mesquita, advisor for this thesis, and toward the professor doctor Hernán Roberto Miranda Henriquez, a great collaborator to this piece of work. This research is mainly the reflection of the intellect, precision and objectivity of such intellectuals.

A warm thank-you to the members of the examining board for affording their priceless time into reading, evaluating and criticizing this investigation.

Finally, genuine acknowledgements to Capes and the CNPq for the financial support, in fact, it is necessary more than gratefulness. It is imperative to praise the competency of such Offices for they were able to masterfully perform their duties. Not even the adversities that came up over the past couple of years, especially financially, result of the intellectual obscurantism experienced in Brazil, were able to thwart these institution from performing their duties for the growth of the Brazilian research.

Resumo

Nesta tese, provamos uma série de resultados relacionados às equações diferenciais funcionais com retardo dependendo do estado. Na primeira parte deste trabalho, apresentamos resultados de existência de soluções fracas para as equações diferenciais funcionais com retardo dependendo do estado usando pontos fixos do operador solução de uma equação diferencial funcional com retardo dependendo do tempo. Também exibimos algumas aplicações dos nossos resultados para as equações diferenciais parciais.

Na segunda parte deste texto, investigamos a classe das equações diferenciais funcionais em medida com retardo dependendo do estado. Para elas, demonstramos resultados de existência e unicidade de soluções, dependência contínua com relação aos parâmetros, o método da média periódico e estabelecemos que as equações dinâmicas funcionais com retardo dependendo do estado em escalas temporais representam um caso particular dessas equações em medida. Além disso, mostramos a relação das suas soluções com as soluções de várias outras classes de equações diferenciais tais como as equações diferenciais funcionais em medida com impulsos e com retardo também dependendo do estado e as equações diferenciais ordinárias generalizadas.

Palavras–chave: Existência e unicidade; dependência contínua; método da média; equações diferenciais funcionais em medida, retardo dependendo do estado; equações diferenciais funcionais abstratas.

Abstract

In this thesis, we prove a series of results related to functional differential equations with state–dependent delay. In the first part of this work, we present results of existence of mild solutions for the delayed functional differential equations with state–dependent delays using fixed points of the solution operator of a functional differential equation with time–dependent delay. We also exhibit some applications of our results for partial differential equations.

In the second part of this text, we investigate the class of measure functional differential equations with state–dependent delay. For them, we demonstrate results of existence and uniqueness of solutions, continuous dependence on the parameters, the periodic averaging method and establish that the functional dynamic equations with state–dependent delay in time scales represent a particular case of these measure equations. In addition, we show the relationship between their solutions to the solutions of several other classes of differential equations such as the impulsive measure functional differential equations with state–dependent delays and the generalized ordinary differential equations.

Keywords: Existence and uniqueness; continuous dependence; averaging method; measure functional differential equations; abstract functional differential equations; state–dependent delays.

CONTENTS

INTRODUCTION

The field of differential equations is certainly one of the most fundamental areas of mathematics, with an extensive, solid and useful theory. Its most common object, the set of ordinary differential equations (ODEs for short), may be used as a tool to describe a large number of real systems. A classical element of this set can be formulated as follows:

$$\begin{aligned} x'(t) &= f(t, x(t)), \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where the symbol $x(t)$ usually represents a present state at a specific time $t \in \mathbb{R}$ of an investigated phenomenon. In addition, any realistic problem governed by (1) does not take the dependence of all previous events into account. However, only in the late 1930s that this dependence was precisely expressed mathematically when Volterra created a realistic predador–prey model in [54] with equations with delayed arguments. It has been considered the starting point of the building of another subfield of differential equations: the field of functional differential equations (simply FDEs).

Despite sparse articles containing differential equations with retarded expressions during the previous years of the 1930s, a consistent content about such equations, and consequently, the development of the FDE subject, has been extensively expanded only recently. In particular, most works about equations with time–dependent delays and state–dependent delays appeared during the past 50 years, with the R. D. Driver's mathematical approach for a two–body problem of classical electrodynamics in [18]. In this formulation, the position $x_i(t)$, $i = 1, 2$, for two charged particles of magnitude q_i moving along the x–axis is analyzed using the Líenard–Wiechert potential, into the Lorentz–Abraham force law. Time delays $\tau_{ji}(t)$, $j = 1, 2$, are incorporated due to the finite speed of propagation of electrical effects. Under a group of suitable conditions and denoting by $v_i(t)$ the velocity of the charges, by $E(t, x)$ the external electric field, by c the speed of light and a_i a constant that depends on the rest mass m_i, the model is the system of differential equations involving time delays below:

$$\begin{aligned} x_i(t) &= v_i(t) \\ \tau'_{ji}(t) &= \frac{(-1)^i v_i(t) - (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))} \\ \frac{v'_i(t)}{(1 - v_i^2(t)/c^2)^{3/2}} &= \frac{c(-1)^i a_i}{\tau_{ji}^2(t)} \left(\frac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))} \right) + q_i E(t, x_i(t))/m_i. \end{aligned}$$

Ordinary differential equations also provide an immediate relation to integration theory. Indeed, under suitable conditions on the function f, the equality

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\mathrm{d}s \tag{2}$$

offers the solution for the initial–value problem (1), where the right–hand side of (2) shows the importance of the integration method chosen. Most mathematicians are familiar with the Riemann integral, an integral that was created in the 1850s by the German mathematician Bernhard Riemann. Its intuitive definition and its large range of applicability are the main reasons that this concept has become notorious and has been extensively used by a considerable group of researchers. At the same time, many other authors have discovered a lot of its drawbacks, inspiring the scientific community to formulate alternative integrations theories not only to solve all problems of the Riemann integral, but to generalize the German mathematician's formulation as well. In 1957, with a slight adjustment of the Riemann's definition, Jaroslav Kurzweil successfully conceived a new type of integration that nowadays, in the literature, has his name. Automatically, it also has led to the concept of the generalized ordinary differential equation, which generalizes, as the name suggests, the notion of ODEs.

Some scientists have obtained few connections between generalized ODEs and FDEs. In 1966, the papers [37, 47] presented the first link between theses subjects for the classical FDEs

$$\begin{aligned} x'(t) &= f(t, x_t), \\ x_{t_0} &= x_0, \end{aligned}$$

where $t \in [t_0, t_0+\sigma]$ and $x_t\colon [-r, 0] \to \mathbb{R}$ is defined by $x_t(\theta) = x(t+\theta)$. Later, Federson and Schwabik in [23] extended this result with a correspondence between the impulsive FDEs and generalized ODEs. This sort of correspondence permits a simultaneous examination of the properties of both types of equations such as existence, uniqueness and regularity of solutions, stability principles and so on.

Inspired by all lines above, this thesis is intended to make a deep investigation on FDEs with state–dependent delays, together with their association to distinct types of FDEs and generalized ODEs. It begins with an introductory chapter with numerous fundamental concepts that may be used as an auxiliary tool to understand all subsequent chapters. It is composed of four sections, where the first section is a brief report about a significant set: the space $G([a, b], \mathbb{R}^n)$ of all regulated functions $f\colon [a, b] \to \mathbb{R}^n$. The subsequent sections are devoted to introducing three different types of integration, namely, the Bochner integral, the Kurzweil integral and the integration on time scales in the sense of Kurzweil–Henstock.

The second chapter, also divided into four sections, deals with abstract FDEs. After an explanation, in the first section, of all attributes that a phase space must have, the following part is concerned with obtaining some existence and uniqueness results of an abstract FDE with time–dependent delay. In the third section, those theorems will be applied to show the existence and uniqueness of solutions of the main equation of the chapter. Finally, an application on diffusion systems is presented to illustrate the significance of all developed concepts.

Chapter 3 is dedicated to measure FDEs with state–dependent delays, the second main goal of this thesis. It is divided into three sections and starts with a presentation of another appropriate Banach space to investigate these equations, which is called phase space as well. Then, existence and uniqueness of solutions are demonstrated for this class of equation. To prove the existence of solutions, we employ the Schauder fixed point theorem and for the uniqueness, we use the generalized Gronwall's inequality. The third part is about a periodic averaging principle for these equations and we also present an example.

Chapter 4 exhibits plenty correspondences between measure FDEs with state–dependent delays and other classes of differential equations. More precisely, a correspondence between measure FDEs with state–dependent delays and generalized ODEs is estabilished. Also, we demonstrate that measure FDEs with state–dependent delays encompass impulsive measure FDEs with state–dependent delays and functional dynamic equations on time scales with state–dependent delays. Using one of the correspondences, we obtain a local existence and uniqueness of solutions for measure FDEs with state–dependent delays. Finally, we prove the results on continuous dependence on parameters for these equations.

All new theorems contained in this PhD thesis have generated three papers (see [31, 32, 33]).

CHAPTER 1

PRELIMINARY CONCEPTS

In this chapter, we exhibit all the fundamental definitions and theorems related to many different concepts needed to develop and comprehend all subsequent chapters. Most of these concepts are natural generalizations of the basic and well–known theory formulated for the set of real numbers, usually explained in graduate courses.

Firstly, we introduce the space of regulated functions. Then, the next three sections are dedicated to show three different types of integrals. The references suggested for those who are interested in the details are [3, 7, 8, 15, 25, 44, 48, 49, 52].

1.1 The space $G([a, b], \mathbb{R}^n)$

Throughout this work, let us consider $(\mathbb{R}, \|\cdot\|)$ and $(\mathbb{R}^n, \|\cdot\|)$.

The set $C([a, b], \mathbb{R}^n)$ of all continuous functions $f\colon [a, b] \to \mathbb{R}^n$ is one of the most important spaces in mathematics. However, in some cases, continuity is considered a strong condition. The existence of a large set of discontinuous functions which are Riemann integrable shows, for instance, how restrictive that assumption can be. Fortunately, researchers have found a set that is as useful as $C([a, b], \mathbb{R}^n)$, has similar characteristics and has some crucial advantages over $C([a, b], \mathbb{R}^n)$. This is the set of all regulated functions, whose definition is given below.

Definition 1.1.1. *Let $a, b \in \mathbb{R}$, $a < b$. A function $f\colon [a, b] \to \mathbb{R}^n$ is called* ***regulated*** *if* *all lateral limits*

$$f(t^-) := \lim_{s\to t^-} f(s), \quad t \in (a,b] \qquad \text{and} \qquad f(t^+) := \lim_{s\to t^+} f(s), \quad t \in [a,b)$$

exist. The space of all regulated functions $f\colon [a,b] \to \mathbb{R}^n$ *will be denoted by* $G([a,b],\mathbb{R}^n)$*. Likewise,* $G((-\infty,0],\mathbb{R}^n)$ *denotes the set of all regulated functions* $f\colon (-\infty,0] \to \mathbb{R}^n$*.*

Similarly to $C([a,b],\mathbb{R}^n)$, the space $G([a,b],\mathbb{R}^n)$ is a Banach space when endowed with the usual supremum norm

$$\|f\|_\infty := \sup_{s\in[a,b]} |f(s)|, \quad f \in G([a,b],\mathbb{R}^n)$$

([44, Theorem 4.2.1]). Also, given $g \in G([a,b],\mathbb{R}^n)$, $\Delta^+ g(t)$ and $\Delta^- g(t)$ will be symbolized by

$$\Delta^+ g(t) := g(t^+) - g(t), \quad t \in [a,b) \quad \text{and} \quad \Delta^- g(t) := g(t) - g(t^-), \quad t \in (a,b].$$

Evidently, $C([a,b],\mathbb{R}^n) \subset G([a,b],\mathbb{R}^n)$. Nevertheless, this inclusion is strict because, for example, the characteristic function $\chi_c\colon [a,b] \to \mathbb{R}$, where $c \in (a,b)$, belongs to $G([a,b],\mathbb{R})$ and is not continuous at c.

Remark 1.1.2. *Usually, the composition of two regulated functions is not a regulated function. Indeed, if we consider the functions* $f, g\colon [0,1] \to \mathbb{R}$ *given by*

$$f(t) = \begin{cases} 0, & t = 0, \\ t\sin(1/t), & t \in (0,1], \end{cases} \qquad g(t) = sgn\ t,$$

then both functions are regulated, but the composition $g \circ f$ *is not. On the other hand, if* g *is continuous and* f *is regulated, then* $g \circ f$ *is regulated. See [13] for more details.*

In the sequel, we present a definition on a family $\mathcal{F} \subset G([a,b],\mathbb{R}^n)$ which resembles the definition of equicontinuity on a family $\mathcal{A} \subset C([a,b],\mathbb{R}^n)$.

Definition 1.1.3 ([44, Definition 4.3.3])**.** *A set* $\mathcal{F} \subset G([a,b],\mathbb{R}^n)$ *has* ***uniform one–sided limits at a point*** $\boldsymbol{t_0 \in [a,b]}$ *if for every* $\varepsilon > 0$*, there is* $\delta > 0$ *such that for every* $x \in \mathcal{F}$ *and* $t \in [a,b]$*, we have:*

(i) If $t_0 < t < t_0 + \delta$*, then* $\|x(t) - x(t_0^+)\| < \varepsilon$*.*

(ii) If $t_0 - \delta < t < t_0$*, then* $\|x(t_0^-) - x(t)\| < \varepsilon$*.*

*The set $\mathcal{F}$ is called **equiregulated** if it has uniform one–sided limits at every point $t_0 \in [a,b]$. Also, $\mathcal{F}$ is called **right–sided** (respectively, **left–sided**) **equiregulated** if condition (i) (respectively, (ii)) holds.*

The next theorem is a similar version of the classical Arzelà–Ascoli Theorem to a family $\mathcal{A} \subset G([a,b],\mathbb{R})$.

Theorem 1.1.4 ([44, Corollary 4.3.7]). *A subset $\mathcal{A} \subset G([a,b],\mathbb{R})$ is relatively compact if and only if it is equiregulated and the set $\{f(t) \colon f \in \mathcal{A}\}$ is bounded for each $t \in [a,b]$.*

Lastly, we present a result that associates relatively compactness of a family $\mathcal{A} \subset G([a,b],\mathbb{R}^n)$ to some properties that all elements of $\mathcal{A}$ must satisfy.

Theorem 1.1.5 ([25, Theorem 2.18]). *The following conditions are equivalent.*

(i) $\mathcal{A} \subset G([a,b],\mathbb{R}^n)$ *is relatively compact.*

(ii) *The set $\{x(a) \colon x \in \mathcal{A}\}$ is bounded and there is an increasing continuous function $\eta \colon [0,\infty) \to [0,\infty)$ with $\eta(0) = 0$ and there is an increasing function $K \colon [a,b] \to \mathbb{R}$ such that*

$$\|x(\tau_2) - x(\tau_1)\| \leqslant \eta(K(\tau_2) - K(\tau_1))$$

for all $x \in \mathcal{A}$ and all $a \leqslant \tau_1 \leqslant \tau_2 \leqslant b$.

1.2 The Bochner integral

In this section, we expose briefly a type of an integral in a Banach space that is an immediate generalization of the Lebesgue integral (see [15], for more details). Throughout it, (X,σ,μ) is a σ–finite measure space on a set X and Y is a Banach space with norm $\|\cdot\|$.

Definition 1.2.1 ([15, Definition 2.1.1]). *A function $s \colon X \to Y$ is called **simple** when its range is a finite set $s(X) = \{y_1,\dots,y_n\}$ and $E_i = s^{-1}(\{y_i\}) \in \sigma$ for all $i \in \{1,2,\dots,n\} \subset \mathbb{N}$. In this case, we write*

$$s(x) = \sum_{i=1}^{n} \chi_{E_i}(x) y_i, \tag{1.1}$$

where $\chi_{E_i} \colon X \to \mathbb{R}$ is the characteristic function on E_i.

Definition 1.2.2 ([15, Definition 2.1.1]). *A function $f\colon X \to Y$ is said to be* ***measurable*** *if there is a sequence of simple functions $s_n\colon X \to Y$ such that $s_n \to f$ for almost every $x \in X$.*

Given a simple function $s\colon X \to Y$ represented as (1.1), we define the **Bochner integral** $\displaystyle\int_X s \,\mathrm{d}\mu$ of s on X as follows:

$$\int_X s \,\mathrm{d}\mu := \sum_{k=1}^{n} \mu\left(E_k\right) y_k. \tag{1.2}$$

It is worth mentioning that the integral (1.2) does not depend on the representation (1.1). In other words, if

$$s(x) = \sum_{j=1}^{m} \chi_{F_j}(x) z_j,$$

where $m \in \mathbb{N}$, $F_j \in \sigma$, $z_j \in Y$ and, for all $i \neq j$, $F_i \cap F_j = \varnothing$, then

$$\sum_{k=1}^{n} \mu\left(E_k\right) y_k = \sum_{j=1}^{m} \mu\left(F_j\right) z_j.$$

Definition 1.2.3 ([15, Definition 2.2.1]). *Let $f\colon X \to Y$ be a measurable function. We say that f is* ***Bochner integrable*** *when there is a sequence of simple functions $(s_n)_{n\in\mathbb{N}}$ whose its limit is f for almost every $x \in X$ and the Lebesgue integral*

$$\int_X \|f - s_n\| \,\mathrm{d}\mu \to 0 \quad \textit{when } n \to \infty. \tag{1.3}$$

In this case, we define the ***Bochner integral*** *of f on X by the equality*

$$\int_X f \,\mathrm{d}\mu := \lim_{n\to\infty} \int_X s_n \,\mathrm{d}\mu. \tag{1.4}$$

Since Y is a Banach space, the limit (1.4) exists for any sequence $(s_n)_{n\in\mathbb{N}}$ as described in Definition 1.2.3. Moreover, if $(t_n)_{n\in\mathbb{N}}$ is a sequence of simple functions converging to f for almost every $x \in X$ and such that

$$\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X t_n \,\mathrm{d}\mu, \tag{1.5}$$

then

$$\lim_{n\to\infty} \int_X t_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X s_n \,\mathrm{d}\mu.$$

It means that (1.4) does not depend on the choice of the sequence $(s_n)_{n\in\mathbb{N}}$.

The theorem below shows an expected property of the Bochner integral: the linearity of the integral. Since it is a direct consequence of all definitions above, we will omit its proof.

Theorem 1.2.4. *If $f, g\colon X \to Y$ are integrable functions and $\alpha \in \mathbb{R}$, then $f + \alpha g$ is integrable and*

$$\int_X (f + \alpha g)\ \mathrm{d}\mu = \int_X f\ \mathrm{d}\mu + \alpha \int_X g\ \mathrm{d}\mu.$$

The next result illustrates a relation between the Bochner integral and the Lebesgue integral. It also enables us to carry over some classical theorems from Lebesgue integral on $\mathbb{R}$ to the vector–valued case.

Theorem 1.2.5 ([15, Theorem 2.2.4])**.** *A function $f\colon X \to Y$ is Bochner integrable if, and only if, the function $|f|\colon X \to \mathbb{R}$ is Lebesgue integrable. In this case,*

$$\left\| \int_X f\ \mathrm{d}\mu \right\| \leqslant \int_X |f|\ \mathrm{d}\mu. \tag{1.6}$$

The upcoming theorem is a version of the Dominated Convergence Theorem for the Bochner integral.

Theorem 1.2.6 ([15, Theorem 2.2.3])**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions from X to Y and let $f\colon X \to Y$ be a measurable function such that $f_n \to f$ for almost every $x \in X$. Furthermore, let $g \in L^1(X)$ be such that $\|f_n\| \leqslant g$ for almost every $x \in X$ and all $n \in \mathbb{N}$. Then, f is Bochner integrable and*

$$\int_X f\ \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n\ \mathrm{d}\mu.$$

From now on, σ will indicate the Borel σ–algebra and μ will denote the Lebesgue measure. Besides, all integrals below are in the sense of Bochner integral.

Definition 1.2.7 ([48, Definition 1.1])**.** *An one parameter family $(T(t))_{t\geqslant 0}$ of bounded linear operators $T(t)\colon Y \to Y$ is a* ***semigroup of bounded linear operators*** *on Y if*

(i) $T(0) = I$, where I is the identity operator,

(ii) $T(s + t) = T(s)T(t)$ for every $t, s \geqslant 0$.

If, in addition, for each $x \in Y$,

$$\lim_{t\to 0^+} T(t)x = x,$$

we say that $(T(t))_{t\geqslant 0}$ is a ***strongly continuous semigroup of bounded linear operators*** *on Y (or a* ***C_0–semigroup*** *on Y). Finally, we say that $(T(t))_{t\geqslant 0}$ is* ***compact*** *if $T(t)$ is a compact operator for each $t \geqslant 0$.*

Definition 1.2.8 ([48, Definition 1.1]). *Let $(T(t))_{t\geqslant 0}$ be a semigroup of bounded linear operators on Y and let*

$$D(A) = \left\{ x \in Y \colon \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

The operator $A\colon D(A) \to Y$ defined by

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t}$$

*is called the **infinitesimal generator** of the semigroup $(T(t))_{t\geqslant 0}$.*

Theorem 1.2.9 ([48, Theorem 2.2]). *If $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup of bounded linear operators on Y, then there exist constants $M \geqslant 1$ and $\omega \geqslant 0$ such that*

$$\|T(t)\| \leqslant Me^{\omega t}, \quad t \geqslant 0.$$

Theorem 1.2.10 ([48, Theorem 2.4]). *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup of bounded linear operators on Y and let A be its infinitesimal generator. Then:*

(i) for $x \in Y$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)x\mathrm{d}s = T(t)x,$$

(ii) for $x \in Y$, $\int_0^t T(s)x\mathrm{d}s \in D(A)$ and

$$A\left(\int_0^t T(s)x\mathrm{d}s\right) = T(t)x - x,$$

(iii) for $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(u)Ax\mathrm{d}u = \int_s^t AT(u)x\mathrm{d}u.$$

1.3 The Kurzweil integral

In this section, we will define the Kurzweil integral. Then, the Kurzweil–Henstock integral and the Kurzweil–Henstock–Stieltjes integral will appear as particular cases. Throughout this section, X will denote a Banach space with norm $\|\cdot\|$.

Let $[a,b]$ be an interval of $\mathbb{R}$ such that $-\infty < a < b < \infty$, $a = s_0 < s_1 < \cdots < s_p = b$ is a finite division of $[a,b]$, $p \in \mathbb{N}$, and $\tau_i \in [s_{i-1}, s_i]$. The collection of point–interval pairs $D = (\tau_i, [s_{i-1}, s_i])$ is called a **tagged division** of $[a,b]$ and we write $p = |D|$.

A **gauge** on a set $B \subset [a,b]$ is any function $\delta\colon B \to (0,\infty)$. Given a gauge δ on $[a,b]$, we say that a tagged division $D = (\tau_i, [s_{i-1}, s_i])$ is $\boldsymbol{\delta}$**–fine** if for every $i \in \{1, 2, \dots, |D|\}$, we have

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

Definition 1.3.1 ([26, Definition 2.1])**.** *A function $U\colon [a,b] \times [a,b] \to X$ is called **Kurzweil integrable** if there is an element $I \in X$ such that for every $\varepsilon > 0$, there is a gauge $\delta\colon [a,b] \to (0,\infty)$ such that*

$$\left\| \sum_{i=1}^{|D|} (U(s_i, \tau_i) - U(s_{i-1}, \tau_i)) - I \right\| < \varepsilon,$$

*for all δ–fine tagged division of $[a,b]$. In this case, I is called **Kurzweil integral** of U and it will be denoted by $\int_a^b DU(t,\tau)$.*

From the definition above, a question about the existence of at least one δ–fine division of $[a,b]$ from a given gauge δ on $[a,b]$ arises. The answer for this question is given by the lemma below, known in the literature as the Cousin Lemma. It ensures that the Kurzweil integral is well–defined.

Lemma 1.3.2 ([52, Lemma 1.4])**.** *Given a gauge δ on $[a,b]$, there is a δ–fine tagged division of $[a,b]$.*

When the function U of the Definition 1.3.1 is given by $U(t,\tau) = tf(\tau)$, where $f\colon [a,b] \to X$ is any function, and it is Kurzweil integrable, then we say that f is **Kurzweil–Henstock integrable**. Additionally, its integral is denoted by $\int_a^b f(s)\mathrm{d}s$. On the other hand, when $U(t,\tau) = g(t)f(\tau)$ is Kurzweil integrable, where $g\colon [a,b] \to \mathbb{R}$ is any other function, we say that f is a **Kurwzeil–Henstock–Stieltjes integrable function with respect to** $\boldsymbol{g}$ and its integral is denoted by $\int_a^b f(s)\mathrm{d}g(s)$.

Theorem 1.3.3 ([52, Theorem 1.9])**.** *If $U, V\colon [a,b] \times [a,b] \to X$ are Kurzweil integrable functions and $c_1, c_2 \in \mathbb{R}$, then $c_1U + c_2V$ is Kurzweil integrable and*

$$\int_a^b D\left(c_1U(t,\tau) + c_2V(t,\tau)\right) = c_1 \int_a^b DU(t,\tau) + c_2 \int_a^b DV(t,\tau).$$

Theorem 1.3.4 ([52, Theorem 1.11])**.** *Let $U\colon [a,b] \times [a,b] \to X$ and $c \in (a,b)$. If both integrals $\int_a^c DU(t,\tau)$ and $\int_c^b DU(t,\tau)$ exist, then the integral $\int_a^b DU(t,\tau)$ exists as well and*

$$\int_a^b DU(t,\tau) = \int_a^c DU(t,\tau) + \int_c^b DU(t,\tau).$$

Surprisingly, as the next example shows, if $U\colon [a,b]\times[a,b]\to X$ is a Kurzweil integrable function and $V\colon [a,b]\times[a,b]\to\mathbb{R}$ is defined by $V(t,\tau)=\|U(t,\tau)\|$, then, in general, V is not Kurzweil integrable. Thus, some care is needed when dealing with inequalities involving the norm of an integral. Specifically, the inequality

$$\left\|\int_a^b DU(t,\tau)\right\| \leqslant \int_a^b DV(t,\tau) \tag{1.7}$$

is not always true.

Example 1.3.5 ([3, Example 10.2.2]). *For $k\in\mathbb{N}$, let $c_k = 1-1/2^k$ and $f\colon [0,1]\to\mathbb{R}$ be the function defined by*

$$f(x)=\begin{cases}\dfrac{(-1)^k}{k}2^k, & c_{k-1}\leqslant x<c_k,\\ 0, & x=1.\end{cases}$$

It is possible to show that f is Kurzweil–Henstock integrable and

$$\int_0^1 f(x)\mathrm{d}x=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}.$$

However, its absolute value $|f|$ is not integrable because, otherwise, it would imply the convergence of the harmonic series.

Despite the inaccuracy of the inequality (1.7) in general, the theorem below illustrates that, with additional hypotheses, (1.7) may be true.

Theorem 1.3.6 ([52, Theorem 1.35]). *Assume that both functions $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ and $V\colon [a,b]\times[a,b]\to\mathbb{R}$ are Kurzweil integrable. If there is a gauge δ on $[a,b]$ such that*

$$|t-\tau|\|U(t,\tau)-U(\tau,\tau)\|\leqslant (t-\tau)(V(t,\tau)-V(\tau,\tau))$$

for every $t\in[\tau-\delta(\tau),\tau+\delta(\tau)]$, then the inequality

$$\left\|\int_a^b DU(t,\tau)\right\| \leqslant \int_a^b DV(t,\tau)$$

holds.

Now, we bring some basic results, particularly for the Kurwzeil–Henstock–Stieltjes integral, that will be applied in future sections.

Theorem 1.3.7 ([52, Corollary 1.34]). *Let $f\colon [a,b]\to\mathbb{R}^n$ be a regulated function and $g\colon [a,b]\to\mathbb{R}$ be a nondecreasing function. Then, the following conditions hold:*

(i) *The integral* $\displaystyle\int_a^b f(s)\mathrm{d}g(s)$ *exists;*

(ii) $\displaystyle\left\|\int_a^b f(s)\mathrm{d}g(s)\right\| \leqslant \int_a^b \|f(s)\|\,\mathrm{d}g(s) \leqslant \|f\|_\infty (g(b)-g(a)).$

The next property is an immediate consequence of Theorems 1.3.3 and 1.3.6. We will omit its proof since it is nearly the same as the demonstration of [4, Corollary 3.3].

Theorem 1.3.8. *Let* $f_1, f_2 \colon [a,b] \to \mathbb{R}$ *be Kurzweil–Henstock–Stieltjes integrable functions on the interval* $[a,b]$ *with respect to a nondecreasing function* $g\colon [a,b]\to\mathbb{R}$ *such that* $f_1(t) \leqslant f_2(t)$, *for* $t\in[a,b]$. *Then*

$$\int_a^b f_1(s)\mathrm{d}g(s) \leqslant \int_a^b f_2(s)\mathrm{d}g(s). \tag{1.8}$$

By analogous arguments of the proof of [4, Theorem 3.2], we may obtain the following corollary.

Corollary 1.3.9. *Let* $f\colon [a,b]\to\mathbb{R}$ *be a Kurzweil–Henstock–Stieltjes integrable function on the interval* $[a,b]$ *with respect to a nondecreasing function* $g\colon[a,b]\to\mathbb{R}$ *and such that* $f(t)\geqslant 0$, *for* $t\in[a,b]$. *Then:*

(i) $\displaystyle\int_a^b f(s)\mathrm{d}g(s) \geqslant 0.$

(ii) *The function* $\xi(t)\colon [a,b]\mapsto\mathbb{R}$ *defined by*

$$\xi(t) = \int_a^t f(s)\mathrm{d}g(s)$$

is nondecreasing.

The next statement is a type of Gronwall–inequality for Kurwzeil–Henstock–Stieltjes integrals.

Theorem 1.3.10 ([44, Theorem 7.5.3], Gronwall Inequality)**.** *Let* $g\colon[a,b]\to[0,\infty)$ *be a nondecreasing and left–continuous function,* $k\geqslant 0$ *and* $l>0$. *Assume that* $\psi\colon[a,b]\to[0,\infty)$ *satisfies*

$$\psi(\xi) \leqslant k + l\int_a^\xi \psi(s)\mathrm{d}g(s), \quad \xi\in[a,b].$$

Then $\psi(\xi)\leqslant ke^{l(g(\xi)-g(a))}$ *for all* $\xi\in[a,b]$.

The following result, which describes some properties of the indefinite Kurzweil–Henstock–Stieltjes integral, is a special case of [52, Theorem 1.16].

Theorem 1.3.11 ([44, Corollary 6.5.5]). *Let $f, g\colon [a,b] \to \mathbb{R}$ be a pair of functions such that g is regulated and $\int_a^b f(s)\mathrm{d}g(s)$ exists. Then the function*

$$h(t) = \int_a^t f(s)\mathrm{d}g(s), \quad t \in [a,b]$$

is regulated on $[a,b]$ and satisfies

$$\begin{aligned} h(t^+) &= h(t) + f(t)\Delta^+ g(t), \quad t \in [a,b), \\ h(t^-) &= h(t) - f(t)\Delta^- g(t), \quad t \in (a,b]. \end{aligned}$$

In the sequel, we bring the definition and some attributes of generalized ordinary differential equations, a class of equations that will be used in the third chapter of this work. From now on, $\mathcal{O} \subset X$ is an open and nonempty subset, $\Omega = [a,b] \times \mathcal{O}$ and $F\colon \Omega \to X$ is a function.

Definition 1.3.12 ([26, Definition 2.5]). *A function $x\colon [a,b] \to X$ is called a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DF(t,x) \tag{1.9}$$

on the interval $[a,b]$ if, for every $t \in [a,b]$, $(t, x(t)) \in \Omega$ and

$$x(d) - x(c) = \int_c^d DF(t, x(\tau)) \tag{1.10}$$

whenever $[c,d] \subset [a,b]$.

Remark 1.3.13. *In general, a solution of the generalized ODE (1.9) does not need to be differentiable at $[a,b]$, although its notation suggests such differentiability. In fact, a continuous function $r\colon [a,b] \to \mathbb{R}$ that has no derivative at any point of $[a,b]$ (an example can be found on [3], page 367) is a solution of the generalized ODE*

$$\frac{dx}{dt} = Dr(t)$$

since, by Definition 1.3.1,

$$\sum_{i=1}^{|D|} (F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))) = \sum_{i=1}^{|D|} (r(s_i) - r(s_{i-1})) = r(d) - r(c),$$

where $F(t,x) = r(t)$ and $D = (\tau_i, [s_{i-1}, s_i])$ is any tagged division of $[c,d] \subset [a,b]$.

If $(s_0, x_0) \in \Omega$ is fixed, then we can define the solution of the generalized ODE (1.9) on the interval $[a,b]$ with initial condition $x(s_0) = x_0$ (we are considering that $s_0 \in [a,b]$), as a function $x\colon [s_0, b] \to X$ such that $(t, x(t)) \in \Omega$ for all $t \in [s_0, b]$ and (1.10) is satisfied for all $[c, d] \subset [s_0, b]$. In the similar way, we can also define a solution of (1.9) for an arbitrary nondegenerate interval I with initial condition $x(s_0) = x_0$.

The next definition is an important prerequisite to estabilish some existence and uniqueness result for generalized ODEs.

Definition 1.3.14 ([26, Definition 2.6]). *We say that* ***F belongs to the class $\mathcal{F}(\Omega, h)$*** *if there exists a nondecreasing function $h\colon [a, b] \to \mathbb{R}$ such that $F\colon \Omega \to X$ satisfies the following conditions:*

(F1) For every $(s_i, x) \in \Omega$, with $i = 1, 2$, we have

$$\|F(s_2, x) - F(s_1, x)\| \leqslant |h(s_2) - h(s_1)|\,.$$

(F2) For every $(s_i, x), (s_i, y) \in \Omega$, with $i = 1, 2$,

$$\|F(s_2, x) - F(s_1, x) - F(s_2, y) + F(s_1, x)\| \leqslant |h(s_2) - h(s_1)|\,\|x - y\|.$$

The upcoming lemma gives us enough requisites to ensure the existence of the Kurzweil integral on the right–hand side of (1.10). When $X = \mathbb{R}^n$, the reader can see a proof of this result in [52, Corollary 3.16]. However, with analogous arguments used for the $\mathbb{R}^n$ case, it is still valid in a more general Banach space as we state below.

Lemma 1.3.15. *Assume $F \in \mathcal{F}(\Omega, h)$. Suppose $x\colon [a, b] \to X$ is a regulated function on $[a, b]$ such that $(s, x(s)) \in \Omega$ for all $s \in [a, b]$. Then the Kurzweil integral $\int_a^b DF(t, x(\tau))$ exists.*

The next conclusion reveals few characteristics of the solutions of the generalized ODEs when F satisfies the condition (F1). The special case $X = \mathbb{R}^n$ is demonstrated in [52, Lemma 3.12]. Since the proof for the general case follows the same steps, we will omit it here.

Lemma 1.3.16. *Let $F\colon \Omega \to X$ be a function that satisfies condition (F1). If $x\colon [a, b] \to X$ is a solution of the generalized ODE (1.9) on the interval $[a, b]$, then x is a regulated function and*

$$\|x(s_2) - x(s_1)\| \leqslant |h(s_2) - h(s_1)|$$

for each pair $s_1, s_2 \in [a, b]$.

Notice that this type of result shows that x has the same discontinuities of h.

1.4 Integration on Time Scales

In this section, we will start with a short exposure of some basic concepts in the theory of time scales. This theory appeared mainly to unify continuous and discrete analysis. As a result, some classical theorems applied to functions defined on $\mathbb{R}$ or defined on $\mathbb{Z}$ can be considered particular cases of a single assertion proved to a function defined on a time scale. Also, this approach can unify other results since there exist many different time scales such as Cantor set, the set $q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$, $q > 1$, among others. This fact makes this theory very alluring due applications, since it is possible to investigate models where the time has hybrid behavior (continuous and discrete) being very useful for population models and it allows us to investigate the theory for quantum calculus, which has applications in quantum physics

This section will be extensively used in the fourth chapter, where we study a relationship between functional dynamic equations on time scales with state–dependent delays and measure FDEs with state–dependent delays.

Definition 1.4.1 ([8, Definition 1.1])**.** *A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. We define, respectively, the **forward jump operator** and the **backward jump operator** $\sigma, \rho \colon \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} \colon\ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} \colon\ s < t\}$. The **graininess function** $\mu \colon \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.*

In this definition, we consider $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$.

Definition 1.4.2 ([8, Definition 1.1])**.** *Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is **right–scattered**. If $t < \sup \mathbb{T}$ and $t = \sigma(t)$, then t is called **right–dense**. We say that t is **left–scattered** if $\rho(t) < t$. Lastly, if $t = \rho(t)$ and $t > \inf \mathbb{T}$, then t is called **left–dense**.*

For a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will denote a closed interval in $\mathbb{T}$, while $[a, b]$ will denote the usual closed interval on the real line. In other words, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} \colon\ a \leqslant t \leqslant b\}$ and $[a, b] = \{t \in \mathbb{R} \colon\ a \leqslant t \leqslant b\}$. Similar notations can be used to numerous other cases such as $(a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ and so on. This notational convention should help the reader to distinguish between ordinary and time scale intervals.

For an arbitrary time scale $\mathbb{T}$, let

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

The next concept can be found in [8, Definition 1.10] for functions taking values in $\mathbb{R}$. The same definition may be extended to functions taking values in $\mathbb{R}^n$ as follows:

Definition 1.4.3 ([8, Definition 1.10])**.** *Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}^{\kappa}$ and $f\colon \mathbb{T} \to \mathbb{R}^n$ be a function. The vector $f^{\Delta}(t)$ is called a* ***Δ–derivative of f at t*** *if it satisfies the following property: for any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\left\| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right\| \leqslant \varepsilon \left| \sigma(t) - s \right| \quad \textit{for all } s \in (t - \delta, t + \delta)_{\mathbb{T}}.$$

Notice that if $\mathbb{T} = \mathbb{R}$, then the definition above is exactly the definition of the usual derivative $f'(t)$ from calculus since, in this case, $\sigma(t) = t$. It is one of the main reasons that several results in the theory of time scales generalize many theorems from the classical calculus.

Definition 1.4.4 ([8, Definition 1.57])**.** *A function $f\colon \mathbb{T} \to \mathbb{R}^n$ is called* ***regulated*** *provided its right–sided limits exist at all right–dense points in $\mathbb{T}$ and its left–sided limits exist at all left–dense points in $\mathbb{T}$. We denote this set by $G(\mathbb{T}, \mathbb{R}^n)$.*

Definition 1.4.5 ([8, Definition 1.58])**.** *A function $f\colon \mathbb{T} \to \mathbb{R}^n$ is called* ***rd–continuous*** *provided it is continuous at right–dense points in $\mathbb{T}$ and its left–sided limits exist at all left–dense points in $\mathbb{T}$. We denote this set by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$.*

If $\mathbb{T} = \mathbb{R}$, then $t \in \mathbb{T}$ is, at the same time, a right–dense and left–dense point, by definition. Thus, in this particular time scale, any regulated function in the sense of Definition 1.1.1 is a regulated function in the sense of the Definition 1.4.4. It shows consistency of Definition 1.4.4. The same applies to the definition of rd–continuous functions.

Next, we present some basic concepts which will allow us to introduce the Kurzweil–Henstock Δ–integral. The definition of such integral was presented for the first time in reference [49].

Definition 1.4.6 ([49, Definition 1.5])**.** *A* ***tagged division*** *of $[a, b]_{\mathbb{T}}$ is a finite collection of point–interval pairs $D_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$, where $a = s_0 < s_1 < \cdots < s_{|D|} = b$ is a division of $[a, b]_{\mathbb{T}}$, $\tau_i \in [s_{i-1}, s_i]_{\mathbb{T}}$ and $\tau_i, s_i \in \mathbb{T}$ $i = 1, 2, \ldots, |D|$, where the symbol $|D|$ denotes the number of subintervals in which $[a, b]_{\mathbb{T}}$ is divided.*

Definition 1.4.7 ([49, Definition 1.4]). *Given two functions* $\delta_L, \delta_R\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$, *we say that the pair* $\delta(t) = (\delta_L(t), \delta_R(t))$ *is a* $\boldsymbol{\Delta}$***–gauge*** *on* $[a,b]_{\mathbb{T}}$ *provided* $\delta_L(t) > 0$ *on* $(a,b]_{\mathbb{T}}$, $\delta_R(t) > 0$ *on* $[a,b)_{\mathbb{T}}$, $\delta_L(a) \geqslant 0$, $\delta_R(b) \geqslant 0$, *and* $\delta_R(t) \geqslant \mu(t)$ *for all* $t \in [a,b)_{\mathbb{T}}$.

Indeed, we can always assume that any Δ–gauge δ on $[a,b]_{\mathbb{T}}$ satisfies $\delta_L(a) \geqslant 0$ and $\delta_R(b) \geqslant 0$, since, otherwise, we can replace it to another Δ–gauge on $[a,b]_{\mathbb{T}}$ with this property.

Definition 1.4.8 ([49, Definition 1.6]). *If* δ *is a* Δ*–gauge on* $[a,b]_{\mathbb{T}}$, *then we say a tagged division* $D_{\mathbb{T}} = (\tau_i, [s_{i-1}, s_i]_{\mathbb{T}})$ *is* $\boldsymbol{\delta}$***–fine*** *if* $\tau_i - \delta_L(\tau_i) \leqslant s_{i-1} < s_i \leqslant \tau_i + \delta_R(\tau_i)$ *for* $1 \leqslant i \leqslant |D|$.

In the sequel, we define the Kurzweil–Henstock Δ–integral.

Definition 1.4.9 ([49, Definition 1.7]). *We say that* $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ *is* ***Kurzweil–Henstock*** $\boldsymbol{\Delta}$***–integrable*** *on* $[a,b]_{\mathbb{T}}$ *with value* I, *provided given any* $\varepsilon > 0$, *there exists a* Δ*–gauge* δ *on* $[a,b]_{\mathbb{T}}$ *such that*

$$\left\| \sum_{i=1}^{n} (s_i - s_{i-1}) f(\tau_i) - I \right\| < \varepsilon$$

for all δ*–fine tagged divisions* $D_{\mathbb{T}}$ *of* $[a,b]_{\mathbb{T}}$.

Once again, from the definition above, a question about the existence of at least one δ–fine division of $[a,b]_{\mathbb{T}}$ from a given Δ–gauge δ on $[a,b]_{\mathbb{T}}$ arises. As the reader may expect, the following lemma answers this question and ensures that the Kurzweil–Henstock Δ–integral is well–defined. It is a type of Cousin Lemma on time scales.

Lemma 1.4.10 ([49, Lemma 1.9]). *If* δ *is a* Δ*–gauge on* $[a,b]_{\mathbb{T}}$, *then there is a* δ*–fine tagged division* $D_{\mathbb{T}}$ *for* $[a,b]_{\mathbb{T}}$.

The same way as the Bochner integral and the Kurzweil–Henstock integral, the Kurzweil–Henstock Δ–integral satisfies the linearity and the additivity properties, both stated below.

Theorem 1.4.11 ([49, Theorem 2.12]). *If* $f, g\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ *are Kurzweil–Henstock* Δ*–integrable functions and* $c_1, c_2 \in \mathbb{R}$, *then* $c_1 f + c_2 g$ *is Kurzweil–Henstock* Δ*–integrable and*

$$\int_a^b c_1 f(t) + c_2 g(t) \Delta t = c_1 \int_a^b f(t) \Delta t + c_2 \int_a^b g(t) \Delta t.$$

Moreover, if $c \in [a,b]_{\mathbb{T}}$ and both $\int_a^c f(t)\Delta t$ and $\int_c^b f(t)\Delta t$ exist, then $\int_a^b f(t)\Delta t$ exists as well and

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$$

Now, we display important definitions which will allow us to obtain a correspondence between Kurzweil–Henstock Δ–integrals and Kurzweil–Henstock–Stieltjes integrals (see [20, 21, 50]).

Firstly, given a real number $t \leqslant \sup \mathbb{T}$, let

$$t^* = \inf\{s \in \mathbb{T} \colon s \geqslant t\}.$$

This definition was first introduced by A. Slavík in [50]. Notice that $t^* \in \mathbb{T}$ since $\mathbb{T}$ is a closed set. Moreover, even though both numbers t^* and $\sigma(t)$ have similar definitions, they may be different depending on the choice of the time scale. For example, if $\mathbb{T} = \mathbb{Z}$ and $t \in \mathbb{Z}$, then it can be shown that $t^* = t$, but $\sigma(t) = t + 1$.

Secondly, given an arbitrary $\mathbb{T}$, define its extension by

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty), & \text{otherwise.} \end{cases}$$

Finally, for a function $f \colon \mathbb{T} \to \mathbb{R}^n$, we consider its extension $f^* \colon \mathbb{T}^* \to \mathbb{R}^n$ given by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

In what follows, we recall some results linking Δ–integrals and Kurzweil–Henstock–Stieltjes integrals.

Theorem 1.4.12 ([20, Theorem 4.2])**.** *Let $f \colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be an arbitrary function. Define $g(t) = t^*$ for every $t \in [a,b]$. Then, the Δ–integral $\int_a^b f(t)\Delta t$ exists if and only if the Kurzweil–Henstock–Stieltjes integral $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists. In this case, both integrals have the same value.*

Lemma 1.4.13 ([20, Lemma 4.4])**.** *Let $a, b \in \mathbb{T}$, $a < b$, $g(t) = t^*$ for every $t \in [a,b]$. If $f \colon [a,b] \to \mathbb{R}^n$ is such that the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists, then*

$$\int_c^d f(t)\,\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\,\mathrm{d}g(t)$$

for every $c, d \in [a,b]$.

According to the following theorem, the Kurzweil–Henstock Δ–integral of a function f defined on $\mathbb{T}$ is, in fact, equivalent to the Kurzweil–Henstock–Stieltjes integral of its extended function f^* for the case $g(t) = t^*$.

Theorem 1.4.14 ([20, Theorem 4.5]). *Let $f\colon \mathbb{T} \to \mathbb{R}^n$ be a function such that the Kurzweil–Henstock Δ–integral $\int_a^b f(s)\Delta s$ exists for every $a, b \in \mathbb{T}$, $a < b$. Define*

$$F_1(t) = \int_a^t f(s)\Delta s, \quad t \in \mathbb{T},$$

$$F_2(t) = \int_a^t f^*(s)\,\mathrm{d}g(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$.

We finish this section showing that the Kurzweil–Henstock–Stieltjes integral $\int_a^b f^*\,\mathrm{d}g$ does not change if f^* is replaced by a function which coincides with f on $[a, b] \cap \mathbb{T}$.

Theorem 1.4.15 ([21, Lemma 4.2]). *Let $\mathbb{T}$ be a time scale, $g(s) = s^*$ for every $s \in \mathbb{T}^*$, $[a, b] \subset \mathbb{T}^*$. Consider a pair of functions $f_1, f_2\colon [a, b] \to \mathbb{R}^n$ such that $f_1(t) = f_2(t)$ for every $t \in [a, b] \cap \mathbb{T}$. If $\int_a^b f_1\,\mathrm{d}g$ exists, then $\int_a^b f_2\,\mathrm{d}g$ exists as well and both integrals have the same value.*

CHAPTER 2

ABSTRACT RETARDED FDE WITH UNBOUNDED STATE–DEPENDENT DELAY

The theory of retarded FDEs with state–dependent delays emerged along with the necessity to obtain more precise mathematical models for a great group of real phenomena. Over the last sixty years, an extensive theory has been developed and many equations with state–dependent delays were used as models in, for example, electrodynamics, neural networks, infectious diseases, among others. Works like [12] and [34] are just a couple of papers that illustrate how relevant to the mathematical community this area has become.

The purpose of this chapter is to study the existence of mild solutions for a class of abstract FDEs with unbounded state–dependent delay specified by

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in [0,a], \\ x_0 &= \varphi, \end{aligned} \tag{2.1}$$

where x, f, and ρ are functions that will be defined later, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ on a Banach space $(X, \|\cdot\|)$ and $x_t\colon (-\infty, 0] \to X$, the segment of x at t, is a function given by $x_t(\theta) = x(t+\theta)$. It is divided into four sections. The introductory section presents all axioms and many other major aspects about the phase space $\mathcal{B}$. The second section is devoted to show the existence and uniqueness of a solution for of a particular FDE with time–dependent delay. These results will be used throughout the third section as an auxiliary tool to derive all crucial characteristics of the main problem (2.1). An application of all the theory developed in former sections will be demonstrated in the last section. It is worth noting that all theorems exposed in this chapter are new in the literature and can be checked in [31]. Lastly, throughout this chapter, we consider the Borel σ–algebra and the Lebesgue measure μ to apply the Bochner integral for functions with range contained in X.

2.1 Phase space

In this section, in order to detail the type of equations to be studied and also to prove our assertions, we consider equations described on a phase space $\mathcal{B}$ defined axiomatically as in Hino et al. [36]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a norm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A1) If $x\colon(-\infty, \sigma+T) \to X$, $T > 0$, is continuous on $[\sigma, \sigma+T)$ and $x_\sigma \in \mathcal{B}$, then the following conditions hold for every $t \in [\sigma, \sigma+T)$:

- (a) $x_t \in \mathcal{B}$.
- (b) $\|x(t)\| \leqslant H\|x_t\|_{\mathcal{B}}$, where $H \geqslant 0$ is a constant and is independent on x.
- (c) $\|x_t\|_{\mathcal{B}} \leqslant K(t-\sigma)\sup\{\|x(s)\|\colon \sigma \leqslant s \leqslant t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $K, M\colon [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded and both functions are independent on x.

(A2) For the function x in (A1), the function $t \mapsto x_t$ is a $\mathcal{B}$–valued continuous function on $[\sigma, \sigma+T)$.

(A3) The space $\mathcal{B}$ is complete.

Throughout this chapter, we always assume that $\mathcal{B}$ is a phase space as described above. Furthermore, for a fixed $a > 0$, let us denote $\widehat{K} = \sup\{K(s)\colon 0 \leqslant s \leqslant a\}$ and $\widehat{M} = \sup\{M(s)\colon 0 \leqslant s \leqslant a\}$.

Next, we denote by C_{00} the space of continuous functions from $(-\infty, 0]$ into X with compact support. It is clear from the axioms of phase space that $C_{00} \subseteq \mathcal{B}$. We also consider the following axiom:

(A4) If a uniformly bounded sequence $(\varphi^n)_{n\in\mathbb{N}}$ in C_{00} converges to a function φ in the compact–open topology, then φ belongs to $\mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.

Remark 2.1.1. *When the axiom (A4) holds, the space $C_b((-\infty, 0], X)$ of all bounded continuous functions $\psi\colon (-\infty, 0] \to X$ is continuously included in $\mathcal{B}$ ([36, Proposition 7.1.1]). Thus, there is a constant $Q > 0$ such that $\|\psi\|_{\mathcal{B}} \leqslant Q\|\psi\|_\infty$ for all $\psi \in C_b((-\infty, 0], X)$.*

Example 2.1.2. Suppose that $1 \leqslant p < \infty$ and g is a nonnegative measurable function on $(-\infty, 0)$ which satisfies the following conditions:

(B1) $\int_s^0 g(\theta)\mathrm{d}\theta < \infty$, for all $s \in (-\infty, 0)$.

(B2) There is a nonnegative function J, which is locally bounded in $(-\infty, 0]$, such that $g(s + \theta) \leqslant J(s)g(\theta)$ for all $s \leqslant 0$ and all $\theta \in (-\infty, 0)\backslash N_s$, where $N_s \subset (-\infty, 0)$ is a set with Lebesgue measure zero, .

The space $C_0 \times L^p(g, X)$ consists of all classes of functions $\varphi\colon (-\infty, 0] \to X$ such that φ is Lebesgue–measurable and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, 0)$. The norm in $C_0 \times L^p(g, X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \|\varphi(0)\| + \left(\int_{-\infty}^0 g(\theta)\|\varphi(\theta)\|^p \mathrm{d}\theta\right)^{1/p}.$$

The space $\mathcal{B} = C_0 \times L^p(g, X)$ satisfies axioms (A1), (A2) and (A3). Moreover, when $p = 2$, we can take $H = 1$, $M(t) = J(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^0 g(\theta)\mathrm{d}\theta\right)^{1/2}$ for $t \geqslant 0$ (see [36, Theorem 1.3.8] for details). Hence, $\widehat{K} = 1 + \left(\int_{-a}^0 g(\theta)\mathrm{d}\theta\right)^{1/2}$ and $\widehat{M} = \sup_{0\leqslant t\leqslant a} J(-t)^{1/2}$. Moreover, if g satisfies

(B3) $\int_{-\infty}^0 g(\theta)\mathrm{d}\theta < \infty$,

then $C_0 \times L^2(g, X)$ satisfies the axiom (A4) and the constant Q is given by

$$Q = 1 + \left(\int_{-\infty}^0 g(\theta)\mathrm{d}\theta\right)^{1/2}.$$

For each $t \geqslant 0$, consider the function $S(t)\colon \mathcal{B} \to \mathcal{B}$ defined by

$$(S(t)\varphi)(\theta) = \begin{cases} \varphi(0), & \theta \in [-t, 0], \\ \varphi(t+\theta), & \theta \in (-\infty, -t]. \end{cases}$$

The family $(S(t))_{t \geqslant 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{B}$. This family is useful to obtain some estimates between segments of x at two different points. Indeed, if $s, t \in \mathbb{R}$ are such that $t \leqslant s$ and the segments x_s, x_t belong to $\mathcal{B}$, then the function $y\colon (-\infty, s] \to X$ defined by

$$y(\theta) = \begin{cases} x(t), & t \leqslant \theta \leqslant s, \\ x(\theta), & \theta < t, \end{cases}$$

is such that $y_t = x_t$ and

$$\begin{aligned} \|x_s - x_t\|_{\mathcal{B}} &\leqslant \|x_s - y_s\|_{\mathcal{B}} + \|y_s - x_t\|_{\mathcal{B}} \\ &\leqslant K(s-t) \sup_{t \leqslant \theta \leqslant s} \|x(\theta) - y(\theta)\| + M(s-t)\|x_t - y_t\|_{\mathcal{B}} + \|S(s-t)x_t - x_t\|_{\mathcal{B}} \\ &\leqslant K(s-t) \sup_{t \leqslant \theta \leqslant s} \|x(\theta) - x(t)\| + \|S(s-t)x_t - x_t\|_{\mathcal{B}}. \end{aligned} \tag{2.2}$$

2.2 Existence of solutions for time–dependent equations

Throughout this section, we assume that $r\colon [0, a] \to \mathbb{R}$ is a regulated function such that $\tau = \inf_{0 \leqslant t \leqslant a}(r(t) - t)$ and, for all $t \in [0, a]$, $r(t) \leqslant t$. We introduce the space $\mathcal{B}_\tau$ consisting of all functions $\varphi \in \mathcal{B}$ such that $\varphi_s \in \mathcal{B}$ for all $\tau \leqslant s \leqslant 0$ and the function $[\tau, 0] \ni s \mapsto \varphi_s \in \mathcal{B}$, is continuous. Endowed with the norm

$$\|\varphi\|_\tau = \sup_{\tau \leqslant s \leqslant 0} \|\varphi_s\|_{\mathcal{B}}, \quad \varphi \in \mathcal{B}_\tau,$$

the space $(\mathcal{B}_\tau, \|\cdot\|_\tau)$ turn into a Banach space as the following lemma shows.

Lemma 2.2.1. The space $(\mathcal{B}_\tau, \|\cdot\|_\tau)$ is complete.

Proof. Let $(\varphi^n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}_\tau$. This implies that $(\varphi^n|_{[\tau,0]})_{n \in \mathbb{N}}$ is a Cauchy sequence in the space $C([\tau, 0], X)$. Therefore, there exists $u \in C([\tau, 0], X)$ such that $\varphi^n(s) \to u(s)$, $n \to \infty$, uniformly for $s \in [\tau, 0]$. On the other hand, $\varphi^n_\tau \to \psi \in \mathcal{B}$, $n \to \infty$.

We define $\varphi\colon (-\infty,0] \to X$ by

$$\varphi(\theta) = \begin{cases} u(\theta), & \tau \leqslant \theta \leqslant 0, \\ \psi(\theta-\tau), & \theta < \tau. \end{cases}$$

It is clear that $\varphi_\tau = \psi$ and φ is continuous on $[\tau, 0]$. Consequently, $\varphi_s \in \mathcal{B}$ for all $s \in [\tau, 0]$. Moreover,

$$\begin{aligned} \|\varphi^n_s - \varphi_s\|_{\mathcal{B}} &\leqslant K(s-\tau) \max_{\tau \leqslant \theta \leqslant s} \|\varphi^n(\theta) - \varphi(\theta)\| + M(s-\tau)\|\varphi^n_\tau - \psi\|_{\mathcal{B}} \\ &\to 0, \ n \to \infty, \end{aligned}$$

and the convergence is uniform for $s \in [\tau, 0]$. This implies that $(\varphi^n)_n$ converges to φ when n goes to infinity in the space $\mathcal{B}_\tau$. □

If $\mathcal{B}$ satisfies axiom (A4), then, as it was pointed before, $C_b((-\infty,0], X) \subseteq \mathcal{B}$ with continuous inclusion. Likewise, for $\varphi \in C_b((-\infty,0], X)$, the function $\varphi_s \in C_b((-\infty,0], X)$ for all $\tau \leqslant s \leqslant 0$. In addition, since $\varphi_\tau \in \mathcal{B}$, it follows from axiom (A2) that the function $s \mapsto \varphi_s$ is continuous. Hence, $C_b((-\infty,0], X) \subseteq \mathcal{B}_\tau$ with continuous inclusion.

The aim of this section is to study the existence of solutions for the abstract FDE with infinite time–dependent delay

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x_{r(t)}), \quad 0 \leqslant t \leqslant a, \\ x_0 &= \varphi \in \mathcal{B}_\tau, \end{aligned} \tag{2.3}$$

where $f\colon [0,a] \times \mathcal{B} \to X$ is a function that satisfies the following Carathéodory condition:

(C1) The function $f(\cdot, \varphi)$ is measurable on $[0,a]$ for each $\varphi \in \mathcal{B}$ and the function $f(t,\cdot)$ is continuous on $\mathcal{B}$ for almost all $t \in [0,a]$.

Our development begins with the following property:

Lemma 2.2.2. *Assume that (C1) is satisfied and $x\colon (-\infty,a] \to X$ is a continuous function on $[0,a]$ such that $x_0 = \varphi \in \mathcal{B}_\tau$. Then, the function $u\colon [0,a] \to X$ given by*

$$u(t) = f(t, x_{r(t)}), \quad 0 \leqslant t \leqslant a,$$

is measurable in the Bochner sense.

Proof. Let $v\colon [\tau,a] \to \mathcal{B}$ be the function given by $v(s) = x_s$. Then, v is a continuous function. This implies that $w\colon [0,a] \to \mathcal{B}$, $w(t) = v(r(t))$, is a regulated function. Consequently, w is a uniform limit of step functions ([44, Theorem 4.1.5]) which implies that w

is a measurable function. Since $u(t) = f(t, w(t))$, it follows from condition (C1) that u is measurable. $\square$

Lemma 2.2.2 enables us to make the definition below.

Definition 2.2.3. *A function $x\colon (-\infty, a] \to X$ is a* ***mild solution*** *of problem* (2.3) *if x is continuous on $[0, a]$, $x_0 = \varphi$ and the integral equation*

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_{r(s)}) \mathrm{d}s, \quad t \in [0, a], \tag{2.4}$$

is satisfied.

The following result gives some conditions on the function f that guarantee the existence and uniqueness of mild solutions for the problem (2.3). In what follows, we denote by $\widetilde{M} = \sup_{0 \leqslant t \leqslant a} \|T(t)\|$ and by $C_\varphi([0, a], X)$ the subset of all $x \in C([0, a], X)$ such that $x(0) = \varphi(0)$. It is clear that $C_\varphi([0, a], X)$ is a closed convex subset of $C([0, a], X)$.

Theorem 2.2.4. *Assume that (C1) is satisfied and suppose that the function $f(\cdot, 0)$ is Bochner integrable. Moreover, consider the existence of a positive function $\eta \in L^1([0, a])$ such that*

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leqslant \eta(t) \|\psi_1 - \psi_2\|_{\mathcal{B}} \tag{2.5}$$

for $\psi_1, \psi_2 \in \mathcal{B}_\tau$ and $t \in [0, a]$. Then the problem (2.3) *has a unique mild solution.*

Proof. The argument to establish this statement is standard, so we will only limit ourselves to present the essential ideas of the proof.

Consider the Banach space

$$Y = \{x\colon (-\infty, a] \to X\colon x_0 \in \mathcal{B}_\tau \text{ and } x|_{[0,a]} \text{ is continuous}\}$$

equipped with the norm

$$\|x\|_Y = \|x_0\|_\tau + \sup_{u \in [0,a]} \|x(u)\|$$

and define the operator $\Gamma\colon Y \to Y$ by the expression

$$\Gamma(x)(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_{r(s)}) \mathrm{d}s, & 0 \leqslant t \leqslant a, \\ \varphi(t), & t \leqslant 0. \end{cases} \tag{2.6}$$

For each $x, y \in Y$ and $t \in (-\infty, a]$, we can estimate

$$\|\Gamma(x)(t) - \Gamma(y)(t)\| \leqslant \widetilde{M} \int_0^t \left\| f(s, x_{r(s)}) - f(s, y_{r(s)}) \right\| \mathrm{d}s$$

$$\leqslant \widetilde{M}\int_0^t \eta(s)\left\|x_{r(s)} - y_{r(s)}\right\|_{\mathcal{B}}\mathrm{d}s$$
$$\leqslant \widetilde{M}\widehat{K}\int_0^t \eta(s)\max_{0\leqslant \xi\leqslant s}\|x(\xi) - y(\xi)\|\mathrm{d}s.$$

Repeating this argument, we achieve that

$$\|\Gamma^n(x)(t) - \Gamma^n(y)(t)\| \leqslant \frac{\widetilde{M}^n\widehat{K}^n}{n!}\left(\int_0^a \eta(s)\mathrm{d}s\right)^n \max_{0\leqslant u\leqslant t}\|x(u) - y(u)\|$$

which shows that Γ^n is a contraction on Y for $n \in \mathbb{N}$ large enough. Hence, we conclude that Γ has a unique fixed point $\overline{x}$ which is the mild solution of the problem (2.3). □

From now on, we will assume that the semigroup $(T(t))_{t\geqslant 0}$ is compact Also, we introduce the following boundedness condition for f.

(C2) There exists a positive function $\mu \in L^1([0,a])$ and a continuous nondecreasing function $\Phi\colon [0,\infty) \to (0,\infty)$ such that

$$\|f(t,\psi)\| \leqslant \mu(t)\Phi(\|\psi\|_{\mathcal{B}})$$

for all $\psi \in \mathcal{B}_\tau$ and a.e. $t \in [0,a]$.

Before we demonstrate another crucial theorem concerning the existence of solutions of the abstract FDEs with state–dependent delays, we recall a classical result which is frequently applied to many other similar theorems. Also, we will need this result in the chapter about measure functional differential equations with state–dependent delays.

Theorem 2.2.5 (Schauder Fixed–Point Theorem). *Let $(E, \|\cdot\|)$ be a normed vector space, S be a nonempty convex and closed subset of E and $T\colon S \to S$ is a continuous function such that $T(S)$ is relatively compact. Then, T has a fixed point in S.*

Now, we are ready to prove our second main result of this chapter.

Theorem 2.2.6. *Assume that the semigroup $(T(t))_{t\geqslant 0}$ is compact. Assume further that f is a locally Lipschitz continuous function that satisfies condition (C2) and suppose that there exists $R > \|\varphi\|_\tau$ for which the following condition holds:*

$$\widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^a \mu(s)\mathrm{d}s\Phi(\widehat{K}R + \widehat{M}\|\varphi\|_\tau) \leqslant R. \tag{2.7}$$

Then, there exists a unique mild solution of the problem (2.3).

Proof. Let $\Psi\colon C_\varphi([0,a],X) \to C_\varphi([0,a],X)$ the map defined as follows

$$\Psi(x)(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, x_{r(s)})\mathrm{d}s, \quad t \in [0,a].$$

By Theorem 1.2.6, we obtain that Ψ is a continuous map. Let B_R be the closed ball with center at 0 and radius R in the space $C_\varphi([0,a],X)$ and let $x \in B_R$. By definition of Ψ, we have

$$\begin{aligned} \|\Psi(x)(t)\| &\leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^t \mu(s)\Phi(\|x_{r(s)}\|_{\mathcal{B}})\mathrm{d}s \\ &\leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^t \mu(s)\Phi(\widehat{K}R + \widehat{M}\|\varphi\|_\tau)\mathrm{d}s \leqslant R, \end{aligned}$$

which shows that $\Psi(B_R) \subseteq B_R$.

Now, let $t \in [0,a]$ be a fixed number and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence on B_R. For all $n \in \mathbb{N}$, without any change of notation, we will extend x_n to $(-\infty, 0]$ by defining $x(\theta) = \varphi(\theta)$ for all $\theta \leqslant 0$. By condition (C2), we have, for any $s \in [0,a]$,

$$\left\|f(s,(x_n)_{r(s)})\right\| \leqslant \mu(s)\Phi\left(\left\|(x_n)_{r(s)}\right\|_{\mathcal{B}}\right) \leqslant \mu(s)\Phi\left(\widehat{K}\sup_{0\leqslant u\leqslant a}\|x_n(u)\| + \widehat{M}\|\varphi\|_{\mathcal{B}}\right),$$

which shows that the sequence $\left(f(s,(x_n)_{r(s)})\right)_{n\in\mathbb{N}}$ is bounded as well. The compactness of T implies the existence of a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $T(t-s)f(s,(x_{n_k})_{r(s)})$ converges to some function $y(s)$. By Theorem 1.2.6, we conclude that $\Psi(x_{n_k})(t)$ converges to some point in X. Hence, the operator $K_t\colon B_R \to X$ given by $K_t(x) = \Psi(x)(t)$ is compact. Therefore, $\Psi(B_R)(t)$ is relatively compact.

Finally, we can show that $\Psi(B_R)$ is an equicontinuous subset of $C([0,a],X)$. In fact, for $x \in B_R$ and $h \geqslant 0$, we can write

$$\begin{aligned} \Psi(x)(t+h) - \Psi(x)(t) &= T(t)(T(h)-I)\varphi(0) + \int_0^t \left(T(t+h-s) - T(t-s)\right)f(s,x_{r(s)})\mathrm{d}s \\ &\quad + \int_t^{t+h} T(t+h-s)f(s,x_{r(s)})\mathrm{d}s \\ &= (T(h)-I)\Psi(x)(t) + \int_t^{t+h} T(t+h-s)f(s,x_{r(s)})\mathrm{d}s. \end{aligned}$$

The first term on the right–hand side $(T(h)-I)\Psi(x)(t) \to 0$ as $h \to 0$ independent of x because the set $\Psi(B_R)(t)$ is relatively compact. In similar way, the second term on the right–hand side

$$\int_t^{t+h} T(t+h-s)f(s,x_{r(s)})\mathrm{d}s \to 0$$

as $h \to 0$ independent of $x \in B_R$. Combining these assertions, we conclude that Ψ is a completely continuous map on B_R. By Theorem 2.2.5, we obtain that Ψ has a fixed point $\overline{x}$. Since f is locally Lipschitz continuous, an standard argument allows us to affirm that $\overline{x}$ is the unique fixed point of Ψ.

To conclude our proof, let $y\colon (-\infty, a] \to X$ be the following function:

$$y(t) = \begin{cases} \overline{x}(t), & t \in [0,a], \\ \varphi(t), & t \in (-\infty, 0]. \end{cases}$$

Since, for all $t \in [0,a]$, $\Gamma y(t) = \Psi(\overline{x})(t) = \overline{x}(t)$, we have that y is the mild solution of problem (2.3). □

Next, for a fixed $\tau \leqslant 0$, we introduce the set $\mathcal{R}_\tau$ consisting of all regulated functions $r\colon [0,a] \to \mathbb{R}$ such that $r(t) \leqslant t$ for all $t \in [0,a]$ and $\inf_{0 \leqslant t \leqslant a}(r(t)-t) \geqslant \tau$. It is not difficult to check that $\mathcal{R}_\tau$ is a closed and convex subset of the Banach space $(G([0,a],\mathbb{R}), \|\cdot\|_\infty)$. For each $r \in \mathcal{R}_\tau$, we denote by $x(\cdot,\varphi,r)$ the unique mild solution of (2.3) whose existence was established in Theorem 2.2.4 or Theorem 2.2.6. Additionally, we denote by $\mathcal{S}\colon \mathcal{R}_\tau \to Y$, $r \mapsto x(\cdot,\varphi,r)$ the segment operator and by $\widetilde{\mathcal{S}}\colon \mathcal{R}_\tau \to C([0,a],\mathcal{B}_\tau)$, $\widetilde{\mathcal{S}}(r)(t) = \mathcal{S}(r)_t$.

Lemma 2.2.7. *Assume that $x \in Y$. Then, the family of functions $\mathcal{R}_\tau \to \mathcal{B}_\tau$, $r \mapsto x_{r(s)}$, is equicontinuous for $s \in [0,a]$. Moreover, if $K \subset C_\varphi([0,a],X)$ is a relatively compact set, then the continuity of $r \mapsto x_{r(s)}$ is independent of x.*

Proof. Let $r^1, r^2 \in \mathcal{R}_\tau$ and $s \in [0,a]$. We can assume, without loss of generality, that $r^1(s) \leqslant r^2(s)$. Using that $\varphi \in \mathcal{B}_\tau$ and from the fact that the set $\{x_t\colon \tau \leqslant t \leqslant a\}$ is relatively compact in $\mathcal{B}$, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|\varphi_{t^1} - \varphi_{t^2}\|_{\mathcal{B}} \leqslant \frac{\varepsilon}{3}, \quad \|(S(h)-I)x_t\|_{\mathcal{B}} \leqslant \frac{\varepsilon}{3}, \quad \|x(s^1)-x(s^2)\| \leqslant \frac{\varepsilon}{3\widehat{K}}$$

for all $s^1, s^2 \in [0,a]$, $t^1, t^2 \in [\tau, 0]$ with $0 \leqslant s^2 - s^1 \leqslant \delta$, $0 \leqslant t^2 - t^1 \leqslant \delta$, and $0 \leqslant h \leqslant \delta$. Assume that $\|r^1 - r^2\|_\infty \leqslant \delta$ and $u \in [\tau, 0]$. This allows us to estimate $\|x_{r^1(s)} - x_{r^2(s)}\|_\tau$ as follows:

(i) If $r^2(s) + u < 0$, then clearly $r^1(s) + u < 0$ and

$$\|x_{r^1(s)+u} - x_{r^2(s)+u}\|_{\mathcal{B}} = \|\varphi_{r^1(s)+u} - \varphi_{r^2(s)+u}\|_{\mathcal{B}}.$$

(ii) If $r^1(s) + u < 0$ and $r^2(s) + u > 0$, then by (2.2),

$$\|x_{r^1(s)+u} - x_{r^2(s)+u}\|_{\mathcal{B}} = \|\varphi_{r^1(s)+u} - x_{r^2(s)+u}\|_{\mathcal{B}}$$

$$
\begin{aligned}
&\leqslant \|\varphi_{r^1(s)+u} - \varphi_0\|_{\mathcal{B}} + \|\varphi_0 - x_{r^2(s)+u}\|_{\mathcal{B}} \\
&\leqslant \|\varphi_{r^1(s)+u} - \varphi_0\|_{\mathcal{B}} + \|x_0 - x_{r^2(s)+u}\|_{\mathcal{B}} \\
&\leqslant \|\varphi_{r^1(s)+u} - \varphi_0\|_{\mathcal{B}} + \|x_0 - S(r^2(s)+u)x_0\|_{\mathcal{B}} + K(r^2(s)+u) \max_{\xi \in [0, r^2(s)+u]} \|x(\xi) - x(0)\| \\
&\leqslant \|\varphi_{r^1(s)+u} - \varphi_0\|_{\mathcal{B}} + \|\varphi_0 - S(r^2(s)+u)\varphi_0\|_{\mathcal{B}} + \widehat{K} \max_{\xi \in [0, r^2(s)+u]} \|x(\xi) - \varphi(0)\|.
\end{aligned}
$$

(iii) If $r^1(s) + u \geqslant 0$, then clearly $r^2(s) + u \geqslant 0$ and, once again, by (2.2),

$$
\begin{aligned}
&\|x_{r^1(s)+u} - x_{r^2(s)+u}\|_{\mathcal{B}} \\
&\quad \leqslant \|(S(r^2(s) - r^1(s)) - I)x_{r^1(s)+u}\|_{\mathcal{B}} + \widehat{K} \max_{r^1(s)+u \leqslant \xi \leqslant r^2(s)+u} \|x(\xi) - x(r^1(s)+u)\|.
\end{aligned}
$$

Combining these estimates with the selection of δ, we can affirm that

$$
\|x_{r^1(s)} - x_{r^2(s)}\|_{\tau} \leqslant \varepsilon
$$

is independent of $s \in [0, a]$, which shows the first assertion.

In addition, this argument also serves to establish the second claim, using, in this case, that the set $\{x(\cdot) \colon x \in Y \text{ and } x|_{[0,a]} \in K\}$ is equicontinuous and the set $\{x_t \colon \tau \leqslant t \leqslant a, x \in Y \text{ and } x|_{[0,a]} \in K\}$ is relatively compact in $\mathcal{B}$. □

Now, we enunciate two important theorems to prove some results of this section.

Theorem 2.2.8 ([29, Lemma 6.2]). *If, for $a \leqslant t \leqslant b$, ψ, α are real valued and continuous functions, $\alpha'(t) \geqslant 0$, $\beta(t) \geqslant 0$ is integrable on $[a, b]$ and*

$$
\psi(t) \leqslant \alpha(t) + \int_a^t \beta(s)\psi(s)\mathrm{d}s, \quad a \leqslant t \leqslant b,
$$

then

$$
\psi(t) \leqslant \alpha(t) \exp\left(\int_a^t \beta(s)\mathrm{d}s\right).
$$

Theorem 2.2.9 ([17, Theorem 69]). *Suppose that the functions $u(t)$ and $\alpha(t)$ are non-negative for $0 < s < t < b$ and $\Phi(s)$ is positive, nondecreasing and continuous for $s > 0$. If*

$$
u(t) \leqslant c + \int_a^t \alpha(s)\Phi(u(s))\mathrm{d}s,
$$

where $c > 0$, then

$$
\int_a^{u(t)} \frac{1}{\Phi(s)}\mathrm{d}s \leqslant \int_a^t \alpha(s)\mathrm{d}s
$$

for all $t \in (a, b)$.

Proposition 2.2.10. *If all hypotheses from Theorem 2.2.4 or Theorem 2.2.6 are satisfied, then $\mathcal{S}\colon \mathcal{R}_\tau \to Y$ is a continuous map.*

Proof. We assume initially that all hypotheses of Theorem 2.2.4 hold. Let $r^1, r^2 \in \mathcal{R}_\tau$. For simplicity, we abbreviate the notation by writing $x = x(\cdot, \varphi, r^1)$ and $y = x(\cdot, \varphi, r^2)$. It follows from (2.4) that

$$y(t) - x(t) = \int_0^t T(t-s)\left(f(s, y_{r^2(s)}) - f(s, x_{r^1(s)})\right) \mathrm{d}s, \quad t \in [0, a].$$

This implies that

$$\begin{aligned} \|y(t) - x(t)\| &\leqslant \widetilde{M} \int_0^t \eta(s) \left\|y_{r^2(s)} - x_{r^1(s)}\right\|_{\mathcal{B}} \mathrm{d}s \\ &\leqslant \widetilde{M} \int_0^t \eta(s) \left\|y_{r^2(s)} - x_{r^2(s)}\right\|_{\mathcal{B}} \mathrm{d}s + \widetilde{M} \int_0^t \eta(s) \left\|x_{r^2(s)} - x_{r^1(s)}\right\|_{\mathcal{B}} \mathrm{d}s \end{aligned} \tag{2.8}$$

for $0 \leqslant t \leqslant a$. Proceeding as in the proof of Lemma 2.2.7, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left\|x_{r^2(s)} - x_{r^1(s)}\right\|_{\mathcal{B}} \leqslant \varepsilon, \quad 0 \leqslant s \leqslant a, \tag{2.9}$$

when $\|r^2 - r^1\|_\infty \leqslant \delta$. Replacing (2.9) in (2.8), we have, for all $t \in [0, a]$,

$$\begin{aligned} \|y(t) - x(t)\| &\leqslant \widetilde{M} \int_0^t \eta(s) \left\|y_{r^2(s)} - x_{r^2(s)}\right\|_{\mathcal{B}} \mathrm{d}s + \widetilde{M} \int_0^a \eta(s) \mathrm{d}s \varepsilon \\ &\leqslant \widetilde{M}\widehat{K} \int_0^t \eta(s) \max_{0 \leqslant \xi \leqslant s} \|y(\xi) - x(\xi)\| \,\mathrm{d}s + \widetilde{M} \int_0^a \eta(s) \mathrm{d}s \varepsilon. \end{aligned}$$

Applying Theorem 2.2.8, we get

$$\|y(t) - x(t)\| \leqslant \widetilde{M}\varepsilon \int_0^a \eta(s) \mathrm{d}s \, e^{\widetilde{M}\widehat{K} \int_0^t \eta(s) \mathrm{d}s}, \quad t \in [0, a].$$

This inequality and the fact that $S(r^2)(t) - S(r^1)(t) = 0$ for all $t \leqslant 0$ imply that S is a continuous map in this case.

We assume now that the hypotheses of Theorem 2.2.6 hold. We first show that the set $\{x(\cdot, \varphi, r)\colon r \in \mathcal{R}_\tau\}$ is bounded. In fact, for $t \in [0, a]$, it follows from (2.4) that

$$\begin{aligned} \|x(t)\| &\leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M} \int_0^t \mu(s) \Phi(\|x_{r(s)}\|_{\mathcal{B}}) \mathrm{d}s \\ &\leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M} \int_0^t \mu(s) \Phi\left(\widehat{K} \max_{0 \leqslant \xi \leqslant s} \|x(\xi)\| + \widehat{M}\|\varphi\|_\tau\right) \mathrm{d}s. \end{aligned}$$

Hence, if $\alpha(t) = \widehat{K} \max_{0 \leqslant \xi \leqslant t} \|x(\xi)\| + \widehat{M}\|\varphi\|_\tau$, then

$$\alpha(t) \leqslant \widetilde{M}\widehat{K}\|\varphi(0)\| + \widehat{M}\|\varphi\|_\tau + \widetilde{M}\widehat{K} \int_0^t \mu(s) \Phi(\alpha(s)) \mathrm{d}s, \quad t \in [0, a].$$

Applying Theorem 2.2.9, we have that $\|x(t)\| \leqslant C$ for some constant $C \geqslant 0$ independent of $r \in \mathcal{R}_\tau$.

We next show that $\{x(t,\varphi,r)\colon r \in \mathcal{R}_\tau\}$ is relatively compact in X for all $0 \leqslant t \leqslant a$. Indeed, for $t > 0$ and $0 < \varepsilon \leqslant t$ sufficiently small, we can write

$$\begin{aligned} x(t) &= T(\varepsilon)T(t-\varepsilon)\varphi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)f(s,x_{r(s)})\mathrm{d}s + \int_{t-\varepsilon}^{t} T(t-s)f(s,x_{r(s)})\mathrm{d}s \\ &= T(\varepsilon)x(t-\varepsilon) + \int_{t-\varepsilon}^{t} T(t-s)f(s,x_{r(s)})\mathrm{d}s. \end{aligned}$$

Using that $T(\varepsilon)$ is a compact operator, the set $\{x(t-\varepsilon,\varphi,r)\colon r \in \mathcal{R}_\tau\}$ is bounded and

$$\int_{t-\varepsilon}^{t} T(t-s)f(s,x_{r(s)})\mathrm{d}s \to 0$$

as $\varepsilon \to 0$, we obtain that $\{x(t,\varphi,r)\colon r \in \mathcal{R}_\tau\}$ is relatively compact in X.

In this step, we show that the set $\{x(\cdot,\varphi,r)\colon r \in \mathcal{R}_\tau\}$ is equicontinuous on $[0,a]$. Proceeding in similar way as above, for $h \geqslant 0$, we can write

$$\begin{aligned} x(t+h)-x(t) &= (T(h)-I)T(t)\varphi(0) + \int_0^{t} (T(h)-I)T(t-s)f(s,x_{r(s)})\mathrm{d}s \\ &\quad + \int_t^{t+h} T(t+h-s)f(s,x_{r(s)})\mathrm{d}s \\ &= (T(h)-I)x(t) + \int_t^{t+h} T(t+h-s)f(s,x_{r(s)})\mathrm{d}s. \end{aligned}$$

Since $\{x(t,\varphi,r)\colon r \in \mathcal{R}_\tau\}$ is relatively compact in X and $f(s,x_{r(s)})$ is bounded independently of r, we conclude that $x(t+h)-x(t) \to 0$ as $h \to 0$ independently of r. Consequently, the set of functions $\{x(\cdot,\varphi,r)|_{[0,a]}\colon r \in \mathcal{R}_\tau\}$ is relatively compact in $C([0,a],X)$.

Let $(r^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}_\tau$ that converges to $r \in \mathcal{R}_\tau$. In what follows, we abbreviate the notation by writing $x^n = x(\cdot,\varphi,r^n)$ and $y = x(\cdot,\varphi,r)$. Thus, there exists a subsequence of $(x^n)_{n\in\mathbb{N}}$, still denoted by the index $n \in \mathbb{N}$, that converges uniformly to z. We extend z to $(-\infty,a]$ by defining the function $\overline{z}\colon (-\infty,a] \to X$ as follows:

$$\overline{z}(t) = \begin{cases} z(t), & t \in [0,a], \\ \varphi(t), & t \in (-\infty,0]. \end{cases}$$

It follows from Lemma 2.2.7 that for each $s \in [0,a]$, $x^n_{r(s)} \to \overline{z}_{r(s)}$ and $f(s,x^n_{r(s)}) \to f(s,\overline{z}_{r(s)})$ as $n \to \infty$. Using now Theorem 1.2.6, we obtain that $\overline{z}$ is a mild solution of problem (2.3). From the uniqueness of mild solutions, we conclude that $\overline{z} = y$, which completes the proof. □

Corollary 2.2.11. *If all the hypotheses of Proposition 2.2.10 hold, then the map* $\widetilde{\mathcal{S}}\colon \mathcal{R}_\tau \to C([0,a],\mathcal{B}_\tau)$ *is continuous.*

Proof. Let $r^1, r^2 \in \mathcal{R}_\tau$, $x = \mathcal{S}(r^1)$ and $y = \mathcal{S}(r^2)$. For every $t \in [0,a]$, and $s \in [\tau, 0]$, we have that $(x_t)_s = x_{t+s}$ and $(y_t)_s = y_{t+s}$. If $t + s < 0$, then $x_{t+s} = y_{t+s} = \varphi_{t+s}$. Hence, we can assume that $t + s \geqslant 0$, which implies that

$$\begin{aligned} \|x_t - y_t\|_\tau &= \max_{\tau \leqslant s \leqslant 0} \|x_{t+s} - y_{t+s}\|_{\mathcal{B}} \\ &\leqslant \max_{\tau \leqslant s \leqslant 0} \widehat{K} \max_{0 \leqslant \xi \leqslant t+s} \|x(\xi) - y(\xi)\| \\ &\leqslant \widehat{K}\|x - y\|_\infty. \end{aligned}$$

The assertion is now a consequence of Proposition 2.2.10. □

2.3 Existence for state–dependent equations

In this section, we apply all results on the existence of solutions for time–dependent delay equations established in Section 2.2 to study the existence of solutions of problem (2.1). Essentially, we do not need to assume the continuity of the function $\rho\colon [0,a] \times \mathcal{B} \to \mathbb{R}$, which allows us to include in the theory those equations in which there is memory loss. Instead, for a fixed $\tau \leqslant 0$, we suppose that ρ satisfies the following conditions:

(D1) For every $\psi \in \mathcal{B}_\tau$, the function $\rho(\cdot, \psi)$ is regulated and, for every compact set $W \subset \mathcal{B}_\tau$, the set of functions $\{\rho(t, \cdot)|_W : t \in [0,a]\}$ is equicontinuous.

(D2) $\tau + t \leqslant \rho(t, \psi) \leqslant t$ for all $t \in [0,a]$ and $\psi \in \mathcal{B}_\tau$.

Moreover, we consider that f satisfies the Carathéodory condition (C1). We begin by establishing our concept of solution. In this statement, we assume that $\varphi \in \mathcal{B}_\tau$.

Definition 2.3.1. *A function* $x\colon (-\infty, a] \to X$ *is a* ***mild solution*** *of the problem* (2.1) *if* x *is continuous on* $[0,a]$, $x_0 = \varphi$ *and the integral equation*

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_{\rho(s,x_s)}) \mathrm{d}s, \quad t \in [0,a], \tag{2.10}$$

is satisfied.

Theorem 2.3.2. *Assume that hypotheses of Theorem 2.2.6 are fulfilled. If hypotheses (D1) and (D2) are satisfied, then the problem* (2.1) *has a mild solution.*

Proof. Since for each $r \in \mathcal{R}_\tau$ there is a unique mild solution of problem (2.3), we define the operator $F\colon \mathcal{R}_\tau \to G([0,a],\mathbb{R})$ given by

$$F(r)(t) = \rho(t, x_t),$$

where $x = \mathcal{S}(r)$. We will show that F has a fixed point, organizing the proof in three steps.

(i) In this first step, we prove that $F(\mathcal{R}_\tau) \subseteq \mathcal{R}_\tau$. Let $r \in \mathcal{R}_\tau$. It follows from (D1) that for every $\psi \in \mathcal{B}_\tau$, the maps

$$\begin{aligned} R^+(t,\psi) &= \rho(t^+,\psi),\ t \in [a,b), \\ R^-(t,\psi) &= \rho(t^-,\psi),\ t \in (a,b], \end{aligned}$$

are well–defined. Let $(t^n)_{n\in\mathbb{N}}$ be a decreasing sequence convergent to $t \in [a,b)$. It follows from Lemma 2.2.7 that $x_{t^n} \to x_t$ as $n \to \infty$. Using (D1) again, we have

$$\rho(t^n, x_{t^n}) - R^+(t,x_t) = \rho(t^n, x_{t^n}) - \rho(t^n, x_t) + \rho(t^n, x_t) - R^+(t, x_t).$$

Hence, $\lim\limits_{n\to\infty} \rho(t^n, x_{t^n}) = R^+(t,x_t)$. Similarly, for an increasing sequence $(t^n)_{n\in\mathbb{N}}$ converging to $t \in (a,b]$, we obtain that $\lim\limits_{n\to\infty} \rho(t^n, x_{t^n}) = R^-(t,x_t)$. Consequently, $F(r)$ is a regulated function. Combining with (D2), we ascertain that $F(r) \in \mathcal{R}_\tau$.

(ii) In this step, we show that F is continuous. We begin with a general remark. Let $W \subseteq \mathcal{B}_\tau$ be a compact set and let $\varepsilon > 0$. It follows from (D1) that for every $\psi \in W$, there exists $\delta(\psi) > 0$ such that

$$|\rho(t,\psi') - \rho(t,\psi)| \leqslant \varepsilon/2$$

for all $\psi' \in B_{\delta(\psi)}(\psi)$ and all $t \in [0,a]$. Let $\delta > 0$ be a Lebesgue number [45, Lemma 27.5] of the covering of W by the open balls $B_{\delta(\psi)}(\psi)$ with $\psi \in W$. For every $\psi^1, \psi^2 \in W$ with $\|\psi^1 - \psi^2\|_\tau < \delta$, there is $\psi^0 \in W$ such that $\psi^1 \in B_{\delta(\psi^0)}(\psi^0)$. This implies that

$$\begin{aligned} |\rho(t,\psi^1) - \rho(t,\psi^2)| &\leqslant |\rho(t,\psi^1) - \rho(t,\psi^0)| + |\rho(t,\psi^0) - \rho(t,\psi^2)| \\ &\leqslant \varepsilon. \end{aligned}$$

Let $(r^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}_\tau$ converging to r. We denote $\mathcal{S}(r^n) = x^n$ and $\mathcal{S}(r) = x$. It follows from Corollary 2.2.11 that $x_t^n \to x_t$ as $n \to \infty$ in the norm in $\mathcal{B}_\tau$ and uniformly for $t \in [0,a]$. This implies that the set

$$W = \{x_t^n : t \in [0,a],\ n \in \mathbb{N}\} \cup \{x_t : t \in [0,a]\}$$

is compact in $\mathcal{B}_\tau$. Applying our previous remark to the set W, we conclude that $\rho(t, x_t^n) \to \rho(t, x_t)$ as $n \to \infty$ uniformly for $t \in [0, a]$.

(iii) In this step, we show that $\{F(r) \colon r \in \mathcal{R}_\tau\}$ is equiregulated. It is sufficient to show that $\{F(r) \colon r \in \mathcal{R}_\tau\}$ is left–sided equiregulated, since the proof for the right–sided equiregulated case is similar. Let $W \subseteq \mathcal{B}_\tau$ be a compact set. It follows from (D1) that for $\varepsilon > 0$ and $\psi^0 \in W$ there exists $\delta(\psi^0) > 0$ such that

$$|\rho(t + h, \psi) - \rho(t + h, \psi^0)| \leqslant \varepsilon/3,$$

for $h > 0$ such that $t + h < a$, all $t \in [0, a)$ and all $\psi \in B_{\delta(\psi^0)}(\psi^0)$. Since

$$\begin{aligned} |R^+(t, \psi) - R^+(t, \psi^0)| &\leqslant |R^+(t, \psi) - \rho(t + h, \psi)| + |\rho(t + h, \psi) - \rho(t + h, \psi^0)| \\ &\quad + |\rho(t + h, \psi^0) - R^+(t, \psi^0)| \end{aligned}$$

taking the limit as $h \to 0^+$, we obtain that

$$|R^+(t, \psi) - R^+(t, \psi^0)| \leqslant \varepsilon/3$$

for all $t \in [0, a)$, and all $\psi \in B_{\delta(\psi^0)}(\psi^0)$.

Let $\psi^i \in W$, $i = 1, \ldots, n$, be such that $\{B_{\delta(\psi^i)}(\psi^i) \colon i = 1, \ldots, n\}$ is an open covering of W. Since the function $\rho(\cdot, \psi^i)$ is left–sided equiregulated, for each $i = 1, \ldots, n$, there exists $\beta^i > 0$ such that

$$|\rho(t + h, \psi^i) - R^+(t, \psi^i)| \leqslant \varepsilon/3,$$

for $0 < h < \beta^i$. Let $\beta = \min\{\beta^i \colon i = 1, \ldots, n\}$. Let $\psi \in W$. We select ψ^i such that $\psi \in B_{\delta(\psi^i)}(\psi^i)$. Combining these estimates, we obtain that

$$\begin{aligned} |\rho(t + h, \psi) - R^+(t, \psi)| &\leqslant |\rho(t + h, \psi) - \rho(t + h, \psi^i)| + |\rho(t + h, \psi^i) - R^+(t, \psi^i)| \\ &\quad + |R^+(t, \psi^i) - R^+(t, \psi)| \\ &\leqslant \varepsilon \end{aligned}$$

for all $0 < h < \beta$. This shows that the set $\{\rho(\cdot, \psi) \colon \psi \in W\}$ is left–sided equiregulated.

On the other hand, proceeding as in the proof of Theorem 2.2.6, we can affirm that $\widetilde{\mathcal{S}}(\mathcal{R}_\tau)$ is a relatively compact set in $C([0, a], \mathcal{B}_\tau)$. We complete the proof of the assertion by applying the above property for the set $W = \{\mathcal{S}(r)_t \colon r \in \mathcal{R}_\tau,\ 0 \leqslant t \leqslant a\}$.

As a consequence of Theorem 1.1.4, we can affirm that the set $\{F(r) \colon r \in \mathcal{R}_\tau\}$ is relatively compact in $G([0, a], \mathbb{R})$. Applying now Theorem 2.2.5, we conclude that F has a fixed point r. It is clear that $x = S(r)$ is a mild solution of problem (2.1). □

Remark 2.3.3. *A relation between state–dependent delays equations and time–dependent delays for the case $r(t) = t$ was already established previously in the literature (see, for example, [30]). However, the relation presented here allows us to consider more general conditions concerning the regularity of the solutions and the considered spaces.*

To establish a result of the same type as Theorem 2.3.2 when the hypotheses of Theorem 2.2.4 are satisfied, we need to restrict the condition (D1). With this object, we introduce the following condition:

(D3) For every $\psi \in \mathcal{B}_\tau$, the function $\rho(\cdot,\psi)$ is regulated. Furthermore, for each $\varphi \in \mathcal{B}_\tau$, there exists a constant $C_\rho \geqslant 0$ such that

$$|\rho(t, x_t) - \rho(t, y_t)| \leqslant C_\rho \max_{0\leqslant s\leqslant t} \|x(s) - y(s)\|$$

for all $x, y \in C_\varphi([0, a], X)$ and $t \in [0, a]$.

Theorem 2.3.4. *Consider that the space $\mathcal{B}$ satisfies axiom (A4). Assume that all hypotheses of Theorem 2.2.4 are satisfied with a bounded function η. Suppose that conditions (D2) and (D3) are fulfilled and the initial condition $\varphi \in C_b((-\infty, 0], X)$ satisfies the Lipschiz condition*

$$\|\varphi(\theta^1) - \varphi(\theta^2)\| \leqslant L_\varphi|\theta^1 - \theta^2| \quad \textit{for all } \theta^1, \theta^2 \leqslant 0.$$

Finally, let $f(\cdot, 0)$ be a bounded function on $[0, a]$ such that $f(t, \psi) \in D(A)$ for all $t \in [0, a]$ and

$$\|Af(t, \psi)\| \leqslant \eta_1\|\psi\|_{\mathcal{B}} \tag{2.11}$$

for some constant $\eta_1 \geqslant 0$. If

$$C_\rho\widetilde{M}Q\max\{L_\varphi, C_4\}\left(\int_0^a \eta(s)\mathrm{d}s\right)\exp\left(\widetilde{M}\widehat{K}\int_0^a \eta(s)\mathrm{d}s\right) < 1, \tag{2.12}$$

then problem (2.1) has a unique mild solution.

Proof. We divide the proof in three steps.

(i) Initially, we will prove that the solutions of (2.3) are bounded independently of $r \in \mathcal{R}_\tau$. Let $r \in \mathcal{R}_\tau$ and $x = \mathcal{S}(r)$. It follows from (2.5) that

$$\|x(t)\| \leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^t \left\|f(s, x_{r(s)})\right\| \mathrm{d}s$$

$$\leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^t \eta(s)\|x_{r(s)}\|_{\mathcal{B}}\mathrm{d}s + \widetilde{M}\int_0^t \|f(s,0)\|\mathrm{d}s$$
$$\leqslant \widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^t \|f(s,0)\|\mathrm{d}s + \widetilde{M}\widehat{M}\|\varphi\|_\tau \int_0^a \eta(s)\mathrm{d}s + \widetilde{M}\hat{K}\int_0^t \eta(s)\max_{0\leqslant\xi\leqslant s}\|x(\xi)\|\mathrm{d}s$$

for all $0 \leqslant t \leqslant a$. Using Theorem 2.2.8, it follows that

$$\begin{aligned} \|x(t)\| &\leqslant C_1 e^{\widetilde{M}\hat{K}\int_0^t \eta(s)\mathrm{d}s} \\ &\leqslant C_2, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \widetilde{M}\|\varphi(0)\| + \widetilde{M}\int_0^a \|f(s,0)\|\mathrm{d}s + \widetilde{M}\widehat{M}\|\varphi\|_\tau \int_0^a \eta(s)\mathrm{d}s, \\ C_2 &= C_1 e^{\widetilde{M}\hat{K}\int_0^a \eta(s)\mathrm{d}s}. \end{aligned}$$

This also implies that

$$\|x_{r(t)}\|_{\mathcal{B}} \leqslant \hat{K}C_2 + \widehat{M}\|\varphi\|_\tau. \tag{2.13}$$

(ii) In this step, we will estimate $\|x_{r^1(s)} - x_{r^2(s)}\|_{\mathcal{B}}$, where $x = \mathcal{S}(r)$, $r, r^1, r^2 \in \mathcal{R}_\tau$ and $0 \leqslant s \leqslant a$. Let $t \in [0,a]$ and $h > 0$ such that $t+h \in [0,a]$. Since

$$x(t+h) - x(t) = \int_0^t T(t-s)(T(h)-I)f(s,x_{r(s)})\mathrm{d}s + \int_t^{t+h} T(t+h-s)f(s,x_{r(s)})\mathrm{d}s,$$

Theorem 1.2.10 and inequalities (2.5), (2.11) and (2.13) imply that

$$\begin{aligned}
&\|x(t+h)-x(t)\| \leqslant \widetilde{M}\int_0^t \left\|(T(h)-I)f(s,x_{r(s)})\right\|\mathrm{d}s + \widetilde{M}\int_t^{t+h}\left\|f(s,x_{r(s)})\right\|\mathrm{d}s \\
&\leqslant \widetilde{M}\int_0^t \left\|\int_0^h T(u)Af(s,x_{r(s)})\mathrm{d}u\right\|\mathrm{d}s + \widetilde{M}\int_t^{t+h}\left\|f(s,x_{r(s)})-f(s,0)\right\|\mathrm{d}s + \widetilde{M}\int_t^{t+h}\left\|f(s,0)\right\|\mathrm{d}s \\
&\leqslant \widetilde{M}\int_0^t\int_0^h \widetilde{M}\eta_1\|x_{r(s)}\|_{\mathcal{B}}\,\mathrm{d}u\,\mathrm{d}s + \widehat{M}\int_t^{t+h}\eta(s)\|x_{r(s)}\|_{\mathcal{B}}\,\mathrm{d}u + \widetilde{M}\int_t^{t+h}\sup_{0\leqslant u\leqslant a}\|f(u,0)\|\mathrm{d}s \\
&\leqslant \widetilde{M}^2\eta_1 h\int_0^t \hat{K}C_2 + \widehat{M}\|\varphi\|_\tau\,\mathrm{d}s + \widehat{M}\int_t^{t+h}\eta(s)\left(\hat{K}C_2 + \widehat{M}\|\varphi\|_\tau\right)\mathrm{d}u + \widetilde{M}h\sup_{0\leqslant u\leqslant a}\|f(u,0)\| \\
&\leqslant \widetilde{M}^2\eta_1 h C_3 a + \widetilde{M}\eta_2 h C_3 + \widetilde{M}h\sup_{0\leqslant u\leqslant a}\|f(u,0)\| \\
&\leqslant C_4 h,
\end{aligned}$$

where we have denoted by

$$\begin{aligned} \eta_2 &= \sup_{t\in[0,a]}\eta(t) \\ C_3 &= \hat{K}C_2 + \widehat{M}\|\varphi\|_\tau, \end{aligned}$$

$$C_4 = \widetilde{M}\left(\widetilde{M}\eta_1 C_3 a + \eta_2 C_3 + \sup_{0\leqslant u\leqslant a}\|f(u,0)\|\right).$$

By Remark 2.1.1, for every $r^1, r^2 \in \mathcal{R}_\tau$ and $0 \leqslant s \leqslant a$, we can estimate

$$\|x_{r^1(s)} - x_{r^2(s)}\|_{\mathcal{B}} \leqslant Q\sup_{\theta\leqslant 0}\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\|.$$

Without loss of generality, we can assume that $r^1(s) \leqslant r^2(s)$. In this case, there are several cases to analyze:

(ii.1) If $r^2(s) \leqslant 0$, then

$$\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| = \|\varphi(r^1(s)+\theta) - \varphi(r^2(s)+\theta)\| \leqslant L_\varphi|r^1(s) - r^2(s)|.$$

(ii.2) If $r^1(s) < 0$, $r^2(s) > 0$ and $\theta \leqslant -r^2(s)$, then

$$\begin{aligned}\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| &= \|\varphi(r^1(s)+\theta) - \varphi(r^2(s)+\theta)\|\\ &\leqslant L_\varphi|r^1(s) - r^2(s)|.\end{aligned}$$

(ii.3) If $r^1(s) < 0$ and $-r^2(s) < \theta \leqslant 0$, then

$$\begin{aligned}\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| &= \|\varphi(r^1(s)+\theta) - x(r^2(s)+\theta)\|\\ &\leqslant \|\varphi(r^1(s)+\theta) - \varphi(0)\| + \|\varphi(0) - x(r^2(s)+\theta)\|\\ &\leqslant L_\varphi|r^1(s)+\theta| + C_4(r^2(s)+\theta)\\ &\leqslant \max\{L_\varphi, C_4\}|r^2(s) - r^1(s)|\end{aligned}$$

(ii.4) If $r^1(s) > 0$ and $-r^1(s) \leqslant \theta \leqslant 0$, then

$$\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| \leqslant C_4|r^2(s) - r^1(s)|.$$

(ii.5) If $-r^2(s) \leqslant \theta \leqslant -r^1(s)$, then

$$\begin{aligned}\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| &= \|\varphi(r^1(s)+\theta) - x(r^2(s)+\theta)\|\\ &\leqslant \|\varphi(r^1(s)+\theta) - \varphi(0)\| + \|\varphi(0) - x(r^2(s)+\theta)\|\\ &\leqslant L_\varphi|r^1(s)+\theta| + C_4||r^2(s)+\theta|\\ &= -L_\varphi(r^1(s)+\theta) + C_4(r^2(s)+\theta)\\ &\leqslant \max\{L_\varphi, C_4\}|r^2(s) - r^1(s)|.\end{aligned}$$

(ii.6) If $\theta \leqslant -r^2(s)$, then

$$\begin{aligned}\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| &= \|\varphi(r^1(s)+\theta) - \varphi(r^2(s)+\theta)\| \\ &\leqslant L_\varphi |r^1(s) - r^2(s)|.\end{aligned}$$

Therefore, in any case, we conclude that

$$\|x(r^1(s)+\theta) - x(r^2(s)+\theta)\| \leqslant \max\{L_\varphi, C_4\}|r^1(s) - r^2(s)|,$$

which is independent of $r \in \mathcal{R}_\tau$.

(iii) Finally, we will conclude that F is a contraction. Let $r^1, r^2 \in \mathcal{R}_\tau$, $x = \mathcal{S}(r^1)$ and $y = \mathcal{S}(r^2)$. For every $t \in [0, a]$, it follows from (2.4) that

$$y(t) - x(t) = \int_0^t T(t-s)\left(f(s, y_{r^2(s)}) - f(s, x_{r^1(s)})\right) \mathrm{d}s.$$

This implies that

$$\begin{aligned}\|y(t) - x(t)\| &\leqslant \widetilde{M} \int_0^t \eta(s) \left\|y_{r^2(s)} - x_{r^1(s)}\right\|_{\mathcal{B}} \mathrm{d}s \\ &\leqslant \widetilde{M} \int_0^t \eta(s) \left\|x_{r^2(s)} - x_{r^1(s)}\right\|_{\mathcal{B}} \mathrm{d}s + \widetilde{M} \int_0^t \eta(s) \left\|y_{r^2(s)} - x_{r^2(s)}\right\|_{\mathcal{B}} \mathrm{d}s \\ &\leqslant \widetilde{M} Q \max\{L_\varphi, C_4\} \int_0^t \eta(s)|r^2(s) - r^1(s)|\mathrm{d}s + \widetilde{M}\widehat{K} \int_0^t \eta(s) \max_{0 \leqslant \xi \leqslant s} \|y(s) - x(s)\|\mathrm{d}s \\ &\leqslant \widetilde{M} Q \max\{L_\varphi, C_4\} \int_0^a \eta(s)\mathrm{d}s \; \|r^2 - r^1\|_\infty + \widetilde{M}\widehat{K} \int_0^t \eta(s) \max_{0 \leqslant \xi \leqslant s} \|y(s) - x(s)\|\mathrm{d}s.\end{aligned}$$

Applying Theorem 2.2.8, we obtain

$$\|y(t) - x(t)\| \leqslant \widetilde{M} Q \max\{L_\varphi, C_4\} \left(\int_0^a \eta(s)\mathrm{d}s\right) \exp\left(\widetilde{M}\widehat{K} \int_0^t \eta(s)\mathrm{d}s\right) \|r^2 - r^1\|_\infty.$$

This implies that

$$\begin{aligned}\|F(r^2) - F(r^1)\|_\infty &= \sup_{0 \leqslant t \leqslant a} |\rho(t, y_t) - \rho(t, x_t)| \\ &\leqslant C_\rho \sup_{0 \leqslant t \leqslant a} \max_{0 \leqslant s \leqslant t} \|y(s) - x(s)\| \\ &\leqslant C_\rho \widetilde{M} Q \max\{L_\varphi, C_4\} \left(\int_0^a \eta(s)\mathrm{d}s\right) \exp\left(\widetilde{M}\widehat{K} \int_0^a \eta(s)\mathrm{d}s\right) \|r^2 - r^1\|_\infty.\end{aligned}$$

Combining with (2.12), the above estimate shows that $F \colon \mathcal{R}_\tau \to \mathcal{R}_\tau$ is a contraction, which in turn implies that F has a unique fixed point. This completes the proof. □

2.4 Applications

The purpose of this section is to apply some of the results of the last sections. In order to do this, we will study the existence of a mild solution for diffusion systems as

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\xi) &= \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + f(t, u_{\rho(t,u_t)}), && 0 \leqslant t \leqslant a, \\ u(\theta,\xi) &= \varphi(\theta,\xi), && -\infty < \theta \leqslant 0, \end{aligned} \tag{2.14}$$

where $0 \leqslant \xi \leqslant \pi$, a is a positive number, $u\colon (-\infty, a]\times[0,\pi] \to \mathbb{R}$ and $\varphi\colon (-\infty,0]\times[0,\pi] \to \mathbb{R}$ is an appropriate function. We model this problem in abstract form on the space $X = L^2([0,\pi])$ endowed the norm $\|\cdot\|_2$ and we take as phase space $\mathcal{B} = C_0 \times L^2(g, X)$, where the function g satisfies the conditions established in Example 2.1.2. We consider the operator $A\colon D(A) \subset X \to X$ defined by

$$Az(\xi) = \frac{d^2 z(\xi)}{d\xi^2}$$

on the domain $D(A) = \{z \in X\colon z'' \in X, z(0) = z(\pi) = 0\}$. It is well–known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geqslant 0}$ on X. Furthermore, A has a discrete spectrum and the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenfunctions $z_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$. Moreover, the set $\{z_n\colon n \in \mathbb{N}\}$ is an orthonormal basis of X. Consequently,

$$Az = \sum_{n=1}^{\infty} -n^2 \langle z, z_n\rangle z_n,$$

for $z \in D(A)$ and

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}\langle z, z_n\rangle z_n,$$

for all $z \in X$. It follows from this expression that $(T(t))_{t\geqslant 0}$ is a compact semigroup with $\|T(t)\| \leqslant 1$ for all $t \geqslant 0$.

We study the problem (2.14) on the interval $[0,a]$. In [30, 46], there are many examples of functions ρ that arise in state–dependent delay problems. We consider the function $\rho(t,\psi) = t - p(t) - q(\psi)$, where $p\colon [0,a] \to \mathbb{R}^+$ is a regulated function and $q\colon \mathcal{B} \to \mathbb{R}^+$ is a bounded continuous function. We denote by

$$\tau = -\sup\{p(t) + q(\psi)\colon 0 \leqslant t \leqslant a,\ \psi \in \mathcal{B}\}.$$

It is clear that ρ satisfies conditions (D1)–(D2).

Let $f_0\colon [0,a] \times X \to X$ be a function that satisfies the Carathéodory condition

(C1), $f_0(\cdot,0) \in L^1([0,a],X)$, and there exists an integrable function L_0 such that

$$\|f_0(t,x) - f_0(t,y)\|_2 \leqslant L_0(t)\|x-y\|_2$$

for all $x, y \in X$. Let $f(t,\psi) = f_0(t,\psi(0))$. Then f satisfies the Carathéodory condition as well, $f(\cdot,0) \in L^1([0,a],X)$ and

$$\begin{aligned} \|f(t,\psi^1) - f(t,\psi^2)\|_2 &= \|f_0(t,\psi^1(0)) - f_0(t,\psi^2(0))\|_2 \\ &\leqslant L_0(t)\|\psi^1(0) - \psi^2(0)\|_2 \\ &\leqslant L_0(t)\|\psi^1 - \psi^2\|_{\mathcal{B}} \end{aligned}$$

for all $\psi^1, \psi^2 \in \mathcal{B}$. In particular, this implies that

$$\|f(t,\psi)\|_2 \leqslant L_0(t)\|\psi\|_{\mathcal{B}} + \|f_0(t,0)\|_2.$$

Moreover, as usual, we abbreviate $\psi(\theta,\xi)$ instead of $\psi(\theta)(\xi)$. With these notations, problem (2.14) is reduced to

$$\begin{aligned} \frac{\partial u}{\partial t}(t,\xi) &= \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + f_0(t,u(t-p(t)-q(u_t)),\xi)), \qquad 0 \leqslant t \leqslant a, \\ u(\theta,\xi) &= \varphi(\theta,\xi), \qquad -\infty < \theta \leqslant 0, \end{aligned} \tag{2.15}$$

for $0 \leqslant \xi \leqslant \pi$, where $u\colon (-\infty,a] \times [0,\pi] \to \mathbb{R}$ is a function such that $u(t,\cdot) \in L^2([0,\pi])$, and $\varphi\colon (-\infty,0] \times [0,\pi] \to \mathbb{R}$ is a function such that $\varphi \in \mathcal{B}_\tau$.

Combining with the previous assertions, problem (2.15) can be modeled in the abstract form (2.1). A direct application of Theorem 2.2.6 allows us to state the following result.

Corollary 2.4.1. *Under the above conditions, if $\varphi(0,\xi) = 0$ for all $\xi \in [0,\pi]$ and*

$$\widehat{K}\int_0^a L_0(s)\mathrm{d}s < 1,$$

then there exists a unique mild solution of problem (2.15).

CHAPTER 3

MEASURE FDES WITH UNBOUNDED STATE–DEPENDENT DELAYS

This chapter is dedicated to obtain a series of properties of the measure functional differential equations with state–dependent delays

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x_{t_0} &= \phi, \end{aligned} \tag{3.1}$$

where the integral of the right–hand side of (3.1) is in the sense of Kurzweil–Henstock–Stieltjes integral, $\sigma > 0$, $t_0 \in \mathbb{R}$, $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function, $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ is an appropriate Banach space, $\phi \in \mathcal{B}$ and $x\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$, $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$ and $f\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$ are functions. In the first section, we introduce an adequate phase space $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ to work with this type of equation. Next, under appropriate conditions, we obtain the existence and uniqueness of solutions of the equation (3.1). The chapter finishes with a periodic averaging theorem for measure functional differential equations with state–dependent delays. All results are new in the literature and are contained in [32].

3.1 Phase Space

For our purposes, we will need a suitable vector space $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ equipped with a norm $\|\cdot\|_{\mathcal{B}}$ which satisfies the following axioms:

(E1) $\mathcal{B}$ is complete.

(E2) If $t_0 \in \mathbb{R}$, $\sigma > 0$, $y\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ is regulated on $[t_0, t_0 + \sigma]$ and $y_{t_0} \in \mathcal{B}$, then there are locally bounded functions $k_1, k_2, k_3\colon [0, \infty) \to (0, \infty)$, all independent of y, t_0 and σ, such that, for every $t \in [t_0, t_0 + \sigma]$:

(a) $y_t \in \mathcal{B}$.

(b) $\|y(t)\| \leqslant k_1(t - t_0) \|y_t\|_{\mathcal{B}}$.

(c) $\|y_t\|_{\mathcal{B}} \leqslant k_2(t - t_0)\|y_{t_0}\|_{\mathcal{B}} + k_3(t - t_0) \sup\limits_{u \in [t_0, t]} \|y(u)\|$.

(E3) For $t \geqslant 0$, let $S(t)\colon \mathcal{B} \to \mathcal{B}$ be the operator defined by

$$(S(t)\varphi)(\theta) = \begin{cases} \varphi(0), & \theta = 0, \\ \varphi(0^-), & -t \leqslant \theta < 0, \\ \varphi(t + \theta), & \theta < -t. \end{cases}$$

Then, there is a continuous function $k\colon [0, \infty) \to [0, \infty)$ such that $k(0) = 0$ and

$$\|S(t)\varphi\|_{\mathcal{B}} \leqslant (1 + k(t)) \|\varphi\|_{\mathcal{B}}, \quad \text{for all } \varphi \in \mathcal{B}.$$

Usually, the motivation on the choice of the phase space lies on the expected properties of the solution of the investigated equation. Therefore, as Theorem 1.3.11 suggests, it seems to be more satisfactory to define phase space for measure FDE to be a particular subset of the set of all regulated functions.

Next, we show some examples of phase spaces.

Example 3.1.1 ([27, Example 3.2]). *Let $\rho\colon (-\infty, 0] \to (0, \infty)$ be a continuous function such that $\rho(0) = 1$ and that the function $p\colon [0, \infty) \to (0, \infty)$ given by*

$$p(t) = \sup_{\theta \leqslant -t} \frac{\rho(t + \theta)}{\rho(\theta)}, \quad t \geqslant 0,$$

is locally bounded. The space

$$\mathcal{B} = BG_\rho((-\infty, 0], \mathbb{R}^n) = \left\{ \varphi \in G((-\infty, 0], \mathbb{R}^n) \colon \frac{\|\varphi(\theta)\|}{\rho(\theta)} \text{ is bounded} \right\}$$

endowed with the norm

$$\|\varphi\|_\rho = \sup_{\theta\leqslant 0}\frac{\|\varphi(\theta)\|}{\rho(\theta)}, \quad \varphi \in BG_\rho((-\infty,0],\mathbb{R}^n),$$

satisfies all the properties (E1)–(E3). Therefore, $BG_\rho((-\infty,0],\mathbb{R}^n)$ *is a phase space.*

Example 3.1.2 ([27, Example 3.6])**.** *Let* $h\colon(-\infty,0]\to(0,\infty)$ *be a continuous function such that* $\int_{-\infty}^0 h(s)\mathrm{d}s = L < \infty$. *Consider the space*

$$\mathcal{B}_h((-\infty,0]),\mathbb{R}^n) = \left\{\varphi \in G((-\infty,0],\mathbb{R}^n)\colon \int_{-\infty}^0 h(s)\sup_{s\leqslant\xi\leqslant 0}|\varphi(\xi)|\mathrm{d}s < \infty\right\},$$

endowed with the norm

$$\|\varphi\|_h = \int_{-\infty}^0 h(s)\sup_{s\leqslant\xi\leqslant 0}|\varphi(\xi)|\mathrm{d}s, \quad \varphi \in \mathcal{B}_h((-\infty,0]),\mathbb{R}^n).$$

Then, $\mathcal{B}_h((-\infty,0]),\mathbb{R}^n)$ *is a phase space as well.*

Contrarily to the phase space H_0 chosen by G. A. Monteiro and A. Slavík in [43], where all functions y_t, $t<0$, belong to H_0 whenever $y\in H_0$, none of the axioms (E1)–(E3) of the phase space introduced here offer much data about the function x_t when $x\in\mathcal{B}$, $t_0 = 0$ and $t<0$. This lack of information permits to deal with more general FDEs with state–dependent delays. It also increases the quantity of sets that may be considered as phase spaces.

Example 3.1.3. *Let* $\tilde{\rho}\colon(-\infty,0]\to(0,\infty)$ *be a function as in Example 3.1.1. Additionally, suppose that*

(a) $\tilde{\rho}(\theta)\to\infty$ *as* $\theta\to-\infty$.

Consider the space

$$\mathcal{B} = BG^0_{\tilde{\rho}}((-\infty,0],\mathbb{R}^n) = \left\{\varphi\in BG_{\tilde{\rho}}((-\infty,0],\mathbb{R}^n)\colon \frac{\|\varphi(\theta)\|}{\tilde{\rho}(\theta)}\to 0, \theta\to-\infty\right\}$$

endowed with the norm

$$\|\varphi\|_{\tilde{\rho}} = \sup_{\theta\leqslant 0}\frac{\|\varphi(\theta)\|}{\tilde{\rho}(\theta)}, \quad \varphi\in BG^0_{\tilde{\rho}}((-\infty,0],\mathbb{R}^n).$$

If $\tilde{\rho}(\theta) = e^{\gamma\theta^2}$ *for* $\theta\leqslant 0$, *then it is possible to show that all conditions (E1)–(E3) are satisfied (see [27] for details).*

On the other hand, let $\varphi\colon (-\infty, 0] \to \mathbb{R}$ *be the function defined by*

$$\varphi(\theta) = \begin{cases} e^{-\gamma}, & \theta \in [-1, 0], \\ e^{\gamma(\theta^2+2\theta)}, & \theta \in (-\infty, -1]. \end{cases}$$

Since $\varphi(\theta)/\tilde{\rho}(\theta) = e^{2\gamma\theta} \to 0$ *as* $\theta \to -\infty$*, we have that* $\varphi \in \mathcal{B}$*. However, if* $\rho(s, \varphi_s) = s - 1$*, then* $\varphi_{\rho(s,\varphi_s)-s} = \varphi_{-1}$ *does not belong to* $\mathcal{B}$*. Indeed,*

$$\lim_{\theta\to-\infty} \frac{|\varphi_{-1}(\theta)|}{\tilde{\rho}(\theta)} = \lim_{\theta\to-\infty} \frac{|\varphi(\theta-1)|}{\tilde{\rho}(\theta)} = \lim_{\theta\to-\infty} \frac{e^{\gamma((\theta-1)^2+2(\theta-1))}}{\tilde{\rho}(\theta)} = e^{-\gamma},$$

which is different from zero. It implies that $\varphi_{-1} \notin \mathcal{B}$*. Thus,* $\mathcal{B}$ *is not a phase space considering the hypotheses from [43], but* $\mathcal{B}$ *is a phase space in our case.*

We finish this section recalling two important properties.

Lemma 3.1.4 ([27, Lemma 3.8])**.** *Assume that* $\mathcal{B}$ *is a phase space. If* $y\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$ *is such that* $y_{t_0} \in \mathcal{B}$ *and* $y|_{[t_0,t_0+\sigma]}$ *is a regulated function, then* $t \mapsto \|y_t\|_{\mathcal{B}}$ *is regulated on* $[t_0, t_0+\sigma]$*.*

Lemma 3.1.5 ([27, Lemma 3.10])**.** *Let* $r\colon [t_0, t_0+\sigma] \to \mathbb{R}$ *be a nondecreasing function such that* $r(s) \leqslant s$ *for all* $s \in [t_0, t_0+\sigma]$*. Assume that* $y\colon (-\infty, r(t_0+\sigma)] \to \mathbb{R}^n$ *is such that* $y_{r(t_0)} \in \mathcal{B}$ *and* $y|_{[r(t_0),r(t_0+\sigma)]}$ *is a regulated function, then* $t \mapsto \|y_{r(t)}\|_{\mathcal{B}}$ *is regulated on* $[t_0, t_0+\sigma]$*.*

Henceforth, until the end of this chapter, $\mathcal{B}$ will always denote a phase space in the sense presented in this section.

3.2 Existence and uniqueness of solutions

Here, we are interested in proving results concerning existence and uniqueness of solutions of measure functional differential equations with state–dependent delays given by

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x_{t_0} &= \phi, \end{aligned} \tag{3.2}$$

where $\sigma > 0$, $t_0 \in \mathbb{R}$, $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function, $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ is a Banach space satisfying axioms (E1)–(E3), $\phi \in \mathcal{B}$ and $x\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$, $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$ and $f\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$ are functions.

To show that the problem (3.2) has a solution, we begin by considering the set

$$X = \{x\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n\colon x_{t_0} \in \mathcal{B} \text{ and } x|_{[t_0,t_0+\sigma]} \text{ is regulated}\}, \tag{3.3}$$

equipped with the norm

$$\|x\|_X = \|x_{t_0}\|_{\mathcal{B}} + \sup_{u\in[t_0,t_0+\sigma]} \|x(u)\|. \tag{3.4}$$

This set is a Banach space. We also assume the following assumptions:

(F1) For all $x \in \mathcal{B}$, the integral $\int_{t_0}^{t_0+\sigma} f(s,x)\mathrm{d}g(s)$ exists in the sense of Kurzweil–Henstock–Stieltjes.

(F2) There exists a Kurzweil–Henstock–Stieltjes integrable function $M\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} f(s,x)\mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} M(s)\mathrm{d}g(s)$$

whenever $x \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0+\sigma]$.

(F3) There exists a Kurzweil–Henstock–Stieltjes integrable function $L\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} (f(s,x) - f(s,y))\,\mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L(s)\,\|x-y\|_{\mathcal{B}}\,\mathrm{d}g(s)$$

whenever $x, y \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0+\sigma]$.

(F4) There exists a Kurzweil–Henstock–Stieltjes integrable function $L_2\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} (f(s,x_u) - f(s,x_v))\,\mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_2(s)\,|u-v|\,\mathrm{d}g(s)$$

for all $x \in X$ and $u_1, u_2, u, v \in [t_0, t_0+\sigma]$.

(F5) For all $x \in X$, the function $t \mapsto \rho(t, x_t)$, $t \in [t_0, t_0+\sigma]$, is nondecreasing, satisfies $\rho(t,x_t) \leqslant t$ and $x_{\rho(t_0,x_{t_0})} \in \mathcal{B}$.

(F6) There exists a Kurzweil–Henstock–Stieltjes integrable function $L_3\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\int_{u_1}^{u_2} |\rho(s,x) - \rho(s,y)|\,\mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} L_3(s)\,\|x-y\|_{\mathcal{B}}\,\mathrm{d}g(s)$$

for all $u_1, u_2 \in [t_0, t_0+\sigma]$ and all $x, y \in \mathcal{B}$.

Remark 3.2.1. *Theorem 1.3.7 and Lemma 3.1.4 guarantee that, whenever* $x\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ *is such that* $x|_{[t_0,t_0+\sigma]}$ *is regulated and* $x_{t_0} \in \mathcal{B}$*, the function* $t \mapsto \|x_t\|_{\mathcal{B}}$ *is Kurzweil–Henstock–Stieltjes integrable with respect to a nondecreasing function* g*.*

Remark 3.2.2. *Notice that condition (F5) is necessary in order to ensure that* $\|x_{\rho(t,x_t)}\|_{\mathcal{B}}$ *is a regulated function (Lemma 3.1.5). Thus, in this case, following the same arguments used in the Remark 3.2.1,* $\|x_{\rho(t,x_t)}\|_{\mathcal{B}}$ *is Kurzweil–Henstock–Stieltjes integrable with respect to a nondecreasing function* g*, whenever* $x\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ *is such that* $x|_{[t_0,t_0+\sigma]}$ *is regulated and* $x_{t_0} \in \mathcal{B}$*.*

We present a result concerning the existence of solutions of measure FDEs with state–dependent delays.

Theorem 3.2.3. *Let* $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ *be a Banach space satisfying axioms (E1)–(E3),* $\phi \in \mathcal{B}$ *and* $g\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ *be a nondecreasing function. If* $f\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}^n$ *and* $\rho\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}$ *are functions that satisfy the properties (F1)–(F6), then the problem* (3.2) *has a solution.*

Proof. Let

$$A = \left\{ x \in X\colon x_{t_0} = \phi \text{ and } \|x(t) - \phi(0)\| \leqslant \int_{t_0}^{t} M(s)\mathrm{d}g(s) \text{ for all } t \in [t_0, t_0 + \sigma] \right\} \tag{3.5}$$

and define the operator $\Gamma\colon A \to X$ by

$$\Gamma x(t) := \begin{cases} \phi(t - t_0), & \text{if } t \leqslant t_0, \\ x(t_0) + \displaystyle\int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s), & \text{if } t_0 \leqslant t \leqslant t_0 + \sigma. \end{cases}$$

We start by proving some statements about A, the operator Γ and $\Gamma(A)$. Then, the theorem will follow as a direct consequence of Theorem 2.2.5 (Schauder Fixed Point Theorem).

Statement 1: The set A is convex. In fact, given $x, y \in A$, $\theta \in (-\infty, 0]$ and $\xi \in (0, 1)$, we have

$$\begin{aligned} (\xi x + (1 - \xi)y)_{t_0}(\theta) &= \xi x(t_0 + \theta) + (1 - \xi)y(t_0 + \theta) \\ &= \xi x_{t_0}(\theta) + (1 - \xi)y_{t_0}(\theta) \\ &= \xi\phi(\theta) + (1 - \xi)\phi(\theta) \\ &= \phi(\theta). \end{aligned}$$

For all $t \in [t_0, t_0 + \sigma]$, we get

$$\begin{aligned} \|(\xi x + (1-\xi)y)(t) - \phi(0)\| &= \|\xi x(t) + (1-\xi)y(t) - \xi\phi(0) - (1-\xi)\phi(0)\| \\ &\leqslant \xi \|x(t) - \phi(0)\| + (1-\xi)\|y(t) - \phi(0)\| \\ &\leqslant \int_{t_0}^{t} M(s)\mathrm{d}g(s), \end{aligned}$$

proving the Statement 1.

Statement 2: $\Gamma(A) \subset A$. Indeed, for $x \in A$, we have $(\Gamma x)_{t_0}(\theta) = (\Gamma x)(t_0 + \theta) = \phi(\theta)$. By (F2), we get

$$\|\Gamma x(t) - \phi(0)\| = \left\| \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s) \right\| \leqslant \int_{t_0}^{t} M(s)\mathrm{d}g(s),$$

for all $t \in [t_0, t_0 + \sigma]$, proving the Statement 2.

Statement 3: The set A is bounded and closed. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A such that converges to x on $\|\cdot\|_X$ norm. Then, for all $n \in \mathbb{N}$, $(x_n)_{t_0} = \phi$,

$$\|x_n(t) - \phi(0)\| \leqslant \int_{t_0}^{t} M(s)\mathrm{d}g(s) \quad \text{for all } t \in [t_0, t_0 + \sigma]$$

and

$$\begin{aligned} \|x_{t_0} - \phi\|_{\mathcal{B}} &= \|x_{t_0} - (x_n)_{t_0}\|_{\mathcal{B}} \\ &\leqslant \|(x - x_n)_{t_0}\|_{\mathcal{B}} + \sup_{u\in[t_0,t_0+\sigma]} \|x(u) - x_n(u)\| \\ &= \|x - x_n\|_X . \end{aligned} \tag{3.6}$$

Thus, passing (3.6) to limit when $n \to \infty$, we obtain $x_{t_0} = \phi$. By (E2), we have

$$\begin{aligned} \|x(t) - \phi(0)\| &\leqslant \|x(t) - x_n(t)\| + \|x_n(t) - \phi(0)\| \\ &\leqslant k_1(t - t_0)\|(x - x_n)_t\|_{\mathcal{B}} + \int_{t_0}^{t} M(s)\mathrm{d}g(s) \\ &\leqslant k_1(t - t_0)k_3(t - t_0) \sup_{u\in[t_0,t]} \|(x - x_n)(u)\| + \int_{t_0}^{t} M(s)\mathrm{d}g(s) \\ &\leqslant \sup_{u\in[0,\sigma]} k_1(u) \sup_{u\in[0,\sigma]} k_3(u) \|x - x_n\|_X + \int_{t_0}^{t} M(s)\mathrm{d}g(s) \end{aligned} \tag{3.7}$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $n \in \mathbb{N}$. If $\sup_{u\in[0,\sigma]} k_1(u) \sup_{u\in[0,\sigma]} k_3(u) > 0$, then let $\varepsilon > 0$ be arbitrary and $n_0 \in \mathbb{N}$ be such that $\|x - x_n\|_X < \varepsilon \left(\sup_{u\in[0,\sigma]} k_1(u) \sup_{u\in[0,\sigma]} k_3(u)\right)^{-1}$ for all $n \geqslant n_0$. By (3.7), we have

$$\|x(t) - \phi(0)\| < \varepsilon + \int_{t_0}^{t} M(s)\mathrm{d}g(s) \quad \text{for all } t \in [t_0, t_0 + \sigma].$$

Since ε is arbitrary, we conclude that

$$\|x(t) - \phi(0)\| \leqslant \int_{t_0}^{t} M(s)\mathrm{d}g(s) \quad \text{for all } t \in [t_0, t_0 + \sigma]. \tag{3.8}$$

Clearly, (3.8) is true if $\sup_{u\in[0,\sigma]} k_1(u) \sup_{u\in[0,\sigma]} k_3(u) = 0$. Thus, we obtain that A is closed. Finally, by (3.8),

$$\begin{aligned}
\|x\|_X &= \|x_{t_0}\|_{\mathcal{B}} + \sup_{u\in[t_0,t_0+\sigma]} \|x(u)\| \\
&\leqslant \|\phi\|_{\mathcal{B}} + \sup_{u\in[t_0,t_0+\sigma]} \left(\|x(u) - \phi(0)\| + \|\phi(0)\|\right) \\
&\leqslant \|\phi\|_{\mathcal{B}} + \sup_{u\in[t_0,t_0+\sigma]} \int_{t_0}^{u} M(s)\mathrm{d}g(s) + \|\phi(0)\| \\
&\leqslant \|\phi\|_{\mathcal{B}} + \int_{t_0}^{t_0+\sigma} M(s)\mathrm{d}g(s) + \|\phi(0)\|.
\end{aligned}$$

Therefore, A is bounded and the statement is proved.

Statement 4: The operator Γ is continuous. Firstly, given $x, y \in A$ and $t \in [t_0, t_0 + \sigma]$, inequalities (F3) and (F4) imply that

$$\begin{aligned}
&\|(\Gamma x - \Gamma y)(t)\| = \left\| \int_{t_0}^{t} f(s, x_{\rho(s,x_s)}) - f(s, y_{\rho(s,y_s)})\mathrm{d}g(s) \right\| \\
&= \left\| \int_{t_0}^{t} f(s, x_{\rho(s,x_s)}) + f(s, y_{\rho(s,x_s)}) - f(s, y_{\rho(s,x_s)}) - f(s, y_{\rho(s,y_s)})\mathrm{d}g(s) \right\| \\
&\leqslant \left\| \int_{t_0}^{t} f(s, x_{\rho(s,x_s)}) - f(s, y_{\rho(s,x_s)})\mathrm{d}g(s) \right\| + \left\| \int_{t_0}^{t} f(s, y_{\rho(s,x_s)}) - f(s, y_{\rho(s,y_s)})\mathrm{d}g(s) \right\| \\
&\leqslant \int_{t_0}^{t} L(s) \left\| x_{\rho(s,x_s)} - y_{\rho(s,x_s)} \right\|_{\mathcal{B}} \mathrm{d}g(s) + \int_{t_0}^{t} L_2(s) \left| \rho(s, x_s) - \rho(s, y_s) \right| \mathrm{d}g(s).
\end{aligned} \tag{3.9}$$

By axiom (E2) and by inequalities (F6) and (3.9), we have

$$\begin{aligned}
&\|(\Gamma x - \Gamma y)(t)\| \\
&\leqslant \int_{t_0}^{t} L(s) k_3(\rho(s, x_s) - t_0) \sup_{u\in[t_0,\rho(s,x_s)]} \|(x - y)(u)\| \mathrm{d}g(s) + \int_{t_0}^{t} L_2(s) L_3(s) \|x_s - y_s\|_{\mathcal{B}} \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t} L(s) k_3(\rho(s, x_s) - t_0) \|x - y\|_X \mathrm{d}g(s) + \int_{t_0}^{t} L_2(s) L_3(s) k_3(s - t_0) \sup_{u\in[t_0,s]} \|(x - y)(u)\| \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t} (L(s) + L_2(s) L_3(s)) \sup_{u\in[0,\sigma]} k_3(u) \mathrm{d}g(s) \|x - y\|_X .
\end{aligned} \tag{3.10}$$

Therefore, by (3.10),

$$\begin{aligned}\|\Gamma x-\Gamma y\|_X &= \left\|(\Gamma x-\Gamma y)_{t_0}\right\|_{\mathcal{B}}+\sup_{u\in[t_0,t_0+\sigma]}\|(\Gamma x-\Gamma y)(u)\| \\ &\leqslant \int_{t_0}^{t_0+\sigma}(L(s)+L_2(s)L_3(s))\sup_{u\in[0,\sigma]}k_3(u)\mathrm{d}g(s)\,\|x-y\|_X\,,\end{aligned}$$

proving the continuity on $\|\cdot\|_X$ norm.

Statement 5: The set $B:=\left\{f\colon[t_0,t_0+\sigma]\to\mathbb{R}^n\colon f=\Gamma x|_{[t_0,t_0+\sigma]}\text{ for some }x\in A\right\}$ is relatively compact on $G([t_0,t_0+\sigma],\mathbb{R}^n)$. Indeed, for all $t\in[t_0,t_0+\sigma]$,

$$\begin{aligned}\|\Gamma x(t)\| &= \left\|x(t_0)+\int_{t_0}^{t}f(s,x_{\rho(s,x_s)})\mathrm{d}g(s)\right\| \\ &\leqslant \|x(t_0)\|+\left\|\int_{t_0}^{t}f(s,x_{\rho(s,x_s)})\mathrm{d}g(s)\right\| \\ &\leqslant \|x(t_0)\|+\int_{t_0}^{t}M(s)\mathrm{d}g(s) \\ &\leqslant \|x(t_0)\|+\int_{t_0}^{t_0+\sigma}M(s)\mathrm{d}g(s).\end{aligned}$$

Furthermore,

$$\begin{aligned}\|\Gamma x(u)-\Gamma x(v)\| &= \left\|\int_{t_0}^{u}f(s,x_{\rho(s,x_s)})\mathrm{d}g(s)-\int_{t_0}^{v}f(s,x_{\rho(s,x_s)})\mathrm{d}g(s)\right\| \\ &= \left\|\int_{v}^{u}f(s,x_{\rho(s,x_s)})\mathrm{d}g(s)\right\|\leqslant\int_{v}^{u}M(s)\mathrm{d}g(s).\end{aligned}$$

By Corollary 1.3.9, the function $h(t)=\int_{t_0}^{t}M(s)\mathrm{d}g(s)$ is nondecreasing. In addition, both functions $K\colon[t_0,t_0+\sigma]\to\mathbb{R}$ and $\eta\colon[0,\infty)\to[0,\infty)$ defined by

$$K(t)=h(t)+t,\quad \eta(t)=t$$

are increasing functions. Moreover, η is continuous and $\eta(0)=0$. By Theorem 1.1.5, B is relatively compact on $G([t_0,t_0+\sigma],\mathbb{R}^n)$.

Statement 6: We conclude that Γ is completely continuous. In fact, let $(x_n)_{n\in\mathbb{N}}\subset A$ be a bounded sequence on $\|\cdot\|_X$ norm and let $t\in[t_0,t_0+\sigma]$. By axiom (E2), we obtain

$$\begin{aligned}\|x_n(t)\| &\leqslant k_1(t-t_0)\,\|(x_n)_t\|_{\mathcal{B}} \\ &\leqslant k_1(t-t_0)\left(k_2(t-t_0)\,\|(x_n)_{t_0}\|_{\mathcal{B}}+k_3(t-t_0)\sup_{u\in[t_0,t]}\|x_n(u)\|\right) \\ &\leqslant \sup_{u\in[0,\sigma]}k_1(u)\left(\sup_{u\in[0,\sigma]}k_2(u)\,\|(x_n)_{t_0}\|_{\mathcal{B}}+\sup_{u\in[0,\sigma]}k_3(u)\sup_{u\in[t_0,t_0+\sigma]}\|x_n(u)\|\right)\end{aligned}$$

$$\leqslant D\|x_n\|_X,$$

where

$$D = \max\left\{\sup_{u\in[0,\sigma]} k_1(u) \sup_{u\in[0,\sigma]} k_2(u), \sup_{u\in[0,\sigma]} k_1(u) \sup_{u\in[0,\sigma]} k_3(u)\right\}.$$

This inequality proves that (x_n) restricted to the interval $[t_0, t_0+\sigma]$ is bounded on the space $G([t_0, t_0+\sigma], \mathbb{R}^n)$. Consequently, by the last statement, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(\Gamma(x_{n_k}))_{k\in\mathbb{N}}$ is convergent on $\|\cdot\|_\infty$ norm. If we denote its limit by y, then the function $\bar{y}\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$ given by

$$\bar{y}(t) = \begin{cases} \phi(t-t_0), & t\in(-\infty, t_0], \\ y(t), & t\in[t_0, t_0+\sigma], \end{cases}$$

is well–defined and is such that

$$\begin{aligned} \|\Gamma(x_{n_k}) - \bar{y}\|_X &\leqslant \|(\Gamma x_{n_k} - \bar{y})_{t_0}\|_{\mathcal{B}} + \sup_{u\in[t_0,t_0+\sigma]} \|(\Gamma(x_{n_k}) - \bar{y})(u)\| \\ &\leqslant \|(x_{n_k})_{t_0} - \phi\|_{\mathcal{B}} + \|\Gamma(x_{n_k}) - y\|_\infty \\ &\leqslant \|\Gamma(x_{n_k}) - y\|_\infty. \end{aligned} \tag{3.11}$$

Passing (3.11) to limit when $k\to\infty$, we conclude that $(\Gamma(x_{n_k}))_k$ converges to $\bar{y}$ on $\|\cdot\|_X$ norm. Since A is closed, $\bar{y}\in A$. We conclude that Γ is completely continuous.

Finally, after all statements together with Theorem 2.2.5, we conclude the desired result. □

In what follows, we present a result which ensures the uniqueness of solutions of (3.2).

Theorem 3.2.4. *Let $\mathcal{B}\subset G((-\infty,0],\mathbb{R}^n)$ be a Banach space satisfying axioms (E1)–(E3), $\phi\in\mathcal{B}$ and $g\colon [t_0, t_0+\sigma]\to\mathbb{R}$ be a nondecreasing and left–continuous function. If $f\colon [t_0,t_0+\sigma]\times\mathcal{B}\to\mathbb{R}^n$ and $\rho\colon [t_0,t_0+\sigma]\times\mathcal{B}\to\mathbb{R}$ are functions that satisfy the properties (F1)–(F6) and L, L_2 and L_3 are regulated functions, then the problem* (3.2) *possesses a unique solution on $(-\infty, t_0+\sigma]$.*

Proof. If x, y are solutions of (3.2), then by following the same steps as in the proof of Theorem 3.2.3, we get

$$\|x(t) - y(t)\| \quad = \quad \|(\Gamma x - \Gamma y)(t)\|$$

$$
\begin{aligned}
&\leqslant \int_{t_0}^{t} L(s) \sup_{u\in[0,\sigma]} k_3(u) \sup_{u\in[t_0,s]} \|(x-y)(u)\| \,\mathrm{d}g(s) + \\
&+ \int_{t_0}^{t} L_2(s)L_3(s)k_3(s-t_0) \sup_{u\in[t_0,s]} \|(x-y)(u)\| \,\mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t} (L(s)+L_2(s)L_3(s)) \sup_{u\in[0,\sigma]} k_3(u) \sup_{u\in[t_0,s]} \|(x-y)(u)\| \,\mathrm{d}g(s).
\end{aligned}
$$

Let $\psi(v) = \sup_{u\in[t_0,v]} \|x(u)-y(u)\|$. Since x, y are regulated functions, it follows that ψ is also regulated, and thus, Kurzweil–Henstock–Stieltjes integrable with respect to the function g. Therefore,

$$
\begin{aligned}
\psi(t) &\leqslant \int_{t_0}^{t} (L(s)+L_2(s)L_3(s)) \sup_{u\in[0,\sigma]} k_3(u)\psi(s)\mathrm{d}g(s) \\
&\leqslant K \int_{t_0}^{t} \psi(s)\mathrm{d}g(s),
\end{aligned}
$$

where

$$
K = \sup_{u\in[t_0,t_0+\sigma]} (L(u)+L_2(u)L_3(u)) \sup_{u\in[0,\sigma]} k_3(u).
$$

Applying Theorem 1.3.10, we get $\psi(t) \leqslant 0$. Since $\psi(t) \geqslant 0$ by definition, it follows the desired result. □

3.2.1 Example

In this subsection, we exemplify three functions f, g and ρ and a phase space that fit all the hypotheses of Theorem 3.2.3.

Let $q\colon (-\infty, 0] \to \mathbb{R}$ be the function $q(\theta) = e^{\theta}$ and choose the phase space $\mathcal{B} = BG_q((-\infty,0],\mathbb{R})$ defined as in Example 3.1.1. Let $T\colon (-\infty,0] \to \mathbb{R}$ be a bounded continuous function such that:

(a) $\dfrac{T(\theta)}{q(\theta)}$ is bounded.

(b) $\int_{-\infty}^{0} |T(\theta)|q(\theta)\mathrm{d}\theta < \infty$.

(c) There exists a constant $D > 0$ such that $\int_{-\infty}^{0} |T(\theta-t_2)-T(\theta-t_1)|\mathrm{d}\theta \leqslant D|t_2-t_1|$ for all $0 \leqslant t_1, t_2$.

Define the functions $f\colon [0,\infty)\times\mathcal{B} \to \mathbb{R}$ and $\rho\colon [0,\infty)\times\mathcal{B} \to [0,\infty)$ by

$$
f(t,x) = \cos^2(t)\int_{-\infty}^{0} T(\theta)\tanh(x(\theta))\mathrm{d}\theta, \qquad \rho(t,x) = t - \int_{-\infty}^{-t} |T(\theta)|\tanh(|x(\theta-t)|)\mathrm{d}\theta.
$$

It is immediate that for all $x \in X$, the function $t \mapsto \rho(t, x_t)$, $t \in [0, \sigma]$, is nondecreasing, satisfies $\rho(t, x_t) \leqslant t$ and $x_{\rho(0,x_0)} \in \mathcal{B}$. In addition, by definition of ρ,

$$\begin{aligned} \rho(s, y) - \rho(s, x) &= \int_{-\infty}^{-s} |T(\theta)| \left(\tanh(|y(\theta - s)|) - \tanh(|x(\theta - s)|)\right) \mathrm{d}\theta \\ &= \int_{-\infty}^{-2s} |T(u + s)| \left(\tanh(|y(u)|) - \tanh(|x(u)|)\right) \mathrm{d}u \end{aligned} \tag{3.12}$$

By (3.12), we have

$$\begin{aligned} |\rho(s, y) - \rho(s, x)| &\leqslant \int_{-\infty}^{0} |T(u + s)| |\tanh(|y(u)|) - \tanh(|x(u)|)| \mathrm{d}u \\ &\leqslant \int_{-\infty}^{0} |T(u + s)| \frac{|y(u) - x(u)|}{q(u)} q(u) \mathrm{d}u \\ &\leqslant \sup_{\theta \leqslant 0} |T(\theta)| \int_{-\infty}^{0} q(u) \mathrm{d}u \|y - x\|_{\mathcal{B}} \end{aligned}$$

Now, since $|\tanh z| \leqslant 1$ for all $z \in \mathbb{R}$ and since there is constant $C > 0$ such that $|T(\theta)|/q(\theta) \leqslant C$ for all $\theta \leqslant 0$, we get

$$|f(t, x)| \leqslant \int_{-\infty}^{0} \frac{|T(\theta)|}{q(\theta)} |\tanh(x(\theta))| q(\theta) \mathrm{d}\theta \leqslant \int_{-\infty}^{0} C q(\theta) \mathrm{d}\theta = C$$

for all $(t, x) \in [0, \infty) \times \mathcal{B}$. Additionally, if $(t, x), (s, y) \in [0, \infty) \times \mathcal{B}$, then

$$\begin{aligned} f(t, x) - f(s, y) &= \cos^2(t) \int_{-\infty}^{0} T(\theta) \tanh(x(\theta)) \mathrm{d}\theta - \cos^2(s) \int_{-\infty}^{0} T(\theta) \tanh(y(\theta)) \mathrm{d}\theta \\ &= \cos^2(t) \int_{-\infty}^{0} T(\theta) (\tanh(x(\theta)) - \tanh(y(\theta))) \mathrm{d}\theta \\ &\quad + (\cos^2(t) - \cos^2(s)) \int_{-\infty}^{0} T(\theta) \tanh(y(\theta)) \mathrm{d}\theta \end{aligned} \tag{3.13}$$

In particular, for all $x \in X$ and all $0 \leqslant b \leqslant a \leqslant \sigma$,

$$\begin{aligned} &f(t, x_a) - f(s, x_b) \\ &\quad = \cos^2(t) \int_{-\infty}^{0} T(\theta) (\tanh(x(\theta + a)) - \tanh(x(\theta + b))) \mathrm{d}\theta \\ &\quad\quad + (\cos^2(t) - \cos^2(s)) \int_{-\infty}^{0} T(\theta) \tanh(x(\theta + b)) \mathrm{d}\theta \\ &\quad = \cos^2(t) \Bigg(\int_{-\infty}^{0} (T(u - a) - T(u - b)) \tanh(x(u)) \mathrm{d}u \\ &\quad\quad + \int_{-b}^{0} (T(u + b - a) - T(u)) \tanh(x(u + b)) \mathrm{d}u + \int_{b-a}^{0} T(u) \tanh(x(u + a)) \mathrm{d}u \Bigg) \\ &\quad\quad + (\cos^2(t) - \cos^2(s)) \int_{-\infty}^{0} T(\theta) \tanh(x(\theta + b)) \mathrm{d}\theta. \end{aligned} \tag{3.14}$$

By (3.13),

$$
\begin{aligned}
|f(s,x)-f(s,y)| &\leqslant \int_{-\infty}^{0} |T(\theta)||\tanh(x(\theta))-\tanh(y(\theta))|\mathrm{d}\theta \\
&\leqslant \int_{-\infty}^{0} |T(\theta)|\frac{|x(\theta)-y(\theta)|}{q(\theta)}q(\theta)\mathrm{d}\theta \\
&\leqslant \int_{-\infty}^{0} |T(\theta)|q(\theta)\mathrm{d}\theta \;\|x-y\|_{\mathcal{B}}.
\end{aligned}
$$

Then,

$$
\left|\int_{u_1}^{u_2} f(s,x)-f(s,y)\mathrm{d}s\right| \leqslant \int_{u_1}^{u_2} \overline{C}\|x-y\|_{\mathcal{B}}\mathrm{d}s,
$$

for all $u_1, u_2 \geqslant 0$, where $\overline{C} = \int_{-\infty}^{0} |T(\theta)|q(\theta)\mathrm{d}\theta$. By (3.14), we obtain

$$
\begin{aligned}
|f(s,x_a)-f(s,x_b)| \leqslant& \int_{-\infty}^{0} |T(u-a)-T(u-b)|\mathrm{d}u \\
&+ \int_{-\infty}^{0} |T(u+b-a)-T(u)|\mathrm{d}u + \int_{b-a}^{0} |T(u)||\tanh(x(u+a))|\mathrm{d}u \\
\leqslant& 2D|a-b| + \overline{D}|a-b|,
\end{aligned}
$$

where $\overline{D} = \sup_{\theta\leqslant 0} |T(\theta)|$.

Therefore, all hypotheses of the Theorem 3.2.3 are satisfied for the case where the function $g\colon [0,\sigma] \to \mathbb{R}$ is given by $g(s) = s$. The continuity of g is enough to conclude that

$$
\begin{aligned}
x(t) &= x(t_0) + \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\
x_{t_0} &= \phi,
\end{aligned}
$$

has a unique solution.

Remark 3.2.5. *Since the example above uses an abstract function $T\colon (-\infty,0] \to \mathbb{R}$ that satisfies some assertions, the question of the existence of such a function arises. Indeed, it is possible to verify that the function $T(\theta) = e^{-\theta^2+\theta}$ answers positively this question.*

3.3 Periodic averaging theorem

In this section, our goal is to prove a periodic averaging theorem for measure FDE with state–dependent delays. This method plays an important role for investigating the asymptotic behavior of the solutions. The classical results on periodic averaging principles for ordinary differential equations ensures that, under certain conditions, the solution of nonlinear differential equations given by

$$\begin{aligned} x'(t) &= \varepsilon f(t,x(t)) + \varepsilon^2 g(s,x(t),\varepsilon), \\ x_{t_0} &= x_0, \end{aligned} \tag{3.15}$$

where $\varepsilon > 0$ is a small parameter and f is T–periodic with respect to the first variable, are close to the solutions of the autonomous differential equation

$$\begin{aligned} y'(t) &= \varepsilon f_0(y(t)), \\ y_{t_0} &= y_0, \end{aligned} \tag{3.16}$$

where

$$f_0(y) = \frac{1}{T}\int_0^T f(s,y)\mathrm{d}s.$$

This type of result allows us to understand the asymptotic behavior of the solutions of equation (3.15) only knowing the information about the solutions of equation (3.16), which is much easier to deal with, since it is an autonomous equation.

It is worth mentioning that it is also possible to have a similar approach when the function f is not periodic with respect to the first variable, but in this case, it is necessary to calculate $f_0(y)$ as the following limit:

$$f_0(y) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t,y)\mathrm{d}t.$$

Therefore, in this case, one needs to ensure the right–hand side is well–defined and this limit exists. This kind of averaging principle was extensively investigated by many authors, including Jaroslav Kurzweil, see [38, 39, 40, 41] and the references therein.

In this work, we are interested to prove that the solutions of the measure FDE with state–dependent delays

$$\begin{aligned} x(t) &= x(0) + \varepsilon\int_0^t f(s, x_{\rho(s,x_s,\varepsilon)})\mathrm{d}h(s) + \varepsilon^2\int_0^t g(s, x_{\rho(s,x_s,\varepsilon)}, \varepsilon)\mathrm{d}h(s), \\ x_0 &= \phi, \end{aligned} \tag{3.17}$$

where f is T–periodic with respect to the first variable, can be approximated by the solutions of the autonomous FDE with state–dependent delays

$$\begin{aligned} y(t) &= y(0) + \varepsilon\int_0^t f_0(y_{\rho(s,y_s,\varepsilon)})\mathrm{d}s, \\ y_0 &= \phi, \end{aligned} \tag{3.18}$$

where

$$f_0(x) = \frac{1}{T}\int_0^T f(s,x)\mathrm{d}h(s),$$

which is known as the **average** of f.

As explained before, the main advantage behind periodic averaging principle is due to the fact that it allows us to investigate the behavior of the solutions of a very complicated equation described by (3.17) only investigating the solutions of a simpler equation given by (3.18), which is autonomous and easier to deal with. In our case, our equation (3.18) is a type of autonomous FDE which is much simpler compared to (3.17). It does not involve measure, for instance.

Let $\varepsilon_0 > 0$, $L > 0$, $T > 0$. Consider that the functions $f\colon [0,\infty)\times\mathcal{B}\to\mathbb{R}^n$, $g\colon [0,\infty)\times\mathcal{B}\times(0,\varepsilon_0]\to\mathbb{R}^n$ are bounded. Suppose $h\colon [0,\infty)\to\mathbb{R}$ is a left–continuous and nondecreasing function and let $\rho\colon [0,\infty)\times\mathcal{B}\times(0,\varepsilon_0]\to[0,\infty)$ be a function. Assume that the following conditions are satisfied:

(G1) For all $x\in\mathcal{B}$, the following integrals

$$\int_{u_1}^{u_2} f(s,x)\mathrm{d}h(s) \text{ and } \int_{u_1}^{u_2} g(s,x,\varepsilon)\mathrm{d}h(s)$$

exist for all $u_1,u_2\in[0,\infty)$ and $\varepsilon\in(0,\varepsilon_0]$ in the sense of Kurzweil–Henstock–Stieltjes.

(G2) Both ρ, f are T–periodic with respect to the first variable.

(G3) There exists a constant $\alpha > 0$ such that $h(t+T)-h(t)=\alpha$ for all $t\geqslant 0$.

(G4) There exists a constant $C > 0$ such that

$$\left\|\int_{u_1}^{u_2}(f(s,x)-f(s,y))\,\mathrm{d}h(s)\right\| \leqslant \int_{u_1}^{u_2} C\,\|x-y\|_{\mathcal{B}}\,\mathrm{d}h(s)$$

for all $x,y\in\mathcal{B}$ and $u_1,u_2\in[0,\infty)$.

(G5) The integral

$$f_0(x) = \frac{1}{T}\int_0^T f(s,x)\mathrm{d}h(s)$$

exists in the sense of Kurzweil–Henstock–Stieltjes for all $x\in\mathcal{B}$.

(G6) There exists a constant $C_2 > 0$ such that

$$\left\|\int_{u_1}^{u_2}(f(s,x_a)-f(s,x_b))\,\mathrm{d}h(s)\right\| \leqslant \int_{u_1}^{u_2} C_2\,|a-b|\,\mathrm{d}h(s)$$

for all $x\in X$, all $a,b\in[0,\sigma]$ and all $u_1,u_2\in[0,\infty)$.

(G7) For all $\varepsilon \in (0, \varepsilon_0]$ and all $x \in X$, the function $t \mapsto \rho(t, x_t, \varepsilon)$, $t \in [0, \sigma]$, is nondecreasing, satisfies $\rho(t, x_t, \varepsilon) \leqslant t$ and $x_{\rho(0,x_0,\varepsilon)} \in \mathcal{B}$.

(G8) There exists a constant $C_3 > 0$ such that

$$|\rho(t, x_a, \varepsilon) - \rho(t, x_b, \varepsilon)| \leqslant \varepsilon C_3 |a - b|$$

for all $a, b \in [0, \sigma]$, all $t \in [0, \infty)$, $x \in X$ and $\varepsilon \in (0, \varepsilon_0]$.

(G9) There exists a constant $C_4 > 0$ such that

$$|\rho(s, y, \varepsilon) - \rho(s, x, \varepsilon)| \leqslant C_4 \|y - x\|_{\mathcal{B}}$$

for all $s \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$ and $x, y \in \mathcal{B}$.

Now, we are ready to prove our periodic averaging theorem for measure FDEs with state–dependent delays. We follow some ideas from [42, Theorem 13]. It is the main result of this section.

Theorem 3.3.1. *Let $\varepsilon_0 > 0$, $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ be a Banach space satisfying axioms (E1)–(E3), $\phi \in \mathcal{B}$ and $h\colon [0, \infty) \to \mathbb{R}$ be a left–continuous nondecreasing function. Assume that $f\colon [0, \infty) \times \mathcal{B} \to \mathbb{R}^n$, $g\colon [0, \infty) \times \mathcal{B} \times (0, \varepsilon_0] \to \mathbb{R}^n$ are bounded functions and $\rho\colon [0, \infty) \times \mathcal{B} \to [0, \infty)$ is a function. Also, suppose that the properties (G1)–(G9) are satisfied. Finally, suppose that, for all $\varepsilon \in (0, \varepsilon_0]$, the initial value problems*

$$\begin{aligned} x(t) &= x(0) + \varepsilon \int_0^t f(s, x_{\rho(s,x_s,\varepsilon)}) \mathrm{d}h(s) + \varepsilon^2 \int_0^t g(s, x_{\rho(s,x_s,\varepsilon)}, \varepsilon) \mathrm{d}h(s), \\ x_0 &= \phi, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} y(t) &= y(0) + \varepsilon \int_0^t f_0(y_{\rho(s,y_s,\varepsilon)}) \mathrm{d}s \\ y_0 &= \phi, \end{aligned} \tag{3.20}$$

have solutions $x^\varepsilon, y^\varepsilon\colon (-\infty, L/\varepsilon] \to \mathbb{R}^n$, respectively. Then, there exists a $J > 0$ such that the inequality

$$\|x^\varepsilon(t) - y^\varepsilon(t)\|_X \leqslant J\varepsilon \tag{3.21}$$

holds for all $t \in (-\infty, L/\varepsilon]$, where X is the Banach space defined on (3.3) *with the norm given by* (3.4).

Proof. Since f and g are bounded functions, there exists $M > 0$ such that $\|f(t, x)\| \leqslant M$ and $\|g(t, x, \varepsilon)\| \leqslant M$ for all $x \in \mathcal{B}$, $t \geqslant 0$ and $\varepsilon \in (0, \varepsilon_0]$. We can assume without loss of generality that M is the same constant for both functions. Then, Theorem 1.3.6 implies that

$$\|f_0(x)\| \leqslant \frac{1}{T}\int_0^T M\mathrm{d}h(s) \leqslant \frac{M}{T}(h(T)-h(0)) = \frac{M}{T}\alpha \tag{3.22}$$

for all $x \in \mathcal{B}$. If x^ε and y^ε are solutions of (3.19) and (3.20) respectively, then

$$\|x^\varepsilon(t)-y^\varepsilon(t)\|_X = \|(x^\varepsilon - y^\varepsilon)_0\|_{\mathcal{B}} + \sup_{t\in[0,L/\varepsilon]}\|x^\varepsilon(t)-y^\varepsilon(t)\| = \sup_{t\in[0,L/\varepsilon]}\|x^\varepsilon(t)-y^\varepsilon(t)\| \tag{3.23}$$

for all $t \in [0, L/\varepsilon]$. On the other hand, given $t \in [0, L/\varepsilon]$, by the conditions (G1)–(G9) and Theorem 1.3.6, we get

$$\begin{aligned}
&\|x^\varepsilon(t)-y^\varepsilon(t)\| = \\
&= \left\|\varepsilon\int_0^t f(s, x^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)})\mathrm{d}h(s) + \varepsilon^2\int_0^t g(s, x^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)},\varepsilon)\mathrm{d}h(s) - \varepsilon\int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| \\
&\leqslant \varepsilon\left\|\int_0^t f(s, x^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)}) - f(s, y^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)})\mathrm{d}h(s)\right\| + \varepsilon\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)}) - f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}h(s)\right\| \\
&\quad + \ \varepsilon\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}h(s) - \int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| + \varepsilon^2 M(h(t)-h(0)) \\
&\leqslant \varepsilon\int_0^t C\left\|x^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)} - y^\varepsilon_{\rho(s,x^\varepsilon_s,\varepsilon)}\right\|_{\mathcal{B}}\mathrm{d}h(s) + \varepsilon\int_0^t C_2\left|\rho(s,x^\varepsilon_s,\varepsilon) - \rho(s,y^\varepsilon_s,\varepsilon)\right|\mathrm{d}h(s) \\
&\quad + \varepsilon\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}h(s) - \int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| + \varepsilon^2 M(h(t)-h(0)) \\
&\leqslant \varepsilon C\int_0^t k_3(s)\sup_{u\in[0,s]}\|x^\varepsilon(u)-y^\varepsilon(u)\|\,\mathrm{d}h(s) + \varepsilon C_2 C_4\int_0^t \|x^\varepsilon_s - y^\varepsilon_s\|_{\mathcal{B}}\,\mathrm{d}h(s) \\
&\quad + \varepsilon\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}h(s) - \int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| + \varepsilon^2 M(h(t)-h(0)) \\
&\leqslant \varepsilon C\int_0^t k_3(s)\sup_{u\in[0,s]}\|x^\varepsilon(u)-y^\varepsilon(u)\|\,\mathrm{d}h(s) + \varepsilon C_2 C_4\int_0^t k_3(s)\sup_{u\in[0,s]}\|x^\varepsilon(u)-y^\varepsilon(u)\|\,\mathrm{d}h(s) \\
&\quad + \varepsilon\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}h(s) - \int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| + \varepsilon^2 M(h(t)-h(0)) \\
&\leqslant \varepsilon(C + C_2C_4)\int_0^t k_3(s)\sup_{u\in[0,s]}\|x^\varepsilon(u)-y^\varepsilon(u)\|\,\mathrm{d}h(s) \\
&\quad + \varepsilon\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}h(s) - \int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| + \varepsilon^2 M(h(t)-h(0)).
\end{aligned} \tag{3.24}$$

Taking p as the largest integer such that $pT \leqslant t$, we have

$$\begin{aligned}
&\left\|\int_0^t f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\,\mathrm{d}h(s) - \int_0^t f_0(y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)})\mathrm{d}s\right\| \\
&\qquad\qquad \leqslant \sum_{i=1}^{p}\left\|\int_{(i-1)T}^{iT} f(s, y^\varepsilon_{\rho(s,y^\varepsilon_s,\varepsilon)}) - f(s, y^\varepsilon_{\rho(s,y^\varepsilon_{(i-1)T},\varepsilon)})\mathrm{d}h(s)\right\|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\left\|\int_{(i-1)T}^{iT} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}h(s)-\int_{(i-1)T}^{iT} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}s\right\| \\
&+\sum_{i=1}^{p}\left\|\int_{(i-1)T}^{iT} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})-f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})\mathrm{d}s\right\| \\
&+\left\|\int_{pT}^{t} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})\mathrm{d}h(s)-\int_{pT}^{t} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})\mathrm{d}s\right\|. \qquad (3.25)
\end{aligned}
$$

For every $i\in\{1,2,\dots,p\}$ and every $s\in[(i-1)T,iT]$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{p}\left\|\int_{(i-1)T}^{iT} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)}) -f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}h(s)\right\| \\
&\leqslant \sum_{i=1}^{p}\int_{(i-1)T}^{iT} C_2|\rho(s,y^{\varepsilon}_{s},\varepsilon)-\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)|\mathrm{d}h(s) \\
&\leqslant \sum_{i=1}^{p} C_2C_3\varepsilon\int_{(i-1)T}^{iT} |s-iT+T|\mathrm{d}h(s) \\
&\leqslant \sum_{i=1}^{p} C_2C_3T\varepsilon(h(iT)-h((i-1)T)) \\
&= C_2C_3T\alpha p\varepsilon. \qquad (3.26)
\end{aligned}
$$

Using this estimate and the fact that $pT\leqslant L/\varepsilon$, we get

$$
\sum_{i=1}^{p}\left\|\int_{(i-1)T}^{iT}\Big(f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})-f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\Big)\,\mathrm{d}h(s)\right\|\leqslant C_2C_3\alpha L. \qquad (3.27)
$$

On the other hand, notice that, for $s\in[(i-1)T,iT]$, we have

$$
\begin{aligned}
\left\|f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})-f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\right\| &= \left\|\frac{1}{T}\int_0^T\Big(f(u,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})-f(u,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\Big)\,\mathrm{d}h(u)\right\| \\
&\leqslant \frac{1}{T}\int_0^T C_2|\rho(s,y^{\varepsilon}_{s},\varepsilon)-\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)|\mathrm{d}h(u) \\
&\leqslant \frac{1}{T}\int_0^T C_2C_3\varepsilon|s-(i-1)T|\mathrm{d}h(u) \\
&\leqslant C_2C_3\varepsilon\alpha.
\end{aligned}
$$

Therefore, it implies that

$$
\begin{aligned}
\sum_{i=1}^{p}\left\|\int_{(i-1)T}^{iT} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{s},\varepsilon)})-f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}s\right\| &\leqslant \sum_{i=1}^{p}\int_{(i-1)T}^{iT} C_2C_3\varepsilon\alpha\mathrm{d}s \\
&\leqslant C_2C_3\varepsilon\alpha pT \\
&\leqslant C_2C_3\alpha L. \qquad (3.28)
\end{aligned}
$$

Since f is T–periodic in the first variable and by the definition of f_0, we obtain

$$\begin{aligned}\sum_{i=1}^{p}&\left\|\int_{(i-1)T}^{iT} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}h(s)-\int_{(i-1)T}^{iT} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}s\right\|\\ &=\sum_{i=1}^{p}\left\|\int_{0}^{T} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})\mathrm{d}h(s)-f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_{(i-1)T},\varepsilon)})T\right\|=0. \qquad (3.29)\end{aligned}$$

Finally, we have

$$\begin{aligned}\left\|\int_{pT}^{t} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_s,\varepsilon)})\mathrm{d}h(s)\right.&\left.-\int_{pT}^{t} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_s,\varepsilon)})\mathrm{d}s\right\|\\ &\leqslant \left\|\int_{pT}^{t} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_s,\varepsilon)})\mathrm{d}h(s)\right\|+\left\|\int_{pT}^{t} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_s,\varepsilon)})\mathrm{d}s\right\|\\ &\leqslant M(h(t)-h(pT))+\frac{M\alpha}{T}(t-pT)\\ &\leqslant M(h((p+1)T)-h(pT))+\frac{M\alpha}{T}T\\ &= M\alpha+M\alpha=2M\alpha. \qquad (3.30)\end{aligned}$$

Combining inequalities (3.25), (3.27), (3.28), (3.29) and (3.30), we get

$$\begin{aligned}\left\|\int_{0}^{t} f(s,y^{\varepsilon}_{\rho(s,y^{\varepsilon}_s,\varepsilon)})\mathrm{d}h(s)-\int_{0}^{t} f_0(y^{\varepsilon}_{\rho(s,y^{\varepsilon}_s,\varepsilon)})\mathrm{d}s\right\| &\leqslant 2M\alpha+C_2C_3\alpha L+C_2C_3\alpha L\\ &\leqslant 2\alpha(M+C_2C_3L). \qquad (3.31)\end{aligned}$$

From inequalities (3.24) and (3.31), we get

$$\begin{aligned}&\|x^{\varepsilon}(t)-y^{\varepsilon}(t)\|\\ &\quad\leqslant \varepsilon(C+C_2C_4)\int_0^t k_3(s)\sup_{u\in[0,s]}\|y^{\varepsilon}(u)-x^{\varepsilon}(u)\|\,\mathrm{d}h(s)+\varepsilon K+\varepsilon^2M(h(t)-h(0)),\end{aligned}$$

where $K=2\alpha(M+C_2C_3L)$. Since k_3 is bounded, there exists $K'>0$ such that $\sup_{s\in[0,t]} k_3(s)\leqslant K'$. It implies that

$$\|x^{\varepsilon}(t)-y^{\varepsilon}(t)\|\leqslant \varepsilon(C+C_2C_4)K'\int_0^t \sup_{u\in[0,s]}\|y^{\varepsilon}(u)-x^{\varepsilon}(u)\|\,\mathrm{d}h(s)+\varepsilon K+\varepsilon^2M(h(t)-h(0)).$$

Notice that

$$\begin{aligned}\varepsilon(h(t)-h(0))&\leqslant \varepsilon\left(h\left(\frac{L}{\varepsilon}\right)-h(0)\right)\\ &\leqslant \varepsilon\left(h\left(\left\lceil\frac{L}{\varepsilon T}\right\rceil T\right)-h(0)\right)\\ &=\varepsilon\left\lceil\frac{L}{\varepsilon T}\right\rceil\alpha\leqslant \varepsilon\left(\frac{L}{\varepsilon T}+1\right)\alpha\leqslant\left(\frac{L}{T}+\varepsilon_0\right)\alpha.\end{aligned}$$

Hence, we have

$$\|x^\varepsilon(t) - y^\varepsilon(t)\| \leqslant \varepsilon K'' \int_0^t \sup_{u\in[0,s]} \|y^\varepsilon(u) - x^\varepsilon(u)\| \,\mathrm{d}h(s) + \varepsilon K + \varepsilon M\left(\frac{L}{T} + \varepsilon_0\right)\alpha,$$

where $K'' = (C + C_2C_4)K'$. If $\psi(s) = \sup\limits_{\tau\in[0,s]} \|x^\varepsilon(\tau) - y^\varepsilon(\tau)\|$, then

$$\psi(t) \leqslant \varepsilon K'' \int_0^t \psi(s)\mathrm{d}h(s) + \varepsilon K + \varepsilon M\left(\frac{L}{T} + \varepsilon_0\right)\alpha.$$

By Theorem 1.3.10, we get

$$\begin{aligned}\psi(t) &\leqslant e^{\varepsilon K''(h(t)-h(0))}\left(K + M\left(\frac{L}{T} + \varepsilon_0\right)\alpha\right)\cdot\varepsilon \\ &\leqslant e^{K''\left(\frac{L}{T}+\varepsilon_0\right)\alpha}\left(K + M\left(\frac{L}{T} + \varepsilon_0\right)\alpha\right)\cdot\varepsilon.\end{aligned}$$

If we define $J := e^{K''\left(\frac{L}{T}+\varepsilon_0\right)\alpha}\left(K + M\left(\frac{L}{T} + \varepsilon_0\right)\alpha\right)$, then we have $\psi(t) \leqslant J\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, L/\varepsilon]$. Therefore, in particular, for $t \in [0, L/\varepsilon]$,

$$\|x^\varepsilon(t) - y^\varepsilon(t)\|_X = \sup_{t\in[0,L/\varepsilon]} \|x^\varepsilon(t) - y^\varepsilon(t)\| = \psi(L/\varepsilon) \leqslant J\varepsilon$$

proving the desired result. $\square$

Remark 3.3.2. *A careful examination in the proof of Theorem 3.3.1 reveals that it is enough to require that conditions (G4), (G6), (G8) and (G9) hold on compact invervals instead of unbounded intervals, since* $t \in [0, L/\varepsilon]$.

CHAPTER 4

MEASURE FDES AND OTHER TYPES OF EQUATIONS: CORRESPONDENCES

In the article [21], M. Federson, J. G. Mesquita and A. Slavík proved that measure FDEs and few others classes of differential equations can be related. One of theses relations is a correspondence between

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x_s)\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{4.1}$$

and the generalized ODE

$$\frac{dx}{d\tau} = DF(t, x), \tag{4.2}$$

where $f\colon [t_0, t_0 + \sigma] \times \Omega \to \mathbb{R}^n$, $\Omega \subset G([-r, 0], \mathbb{R}^n)$, x_t the function $x_t\colon [-r, 0] \to \mathbb{R}^n$ defined by $x_t(\theta) = x(t + \theta)$, $r > 0$ is fixed, the domain of F is a subset of $[t_0, t_0 + \sigma] \times G([-r, 0], \mathbb{R}^n)$ and its expression is given by

$$F(t, y)(\xi) = \begin{cases} 0, & \xi \in [t_0 - r, t_0], \\ \displaystyle\int_{t_0}^{\xi} f(s, y_s)\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f(s, y_s)\mathrm{d}g(s), & \xi \in [t, t_0 + \sigma]. \end{cases} \tag{4.3}$$

With minor modifications on the domain of the functions f, x_t and F, in [51], A. Slavík, applying analogous techniques found in [21], proved a correspondence between (4.1) and (4.2) for infinite delays. C. Gallegos, H. Henríquez, and J.G. Mesquita generalized Slavík's results in [27], obtaining a relation between the generalized ODE (4.2), where F is defined on a suitable Banach space and has the expression

$$F(t,y)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f(s, y_{r(s)}) \mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f(s, y_{r(s)}) \mathrm{d}g(s), & \xi \in [t, t_0+\sigma], \end{cases} \tag{4.4}$$

and the measure FDE with time–dependent delays

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x_{r(s)}) \mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma],$$

where $r\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function that satisfies $r(t) \leqslant t$ for all $[t_0, t_0+\sigma]$, $f\colon [t_0, t_0+\sigma] \times Q \to \mathbb{R}^n$, $Q \subset G((-\infty, 0], \mathbb{R}^n)$ and the segment x_t is defined on $(-\infty, 0]$.

Those papers and theorems reveal how meaningful is the task to find a connection between measure FDEs and other classes of differential equations. In this chapter, we show correspondences between measure FDE with state dependent delays

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(s, x_{\rho(s,x_s)}) \mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x_{t_0} &= \phi, \end{aligned} \tag{4.5}$$

and many other categories of differential equations.

4.1 Measure FDEs and generalized ODEs

Once again, we consider the phase space $\mathcal{B}$ and the set

$$X = \left\{ x\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n : x_{t_0} \in \mathcal{B} \text{ and } x|_{[t_0, t_0+\sigma]} \text{ is regulated} \right\}$$

as defined in the proof of the Theorem 3.2.3. In other words, $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ and X are normed spaces equipped, respectively, with the norm $\|\cdot\|_{\mathcal{B}}$ and

$$\|x\|_X = \|x_{t_0}\|_{\mathcal{B}} + \sup_{u \in [t_0, t_0+\sigma]} \|x(u)\|. \tag{4.6}$$

Additionally, $\mathcal{B}$ satisfies the following axioms:

(E1) $\mathcal{B}$ is complete.

(E2) If $t_0 \in \mathbb{R}$, $\sigma > 0$, $y\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ is regulated on $[t_0, t_0 + \sigma]$ and $y_{t_0} \in \mathcal{B}$, then there are locally bounded functions $k_1, k_2, k_3\colon [0, \infty) \to (0, \infty)$, all independent of y, t_0 and σ, such that the following conditions hold for every $t \in [t_0, t_0 + \sigma]$:

(a) $y_t \in \mathcal{B}$.

(b) $\|y(t)\| \leqslant k_1(t - t_0)\, \|y_t\|_{\mathcal{B}}$.

(c) $\|y_t\|_{\mathcal{B}} \leqslant k_2(t - t_0)\|y_{t_0}\|_{\mathcal{B}} + k_3(t - t_0) \sup\limits_{u \in [t_0, t]} \|y(u)\|$.

(E3) For $t \geqslant 0$, let $S(t)\colon \mathcal{B} \to \mathcal{B}$ be the operator defined as follows:

$$(S(t)\varphi)(\theta) = \begin{cases} \varphi(0), & \theta = 0, \\ \varphi(0^-), & -t \leqslant \theta < 0, \\ \varphi(t + \theta), & \theta < -t. \end{cases}$$

Then, there is a continuous function $k\colon [0, \infty) \to (0, \infty)$ such that $k(0) = 0$ and

$$\|S(t)\varphi\|_{\mathcal{B}} \leqslant (1 + k(t))\, \|\varphi\|_{\mathcal{B}}, \quad \text{for all } \varphi \in \mathcal{B}.$$

For functions $x\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$, $\rho\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}$, $f\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}^n$, $\sigma > 0$, $t_0 \in \mathbb{R}$, and a nondecreasing function $g\colon [t_0, t_0 + \sigma] \to \mathbb{R}$, we also recall assumptions (F1)–(F6) from Chapter 3:

(F1) For all $x \in \mathcal{B}$, the integral $\int_{t_0}^{t_0+\sigma} f(s, x)\mathrm{d}g(s)$ exists in the sense of Kurzweil–Henstock–Stieltjes.

(F2) There exists a Kurzweil–Henstock–Stieltjes integrable function $M\colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} f(s, x)\mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} M(s)\mathrm{d}g(s)$$

whenever $x \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0 + \sigma]$.

(F3) There exists a Kurzweil–Henstock–Stieltjes integrable function $L\colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} (f(s, x) - f(s, y))\, \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L(s)\, \|x - y\|_{\mathcal{B}}\, \mathrm{d}g(s)$$

whenever $x, y \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0 + \sigma]$.

(F4) There exists a Kurzweil–Henstock–Stieltjes integrable function $L_2\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that
$$\left\| \int_{u_1}^{u_2} (f(s, x_u) - f(s, x_v))\, \mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_2(s)\, |u - v|\, \mathrm{d}g(s)$$
for every $x \in X$ and $u_1, u_2, u, v \in [t_0, t_0 + \sigma]$.

(F5) For every $x \in X$ and $u_1, u_2, u, v \in [t_0, t_0+\sigma]$, the function $t \mapsto \rho(t, x_t)$, $t \in [t_0, t_0+\sigma]$, is nondecreasing, satisfies $\rho(t, x_t) \leqslant t$ and $x_{\rho(t_0, x_{t_0})} \in \mathcal{B}$.

(F6) There exists a Kurzweil–Henstock–Stieltjes integrable function $L_3\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that
$$\int_{u_1}^{u_2} |\rho(s, x) - \rho(s, y)|\, \mathrm{d}g(s) \leqslant \int_{u_1}^{u_2} L_3(s)\, \|x - y\|_{\mathcal{B}}\, \mathrm{d}g(s)$$
for all $u_1, u_2 \in [t_0, t_0 + \sigma]$ and all $x, y \in \mathcal{B}$.

In sequel, for any subset $\mathcal{O}$ of X, let us define the function $F\colon [t_0, t_0 + \sigma] \times \mathcal{O} \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ by
$$F(t, y)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f(s, y_{\rho(s, y_s)})\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f(s, y_{\rho(s, y_s)})\mathrm{d}g(s), & \xi \in [t, t_0 + \sigma]. \end{cases} \tag{4.7}$$
The next result describes a relation between the regularity of measure FDEs with state–dependent delays and generalized ODEs. The proof is analogous to the one found in [27], but we will repeat it here to show the particularities of the state–dependent delays.

Lemma 4.1.1. *Let $\mathcal{O} \subset X$ and assume that (E1)–(E3) are satisfied and $f\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}^n$, $\rho\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}$ and $g\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ satisfy the conditions (F1)–(F6). If $F\colon [t_0, t_0 + \sigma] \times \mathcal{O} \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ is a function defined in (4.7), then F belongs to the class $\mathcal{F}([t_0, t_0 + \sigma] \times \mathcal{O}, h)$, where*
$$h(t) = \int_{t_0}^{t} (M(s) + (L(s) + L_2(s)L_3(s))K_\sigma)\mathrm{d}g(s) \tag{4.8}$$
and $K_\sigma = \max\left(\sup_{\xi \in [0,\sigma]} k_2(\xi), \sup_{\xi \in [0,\sigma]} k_3(\xi) \right)$.

Proof. Since the integrand of the expression (4.8) is a positive function and g is a nondecreasing function, Corollary 1.3.9 implies that h is a nondecreasing function. Let $y, z \in \mathcal{O}$

and $s_1, s_2 \in \mathbb{R}$ be such that $t_0 \leqslant s_1 < s_2 \leqslant t_0 + \sigma$. Since

$$(F(s_2, y) - F(s_1, y))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_1], \\ \displaystyle\int_{s_1}^{\xi} f(s, y_{\rho(s,y_s)})\mathrm{d}g(s), & \xi \in [s_1, s_2], \\ \displaystyle\int_{s_1}^{s_2} f(s, y_{\rho(s,y_s)})\mathrm{d}g(s), & \xi \in [s_2, t_0 + \sigma], \end{cases}$$

by definition, we have that condition (F1), Theorem 1.3.8 and Corollary 1.3.9 imply that

$$\begin{aligned} \|F(s_2, y) - F(s_1, y)\|_X &= \sup_{\xi \in [t_0, t_0 + \sigma]} \|(F(s_2, y) - F(s_1, y))(\xi)\| + \left\|(F(s_2, y) - F(s_1, y))_{t_0}\right\|_{\mathcal{B}} \\ &= \sup_{\xi \in [s_1, s_2]} \left\| \int_{s_1}^{\xi} f(s, y_{\rho(s,y_s)})\mathrm{d}g(s) \right\| \\ &\leqslant \int_{s_1}^{s_2} M(s)\mathrm{d}g(s) \leqslant h(s_2) - h(s_1). \end{aligned}$$

By conditions (F3), (F4), we have

$$\begin{aligned} &\left\| \int_{s_1}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, z_{\rho(s,z_s)})\mathrm{d}g(s) \right\| \\ &\quad \leqslant \left\| \int_{s_1}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, z_{\rho(s,y_s)})\mathrm{d}g(s) \right\| + \left\| \int_{s_1}^{\xi} f(s, z_{\rho(s,y_s)}) - f(s, z_{\rho(s,z_s)})\mathrm{d}g(s) \right\| \\ &\quad \leqslant \int_{s_1}^{\xi} L(s) \left\| y_{\rho(s,y_s)} - z_{\rho(s,y_s)} \right\|_{\mathcal{B}} \mathrm{d}g(s) + \int_{s_1}^{\xi} L_2(s) \left| \rho(s, y_s) - \rho(s, z_s) \right| \mathrm{d}g(s) \end{aligned} \tag{4.9}$$

By axiom (E2), by condition (F6) and by inequality (4.9), we get

$$\begin{aligned} &\left\| \int_{s_1}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, z_{\rho(s,z_s)})\mathrm{d}g(s) \right\| \\ &\leqslant \int_{s_1}^{\xi} L(s) \Big(k_2(\rho(s, y_s) - t_0) \|(y - z)_{t_0}\|_{\mathcal{B}} + k_3(\rho(s, y_s) - t_0) \sup_{u \in [t_0, \rho(s,y_s)]} \|(y - z)(u)\| \Big) \mathrm{d}g(s) \\ &\quad + \int_{s_1}^{\xi} L_2(s) L_3(s) \left\| y_s - z_s \right\|_{\mathcal{B}} \mathrm{d}g(s) \\ &\leqslant \int_{s_1}^{\xi} L(s) K_\sigma \|y - z\|_X \mathrm{d}g(s) + \int_{s_1}^{\xi} L_2(s) L_3(s) K_\sigma \|y - z\|_X \mathrm{d}g(s). \end{aligned} \tag{4.10}$$

By inequality (4.10) and Corollary 1.3.9, we conclude that

$$\begin{aligned} \|F(s_2, y) - F(s_1, y) - F(s_2, z) + F(s_1, z)\|_X &= \sup_{\xi \in [s_1, s_2]} \left\| \int_{s_1}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, z_{\rho(s,z_s)})\mathrm{d}g(s) \right\| \\ &\leqslant \sup_{\xi \in [s_1, s_2]} \int_{s_1}^{\xi} \left(L(s) + L_2(s) L_3(s) \right) K_\sigma \mathrm{d}g(s) \, \|y - z\|_X \\ &\leqslant \int_{s_1}^{s_2} \left(M(s) + \left(L(s) + L_2(s) L_3(s) \right) K_\sigma \right) \mathrm{d}g(s) \, \|y - z\|_X \\ &= \left(h(s_2) - h(s_1) \right) \|y - z\|_X, \end{aligned}$$

which completes the proof. □

The next lemma establishes an important property of the solutions of the generalized ODEs. Since its proof follows analogously to the one found in [51], we will omit it here.

Lemma 4.1.2. *Assume that $\mathcal{O} \subset X$, (E1)–(E3) are satisfied, $\phi \in \mathcal{B}$, and that $F\colon \mathcal{O} \times [t_0, t_0+\sigma] \to X$ is the function given by (4.7). Assume further that $f\colon [t_0, t_0+\sigma]\times\mathcal{B} \to \mathbb{R}^n$, $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$ and $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ satisfy the conditions (F1)–(F6). If $x\colon [t_0, t_0+\sigma] \to \mathcal{O}$ is a solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(t, x). \tag{4.11}$$

on the interval $[t_0, t_0+\sigma]$ and $x(t_0)$ is a function which is constant on $[t_0, t_0+\sigma]$, then

$$x(v)(\xi) = x(v)(v), \quad t_0 \leqslant v \leqslant \xi \leqslant t_0+\sigma,$$
$$x(v)(\xi) = x(\xi)(\xi), \quad t_0 \leqslant \xi \leqslant v \leqslant t_0+\sigma.$$

Next, we will define an important property of subsets of X that will allow us to obtain a well–defined correspondence between solutions of (4.5) and solutions of (4.11). See [21] for instance.

Definition 4.1.3. *Let $I \subset \mathbb{R}$ be an interval, $t_0 \in I$ and $\mathcal{O}$ be a set whose elements are functions $f\colon I \to \mathbb{R}^n$. We say that $\mathcal{O}$ has the* ***prolongation property*** *for $t \geqslant t_0$ if for every $y \in \mathcal{O}$ and every $t \in I \cap [t_0, \infty)$, the function $\overline{y}\colon I \to \mathbb{R}^n$ given by*

$$\overline{y}(s) = \begin{cases} y(s), & s \in (-\infty, t] \cap I, \\ y(t), & s \in [t, \infty) \cap I, \end{cases}$$

is an element of $\mathcal{O}$.

The next two theorems establish a relation between solutions of measure FDEs with state–dependent delay and generalized ODEs.

Theorem 4.1.4. *Let $\mathcal{O}$ be a subset of X having the prolongation property for $t \geqslant t_0$. Assume that (E1)–(E3) are satisfied, $\phi \in \mathcal{B}$, and that $F\colon [t_0, t_0+\sigma] \times \mathcal{O} \to X$ is the function given by (4.7). Assume further that $f\colon [t_0, t_0+\sigma]\times\mathcal{B} \to \mathbb{R}^n$, $\rho\colon [t_0, t_0+\sigma]\times\mathcal{B} \to \mathbb{R}$ and $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ satisfy the conditions (F1)–(F6). If $y \in \mathcal{O}$ is a solution of the* *equation* (4.5), *then the function* $x\colon [t_0, t_0+\sigma] \to \mathcal{O}$ *given by*

$$x(t)(\xi) = \begin{cases} y(\xi), & \xi \in (-\infty, t], \\ y(t), & \xi \in [t, t_0+\sigma], \end{cases} \tag{4.12}$$

is a solution of the generalized ODE (4.11) *on the interval* $[t_0, t_0+\sigma]$.

Proof. Let $\varepsilon > 0$ be fixed and consider the function $q\colon [t_0, t_0+\sigma] \to [0, \infty)$ defined by

$$q(t) = \int_{t_0}^{t} M(s)\mathrm{d}g(s).$$

Since q is nondecreasing (Corollary 1.3.9), there exists only a finite number of points $\{t_1, \ldots, t_m\} \subset [t_0, v]$ such that $\Delta^+ q(t_k) \geqslant \varepsilon$ for all $k \in \{1, \ldots, m\}$ ([44, Theorem 4.1.7]). Now, choose a gauge $\delta\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

(i) $\delta(\tau) < \min\limits_{2 \leqslant k \leqslant m} \left\{\dfrac{t_k - t_{k-1}}{2}\right\}, \tau \in [t_0, t_0+\sigma]$,

(ii) $\delta(\tau) < \min\limits_{1 \leqslant k \leqslant m} \{|\tau - t_k|\}, \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}$.

These conditions imply that, if a point–interval pair $(\tau, [c,d])$ satisfies $[c,d] \subset (\tau - \delta(\tau), \tau + \delta(\tau))$, then $[c,d]$ contains at most one of the points $t_1, \ldots, t_m$. Moreover, $\tau = t_k$ whenever $t_k \in [c,d]$. From (4.12) and Theorem 1.3.11, $y_{t_k} = x(t_k)_{t_k}$, $y_{\rho(t_k, y_{t_k})} = x(t_k)_{\rho(t_k, y_{t_k})}$, this implies the following equalities:

$$\lim_{t \to t_k^+} \int_{t_k}^{t} L_2(s) L_3(s) \|y_s - x(t_k)_s\|_{\mathcal{B}} \mathrm{d}g(s) = L_2(t_k) L_3(t_k) \|y_{t_k} - x(t_k)_{t_k}\|_{\mathcal{B}} \Delta^+ g(t_k) = 0$$

and

$$\lim_{t \to t_k^+} \int_{t_k}^{t} L(s) \|y_{\rho(s, y_s)} - x(t_k)_{\rho(s, y_s)}\|_{\mathcal{B}} \mathrm{d}g(s) = L(t_k) \|y_{\rho(t_k, y_{t_k})} - x(t_k)_{\rho(t_k, y_{t_k})}\|_{\mathcal{B}} \Delta^+ g(t_k) = 0$$

for all $k \in \{1, \ldots, m\}$. In consequence, we may choose a gauge δ in such a way that

$$\int_{t_k}^{t_k + \delta(t_k)} L_2(s) L_3(s) \|y_s - x(t_k)_s\|_{\mathcal{B}} \mathrm{d}g(s) \leqslant \frac{\varepsilon}{4m+2}, \quad k \in \{1, \ldots, m\},$$

and

$$\int_{t_k}^{t_k + \delta(t_k)} L(s) \|y_{\rho(s, y_s)} - x(t_k)_{\rho(s, y_s)}\|_{\mathcal{B}} \mathrm{d}g(s) \leqslant \frac{\varepsilon}{4m+2}, \quad k \in \{1, \ldots, m\}.$$

By (F2), we have

$$\|y(\tau + t) - y(\tau)\| = \left\| \int_{\tau}^{\tau + t} f(s, y_{\rho(s, y_s)}) \mathrm{d}g(s) \right\| \leqslant \int_{\tau}^{\tau + t} M(s) \mathrm{d}g(s) \leqslant q(\tau + t) - q(\tau),$$

which implies

$$\|y(\tau^+) - y(\tau)\| \leqslant \Delta^+ q(\tau) < \varepsilon, \quad \tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}.$$

Thus, we can select the gauge δ such that

$$\|y(\rho) - y(\tau)\| \leqslant \varepsilon \tag{4.13}$$

for all $\tau \in [t_0, t_0 + \sigma] \backslash \{t_1, \ldots, t_m\}$ and $\rho \in [\tau, \tau + \delta(\tau))$.

Let $\{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{l}$ be a δ–fine tagged division of $[t_0, v]$. By relations (4.7) and (4.12) and from the fact that y is a solution of the problem (4.5), we obtain

$$(x(s_i) - x(s_{i-1}))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \displaystyle\int_{s_{i-1}}^{\xi} f(s, y_{\rho(s, y_s)}) \mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \displaystyle\int_{s_{i-1}}^{s_i} f(s, y_{\rho(s, y_s)}) \mathrm{d}g(s), & \xi \in [s_i, v] \end{cases}$$

and

$$(F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i)))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \displaystyle\int_{s_{i-1}}^{\xi} f(s, x(\tau_i)_{\rho(s, x(\tau_i)_s)}) \mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \displaystyle\int_{s_{i-1}}^{s_i} f(s, x(\tau_i)_{\rho(s, x(\tau_i)_s)}) \mathrm{d}g(s), & \xi \in [s_i, v] \end{cases}$$

for all $i \in \{1, \ldots, l\}$. The combination of both expressions give us

$$\begin{aligned} (x(s_i) - &x(s_{i-1}) - F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i)))(\xi) \\ &= \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \displaystyle\int_{s_{i-1}}^{\xi} f(s, y_{\rho(s, y_s)}) - f(s, x(\tau_i)_{\rho(s, x(\tau_i)_s)}) \mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \displaystyle\int_{s_{i-1}}^{s_i} f(s, y_{\rho(s, y_s)}) - f(s, x(\tau_i)_{\rho(s, x(\tau_i)_s)}) \mathrm{d}g(s), & \xi \in [s_i, v]. \end{cases} \end{aligned}$$

Consequently,

$$\begin{aligned} &\|x(s_i) - x(s_{i-1}) - F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))\|_X \\ &\qquad = \sup_{\xi \in [s_{i-1}, s_i]} \left\| \int_{s_{i-1}}^{\xi} f(s, y_{\rho(s, y_s)}) - f(s, x(\tau_i)_{\rho(s, x(\tau_i)_s)}) \mathrm{d}g(s) \right\|. \end{aligned} \tag{4.14}$$

By (4.12), if $s \leqslant \tau_i$, then $x(\tau_i)_s = y_s$, $x(\tau_i)_{\rho(s,y_s)} = y_{\rho(s,y_s)}$ and

$$\begin{aligned}
&\int_{s_{i-1}}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, x(\tau_i)_{\rho(s,x(\tau_i)_s)})\mathrm{d}g(s) \\
&= \int_{s_{i-1}}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, x(\tau_i)_{\rho(s,y_s)})\mathrm{d}g(s) + \int_{s_{i-1}}^{\xi} f(s, x(\tau_i)_{\rho(s,y_s)}) - f(s, x(\tau_i)_{\rho(s,x(\tau_i)_s)})\mathrm{d}g(s) \\
&= \int_{\tau_i}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, x(\tau_i)_{\rho(s,y_s)})\mathrm{d}g(s) + \int_{\tau_i}^{\xi} f(s, x(\tau_i)_{\rho(s,y_s)}) - f(s, x(\tau_i)_{\rho(s,x(\tau_i)_s)})\mathrm{d}g(s).
\end{aligned}$$

The last equality together with conditions (F3), (F4) and (F6) imply

$$\begin{aligned}
&\left\| \int_{s_{i-1}}^{\xi} f(s, y_{\rho(s,y_s)}) - f(s, x(\tau_i)_{\rho(s,x(\tau_i)_s)})\mathrm{d}g(s) \right\| \\
&\leqslant \int_{\tau_i}^{\xi} L(s)\|y_{\rho(s,y_s)} - x(\tau_i)_{\rho(s,y_s)}\|_{\mathcal{B}}\mathrm{d}g(s) + \int_{\tau_i}^{\xi} L_2(s)|\rho(s, y_s) - \rho(s, x(\tau_i)_s)|\mathrm{d}g(s) \\
&\leqslant \int_{\tau_i}^{\xi} L(s)\|y_{\rho(s,y_s)} - x(\tau_i)_{\rho(s,y_s)}\|_{\mathcal{B}}\mathrm{d}g(s) + \int_{\tau_i}^{\xi} L_2(s)L_3(s)\,\|y_s - x(\tau_i)_s\|_{\mathcal{B}}\,\mathrm{d}g(s). \qquad (4.15)
\end{aligned}$$

Given a particular point–interval pair $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(a) The intersection between $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ contains a single point t_k. In this case, it follows from condition (ii) at the beginning of this proof that $t_k = \tau_i$.

(b) The intersection between $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ is empty.

If (a) happens, then from the construction of the gauge δ, we get

$$\int_{\tau_i}^{s_i} L_2(s)L_3(s)\|y_s - x(\tau_i)_s\|_{\mathcal{B}}\mathrm{d}g(s) \leqslant \frac{\varepsilon}{4m+2} \qquad (4.16)$$

and

$$\int_{\tau_i}^{s_i} L(s)\|y_{\rho(s,y_s)} - x(\tau_i)_{\rho(s,y_s)}\|_{\mathcal{B}}\mathrm{d}g(s) \leqslant \frac{\varepsilon}{4m+2}. \qquad (4.17)$$

From relations (4.14), (4.15) (4.16) and (4.17), it follows that

$$\|x(s_i) - x(s_{i-1}) - F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))\|_X \leqslant \frac{\varepsilon}{2m+1}.$$

Assume now case (b), and let $s \in [\tau_i, s_i]$. If $\rho(s, y_s) \in [\tau_i, s_i]$, then

$$\begin{aligned}
\|y_s - x(\tau_i)_s\|_{\mathcal{B}} &\leqslant k_2(s - t_0)\|y_{t_0} - x(\tau_i)_{t_0}\|_{\mathcal{B}} + k_3(s - t_0) \sup_{\xi\in[t_0,s]} \|(y - x(\tau_i))(\xi)\| \\
&\leqslant K_\sigma \sup_{\xi\in[\tau_i,s]} \|y(\xi) - y(\tau_i)\| \\
&\leqslant K_\sigma \varepsilon \qquad (4.18)
\end{aligned}$$

and

$$
\begin{aligned}
\|y_{\rho(s,y_s)} - & x(\tau_i)_{\rho(s,y_s)}\|_{\mathcal{B}} \\
&\leqslant k_2(\rho(s,y_s)-t_0)\|y_{t_0}-x(\tau_i)_{t_0}\|_{\mathcal{B}} + k_3(\rho(s,y_s)-t_0) \sup_{\xi\in[t_0,\rho(s,y_s)]} \|(y-x(\tau_i))(\xi)\| \\
&\leqslant K_\sigma \sup_{\xi\in[\tau_i,\rho(s,y_s)]} \|y(\xi)-y(\tau_i)\| \\
&\leqslant K_\sigma \varepsilon, \qquad (4.19)
\end{aligned}
$$

where (4.18) and (4.19) follow from (4.13). On the other hand, if $\rho(s,y_s) \leqslant \tau_i$, then $\|y_{\rho(s,y_s)} - x(\tau_i)_{\rho(s,y_s)}\|_{\mathcal{B}} = 0$ by (4.12). Thus,

$$
\int_{\tau_i}^{s_i} L_2(s)L_3(s)\|y_s - x(\tau_i)_s\|_{\mathcal{B}}\mathrm{d}g(s) \leqslant K_\sigma\varepsilon \int_{\tau_i}^{s_i} L_2(s)L_3(s)\mathrm{d}g(s) \qquad (4.20)
$$

by (4.18) and

$$
\int_{\tau_i}^{s_i} L(s)\|y_{\rho(s,y_s)} - x(\tau_i)_{\rho(s,y_s)}\|_{\mathcal{B}}\mathrm{d}g(s) \leqslant K_\sigma\varepsilon \int_{\tau_i}^{s_i} L(s)\mathrm{d}g(s) \qquad (4.21)
$$

by (4.19). Relations (4.14), (4.15) and the inequalities (4.20) and (4.21) imply that

$$
\|x(s_i) - x(s_{i-1}) - F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))\|_X \leqslant K_\sigma\varepsilon \int_{\tau_i}^{s_i} (L_2(s)L_3(s) + L(s))\mathrm{d}g(s).
$$

Combining cases (a) and (b) and using the fact that case (a) occurs at most $2m$ times, it follows that

$$
\begin{aligned}
\left\| x(v) - x(t_0) - \sum_{i=1}^{l} F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i)) \right\|_X & \\
&\leqslant \varepsilon \left(K_\sigma \int_{t_0}^{t_0+\sigma} (L_2(s)L_3(s) + L(s))\mathrm{d}g(s) + \frac{2m}{2m+1} \right) \\
&< \varepsilon \left(K_\sigma \int_{t_0}^{t_0+\sigma} (L_2(s)L_3(s) + L(s))\mathrm{d}g(s) + 1 \right).
\end{aligned}
$$

By definition of Kurzweil integral, $\displaystyle\int_{t_0}^{v} DF(x(\tau), t)$ exists and

$$
x(v) - x(t_0) = \int_{t_0}^{v} DF(x(\tau), t)
$$

for all $v \in [t_0, t_0 + \sigma]$, which completes the proof. □

Theorem 4.1.5. *Let $\mathcal{O}$ be a subset of X having the prolongation property for $t \geqslant t_0$. Assume that (E1)–(E3) are satisfied, $\phi \in \mathcal{B}$, and that $F\colon \mathcal{O} \times [t_0, t_0 + \sigma] \to X$ is the function given by (4.7). Assume further that $f\colon [t_0, t_0+\sigma]\times\mathcal{B} \to \mathbb{R}^n$, $\rho\colon [t_0, t_0+\sigma]\times\mathcal{B} \to \mathbb{R}$ and $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ satisfy the conditions (F1)–(F6). If $x\colon [t_0, t_0+\sigma] \to \mathcal{O}$ is a solution* *of the generalized ODE (4.11) on the interval* $[t_0, t_0+\sigma]$ *with the initial condition*

$$x(t_0)(\xi) = \begin{cases} \phi(\xi - t_0), & \xi \in (-\infty, t_0], \\ \phi(0), & \xi \in [t_0, t_0+\sigma], \end{cases} \tag{4.22}$$

then the function $y \in \mathcal{O}$ *defined by*

$$y(\xi) = \begin{cases} x(t_0)(\xi), & \xi \in (-\infty, t_0], \\ x(\xi)(\xi), & \xi \in [t_0, t_0+\sigma], \end{cases} \tag{4.23}$$

is a solution of the measure FDE with state–dependent delay (4.5).

Proof. The equality $y_{t_0} = \phi$ follows directly from the definition of y and $x(t_0)$. It remains to prove that

$$y(v) - y(t_0) = \int_{t_0}^{v} f(s, y_{\rho(s,y_s)})\mathrm{d}g(s)$$

for all $v \in [t_0, t_0+\sigma]$. By Lemma 4.1.2, we obtain

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(t_0) = x(v)(v) - x(t_0)(v) = \left(\int_{t_0}^{v} DF(t, x(\tau))\right)(v). \tag{4.24}$$

Let $\varepsilon > 0$ be fixed. By Lemma 4.1.1, since the conditions (F1)–(F6) are satisfied, F belongs to the class $\mathcal{F}([t_0, t_0+\sigma] \times \mathcal{O}, h)$, where h is a nondecreasing function given by (4.8). Now, we can argue as in the proof of Theorem 4.1.4 to get that there exists a finite quantity of points $t_1, \ldots, t_m$ in $[t_0, v]$ such that $\Delta^+ h(t_k) \geqslant \varepsilon$. Also, in the same way as before, we can find a gauge $\delta\colon [t_0, t_0+\sigma] \to (0, \infty)$ that satisfies the following conditions:

(i) $\delta(\tau) < \min\limits_{2 \leqslant k \leqslant m} \left\{\dfrac{t_k - t_{k-1}}{2}\right\}, \quad \tau \in [t_0, t_0+\sigma],$

(ii) $\delta(\tau) < \min\limits_{1 \leqslant k \leqslant m} \{|\tau - t_k|\}, \quad \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}$

(iii) $\displaystyle\int_{t_k}^{t_k+\delta(t_k)} L(s) \left\|y_{\rho(s,y_s)} - x(t_k)_{\rho(s,y_s)}\right\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{4m+2}, \quad k \in \{1, \ldots, m\}$

(iv) $\displaystyle\int_{t_k}^{t_k+\delta(t_k)} L_2(s)L_3(s) \left\|y_s - x(t_k)_s\right\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{4m+2}, \quad k \in \{1, \ldots, m\},$

(v) $\|h(u) - h(\tau)\| \leqslant \varepsilon, \quad \tau \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}, u \in [\tau, \tau+\delta(\tau)).$

Let $\{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{l}$ be a δ–fine tagged division of $[t_0, v]$ such that

$$\left\|\int_{t_0}^{v} DF(t, x(\tau)) - \sum_{i=1}^{l} F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))\right\|_X < \varepsilon. \tag{4.25}$$

By equality (4.24), we obtain

$$\begin{aligned}
&\left\|y(v)-y(t_0)-\int_{t_0}^{v}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| = \left\|\left(\int_{t_0}^{v}DF(t,x(\tau))\right)(v)-\int_{t_0}^{v}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&\leqslant \left\|\left(\int_{t_0}^{v}DF(t,x(\tau))\right)(v)-\sum_{i=1}^{l}\left(F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i))\right)(v)\right\| \\
&+\left\|\sum_{i=1}^{l}(F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i)))(v)-\int_{t_0}^{v}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\|. \qquad (4.26)
\end{aligned}$$

Axiom (E2) (a), (b) and inequalities (4.25) and (4.26) imply that

$$\begin{aligned}
&\left\|y(v)-y(t_0)-\int_{t_0}^{v}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&\leqslant k_1(v-t_0)\left\|\left(\int_{t_0}^{v}DF(t,x(\tau))\right)_v-\left(\sum_{i=1}^{l}F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i))\right)_v\right\|_{\mathcal{B}} \\
&\quad+\left\|\sum_{i=1}^{l}(F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i)))(v)-\int_{t_0}^{v}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&\leqslant C_\sigma K_\sigma\varepsilon+\sum_{i=1}^{l}\left\|(F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i)))(v)-\int_{s_{i-1}}^{s_i}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \qquad (4.27)
\end{aligned}$$

where $C_\sigma = \sup\limits_{\xi\in[0,\sigma]} k_1(\xi)$. Also, by the definition of F, we have

$$\left(F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i))\right)(v)=\int_{s_{i-1}}^{s_i}f(s,x(\tau_i)_{\rho(s,x(\tau_i)_s)})\mathrm{d}g(s),$$

which implies

$$\begin{aligned}
&\left\|\left(F(s_i,x(\tau_i))-F(s_{i-1},x(\tau_i))\right)(v)-\int_{s_{i-1}}^{s_i}f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&=\left\|\int_{s_{i-1}}^{s_i}f(s,x(\tau_i)_{\rho(s,x(\tau_i)_s)})-f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&\leqslant\left\|\int_{s_{i-1}}^{s_i}f(s,x(\tau_i)_{\rho(s,x(\tau_i)_s)})-f(s,x(\tau_i)_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&\quad+\left\|\int_{s_{i-1}}^{s_i}f(s,x(\tau_i)_{\rho(s,y_s)})-f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\
&\leqslant\int_{s_{i-1}}^{s_i}L_2(s)L_3(s)\left\|x(\tau_i)_s-y_s\right\|_{\mathcal{B}}\mathrm{d}g(s)+\int_{s_{i-1}}^{s_i}L(s)\left\|x(\tau_i)_{\rho(s,y_s)}-y_{\rho(s,y_s)}\right\|_{\mathcal{B}}\mathrm{d}g(s), \qquad (4.28)
\end{aligned}$$

where we employ the conditions (F3), (F4) and (F6) in (4.28). By Lemma 4.1.2, we get, for all $\theta\leqslant 0$, the following equalities:

(a) If $s\in[s_{i-1},\tau_i]$, then

$$x(\tau_i)_{\rho(s,y_s)}(\theta)=x(\tau_i)(\rho(s,y_s)+\theta)=x(\rho(s,y_s)+\theta)(\rho(s,y_s)+\theta)$$

$$= y(\rho(s, y_s) + \theta) = y_{\rho(s,y_s)}(\theta)$$

(b) If $s \in [\tau_i, s_i]$, then

$$\begin{aligned} x(s_i)_{\rho(s,y_s)}(\theta) &= x(s_i)(\rho(s, y_s) + \theta) = x(\rho(s, y_s) + \theta)(\rho(s, y_s) + \theta) \\ &= y(\rho(s, y_s) + \theta) = y_{\rho(s,y_s)}(\theta). \end{aligned}$$

In other words,

$$y_{\rho(s,y_s)} = \begin{cases} x(\tau_i)_{\rho(s,y_s)}, & s \in [s_{i-1}, \tau_i], \\ x(s_i)_{\rho(s,y_s)}, & s \in [\tau_i, s_i] \end{cases} \tag{4.29}$$

and analogously, we can show

$$y_s = \begin{cases} x(\tau_i)_s, & s \in [s_{i-1}, \tau_i] \\ x(s_i)_s, & s \in [\tau_i, s_i]. \end{cases} \tag{4.30}$$

Relations (4.28), (4.29) and (4.30) imply that

$$\begin{aligned} &\left\| \left(F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))\right)(v) - \int_{s_{i-1}}^{s_i} f(s, y_{\rho(s,y_s)}) \mathrm{d}g(s) \right\| \\ &\leqslant \int_{\tau_i}^{s_i} L_2(s) L_3(s) \left\| x(\tau_i)_s - y_s \right\|_{\mathcal{B}} \mathrm{d}g(s) + \int_{\tau_i}^{s_i} L(s) \left\| x(\tau_i)_{\rho(s,y_s)} - y_{\rho(s,y_s)} \right\|_{\mathcal{B}} \mathrm{d}g(s). \end{aligned} \tag{4.31}$$

We distinguish two cases:

(a) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ contains a single point $t_k = \tau_i$.

(b) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \ldots, t_m\}$ is empty.

If (a) happens, then from the definition of δ and (iv), we have

$$\int_{\tau_i}^{s_i} L_2(s) L_3(s) \left\| x(\tau_i)_s - y_s \right\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{4m + 2}$$

and by (iii), it follows that

$$\int_{\tau_i}^{s_i} L(s) \left\| x(\tau_i)_{\rho(s,y_s)} - y_{\rho(s,y_s)} \right\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{4m + 2}.$$

These two inequalities together with (4.31) imply

$$\left\| \left(F(s_i, x(\tau_i)) - F(s_{i-1}, x(\tau_i))\right)(v) - \int_{s_{i-1}}^{s_i} f(s, y_{\rho(s,y_s)}) \mathrm{d}g(s) \right\| < \frac{\varepsilon}{2m + 1}.$$

We now consider the case (b). It follows from the definition of F and from the fact that x is a solution of (4.5), the following equality that $x(s_i)(\xi) - x(\tau_i)(\xi) = 0$, $\xi \in (-\infty, t_0]$.

Using Lemma 1.3.16 and axiom (E2), for $s \in [\tau_i, s_i]$, we obtain the estimate

$$\begin{aligned}\|x(s_i)_s - x(\tau_i)_s\|_{\mathcal{B}} &\leqslant k_2(s-t_0)\|x(s_i)_{t_0} - x(\tau_i)_{t_0}\|_{\mathcal{B}} + k_3(s-t_0)\sup_{\xi\in[t_0,s]}\|(x(s_i)-x(\tau_i))(\xi)\| \\ &\leqslant K_\sigma\|x(s_i)-x(\tau_i)\|_X \\ &\leqslant K_\sigma(h(s_i)-h(\tau_i)) \\ &\leqslant K_\sigma\varepsilon, \end{aligned} \tag{4.32}$$

where (4.32) follows from (v). Also, the same way as before, we get

$$\begin{aligned}&\left\|x(s_i)_{\rho(s,y_s)} - x(\tau_i)_{\rho(s,y_s)}\right\|_{\mathcal{B}} \\ &\leqslant k_2(\rho(s,y_s)-t_0)\|x(s_i)_{t_0} - x(\tau_i)_{t_0}\|_{\mathcal{B}} + k_3(\rho(s,y_s)-t_0)\sup_{\xi\in[t_0,\rho(s,y_s)]}\|(x(s_i)-x(\tau_i))(\xi)\| \\ &\leqslant K_\sigma\|x(s_i)-x(\tau_i)\|_X \leqslant K_\sigma(h(s_i)-h(\tau_i)) \leqslant K_\sigma\varepsilon.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left\|\left(F(s_i,x(\tau_i)) - F(s_{i-1},x(\tau_i))\right)(v) - \int_{s_{i-1}}^{s_i} f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\ &\leqslant \int_{\tau_i}^{s_i} L_2(s)L_3(s)\left\|x(\tau_i)_s - x(s_i)_s\right\|_{\mathcal{B}}\mathrm{d}g(s) + \int_{\tau_i}^{s_i} L(s)\left\|x(\tau_i)_{\rho(s,y_s)} - x(s_i)_{\rho(s,y_s)}\right\|_{\mathcal{B}}\mathrm{d}g(s) \\ &\leqslant K_\sigma\varepsilon\left(\int_{\tau_i}^{s_i} L_2(s)L_3(s) + L(s)\mathrm{d}g(s)\right).\end{aligned}$$

Combining the cases (a) and (b), as well as by using the fact that the case (a) occurs at most $2m$ times, we obtain

$$\begin{aligned}&\sum_{i=1}^{l}\left\|F(s_i,x(\tau_i)) - F(s_{i-1},x(\tau_i)) - \int_{s_{i-1}}^{s_i} f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\ &\qquad\leqslant \varepsilon\left(K_\sigma\int_{t_0}^{t_0+\sigma} L_2(s)L_3(s) + L(s)\mathrm{d}g(s) + \frac{2m}{2m+1}\right)\end{aligned} \tag{4.33}$$

and replacing (4.33) in (4.27), we get

$$\begin{aligned}&\left\|y(v) - y(t_0) - \int_{t_0}^{v} f(s,y_{\rho(s,y_s)})\mathrm{d}g(s)\right\| \\ &\qquad< \varepsilon\left(C_\sigma K_\sigma + K_\sigma\int_{t_0}^{t_0+\sigma}(L_2(s)L_3(s) + L(s))\mathrm{d}g(s) + 1\right)\end{aligned}$$

which completes the proof. □

4.2 Functional dynamic equations on time scales

In this section, we will describe a correspondence between functional dynamic equations on time scales with state–dependent delays and measure FDEs with state–dependent delays. Recall from the Chapter 1 that for a given real number $t \leqslant \sup \mathbb{T}$, we have defined

$$t^* = \inf\{s \in \mathbb{T} \colon s \geqslant t\}.$$

We also define

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty), & \text{otherwise.} \end{cases}$$

Finally, recall the extension $f^* \colon \mathbb{T}^* \to \mathbb{R}^n$ of a given function $f \colon \mathbb{T} \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

The same way, we can define $f^* \colon \mathbb{T}^* \times \mathcal{B} \to \mathbb{R}^n$ as the extension of a given function $f \colon \mathbb{T} \times \mathcal{B} \to \mathbb{R}^n$ by

$$f^*(t, x) = f(t^*, x), \quad t \in \mathbb{T}^* \text{ and } x \in \mathcal{B}.$$

To have a satisfactory framework for functional dynamic equation with state–dependent delays on time scales, the function ρ will have domain $\mathbb{T} \times \mathcal{B}$ and takes values in $\mathbb{T}$. Besides, we need to extend both functions ρ and x. Therefore,

$$x^*_{\rho^*(t,x_t)} \text{ means } (x^*)_{\rho^*(t,x_t)},$$

and consequently,

$$x^*_{\rho^*(t,x_t)} = x^*_{\rho(t^*,x_{t^*})}.$$

From this, we can determine a correspondence between measure FDEs with state–dependent delays and functional dynamic equations with state–dependent delays on time scales, since $x^*_{\rho(t^*,x_{t^*})}$ contains the same information as $x_{\rho(t,x_t)}$. Notice that $x^*_{\rho(t^*,x_{t^*})}$ can be regarded as an extension of $x_{\rho(t,x_t)}$ because it is defined in the whole interval $(-\infty, 0]$.

The next result shows that it is possible to translate all results from measure FDEs with state–dependent delays to functional dynamic equations on time scales with state–dependent delays. Some ideas of its proof are inspired in [21].

Theorem 4.2.1. *Let $(-\infty, t_0 + \sigma]_{\mathbb{T}}$ be a time scale interval, $t_0 \in \mathbb{T}$, $f \colon [t_0, t_0 + \sigma]_{\mathbb{T}} \times$*

$\mathcal{B} \to \mathbb{R}^n$, $\phi \in \mathcal{B}$ and $\rho\colon \mathbb{T} \times \mathcal{B} \to \mathbb{T}$. Define $g(s) = s^*$ for every $s \in [t_0, t_0+\sigma]$. If $x\colon (-\infty, t_0+\sigma]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional dynamic equation on time scales with state–dependent delays

$$\begin{aligned} x^{\Delta}(t) &= f(t, x^*_{\rho(t,x^*_t)}), \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}}, \\ x(t) &= \phi(t), \qquad\qquad t \in (-\infty, t_0]_{\mathbb{T}}, \end{aligned} \tag{4.34}$$

then $x^*\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$ satisfies

$$\begin{aligned} x^*(t) &= x^*(t_0) + \int_{t_0}^{t} f(s, x^*_{\rho(s,x^*_s)})\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x^*_{t_0} &= \phi^*_{t_0}. \end{aligned} \tag{4.35}$$

Conversely, if $y\colon (-\infty, t_0+\sigma] \to \mathcal{O}$ is a solution of the measure functional differential equation with state–dependent delays

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(s^*, y_{\rho(s^*,y_{s^*})})\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ y_{t_0} &= \phi^*_{t_0}, \end{aligned}$$

then $y = x^*$, where $x\colon (-\infty, t_0+\sigma]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies (4.34).

Proof. If $x\colon (-\infty, t_0+\sigma]_{\mathbb{T}} \to \mathbb{R}^n$ satisfies (4.34), then

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x^*_{\rho(s,x^*_s)})\,\Delta s, \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}}.$$

By Theorem 1.4.14,

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} f^*(s, x^*_{\rho(s,x^*_s)})\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma].$$

It implies that

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} f(s^*, x^*_{\rho(s^*,x^*_{s^*})})\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma].$$

Since $f(s^*, x^*_{\rho(s^*,x^*_{s^*})}) = f(s, x^*_{\rho(s,x^*_s)})$ for every $s \in \mathbb{T}$, we apply Theorem 1.4.15 to achieve that

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} f(s, x^*_{\rho(s,x^*_s)})\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma].$$

Thus, for $y = x^*$, we conclude that

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y_{\rho(s^*,y_s)})\,\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma],$$

which is a solution of (4.35).

Reciprocally, assume that y satisfies (4.35). Since g is constant on every interval

$(\alpha,\beta]$, where $\beta \in \mathbb{T}$ and $\alpha = \sup\{\tau \in \mathbb{T} \colon \tau < \beta\}$, y inherits the same property and it follows that $y = x^*$ for some $x \colon (-\infty, t_0+\sigma]_{\mathbb{T}} \to \mathcal{O}$. Using all previous arguments, we conclude that x satisfies (4.34). □

4.3 Impulsive measure FDEs

In this section, we will show that impulsive measure FDEs with state–dependent delays with pre–assigned moments of impulses are a special case of measure FDEs with state–dependent delays. In other words, we wil show that it is possible to investigate impulsive measure FDEs with state–dependent delays by using these equations without impulses. To prove all results of this section, we use some ideas from [20].

Let us consider the following type of impulsive measure FDEs with state–dependent delays:

$$\begin{aligned} x(v) - x(u) &= \int_u^v f(s, x_{\rho(s,x_s)})\,\mathrm{d}g(s), \text{ whenever } u, v \in J_k \text{ for some } k \in \{0,\dots,m\}, \\ \Delta^+ x(t_k) &= I_k(x(t_k)), \quad k \in \{1,\dots,m\}, \qquad (4.36) \\ x_{t_0} &= \phi, \end{aligned}$$

where $t_0 \leqslant t_1 < \ldots < t_m < t_0+\sigma$ are the moments of impulses, $I_k \colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,m$ are the operators of impulses, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k \in \{1,\dots,m-1\}$, and $J_m = (t_m, t_0+\sigma]$. Here, we are assuming that the integral in the right–hand side of the first equality in (4.36) exists in the sense of Kurzweil–Henstock–Stieltjes and the function g is nondecreasing and left–continuous. By the properties of this type of integral, the value of the integral $\int_u^v f(s, x_{\rho(s,x_s)})\,\mathrm{d}g(s)$, where $u, v \in J_k$, does not change if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k. The same way as in [20], this fact allows us to suppose, without loss of generality, that g is such that $\Delta^+ g(t_k) = 0$ for every $k \in \{1,\dots,m\}$. From this property and using the fact that g is a left–continuous function, we conclude that g is continuous at $t_1,\dots,t_m$ and thus, function $t \mapsto \int_{t_0}^t f(s, x_{\rho(s,x_s)})\,\mathrm{d}g(s)$ is continuous at $t_1,\dots,t_m$. In other words, we can have the following formulation for our problem

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t f(s, x_{\rho(s,x_s)})\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma], \qquad (4.37) \\ x_{t_0} &= \phi. \end{aligned}$$

Notice that if $g(s) = s$, then equation (4.37) is the classical impulsive functional differential equation with state–dependent delays which was investigated by many authors (see [2, 5, 11, 14] and the references therein), showing consistency and relation between both equations.

The following lemma will be employed as an auxiliary tool to obtain, in the subsequent theorem, the correspondence mentioned in the beginning of this section.

Lemma 4.3.1 ([20, Lemma 2.4]). *Let $m \in \mathbb{N}$, $a \leqslant t_1 < t_2 < \cdots < t_m \leqslant b$. Consider a pair of functions $f, g\colon [a,b] \to \mathbb{R}$, where g is regulated, left–continuous on $[a,b]$ and continuous at $t_1, \ldots, t_m$. Let $\tilde{f}, \tilde{g}\colon [a,b] \to \mathbb{R}$ be such that $\tilde{f}(t) = f(t)$ for every $t \in [a,b] \backslash \{t_1, \ldots, t_m\}$ and $\tilde{g} - g$ is constant on each of the intervals $[a, t_1]$, $(t_1, t_2], \ldots, (t_{m-1}, t_m]$, $(t_m, b]$. Then, the integral $\int_a^b \tilde{f}\,\mathrm{d}\tilde{g}$ exists if and only if the integral $\int_a^b f\,\mathrm{d}g$ exists. In this case, we have*

$$\int_a^b \tilde{f}\,\mathrm{d}\tilde{g} = \int_a^b f\,\mathrm{d}g + \sum_{\substack{k \in \{1,\ldots,m\},\\ t_k < b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

Theorem 4.3.2. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < t_0 + \sigma$, $I_1, \ldots, I_m\colon \mathbb{R}^n \to \mathbb{R}^n$, $f\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$. Assume that $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a regulated and left–continuous function which is continuous at $t_1, \ldots, t_m$. For every $y \in \mathcal{B}$, define*

$$\tilde{f}(t,y) = \begin{cases} f(t,y), & t \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1, \ldots, m\}. \end{cases}$$

Moreover, let $c_1, \ldots, c_m \in \mathbb{R}$ be constants such that the function $\tilde{g}\colon [t_0, t_0+\sigma] \to \mathbb{R}$ given by

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + c_k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1, \ldots, m-1\}, \\ g(t) + c_m, & t \in (t_m, t_0+\sigma] \end{cases}$$

satisfies $\Delta^+\tilde{g}(t_k) = 1$ for every $k \in \{1, \ldots, m\}$. Also, suppose that $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$ satisfies $\rho(t_k, x_{t_k}) = t_k$, for each $k \in \{1, \ldots, m\}$. Then, $x \in G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is a solution of

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t f(s, x_{\rho(s,x_s)})\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\ldots,m\},\\ t_k < t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma], \quad (4.38) \\ x_{t_0} &= \phi \end{aligned}$$

if and only if

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} \tilde{f}(s, x_{\rho(s,x_s)})\, \mathrm{d}\tilde{g}(s), \quad t \in [t_0, t_0+\sigma], \qquad (4.39) \\ x_{t_0} &= \phi. \end{aligned}$$

Proof. According to Lemma 4.3.1 and by hypotheses, we have

$$\begin{aligned} \int_{t_0}^{t} \tilde{f}(s, x_{\rho(s,x_s)})\, \mathrm{d}\tilde{g}(s) &= \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\, \mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} \tilde{f}(t_k, x_{\rho(t_k,x_{t_k})})\Delta^+\tilde{g}(t_k) \\ &= \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\, \mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} \tilde{f}(t_k, x_{t_k})\Delta^+\tilde{g}(t_k) \\ &= \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\, \mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \end{aligned}$$

proving the desired result. □

The proof of the next result follows similarly to [1] and [20] and thus, we omit it here.

Theorem 4.3.3. *Let $m \in \mathbb{N}$, $t_0 \leqslant t_1 < \cdots < t_m < t_0+\sigma$, $I_1, \dots, I_m \colon \mathbb{R}^n \to \mathbb{R}^n$, $f\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$. Assume that $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing and left–continuous function. Let $f\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$ be an arbitrary function. Define, for every $y \in \mathcal{B}$,*

$$\tilde{f}(t,y) = \begin{cases} f(t,y), & t \in [t_0, t_0+\sigma]\backslash\{t_1,\dots,t_m\}, \\ I_k(y(0)), & t = t_k \text{ for some } k \in \{1,\dots,m\}. \end{cases}$$

Moreover, let $c_1, \dots, c_m \in \mathbb{R}$ be constants such that the function $\tilde{g}\colon [t_0, t_0+\sigma] \to \mathbb{R}$ given by

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + c_k, & t \in (t_k, t_{k+1}] \text{ for some } k \in \{1,\dots,m-1\}, \\ g(t) + c_m, & t \in (t_m, t_0+\sigma] \end{cases}$$

satisfies $\Delta^+\tilde{g}(t_k) = 1$ for every $k \in \{1,\dots,m\}$. Also, suppose that $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$ satisfies $\rho(t_k, x_{t_k}) = t_k$, for each $k \in \{1,\dots,m\}$. Then, the following statements hold:

(i) The function $\tilde{g}$ is nondecreasing and left–continuous.

(ii) If the Kurzweil–Henstock–Stieltjes integral $\int_{u_1}^{u_2} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s)$ exists for every $x \in$

X and every $u_1, u_2 \in [t_0, t_0 + \sigma]$, then the Kurzweil–Henstock–Stieltjes integral $\int_{u_1}^{u_2} \tilde{f}(s, x_{\rho(s,x_s)})\mathrm{d}\tilde{g}(s)$ also exists.

(iii) If there exists a Kurzweil–Henstock–Stieltjes integrable function $M_1 \colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} f(s,x)\mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} M_1(s)\mathrm{d}g(s)$$

for all $x \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0 + \sigma]$, then there exists a Kurzweil–Henstock–Stieltjes integrable function $M \colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} \tilde{f}(s,x)\mathrm{d}\tilde{g}(s) \right\| \leqslant \int_{u_1}^{u_2} M(s)\mathrm{d}\tilde{g}(s)$$

for all $x \in \mathcal{B}$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$.

(iv) If there exists a Kurzweil–Henstock–Stieltjes integrable function $L_1 \colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} (f(s,x) - f(s,y))\mathrm{d}g(s) \right\| \leqslant \int_{u_1}^{u_2} L_1(s)\|x - y\|_{\mathcal{B}}\mathrm{d}g(s)$$

for all $x, y \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0 + \sigma]$, then there exists a Kurzweil–Henstock–Stieltjes integrable function $L \colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{u_1}^{u_2} (\tilde{f}(s,x) - \tilde{f}(s,y))\mathrm{d}\tilde{g}(s) \right\| \leqslant \int_{u_1}^{u_2} L(s)\|x - y\|_{\mathcal{B}}\mathrm{d}\tilde{g}(s)$$

for all $x, y \in \mathcal{B}$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$.

(v) If there exists a Kurzweil–Henstock–Stieltjes integrable function $\tilde{L}_1 \colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{u_1}^{u_2} |\rho(s,x) - \rho(s,y)|\mathrm{d}g(s) \right| \leqslant \int_{u_1}^{u_2} \tilde{L}_1(s)\|x - y\|_{\mathcal{B}}\mathrm{d}g(s)$$

for all $x, y \in \mathcal{B}$ and $u_1, u_2 \in [t_0, t_0 + \sigma]$, then there exists a Kurzweil–Henstock–Stieltjes integrable function $\tilde{L} \colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{u_1}^{u_2} |\rho(s,x) - \rho(s,y)|\mathrm{d}\tilde{g}(s) \right| \leqslant \int_{u_1}^{u_2} \tilde{L}(s)\|x - y\|_{\mathcal{B}}\mathrm{d}\tilde{g}(s)$$

for all $x, y \in \mathcal{B}$ and all $u_1, u_2 \in [t_0, t_0 + \sigma]$.

Remark 4.3.4. *It is worth mentioning that if ρ is a function satisfying the condition (F5), then if we change the definition of ρ at t_k for each $k = 1, \ldots, m$ in order to satisfy $\rho(t_k, x_{t_k}) = t_k$, we obtain that ρ keeps satisfying the condition $(F5)$. This fact ensures that the last part of the previous result could be extended for a $\tilde{\rho}$ which can be changed at*

(t_k, x_{t_k}) *directly.*

4.4 Local existence and uniqueness

Here, we illustrate how the correspondence presented in Section 4.1 and a known fact for generalized ODEs can lead to a local existence and uniqueness of solutions for measure FDEs with state–dependent delays.

The following existence–uniqueness theorem for generalized ODEs can be found in [22, Theorem 2.16].

Theorem 4.4.1. *Assume that X is a Banach space, $\mathcal{O} \subset X$ is an open set and $F\colon [t_0, t_0+\sigma] \times \mathcal{O} \to X$ belongs to the class $\mathcal{F}([t_0, t_0+\sigma] \times \mathcal{O}, h)$, where $h\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a left–continuous nondecreasing function. If $x_0 \in \mathcal{O}$ is such that $x_0 + F(t_0^+, x_0) - F(t_0, x_0) \in \mathcal{O}$, then there exists a $\delta > 0$ and a function $x\colon [t_0, t_0+\delta] \to X$ which is the unique solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DF(t, x), \quad x(t_0) = x_0.$$

In what follows, we provide an existence and uniqueness theorem for measure functional differential equations with state–dependent delays.

Theorem 4.4.2. *Let X be the Banach space given by (4.6), $\mathcal{B}$ be the phase space that satisfies the axioms (E1)–(E3), $\phi \in \mathcal{B}$ and $\mathcal{O} \subset X$ be an open subset having the prolongation property for $t \geqslant t_0$. Assume that $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is left–continuous nondecreasing function, $f\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$ and $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$ satisfy the conditions (F1)–(F6) and $z\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$ is the function*

$$z(t) = \begin{cases} \phi(t - t_0), & t \in (-\infty, t_0], \\ \phi(0) + f(t_0, \varphi)\Delta^+ g(t_0), & t \in (t_0, t_0+\sigma], \end{cases}$$

in $\mathcal{O}$, where φ is defined by $\varphi(\theta) = \phi(\theta + \rho(t_0, \phi) - t_0)$, $\theta \in (-\infty, 0]$. Then, there exist $\beta > 0$ and a function $y\colon (-\infty, t_0+\beta] \to \mathbb{R}^n$ which is the unique solution of the initial value problem

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f(s, x_{\rho(s,x_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ x_{t_0} &= \phi. \end{aligned} \tag{4.40}$$

on $(-\infty, t_0+\beta]$.

Proof. Lemma 4.1.1 shows that $F\colon [t_0, t_0+\sigma]\times\mathcal{O}\to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ defined by (4.7) belongs to $\mathcal{F}([t_0, t_0+\sigma]\times\mathcal{O}, h)$, where h is given by (4.8). Let $x_0 = x(t_0)$ defined by (4.22). We will prove that $x_0 + F(t_0^+, x_0) - F(t_0, x_0) \in \mathcal{O}$. Firstly, it is straightforward that $F(t_0, x_0) = 0$. Secondly, the limit $F(t_0^+, x_0)$, taken with respect to the supremum norm, exists since F is a regulated function with respect to the first variable. Lastly, by definition of F and by Theorem 1.3.11,

$$F(t_0^+, x_0)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ f(t_0, \varphi)\Delta^+ g(t_0), & \xi \in (t_0, t_0+\sigma]. \end{cases}$$

Therefore, by hypotheses, it follows that $x_0 + F(t_0^+, x_0) - F(t_0, x_0) \in \mathcal{O}$. Consequently, all hypotheses of Theorem 4.4.1 are satisfied, which implies the existence of a number $\beta > 0$ and a unique solution $x\colon [t_0, t_0+\beta]\to X$ of the generalized ODE

$$\frac{dx}{d\tau} = DF(t, x), \quad x(t_0) = x_0.$$

If we define the function $y\colon (-\infty, t_0+\beta]\to\mathbb{R}^n$ by

$$y(\xi) = \begin{cases} x(t_0)(\xi), & \xi \in (-\infty, t_0], \\ x(\xi)(\xi), & \xi \in [t_0, t_0+\beta]. \end{cases}$$

Theorem 4.1.5 guarantees that y is the unique solution of initial value problem (4.40) on $(-\infty, t_0+\beta]$. $\square$

Remark 4.4.3. *It is also possible to prove local existence and uniqueness of solutions for impulsive measure FDEs with state–dependent delays and functional dynamic equations with state–dependent delays, by means of the correspondences previously presented, but we omit these results here, since they follow directly by the application of the respectives correspondences.*

4.5 Continuous Dependence on Parameters

In this section, our goal is to prove results on continuous dependence on parameters for measure FDEs with state–dependent delays, via, once more, the correspondence presented in Section 4.1 and another known fact for generalized ODEs. We begin presenting a continuous dependence on parameters for generalized ODE which can be found in [24, Theorem 2.4] for the case $Y = \mathbb{R}^n$. Nonetheless, a version for an arbitrary Banach space follows similarly.

Theorem 4.5.1. *Let Y be a Banach space, $\mathcal{O} \subset Y$ be an open subset and $h_k\colon [a,b] \to \mathbb{R}$, $k \in \mathbb{N}$, be a sequence of nondecreasing left–continuous functions such that $h_k(b)-h_k(a) \leqslant c$, for some $c > 0$ and all $k \in \mathbb{N}_0$. Assume that for every $k \in \mathbb{N}_0$, $F_k\colon [a,b] \times \mathcal{O} \to Y$ belongs to the class $\mathcal{F}([a,b] \times \mathcal{O}, h_k)$ and that*

$$\lim_{k\to\infty} F_k(t,x) = F_0(t,x), \quad x \in \mathcal{O}, t \in [a,b],$$

$$\lim_{k\to\infty} F_k(t^+,x) = F_0(t^+,x), \quad x \in \mathcal{O}, t \in [a,b).$$

For every $k \in \mathbb{N}$, let $x_k\colon [a,b] \to \mathcal{O}$ be a solution of generalized ODE

$$\frac{dx}{d\tau} = DF_k(t,x), \quad t \in [a,b].$$

If there exists a function $x_0\colon [a,b] \to \mathcal{O}$ such that $\lim_{k\to\infty} x_k(t) = x_0(t)$ uniformly for $t \in [a,b]$, then x_0 is a solution of

$$\frac{dx}{d\tau} = DF_0(t,x), \quad t \in [a,b].$$

Next, we present a continuous dependence on parameters for measure FDEs with state–dependent delays, which is obtained by means of the correspondence between generalized ODEs.

Theorem 4.5.2. *Let X be the Banach space defined by (3.3), $\mathcal{B}$ be a phase space satisfying the axioms $(E1)$–$(E3)$ and $\mathcal{O}$ be an open subset of X having the prolongation property for $t \geqslant t_0$. Suppose that $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a left–continuous nondecreasing function, $f_k\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, and $\rho\colon [t_0, t_0+\sigma] \times \mathcal{B} \to \mathbb{R}$, satisfy the conditions (F1)–(F6). Assume that for every $y \in X$,*

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k(s, y_{\rho(s,y_s)})\mathrm{d}g(s) = \int_{t_0}^{t} f_0(s, y_{\rho(s,y_s)})\mathrm{d}g(s)$$

uniformly with respect to $t \in [t_0, t_0+\sigma]$. For every $k \in \mathbb{N}$, $z \in \mathcal{O}$ and $t \in [t_0, t_0+\sigma]$, assume that $F_k\colon [t_0, t_0+\sigma] \times \mathcal{O} \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is the function defined by

$$F_k(t,z)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f_k(s, z_{\rho(s,z_s)})\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f_k(s, z_{\rho(s,z_s)})\mathrm{d}g(s), & \xi \in [t, t_0+\sigma] \end{cases}$$

and is an element of X. Let $\phi_k \in \mathcal{B}$, $k \in \mathbb{N}$, be a sequence of functions such that $\lim_{k\to\infty} \phi_k = \phi_0$ *uniformly on* $(-\infty, 0]$. *Let $y_k \in \mathcal{O}$, $k \in \mathbb{N}$, be the solution of*

$$\begin{aligned} y_k(t) &= y_k(t_0) + \int_{t_0}^{t} f_k(s, (y_k)_{\rho(s,(y_k)_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_k)_{t_0} &= \phi_k. \end{aligned}$$

If there exists a function $y_0 \in X$ such that $\lim_{k\to\infty} y_k = y_0$ pointwisely on $(-\infty, t_0 + \sigma]$, then y_0 is a solution of

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f_0(s, (y_0)_{\rho(s,(y_0)_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned}$$

Proof. By hypotheses, for every $x \in \mathcal{O}$, we have

$$\lim_{k\to\infty} F_k(t, x) = F_0(t, x)$$

uniformly on $t \in [t_0, t_0 + \sigma]$, where F_0 is given by

$$F_0(t, z)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f_0(s, z_{\rho(s,z_s)})\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f_0(s, z_{\rho(s,z_s)})\mathrm{d}g(s), & \xi \in [t, t_0 + \sigma]. \end{cases}$$

By Moore–Osgood Theorem (see [6]), we obtain

$$\lim_{k\to\infty} F_k(t^+, x) = F_0(t^+, x)$$

for all $x \in \mathcal{O}$ and $t \in [t_0, t_0 + \sigma]$. Also, F_0 takes value in X because X is complete. By conditions (F1)–(F6) and following the same steps as the Lemma 4.1.1, F_k belongs to the class $\mathcal{F}([t_0, t_0 + \sigma] \times \mathcal{O}, h)$ for all $k \in \mathbb{N}$, where the function h is given by (4.8). Since $\lim_{k\to\infty} F_k(t, x) = F_0(t, x)$ uniformly, it is not difficult to see that $F_0 \in \mathcal{F}([t_0, t_0+\sigma]\times\mathcal{O}, h)$ as well.

Now, for every $k \in \mathbb{N}_0$ and $t \in [t_0, t_0 + \sigma]$, we define

$$x_k(t)(\xi) = \begin{cases} y_k(\xi), & \xi \in (-\infty, t], \\ y_k(t), & \xi \in [t, t_0 + \sigma]. \end{cases}$$

By Theorem 4.1.4, x_k is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF_k(t, x), \quad t \in [t_0, t_0 + \sigma].$$

If $k \in \mathbb{N}$ and $t_0 \leqslant t_1 \leqslant t_2 \leqslant t_0 + \sigma$, then

$$\begin{aligned} \|y_k(t_2) - y_k(t_1)\| &= \left\| \int_{t_1}^{t_2} f_k(s, (y_k)_{\rho(s,(y_k)_s)}, s) \mathrm{d}g(s) \right\| \leqslant \int_{t_1}^{t_2} M(s) \mathrm{d}g(s) \\ &\leqslant K(t_2) - K(t_1) = \eta \left(K(t_2) - K(t_1) \right), \end{aligned}$$

where $K(t) = t + \int_{t_0}^{t} M(s)\mathrm{d}g(s)$ is an increasing function and $\eta(t) = t$. Besides, $(y_k(t_0))_{k\in\mathbb{N}}$ is bounded. Therefore, by Theorem 1.1.5, $\left(y_k|_{[t_0,t_0+\sigma]}\right)_{k\in\mathbb{N}}$ contains a subsequence which is uniformly convergent on $[t_0, t_0+\sigma]$. Without loss of generality, we denote this subsequence again by $(y_k)_{k\in\mathbb{N}}$. Since $(y_k)_{t_0} = \phi_k$ for $\theta \in (-\infty, 0]$, we get that $(y_k)_{k\in\mathbb{N}}$ is in fact uniformly convergent on $(-\infty, t_0 + \sigma]$. By definition of x_k, $\lim_{k\to\infty} x_k(t) = x_0(t)$ uniformly with respect to $t \in [t_0, t_0 + \sigma]$. Theorem 4.5.1 yields that x_0 is a solution of

$$\frac{dx}{d\tau} = DF_0(t, x), \quad t \in [t_0, t_0 + \sigma].$$

Using Theorem 4.1.5, we conclude that y_0 is a solution of

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f(s, (y_0)_{\rho(s,(y_0)_s)}) \mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_0)_{t_0} &= \phi_0, \end{aligned}$$

obtaining the desired result. □

In the sequel, we have another type of result on continuous dependence on parameters for generalized ODEs on Banach spaces. It can be found in [10]. This theorem brings a very special result concerning the continuous dependence on parameters for these equations.

Theorem 4.5.3. *Let $C \subset \mathcal{O}$ be a closed set. Assume that, for each $k \in \mathbb{N}$, $F_k\colon [t_0, t_0 + \sigma] \times C \to X$ belongs to the class $\mathcal{F}([t_0, t_0 + \sigma] \times C, h)$, where $h\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ is a nondecreasing and left–continuous function, and $(F_k)_{k\in\mathbb{N}}$ converges pointwisely to F_0 for each $(t, x) \in [t_0, t_0 + \sigma] \times C$. Let $x_0\colon [t_0, t_0 + \sigma] \to X$ be the solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF_0(t, x) \tag{4.41}$$

on $[t_0, t_0 + \sigma]$ satisfying the following uniqueness property:

(U) If $z\colon [t_0, \gamma] \to X$, $[t_0, \gamma] \subset [t_0, t_0 + \sigma]$, is a solution (4.41) *such that $z(t_0) = x_0(t_0)$, then $z(t) = x_0(t)$ for every $t \in [t_0, \gamma]$.*

Assume further that there is a $\lambda > 0$ such that if $s \in [t_0, t_0 + \sigma]$ and $\|y - x_0(s)\| < \lambda$, then $(s, y) \in [t_0, t_0 + \sigma] \times C$, and let $(y_k)_{k\in\mathbb{N}} \subset C$ satisfying $\lim_{k\to\infty} y_k = x_0(t_0)$. Then, there exists a positive integer k_0 such that, for all $k > k_0$, there exists a solution $x_k\colon [t_0, t_0+\sigma] \to X$ of the generalized ODE

$$\frac{dx}{d\tau} = DF_k(t, x) \tag{4.42}$$

with $x_k(t_0) = y_k$ and $(x_k)_{k\in\mathbb{N}}$ converges uniformly to x_0 on $[t_0, t_0 + \sigma]$.

Next, we present a result of continuous dependence on parameters for measure FDEs with state–dependent delays as a consequence of the previous theorem. Some steps of its proof are inspired by [10].

Theorem 4.5.4. *Let X be the Banach space defined by (3.3) and $\mathcal{B}$ be a phase space satisfying the axioms (E1)–(E3). Assume $g\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ is a left–continuous nondecreasing function, $f_k\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, and $\rho\colon [t_0, t_0 + \sigma] \times \mathcal{B} \to \mathbb{R}$, satisfy the conditions (F1)–(F6). Suppose, further, that for every $y \in X$,*

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k(s, y_{\rho(s,y_s)})\mathrm{d}g(s) = \int_{t_0}^{t} f_0(s, y_{\rho(s,y_s)})\mathrm{d}g(s)$$

for $t \in [t_0, t_0 + \sigma]$. Consider that $y_0 \in X$ is the unique solution of

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f_0(s, (y_0)_{\rho(s,(y_0)_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned} \tag{4.43}$$

where $\phi_0 \in \mathcal{B}$. Let $(\phi_k)_{k\in\mathbb{N}}$ be a sequence of functions in $\mathcal{B}$ such that $\lim_{k\to\infty} \phi_k = \phi_0$ uniformly on $(-\infty, 0]$. Assume further that there is a $\lambda > 0$ such that if $s \in [t_0, t_0 + \sigma]$ and $\|z - y_0(s)\| < \lambda$, then $(s, z) \in [t_0, t_0+\sigma] \times X$ and let $(z_k)_{k\in\mathbb{N}} \subset X$ satisfying $\lim_{k\to\infty} z_k = y_0(t_0)$. Then for sufficiently large $k \in \mathbb{N}$, there exists a solution y_k of

$$\begin{aligned} y_k(t) &= y_k(t_0) + \int_{t_0}^{t} f_k(s, (y_k)_{\rho(s,(y_k)_s)})\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \\ (y_k)_{t_0} &= \phi_k. \end{aligned} \tag{4.44}$$

Also, the sequence $(y_k)_{k\in\mathbb{N}}$ converges uniformly to y_0 on $(-\infty, t_0 + \sigma]$.

Proof. For each $k \in \mathbb{N}$, $z \in X$ and $t \in [t_0, t_0 + \sigma]$, define the function $F_k\colon [t_0, t_0 + \sigma] \times X \to$

$G((-\infty, t_0+\sigma], \mathbb{R}^n)$ by

$$F_k(t,z)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f_k(s, z_{\rho(s,z_s)})\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f_k(s, z_{\rho(s,z_s)})\mathrm{d}g(s), & \xi \in [t, t_0+\sigma]. \end{cases}$$

Hypotheses imply that $(F_k)_{k\in\mathbb{N}}$ converges pointwisely to F_0 for every $(t,x) \in [t_0, t_0+\sigma]\times X$. By Lemma 4.1.1, it follows that $F_k \in \mathcal{F}([t_0, t_0+\sigma] \times X, h)$ for every $k \in \mathbb{N}$, where h is given by (4.8).

Let y_0 be the unique solution of (4.43). Defining $x_0 \colon [t_0, t_0+\sigma] \to X$ by

$$x_0(t)(\xi) = \begin{cases} y_0(\xi), & \xi \in (-\infty, t], \\ y_0(t), & \xi \in [t, t_0+\sigma], \end{cases}$$

We have, by Theorem 4.1.4, that x_0 is the solution of (4.41) on $[t_0, t_0+\sigma]$. Since y_0 is the unique solution of (4.43) on $[t_0, t_0+\sigma]$, applying again Theorem 4.1.4, we obtain that x_0 is the unique solution of (4.41).

Assume further that there is $\lambda > 0$ such that if $s \in [t_0, t_0+\sigma]$ and $\|z - x_0(s)\| < \lambda$, then $(s,z) \in [t_0, t_0+\sigma] \times X$, and let $(z_k)_{k\in\mathbb{N}} \subset X$ satisfy $\lim_{k\to\infty} z_k = x_0(t_0)$. Therefore, all the hypotheses from Theorem 4.5.3 are satisfied. It implies that there exists a positive integer k_0 such that for all $k > k_0$, there exists a solution x_k of the generalized ODE (4.42) on $[t_0, t_0+\sigma]$ such that $x_k(t_0) = x_0(t_0)$ and $\lim_{k\to\infty} x_k(s) = x_0(s)$ where x_0 is the solution of (4.41) by the uniqueness. Therefore, define for $k > k_0$ and $t \in [t_0, t_0+\sigma]$, the function

$$y_k(\xi) = \begin{cases} x_k(t_0)(\xi), & \xi \in (-\infty, t_0], \\ x_k(\xi)(\xi), & \xi \in [t_0, t_0+\sigma]. \end{cases}$$

According to Theorem 4.1.5, y_k is a solution of the measure functional differential equations with state–dependent delays (4.44) on $(-\infty, t_0+\sigma]$. Thus, as a consequence, by definition of $(y_k)_{k\in\mathbb{N}}$ and by hypotheses, we get

$$\lim_{k\to\infty} (y_k)_{t_0}(\theta) = \lim_{k\to\infty} \phi_k(\theta) = \phi_0(\theta) = (y_0)_{t_0}(\theta) \tag{4.45}$$

for $\theta \in (-\infty, 0]$. It implies that $\lim_{k\to\infty} y_k(s) = y_0(s)$ for $s \in (-\infty, t_0]$. On the other hand, for $t \in [t_0, t_0+\sigma]$, we have by definition of y_k,

$$\lim_{k\to\infty} y_k(t) = \lim_{k\to\infty} x_k(t)(t) = x_0(t)(t) = y_0(t). \tag{4.46}$$

In consequence, combining (4.45) and (4.46), we have $\lim_{k\to\infty} y_k(t) = y_0(t)$ for $t \in (-\infty, t_0 + \sigma]$, getting the desired result. □

Remark 4.5.5. *Using Theorems 4.2.1 and 4.3.2, it is possible to prove similar results on continuous dependence on parameters to impulsive measure FDEs with state–dependent delays and functional dynamic equations on time scales with state–dependent delays. Then again, we omit them here since they follow as an immediate consequence of both correspondences.*

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LIST OF SYMBOLS

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