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Dissertação submetida à Universidade de Brasília de acordo com os requisitos básicos de aquisição do título de MESTRE EM FÍSICA pelo instituto de física.

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14 DE AGOSTO DE 2019

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A Non-Commutative Model for AdS/CFT Correspondence

Instituto de Física - Universidade de Brasília

September 13, 2019

Dedico esse trabalho à Iêda Rodrigues, sem a sua presença eu não chegaria onde estou.

Acknowledgements

Agradeço à todas as pessoas que participaram de forma direta e indireta na produção deste trabalho. Primeiramente, agradeço à minha esposa, Tamine, por ter permanecido ao meu lado, me apoiando enquanto eu produzia esta dissertação. Agradeço também aos professores Aleksandr Nikolaievich Pinzul, Arsen Melikyan e Marco Cezar pelas frutíferas discussões que me ajudaram muito no entendimento das estruturas matemáticas utilizadas neste trabalho. Gostaria de agradecer também às várias pessoas que me auxiliaram em cálculos, discussões, debates e risadas, estes são: O nosso eterno amigo Arlon Fernando, Ravi Misty, Caio Licínio, Davi Feliciano, Daniel Andrada, Matheus Aguilar, Lorena do Carmo, Luis Gustavo, Diogo Colen, João Gilberto Fernandes, Lucas Loureiro, Juliana Loureiro, Luís Gomes, Caio Matheus, Rafael Ribeiro, Lucas Camacho, Gabriel Aleixo, Victor Rebeque, Caio César, Vitor Bruno, João Gabriel, Bruno Canongia, Lucas Nixxon, Bruno Simões e os demais colegas do Departamento de Física da Universidade de Brasília que participaram de alguma maneira deste trabalho. E gostaria de agradecer finalmente à minha mãe, Iêda, por ter se esforçado muito para garantir uma boa criação para seus filhos, esse resultado é um fruto de seu esforço e dedicação.

Contents

0.1 Introduction

O objetivo deste trabalho é cobrir, de maneira concisa e pragmática, alguns dos aspéctos gerais do uso de geometrias não comutativas no estudo das teorias em espaços AdS. A não comutatividade surge como uma forma natural de acoplar, de maneira intrínseca, as relações de incerteza na estrutura fundamental da teoria, visando evitar alguns problemas, quando queremos, por exemplo, considerar efeitos relativísticos nestes espaços. Além disso, tentaremos verificar a validade da correspondência AdS/CFT para o caso mais geral, aplica-la no contexto de alguns *toy models* visando mapear as propriedades que podem ser recuperadas no limite comutativo da teoria. Seguirei como fonte principal o artigo [1] e buscarei generalizar alguns aspectos para formular uma análise aplicável à correspondência AdS^{d+1}/CFT_d e suas respectivas correções quânticas devido a não comutatividade, incorporando os efeitos quase-clássicos da gravitação quântica.

As justificativas para este trabalho remontam à época de Heisenberg. Este acreditava que, por meio da geometria não comutativa, era possível remover algumas quantidades infinitas antes da aplicação da renormalização. Poucos pesquisadores compraram esta ideia, pois a grande maioria dos cientistas da época percebeu que a renormalização, de fato, funcionava e conseguia render ótimos resultados. Isto mudou na década de 90 quando os matemáticos conseguiram estruturar uma teoria formal para aneis e algebras não comutativas e algumas possíveis interpretações de efeitos provenientes da gravitação quântica poderiam ser mais facilmente acoplados à teoria por meio de relações de comutação oriundas de geometrias não comutativas. Na ausência de uma teoria completa de gravitação quântica, o caso não comutativo é o regime quase-clássico de qualquer teoria quântica de campos. (Veja [22])

Para introduzir de forma única a não comutatividade no espaço AdS_2 podemos impor que ele preserve as isometrias do grupo $SO(2,1)$. Este objetivo é alcançado quando construímos os vetores de Killing do AdS_2 no espaço não comutativo. Uma forma natural de verificar a validade dessa formulação seria analisar a aplicação do *ncAdS* para as partículas livres. Começamos por quantizar a variedade de Poisson que define o AdS_2 comutativo por promover as variáveis de imersão X^μ, que definem a métrica nesta variedade, à operadores Hermitianos:

$$\hat{X}^\mu \hat{X}_\mu = -\ell^2 \mathbb{1} \tag{0.1}$$

Onde $\ell^2 < 0$ se associa com o vínculo definido no caso comutativo para que esta restrição defina um hiperboloide de duas folhas, representado pelos geradores do grupo $SU(1,1)$. Seguindo os procedimentos habituais, promovemos o parentese de Poisson para comutadores e estes satisfazem as seguintes relações de comutação:

$$[\hat{X}^\mu, \hat{X}^\nu] = i\alpha\varepsilon^{\mu\nu\rho}\hat{X}_\rho \tag{0.2}$$

Ambas relações (0.1) e (0.2) são preservadas pela ação do grupo $SO(2,1)$ que é isomorfo localmente ao grupo $SU(1,1)$, sendo que estes elementos geram a algebra $so(2,1)$. Os estados do $ncAdS_2$ pertencem as grupo de recobrimento universal $SU(1,1)$, que geram as séries principais, suplementares e discretas. Como o nosso interesse é recuperar o AdS comutativo no limite $\alpha \to 0$, utilizaremos apenas a série discreta, pois esta possui esta propriedade. Os estados podem ser representados como auto-valores do operador $\hat{r}$ definido como:

$$\hat{r} = \frac{\hat{X}^1 - \hat{X}^2}{\ell} \tag{0.3}$$

Esses estados nesta representação dependem de polinômios de Laguerre que são representados como os operadores diferenciais, agindo no espaço de funções $L^2(\mathbb{R}, dx)$, considere $i = 1,2$:

$$\tilde{\pi}(\hat{X}^0) = \hat{y}$$
$$\tilde{\pi}(\hat{X}^i) = -\frac{1}{2\ell}\hat{y}e^{-\hat{x}}\hat{y} - \frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}} + (-1)^{i+1}\frac{\ell}{2}e^{\hat{x}} \tag{0.4}$$

Sendo x e y as coordenadas canônicas do caso comutativo representadas como operadores que satisfazem as relações de comutação de Heisenberg. Como estes operadores satisfazem as relações de comutação, estes podem ser mapeados para seus respectivos símbolos no plano de Moyal-Weyl, que é gerado pelas variaveis x e y, que comutam entre si. A continuidade no processo se da por definir a fronteira do plano de Moyal-Weyl que coincida com a fronteira do $ncAdS_2$, no limite comutativo. Por utilizar o produto estrela no espaço de MW, podemos construir os vetores de Killing que preservam paricialmente as isometrias do caso não comutativo e com isto estudar quais características da correspondência AdS/CFT serão preservadas neste contexto. Um dos resultados demonstrados é que as simetrias conformes são preservadas, até determinada ordem, na passagem para o caso não comutativo, revelando que as propriedades intrísecas da geometria local estão sendo deformadas, porém, mantendo algumas estruturas invariantes durante estas transformações. Finalmente, verificaremos algumas propriedades como o Limite de Breitenlohner-Freedman para casos massivos e analisaremos o caso massivo com interação com o objetivo de encontrar uma teoria consistente, para tal, o cálculo da função de três pontos poderá auxiliar nesta empreitada.

No capítulo 1 introduzimos as principais coordenadas utilizadas na imersão do espaço AdS e como podemos observar suas propriedades analisando alguns exemplos como o caso de uma partícula escalar em um espaço-tempo AdS. Discutimos também as principais características dos propagadores da teoria e apresentamos de forma detalhada os passos necessários para a construção dos propagadores e campos. No capítulo 2 fazemos uma breve apresentação da teoria conforme de campos introduzindo o grupo conforme, campos primários e as funções de correlação. No capítulo 3 as principais ideias por trás do princípio da correspondência AdS/CFT são apresentados de forma concisa e revisitamos o caso $d = 1$ apresentando o modelo dAFF e sua relação com o grupo $SU(1,1)$.

Apresentamos no capítulo 4 as motivações que levam a adoção de modelos não comutativos em teorias quânticas, exploramos a quantização no espaço de fase e as ideias principais que subsidiam a correspondência de Weyl e a definição do produto estrela. Terminamos o capítulo construindo as representações e os vetores de Killing do grupo $SU(1,1)$. O objetivo deste trabalho começa a ser desenvolvido no capítulo 5 e é finalizado no capítulo 6, onde iniciamos a análise dos modelos AdS/CFT não comutativos nos casos com e sem massa e adicionamos a interação no capítulo 6 quando introduzimos um termo de interação proporcional a $\Phi\star\Phi\star\Phi$ na ação, comparando os resultados comutativos com os não comutativos. Devido a dificuldades técnicas, a comparação dos resultados comutativos e não comutativos se torna demasiadamente complicada e recorremos a análise do efeito da deformação da quantização nos vetores de Killing da teoria e isso nos mostra que essas transformações são, de fato, não triviais, gerando um conjunto de vetores de Killing que carregam as simetrias do espaço comutativo para a teoria não comutativa.

Brasília, Agosto 2019 *Felipe Rodrigues de Almeida Araújo*

Chapter 1

AdS Spacetime

In this chapter we are going to present the main properties of the AdS spacetime. We are following the presentation given in [2], [15], [17], [25] and [28]. As a first step we will study the most useful embeddings of the AdS space in a most general spacetime. After this, we discuss the two-dimensional case, since it is the focus of this work. Taking this example as a starting point, we continue deriving the equations of motion for the general case and in the next step we apply the separation of variables in order to find a solution to these equations. As the final part of this chapter we follow the construction given in [12] and [25] to define the boundary-to-bulk and the boundary-to-boundary propagator.

1.1 The AdS Spacetime

Anti-deSitter spacetime is a non-compact, maximally symmetric spacetime with constant negative curvature. By maximally symetric, we mean that it has the maximal number of symmetries for $d+1$ dimensions, from now on, we will call it AdS_{d+1}. The AdS_{d+1} has $\frac{1}{2}(d+1)(d+2)$ symmetries, that is the same number of the flat spacetime symmetries related to $(d+1)$ translations, d boosts and $\frac{1}{2}d(d-1)$ rotations. Usually we study $(d+1)$-dimensional AdS spaces because the CFT dual of AdS_{d+1} have d spacetime dimensions. It's a solution to Einstein's equations with negative cosmological constant. There are a variety of coordinate systems for it and they satisfies the equation of the hyperboloid:

$$X_A X^A = X_0^2 + X_{d+1}^2 - \sum_{n=1}^{d} X_n^2 = \ell^2 \ . \tag{1.1}$$

And it can be embedded in a $(d+2)$-dimensional space as:

$$\begin{aligned} X_0 &= \ell \frac{\cos(t)}{\cos(r)} \\ X_{d+1} &= \ell \frac{\sin(t)}{\cos(r)} \\ X_n &= \ell \frac{\sin(r)}{\cos(r)} \hat{\Omega}_n \ , \end{aligned} \tag{1.2}$$

this embedding defines the Minkowskian AdS_{d+1} which has the following metric

$$ds^2 = \frac{1}{\cos^2\left(\frac{r}{\ell}\right)}\left(dt^2 - dr^2 - \sin^2\left(\frac{r}{\ell}\right)d\Omega_{d-1}^2\right) \ . \tag{1.3}$$

Here, ℓ is the length scale, which will be chosen in a convenient way in order to make the measurements of the energies be in the right scale, that is, unless specified diferently we are taking from now $\ell = 1$, r is the radial coordinate $r \in [0, \frac{\pi}{2})$, while $t \in (-\infty, \infty)$ and the angular coordinate Ω defines a $(d-1)$-dimensional sphere S^{d-1}. In global coordinates we can imagine AdS as the interior part of a infinitely long cylinder. In order to see the symmetries of AdS we can represent them directly by:

$$L_{AB} = X_A \frac{\partial}{\partial X_B} - X_B \frac{\partial}{\partial X_A} \ , \tag{1.4}$$

which generate the group $SO(d,2)$ that leaves the equation (1.2) invariant. We will refer to this group as the conformal group because it's the same isometry group for CFT_d. Note that the generator of translations in the t-direction is easily obtained by:

$$\begin{aligned} L_{(d+1),0} = X_{d+1}\frac{\partial}{\partial X_0} - X_0 \frac{\partial}{\partial X_{d+1}} &= \ell\frac{\sin(t)}{\cos(r)}\frac{\partial}{\partial X_0} - \ell\frac{\cos(t)}{\cos(r)}\frac{\partial}{\partial X_{d+1}} = \\ &= -\frac{\partial X_0}{\partial t}\frac{\partial}{\partial X_0} - \frac{\partial X_{d+1}}{\partial t}\frac{\partial}{\partial X_{d+1}} = -\frac{\partial}{\partial t} \ . \end{aligned} \tag{1.5}$$

The other generators are the usual isometries of the sphere S^{d-1} that form the group $SO(d)$. One can choose also another parametrization

$$X_0 = \sqrt{\ell^2 + r^2}\sin\left(\frac{t}{\ell}\right) \quad X_{d+1} = \sqrt{\ell^2 + r^2}\cos\left(\frac{t}{\ell}\right) \quad X_n = r\hat{x}_n \ , \tag{1.6}$$

with

$$r \in [0, \infty) \ , \quad t \in [0, 2\pi\ell] \ , \quad \hat{x}^n\hat{x}_n = 1 \ , \tag{1.7}$$

giving the following metric

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2 \ . \tag{1.8}$$

1.2 The Poincaré Patch

The Euclidean AdS and the Euclidean conformal group which is $SO(d+1,1)$ can be better studied in this embedding space:

$$X_0^2 - \sum_{j=1}^{d+1} X_j^2 = \ell^2 \, . \tag{1.9}$$

When we consider the global coordinates, the t term of the metric (1.3) changes the sign and it will just swap the trigonometric functions for the hyperbolic trigonometric ones in the global mapping (1.2), giving for $\tau = it$:

$$\begin{aligned} X_0 &= \ell \frac{\cosh(\tau)}{\cos(r)} \\ X_{d+1} &= \ell \frac{\sinh(\tau)}{\cos(r)} \\ X_n &= \ell \frac{\sin(r)}{\cos(r)} \hat{\Omega} \, . \end{aligned} \tag{1.10}$$

This embedding defines the Euclidean AdS_{d+1}. There is a coordinate system that makes the d-dimensional Poincaré subgroup of the conformal group clear and manifest, we call it Poincaré Patch (PP). The relation between the Euclidean, Poicaré patch and global coordinates, respectively, is:

$$\begin{aligned} X_0 &= \frac{z^2 + x^i x_i + \ell^2}{2z} = \ell \frac{\cosh(\tau)}{\cos(r)} \\ X_{d+1} &= \frac{z^2 + x^i x_i - \ell^2}{2z} = \ell \frac{\sinh(\tau)}{\cos(r)} \\ X_n &= \frac{\ell}{z} x_i = \ell \frac{\sin(r)}{\cos(r)} \hat{\Omega} \, , \end{aligned} \tag{1.11}$$

where x is a d-dimensional space vector, z runs from 0 to ∞ and τ is the global "time" coordinate, this fix the signal of X_0. The dilatations can be obtained by direct calculation of $L_{0,(d+1)}$:

$$\begin{aligned} z(X_0, X_{d+1}, X_i) &= \frac{\ell^2}{(X_0 - X_{d+1})^2} \\ x_i(X_0, X_{d+1}, X_i) &= \frac{\ell X_i}{(X_0^2 - X_{d+1})^2} \, , \end{aligned} \tag{1.12}$$

with this, one can show that:

$$\begin{aligned} L_{0,(d+1)} &= (X_0 - X_{d+1}) \frac{\ell^2 \partial_z + \ell X_i \partial_{x_i}}{(X_0 - X_{d+1})^2} \\ &= \frac{\ell^2}{z} \left(\frac{z^2 \partial_z + z x_i \partial_{x_i}}{\ell^2} \right) = z \partial_z + x_i \partial_{x_i} \, . \end{aligned} \tag{1.13}$$

Note that the dilatation generator acts in the same way $L_{0,(d+1)}$ does, this means that the Hamiltonian will be associated to the dilatation operator. This operator acts on x, stretching the space, generating the "time" evolution.

The Lorentzian case is the analytic continuation of the $(d+1)$-dimensional version of the Lobachevski space (via Wick rotation). The Euclidean Poincaré Patch covers the entire AdS space, however, in the Lorentzian case, solving (t,z,x_i) in terms of the global coordinates (τ,r,Ω_i) we find

$$
\begin{aligned}
t &= \ell\frac{\sin(t)}{\cos(\tau)-\Omega_d\sin(r)}\\
z &= \ell\frac{\cos(r)}{\cos(\tau)-\Omega_d\sin(r)}\\
\hat{x}_i &= \ell\frac{\sin(r)\hat{\Omega}_i}{\cos(\tau)-\Omega_d\sin(r)}\ ,
\end{aligned}
$$

it shows us that the Lorentzian PP only cover a small region of the AdS spacetime. It's convenient to switch the labels, giving:

$$
\begin{aligned}
X_0 &= \frac{z^2+(x_i)^2-t^2+\ell^2}{2z} = \ell\frac{\cos(\tau)}{\cos(r)}\\
X_d &= \frac{z^2+(x_i)^2-t^2-\ell^2}{2z} = \ell\frac{\sin(\tau)}{\cos(r)}\\
X_{i\neq d} &= \frac{\ell}{z}x_i = \ell\tan(r)\Omega_i\\
X_{d+1} &= \frac{\ell}{z}t = \ell\tan(r)\Omega_d\ .
\end{aligned}
\tag{1.14}
$$

In this setting the global coordinates makes the sub-group $SO(2)\times SO(d)$ of the conformal group clearly manifest in the embedding of the coordinates and the Poincaré Patch turns out to show us the Poincaré symmetry of the AdS spacetime. The usual metric for the Poincaré Patch in these coordinates is

$$
ds^2 = \frac{1}{z^2}\left(dt^2-dz^2-\sum_{i=1}^{d-1}dx_i^2\right)\ . \tag{1.15}
$$

1.3 The d=1 Case

In this dissertation we will focus on the AdS_2 case and the main reason for this is that after constructing the causal structure by wrapping the τ-circle $\mathbb{S}^1$ and taking the universal covering of the hyperboloid, the AdS_2 exhibts two timelike boundaries that makes the dual CFT live in a disconnected manifold. Another remarkable fact is that all theories of two dimensional quantum gravity are conformal field theories [32]. To see this, we begin with the constraint equation for the Euclidean case

$$
\ell^2 = X_0^2-X_1^2-X_2^2\ , \tag{1.16}
$$

and the metric is

$$
ds^2 = \frac{\ell^2}{z^2}(dz^2+dt^2)\ , \tag{1.17}
$$

with the embedding

$$
\begin{aligned}
X_0 &= \frac{z^2+t^2+\ell^2}{2z} \\
X_1 &= \frac{z^2+t^2-\ell^2}{2z} \\
X_2 &= \frac{\ell}{z}t \;.
\end{aligned}
\tag{1.18}
$$

One can verify (1.16) by direct calculation. Clearly, one can obtain the two boundaries just by taking $z \to 0$. Since the limits for X_i are not the same, we get into two different regions and one should expect to find different CFT duals for each boundary. One would like to study a single particle on AdS_2 quantum mechanically by naively trying to derive the AdS (written in the coordinates (1.16)) Schrodinger equation quantising the classical action for a free particle with mass m

$$
S = m\int d\tau = m\int dt\sqrt{g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} = \int dt\frac{m}{\cos(r)}\sqrt{1-\dot{r}^2}\;, \tag{1.19}
$$

with

$$
\mathscr{L} = \frac{m}{\cos(r)}\sqrt{1-\dot{r}^2}\;. \tag{1.20}
$$

Clearly the momentum canonically conjugate to r is

$$
P_r = -\frac{m\dot{r}}{\cos(r)\sqrt{1-\dot{r}^2}} \quad \to \quad \dot{r}^2 = \frac{P_r^2}{P_r^2+\frac{m^2}{\cos^2(r)}}\;. \tag{1.21}
$$

Using this results one obtains the Hamiltonian

$$
H = \sqrt{P_r^2+\frac{m^2}{\cos^2(r)}}\;. \tag{1.22}
$$

Proceeding with the with the canonical quantization, we must impose the commutation relation $[r,P]=i$, taking $\hbar=1$. This can be achieved by taking $P=-i\partial_r$ acting on the r-basis states. Looking to the equation (1.22) it's clear that we should consider the equation for $-\partial_t^2\psi(t,r) = H^2\psi(t,r)$, which gives

$$
-\partial_t^2\psi(r,t) = \left(\frac{m^2}{\cos^2(r)}-\partial_r^2\right)\psi(r,t)\;. \tag{1.23}
$$

We will see later that the equation (1.23) is equivalent to the equation obtained via relativistic field theory for AdS_{d+1}. One also can solve (1.23) and find an answer that depends on hypergeometric functions as will be shown later.

1.3.1 AdS (d+1) Action and equations of motion

In this subsection we will use μ for d-dimensional Minkowski space index, A for $(d+1)$-dimensional AdS index and i for $(d-1)$-dimensional space index. First we start with the Einstein-Hilbert action in vacuum with cosmological constant Λ, considering all other constants to be equal to 1, we write

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{-g}(R-\Lambda)\ , \tag{1.24}$$

where R is the Ricci scalar, and we are taking the metric in the form corresponding to any of the discussed coordinates. If one consider that

$$\Lambda = -\frac{1}{\ell^2}d(d-1)\ , \tag{1.25}$$

one can calculate the equations of motion

$$\begin{aligned} 0 = \delta S &= \frac{1}{2}\int d^{d+1}x\frac{\delta}{\delta g_{\mu\nu}}\left(\sqrt{-g}\left[R+\frac{1}{\ell^2}d(d-1)\right]\right)\delta g_{\mu\nu} \\ &= \frac{1}{2}\int d^{d+1}x\sqrt{-g}\delta g_{\mu\nu}\left(\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}}\left[\frac{R}{\sqrt{-g}}+\frac{1}{\ell^2\sqrt{-g}}d(d-1)\right]+\frac{\delta R}{\delta g_{\mu\nu}}\right)\ , \end{aligned} \tag{1.26}$$

since the equation above is zero for any variation $\delta g_{\mu\nu}$, it means that the integrand is zero also and taking the usual boundary conditions ($\delta g_{\mu\nu}$ vanishes near the boundary) we get

$$\frac{g_{\mu\nu}}{2}\left[R+\frac{1}{\ell^2}d(d-1)\right] = \frac{\delta R}{\delta g_{\mu\nu}} = R_{\mu\nu}\ . \tag{1.27}$$

One can readly recognize the Einstein equation from the expression above and this means that the space is an Einstein manifold, i.e. the Ricci tensor is proportional to the metric tensor and the maximal symmetry of the AdS_{d+1} space can be verified in the expressions below

$$R_{\mu\nu} = -\frac{d}{\ell^2}g_{\mu\nu}\quad ,\qquad R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})\ . \tag{1.28}$$

Considering a scalar field $\phi(X)$ on an $\mathscr{M} = AdS_{d+1}$ background, one can write the action

$$S = -\frac{1}{2}\int_{\mathscr{M}} d^{d+1}X\sqrt{g}(g^{AB}(\partial_A\phi)(\partial_B\phi)+m^2\phi^2)\ , \tag{1.29}$$

integrating by parts and taking $\phi(X)|_{\partial\mathscr{M}}$ decays exponentially as $z\to 0$, we obtain

$$\begin{aligned} S = &-\frac{1}{2}\int_{\mathscr{M}} d^{d+1}X\sqrt{g}\phi(X)\left(-\frac{1}{\sqrt{g}}\partial_A(\sqrt{g}g^{AB}\partial_B)+m^2\right)\phi(X)+ \\ &+\int_{\partial\mathscr{M}} d^d\sigma^A\sqrt{\gamma}\phi\partial_A\phi\ . \end{aligned} \tag{1.30}$$

where $\gamma_{\mu\nu}$ is the induced metric on the boundary of AdS, σ_A is the unit normal vector to the boundary. Using the Poincaré Patch coordinates and the following metric

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + \eta^{\mu\nu} dx^\mu dx^\nu) \,, \tag{1.31}$$

one can calculate $\sqrt{g}$

$$\sqrt{g} = \sqrt{|det(g^{\mu\nu})|} = \sqrt{\frac{\ell^{2(d+1)}}{z^{2(d+1)}}} = \frac{\ell^{d+1}}{z^{d+1}} \,, \tag{1.32}$$

and so the Laplacian Δ

$$\begin{aligned}\Delta = \nabla_A \nabla^A &= \frac{1}{\sqrt{g}} \partial_A (\sqrt{g} g^{AB} \partial_B) \\ &= \frac{z^{d+1}}{\ell^{d+1}} \left[\partial_z \left(\frac{\ell^{d+1}}{z^{d+1}} \frac{z^2}{\ell^2} \partial_z \right) + \eta^{\mu\nu} \partial_\mu \frac{\ell^{d+1}}{z^{d+1}} \frac{z^2}{\ell^2} \partial_\nu \right] \\ &= \frac{z^2}{\ell^2} \left(\partial_z^2 - \frac{(d-1)}{z} \partial_z + \eta^{\mu\nu} \partial_\mu \partial_\nu \right) \,.\end{aligned} \tag{1.33}$$

Assuming that ϕ satisfies the equations of motion obtained by the variation of the action and evaluating the action for this field, only the boudary term of (1.30) remain. Decomposing the surface of the boundary in two parts, whose are normal to x and z respectively. Since the x part is related to the Minkowski part itself ϕ must vanish as $x = \rightarrow \pm\infty$ then we only consider the part of the boundary that is normal to z knowing that $\partial\mathscr{M}$ is just the usual Minkowski space

$$S_{\partial\mathscr{M}} = \int_{\partial\mathscr{M}} d^d x \sqrt{\gamma} \phi \partial_z \phi \Big|_{z=\varepsilon} \,. \tag{1.34}$$

As the can be seen, the induced metric γ diverges as $z \to 0$, so we introduced a cut-off ε to avoid any problem for now. Solving (1.34) trivially gives

$$S_{\partial\mathscr{M}} = \int_{\partial\mathscr{M}} d^d x \phi \partial_z \phi \Big|_{z=\varepsilon} \,, \tag{1.35}$$

and the equation of motion is

$$(\Delta - m^2)\phi = 0 = \frac{\partial^2 \phi}{\partial z^2} - \frac{(d-1)}{z} \frac{\partial \phi}{\partial z} + \eta^{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \phi - \frac{m^2 \ell^2}{z^2} \phi \tag{1.36}$$

1.4 Solutions of the equations of motion

We are going to discuss different aspects of the solutions to the Klein-Gordon equation (1.36).

1.4.1 Separation of variables

Looking for a solution to (1.36), we hope that we can separate variables because of the translation invariance in the x direction due to the symmetries of AdS, we can write the fields as

$$\phi(x,z) = \psi(z)\Phi(x) \ . \tag{1.37}$$

Substituting (1.37) in (1.36) and dividing the whole equation by $\phi(x,z)$

$$\frac{1}{\psi(z)}\left(\partial_z^2\psi(z) - \frac{(d-1)}{z}\partial_z\psi(z) - \frac{m^2\ell^2}{z^2}\right) = -\frac{\partial_\eta^2\Phi(x)}{\Phi(x)} = k^2 \ , \tag{1.38}$$

where the notation ∂_η^2 refers to the Laplacian on Minkowski space and k^2 is the norm of a d-dimensional vector $k^\mu \in M_\eta$. This separation gives us two equations

$$(\partial_\eta^2 + k^2)\Phi(x) = 0 \quad , \qquad \left(\partial_z^2 - \frac{(d-1)}{z}\partial_z - \frac{m^2\ell^2}{z^2} - k^2\right)\psi(z) = 0 \ . \tag{1.39}$$

Hence the solutions for Φ will be depending on the choice of k and the k^2 sign, we must consider the consequences for each case:

- $k^2 > 0$ with $k_0 = 0$ (Euclidean): This will lead to the Euclidean Green function, giving a real exponential in z direction.
- $k^2 > 0$ (Spacelike Minkowskian): The momentum is off-shell. The solution for z is again a real exponential.
- $k^2 < 0$ (Timelike Minkowskian): On-shell mass condition for the momentum. The z equation will lead to a imaginary exponential. (Advanced and retarded Green functions)

If we want to define mass of a particle in the d-dimensional space we should expect discrete values for k^2 in the spectrum of the z equation, that is not the case here, but if z is bound from above in the interior of $\mathscr{M}$ then the conformal symmetry will be broken and k^2 will assume discrete values. We will mostly work in euclidean space without loss of generality. One can easily recognize the d-dimensional equation of (1.39) as the Klein-Gordon equation, which gives plane-waves as solutions. Superposing them for all possible k gives

$$\phi(z,x) = \int \frac{d^d k}{(2\pi)^d} \psi(z,k) e^{ik^\mu x_\mu} \ . \tag{1.40}$$

Due to the translational invariance we get that $\phi(z,x)$ is just the Fourier transform of $\psi(z,k)$ and by inverting (1.40) we see that $\psi(z,k)$ is the solution in momentum space. From now we will denote $\psi(z,k)$ as $\psi_k(z)$. In order to solve the equation for z we can make the following change of variables $\psi_k(z) = z^{d/2} f_k(z)$ to get

$$\frac{d}{2z^2}\left(\frac{d}{2}-1\right) z^{d/2} f_k + \frac{d}{z} z^{d/2} f_k' + z^{d/2} f_k'' - \frac{(d-1)}{z}\left(\frac{d}{2z} z^{d/2} f_k + z^{d/2} f_k'\right) - \left(\frac{m^2\ell^2}{z^2} + k^2\right) z^{d/2} f_k = 0 \ . \tag{1.41}$$

Cancelling all the terms and taking f as a function of $z|k|$ instead of z alone, i.e. $z \to |k|z$, we obtain the following

$$(|k|z)^2 f_k''(|k|z) + |k|z f_k'(|k|z) - \left(\frac{d^2}{4} + k^2 z^2 + m^2\ell^2\right) f_k(|k|z) = 0 \ , \tag{1.42}$$

which is the modified Bessel equation. The general solution of (1.42) is given by

$$f_k(|k|z) = A_k K_\nu(|k|z) + B_k I_\nu(|k|z) \qquad \nu = \sqrt{\frac{d^2}{4} + m^2\ell^2} \ . \tag{1.43}$$

Note that imposing that $\nu \in \mathbb{R}$ in (1.43) gives the Breitenlohner-Freedman bound (see [7])

$$m^2\ell^2 > -\frac{d^2}{4} \ . \tag{1.44}$$

In order to avoid any divergences we should analyse the assymptotic behavior of the modified Bessel functions of first and second kind

$$z \to \infty \ , \qquad I_\mu(z) \sim e^{kz} \ , \qquad K_\nu(z) \sim e^{-kz} \ , \tag{1.45}$$

imposing $B_k = 0$ we finnaly get the solution for $\psi_k(z)$

$$\psi_k(z) = A_k (|k|z)^{d/2} K_\nu(|k|z) \ . \tag{1.46}$$

Near the boundary $z = 0$ the behavior of the solution can be analyzed using

$$K_\nu(|k|z) = \frac{\pi}{2\sin(\pi\nu)}\left(I_{-\nu}(|k|z) - I_\nu(|k|z)\right) , \tag{1.47}$$

using

$$\frac{\pi}{\sin(\nu\pi)} = \Gamma(1-\nu)\Gamma(\nu) , \tag{1.48}$$

and

$$I_{\pm\nu}(|k|z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n\pm\nu+1)} \left(\frac{|k|z}{2}\right)^{2n} \left(\frac{|k|z}{2}\right)^{\pm\nu} , \tag{1.49}$$

for $z \to 0$ the main contribution comes from the $n = 0$ term in the summation

$$I_{\pm\nu}(|k|z)\Big|_{z\to 0} \sim \frac{1}{\Gamma(\pm\nu+1)} \left(\frac{|k|z}{2}\right)^{\pm\nu} , \tag{1.50}$$

substituting (1.50) and (1.48) in (1.47) gives

$$\begin{aligned} K_\nu(|k|z) &\sim \frac{\Gamma(1-\nu)\Gamma(\nu)}{2} \left[\frac{1}{\Gamma(-\nu+1)} \left(\frac{|k|z}{2}\right)^{-\nu} - \frac{1}{\Gamma(\nu+1)} \left(\frac{|k|z}{2}\right)^{\nu}\right] \\ K_\nu(|k|z) &\sim \left[\frac{\Gamma(\nu)}{2} \left(\frac{|k|z}{2}\right)^{-\nu} + \frac{\Gamma(-\nu)}{2} \left(\frac{|k|z}{2}\right)^{\nu}\right] , \end{aligned} \tag{1.51}$$

which gives

$$\psi_k(z) \sim z^{d/2} A_k \left[\frac{\Gamma(\nu)}{2} \left(\frac{2}{|k|z}\right)^{\nu} + \frac{\Gamma(-\nu)}{2} \left(\frac{|k|z}{2}\right)^{\nu}\right] . \tag{1.52}$$

Defining

$$\begin{gathered} \Delta_\pm = \frac{d}{2} \pm \nu , \\ \phi_0(k) = A_k 2^{\nu-1}\Gamma(\nu)|k|^{\Delta_-}, \qquad \phi_1(k) = A_k 2^{-(\nu+1)}\Gamma(-\nu)|k|^{\Delta_+} . \end{gathered} \tag{1.53}$$

We get near the boundary

$$\psi(z,k) \sim \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+} . \tag{1.54}$$

Note that we can deduce these results by plugging the ansatz $\psi_k(z) = z^\Delta$ in (1.39) and find

$$\Delta(\Delta - d) - m^2\ell^2 - k^2 z^2 = 0 . \tag{1.55}$$

Since $z \to 0$ one can ignore the z^2 term and solve this quadratic equation to recover $\Delta_\pm$. The solution for Δ_+ corresponds to a bulk excitation that vanishes on the boundary giving a normalizable solution ($\Delta_+ > 0$). The solution for Δ_- does not decay and represents a field on the boundary

$$\phi_0(k) = \lim_{z\to 0} z^{-\Delta_-} \psi_k(z) \ , \tag{1.56}$$

now we use a cut-off to remove the limit

$$\psi_k(\varepsilon) = \varepsilon^{\Delta_-} \phi_0(k) \ . \tag{1.57}$$

One remarkable fact is that we need the non-normalizable modes in order to construct the Hilbert space of the theory. The normalizable modes, which define the Hilbert space itself, are propagated in the bulk and the non-normalizable modes are necessary to specify the boundary conditions of the background where the normal modes propagate. For this we will use the method of Green's functions to specify these propagators.

1.4.2 Free solution

Substituting (1.46) in (1.40), using the notation $|k^\mu| = k$ and $e^{ik^\mu x_\mu} = e^{ikx}$

$$\phi(z,x) = \int \frac{d^dk}{(2\pi)^d} A_k (kz)^{d/2} K_\nu(kz) e^{ik^\mu x_\mu} \ , \tag{1.58}$$

using the cutoff to avoid any divergence and inverting (1.57)

$$A_k = \frac{2^{1-\nu} k^{-\Delta_-} \phi_0(k)}{\Gamma(\nu)} \ , \tag{1.59}$$

substituting (1.59) in (1.58) and taking the Fourier transform of $\phi_0(k)$, one finds the free solution on position space

$$\phi(z,x) = \frac{2^{1-\nu} z^{d/2}}{\Gamma(\nu)} \int d^dx' \frac{d^dk}{(2\pi)^d} k^\nu K_\nu(kz) \phi_0(x') e^{ik(x-x')} \ . \tag{1.60}$$

In the following topics we will mainly focus on Δ_+ boundary condition since ϕ_0 can be interpreted as a source on the boundary and when considering the Hamiltonian analysis ϕ_1 is the canonical momentum associated to ϕ_0.

1.5 AdS propagators

We will study in this section the general properties of the three propagators defined for the AdS background.

1.5.1 Bulk-to-bulk propagator

The bulk-to-bulk propagator is defined, for Euclidean time, as follows

$$(-\partial_X^2+m^2)G(X;X')=\frac{1}{\sqrt{g}}\delta^{d+1}(X-X')\ . \tag{1.61}$$

The delta function satisfies

$$\int d^{d+1}X'\delta^{d+1}(X-X')\phi(X')=\phi(X)\ . \tag{1.62}$$

With this we can solve the inhomogeneous Klein-Gordon equation with the source $J(X)$ by using the Green's function convoluted with the source

$$\phi(X)=\int d^{d+1}X'\sqrt{g}G(X;X')J(X')\ . \tag{1.63}$$

Really, applying the Klein-Gordon operator on $\phi(X)$ we get

$$\begin{aligned}(-\partial_X^2+m^2)\phi(X)&=(-\partial_X^2+m^2)\int d^{d+1}X'\sqrt{g}G(X;X')J(X')\\&=\int d^{d+1}X'\sqrt{g}(-\partial_X^2+m^2)G(X;X')J(X')\\&=\int d^{d+1}X'\delta^{d+1}(X-X')J(X')=J(X)\ ,\end{aligned} \tag{1.64}$$

for any $\phi_h(X)$ that satisfies the homogeneous equation

$$(-\partial_X^2+m^2)\phi_h(X)=0\ , \tag{1.65}$$

we can write it as

$$\phi_h(X)=\int d^{d+1}X'\delta^{d+1}(X-X')\phi_h(X')\ , \tag{1.66}$$

using (1.61)

$$\phi_h(X) = \int d^{d+1}X' \sqrt{g}\phi_h(X')(-\partial^2_{X'} + m^2)G(X;X') \,, \tag{1.67}$$

integrating by parts and using e.o.m (1.65) we get

$$\phi_h(X) = -\int_{\partial\mathscr{M}} d^d\sigma^A \sqrt{\gamma}\Big(\phi_h(X')\partial_A G(X;X') - G(X;X')\partial_A \phi_h(X')\Big) \,, \tag{1.68}$$

there are several possibilities for the boundary conditions:

- If $G(X;X')|_{\partial\mathscr{M}} = 0$ then $\phi(X)$ is given by Dirichlet boundary conditions on $\partial\mathscr{M}$.
- If $\partial_A G(X;X')|_{\partial\mathscr{M}} = 0$ then $\phi(X)$ is given by Neumann conditions on $\partial\mathscr{M}$
- If none vanishes then we have mixed conditions for the boundary.

We are interested in the Dirichlet boundary conditions since we have found that our solutions approaches ϕ_0 on the boundary after rescaling the field. In addition to this, if we have Dirichlet conditions on boundary the solution for $|k| > 0$ is unique (see [5]). Now we can review the generic treatment employed in solving field theories with the Green's functions. Let's start with the action for an interacting, massive scalar field living in some manifold $\mathscr{M}$

$$S[\phi] = \frac{1}{2}\int_{\mathscr{M}} d\mu(\mathscr{M})\Big(D_\mu\phi D^\mu\phi + m^2\phi^2\Big) + S_{int}[\phi] \,, \tag{1.69}$$

where $d\mu(\mathscr{M})$ is the invariant volume measure on the manifold, D^μ is the covariant derivative and S_{int} is the part of the action that contains all interaction terms and assuming that x and y are coordinates on $\mathscr{M}$ and n^μ is the normal vector to the boundary $\partial\mathscr{M}$. Making the variation of the action we get the equation of motion

$$\Big(-D^\mu D_\mu + m^2\Big)\phi(x) = \frac{\delta S_{int}}{\delta\phi(x)} \,. \tag{1.70}$$

Defining the Green's function as the solution to the equation (1.61) with the Dirichlet boundary conditions, we must have that

$$G(x-x')D_\mu\phi(x')\Big|_{x'\in\partial\mathscr{M}} = 0 \quad \longrightarrow \quad G(\mathscr{M})|_{\partial\mathscr{M}} = 0 \,, \tag{1.71}$$

we can write the solution in the general form

$$\phi(x) = \int_{\partial\mathscr{M}} dy' \frac{\partial}{\partial y'^\mu} G(x,y')\Big|_{y'\in\partial\mathscr{M}} n^\mu(y')\phi(y') + \int_{\mathscr{M}} dy\, G(x,y)\frac{\delta S_{int}}{\delta\phi(y)} \,, \tag{1.72}$$

here, S_{int} is to be understood as pertubative series in ϕ_0 and depends on the boundary conditions. From equation (1.43) we have two linearly independent solutions which satisfy the homogeneous equation and we must take into account the time ordering to avoid any divergence. Now we can construct (ansatz) the Green's function

$$G_0(X,X') = \int \frac{d^d k}{(2\pi)^d}(zz')^{d/2} e^{-ik^\mu(x_\mu - x'_\mu)} \Big(\theta(z-z')K_\nu(kz)I_\nu(kz') + \theta(z'-z)I_\nu(kz)K_\nu(kz')\Big) \ . \tag{1.73}$$

Defining

$$\xi = \frac{2zz'}{z^2+(z')^2+(x-x')^2} \ , \tag{1.74}$$

one can represent (1.73) in terms of the hypergeometric function:

$$G_0(X,X') = \frac{2C_{\Delta_+}}{\nu}\left(\frac{\xi}{2}\right)^{\Delta_+} {}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2}+\frac{1}{2};\nu+1;\xi^2\right) \ , \tag{1.75}$$

with C_{Δ_+} to be defined later.

To incorporate the boundary condition at $z=\varepsilon$ we can add to (1.74) a solution to the homogeneous equation that satisfies $\lim_{\varepsilon\to 0} G_\varepsilon = G_0$

$$G_\varepsilon(X;X') = G_0(X;X') + \int \frac{d^d k}{(2\pi)^d}(zz')^{d/2} e^{ik^\mu(x_\mu - x'_\mu)} K_\nu(zk)K_\nu(kz')\frac{I_\nu(k\varepsilon)}{K_\nu(k\varepsilon)} \ , \tag{1.76}$$

it's not necessary to perform the integration of (1.80), taking the derivative of G_ε on the boundary

$$n^\mu \partial_\mu G_\varepsilon(X;X')\Big|_{z'=\varepsilon} = \frac{z^{d/2}\varepsilon^{\Delta_+}2^{1-\nu}}{\Gamma(\nu)}\int \frac{d^d k}{(2\pi)^d} e^{ik^\mu(x_\mu - x'_\mu)} K_\nu(kz)K_\nu(k\varepsilon) \ , \tag{1.77}$$

noting that

$$d\mu(\partial\mathscr{M})\Big|_{z=\varepsilon} = \varepsilon^{-d} d^d X, \qquad n^\mu = (-\varepsilon, \mathbf{0}) \ . \tag{1.78}$$

Subistituting the expression (1.78) in the equation (1.72) and considering $S_{int}=0$ for a free field, we recover the equation (1.60) as expected.

1.5.2 Boundary-to-bulk propagator

When the point source is located at the boundary we must use the boundary to bulk propagator in this situation with the following property for $z \to 0$

$$K(z,x;x') \longrightarrow z^{\Delta_-}\delta^d(x-x') \ . \tag{1.79}$$

In general the solution is given by the convolution of the propagator with a source

$$\phi(z,x) = \int d^dx' K(z,x;x')\phi_0(x') \ . \tag{1.80}$$

In order to find an expression depending on $G(X;X')$ we must take $\phi(x')$ to be located on the boundary and use the equations (1.80) and (1.68) with the homogeneous field satisfying $\phi_h(X')\Big|_{\partial\mathscr{M}} = z^{\Delta_-}\phi_0(x')$

$$\int d^dx' K(z,x;x')\phi_0(x') = \int_{\partial\mathscr{M}} d^dx' \left(\frac{\ell^d}{z'^d}\right)\left(\frac{z}{\ell}\right)\phi_h(x')\partial_{z'}G(z,x;z'x') \ , \tag{1.81}$$

where ℓ comes from the metric. (1.85) implies that

$$K(z,x;x') = \ell^{d-1}\lim_{z'\to 0}(z')^{\Delta_+ +1}\partial_{z'}G(z,x;z',x') \ . \tag{1.82}$$

In order to get an expression without derivatives we use the Green's theorem for $G(X'';X)$ and $K(X';X'')$ and introducing a new notation $\square = (-\partial^2 + m^2)$

$$\int d^{d+1}X''\sqrt{g}\Big(G\square K - K\square G\Big) = -\int_\partial d^dx\sqrt{\gamma}\Big(Gn\cdot\partial K - Kn\cdot\partial G\Big)\Big|_{z''=\varepsilon} \tag{1.83}$$

Clearly, by definition

$$\square K(z,x;x') = 0 \tag{1.84}$$

The left hand side of the equation gives

$$-\int d^{d+1}X''\sqrt{g}K(X'';X')\Box G(X'';X) = \int d^{d+1}X''\delta(X-X'')K(X'';X') \\ = -K(X;X') \tag{1.85}$$

with this, we have

$$z\partial_z G(X;X') = \Delta_+ G(X;X') \tag{1.86}$$

Solving the right hand side

$$\begin{aligned} -K(X;X') &= -\int_\partial d^dx''\sqrt{\gamma}\Big(Gn^{z''}\cdot\partial_{z''}K - Kn^{z''}\cdot\partial_{z''}G\Big)\Big|_{z''=\varepsilon} \\ &= -\ell^{d-1}\int_\partial d^dx''z^{-d}\sqrt{\gamma}\Big(Gz''\partial_{z''}K - Kz''\partial_{z''}G\Big)\Big|_{z''=\varepsilon} \\ &= -\ell^{d-1}\int_\partial d^dx''(z'')^{-d}\Big((\Delta_- - \Delta_+)G(X';X'')z^{\Delta_-}\delta^d(x'-x'')\Big)\Big|_{z''=\varepsilon}\ . \end{aligned} \tag{1.87}$$

equalling both sides and using

$$_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2}+\frac{1}{2};\nu+1;0\right) = 1\ . \tag{1.88}$$

We finally get

$$K(z,x;x') = \lim_{z'=\to 0}\frac{2\nu}{(z')^{\Delta_+}}G(z,x;z'x') = C_{\Delta_+}\left(\frac{z}{z^2+(x-x')^d}\right)^{\Delta_+}\ . \tag{1.89}$$

1.5.3 The Witten's approach

First, we will derive the coefficient C_Δ.If we compactify $\mathbb{R}^d$ to $\mathbb{S}^d$ by adding the point $z=\infty$ the whole x space shrinks to a point and the Green's function becomes x-independent

$$\lim_{z\to\infty}\gamma_{\mu\nu} = 0\ . \tag{1.90}$$

Taking the Green's equation in the neighbourhood of $z=\infty$

$$\Big(z^{d+1} - \partial_z(z^{-d+1}\partial_z) + m^2\ell^2\Big)K(z) = 0 \ . \tag{1.91}$$

We have seen before that the solutions for this equation are power-law

$$K(z) = C_\Delta z^\Delta \quad \longrightarrow \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \ . \tag{1.92}$$

By the third expression in (1.91) we must take the bigger root. In order to map the point $z = \infty$ to finite $x = 0$ we take two AdS isometries, an inversion followed by a translation. Let's do first the inversion

$$X'^A = \frac{X^A}{X^B X_B} = \left\{ \frac{z}{z^2 + x^2} \ , \ \frac{x_\mu}{z^2 + x^2} \right\} \ , \tag{1.93}$$

where $x^2 = \eta^{\mu\nu} x_\mu x_\nu$. Now we follow with a translation $x \to x - x'$

$$X'^A = \left\{ \frac{z}{z^2 + (x - x')^2} \ , \ \frac{x_\mu - x'_\mu}{z^2 + (x - x')^2} \right\} \ , \tag{1.94}$$

since it is an isometry of AdS it is still a solution in these coordinates for (1.95), and clearly when $z = \infty \longrightarrow X^A = (z, 0)$. Note that the limit of K on the boundary $z = 0$ is

$$K(z, x; x') = \begin{cases} C_{\Delta_+} z^{\Delta_+} \to 0, & \text{if } x \neq x' \\ C_{\Delta_+} z^{-\Delta_+} \to \infty, & \text{if } x = x' \end{cases} . \tag{1.95}$$

In order to check that $K(z)$ has a finite measure in the neighbourhood of $x = x'$ we compute the integral

$$\begin{aligned} \int d^d x K(z, x) &= C_{\Delta_+} z^{\Delta_+} \int d^d x \frac{1}{(z^2 + x^2)^{\Delta_+}} \\ d\mu(\mathbb{R}^d) \to d\mu(\mathbb{S}^d) &\to C_{\Delta_+} z^{\Delta_+} \Omega_{d-1} \int_0^\infty dr \frac{r^{d-1}}{(z^2 + r^2)^{\Delta_+}} \\ \text{Taking } y = r/z &\to C_{\Delta_+} \Omega_{d-1} \frac{z^{\Delta_+ + d}}{z^{2\Delta_+}} \int_0^\infty dy \frac{y^{d-1}}{(1 + y^2)^{\Delta_+}} \\ \text{Taking } x = y^2 &\to \frac{1}{2} C_{\Delta_+} \Omega_{d-1} z^{d - \Delta_+} \int_0^\infty dx \frac{x^{d/2 - 1}}{(1 + x)^{\Delta_+}} \end{aligned} . \tag{1.96}$$

To evaluate the last integral, we just use the definition of the Beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^\infty da \frac{a^x}{(1 + a)^{x + y}} \ , \tag{1.97}$$

and the formula for the surface of n-sphere

$$\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \ , \tag{1.98}$$

to get that

$$\int d^d x K(z,x) = \frac{1}{2} C_{\Delta_+} \frac{2\pi^{d/2}}{\Gamma(d/2)} z^{d/2-\nu} \frac{\Gamma(d/2)\Gamma(\Delta_+ - d/2)}{\Gamma(\Delta_+)} \ , \tag{1.99}$$

remembering that $\Delta_- = d/2 - \nu$ and defining

$$C_{\Delta_+} = \frac{\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\nu)} \ . \tag{1.100}$$

Now, without loss of generality, consider a free scalar field ϕ on AdS_2

$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}_+} dz \ dt \left\{ (\partial_z \phi)^2 + (\partial_t \phi)^2 + \left(\frac{m\ell}{z}\right)^2 \phi^2 \right\} \ , \tag{1.101}$$

and with the e.o.m

$$\partial_t^2 \phi^2 + \partial_z^2 \phi^2 = \left(\frac{m\ell}{z}\right)^2 \phi^2 \ . \tag{1.102}$$

If we set new coordinates from $X^A = (z,t)$ as (1.93) it is clear that the e.o.m will have the same form in terms of $X'^A = \frac{X^A}{X^B X_B}$ because

$$\frac{dX'^A dX'_A}{X'^2} = \frac{dX^A dX_A}{X^2} \ . \tag{1.103}$$

Let us now look for t'-independent solution, $K_\Delta(z') = C_\Delta z'^\Delta$, applying the Klein-Gordon operator to it gives

$$\Delta(\Delta-1) - (m\ell)^2 = 0 \to \Delta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (m\ell)^2} \ . \tag{1.104}$$

Then, the independent solution is of the form

$$K_\Delta(z,t) = C_\Delta \left(\frac{z}{z^2+t^2}\right)^\Delta \ . \tag{1.105}$$

Let ϕ_0 be a field on the boundary: $\phi_0 = \phi_0(t)$, then

$$\phi(z,t) := \int_{\mathbb{R}} K_{\Delta}(z,t-t')\phi_0(t')dt' = \int_{\mathbb{R}} \left(\frac{1}{1+\left(\frac{t-t'}{z}\right)^2} \right) \phi_0(t')dt' \ , \tag{1.106}$$

is a solution. Changing the variables $x = \frac{t-t'}{z}$ and requiring that $z \to +0$ implies that $\phi \to z^{1-\Delta}\phi_0$ and we get to the following

$$\phi(z,t) = z^{1-\Delta} C_{\Delta} \int_{\mathbb{R}} \frac{1}{1+x^2} \phi_0(xz+t) \ . \tag{1.107}$$

which gives the solution (1.99) and taking the cutoff clearly $\phi(z,t)|_{z=\varepsilon} = \varepsilon^{1-\Delta}\phi_0(t)$. Because $\Delta_+ > \Delta_-$, the dominating solution corresponds to $\Delta = \Delta_+$ and

$$K_{\Delta_+}(z,t) := K(z,t) = \frac{\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma\left(\Delta_+ - \frac{1}{2}\right)} \left(\frac{z}{z^2+t^2} \right)^{\Delta_+} \ . \tag{1.108}$$

With this last expression we constructed both propagators that we will need in the next sections.

Chapter 2

Basics of Conformal Field Theory

In this chapter we expose the basics of CFT in a standard way following some well established material of [32], [33] and [28]. The main objective here is to show the motivation behind the studies of the correlators and OPE's when we try to quantise some general background spacetime. The CFT's are the other side of the AdS/CFT correspondence, we will start discussing the 'axiomatic' point of view from an AdS viewpoint, stating the main ideas behind their relations.

- The conformal transformations consist of the Poincaré group plus scale transformations and special conformal transformations(SCT). One can derive the aditional symmetries by demanding Poincaré invariance plus an additional symmetry under inversions i.e. $x^i \to x^i/x^2$. This coordinate transformations leave the metric locally invariant rescaling it by an overall spacetime-dependent factor i.e. $g^{\mu\nu} \to \Lambda(x)g^{\mu\nu}$.

- The quantization proccess is done in the standard way along flat spacelike surfaces that evolve with time. In the Euclidean space one can introduce an alternative quantization scheme, the radial quantization. In this quantization one starts at a point and evolve outward on expanding speres.

- Using conformal symmetry we can classify states according to irreducible representations of the conformal group Conf($\mathbb{R}^{1,d+1}$) $\simeq SO(1,d+1)$ in Euclidean space and describe every state as a linear combination of primary states.

- The conformal symmetry also allow us to move operators, so that from $\mathscr{O}(0)$ at the origin we obtain $\mathscr{O}(x)$ at any point x. The correlation function for any pair or trio of operators is fixed by symmetry, up to a finite set of constants.

- We can multiply any two operators in order to obtain a new one. The operator state correspondence applied to a product of operators leads us to the operator product expansion (OPE), which has a finite radius of convergence in any CFT. This can be made explicitly if there is a path integral description of the theory, but it is not necessary.

- Local conserved currents are extremely important, since they generate global symmetries. Conventionally a theory is only defined to be a CFT if there exist a spin-2 conserved current $T^{\mu\nu}$, if that is not the case the theory is called non-local.

- The special feature of CFT, for $d > 2$, is the existence of an infinite number of independent symmetries of the system, leading to corresponding invariants of motion, reducing drastically the number of degrees of freedom from a classical point of view.

2.1 Basics of CFT in *d* Dimensions

2.1.1 Conformal transformations

Let $g^{\mu\nu}$ be a metric tensor on some d-dimensional manifold with respect to some set of coordinates x^{μ}. We define a conformal transformation as follows

$$g'^{\mu\nu}(x) = \Lambda(x) g^{\mu\nu}(x) \ . \tag{2.1}$$

For flat spaces the scale factor $\Lambda(x) = 1$ corresponding to the Poincaré group of translations, rotations and boosts in Minkowski space.

2.1.2 Conformal group

For $d \geq 3$ all the transformations satisfying (2.1) and the respective generators, are given by

- Translations: $x'^{\mu} = x^{\mu} + a^{\mu}$, Generator: $P_{\mu} = -i\partial_{\mu}$,
- Dilatation: $x'^{\mu} = \lambda x^{\mu}$, Generator: $D = -ix^{\mu}\partial_{\mu}$,
- Rotation: $x'^{\mu} = M^{\mu}_{\nu} x^{\nu}$, Generator: $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
- SCT: $x'^{\mu} = \frac{x^{\mu} - x^2 b^{\mu}}{1 - 2b^{\mu}x_{\mu} + b^2 x^2}$, Generator: $K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$.

The special conformal transformation is to be understood as an inversion followed by a translation and followed by other inversion at the end. We observe that the SCT as defined above is not globally defined. In order to avoid it, one considers a conformal compactification that adds the infity as a point of the manifold. Defining new generators

$$\begin{aligned} J_{\mu,\nu} &= L_{\mu\nu} \ , \qquad J_{-1,\mu} = \tfrac{1}{2}(P_{\mu} - K_{\mu}) \\ J_{-1,0} &= D \ , \qquad J_{0,\mu} = \tfrac{1}{2}(P_{\mu} + K_{\mu}) \end{aligned} \ , \tag{2.2}$$

which satisfiy

$$[J_{mn}, J_{rs}] = i\Big(\eta_{nr}J_{ms} - \eta_{mr}J_{ns} + \eta_{ms}J_{nr} - \eta_{ns}J_{mr}\Big) \ , \tag{2.3}$$

we obtain that in the case of dimensions $p+q \geq 3$ the conformal group is isomorphic to $SO(p+1, q+1)$. In the $d = 2$ case the conformal group is the set of all orientation preserving global conformal transformations ,Möbius group $SL(2,\mathbb{C})/\mathbb{Z}_2 = PSL(2,\mathbb{C})$, that forms a group with respect to composition and satisfies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \;, \tag{2.4}$$

such that the action on the complex plane is given $\forall\ \phi \in SL(2\mathbb{C})$ and $\forall\ z \in \mathbb{C}$ by $\phi(z) = \frac{az+b}{cz+d}$ with $cz + d \neq 0$. If one looks at w the conformal Killing fields, i.e. the vector fields whose flows define conformal transformations preserving $g^{\mu\nu}$ and the conformal structure. In this context, the Witt algebra $\mathscr{W}$ is the complex vector space with basis $\{l_n\}_{n\in\mathbb{Z}}, \quad l_n := -z^{n+1}\partial_z$ or $l_n := z^{1-n}\partial_z$ with the respective $\bar{z}$ copy and the Lie bracket

$$[l_n, l_m] = (n-m)l_{n+m} \;. \tag{2.5}$$

We will effectively treat z and $\bar{z}$ as independent variables on $\mathbb{C}^2$. The globally defined conformal transformations on the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$ are generated by $\{l_{-1}, l_0, l_{-1}\}$. The global generators will be associated to the conformal group, for $z = re^{i\theta}$

- Generator of translations: $l_{-1} = -\partial_z$
- Generator of dilatations: $l_0 + \bar{l}_0 = r\partial_r$
- Generator of rotations: $i(l_0 - \bar{l}_0) = -\partial_\theta$
- Generator of Special conformal transformations: $l_1 = \partial_{\frac{1}{z}} = -z^2\partial_z$.

The central extension of the Witt algebra is called Virasoro algebra with the central charge c. It satisfies the following commutation relation

$$[l_n, l_m] = (n-m)l_{m+n} + \frac{c}{12}(n^3 - n)\delta_{(m+n),0} \;. \tag{2.6}$$

2.2 Chiral and Primary fields

Fields depending only on z are called *chiral fields* and fields depending only on $\bar{z}$ are called *anti-chiral* fields. Let $\phi(z,\bar{z})$ be a field that transforms $z \to \lambda z$ as

$$\phi(z,\bar{z}) \quad \longrightarrow \quad \phi'(z,\bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}) \;. \tag{2.7}$$

Then it is said to have *conformal dimensions* $(h,\bar{h})$. If a field transforms under conformal transformations $z \longrightarrow f(z)$ as

$$\phi(z,\bar{z}) \quad \longrightarrow \quad \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \;, \tag{2.8}$$

it is called a *primary field* of concormal dimension $(h,\bar{h})$. If the the equation (2.8) holds only for $f(z) \in SL(2,\mathbb{C})/\mathbb{Z}_2$, then ϕ is called a *quasi-primary field*. If a field is not a primary or quasi-primary field, then we call it a *secondary field*.

2.3 Energy-momentum tensor, radial quantisation and the OPE

The energy-momentum tensor $T^{\mu\nu}$ can be deduced from the variation of the action with respect to the metric, encoding this way the behaviour of the theory under infinitesimal transformations. To make all this clearer, let us take a free massless scalar field and make a variation $\phi(x) \longrightarrow \phi(z+\varepsilon) \sim \phi(x) + \varepsilon_\mu(x)\partial^\mu\phi(x)$ the Lagrangian transforms as

$$\delta L = \partial_\mu\phi(x)\partial^\mu(\varepsilon(x)_\alpha\partial^\alpha\phi(x)) = \partial_\mu\varepsilon_\alpha\left(\partial^\mu\phi\partial^\alpha\phi - \frac{1}{2}\eta^{\mu\alpha}(\partial\phi)^2\right)\partial_\mu\varepsilon_\alpha T^{\mu\alpha}\ . \tag{2.9}$$

When we take $\varepsilon_\alpha = \lambda x_\alpha$ (dilatation), for example, we find that $\delta L \propto T^\mu_\mu$. In general, the energy-momentum tensor is traceless for $d=2$ dimensions. For a theory to be a CFT it must contain in the spectrum of operators a spin 2 tensor with conformal dimension $\Delta = d$, the space-time dimension, or equivalently, an energy-momentum tensor that is conserved, manifesting the conformal symmetries as local space-time symmetries.

The process of radial quantisation is related to the choice of a foliation of space-time with fixed time surfaces. In this formalism, the time evolution operator connects states in different surfaces. Each leaf of this folation has its own Hilbert space. If we take $w = \ln(z)$ with $w = x_0 + ix_1$ relating x_0 to the Euclidean time, one can map an infinite cylinder to the entire complex plane. Clearly in this context, the generator of time translations is related to the dilation operator and the generator of space translations is the momentum operator corresponding to rotations on the complex plane. In order to define the *in-out states*, we must expand an arbitrary field and take $x_0 \to \pm\infty$ and promote its Laurent modes to operators

$$\phi(z,\bar{z}) = \sum_{n,m\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{h}-m}\phi_{n,m}\ , \tag{2.10}$$

taking $x_0 \to -\infty$ on the infinite past we get z and $\bar{z}$ mapped to zero. With it we define the *in-state*

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle, \quad \text{with} \quad \phi_{n,m}|0\rangle = 0, \quad \text{for} \quad n > -h \quad \text{and} \quad m > -\bar{h}\ , \tag{2.11}$$

considering that, by definition, the hermitian conjugate of ϕ that corresponds to $z \to 1/\bar{z}$, is given by

$$\phi^\dagger(z,\bar{z}) = z^{-2\bar{h}}\bar{z}^{-2h}\phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right) \quad \text{with} \quad (\phi_{n,m})^\dagger = \phi_{-n,-m}\ . \tag{2.12}$$

With this one can make a simmilar approach for the *out-state*

$$\langle\phi_{out}| = \langle 0|\lim_{z,\bar{z}\to 0}\phi^{\dagger}(z,\bar{z}), \quad \text{with} \quad \langle 0|\phi_{-n,-m} = 0, \quad \text{for} \quad n < -h \quad \text{and} \quad m < \bar{h}\,. \tag{2.13}$$

Since all momenta commute one can simultaneously diagonalize D, labeling its eigenstates by Δ and we get

$$D|\Delta\rangle = i\Delta|\Delta\rangle\;, \tag{2.14}$$

while P_μ will be the raising and K_μ the lowering operator with respect to the eigenvalues of D. One can define as the vacuum $|0\rangle$ the state killed the conformal generators, associated to each primary state (states anihilated by K^μ), there is a primary operator $O(0)$ and with it we get a discrete spectrum, with the unitary time evolution operator obtained by taking $e^{iD\tau}$ with $\tau = e^r$. Associated to each primary state is a primary operator $\mathscr{O}(0)$ with a scaling parameter τ and angular momentum ℓ. But, what do we mean by a state? We defined it as $|\phi_{in}\rangle$ state if its foliation is defined for a past time. For the converse we defined it as $|\phi_{out}\rangle$ state, in which all foliations are constructed for times on the future. The overlap of *in* and *out* states is defined by the two point function $\langle\phi_{in}|\phi_{out}\rangle$. In a scale invariant theory we can make the *radial quantization* by taking concentric foliations of the spacetime varying only the radius, as we showed in the example explained above.

When the *in* and *out* states live on different foliations there exists some unitary operator connecting them. The associated correlating functions is easily defined as $\langle\phi_{in}|\mathscr{O}|\phi_{out}\rangle$. For theories with Poincaré invariance, the states on the foliations are defined by their 4-momenta and the Hamiltonian moves us between the foliations of different radii. So considering a state that satisfies

$$\mathscr{O}(0)|0\rangle = |\tau,\ell\rangle = |\Delta\rangle\;, \tag{2.15}$$

this gives us the local state-operator correspondence (isomorphism). Note that if we analyse the path integral formalism by defining the states as wave functionals on Cauchy surfaces, the path integral allows us to evolve from one Cauchy surface to another. With it, the radial quantisation is immediate and the state-operator correspondence follows if we evolve back in time towards a point, where the operator is obtained as the insertion of a functional of fields directly in the path integral. Let's take the Hilbert space spanned by the eigenfunctions $|\phi_{\partial R}\rangle$ on the Cauchy surface. One can take the path integral over the region R

$$\langle 0|\phi_{\partial R}\rangle = \int_{\phi\in R} D\phi(r)e^{S[\phi]}\;. \tag{2.16}$$

If we insert a unitary, primary operator inside this region we can prepare a diferent states on the boundary ∂R. Clearly we have obtained the state $|\Delta\rangle = \mathscr{O}_\Delta(0)|0\rangle$ with scaling dimension Δ, this state at the origin defines the absolute past on our foliation. One can finaly construct the operators A^- and A^+ that act as ladder operators for the scaling dimensions. If we insert an operator somewhere other than origin we should have:

$$|\psi\rangle\; = e^{-\Delta A^+(x)}\mathscr{O}_\Delta(0)e^{\Delta A^+(x)}|0\rangle\; = \mathscr{O}_\Delta(x)|0\rangle\;,$$

as we have on the quantum oscillator, once we hit the vacuum state, operating the descendant operator will give us the null eigenvalue. This is the *state operator correspondence* , in which every state has a $1-2-1$ correspondence to an unitary operator. With the assumption that

$$[P_\mu, \mathscr{O}(x)] = -i\partial_\mu \mathscr{O}(x) , \tag{2.17}$$

one entirely determines the action of the conformal algebra on the local operator $\mathscr{O}(x)$, assuming the commutation relations of the conformal algebra. At a general point in flat Euclidean space, we have

$$\begin{aligned}
[D, \mathscr{O}(x)] &= -i(\Delta + x^\mu \partial_\mu)\mathscr{O}(x) \\
[P_\mu, \mathscr{O}(x)] &= -i\partial_\mu \mathscr{O}(x) \\
[K_\mu, \mathscr{O}(x)] &= -i(2x_\mu \Delta + 2x^\nu \Sigma_{\nu\mu} + 2x^\nu x_\mu \partial_\nu - x^2 \partial_\mu)\mathscr{O}(x) \\
[L_{\mu\nu}, \mathscr{O}(x)] &= -i(\Sigma_{\mu\nu} + x_\mu \partial_\nu - x_\nu \partial_\mu)\mathscr{O}(x) ,
\end{aligned} \tag{2.18}$$

where $\Sigma_{\mu\nu}$ is a finite dimensional spin matrix of the angular momentum representation of $\mathscr{O}(x)$. A key property of all local quantum field theories is the existence of the Operator Product Expansion (OPE), for CFTs this expansion converges in a finite region, so it can be used to make exact statements. The OPE says that for any complete set of local field operators we have

$$\phi_1(x)\phi_2(0) = \sum_{\mathscr{O}} C(x,\partial)\mathscr{O}(0) , \tag{2.19}$$

in the CFT case one can derive it without any explicit action. Consider the operators $\mathscr{O}_1(x_1)$ and $\mathscr{O}_2(x_2)$ where there is some circle of radius r located at some point x in which both x_1 and x_2 are inside this circle and there aren't any other operators defined in this region. Let us imagine the radial evolution from x outwards. We start with the vacuum and as we evolve this state, it eventually hit the operators $\mathscr{O}_1$ and $\mathscr{O}_2$. It means that there exists some state

$$|\phi_{12}(r)\rangle = \mathscr{O}_1(x_1)\mathscr{O}_2(x_2)|0\rangle , \tag{2.20}$$

which will be some linear combination of all states in the Hilbert space and by the state-operator correspondence there exists some operator such that

$$\mathscr{O}_{12}(x)|0\rangle = |\phi_{12}(r)\rangle , \tag{2.21}$$

with it one can express $\mathscr{O}_{12}$ as a sum over all primary operators of the theory and their descendants

$$\mathscr{O}_1(x_1)\mathscr{O}_2(x_2) = \sum_{\Delta,\ell} \lambda_{\Delta,\ell} C_{\Delta,\ell}(x_1 - x, x_2 - x, \partial_x)\mathscr{O}_{\Delta,\ell}(x) . \tag{2.22}$$

This function is entirely determined by conformal symmetry and the only non-trivial information in the OPE is the value of the coefficients, which are labelled by the spin and conformal dimension of the primary operators.

2.4 Two- and Three-Point Functions

Employing the global conformal symmetry we can derive both of the correlators. Lets start with the two-point function of two chiral quasi-primary fields

$$\langle\phi_1(z)\phi_2(w)\rangle = f(z,w)\ . \tag{2.23}$$

The invariance under translations implies that $f(z,w) = f(z-w)$ and the invariance under rescalings implies that

$$f(z-w) = \lambda^{h_1+h_2} f(\lambda(z-w))\ . \tag{2.24}$$

In addition to these properties, the two-point function must be invariant under inversions

$$f(z-w) = \lambda^{-h_1-h_2} f(\frac{1}{z}-\frac{1}{w})\ . \tag{2.25}$$

Clearly (2.25) and (2.24) can only be satisfied simultaneously if $h_1 = h_2$. From these three assumptions one can make the ansatz, for an structure constant c_{12}

$$f(z-w) = \frac{c_{12}}{(z-w)^{2h}}\ , \tag{2.26}$$

which gives in general

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{c_{ij}\delta_{h_i,h_j}}{(z-w)^{2h}}\ . \tag{2.27}$$

For the three-point function imposing the same requirements one find that

$$\langle\phi_1(z)\phi_2(w)\phi_3(t)\rangle = \frac{c_{123}}{(z-w)^{h_1+h_2-h_3}(w-t)^{h_2+h_3-h_1}(t-z)^{h_3+h_1-h_2}}\ . \tag{2.28}$$

Since the results for the correlators have been derived using only the global conformal symmetries one can extend these results to $d > 2$ dimensions, but by analogous reasoning, they have the same form as for $d = 2$. In order to have a single-valued two-point function on the complex plane, the dimension of a chiral quasi-primary field must be a integer or a half-integer.

2.5 The d=1 case revisited

The most studied case of the correspondence is the $d = 2$, i.e. AdS_3/CFT_2, this is due to the fact that the conformal symmetry in two dimensions is infinite dimensional and highly constraints the dynamics of the fields, as consequence, this is very well studied CFT. For the $d = 1$ we have a special feature. Only in this case the Lorentzian and the Euclidean AdS are defined by the same group of symmetries, and, in first place, one can naively thinks that the case for d=1 is easier than the others, but there are some subtleties concerning the boundaries and the vacuum of the theory that makes this case the most elusive and the least understood. There is a realization of CFT_1, known as the de Alfaro-Fubini-Furlan model (dAFF), which is actually conformal quantum mechanics. We will briefly recall some aspects of this model

2.5.1 de Alfaro-Fubini-Furlan model

The simplest massless dilatation-invariant Lagrangian for a scalar field ϕ has the general form

$$\mathscr{L} = \frac{1}{2}(\partial^{\mu}\phi)(\partial_{\mu}\phi) - \frac{1}{2}g\phi^{2d/(d-2)} , \tag{2.29}$$

where d is the number of space-time dimensions. Our main interest is for $d = 1$, which corresponds to a single physical operator depending only upon time t, so setting the dimension in (2.29) the Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}\left(\left(\frac{dQ}{dt}\right)^2 - \frac{g}{Q^2}\right) , \tag{2.30}$$

the respective equation of motion is

$$\frac{d^2Q}{dt^2} = \frac{g}{Q} . \tag{2.31}$$

Considering $g > 0$ without loss of generality, and noting that the coupling constant is dimensionless implies that the action S has invariance properties larger than just time translation. In fact, the set of transformations that leaves the action invariant is $SL(2,\mathbb{R})$ such that

$$t' = \frac{at+b}{ct+d} := \omega[t] , \tag{2.32}$$

note that

$$\frac{d\omega}{dt} = \frac{ad-bc}{(ct+d)^2} = (ct+d)^{-2} , \tag{2.33}$$

and hence, the transformation properties of Q are

$$Q'(t') = (ct+d)^{-1}Q(t) \ . \tag{2.34}$$

Suppose that there exists a state vector $|\psi\rangle$, the action of the symmetries of the action on these states are the symmetries of $SL(2,\mathbb{R})$, for the time translation symmetry

$$H|\psi\rangle = i\frac{d}{dt}|\psi\rangle \ , \tag{2.35}$$

for dilations

$$D|\psi\rangle = it\frac{d}{dt}|\psi\rangle \ , \tag{2.36}$$

and for special conformal transformations

$$K|\psi\rangle = it^2\frac{d}{dt}|\psi\rangle \ , \tag{2.37}$$

whose elements of the algebra satisfies the respective commutation relations

$$[H,D] = iH \ , \quad [K,D] = -iK \ , \quad [H,K] = 2iD \ . \tag{2.38}$$

We can straightforwardly get the explicit symmetrized expressions of the generators H,K and D in terms of the field operators $Q(t)$ and $\dot{Q}(t)$

$$\begin{cases} H = \frac{1}{2}\left(\left(\frac{dQ}{dt}\right)^2 + \frac{g}{Q^2}\right) \\ D = tH - \frac{1}{4}\left(Q\dot{Q}+\dot{Q}Q\right) \\ K = t^2H - \frac{t}{2}\left(Q\dot{Q}+\dot{Q}Q\right) + \frac{1}{2}Q^2 \ . \end{cases} \tag{2.39}$$

Writing these operators in the Cartan basis, we have

$$\begin{cases} R := \frac{1}{2}\left(\frac{K}{a} + aH\right) \\ L_\pm := \frac{1}{2}\left(\frac{K}{a} - aH\right) \pm iD \ , \end{cases} \tag{2.40}$$

satisfying

$$[R, L_\pm] = \pm L_\pm \ , \qquad [L_-, L_+] = 2R \ . \tag{2.41}$$

2.5.2 The group SU(1,1)

The algebra (2.41) is related to the group SO(2,1), which is locally isomorphic to SU(1,1), we will briefly recall some properties of this group. The group SU(1,1) is locally isomorphic to SO(2,1), being more precise, SO(2,1) = SU(1,1)/C_2, where C_2 is the cyclic group containing only two elements. In addition, SU(1,1) is locally isomorphic to Sp(2,$\mathbb{R}$). Unless SU(2), SU(1,1) is noncompact and it is not simply connected, having the topological structure of the direct product between the planar disk bounded with the unit circle and the sphere S^1. This space is the upper sheet of the two-sheet hyperboloid embedded on the three-dimensional with the pseudo-Euclidean metric tensor $\eta^{\mu\nu}$, one can still obtain this through the stereographical projection.

For example, we can foliate the complex plane into three orbits via stereographical projection, i.e., the interior of the unit circle, the circle itself and the exterior region of the circle. One can identify the group G_+, with $\gamma \in \mathbb{C}, \delta \in \mathbb{R}$ defined by:

$$g = \begin{bmatrix} \delta & \gamma \\ \bar{\gamma} & \delta \end{bmatrix} , \qquad \delta^2 - |\gamma|^2 = 1 , \tag{2.42}$$

setting:

$$\delta = \cosh\left(\frac{\theta}{2}\right) , \qquad \gamma = \sinh\left(\frac{\theta}{2}\right) e^{-i\phi} , \qquad \theta > 0 .$$

We see that G_+ is isomorphic to $\mathbb{H} = \{(\delta, \gamma_1, \gamma_2) \quad | \quad \delta^2 - \gamma_1^2 - \gamma_2^2 = 1\}$, the set of unit vectors of the form:

$$m = (\cosh(\theta), \sinh(\theta)\cos(\phi), \sinh(\theta)\sin(\phi)) \qquad , \tag{2.43}$$

finally, one can write the elements of G_+ as:

$$g_+ = exp\left(\frac{\theta}{2} u^\mu \sigma^\nu \delta_{\mu\nu}\right) , \tag{2.44}$$

where $u = (\sin(\phi), -\cos(\phi))$ and σ^ν are the Pauli matrices. The matrix g_+ describes a hyperbolic rotation around the vector m with the rotation angle being θ. One can define an isomorphism between the unit circle and the upper sheet of the hyperboloid just by setting:

$$\eta = \tanh\left(\frac{\theta}{2}\right) e^{-i\phi} .$$

It's well known that there are three different types of nontrivial unitary irreducible representations of SU(1,1), as is shown in [2], we have: the supplemental, principal and discrete series representations. Following [1], the Lorentzian AdS_2 is obtained by the principal series representation, and by the physical constraints assigned to ℓ, we shall not consider the supplemental case. For the discrete series representations we should recover the Euclidean AdS_2. From now on, we will be considering the representation $D^+(\varepsilon_0)$. Generally, we label the representation by two parameters ε_0 and m, taking the the Hilbert space spanned by the states $|\varepsilon_0, m\rangle$, where ε_0 is the eigenvalue of the lowest state, the vacuum of R with respect to the raising and lowering operators, $L_\pm$. Associating an integer m to the raising and lowering operators one can calculate the spectrum of R and the action of $L_\pm$ in these states. We begin with

$$\begin{cases} [L_-, L_+] = 2R \,, \\ L_+|\varepsilon_0, m\rangle = c_m|\varepsilon_0, m+1\rangle \,, \\ L_-|\varepsilon_0, m\rangle = c_{m-1}|\varepsilon_0, m-1\rangle \,, \\ R|\varepsilon_0, m\rangle = (\varepsilon_0 + m)|\varepsilon_0, m\rangle \,, \\ \mathscr{C}|\varepsilon_0, m\rangle = \varepsilon_0(\varepsilon_0 - 1)|\varepsilon_0, m\rangle \,, \end{cases} \tag{2.45}$$

where $\mathscr{C}$ stands for the Casimir element, let's use the Casimir element, defined as $C = \frac{1}{2}(HK + KH) - D^2 = R^2 - L_+L_-$. Using the Casimir element, we can normalize these states by taking

$$\begin{aligned} \mathscr{C}|\varepsilon_0, m\rangle = \varepsilon_0(\varepsilon_0 - 1)|\varepsilon_0, m\rangle = (R^2 - L_+L_-)|\varepsilon_0, m\rangle \,, \\ \varepsilon_0(\varepsilon_0 - 1)|\varepsilon_0, m\rangle = ((\varepsilon_0 + m)^2 - |c_{m-1}|^2)|\varepsilon_0, m\rangle \,, \end{aligned} \tag{2.46}$$

with this one find that the ladder operators act as

$$c_m = \sqrt{(\varepsilon_0 + m)(\varepsilon_0 + m + 1) - \varepsilon_0(\varepsilon_0 - 1)} \,, \tag{2.47}$$

implying that

$$|\varepsilon_0, m\rangle = \sqrt{\frac{\Gamma(2\varepsilon_0)}{m!\,\Gamma(2\varepsilon_0 + m)}}(L_+)^m|\varepsilon_0, 0\rangle \,. \tag{2.48}$$

satisfying the orthonormality condition:

$$\langle\varepsilon_0, m|\varepsilon_0, n\rangle = \delta_{m,n} \,. \tag{2.49}$$

Introducing the conjugate momentum

$$p = \frac{\partial \mathscr{L}}{\partial \dot{Q}} = \dot{Q} \,, \qquad \text{with} \qquad [Q(t), p(t)] = i \,, \tag{2.50}$$

one can realize the dAFF model with

$$
\begin{aligned}
H &= \frac{1}{2}\left(p^2 + \frac{g}{Q^2}\right) , \qquad g > 0 , \\
D &= \frac{t}{2}\left(p^2 + \frac{g}{Q^2}\right) - \frac{1}{4}(pq + qp) , \\
K &= -t^2 H + 2tD + \frac{1}{2}Q^2 , \\
\mathscr{C} &= \frac{g}{4} - \frac{3}{16} , \\
\varepsilon_0 &= \frac{1}{2}\left(1 + \sqrt{g + \frac{1}{4}}\right) .
\end{aligned}
\tag{2.51}
$$

There are a lot of puzzles of this realization, the first one is that there is not an invariant vacuum of the theory. Another fact is that in this model all the invariant states are not normalizable, so forming the diagonal matrix elements turns out to be a very dificult task. Another problem is that no state is invariant under all three isometries of the conformal group. On the other hand, if one constructs non-invariant normalizable states, these interfere with derivations of conformal constraints, and lastly looking to the AdS_2 averaged operators, they carry arbitrary dimensions while the canonical CFT models involves operators with fixed conformal dimensions. To dodge these problems we have to modify the usual operator-state correspondence taking a non-primary operator O_Δ such that when we calculate the correlation functions with respect to a specific state $|\Omega\rangle$ one obtains the equation (2.27) for the two-point function. One possible explanation to this is since the operator itself is not well suited to give the right results for the correlation function, one chooses a specific vacuum state in order to correct the conformal dimension of this operator, showing in the end the expected behaviour for these functions (see [9]).

Chapter 3

The AdS/CFT Correspondence

In this chapter we introduce the primordial ideas which conducted to the conjecture of a strong/weak duality and the relation between the AdS and some CFT on the boundary. After this exposition we study the case $d = 1$ in which the natural choice of a quantum mechanical system is the dAFF model. We end this chapter discussing what are the real challenges in constructing a quantum mechanical system for AdS_2/CFT_1 and the possible ways to avoid any undesired phenomena that may appear. We are following [3], [5], [9], [13], [17] and [20].

Juan Maldacena conjectured in 1998 that a certain classical type IIB super-string theory on $AdS_5 \times S^5$ is dual to a highly symmetric $\mathscr{N} = 4$ super Yang-Mills theory in the large N limit. We are not going to explain in detail what are the properties of this two theories, from now on, we are assuming that the reader is familiarized with both theories. Maldacena demanded that the 't Hooft limit coupling be large compared with r dependent term in the metric in units of string lenght, turning the metric of a type IIB supergravity into

$$ds^2 = \frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + \ell^2 d\Omega_5^2 \tag{3.1}$$

where t denotes the world volume coordinates of the black 3-brane solution. The form of the metric shows that near the horizon the supergravity solution is $AdS_5 \times S^5$ with the lenght scale ℓ playing the role of the radius of the five-spehe and the 'radius' of AdS^5. This duality has some intriguing properties. First, it is a strong-weak coupling duality, secondly, it is non pertubative in the string coupling and also in the Yang-Mills coupling g_{YM} and lastly, its a classical-quantum duality, because that classical supergravity is conjectured to be dual to a quantum gauge theory, being supressed by powers of $1/N$.[5] The general correspondence formula is

$$\int_{\partial_{\phi \in \partial R}} D\phi_0 e^{-iS_{AdS}[\phi]} = \left\langle \exp \int d^d x \mathscr{O}(x)\phi_0(x) \right\rangle , \tag{3.2}$$

where $\mathscr{O}$ denotes the conformal primary operators on the boundary and the left integral is over all fields whose the assymptotic boundary values are ϕ_0. In the classical limit one can make the saddle-point approximation and find that

$$S_{AdS}[\phi_0] = W_{CFT}[\phi_0] , \tag{3.3}$$

where S_{AdS} is the classical on-shell of an AdS theory and W_{CFT} is the effective action given by minus the logarithm of the right hand side of (3.2). Since the metric of AdS is divergent, one expect that the classical action is also divergent. In order to extract the physical information from it, one must renormalize the on-shell action by adding counter terms which cancel the infinities, giving

$$S_R = W_{CFT} \ , \tag{3.4}$$

in the expression above S_R stands for the renormalized on-shell action for AdS. Any field theory on the AdS space, which includes gravity, as the boundary value of gravitons couples to the energy-momentum tensor, which is an standard feature of any CFT, has a corresponding counter part CFT. Thus, the AdS/CFT correspondence is an important tool for formulating non-trivial CFTs.

3.1 The Euclidean AdS_2

Lets start with AdS_2 spacetime with the metric defined by

$$ds^2 = \frac{dz^2 + dt^2}{z^2} \ , \tag{3.5}$$

the embedding coordinates are X^μ with $\mu = 0,1,2$ and signature $\mathrm{diag}(1,1,-1)$ on the ambient space. This space has three Killing vectors that generate $SO(2,1)$ isometry group. These coordinates and the Killing vector fields on AdS_2 satisfies:

$$\{X^\mu, X^\nu\} = \varepsilon^{\mu\nu\gamma} X \ , \tag{3.6}$$

$$[K^\mu, K^\nu] = \varepsilon^{\mu\nu\gamma} K_\gamma \ , \tag{3.7}$$

$$(K^\mu X^\nu) = \varepsilon^{\mu\nu\gamma} X_\gamma \ , \tag{3.8}$$

$$X^\mu X_\mu = -\ell_0^2 \ , \tag{3.9}$$

where the bracket on (3.6) is the Poisson bracket, on (3.7) is a commutation relation and (3.8) denote the action of the Killing vector on the embedding coordinate. One can note that there is a direct correspondence between the Killing vector field acting on a function and the Poisson bracket taken with respect to the embedding coordinates. We summarize this below:

$$(K^\mu, \circ) = \{X^\mu, \circ\} \ .$$

With this fact, it's the right time to show an algebraic way to calculate the action of AdS_2 using only the embedding coordinates, without concerning about on whose coordinates the Poisson bracket is defined. First, consider the embedding coordinates and Killing vectors :

$$X^0 = -\frac{\ell_0 t}{z} \ , \qquad X^1 = -\frac{\ell_0}{2z}(z^2 + t^2 - 1) \ , \qquad X^2 = -\frac{\ell_0}{2z}(z^2 + t^2 - 1) \ , \tag{3.10}$$

$$K^0 = -t\partial_t - z\partial_z \ , \qquad K^1 = \frac{1}{2}(z^2 - t^2 + 1)\partial t - zt\partial_z \ , \qquad K^2 = \frac{1}{2}(z^2 - t^2 - 1)\partial t - zt\partial z \ ,$$

these are the Fefferman-Graham coordinates, covering the half plane $\{(z,t) \in \mathbb{R}^2 \mid z \geq 0 \ , \ -\infty < t < \infty)\}$. The action for a massless scalar field in the AdS_2 background is

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} dtdz\Big[(\partial_z\phi)^2 + (\partial_t\phi)^2\Big] \ , \tag{3.11}$$

note that

$$\begin{aligned}(K^\mu\phi)(K_\mu\phi) &= (K^0\phi)^2 + (K^1\phi)^2 - (K^2\phi)^2 \\ &= t^2(\partial_t\phi)^2 + 2zt(\partial_t\phi)(\partial_z\phi) + z^2(\partial_z\phi)^2 + \\ &\quad (z^2 - t^2)(\partial_t\phi)^2 - 2zt(\partial_z\phi)(\partial_t\phi) \\ &= z^2 \ ,\end{aligned} \tag{3.12}$$

using this result on (3.11) one can rewrite the action in terms of Killing vectors as follows:

$$S = \frac{1}{2}\int_{AdS_2} d^2x\sqrt{g}\{X^\mu,\phi\}\{X_\mu,\phi\} \ , \tag{3.13}$$

since

$$(K^\mu\phi)(K_\mu\phi) = \{X^\mu,\phi\}\{X_\mu,\phi\} \ , \quad \text{with} \quad \sqrt{g} = \frac{1}{z^2} \ . \tag{3.14}$$

As expected, one can obtain the equations of motion by calculating the variation of the action directly in (3.13). All these equations are written on the Fefferman-Graham coordinates. The canonical laws of transformation between these coordinates are:

$$x = -\ln(z) \ , \qquad y = \frac{\ell_0 t}{z} \ , \tag{3.15}$$

the Poisson bracket can be easily calculated for each set of coordinates

$$\{x,y\} = 1 \ , \qquad \{t,z\} = \frac{z^2}{\ell_0} \ . \tag{3.16}$$

To calculate the variation of the action we note that $\{X^\mu,\phi\} = \{t,z\}\varepsilon^{ij}\partial_i X^\mu \partial_j\phi$, with ε^{ij} being the Levi-Civita symbol of rank two. For a scalar field with mass, the variation of the action can be written as:

$$\delta S = 0 = \int_{AdS_2} d^2x\sqrt{g}(\{X^\mu,\phi\}\{X_\mu,\delta\phi\} + m^2\phi\delta\phi) \ . \tag{3.17}$$

Note that we have ignored the variations $\delta\sqrt{-g}$ and δX^μ because they don't influence directly the dynamics of the motion, since they are the kinetic terms. Calculating the term of Poisson brackets one can show that

$$\int_{AdS_2} d^2x\sqrt{g}\{X^\mu,\phi\}\{X_\mu,\delta\phi\} = \{t,z\}^2 \int_{AdS_2} d^2x\sqrt{g}\varepsilon^{\gamma\nu}\varepsilon^{\alpha\beta}\partial_\gamma X^\mu\partial_\nu\phi\partial_\alpha X_\mu\partial_\beta\delta\phi \ , \tag{3.18}$$

contracting the skew-symmetric matrices on the RHS

$$\{t,z\}^2 \int_{AdS_2} d^2x\sqrt{g}(\delta^\alpha_\gamma\delta^\beta_\nu - \delta^\beta_\gamma\delta^\alpha_\nu)\partial_\gamma X^\mu\partial_\nu\phi\partial_\alpha X_\mu\partial_\beta\delta\phi \ , \tag{3.19}$$

integrating by parts and relabelling the terms

$$\begin{aligned}\{t,z\}^2 \Bigg[&\int_{\partial AdS_2} d\sigma^\gamma\sqrt{g}\partial_\gamma X^\mu(\varepsilon^{\gamma\nu}\partial_\gamma X_\mu\partial_\nu\phi)\delta\phi \\ - &\int_{AdS_2} dx^2\sqrt{g}\partial_\nu(\partial_\gamma X^\mu(\varepsilon^{\gamma\nu}\partial_\gamma X_\mu\partial_\nu\phi))\delta\phi \Bigg] \ ,\end{aligned} \tag{3.20}$$

applying the condition for the boundary of the AdS to cancel one of the terms and returning to the Poisson bracket form, the equation of motion reads

$$0 = \{X^\mu,\{X_\mu,\phi\}\} - m^2\phi \ . \tag{3.21}$$

To see the direct correspondence between the Poisson brackets and the action of the Killing vector fields, one can calculate

$$K^\mu(K_\mu\phi) - m^2\phi = K^0(K^0\phi) + K^1(K^1\phi) - K^2(K^2) - m^2\phi \ . \tag{3.22}$$

Defining $K^\pm = K^2 \pm K^1$, clearly one can show that:

$$K^+ = (z^2-t^2)\partial_t - 2zt\partial_z \qquad K^- = -\partial_t \ , \tag{3.23}$$

and with this two new vector fields, we can calculate:

$$K^1(K^1\phi) = \frac{1}{4}\left(K^+(K^+\phi) - K^-(K^+\phi) - K^+(K^-\phi) + K^-(K^-\phi)\right) \ , \tag{3.24}$$

for K^2 we have

$$K^2(K^2\phi) = \frac{1}{4}\left(K^+(K^+\phi) + K^-(K^+\phi) + K^+(K^-\phi) + K^-(K^-\phi)\right) \ , \tag{3.25}$$

substituting these two terms on (3.22) we get to the main equation:

$$K^\mu(K_\mu\phi) - m^2\phi = K^0(K^0\phi) - K^+(K^-\phi) - \frac{1}{2}[K^+,K^-]\phi - m^2\phi \ , \tag{3.26}$$

using the commutation relations defined on (3.7), the definition of the vector fields K^+ and K^- one can easily show that $[K^+, K^-] = -2K^0$, applying this on the equation above, we obtain:

$$K^0(K^0\phi) - K^+(K^-\phi) + K^0\phi - m^2\phi = 0 \ . \tag{3.27}$$

One can verify immediately that the action of the Killing vectors on the field yields an equation that is very similar to the Klein-Gordon equation (Equation 1.36 for d=1):

$$K^\mu(K_\mu\phi) - m^2\phi = 0 = \Box\phi - \frac{\ell_0^2 m^2\phi}{z^2} \ . \tag{3.28}$$

A deeper analysis on these results can be made by examining the behavior of the solutions. Any isometry is generated by a Killing vector field K^ν, with it we should have the conserved current $j^\mu = T^{\mu\nu}K_\nu$, where $T^{\mu\nu}$ is the energy-momentum tensor which is covariantly conserved since $\nabla_\mu j^\mu = 0$ implying the conservation of the energy, there is not any energy leaving the AdS boundaries. Taking the variation of the action and using the conservation equations, one can find that:

$$0 = \int_{AdS} d^2x\sqrt{g}\nabla_\mu j^\mu = \int_{\partial AdS} \sqrt{\gamma}\Big|_{\partial AdS} d\sigma^\nu K^\mu T_{\mu\nu} \ , \tag{3.29}$$

where the boundary induced metric is considered on the second integral. If we let our killing vector be $-K^-$ and we can take a big chunk of AdS restricted into two spacelike slices at $t \in [t_0, t_f]$ extending across all space, which means $z \in [0, \infty)$. We should assume that all fields should decay exponentially when $z \to \infty$. After all these considerations, the integral becomes:

$$\int_0^\infty dz T_{tt}\Big|_{t=t0}^{t=t_f} - \int_{t=t0}^{t=t_f} dt T_{tz}\Big|_{z=0} \ , \tag{3.30}$$

the first term is the energy inside the AdS, the second term is the flux of energy-momentum out the AdS boundary. If we show that the second term is zero, then this equation implies a kind of energy conservation, since the fist integral should be constant for all spacelike slices at any arbitrary time interval. But, to calculate this we should get an expression of the energy momentum tensor, it's easy to do, we just take the variation of the action with respect to the metric of the AdS (defined by canonical coordinates).

$$\frac{\delta S}{\delta g^{\mu\nu}} = T^{\mu\nu} \ , \tag{3.31}$$

one can calculate it with respect to the Poisson brackets:

$$T_{\mu\nu} = \sqrt{g}\{X_\mu, \phi\}\{X_\nu, \phi\} - \frac{g_{\mu\nu}}{2\sqrt{g}}\left(\{X_\alpha, \phi\}\{X^\alpha, \phi\} + m^2\phi^2\right) \ . \tag{3.32}$$

Applying this on the integral of conservation one can deduce directly the conditions discussed on the section 1.5 by other ways. We see clearly that when we take $z \longrightarrow 0$, the Killing vectors take the form

$$K^{-} \to -\partial_t \ , \quad K^0 \to -t\partial_t \ , \quad K^{+} \to -t^2\partial_t \ , \tag{3.33}$$

they correspond to the generators explained on the section 2.1.2 showing that one recover the symmetries of the conformal group as the killing vector fields on the boundary of the AdS space, giving some hints that it is possible to realise this correspondence, that was once conjectured by Juan Maldacena for other specific cases, and this realisation extends to AdS_2/CFT_1.

Chapter 4

Non-Commutative AdS/CFT

In this chapter we start by briefly reviewing some aspects of the non-commutative differential geometry applied in physics following [1], [6], [21], [22], [23] and [24]. The main objective here is to give a real meaning of what is the non-commutative geometry and why we would use it in physics to achieve some goals that the standard ways of quantum field theory could not work.After, we discuss the one dimensional case and we construct some representations to it and finish this chapter by constructing the set of Killing vectors that preserves the SO(2,1) symmetries after the deformation quantization, following in the main part [11] and [1].

4.0.1 Motivation

In the Euclidean geometry the notion of a point is ubiquitous and also necessary for most of it's results and applications. Extending this basic interpretation, for example, one can consider any finite-dimensional commutative algebra which is a C^*-algebra can be as an algebra of functions on a finite set of points, in which a C^*-algebra is a Banach algebra B over $\mathbb{C}$ with an *involution* that takes $f \to f^*$ and satisfies:

Let a and a^* be elements of B and let the map $* : B \longrightarrow B$ be defined with the properties

(i) $a^{**} = (a^*)^* = a \,, \qquad \forall\, a \in B \,.$

(ii) $(ab + \lambda c)^* = b^* a^* + \overline{\lambda} c^* \,, \qquad \forall\, a,b,c \in B$ and $\lambda, \overline{\lambda} \in \mathbb{C}$.

(iii) $\parallel a^* a \parallel = \parallel a \parallel \cdot \parallel a^* \parallel \,, \qquad \forall\, a \in B \,.$

Clearly, the notion of a point can be extended to various types of spaces and situations, but this turn out to be problematic when someone tries to quantize the classical mechanics. The standard procedure of quantization can be naively described as the correspondence between the classical observables with operators that acts in some separable Hilbert space of states, which don't pose a real huge problem until someone attempts to measure the amplitude of some quantum field at a precise given point in space-time, resulting in a series of ultraviolet divergences. The aim of the non-commutative geometry is to rebuild the geometry of manifolds in terms of an algebra of functions on it and then generalize the differential geometry results to the case of a non-commutative algebra, causing loss of the notion of a point in space. Dirac in his first papers was aware of the absence of localization, as pointed by the Heisenberg uncertainty principle and several physicists over the decades studied the algebra of observables considering the states as secondary derived objects, which constitutes the transition to the non-commutative case where the notion of pure states replace that of a point and vector fields are replaced by derivations of the algebra. The interest in this subject increased lately because of the possibility that at very small length scales the space-time does not behave as a differentiable manifold, pointing to the alternative formulation of the non-commutative quantum field theory.

4.1 Phase-space quantization

There are three main alternative paths to quantization. The standard formulation that uses operators in Hilbert space, the path integral formulation and the phase-space formulation based on Wigner's quasi-probability distribution function in phase space (WF) and Weyl's correspondence between quantum operators and ordinary c-numbers phase-space functions, that relies on the star-product, that was fully understood by Groenewold together with Moyal, which maps products of operators that act in some Hilbert space to product of functions on the phase space, giving an alternative procedure to achieve the quantization.

4.1.1 The Wigner's Function

The WF is defined as

$$f(x,p) = \frac{1}{2\pi}\int dy\ \psi^*\left(x-\frac{\hbar}{2}y\right) e^{-iyp}\psi\left(x+\frac{\hbar}{2}y\right)\ . \tag{4.1}$$

If $\psi(x) \in L^2(\mathbb{R})$, i.e. if ψ is a Lebesgue square-integrable complex-valued function on $\mathbb{R}$ satisfying $|\psi|^2 = 1$, obviously the WF is normalized

$$\begin{aligned}\int dp\ dx f(x,p) &= \frac{1}{2\pi}\int dy \int dp\ dx\ \psi^*\left(x-\frac{\hbar}{2}y\right) e^{-iyp}\psi\left(x+\frac{\hbar}{2}y\right)\\ &= \int dy\ dx\ \psi^*\left(x-\frac{\hbar}{2}y\right)\delta(y)\psi\left(x+\frac{\hbar}{2}y\right)\\ &= \int dx\ |\psi(x)|^2 = 1\ .\end{aligned} \tag{4.2}$$

In the classical limit as $\hbar \to 0$, it reduces to the probability density in coordinate space. The usual x- or p-projections leads to probability densities in momentum or coordinate space. WF cannot be interpreted as a probability distribution, it is therefore a quasi-probability distribution because it can assume negative values for an arbitrary open set in the phase-space, but it leads to correct position and momentum probability distributions given by quantum mechanics, replacing the wave-function in this formulation. It also provides the integration measure for functions on phase space that represents classical quantities in general. These functions are associated to ordered operators upon quantization through the Weyl's correspondence.

4.1.2 The Weyl's Correspondence

The Weyl correspondence is the association of a quantum-mechanical operator $W(g)$ in a given ordering prescription with the classical c-number Fourier transformed function g on phase-space. This correspondence reads

$$W(g) = \mathfrak{G}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2}\int dp\, dx\, d\alpha\, d\beta\; g(x,p)\exp\left(i\alpha(\mathfrak{p}-p)+i\beta(\mathfrak{x}-x)\right)\,, \tag{4.3}$$

where $g(x,p)$ is the corresponding phase-space function, and $\mathfrak{x}$ and $\mathfrak{p}$ are the respective quantum operators associated to x and p. The ordering prescription requires that an arbitrary operator written as a power series of $\mathfrak{x}$ and $\mathfrak{p}$ be ordered in a completely symmetrized expression by use of Heisenberg's commutation relations, $[\mathfrak{x},\mathfrak{p}] = i\hbar$. Finally, Groenewold worked out how two classical c-number functions $f(x,p)$ and $g(x,p)$ must compose in order to yield the product of operators $\mathfrak{G}$ and $\mathfrak{H}$:

$$\mathfrak{G}\mathfrak{H} = \frac{1}{(2\pi)^2}\int d\alpha\, d\beta\, dx\, dp\; \exp(i\alpha(\mathfrak{p}-p)+i\beta(\mathfrak{x}-x))(f\star g)(x,p)\,, \tag{4.4}$$

here $\star$ stands for the star product. This is the original definition of the star product and it enables the formulation of quantum mechanics in the phase-space.

4.1.3 Star Product

The star product a associative pseudo-differential deformation of ordinary products of phase-space c-number functions. It is defined as

$$\star := \exp\left[\frac{i\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)\right]\,. \tag{4.5}$$

It can be written in an expanded form as

$$F(x,p)\star G(x,p) = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \varepsilon^{i_1 j_1}\ldots\varepsilon^{i_n j_n}(\partial_{i_1}\ldots\partial_{i_n}F)(\partial_{j_1}\ldots\partial_{j_n}G)\,, \tag{4.6}$$

where i,j stands for x,p and the matrices ε^{ij} are the Levi-Civita symbols of rank two. Since it involves exponential of derivatives, it can be easily evaluated through translation of function arguments

$$F(x,p)\star G(x,p) = F\left(x+\frac{i\hbar}{2}\overrightarrow{\partial}_p,\ p-\frac{i\hbar}{2}\overrightarrow{\partial}_x\right)G(x,p)\ . \tag{4.7}$$

If one uses the Fourier representation of the star product as an integral kernel

$$F\star G = \frac{1}{(\hbar\pi)^2}\int dp'\ dp''\ dx'\ dx''\ f(x',p')g(x'',p'')$$
$$\times\exp\left(-\frac{2i}{\hbar}\Big(p(x'-x'')+p'(x''-x)+p''(x-x')\Big)\right)\ , \tag{4.8}$$

the expression on the exponent is twice the area of the phase-space triangle determined by the points $(x,p'),\ (x',p'),$ and (x'',p''), simplifying the calculation of multiple star products. For more multiplications one can use the Almeida's polygon theorem ([22],[24]), it associates multiple star products with the sum of areas of triangles on phase space.

One can also define the star product using coherent states (CS). With the completeness relation, we can assume that the CS form an overcomplete basis for the quantum-mechanical Hilbert space spanned by the eigenvectors $|\zeta\rangle$, labelled by complex numbers ζ and usually satisfying $\langle\zeta|\zeta\rangle = 1$. To every operator $\hat{O}$ acting on the Hilbert space, we can associate a function $O(\zeta,\overline{\zeta})$, by definition:

$$O(\zeta,\overline{\zeta}) := \langle\zeta|\hat{O}|\zeta\rangle\ . \tag{4.9}$$

Using the resolution of identity, one can define an associative product for two of such functions:

$$O(\zeta,\overline{\zeta})\star P(\zeta,\overline{\zeta}) := \int d\mu(\gamma,\overline{\gamma})\langle\zeta|\hat{O}|\gamma\rangle\langle\gamma|\hat{P}|\zeta\rangle = \langle\zeta|\hat{O}\hat{P}|\zeta\rangle\ . \tag{4.10}$$

Using the normal (anti-normal) representations of the operator defined above, these functions are analytic in ζ $(\overline{\zeta})$and acting the translation operator twice in these states, we can construct directly the function $\langle\zeta|\hat{O}|\gamma\rangle$ by the action of the *ordered exponential* upon $(\zeta,\overline{\zeta})$ depending functions:

$$:\exp(\gamma-\zeta)\overrightarrow{\partial}_\zeta : O(\zeta,\overline{\zeta}) := \frac{\langle\zeta|\hat{O}|\gamma\rangle}{\langle\zeta|\gamma\rangle}\ , \tag{4.11}$$

the ordered derivatives acts to the right in each term of the Taylor expansion, we can similarly define an ordered exponential that acts to the left (on the anti-analytic part of the functions) and finally substitute this on the definition of the star product:

$$O(\zeta,\overline{\zeta})\star P(\zeta,\overline{\zeta}) = \int d\mu(\gamma,\overline{\gamma})O(\zeta,\overline{\zeta}) : \exp(\gamma-\zeta)\overleftarrow{\partial}_\zeta : |\langle\zeta|\gamma\rangle|^2 \!: \exp(\overline{\gamma}-\overline{\zeta})\overrightarrow{\partial}_{\overline{\zeta}} : P(\zeta,\overline{\zeta})\ . \tag{4.12}$$

Consider the case that the CS are eigenvectors of some operator $\hat{x}$. Trivially, the star product of two analytic functions is the same as the ordinary product. For the anti-analytic functions we take the action of the adjoint operator $\langle\zeta|\hat{x}^\dagger = \overline{\zeta}\langle\zeta|$, and we recover trivially the same property for analytic functions. For non-trivial results, we must consider the product:

$$O(\zeta) \star P(\overline{\zeta}) = O(\zeta)P(\overline{\zeta}) + \langle \zeta | [\hat{O}(\hat{x}), \hat{P}(\hat{x}^\dagger)] | \zeta \rangle \ . \tag{4.13}$$

Once we know the commutation relations for the operators $\hat{x}$ and $\hat{x}^\dagger$ we can evaluate these products. Demanding that the commutation relations between these two operators could be expanded with respect to $\hbar$ and for the classical limit$\hbar \to 0$

$$O(\zeta,\overline{\zeta}) \star P(\zeta,\overline{\zeta}) = O(\zeta,\overline{\zeta})P(\zeta,\overline{\zeta}) \tag{4.14}$$

$$O(\zeta,\overline{\zeta}) \star P(\zeta,\overline{\zeta}) - P(\zeta,\overline{\zeta}) \star O(\zeta,\overline{\zeta}) = \mathscr{O}(\hbar) \ , \tag{4.15}$$

where the LHS of (4.15) is related to the classical Poisson bracket. Now we define the Moyal Bracket

$$\{O,P\}_\star = O \star P - P \star O \ , \tag{4.16}$$

with, for the CS case

$$\star = \exp\left(\frac{\hbar}{2} \overleftarrow{\partial}_\zeta \overrightarrow{\partial}_{\overline{\zeta}}\right) \ . \tag{4.17}$$

Any function $\hat{F}(\hat{x},\hat{y})$ of canonical conjugate operators that satisfies $[\hat{x},\hat{y}] = \mathbb{1}$ can be mapped to the Moyal-Weyl plane spanned by the commuting coordinates (x,y) and any product of functions of operators is mapped to the star products of the symbols $F(x,y)$.

4.2 The non-commutative d=1 AdS

Let's start with the canonical coordinates (x,y) that satisfies (3.15) and (3.16). The embedding (3.10) and the Killing vectors are written in these coordinates as

$$\begin{gathered} X^0 = -y \ , \quad X^1 = -\frac{1}{2\ell_0} e^{-x} y^2 + \ell_0 \sinh(x) \ , \quad X^2 = -\frac{1}{2\ell_0} e^{-x} y^2 - \ell_0 \cosh(x) \\ K^0 = \partial_x \ , \quad K^1 = \frac{1}{\ell_0} e^{-x} y \partial_x - X^2 \partial_y \ , \quad K^2 = \frac{1}{\ell_0} e^{-x} y \partial_x - X^1 \partial_y \ . \end{gathered} \tag{4.18}$$

Following the usual procedure for quantization, we replace the three embedding coordinates X^μ by Hermitian operators satisfying the analogues of equations (3.6) and (3.9), promoting Poisson brackets to commutation relations

$$\hat{X}^\mu \hat{X}_\mu = -\ell^2 \mathbb{1} \qquad [\hat{X}^\mu, \hat{X}^\nu] = i\alpha \varepsilon^{\mu\nu\rho} \hat{X}_\rho \ , \tag{4.19}$$

where α stands for the non-commuting parameter with units of length. To recover the commutative AdS_2 we just take the commutative limit $\alpha \to 0$ and $\ell \to \ell_0$, with α playing the role of$\hbar$. This limit is achieved by taking the coordinates to the boundary making $\hat{r} \to \infty$ i.e. $\hat{z} \to 0$. These equations define the non-commutative AdS_2 satisfying the $\mathfrak{so}(2,1)$ algebra. One can also define the radial operator

$$\hat{r} = \hat{z}^{-1} = \frac{1}{\ell}(\hat{X}^1 - \hat{X}^2) \ , \tag{4.20}$$

which will have a huge importance in the non-commutative AdS/CFT correspondence. The states of the $ncAdS_2$ belong to the universal cover of the group SU(1,1) and the algebra of the operators sastisfies (4.19). Taking a slightly different approach from (3.8), one can take a basis in a given representation of the X^2 eigenvectors

$$\hat{X}_+|\varepsilon_0, k, m\rangle = -\alpha\ c_m\ |\varepsilon_0, k, m+1\rangle \ , \tag{4.21}$$

$$\hat{X}_-|\varepsilon_0, k, m\rangle = -\alpha\ c_{m-1}\ |\varepsilon_0, k, m-1\rangle \ , \tag{4.22}$$

$$\hat{X}^2|\varepsilon_0, k, m\rangle = -\alpha\ (\varepsilon_0 + m)\ |\varepsilon_0, k, m+1\rangle \ , \tag{4.23}$$

$$\hat{X}_\mu \hat{X}^\mu|\varepsilon_0, k, m\rangle = -\alpha^2\ k(k+1)\ |\varepsilon_0, k, m+1\rangle \ , \tag{4.24}$$

where the coefficient is

$$c_m = \sqrt{(k + \varepsilon_0 + m + 1)(\varepsilon_0 - k + m)} \ . \tag{4.25}$$

4.2.1 Representations

As discussed in the section 3.1.2, we will use the generalized Laguerre polynomials to construct a differential representation for the embedding coordinates for the discrete representation $D^+(k)$. Setting the lowest state as $|k, 0\rangle$ which is anihhilated by $\hat{X}_-$, since we are assuming $\varepsilon_0 = -k > 0$, we expand the eigenvectors of the radial operator $|r, k\rangle$ in terms of the $\hat{X}^2$ eigenbasis

$$|r, k\rangle = \sum_{m=0}^{\infty} \psi^+_{k,m}(r)|k, m\rangle \ . \tag{4.26}$$

Writing the radial operator in terms of the raising and lowering operators

$$\hat{r} = \frac{1}{2\ell}(\hat{X}_+ - \hat{X}_- - 2\hat{X}^2) \ , \tag{4.27}$$

one can write the eigenvalue equation

$$\hat{r}|r, k\rangle = \frac{1}{2\ell}(\hat{X}_+ - \hat{X}_- - 2\hat{X}^2)|r, k\rangle = r|r, k\rangle \ . \tag{4.28}$$

Using (4.21) and (4.23) we get to the following

$$
\begin{aligned}
&-\sqrt{(m+1)(m-2k)}\,\psi^{+}_{m+1}(r)-\sqrt{m(m-1-2k)}\,\psi^{+}_{m-1}(r)\\
&+2(k-m)\psi^{+}_{k,m}(r)=\frac{\ell r}{\alpha}\psi^{+}_{k,m}(r)\ ,
\end{aligned}
\tag{4.29}
$$

all the coeficients are determined by the recursion relation (4.29), agreeing with the generalized Laguerre polynomials for $m > 0$

$$
\psi^{+}_{k,m}(r)=\sqrt{\frac{m!}{(m-2k-1)!}}L_m^{-2k-1}\left(\frac{2\ell r}{\alpha}\right)\ . \tag{4.30}
$$

These relations has the half line $r > 0$ as domain, this means that these states picks one of the boundaries of the AdS_2 to the non-commutative case finishing the boundary ambiguities for this model as $r \to \infty$. In order to simplify the coefficients, lets use the orthogonality conditions of $L_m^{\alpha}(x)$

$$
\int_{\mathbb{R}_+} dx\, x^{\beta} e^{-x} L_m^{\beta}(x) L_n^{\beta}(x)=\frac{\delta_{n,m}}{m!}\Gamma(m+\beta+1)\ , \tag{4.31}
$$

and defining

$$
C_m=\sqrt{\frac{m!}{(m-2k-1)!}}\ , \tag{4.32}
$$

we can reorder (4.30)

$$
L_m^{-2k-1}\left(\frac{2\ell r}{\alpha}\right)=\frac{\psi^{+}_{k,m}(r)}{C_m}\ , \tag{4.33}
$$

substituting (4.33) in (4.31) we get to the following

$$
\int_{\mathbb{R}_+} dr\,\left(\frac{2\ell}{\alpha}\right)^{-2k} e^{-2\ell r/\alpha} r^{-2k-1}\frac{\psi^{+}_{k,m}(r)}{C_m}\frac{\psi^{+}_{k,n}(r)}{C_n}=\frac{\delta_{n,m}}{m!}(m-2k-1)!\ , \tag{4.34}
$$

since (4.34) will not vanish for $m = n$ we can impose that exists a function $u^{+}_{k,m}(r)$ such that

$$
\int_{\mathbb{R}} dr\, u^{+}_{k,m}(r)u^{+}_{k,n}(r)=\delta_{m,n}\ , \tag{4.35}
$$

it implies that

$$
u^{+}_{k,m}(r)=\left(\frac{2\ell}{\alpha}\right)^{-k} e^{-\ell r/\alpha} r^{-k-1/2}\ \psi^{+}_{k,m}(r)\ . \tag{4.36}
$$

Now, to get a representation of the differential operator $\hat{D} = (\hat{r} - r)$ satisfying $\hat{D}\psi^{+}_{k,m}(r) = 0$ we must use the differential equation that defines the generalized Laguerre polynomials

$$x\frac{d^2}{dx^2}L^{\beta}_{m}(x) + (\beta + 1 - x)\frac{d}{dx}L^{\beta}_{m}(x) + mL^{\beta}_{m}(x) = 0 \ . \tag{4.37}$$

Substituting (4.33) and (4.36) in (4.37) and using $x = 2\ell r/\alpha$ and $\beta = -2k - 1$ one finds that

$$\left(\frac{\alpha(k+\frac{1}{2})^2}{2\ell r} + \frac{\ell r}{2\alpha} + \frac{\alpha(m-k)}{2\ell} - \frac{d}{dr}\left(\frac{\alpha r}{2\ell}\frac{d}{dr}\right)\right)u^{+}_{k,m}(r) = 0 \ . \tag{4.38}$$

Comparing (4.38) with (4.23) we just found a differential representation π^k of X^2 on $L^2(\mathbb{R}_+, dr)$ by multiplying the equation (4.38) by $-\alpha$. To find the representations of the other operators lets calculate $\pi^k([\hat{r}, \hat{X}^2])$

$$\left(r\pi^k(\hat{X}^2) - \pi^k(\hat{X}^2)r\right)[\psi(r)] = \pi^k([\hat{r}, \hat{X}^2])[\psi(r)] \ , \tag{4.39}$$

using (4.19) and (4.20) i.e. $[\hat{r}, \hat{X}^2] = \frac{1}{\ell}[\hat{X}^1, \hat{X}^2] = \frac{i\alpha}{\ell}$

$$\frac{\alpha^2}{2\ell}\left(r\frac{d}{dr}\left[r\frac{d\psi}{dr}\right] - \frac{d}{dr}\left[r\psi + r^2\frac{d\psi}{dr}\right]\right) = \frac{i\alpha}{\ell}\pi^k(\hat{X}^0)[\psi(r)] \ . \tag{4.40}$$

which simplifies to

$$\pi^k(\hat{X}^0) = i\alpha\left(r\frac{d}{dr} + \frac{1}{2}\right) \ . \tag{4.41}$$

doing the same procedure for $\hat{X}^1$ using the commutator between X^2 and X^0 one finds that

$$\pi^k(\hat{X}^2) = -\frac{\alpha^2}{2\ell}\left(\frac{(k+\frac{1}{2})^2}{r} + \frac{\ell^2 r}{\alpha^2} - \frac{d}{dr}\left(r\frac{d}{dr}\right)\right) \ , \tag{4.42}$$

$$\pi^k(\hat{X}^1) = -\frac{\alpha^2}{2\ell}\left(\frac{(k+\frac{1}{2})^2}{r} - \frac{\ell^2 r}{\alpha^2} - \frac{d}{dr}\left(r\frac{d}{dr}\right)\right) \ . \tag{4.43}$$

These operators acts linearly on $L^2(\mathbb{R}_+, dr)$, the space of square-integrable functions on the half real line. Replacing $r = e^x$ we can recover the linear operators $\tilde{\pi}(\hat{X}^\mu)$ that acts on $L^2(\mathbb{R}, dx)$ spanned by $\{f(x) = e^{x/2}\psi(e^x)\}$, in terms of the self-adjoint operators $\hat{x}$ and $\hat{y}$, satisfying

$$[\hat{x}, \hat{y}] = i\alpha\mathbb{1} \ . \tag{4.44}$$

In these coordinates the operators X^μ acting on $\{f(x)\}$ are

$$\tilde{\pi}^k(\hat{X}^0) = -\hat{y} \ , \tag{4.45}$$

$$\tilde{\pi}^k(\hat{X}^1) = -\frac{1}{2\ell}\hat{y}e^{\hat{x}}\hat{y} - \frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}} + \frac{\ell}{2}e^{\hat{x}} \,, \tag{4.46}$$

$$\tilde{\pi}^k(\hat{X}^2) = -\frac{1}{2\ell}\hat{y}e^{\hat{x}}\hat{y} - \frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}} - \frac{\ell}{2}e^{\hat{x}} \,. \tag{4.47}$$

The operators $\hat{x}$ and $\hat{y}$ satisfy the canonical commutation relations and can be mapped to their respective symbols on the Moyal-Weyl plane spanned by coordinates (x,y). This mapping is an isomorphism and by the Weyl correspondence, the product of functions of the operators $\mathfrak{F}\mathfrak{G}(\hat{x},\hat{y})$ is mapped to the star product on the Moyal-Weyl plane $\mathscr{F}\star\mathscr{G}(x,y)$ defined by (4.5) substituting $\alpha\rightarrow\hbar$ and $p\rightarrow y$. The symbols of $\tilde{\pi}^k(\hat{X}^\mu)$ are denoted by $\mathscr{X}^\mu$ and take the form

$$\mathscr{X}^0 = -y \,, \tag{4.48}$$

$$\mathscr{X}^1 = -\frac{1}{2\ell}y\star e^{-x}\star y - \frac{\alpha^2}{2\ell}k(k+1)e^{-x} + \frac{\ell}{2}e^{x} \,, \tag{4.49}$$

$$\mathscr{X}^2 = -\frac{1}{2\ell}y\star e^{-x}\star y - \frac{\alpha^2}{2\ell}k(k+1)e^{-x} - \frac{\ell}{2}e^{x} \,. \tag{4.50}$$

These functions satisfy the same relations (4.19) when mapping the usual point-wise product to the star product on the moyal plane.

$$\mathscr{X}^\mu\star\mathscr{X}_\mu = -\ell^2 \,, \tag{4.51}$$

$$[\mathscr{X}^\mu,\mathscr{X}^\nu]_\star = \mathscr{X}^\mu\star\mathscr{X}^\nu - \mathscr{X}^\nu\star\mathscr{X}^\mu = i\alpha\varepsilon^{\mu\nu\rho}\mathscr{X}_\rho \,. \tag{4.52}$$

Clearly, taking $\alpha\rightarrow 0$ we recover the point-wise product and, as explained before, the leading term on the α expansion in the star commutator is the Poisson bracket (3.39). For some calculations we will need to write the embedding coordinates of $ncAdS_2$ as functions of z and t, for this we must verify if this transformation spoils the algebra on the Moyal-Weyl plane for some order in α. Starting with $[\hat{x},\hat{y}] = i\alpha\hat{I}$, we want to define new parameters as functions of x̂ and ŷ satisfying the ordering prescription as follows:

$$\hat{t} = \frac{1}{2\ell_0}\left(\hat{y}e^{-\hat{x}} + e^{-\hat{x}}\hat{y}\right) \,, \qquad \hat{z} = e^{-\hat{x}} \,. \tag{4.53}$$

We can calculate the commutator of the new operators following the relation which consider that x and y are canonically conjugate. One can find that

$$[f(\hat{x}),\hat{y}] = i\alpha\frac{\partial f(\hat{x})}{\partial x} \,, \tag{4.54}$$

one can easily show by induction

$$[\hat{x}^n,\hat{y}] = n\hat{x}^{n-1}[\hat{x},\hat{y}] \,, \tag{4.55}$$

expanding on Taylor's series

$$[f(\hat{x}),\hat{y}] = \left[\sum_{n=1}^{\infty} \frac{\partial^n f}{\partial x^n} \frac{\hat{x}^{n-1} i\alpha}{(n-1)!}, \hat{y}\right] = i\alpha \frac{\hat{\partial f}}{\partial x} \ . \tag{4.56}$$

With this useful result, we can calculate the commutator of z and t. Using $[f(\hat{x}),g(\hat{x})] = 0$, one can find that by direct substitution:

$$[\hat{z},\hat{t}] = \frac{1}{2l_0}([e^{-\hat{x}},\hat{y}e^{-\hat{x}}] + [e^{-\hat{x}},e^{-\hat{x}}\hat{y}]) = -\frac{i\alpha\hat{z}^2}{\ell_0} \ . \tag{4.57}$$

In order to verify if the mapping to the Moyal-Weyl plane preserves this commutator, we must calculate the Moyal-commutator of the symbols z and t, denoting as $\star_{x,y}$ the star product for the canonical coordinates and $\star_{z,t}$ the transformed star product, one can calculate

$$[z,t]_{\star_{x,y}}(x,y) = z(x,y) \star_{x,y} t(x,y) - t(x,y) \star_{x,y} z(x,y) \ , \tag{4.58}$$

applying the definition of the star product

$$[z,t]_{\star_{x,y}}(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i\alpha}{2}\right)^n \varepsilon^{i_1 j_1} \dots \varepsilon^{i_n j_n} (\partial_{i_1} \dots \partial_{i_n}) e^{-x} (\partial_{j_1} \dots \partial_{j_n}) \Big(\frac{y}{\ell_0} e^{-x}\Big) \ , \tag{4.59}$$

calculating up to $\mathscr{O}(1)$

$$[z,t]\star_{x,y}(x,y) \simeq e^{-2x}y - e^{-2x}y = 0 \ , \tag{4.60}$$

the term proportional to α is

$$[z,t]_{\star_{x,y}}(x,y) \simeq \frac{i\alpha}{2}\left[\partial_x(e^{-x})\partial_y\Big(\frac{e^{-x}y}{\ell_0}\Big) + \partial_y\Big(\frac{e^{-x}y}{\ell_0}\Big)\partial_x(e^{-x})\right] = -\frac{e^{-2x}i\alpha}{\ell_0} \ , \tag{4.61}$$

the term proportional to α^2 consist in products of two derivatives acting on z and t, clearly for any y-derivative acting on z the respective term will be zero. Since all terms have at least one y derivative on z, all of them are zero except the term that has two x-derivatives on z, but it's clear that $\partial_y^2 t(x,y) = 0$. With this analysis, it's clear that the only non-vanishing term of the commutator is

$$[z,t]_{\star_{x,y}}(x,y) = -\frac{e^{-2x}i\alpha}{\ell_0} = -\frac{i\alpha}{\ell_0}z^2 \ , \tag{4.62}$$

which is equivalent to (4.57). To see if the (z,t) star commutator of the new coordinates is preserved under this transformation we must calculate the transformation rule for the derivatives

$$\frac{\partial}{\partial x} = -t\frac{\partial}{\partial t} - z\frac{\partial}{\partial z} \ , \qquad \frac{\partial}{\partial y} = -\frac{z}{\ell_0}\frac{\partial}{\partial t} \ , \tag{4.63}$$

calculating the commutator for (z,t) by definition:

$$[z,t]_{\star_{z,t}} = z \star_{z,t} t - t \star_{z,t} z \ , \tag{4.64}$$

we can apply a change of variables and using this notation, with n being the n-th derivative:

$$\partial_\nu = \frac{\partial x^\mu}{\partial x^\nu}\partial_\mu \ , \qquad (\partial_\nu)^{(n)} = \left(\frac{\partial x^\mu}{\partial x^\nu}\partial_\mu\right)^{(n)} \ . \tag{4.65}$$

Using this on the star product we obtain:

$$F\star_{z,t} G = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i\alpha}{2}\right)^n \varepsilon^{\mu_1\nu_1}\dots\varepsilon^{\mu_n\nu_n}\left(\frac{\partial x^{i_1}}{\partial x^{\mu_1}}\partial_{i_1}\dots\frac{\partial x^{i_n}}{\partial x^{\mu_n}}\partial_{i_n}F\right)\left(\frac{\partial x^{j_1}}{\partial x^{\nu_1}}\partial_{j_1}\dots\frac{\partial x^{j_n}}{\partial x^{\nu_n}}\partial_{j_n}G\right) \ , \tag{4.66}$$

one can easily find these relations

$$(\partial_x)^{(n)} = \sum_{l=0}^{n}\binom{l}{k}(-z\partial_z)^{(k-l)}(-t\partial_t)^{(l)} \ , \qquad (\partial_y)^{(n)} = \left(\frac{z}{\ell^0}\partial_t\right)^{(n)} \ . \tag{4.67}$$

For $n>1$ the successive application of either $(\partial_x)^{(n)}$ or $(\partial_y)^{(n)}$ is zero because the remaining term will be z and it will always be differentiated under respect to t at least one time resulting in zero for all terms beyond the second order, so, as expected, we obtain from the non-vanishing high order terms the same result

$$[z,t]_{\star_{z,t}} = -\frac{i\alpha}{\ell_0}z^2 = -\frac{i\alpha}{\ell_0}z\star z \ . \tag{4.68}$$

Therefore, we can't calculate so easily these commutators on the new variables because after the order α^2 the differential operators turn out to be very complicated, leading to some tedious calculations as the one that follows

$$\begin{aligned}\star_{x,y} = 1 + \frac{i\alpha}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}) + \\ \frac{1}{2}\left(\frac{i\alpha}{2}\right)^2[\overleftarrow{\partial_x}\overleftarrow{\partial_x}\overrightarrow{\partial_y}\overrightarrow{\partial_y} + \overleftarrow{\partial_y}\overleftarrow{\partial_y}\overrightarrow{\partial_x}\overrightarrow{\partial_x} - \overleftarrow{\partial_x}\overleftarrow{\partial_y}\overrightarrow{\partial_y}\overrightarrow{\partial_x} - \overleftarrow{\partial_y}\overleftarrow{\partial_x}\overrightarrow{\partial_x}\overrightarrow{\partial_y}] + O(\alpha^3) \ ,\end{aligned} \tag{4.69}$$

using the definitions we can show that:

$$\begin{cases}\partial_x^2 = (t+z)(z\partial_z+1)\partial_t + t^2\partial_t^2 \\ \partial_y^2 = \frac{z^2}{\ell^2}\partial_t^2 \\ \partial_x\partial_y = \partial_y\partial_x = -\frac{z}{\ell}(z\partial_z + 1 + t\partial_t)\partial_t \ .\end{cases} \tag{4.70}$$

Clearly we can see that we have t and z dependence on order α^2, generating non trivial terms on the star product. The transformed star product up to order α is

$$\star_{z,t} = 1 - \frac{i\alpha}{2}\left(\overleftarrow{\partial_z}z^2\overrightarrow{\partial_t} - \overleftarrow{\partial_t}z^2\overrightarrow{\partial_z}\right) + \mathscr{O}\left(\alpha^2\right) \ . \tag{4.71}$$

4.2.2 Killing vectors

From (3.21) the isometry transformations of AdS_2 on a scalar field can be obtained by taking the Poisson bracket of this field with the embedding coordinates. In the non-commutative case for a function $\hat{\Phi}$ the action of the $SO(2,1)$ isometry group will induce an infinitesimal variation of the form

$$\delta_{nc}\hat{\Phi} = \varepsilon_\mu(\hat{K}^\mu\hat{\Phi}) = i\varepsilon_\mu[\hat{X}^\mu,\hat{\Phi}] \ , \tag{4.72}$$

for some infinitesimal parameter ε_μ. A natural step is map these Killing vectors to the Moyal-Weyl plane. From now on, the functions without $\wedge$ will denote the symbols of the operators. Then, the equation (4.72) becomes

$$\delta_{nc}\Phi = \varepsilon_\mu(K_\star^\mu\Phi) = i\varepsilon_\mu[\mathscr{X}^\mu,\Phi]_\star \ , \tag{4.73}$$

where $(K_\star^\mu\Phi)$ is the symbol of $(\hat{K}^\mu\hat{\Phi})$. To evaluate (4.73) we can use the identity (4.7) and do first the minor steps

$$[e^{\pm x},\Phi]_\star = e^{\pm x}\exp\left(\frac{i\alpha}{2}\varepsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)\Phi(x,y) - \Phi(x,y)\exp\left(\frac{i\alpha}{2}\varepsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)e^{\pm x} \ , \tag{4.74}$$

using the translation property and the fact that the y-derivative on $e^{\pm x}$ vanishes, one gets

$$[e^{\pm x},\Phi]_\star = \pm e^{\pm x}e^{(i\alpha/2)\partial_y}\Phi(x,y) \mp e^{\pm x}e^{-(i\alpha/2)\partial_y}\Phi(x,y) \ , \tag{4.75}$$

and hence

$$[e^{\pm x},\Phi]_\star = \pm e^{\pm x}\left(\Phi\left(x,y+\frac{i\alpha}{2}\right) - \Phi\left(x,y-\frac{i\alpha}{2}\right)\right) \ , \tag{4.76}$$

reorganizing (4.75) one can rewrite this as

$$\pm e^{\pm x}\left(e^{(i\alpha/2)\partial_y} - e^{-(i\alpha/2)\partial_y}\right)\Phi(x,y) = \pm 2i\, e^{\pm x}\sin\left(\frac{\alpha}{2}\partial_y\right)\Phi(x,y) \ . \tag{4.77}$$

Defining

$$\Delta_y = \frac{2}{\alpha}\sin\left(\frac{\alpha}{2}\partial_y\right) \ , \tag{4.78}$$

the star commutator of (4.74) is

$$[e^{\pm x},\Phi]_\star = \pm i\alpha\, e^{\pm x}\Delta_y\Phi \ . \tag{4.79}$$

To start the second step, we can calculate

$$y\star(e^{-x}\star y) = y\star e^{-x}\left[y-\frac{i\alpha}{2}\right] = y\left(e^{-x+(i\alpha/2)\overleftarrow{\partial}_y}\left[y-\frac{i\alpha}{2}\overleftarrow{\partial}_x\right]\right) \ , \tag{4.80}$$

$$y\star(e^{-x}\star y)=e^{-x}\left(y^2+\frac{\alpha^2}{4}\right)\ .$$

After this, we proceed as the first step

$$[y\star e^{-x}\star y,\Phi]_\star=\left[e^{-x}\left(y^2+\frac{\alpha^2}{4}\right),\Phi\right]\ , \tag{4.81}$$

using the identities

$$e^{-x}\left(y^2+\frac{\alpha^2}{4}\right)\star\Phi=e^{-x+(i\alpha/2)\partial_y}\left(\left(y-\frac{i\alpha}{2}\partial_x\right)^2y^2+\frac{\alpha^2}{4}\right)\Phi(x,y)\ , \tag{4.82}$$

and

$$\Phi\star e^{-x}\left(y^2+\frac{\alpha^2}{4}\right)=e^{-x-(i\alpha/2)\partial_y}\left(\left(y+\frac{i\alpha}{2}\partial_x\right)^2y^2+\frac{\alpha^2}{4}\right)\Phi(x,y)\ , \tag{4.83}$$

it gives

$$\begin{aligned}[y\star e^{-x}\star y,\Phi]_\star=&\,e^{-x}\left(y^2+\frac{\alpha^2}{4}(1-\partial_x^2)-i\alpha\, y\partial_x\right)\Phi\left(x,y-\frac{i\alpha}{2}\right)-\\ &-e^{-x}\left(y^2+\frac{\alpha^2}{4}(1-\partial_x^2)+i\alpha\, y\partial_x\right)\Phi\left(x,y+\frac{i\alpha}{2}\right)\ ,\end{aligned} \tag{4.84}$$

using (4.78) and defining

$$S_y\Phi(x,y)=\frac{\Phi\left(x,y+\frac{i\alpha}{2}\right)+\Phi\left(x,y-\frac{i\alpha}{2}\right)}{2}=\cos\left(\frac{\alpha}{2}\partial_y\right)\Phi(x,y)\ , \tag{4.85}$$

we finally get the second commutator

$$[y\star e^{-x}\star y,\Phi]_\star=-i\alpha\ e^{-x}\left(y^2\Delta_y+2y\partial_xS_y+\frac{\alpha^2}{4}(1-\partial_x^2)\Delta_y\right)\Phi(x,y)\ . \tag{4.86}$$

Now, with these expressions we can take the commutator of the symbols (4.48) - (4.50) with Φ to evaluate the non-commutative variation of the field

$$\delta_{nc}\Phi=\alpha\varepsilon_0\partial_x\Phi+\frac{i\varepsilon_+}{2\ell}[y\star e^{-x}\star y,\Phi]_\star+\frac{i\varepsilon_+\alpha^2}{2\ell}k(k+1)[e^{-x},\Phi]_\star+\frac{i\varepsilon_-\ell}{2}[e^x\Phi]_\star\ . \tag{4.87}$$

Following the definition of the raising and lowering operators, the non-commutative variation can be written as $\delta_{nc}=\frac{\alpha}{2}\left(\varepsilon_-K_\star^-+\varepsilon_+K_\star^+\right)+\varepsilon_0K_\star^0$, with this the analogues of the Killing vectors are

$$K_\star^- = -\ell e^x \Delta_y \qquad , \qquad K_\star^0 = \partial_x$$
$$K_\star^+ = \frac{e^{-x}}{\ell}\left(2y\partial_x S_y + \left(y^2+\ell^2+\frac{\alpha^2}{4}(1-\partial_x^2)\right)\right) . \tag{4.88}$$

In the commutative limit, these differential operators agree with (3.10) and they indeed satisfy the $\mathfrak{so}(2,1)$ algebra. $K_\star^0$ is equivalent to K^0, while the others are deformations containing infinite order polynomials in ∂_y. We can re-express (4.88) in terms of the Fefferman-Graham coordinates doing the same change of variables of the section (4.3.1)

$$K_\star^- = -\frac{\ell}{z}\Delta_t \qquad , \qquad K_\star^0 = -t\partial_t - z\partial_z$$
$$K_\star^+ = -2t(t\partial_t + z\partial_z)S_t + \frac{\ell}{z}\left(t^2+\left(1+\frac{\alpha^2}{4\ell}\right)z^2\right)\Delta_t - \frac{\alpha^2 z}{4\ell}(t\partial_t+z\partial_z)^2\Delta_t \tag{4.89}$$

These expressions can be taken to the boundary by taking $z \to 0$, which corresponds exactly (3.33), showing that the $ncAdS_2$ is asymptotically AdS_2, now we may try, in principle, to apply the AdS/CFT correspondence. In the next sections we will explore briefly the massless scalar case, since it is deeply discussed in [1], and we will be mainly interested in the case with mass and interactions. There are some works (see [10] for example) pointing that the AdS/CFT correspondence holds for some nearly conformal field theories on the boundary, since the whole symmetry of the conformal group cannot be attained by some fields on AdS.

Chapter 5

Non-Commutative Field theories

In this section we are following [1]. Here we will write an expression for the massless scalar and the massive field on $ncAdS_2$, a similar discussion can be found in [35]. As an important feature, we find that after the mapping to the non-commutative space, the field will present nontrivial nonlocal interactions that desapears near the boundary. In sequence, we calculate the correlators of the theory and we show that after the deformation, the usual two point correlator gain a non-commutative correction that depends on the non-commutative scale factor.

5.1 Free Massless Scalar Field

Let $\Phi^{(0)}$ be a massless scalar field on AdS_2. The invariant $SO(2,1)$ standard action can be written as in (3.13)

$$S[\Phi^{(0)}] = \frac{1}{2\ell_0}\int_{AdS_2} d\mu\ \{X^\mu,\Phi^{(0)}\}\{X_\mu,\Phi^{(0)}\}\ , \tag{5.1}$$

where $d\mu$ is an invariant measure on AdS_2. Using the canonical coordinates defined on section 3, the equation reads

$$S[\Phi^{(0)}] = \frac{1}{2\ell_0}\int_{\mathbb{R}^2} dx\ dy\Big[\Big(y\partial_y\Phi^{(0)} + \partial_x\Phi^{(0)}\Big)^2 + \ell_0^2(\partial_y\Phi^{(0)})^2\Big]\ , \tag{5.2}$$

promoting $\Phi^{(0)}$ to a field in $ncAdS_2$ (see [1]), we generalize (5.1)

$$S_{nc}[\hat{\Phi}] = -\frac{1}{2\ell}\mathrm{Tr}[\hat{X}^\mu,\hat{\Phi}][\hat{X}_\mu,\hat{\Phi}]\ , \tag{5.3}$$

where Tr denotes the trace operation. Assuming that the scale parameter is the same for the two cases, we can map the action to the Moyal-Weyl plane by replacing Tr$\to \frac{1}{\alpha^2}\int_{\mathbb{R}^2} dx\, dy$ as follows

$$S_{nc}[\Phi] = -\frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dx\, dy\, [\mathscr{X}^{\mu},\phi]_\star \star [\mathscr{X}_{\mu},\phi]_\star \,. \tag{5.4}$$

Without taking account of the boundary terms, we can express the action for the bulk field applying (4.48) - (4.50), (4.79) and (4.86) to (5.4)

$$\begin{aligned} S_{nc}[\Phi] = \frac{1}{2\ell}\int_{\mathbb{R}^2} dx\, dy \Big\{ &(\partial_x\Phi)^2 + \Delta_y\Phi\,\Big(y^2\Delta_y\Phi + 2y\partial_x S_y\Phi - \frac{\alpha^2}{4}\partial_x^2\Delta_y\Phi\Big) \\ &+\alpha^2\Big(k+\frac{1}{2}\Big)^2(\Delta_y\Phi)^2\Big\}\,, \end{aligned} \tag{5.5}$$

note that

$$\begin{aligned} &\int_{\mathbb{R}^2} dx\, dy \Big\{ \Big(\partial_x S_y\Phi\Big)^2 - \frac{\alpha^2}{4}\Big(\partial_x\Delta_y\Phi\Big)^2 - (\partial_x\Phi)^2\Big\} = \\ &= \int_{\mathbb{R}^2} dx\, dy \Big\{ \Big(\cos^2\Big(\frac{\alpha}{2}\partial_y\Big) + \sin^2\Big(\frac{\alpha}{2}\partial_y\Big)\Big)(\partial_x\Phi)^2 - (\partial_x\Phi)^2\Big\} = 0\,, \end{aligned} \tag{5.6}$$

integrating (5.5) by parts under respect to x, using (4.19) and (4.24) we can show that $\alpha^2 k(k+1) = \ell^2$ in addition to (5.6) one can simplify (5.5)

$$S_{nc}[\Phi] = \frac{1}{2\ell}\int_{\mathbb{R}^2} dx\, dy \Big\{ \Big(y\Delta_y\Phi + \partial_x S_y\Phi\Big)^2 + \Big(\frac{\alpha^2}{4}+\ell^2\Big)(\Delta_y\Phi)^2\Big\}\,. \tag{5.7}$$

In order to see the behaviour of the action on the boundary, we express (5.7) in terms of Fefferman-Graham coortinates

$$S_{nc}[\Phi] = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} dt\, dz \Big\{ \Big(\frac{\ell t}{z}\Delta_t\Phi - (t\partial_t + z\partial_z)S_t\Phi\Big)^2 + \Big(\frac{\alpha^2}{4}+\ell^2\Big)(\Delta_t\Phi)^2\Big\}\,, \tag{5.8}$$

as z goes to zero $\Delta_t\Phi \to \frac{z}{\ell}\partial_t\Phi|_{z=0}$ and $S_t\Phi \to \Phi|_{z=0}$, the integrand goes to the action density of a massless scalar field on commutative AdS_2 with a rescaled time parameter t, satisfying the equation for a massless scalar field on an asymptotically AdS_2 space

$$\Big(1+\frac{\alpha^2}{4\ell^2}\Big)(\partial_t\Phi)^2 + (\partial_z\Phi)^2\,. \tag{5.9}$$

Taking the variation of the action (5.4) under respect to Φ

$$\begin{aligned} \delta S_{nc}[\Phi] &= -\frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dx\, dy \Big([\mathscr{X}^{\mu},\delta\Phi]_\star \star [\mathscr{X}_{\mu},\Phi]_\star + [\mathscr{X}^{\mu},\Phi]_\star \star [\mathscr{X}_{\mu},\delta\Phi]_\star\Big) \\ &= \frac{1}{\ell\alpha^2}\int_{\mathbb{R}^2} dx\, dy \Big(\delta\Phi \star [\mathscr{X}^{\mu},[\mathscr{X}_{\mu},\Phi]_\star]_\star - \Big([\mathscr{X}^{\mu},\delta\Phi \star [\mathscr{X}_{\mu},\Phi]_\star]_\star \\ &\qquad - [[\mathscr{X}^{\mu},\Phi]_\star,[\mathscr{X}_{\mu},\delta\Phi]_\star]_\star\Big)\Big)\,, \end{aligned} \tag{5.10}$$

from the first term, the field equation in the bulk is

$$[\mathscr{X}^\mu, [X_\mu, \Phi]_\star]_\star = 0 \; . \tag{5.11}$$

The remaining two terms are only defined on the boundary since the Moyal star commutator of any two functions on the Moyal-Weyl plane is a total divergence. In order to show this let $\mathscr{F}$ and $\mathscr{G}$ be two arbitrary functions on the Moyal-Weyl plane, it's easy to see that

$$\int_{\mathbb{R}^2} dx\, dy\; \mathscr{F} \star \mathscr{G}(x,y) = \int_{\mathbb{R}^2} dx\, dy\; \mathscr{F}\mathscr{G}(x,y) + \text{boundary terms} \; , \tag{5.12}$$

evaluating the star commutator on a domain D

$$\int_D dx\, dy\; [\mathscr{F}, \mathscr{G}]_\star(x,y) = \int_D dx\, dy\; i\alpha \left[\left(\partial_x \mathscr{F} \partial_y \mathscr{G} - \partial_x \mathscr{G} \partial_y \mathscr{F} \right) + \mathscr{O}(\alpha^2) \right] \; , \tag{5.13}$$

we can rearrange the terms using the symmetry of the equation

$$\int_D dx\, dy\; [\mathscr{F}, \mathscr{G}]_\star(x,y) = \int_D dx\, dy\; i\alpha \left[\left(\partial_y (\mathscr{G} \partial_x \mathscr{F}) - \partial_x (\mathscr{G} \partial_y \mathscr{F}) \right) + \mathscr{O}(\alpha^2) \right] \; . \tag{5.14}$$

Clearly the order α term is a total divergence. Up to order α^3, we define

$$\mathscr{V}_x = -i\alpha \left(\mathscr{G} \partial_x \mathscr{F} + \frac{\alpha^2}{24} \left(\partial_x^3 \mathscr{F} \partial_y^2 \mathscr{G} + \partial_x \partial_y^2 \mathscr{F} \partial_x^2 \mathscr{G} - 2\partial_x^2 \partial_y \mathscr{F} \partial_x \partial_y \mathscr{G} \right) + \mathscr{O}(\alpha^4) \right) \tag{5.15}$$

$$\mathscr{V}_y = -i\alpha \left(\mathscr{G} \partial_y \mathscr{F} + \frac{\alpha^2}{24} \left(\partial_y^3 \mathscr{F} \partial_x^2 \mathscr{G} + \partial_y \partial_x^2 \mathscr{F} \partial_y^2 \mathscr{G} - 2\partial_y^2 \partial_x \mathscr{F} \partial_y \partial_x \mathscr{G} \right) + \mathscr{O}(\alpha^4) \right) \; , \tag{5.16}$$

then the integral of the star commutator can be written as a boundary integral

$$\int_D dx\, dy\; [\mathscr{F}, \mathscr{G}]_\star(x,y) = \int_D [\partial_x \mathscr{V}_y - \partial_y \mathscr{V}_x](x,y) = \int_{\partial D} (\mathscr{V}_x dx + \mathscr{V}_y dy) \; . \tag{5.17}$$

For us, the boundary is located at $z = 0$ which implies that

$$\int_{\partial D} (\mathscr{V}_x dx + \mathscr{V}_y dy) = \int \mathscr{V}_t|_{z=0}\; dt \; , \tag{5.18}$$

where $\mathscr{V}_t = \frac{\ell}{z} \mathscr{V}_y$. Now we set $\mathscr{F} = \mathscr{X}^\mu$ and $\mathscr{G} = \delta\Phi \star [\mathscr{X}_\mu, \Phi]_\star$ in (5.10) to get up to order α

$$\begin{aligned} \frac{1}{\ell\alpha^2} \int_D dx\, dy [\mathscr{X}^\mu, \delta\Phi \star [\mathscr{X}_\mu, \Phi]_\star]_\star = -\frac{1}{\ell\alpha^2} \int_{\partial D} \Big(\delta\Phi \star [\mathscr{X}_\mu, \Phi]_\star \partial_x \mathscr{X}^\mu dx \\ + \mathscr{X}^\mu \partial_y (\delta\Phi \star [\mathscr{X}_\mu, \Phi]_\star) dy \Big) = -\int_{\partial D} \delta\Phi \partial_z \Phi|_{z=0}\; dt \; , \end{aligned} \tag{5.19}$$

which is the commutative result. As explained in [1], the α^2 corrections to this go like z^n for $n \in \mathbb{N}$, which vanish on the boundary. To evaluate (5.10)for the second boundary term we just set $\mathscr{F} = [\mathscr{X}^\mu, \Phi]_\star$ and $\mathscr{G} = [\mathscr{X}_\mu, \delta\Phi]_\star$ and then sum over μ. This procedure shows that all contributions to $\mathscr{V}_t$ go like z^n for $n \in \mathbb{N}$ vanishing on the boundary. Since for all orders greater than α^2 they involve higher derivatives, it will produce higher powers of z meaning that all terms must vanish for $\mathscr{O}(\alpha^n)$ with $n \geq 3$. The last equality on (5.19) means that we can fix the boundary value of the field

$$\phi_0(t) = \Phi(0,t) \ , \tag{5.20}$$

and the variational problem is well defined for Dirichlet boundary conditions. Then, the field equation of motion following from (5.8) is

$$\Big(\ell\Delta_t \frac{t}{z} - (t\partial_t + z\partial_z)S_t\Big)\Big(\ell\Delta_t \frac{t}{z} - (t\partial_t + z\partial_z)S_t\Big)\Phi + \Big(\frac{\alpha^2}{4} + \ell^2\Big)\Delta_t^2\Phi = 0 \ . \tag{5.21}$$

The expression (5.21) reduces to second order differential equations in both commutative and asymptotic limits, then it can be solved given sufficient data at the AdS_2 boudary. Using standard techniques (see [1], [5] and [7]) and with the help of the Propagators defined in the first chapter, the on-shell action merely undergoes an overall rescaling when extended to the non-commutative case.

$$S_{nc}[\Phi_{sol}] = -\frac{1}{2\pi}\int dt \int dt' \ \phi_0(t)\phi_0(t')\bigg(\Big(1 - \frac{\alpha^2}{8\ell^2}\Big)\frac{1}{(t-t')^2} + \mathscr{O}(\alpha^4)\bigg) \ . \tag{5.22}$$

The usual AdS/CFT correspondence, for the commutative case $(\alpha = 0)$ comes from the equation of motion

$$\Box\Phi = (\partial_z^2 + \partial_t^2)\Phi = 0 \ . \tag{5.23}$$

The solutions for (5.23) which are everywhere regular can be expressed in terms of the boundary value of the field using the boundary-to-bulk propagator

$$\Phi(z,t) = \int_{\mathbb{R}} dt' \ K(z,t;t')\phi_0(t') \ , \tag{5.24}$$

substituting (5.24) on the usual commutative action, this will leave only the boundary term

$$S[\Phi(z,t)] = -\frac{1}{2\pi}\int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \frac{\phi_0(t)\phi_0(t')}{(t-t')^2} \ . \tag{5.25}$$

In the correspondence, one indentifies the on-shell action with the generating functional of the connected correlation functions for the operators $\mathscr{O}$ associated with ϕ_0 at the boundary. The n-point function is

$$\langle \mathscr{O}(t_1)...\mathscr{O}(t_n)\rangle = \frac{\delta^n S[\Phi[\phi_0]]}{\delta\phi_0(t_1)...\delta\phi_0(t_n)}\bigg|_{\phi_0=0} \ . \tag{5.26}$$

For the two-point function, for example

$$\langle \mathscr{O}(t)\mathscr{O}(t')\rangle = -\frac{1}{2\pi}\frac{1}{(t-t')^2} \; . \tag{5.27}$$

In the non-commutative case, using (5.22), we see that the two-point function is just multiplied by a rescaling factor at the leading order in α

$$\langle \mathscr{O}(t)\mathscr{O}(t')\rangle = -\frac{1}{2\pi}\Big(1-\frac{\alpha^2}{8\ell^2}\Big)\frac{1}{(t-t')^2} \; , \tag{5.28}$$

if ℓ depends on α, we should replace ℓ in the leading order correction by ℓ_0, as will be shown in the next sections. From all the general arguments used in this section, we showed that the AdS/CFT correspondence is applicable in the non-commutative case. The dynamics for the scalar field contains non-trivial non-local interactions without seriously affecting the boundary conformal theory.

5.2 Massive Case

The whole discussion that will be conducted in the following sections is contained in the paper [35]. Starting with the commutative case, we can use the Fefferman-Graham coordinates of the lower hyperboloid, which in the Euclidean case, gives us the Laplacian

$$\mathscr{L}^{(0)} = z^2(\partial_z^2+\partial_t^2) = K^\mu K_\mu \tag{5.29}$$

Where the K is the Killing vectors defined in the last chapter, which are the quadratic casimir of $\mathfrak{so}(2,1)$.The action for a real massive scalar field $\Phi^{(0)}$ in the Euclidean AdS_2 is

$$S[\Phi^{(0)}] = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} dt\; dz \Big\{ (\partial_z\Phi^{(0)})^2 + (\partial_t\Phi^{(0)})^2 + \Big(\frac{m_0\ell_0\Phi^{(0)}}{z}\Big)^2 \Big\} \; , \tag{5.30}$$

where, m_0 is the mass satisfying the Beitenlohner-Freedman Bound and the superscript (0) denotes the commutative theory. The equations of motion for the action are

$$\mathscr{L}^{(0)} = (m_0\ell_0)^2\Phi^{(0)} \; . \tag{5.31}$$

Near the boundary, the solutions will behave as (1.54), the leading term for $z\to 0$ is the power of Δ_-. Assuming that this solution is non-vanishing, the field will be singular in the limit $(m_0\ell_0)^2 > 0$. Away from the boundary, the solutions can be expressed using the boundary to bulk propagator as (1.108), denoting these solutions as $\Phi^{(0)}_{sol}[\phi_0]$ and substituting this in the action (5.30) it gives for $|t-t'| >> \varepsilon$

$$S\left[\Phi^{(0)}_{sol}[\phi_0]\right] = -\frac{1}{2}\int_{\mathbb{R}} dt\Phi^{(0)}_{sol}[\phi_0]\partial_z\Phi^{0}_{sol}[\phi_0]\bigg|_{z=\varepsilon} \tag{5.32}$$
$$= \frac{\Delta_+\Gamma(\Delta_+)}{2\sqrt{\pi}\Gamma(\nu)}\int_{\mathbb{R}^2} dt\ dt' \frac{\phi_0(t)\phi_0(t')}{|t-t'|^{2\Delta_+}} \ ,$$

using the correspondence, lets associate the on-shell action with the generating functional of the n-point connected correlation function for some operator $\mathscr{O}$ defined on the boundary of this space, identifying the source on the boundary with the field ϕ_0. Taking the functional derivative of (5.32)

$$\langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(0)} = \frac{\Delta_+\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma(\nu)|t-t'|^{2\Delta_+}} \ . \tag{5.33}$$

Now, we are going to try the generalization of this method to the non-commutative theory ($ncAdS_2$). Starting with the quantization that preserves the full isometry group, we get into the action

$$S_{nc}[\hat{\Phi}] = -\frac{1}{2\ell}Tr\Big\{[\hat{X}^\mu,\hat{\Phi}][\hat{X}_\mu,\hat{\Phi}] - (\alpha\ell m)^2\Big\} \ . \tag{5.34}$$

The commutative limit corresponds to $(\alpha,\ell,m)\to(0,\ell_0,m_0)$ and m is the mass of the scalar field. This action can be mapped to the Moyal-Weyl plane as we did before in (5.4) adding the mass term to it. After writing the action explicitly as (5.5), we find that

$$S_{nc}[\Phi] = \frac{1}{2\ell}\int_{\mathbb{R}^2} dx\, dy\Big\{(\partial_x\Phi)^2 + \Delta_y\Phi\ \Big(y^2\Delta_y\Phi + 2y\partial_x S_y\Phi - \frac{\alpha^2}{4}\partial_x^2\Delta_y\Phi\Big) \tag{5.35}$$
$$+\ \alpha^2\Big(k+\frac{1}{2}\Big)^2(\Delta_y\Phi)^2 + (m\ell)^2\Phi^2\Big\} \ ,$$

in which we write the field equation for the variables (z,t)

$$\Big(\ell\Delta_t\frac{t}{z} - (t\partial_t + z\partial_z)S_t\Big)\Big(\ell\Delta_t\frac{t}{z} - (t\partial_t + z\partial_z)S_t\Big)\Phi + \Big(\frac{\alpha^2}{4} + \ell^2\Big)\Delta_t^2\Phi = (m\ell)^2\Phi \ . \tag{5.36}$$

We search for pertubative solutions to the field equations by expanding the non-commutative Laplacian, as well ℓ, in powers of α^2

$$\mathscr{L} = \mathscr{L}^{(0)} + \alpha^2\mathscr{L}^{(1)} + \mathscr{O}(\alpha^4) \ , \tag{5.37}$$

using the expression for the non-commutative Killing vectors $K_\star$ we can write $\mathscr{L}^{(1)}$ in terms of (x,y) variables

$$\mathscr{L}^{(1)} = \frac{1}{12}\Big\{-(\ell_0^2+y^2)\partial_y^4 + 24\frac{\ell_1}{\ell_0}\partial_y^2 - 4y\partial_y^3(1+\partial_x) - 3\partial_x\partial_y^2(2+\partial_x)\Big\} \ , \tag{5.38}$$

for

$$\ell = \ell_0 + \frac{\alpha^2}{\ell_0^2}\ell_1 + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) , \tag{5.39}$$

we can simplify (5.38) by taking a similarity transformation (see Appendix A.1.)

$$U\mathscr{L}U^{-1} = \mathscr{L}^{(0)} + \alpha^2 \mathscr{L}_U^{(1)} , \tag{5.40}$$

upon expanding U in powers of α^2

$$U = \mathbb{1} + \alpha^2 G + \mathscr{O}(\alpha^4) , \tag{5.41}$$

we get

$$\mathscr{L}_U^{(1)} = \mathscr{L}^{(1)} + [G, \mathscr{L}^{(0)}] . \tag{5.42}$$

For a particular choice of G

$$G = \frac{1}{96}\Big(3 + 2y\partial_y + 6\partial_x\Big)\partial_y^2 + \frac{1}{\ell_0^2}\Big(\frac{32\ell_1 + 3}{32\ell_0}\Big)(y\partial_y + \partial_x) , \tag{5.43}$$

we get the simple result for the transformed Laplace operator on the Fefferman-Graham Coordinates

$$U\mathscr{L}U^{-1} = z^2(\partial_z^2 + \partial_t^2) - \alpha^2 \frac{1}{8\ell_0^2} z^4 \partial_t^4 + \mathscr{O}(\alpha^4) . \tag{5.44}$$

By defining the transformed field $\Phi_U = U\Phi$ we get to the modified field equation

$$(U\mathscr{L}U^{-1})\Phi_U = (m_0\ell_0)^2 \Phi_U . \tag{5.45}$$

For simplicity, we set $m = m_0 \frac{\ell_0}{\ell}$ in order to get $(m\ell)^2 = (m_0\ell_0)^2$. By direct calculation of (5.42), we find that

$$U = 1 - \frac{\alpha^2}{\ell_0^2}\Big(\frac{z^2}{96}(9 + 4t\partial_t + 6z\partial_z)\partial_t^2 + cz\partial_z\Big) + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) , \tag{5.46}$$

with $c = \frac{\ell_1}{\ell_0} + \frac{3}{32}$. By direct verification, we can apply the inverse map on Φ_U

$$\Phi(z,t) = U^{-1}\Phi_U(z,t) = \left(1 + \frac{\alpha^2}{\ell_0^2}\Big(\frac{z^2}{96}(9 + 4t\partial_t + 6z\partial_z)\partial_t^2 + cz\partial_z\Big) + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right)\right)\Phi_U(z,t) , \tag{5.47}$$

note that the leading corrections of U^{-1} vanishes as we get closer to the boundary

$$\lim_{\varepsilon\to 0}\Phi\Big|_{z=\varepsilon} = \lim_{\varepsilon\to 0}\Phi_U\Big|_{z=\varepsilon} + \mathscr{O}\left(\frac{\alpha}{\ell_0^4}\right) , \tag{5.48}$$

the behaviour of $\partial_z\Phi$ near the boundary is

$$\lim_{\varepsilon\to 0}\partial_z\Phi\Big|_{z=\varepsilon} = \left(1+\frac{\alpha^2}{\ell_0^2}c\right)\lim_{\varepsilon\to 0}\partial_z\Phi\Big|_{z=\varepsilon} + \mathscr{O}\left(\frac{\alpha}{\ell_0^4}\right) . \tag{5.49}$$

In order to simplify the problem, we assume that the source is independent of the perturbation parameter. Using the boundary to bulk propagator defined in the chapter 1, we expand the field in even powers of the pertubation parameters

$$\Phi_U = \Phi^{(0)} + \frac{\alpha^2}{\ell_0^2}\Phi^{(1)} + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) . \tag{5.50}$$

The field $\Phi^{(0)}$ satisfies the free equation. The field $\Phi^{(1)}$ satisfies

$$\left(\mathscr{L}^{(0)} - (m_0\ell_0)^2\right)\Phi^{(1)}(z,t) = \frac{1}{8}z^4\partial_t^4\Phi^{(0)}(z,t) , \tag{5.51}$$

using the expression

$$\Phi_U = \int_{\mathbb{R}} dt' K_{nc}(z;t,t')\phi_0(t') , \tag{5.52}$$

with the use of bulk to bulk propagator $G(z,t;z',t')$ we can derive a expression for $\Phi^{(1)}$

$$\Phi^{(1)}(z,t) = \frac{1}{8}\int_0^\infty dz' z'^2 \int_{\mathbb{R}} dt' G(z,t;z',t') \int_{\mathbb{R}} dt'' \partial_{t'}^4 K(z',t';t'')\phi_0(t'') , \tag{5.53}$$

from the first order terms of the solutions (1.84) anolgue for the non commutative case and

$$K(z,t;t') = \frac{\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma(\Delta_+ - \frac{1}{2})}\left(\frac{z}{z^2+(t-t')^2}\right)^{\Delta_+} , \tag{5.54}$$

we can write down the expression for the non-commutative boundary to bulk propagator up to order α^2

$$K_{nc}(z,t;t') = K(z,t;t') - \frac{\alpha^2}{8\ell_0^2}\int_0^\infty dz' z'^2 \int_{\mathbb{R}} dt'' G(z,t;z',t'')\partial_{t''}^4 K(z',t';t'') + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) . \tag{5.55}$$

Using the assymptotic behaviour of the propagators that we have already calculated on the chapter 1, it follows that $K_{nc}(\varepsilon,t;t') \to K(\varepsilon,t;t')$ as $\varepsilon \to 0$. From the solution of Φ_U we can denote them by $\Phi[\phi_0]$ since they are functionals of ϕ_0. Substituting the solution in the non-commutative action S_{nc} we should obtain the on-shell action. In order to facilitate the next step it is convenient to re-express the action as

$$S_{nc}[\Phi] = \frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dxdy \left\{\Phi\star\left([\mathscr{X}^\mu,[\mathscr{X}_\mu,\Phi]_\star]_\star + (\alpha\ell m)^2\Phi\right) - [\mathscr{X}^\mu,\Phi\star[\mathscr{X}_\mu,\Phi]_\star]_\star\right\} . \tag{5.56}$$

Clearly the quantity in parenthesis vanishes on-shell from the field equation. The remaining term is only defined on the boundary $z=\varepsilon$, since the Moyal star commutator is a total divergence. Thus

$$S_{nc}\left[\Phi[\phi_0]\right] = -\frac{\ell_0}{2\ell}\int_{\mathbb{R}} dt \Phi[\phi_0]\partial_z\Phi[\phi_0]\Big|_{z=\varepsilon} , \tag{5.57}$$

using the boundary behaviour (5.48) and (5.49) one can express the on-shell action up to order $\frac{\alpha^2}{\ell_0^2}$.

$$S_{nc}\Big[\Phi[\phi_0]\Big] = -\frac{\ell_0}{2\ell}\left(1+\frac{\alpha^2}{\ell_0^2}c\right)\int_{\mathbb{R}} dt\Big(\Phi_U\partial_z\Phi_U\Big)\Big|_{z=\varepsilon} + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) , \tag{5.58}$$

writing it in terms of the Green's function

$$-\frac{\ell_0}{2\ell}\left(1+\frac{\alpha^2}{\ell_0^2}c\right)\int_{\mathbb{R}} dt\int_{\mathbb{R}} dt'\int_{\mathbb{R}} dt'' K_{nc}(\varepsilon,t;t')\partial_z K_{nc}(z,t;t'')\Big|_{z=\varepsilon}\phi_0(t')\phi_0(t'') + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) . \tag{5.59}$$

Now we can use this expression to calculate the two-point correlator

$$\begin{aligned}\langle\mathscr{O}(t)\mathscr{O}(t')\rangle = -\frac{\ell_0}{2\ell}\left(1+\frac{\alpha^2}{\ell_0^2}c\right)\int_{\mathbb{R}} dt\int_{\mathbb{R}} dt'\int_{\mathbb{R}} dt''\Big[K_{nc}(\varepsilon,t;t')\partial_z K_{nc}(z,t;t'')\Big|_{z=\varepsilon} \\ +K_{nc}(\varepsilon,t;t')\partial_z K_{nc}(z,t;t'')\Big|_{z=\varepsilon}\Big] + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) ,\end{aligned} \tag{5.60}$$

expanding it in powers of $\frac{\alpha^2}{\ell_0^2}$ and substituting the boundary behaviour of the propagator

$$\langle\mathscr{O}(t)\mathscr{O}(t')\rangle = \langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(0)} + \frac{\alpha^2}{\ell_0^2}\langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(1)} + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) , \tag{5.61}$$

$$\begin{aligned}\langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(1)} = \frac{1}{16}\int_{\mathbb{R}^+} dz'(z')^2\int_{\mathbb{R}} dt''\int_{\mathbb{R}} dt'''\Big\{\partial_z\Big(K(z,t'';t)G(z,t'';z',t''')\Big)\Big|_{z=\varepsilon}\partial_{t'''}^4 K(z',t''';t') \\ +\partial_z\Big(K(z,t'';t')G(z,t'';z',t''')\Big)\Big|_{z=\varepsilon}\partial_{t'''}^4 K(z',t''';t)\Big\} + \Big(c-\frac{\ell_1}{\ell_0}\Big)\langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(0)} ,\end{aligned} \tag{5.62}$$

using the assymptotic behaviour of the propagators and the fact that two Green functions in ε commutes with the limit, one can show that

$$\int dt''\partial_z\Big(K(z,t'';t)G(z,t'';z',t''')\Big)\Big|_{z=\varepsilon} = \frac{1}{2\nu}K(z',t;t''') , \tag{5.63}$$

then, the first order corrections becomes

$$\langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(1)} = \frac{1}{32\nu}\left(I_{\Delta_+}(t,t') + I_{\Delta_+}(t',t)\right) + \frac{3}{32}\langle\mathscr{O}(t)\mathscr{O}(t')\rangle^{(0)} , \tag{5.64}$$

where

$$I_{\Delta_+}(t,t') = \int_{\mathbb{R}^2} dt'' dz\, z^2 K(z,t;t'')\partial^4_{t''} K(z,t;t'') \ . \tag{5.65}$$

Since $K(z,t;t'')$ is only a function of $t-t''$, we can bring the derivative outside the integral. To evaluate the correlation function of two propagators we can make use of the conformal isometries that K has, using the inversion to take this integral to the form (1.106) for a scaling dimension equal to $1-\Delta_+$. It bring us to the expression (5.32). After this, the expression simplifies

$$I_{\Delta_+}(t,t') = \partial^4_{t'}\left(\frac{C^2_{\Delta_+}\sqrt{\pi}\Gamma(\Delta_+ + \frac{3}{2})\Gamma(\Delta_+ - 2)}{\Gamma(\Delta_+ + \frac{1}{2})^2|t-t'|^{2-4\Delta_+}}\right) \ , \tag{5.66}$$

using properties of the Gamma functions we can calculate I_{Δ_+}

$$I_{\Delta_+}(t,t') = \frac{4}{3}\frac{(\Delta_+ + \frac{1}{2})(\Delta_+ - \frac{1}{2})^3(\Delta_+ - \frac{3}{2})}{\sqrt{\pi}\Gamma(\Delta_+ + \frac{1}{2})}\frac{1}{|t-t'|^{2\Delta_+}} \ , \tag{5.67}$$

substituting in (5.64)

$$\langle \mathscr{O}(t)\mathscr{O}(t')\rangle^{(1)} = \frac{\Gamma(\Delta_+)}{32\sqrt{\pi}\Gamma\Delta_+ - \frac{1}{2}}\left[\frac{8}{3}\left(\Delta_+^2 - \frac{1}{4}\right)\left(\Delta_+ - \frac{3}{2}\right) - 3\Delta_+\right]\frac{1}{|t-t'|^{2\Delta_+}} \ . \tag{5.68}$$

As we can see from the equation above, the leading order of the non-commutative correction to the two point function is just a rescaling of the usual two-point correlator for the commutative case. Only the conformal weight factor Δ_+ depends on ℓ_0 and m_0, and for the commutative limit, it doesn't receive any correction, as can be seen if we set $\Delta_+ = 1$ one should find the same result as using standard procedures.

Chapter 6

Interacting Theory

In this section we present the main results of [35] in a deeper level of detail, deducing most of the skiped steps. First we analyse the usual interacting commutative theory and calculate the three point correlator. After, we map the action to the Moyal-Weyl plane and separate the boundary and the bulk parts of the action and solve them using the non-commutative propagators. We finish this chapter showing that the undeformed conformal symmetries are preserved after the MW mapping.

6.1 The commutative case

We start this section by making a brief review of the interacting commutative theory. Initially, we add a cubic term to the free scalar field action (5.1) to get

$$S[\Phi^{(0)}] = \frac{1}{2}\int_{R\times R_+} dt dz \left\{ (\partial_z \Phi^{(0)})^2 + (\partial_t \Phi^{(0)})^2 + \left(\frac{m_0 \ell_0}{z}\right)^2 \Phi^{(0)\,2} + \frac{2\lambda}{3z^2}\Phi^{(0)\,3} \right\} , \tag{6.1}$$

where λ is a real parameter, and the superscript still indicates that we are analising the commutative case. The field equation is then

$$\left(\mathscr{L}^{(0)} - (m_0\ell_0)^2\right)\Phi^{(0)} = \lambda\,\Phi^{(0)\,2} , \tag{6.2}$$

with the same $\mathscr{L}^{(0)}$ as in the last section. We still assume the asymptotic behavior in order to solve (6.2) perturbatively in λ using the boundary-to-bulk and bulk-to-bulk propagators, $K(z,t;t')$ and $G(z,t;z',t')$, defined on the other sections. At zeroth order in λ the solution is (5.54). Up to first order one has

$$\begin{aligned} \Phi^{(0)}(z,t) = &\int dt'\, K(z,t;t')\phi_0(t') \\ &- \lambda \int \frac{dz'dt'}{z'^2}\, G(z,t;z',t') \int dt_1 \int dt_2\, K(z',t';t_1)K(z',t';t_2)\,\phi_0(t_1)\phi_0(t_2) + \mathscr{O}(\lambda^2) . \end{aligned} \tag{6.3}$$

We again denote the solution by $\Phi[\phi_0]$. The on-shell action now includes a bulk term, as well as a boundary term

$$S[\Phi] = S^{bdy}[\Phi] + S^{blk}[\Phi] , \tag{6.4}$$

$$S^{bdy}[\Phi] = -\frac{1}{2}\int_R dt\ \Phi\ \partial_z\Phi\Big|_{z=0}\ ,$$

$$S^{blk}[\Phi] = \frac{\lambda}{3}\int_{R\times R_+}\frac{dtdz}{z^2}\Phi^3\quad .$$

Substituting the solution $\Phi = \Phi^{(0)}[\phi_0]$ on the boundary term we will get

$$\begin{aligned} S^{bdy}\left[\Phi^{(0)}[\phi_0]\right] =& -\frac{1}{2}\int dtdt'dt''\left[K(z,t;t')\partial_z K(z,t;t'')\right]_{z=0}\phi_0(t')\phi_0(t'') \\ &+\frac{\lambda}{2}\int dt\frac{dz'dt'}{z'^2}dt_1dt_2dt_3\partial_z\left(K(z,t;t_1)G(z,t;z',t')\right)\Big|_{z=0}\times \\ &\times K(z',t';t_2)K(z',t';t_3)\phi_0(t_1)\phi_0(t_2)\phi_0(t_3)+\mathscr{O}(\lambda^2) \\ &-\frac{\Delta_+\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma(\Delta_+-\frac{1}{2})}\int dt'dt''\frac{\phi_0(t')\phi_0(t'')}{|t'-t''|^{2\Delta_+}} \\ &+\frac{\lambda\Delta_+}{4\nu}\int\frac{dz'dt'}{z'^2}dt_1dt_2dt_3K(z',t';t_1)K(z',t';t_2)K(z',t';t_3)\times \\ &\times\phi_0(t_1)\phi_0(t_2)\phi_0(t_3)+\mathscr{O}(\lambda^2) \\ &, \end{aligned} \tag{6.5}$$

where we used the asymptotic expressions from the chapter 1. While the first term is exactly (5.32) and will lead to the same 2-point function, the second term will give a non-trivial contribution to the 3-point function. This should be combined with the bulk term (6.4), which after substitution of $\Phi^{(0)} = \Phi^{(0)}_{sol}[\phi_0]$ takes the form

$$\begin{aligned} S^{blk}\left[\Phi^{(0)}[\phi_0]\right] =& \frac{\lambda}{3}\int\frac{dtdz}{z^2}\int dt_1dt_2dt_3\ \times \\ &\times K(z,t;t_1)K(z,t;t_2)K(z,t;t_3)\,\phi_0(t_1)\phi_0(t_2)\phi_0(t_3)+\ \mathscr{O}(\lambda^2)\ . \end{aligned} \tag{6.6}$$

Combining this with (6.5) and using the definition (5.26), the three-point function is

$$<\mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3)>^{(0)} = \lambda\left(\frac{3\Delta_+}{2\nu}+2\right)\int\frac{dzdt}{z^2}\ K(z,t_1;t)K(z,t;t_2)K(z,t;t_3)\ . \tag{6.7}$$

Since these correlator functions are conformally covariant they must depend on the diferences t_1, t_2 and t_3 which set the form of the expression up to a constant factor

$$<\mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3)>^{(0)} = \lambda\left(\frac{3\Delta_+}{2\nu}+2\right)\frac{a_{\Delta_+}}{|t_1-t_2|^{\Delta_+}|t_2-t_3|^{\Delta_+}|t_3-t_1|^{\Delta_+}}\ . \tag{6.8}$$

To compute this coefficient, as done in [ref] we use the inversion $t' = \frac{1}{t}$ as change of variables and after this we use the translational symmetry to take the boundary at the point 0, in order to use the assymptotics for the propagator. The last integral can be calculated using the Feynman parameter method. Finally the coeficient comes to be

$$a_{\Delta_+} = \frac{\Gamma(\Delta_+/2)^3\,\Gamma\Big((3\Delta_+ - 1)/2\Big)}{2\pi\,\Gamma(\nu)^3} \,. \tag{6.9}$$

6.2 Non-commutative Interacting theory

Next we will try generalize all the results obtained on the last section to the non-commutative case. As expected, we add a cubic term to the action (5.30)

$$S_{nc}[\hat{\Phi}] = -\frac{1}{2\ell}\mathrm{Tr}\left\{[\hat{X}^\mu,\hat{\Phi}][\hat{X}_\mu,\hat{\Phi}] - (\alpha\ell m)^2\hat{\Phi}^2 - \frac{2}{3}\alpha^2\lambda\hat{\Phi}^3\right\} \,. \tag{6.10}$$

Now, mapping this action to the Moyal-Weyl plane

$$S_{nc}[\Phi] = -\frac{1}{2\ell\alpha^2}\int_{R^2} dxdy\left\{[\mathscr{X}^\mu,\Phi]_\star\star[\mathscr{X}_\mu,\Phi]_\star - (\alpha\ell m)^2\Phi\star\Phi - \frac{2}{3}\alpha^2\lambda\,\Phi\star\Phi\star\Phi\right\} \,. \tag{6.11}$$

The field equation following from (6.11) is

$$\mathscr{L}\Phi - (\ell m)^2\Phi = \lambda\,\Phi\star\Phi \,, \tag{6.12}$$

where $\mathscr{L}$ is the noncommutative Laplace operator, defined in the last sub-section. We will re-express the field equation for $\Phi_U = U\Phi$ with U defined in (5.46). The resulting equation is

$$\Big[U\mathscr{L}U^{-1} - \ell^2 m^2\Big]\Phi_U = \lambda U\Big[(U^{-1}\Phi_U)\star(U^{-1}\Phi_U)\Big] \,. \tag{6.13}$$

For simplicity, we set $m = \frac{m_0\ell_0}{\ell}$. When we set $\lambda = 0$ the field equation reduces to the free non-commutative one, and the solution is given by

$$\Phi_U(z,t) = \int_{\mathscr{R}} dt'\; K_{nc}(z,t;t')\phi_0(t') \,, \tag{6.14}$$

taking small λ we can solve this equation by replacing the propagators by it's non-commutative analogues, K_{nc} and G_{nc}. The main solution (6.14) expressed in terms of Fefferman-Graham coordinates with K_{nc} expanded up to order α^2 is

$$\begin{aligned} K_{\rm nc}(z,t;t') &= U_{z,t}^{-1}K_{\rm nc}^U(z,t;t') \\ &= K_{\rm nc}^U(z,t;t') + \frac{\alpha^2}{\ell_0^2}\mathscr{D}_{z,t}K(z,t;t') + \mathscr{O}\left(\alpha^4\right) \,, \end{aligned} \tag{6.15}$$

where

$$\mathscr{D}_{z,t} = \frac{1}{8}z^4\partial_t^4 z \,. \tag{6.16}$$

In order to define the bulk-to-bulk propagator, we require that it satisfies

$$\left[(U\mathscr{L}U^{-1})_{z,t}-(\ell m)^2\right]G^U_{\rm nc}(z,t;z',t') = -z^2\,\delta(z-z')\delta(t-t')\;, \tag{6.17}$$

so $G^U_{\rm nc}(z,t;z',t')\to G(z,t;z',t')$ when $\alpha\to 0$ in the commutative limit. Upon expanding in λ, and substituting (6.14) and (6.17) into the field equation the solution to (6.12) is

$$\begin{aligned}\Phi(z,t) =& \int dt'\,K_{\rm nc}(z,t;t')\,\phi_0(t')\\ &-\lambda\int\frac{dz'dt'}{z'^2}\,U^{-1}_{z,t}G^U_{\rm nc}(z,t;z',t')\int dt_1dt_2\,U_{z',t'}[K^{(1)}_{\rm nc}\star K^{(2)}_{\rm nc}](z',t')\,\phi_0(t_1)\phi_0(t_2)\\ &+\mathscr{O}(\lambda^2)\;,\end{aligned} \tag{6.18}$$

where $K^{(n)}_{\rm nc}(z,t)$ denotes the function $K_{\rm nc}(z,t;t_n)$ and the star-product is with respect to the explicitly shown variables. The non-commutative corrections on $G^U_{\rm nc}(z,t;z',t')$ can be computed perturbatively in powers of α^2. If we write

$$G^U_{\rm nc}(z,t;z',t') = G(z,t;z',t')+\alpha^2G^{(1)}(z,t;z',t')+\mathscr{O}(\alpha^4)\;, \tag{6.19}$$

then the leading order non-commutative correction $G^{(1)}(z,t;z',t')$ satisfies

$$\left[\mathscr{L}^{(0)}_{z,t}-(\ell m)^2\right]G^{(1)}(z,t;z',t') = \frac{1}{\ell_0^2}\,\mathscr{D}_{z,t}\,G(z,t;z',t')\;, \tag{6.20}$$

The solution is

$$G^{(1)}(z,t;z',t') = -\frac{1}{8\ell_0^2}\int dz''dt''z''^2G(z,t;z'',t'')\,\partial^4_{t''}\,G(z'',t'';z',t')\;. \tag{6.21}$$

Upon substituting the solution to (6.18) in (6.11) we get the on-shell action. After this we can divide the latter in two main contributions as we did in the commutative case

$$S_{nc}[\Phi] = S^{bdy}_{nc}[\Phi]+S^{blk}_{nc}[\Phi]\;, \tag{6.22}$$

where $S^{bdy}_{nc}[\Phi]$ was defined in (5.56) as the remaining term after the cutoff and for the bulk term

$$S^{blk}_{nc}[\Phi] = \frac{\lambda}{3\ell}\int_{R^2}dxdy\,\Phi\star\Phi\star\Phi\;. \tag{6.23}$$

Substituting the solution into the bulk and the boundary terms $S^{blk}_{nc}[\Phi]$ and $S^{bdy}_{nc}[\Phi]$, respectively, collecting the third order terms in the solution and converting to Fefferman-Graham coordinates gives

$$S^{blk}_{nc} = \frac{\lambda}{3}\int\frac{dzdt}{z^2}\,dt_1\,dt_2\,dt_3\;[K^{(1)}_{\rm nc}\star K^{(2)}_{\rm nc}\star K^{(3)}_{\rm nc}](z,t)\,\phi_0(t_1)\phi_0(t_2)\phi_0(t_3)\;. \tag{6.24}$$

and

$$S^{bdy}_{nc} = \frac{\lambda}{2}\int \frac{dz'dt'}{z'^2} dt\, dt_1\, dt_2\, dt_3\; \partial_z\Big(K_{nc}(z,t;t_1)U^{-1}_{z,t}G^U_{nc}(z,t;z',t')\Big)\Big|_{z=0}$$
$$\times\; U_{z',t'}[K^{(2)}_{\rm nc}\star K^{(3)}_{\rm nc}](z',t')\,\phi_0(t_1)\phi_0(t_2)\phi_0(t_3)\;. \qquad (6.25)$$

In the commutative limit, we recover the commutative boundary term (6.2). If we take $\alpha \to 0$ all the non-commutative corrections vanish and we have $K_{\rm nc}(z,t;t') \to K(z,t;t')$, $G^U_{\rm nc}(z,t;z',t') \to G(z,t;z',t')$ and $U_{z,t} \to 1$. Now we evaluate some assymptotics in order to calculate explicitly the solutions of the field equations. Using the definitions for $K(z,t;t') = K_\Delta(z,t;t')$ and $G(\xi)$ given by (5.54) and (1.79), respectively, and we defined $\Delta_+ = \Delta$, $\Delta_- = 1-\Delta$. Taking the derivative of K_Δ with respect to z, one finds

$$\partial_z K_\Delta(z,t;t') = \frac{\Delta}{z}K_\Delta(z,t;t') - (2\Delta-1)K_{\Delta+1}(z,t;t')\;.$$

Combining this with the asymptotics for $K(z,t;t')$, which trivially follows from (1.83) and (5.54) one gets

$$\begin{aligned} K(z,t;t') &\underset{z\to 0}{\longrightarrow} z^{1-\Delta}\delta(t-t')\;,\\ \partial_z K(z,t;t') &\underset{z\to 0}{\longrightarrow} (1-\Delta)z^{-\Delta}\delta(t-t')\;. \end{aligned} \qquad (6.26)$$

Before taking the limit $z \to 0$ for $K(z,t;t')\partial_z K(z,t;t'')$ first assume that $|t'-t''| \gg \varepsilon > 0$. From (5.54) one gets

$$K(z,t;t')\partial_z K(z,t;t'') = z^{\Delta-1}K(z,t;t')\Delta C_\Delta\left(\frac{1}{z^2+(t-t'')^2}\right)^{\Delta}\frac{-z^2+(t-t'')^2}{z^2+(t-t'')^2}\;. \qquad (6.27)$$

Using (6.26) one more time and taking into account that $|t'-t''| \gg \varepsilon > 0$ we obtain

$$K(z,t;t')\partial_z K(z,t;t'') \underset{z\to 0}{\longrightarrow} \Delta C_\Delta \delta(t-t')\frac{1}{|t'-t''|^{2\Delta}}\;, \qquad (6.28)$$

which is now valid for any $t' \neq t''$ on the boundary.

To get the analogous results for the bulk-to-bulk propagator we just use the expansion of the hypergeometric function in the definition (1.79)

$$G(\xi) = \frac{C_\Delta}{2\Delta-1}\left(\frac{\xi}{2}\right)^{\Delta}\left(1 + \frac{\frac{\Delta}{2}\left(\frac{\Delta}{2}+\frac{1}{2}\right)}{\Delta+\frac{1}{2}}\xi^2 + \mathscr{O}(\xi^4)\right)\;.$$

Then taking into account

$$\xi = \frac{2zz'}{z'^2 + (t-t')^2} + \mathscr{O}(z^3) \quad \text{and} \quad \partial_z \xi = \frac{1}{z}\xi - \frac{1}{z'}\xi^2 \;, \tag{6.29}$$

we immediately get

$$\begin{aligned}
G(\xi) &\xrightarrow[z\to 0]{} \frac{C_\Delta}{\Delta - 1}\left(\frac{z'}{z'^2 + (t-t')^2}\right)^{\Delta} z^{\Delta} \equiv \frac{1}{2\Delta - 1} z^{\Delta} K(z',t';t) \;, \\
\partial_z G(\xi) &\xrightarrow[z\to 0]{} \frac{\Delta}{2\Delta - 1} z^{\Delta - 1} K(z',t';t) \;.
\end{aligned} \tag{6.30}$$

We also need to evaluate $z \to 0$ behaviour of $\partial_z \left(K(z,t;t')G(z,t;z'',t'')\right)$ which can be found using (6.26) and (6.30)

$$\partial_z \left(K(z,t;t')G(z,t;z'',t'')\right) \xrightarrow[z\to 0]{} \frac{1}{2\Delta - 1}\delta(t-t')K(z'',t'';t') \;. \tag{6.31}$$

Now we return to the evaluation of the three point function. The sum of (6.25) and (6.24) gives all the ϕ_0^3 terms in the non-commutative on-shell action. So the expression for the three-point function is easily calculated from

$$\begin{aligned}
< \mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3) > = &\left.\frac{\delta^3 S_{nc}\Big[\Phi_{sol}[\phi_0]\Big]}{\delta\phi_0(t_1)\delta\phi_0(t_2)\delta\phi_0(t_3)}\right|_{\phi_0 = 0} = \\
= \frac{\lambda}{2}\int \frac{dz dt}{z^2}\Bigg\{ &\int dt' \, \partial_{z'}\Big(K_{\rm nc}(z',t';t_1)U^{-1}_{z',t'}G^U_{\rm nc}(z',t';z,t)\Big)\Big|_{z'=0} \cdot U_{z,t}[K^{(2)}_{\rm nc} \star K^{(3)}_{\rm nc}](z,t) \\
&+ \frac{2}{3}\,[K^{(1)}_{\rm nc} \star K^{(2)}_{\rm nc} \star K^{(3)}_{\rm nc}](z,t)\Bigg\} + \text{all permutations of } (t_1,t_2,t_3) \;.
\end{aligned} \tag{6.32}$$

Near the boundary the first term in the integrand can be expanded in α^2 using the assymptotics calculated above. Using these asymptotics in the definitions (5.55), (6.15) and (6.19) one easily establishes the following asymptotic formulas:

$$\begin{aligned}
K_{\rm nc}(z,t;t') &\to z^{1-\Delta_+}\Big(1 + \frac{3}{32}\frac{\alpha^2}{\ell^2}(1-\Delta_+)\Big)\delta(t-t') \;+\; \mathscr{O}\left(\alpha^4\right) \;, \\
\partial_z K_{\rm nc}(z,t;t') &\to z^{-\Delta_+}\Big(1 - \Delta_+ + \frac{3}{32}\frac{\alpha^2}{\ell^2}(1-\Delta_+)^2\Big)\delta(t-t') \;+\; \mathscr{O}\left(\alpha^4\right) \;, \\
U^{-1}_{z,t}G^U_{\rm nc}(z,t;z',t') &\to \frac{1}{2\Delta_+ - 1} z^{\Delta_+} K^U_{\rm nc}(z',t';t)\Big(1 + \frac{3}{32}\frac{\alpha^2}{\ell^2}\Delta_+\Big) + \mathscr{O}(\alpha^4) \;, \\
\partial_z(U^{-1}_{z,t}G^U_{\rm nc}(z,t;z',t')) &\to \frac{\Delta_+}{2\Delta_+ - 1} z^{\Delta_+ - 1} K^U_{\rm nc}(z',t';t)\Big(1 + \frac{3}{32}\frac{\alpha^2}{\ell^2}\Delta_+\Big) + \mathscr{O}(\alpha^4) \;,
\end{aligned} \tag{6.33}$$

which leads to the $z \to +0$ value for the relevant term in

(6.32)

$$\partial_z\Big(K_{\rm nc}(z,t;t_1)U^{-1}_{z,t}G^U_{\rm nc}(z,t;z',t')\Big)\Big|_{z=0} \to \frac{1}{2\Delta_+-1}\left(1+\frac{3}{32}\frac{\alpha^2}{\ell^2}\right)K^U_{\rm nc}(z',t';t)\delta(t-t_1)+\mathscr{O}(\alpha^4)\ . \tag{6.34}$$

Substituting into (6.32) gives

$$\begin{aligned}&< \mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3) >\\ &= \frac{\lambda}{2}\int\frac{dz dt}{z^2}\Bigg\{\frac{1}{2\Delta_+-1}\Big(1+\frac{3}{32}\frac{\alpha^2}{\ell^2}\Big)U_{z,t}K^{(1)}_{\rm nc}(z,t)\cdot U_{z,t}[K^{(2)}_{\rm nc}\star K^{(3)}_{\rm nc}](z,t)\\ &+\frac{2}{3}\,[K^{(1)}_{\rm nc}\star K^{(2)}_{\rm nc}\star K^{(3)}_{\rm nc}](z,t)\Bigg\} + \text{all permutations of } (t_1,t_2,t_3)\\ &+\mathscr{O}\left(\alpha^4\right)\ .\end{aligned} \tag{6.35}$$

In the next sections we will analyze the result (6.35) and it would be expected that it has the same conformal properties as the commutative 3−point function (6.8) (at least up to leading order in α^2). In addition to this we will demonstrate that the non-commutative 3-point function has the scaling and translational invariance.

6.3 Three Point Function and Conformal Invariance

6.3.1 Translational Invariance

Let's try to show that the three point function has the translational symmetry, i.e.:

$$\langle\mathscr{O}(t_1+a)\mathscr{O}(t_2+a)\mathscr{O}(t_3+a)\rangle^{(0)} = \langle\mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3)\rangle^{(0)}\ . \tag{6.36}$$

In order to do this, first I will show what is the result of the translation on every term that belongs to the three point function. Starting with

$$K(z,t;t') = \left(\frac{z}{z^2+(t-t')^2}\right)^{\Delta_+} \tag{6.37}$$

Making the translation in t and t', one can show that

$$K(z,t+a;t'+a) = \left(\frac{z}{z^2+((t+a)-(t'+a))^2}\right)^{\Delta_+} = K(z,t;t')\ . \tag{6.38}$$

Clearly, the Green Function $G(z,t;z',t')$ is also translational invariant because of it's definition, i.e. that depends on $\delta(t-t')$:

$$G(z,t+a;z',t'+a)=\frac{C_{\Delta_+}}{2\nu}\left(\frac{\xi}{2}\right)^{\Delta_+}{}_2F_1\left(\frac{\Delta_+}{2},\frac{\Delta_+}{2}+\frac{1}{2};\nu+1;\xi^2\right)\ , \tag{6.39}$$

since, for the translated case

$$\xi(t+a,t'+a)=\frac{2z'z}{z^2+(z')^2+((t-a)-(t'-a))^2}=\xi(t,t') \tag{6.40}$$

Now, lets use the definition of the non-commutative Boundary to Boundary Propagator:

$$K_{nc}(z,t;t')=K(z,t;t')-\frac{\alpha^2}{8\ell_0^2}\int_{\mathbb{R}_+}(z')^2dz'\int_{\mathbb{R}}dt''G(z,t;z',t'')\partial_{t''}^4K(z',t';t'')\ , \tag{6.41}$$

Using (6.38), we can show that the same procedure applies to K_{nc}:

$$K_{nc}(z,t+a;t'+a)=K(z,t+a;t'+a)-\frac{\alpha^2}{8\ell_0^2}\int_{\mathbb{R}_+}(z')^2dz'\int_{\mathbb{R}}dt''G(z,t+a;z',t'')\partial_{t''}^4K(z',t'+a;t'')z, \tag{6.42}$$

calling $t''=\tilde{t}+a$, clearly $\partial_{t''}=(\partial_{t''}\tilde{t})\partial_{\tilde{t}}=\partial_{\tilde{t}}$, and $dt''=d\tilde{t}$, we have:

$$K_{nc}(z,t+a;t'+a)=K(z,t+a;t'+a)-\frac{\alpha^2}{8\ell_0^2}\int_{\mathbb{R}_+}(z')^2dz'\int_{\mathbb{R}}d\tilde{t}G(z,t+a;z',\tilde{t}+a)\partial_{\tilde{t}}^4K(z',t'+a;\tilde{t}+a)\ , \tag{6.43}$$

since $\tilde{t}$ is a dummy variable and renaming the variables, i.e. $\bar{t}\to t$, we have the following:

$$K_{nc}(z,t+a;t'+a)=K_{nc}(z,t;t')\ . \tag{6.44}$$

To go further ahead, lets apply $U_{z,t}^{-1}$ to K_{nc}. Using the definition:

$$U_{z,t}^{-1}=\left(1+\frac{\alpha^2}{\ell_0^2}\left(\frac{z^2}{96}(9+4t\partial_t+6z\partial_z)\partial_t^2+cz\partial_z\right)\right)+\mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right)\ , \tag{6.45}$$

and translating this operator, we find that:

$$U_{z,t+a}^{-1}=U_{z,t}^{-1}+\frac{a\alpha^2z^2}{24\ell_0}\partial_t^3=U_{z,t}^{-1}+J(a,t)\ , \tag{6.46}$$

where

$$J(a,t)=\frac{a\alpha^2z^2}{24\ell_0^2}\partial_t^3\ . \tag{6.47}$$

With this result, we just use the definition $\tilde{K}_{nc} = U_{z,t}^{-1} K_{nc}$ as follows:

$$\tilde{K}_{nc}(z,t+a;t'+a) = U_{z,t+a}^{-1} K_{nc}(z,t+a;t'+a) = \tilde{K}_{nc}(z,t;t') + J(a,t) K_{nc}(z,t;t') \ . \tag{6.48}$$

In this step we found the first non-trivial term. From now, we will denote $J(a,t) = \mathscr{J}_{z,t}(a)$ and up to order α^2 we will have

$$\tilde{K}_{nc}(z,t+a;t'+a) = \tilde{K}_{nc}(z,t;t') + \mathscr{J}_{z,t}(a) K_{nc}(z,t;t') \ . \tag{6.49}$$

Clearly, when writing the star product for x,t variables, we expect that at least up to order 1 the star product remains invariant on translations of the parameter t, up to order α^2 as demonstrated on the equations (4.69) and (4.70) and the non-trivial terms comes after the translation of the equation (4.71). To calculate the $\mathscr{O}(\alpha^2)$ term lets explicitly calculate each product

$$\begin{aligned}(\partial_x^2 A)(\partial_y^2 B) + (\partial_x^2 B)(\partial_y^2 A) = \frac{z^2}{\ell^2}\Big[& z^2(\partial_t^2 A \partial_z^2 B + \partial_t^2 B \partial_z^2 A) + \\ &+ 2tz(\partial_z \partial_t A \partial_t^2 B + \partial_z \partial_t B \partial_t^2 A) + 2t^2 \partial_t^2 A \partial_t^2 B + \\ &+ \Big(t(\partial_t A \partial_t^2 B + \partial_t B \partial_t^2 A) + z(\partial_z A \partial_t^2 B + \partial_z B)\partial_t^2 A\Big)\Big] \ ,\end{aligned} \tag{6.50}$$

the other product is

$$\begin{aligned}2(\partial_x \partial_y A)(\partial_y \partial_x B) = \frac{2z^2}{\ell^2}\Big[& z^2(\partial_t \partial_z A \partial_t \partial_z B) + zt(\partial_t^2 A \partial_z \partial_t B + \partial_t^2 B \partial_t \partial_z A) \\ &+ t^2(\partial_t^2 A \partial_t^2 B) + z(\partial_z \partial_t A \partial_t Ba + \partial_z \partial_t B \partial_t A) + \partial_t A \partial_t B \\ &+ t(\partial_t A \partial_t^2 B + \partial_t B \partial_t^2 A)\Big] \ ,\end{aligned} \tag{6.51}$$

subtracting (6.50) from (6.51) and substituting it on the star product, the order α^2 term is

$$\begin{aligned}\mathscr{O}(\alpha^2) = \frac{\alpha^2 z^2}{8\ell^2}\Big(& z^2(2\partial_z \partial_t A \partial_z \partial_t B - \partial_z^2 A \partial_t^2 B - \partial_z^2 B \partial_t^2 A) + 2\partial_t A \partial_t B \\ &+ z(2\partial_z \partial_t A \partial_t B + 2\partial_z \partial_t B \partial_t A - \partial_z A \partial_t^2 B - \partial_z B \partial_t^2 A) \\ &+ t(\partial_t^2 A \partial_t B + \partial_t^2 B \partial_t A)\Big) \ .\end{aligned} \tag{6.52}$$

Up to order α^2 the translated star product has a deformation

$$\star_{z,t+a} = \star_{z,t} + \mathscr{S}_{(z,t)}(a) \ , \tag{6.53}$$

where

$$\mathscr{S}_{(z,t)}(a) = \frac{a\alpha^2 z^2}{8\ell^2}\left(\overleftarrow{\partial}_t^2\overrightarrow{\partial}_t + \overleftarrow{\partial}_t\overrightarrow{\partial}_t^2\right) . \tag{6.54}$$

Now, lets use (6.35) written using the functions defined above:

$$\begin{aligned}(6.35) = \frac{F(\alpha,\ell,\Delta_+)\lambda}{2}\int_{\mathbb{R}\times\mathbb{R}_+}\frac{dzdt}{z^2}\Bigg\{\int_{\mathbb{R}} dt'\partial_{z'}\left(\tilde{K}_{nc}(z',t'+a;t_1+a)\tilde{G}_{nc}(z',t'+a;z,t+a)\right)\Bigg|_{z'=\varepsilon}\\ \times\left[\tilde{K}_{nc}(z,t+a;t_2+a)\star_{z,t+a}\tilde{K}_{nc}(z,t+a;t_3+a)\right] - \\ +\frac{2}{3}\tilde{K}_{nc}(z,t+a;t_1+a)\star_{z,t+a}\tilde{K}_{nc}(z,t+a;t_2+a)\star_{z,t+a}\tilde{K}_{nc}(z,t+a;t_3+a)\Bigg\} ,\end{aligned} \tag{6.55}$$

with

$$F(\alpha.\ell,\Delta_+) = \frac{1}{2\Delta_+ - 1}\left(1+\frac{3}{32}\frac{\alpha^2}{\ell^2}\right) . \tag{6.56}$$

The G_{nc} satisfies the following equation

$$\left[(U\mathscr{L}U^{-1})_{z,t} - (\ell_0 m_0)^2\right]G_{nc}(z,t;z',t') = -z^2\delta(t-t')\delta(z-z') . \tag{6.57}$$

This Green's function can be computed perturbatively by taking

$$G_{nc}(z,t;z',t') = G(z,t;z',t') + \alpha^4 G^{(1)}(z,t;z',t') + \mathscr{O}\left(\frac{\alpha^2}{\ell_0^4}\right) , \tag{6.58}$$

using (6.57), (6.58) and the expression of the Lagrangian on (6.58) in written in corrdinates (z,t) one can show that

$$G_{nc}(z,t;z',t') = G(z,t;z',t') - \frac{\alpha^2}{8\ell_0^2}\int dz''dt''(z'')^2 G(z,t;z'',t'')\partial_{t''}^4 G(z'',t'';z',t') , \tag{6.59}$$

upon transforming we obtain

$$\tilde{G}_{nc}(z,t;z',t') = U_{z,t}^{-1}G_{nc}(z,t;z',t') , \tag{6.60}$$

with this, we can translate $\tilde{G}$ up to order α^2

$$\tilde{G}_{nc}(z,t+a;z',t'+a) = \tilde{G}_{nc}(z,t;z',t') + \mathscr{J}_{z,t}(a)G_{nc}(z,t;z',t') . \tag{6.61}$$

Finally we will verify if the translational invariance holds for t_i, to the other terms we just make permutations over indices. Making $t_i \to t_i + a$, $i = \overline{1,3}$ and $t \to t + a$. To simplify the notation I will write only the specific time dependence, i.e. $\tilde{K}_{nc}(z,t'+a;t_1+a) = \tilde{K}_{nc}(t_1)$, if I need to specify the other variables I will return to the original notation. Taking the first term on (6.55), using (6.44) and (6.47) the RHS up to order α^2 is

$$\tilde{K}_{nc}(t_1+a)\tilde{G}_{nc}(t+a,t'+a) = \Big(\tilde{K}_{nc}(t_1) + \mathscr{J}_{z,t'}K_{nc}(t_1)\Big) \times\Big(\tilde{G}_{nc}(t,t') + \mathscr{J}_{z,t}G_{nc}(t,t')\Big) \;, \tag{6.62}$$

the translation of the second part of the first term is (Here, the other dependence is on t, not t'):

$$\tilde{K}_{nc}(t_2+a)\star_{z,t+a}\tilde{K}_{nc}(t_3+a) = \Big(\tilde{K}_{nc}(t_2) + \mathscr{J}_{t}K_{nc}(t_2)\Big) \times\Big(\star_{z,t} + \mathscr{S}_{(z,t)}\Big)\times\Big(\tilde{K}_{nc}(t_3) + \mathscr{J}_{t}K_{nc}(t_3)\Big) \;. \tag{6.63}$$

In the last equation we omited the dependence of z of $\mathscr{J}_{z,t}$. Now we define equations (A) and (B) up to order α^2 as

$$(A) = \tilde{K}_{nc}(t_1)\tilde{G}_{nc}(t,t') + G_{nc}(t,t')\Big(\mathscr{J}_{t'}K_{nc}(t_1)\Big) + K_{nc}(t_1)\mathscr{J}_{t}G_{nc}(t,t') \;,$$
$$(B) = \tilde{K}_{nc}(t_2)\star_{z,t}\tilde{K}_{nc}(t_3) + K_{nc}(t_2)\mathscr{S}_{(z,t)}K_{nc}(t_3) + K_{nc}(t_3)\mathscr{J}_{t}K_{nc}(t_2) + K_{nc}(t_2)\mathscr{J}_{t}K_{nc}(t_3) \;.$$

The first term on (6.55) will be given by the product of (A) and (B). Doing the same for the second term one gets

$$\begin{aligned}\tilde{K}_{nc}(t_1+a)\star_{z,t+a}\tilde{K}_{nc}(t_2+a)\star_{z,t+a}\tilde{K}_{nc}(t_3+a) &= \tilde{K}_{nc}(t_1)\star_{z,t}\tilde{K}_{nc}(t_2)\star_{z,t}\tilde{K}_{nc}(t_3)\\ &+\Big(K_{nc}(t_1)\mathscr{S}_{(z,t)}K_{nc}(t_2)\Big)K_{nc}(t_3) + K_{nc}(t_1)\Big(K_{nc}(t_2)\mathscr{S}_{(z,t)}K_{nc}(t_3)\Big) + K_{nc}(t_2)\Big(K_{nc}(t_1)\mathscr{S}_{(z,t)}K_{nc}(t_3)\Big)\\ &+\Big(\mathscr{J}_{t}K(t_1)\Big)K(t_2)K(t_3) + \Big(\mathscr{J}_{t}K(t_3)\Big)K(t_1)K(t_2) + \Big(\mathscr{J}_{t}K(t_1)\Big)K(t_2)K(t_3) \;,\end{aligned} \tag{6.64}$$

where we can use (6.54) to further simplify (6.64)

$$\Big(K_{nc}(t_i)\star_{(z,t)}K_{nc}(t_j)\Big)\star_{z,t}\tilde{K}_{nc}(t_k) = \Big(K(t_i)\mathscr{S}_{(z,t)}K(t_j)\Big)K(t_k) + \Big(K(t_i)K(t_j)\Big)\mathscr{S}_{(z,t)}K(t_k) + \mathscr{O}(\alpha^3) \;, \tag{6.65}$$

note that using (6.47) and (6.41) we have:

$$\mathscr{J}_{t}K_{nc}(t_i) = \frac{a\alpha^2 z^2}{24\ell_0^2}\partial_t^3 K(z,t;t_i) + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) \;. \tag{6.66}$$

Using (6.54) and (6.41) we also have

$$K_{nc}(t_i)\mathscr{S}_{(z,t)}K_{nc}(t_j) = \frac{az^2\alpha^2}{8\ell^0}\Big((\partial_t^2 K(z,t;t_i))\partial_t K(z,t,t_j) + (\partial_t^2 K(z,t;t_j))\partial_t K(z,t,t_i)\Big) \tag{6.67}$$

Similarly we can simplify the terms containing any combination of G_{nc} and $\mathscr{J}$ or $\mathscr{S}$. On the translations of G we used the fact that G_{nc} is translationaly invariant up to order α^2 as can be seen in (6.40) in addition to (6.61). Note that, by simply differentiation

$$\frac{\alpha^2 z^2 a}{4!\,\ell_0^2}\partial_t^3\Big(K(t_2)K(t_3)\Big) = K(t_2)\mathscr{S}_{(z,t)}K(t_3) + K(t_2)\mathscr{J}_t K(t_3) + K(t_3)\mathscr{J}_t K(t_2)\ , \tag{6.68}$$

and

$$\begin{aligned}
\frac{\alpha^2 z^2 a}{4!\,\ell_0^2}\partial_t^3\Big(K(t_1)K(t_2)K(t_3)\Big) &= \Big(K_{nc}(t_1)\mathscr{S}_{(z,t)}K_{nc}(t_2)\Big)K_{nc}(t_3) \\
K_{nc}(t_1)\Big(K_{nc}(t_2)\mathscr{S}_{(z,t)}K_{nc}(t_3)\Big) &+ K_{nc}(t_2)\Big(K_{nc}(t_1)\mathscr{S}_{(z,t)}K_{nc}(t_3)\Big) \\
+\Big(\mathscr{J}_t K(t_1)\Big)K(t_2)K(t_3) &+ \Big(\mathscr{J}_t K(t_3)\Big)K(t_1)K(t_2) + \Big(\mathscr{J}_t K(t_2)\Big)K(t_3)K(t_1) \\
+\Big(K(t_1)K(t_2)\Big)\mathscr{S}_{(z,t)}K(t_3) &+ \Big(K(t_3)K(t_1)\Big)\mathscr{S}_{(z,t)}K(t_2) + \Big(K(t_2)K(t_3)\Big)\mathscr{S}_{(z,t)}K(t_1)\ ,
\end{aligned} \tag{6.69}$$

clearly

$$\frac{\alpha^2 z^2 a}{4!\,\ell^2}\partial_t^3 = \mathscr{J}_t(a)\ , \tag{6.70}$$

with this, the fist term of (6.55), obtained by the product of (A) and (B) is the usual $\langle \mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3)\rangle$ acted by the operator $\mathscr{J}_t(a)$. Using the most simplified expression (6.35) applying the result found in (6.69) we finally find that

$$\begin{aligned}
&< \mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3) > + \frac{\lambda}{2}\int \frac{dz\,dt}{z^2}\Big\{F(\alpha,\ell,\Delta_+)\Big(\mathscr{J}_t K^{(1)}(z,t)[K^{(2)}K^{(3)}](z,t) + K^{(1)}(z,t)\mathscr{J}_t[K^{(2)}K^{(3)}_{\rm nc}](z,t)\Big) \\
&+ \frac{2}{3}\,\mathscr{J}_t[K^{(1)}K^{(2)}K^{(3)}](z,t)\Big\} + \text{all permutations of } (t_1,t_2,t_3) \\
&+ \mathscr{O}\left(\alpha^4\right)\ .
\end{aligned} \tag{6.71}$$

we can see that none of the non-commutative terms influenced the final result since all the corrections are of order α^2 at least. Combining all the terms and using the translational invariance of $K^U_{\rm nc}(z,t;t')$, we see that the contribution of the boundary term to the 3-point function is explicitly translationally invariant. The term corresponding to the bulk contribution to the correlator is also translationally invariant due to the fact that the non-invariant term coming from (6.71) is given by integral of a total derivative, which will drop out even without any symetrization.

6.3.2 Scaling Invariance

First, we will establish the behaviour of (6.71) under the simultaneous scaling of t_i, $i = 1,2,3$: $t_i \to \mu t_i$, where μ is a constant parameter. Using (1.79), (5.54), (5.46), and (6.15) one can easily see that under the simultaneous rescalling of *all* the variables the relevant quantities have the following behaviour, starting by the propagator $K(z,t;t')$

$$K(\mu z, \mu t; \mu t') = \mu^{-\Delta_+} \left(\frac{z}{z^2 + (t-t')^2} \right)^{\Delta_+} .$$

Now, i will show that the transformed $\mathscr{L}$ operator doesn't change under rescaling:

$$U\mathscr{L}U^{-1}(\mu z, \mu t) = \left(\cancel{\mu^2} z^2 (\cancel{\mu^{-2}}(\partial_t^2 + \partial_z^2) - \frac{\alpha^2 \cancel{\mu^4} z^4}{8\ell_0^4 \cancel{\mu^4}}) \partial_t^4 + \mathscr{O}\left(\frac{\alpha^4}{\ell_0^4}\right) \right) = U\mathscr{L}U^{-1}(z,t) ,$$

since ξ does not change under rescaling, we have that G is invariant.

$$G(\mu z, \mu t; \mu z', \mu t') = \frac{C_{\Delta_+}}{2\nu} \left(\frac{\xi}{2}\right)^{\Delta_+} {}_2F_1\left(\frac{\Delta_+}{2}, \frac{\Delta_+}{2} + \frac{1}{2}; \nu + 1; \xi^2\right), \quad \xi = \frac{2z'z\cancel{\mu^2}}{\cancel{\mu^2}(z^2 + (z')^2 + (t-t')^2)}$$

$$[U\mathscr{L}U^{-1}_{\mu z, \mu t} - (\ell_0 m_0)^2] G_{nc}(\mu z, \mu t; \mu z', \mu t') = -\frac{\cancel{\mu^2}}{\cancel{|\mu|^2}} z^2 \delta(z - z') \delta(t - t') .$$

For the star product, lets define it this way, with i, j running between x, y:

$$\star_{x,y} = \exp\left(\overleftarrow{\partial_i} \frac{i\alpha\varepsilon^{ij}}{2} \overrightarrow{\partial_j} \right) \qquad (\partial_x)^n = \sum_{l=0}^{n} \begin{pmatrix} l \\ k \end{pmatrix} (-1)^n (z\partial_z)^{k-l} (t\partial_t)^l , \quad ,(\partial_y)^n = \left(\frac{z}{\ell}\partial_t\right)^n ,$$

changing the variables to z and t and expanding this up to order $\mathscr{O}(\alpha^2)$, one should find that

$$\star_{z,t} = 1 - \frac{i\alpha}{2} \left(\overleftarrow{\partial_z} z^2 \overrightarrow{\partial_t} - \overleftarrow{\partial_t} z^2 \overrightarrow{\partial_z} \right) + \mathscr{O}\left(\alpha^2\right) ,$$

note that, for every combinations of x and y derivatives, the scaling factors cancel out:

$$\partial_x = -z\partial_z - t\partial t, \quad \partial_y = \frac{z}{\ell}\partial_t .$$

So, for all orders, the star commutator is invariant under the rescaling. Before tackling the main equation, note that the U operator is also invariant under rescaling, by direct inspections is clear to see it. All the other elements of the main equations are considered below:

$$K_{nc}(\mu z,\mu t;\mu t') = \mu^{-\Delta_+}K(z,t;t') - \frac{\alpha^2}{8\ell_0^2}\int_{\mathbb{R}_+}(z')^2dz'\int_{\mathbb{R}}dt''G(\mu z,\mu t;z',\mu t')\partial_{t''}^4 K(z',\mu t';t'')\ ,$$

taking $\tilde{z} = \frac{z'}{\mu}$, $\tilde{t} = \frac{t''}{\mu}$, $\mu d\tilde{z} = dz'$, $\mu d\tilde{t} = dt''$ and $\partial_{t''}^4 = \mu^{-4}\partial_{\tilde{t}}^4$ we get to the following:

$$K_{nc}(\mu z,\mu t;\mu t') = \mu^{-\Delta_+}K(z,t;t') - \frac{\alpha^2}{8\ell_0^2}\int_{\mathbb{R}_+}\cancel{\mu^2}\tilde{z}^2\cancel{\mu}d\tilde{z}\int_{\mathbb{R}}\cancel{\mu}d\tilde{t}G(z,t;\tilde{z},t')\cancel{\mu^{-4}}\partial_{\tilde{t}}^4 K(\mu\tilde{z},\mu t';\mu\tilde{t})\ ,$$

$$K_{nc}(\mu z,\mu t;\mu t') = \mu^{-\Delta_+}\left\{K(z,t;t') - \frac{\alpha^2}{8\ell_0^2}\int_{\mathbb{R}_+}\tilde{z}^2d\tilde{z}\int_{\mathbb{R}}d\tilde{t}G(z,t;\tilde{z},t')\partial_{\tilde{t}}^4 K(\tilde{z},t';\tilde{t})\right\} + \mathscr{O}(\alpha^4)\ ,$$

which implies that:

$$\tilde{K}_{nc}(\mu z,\mu t;\mu t') = U^{-1}_{\mu z,\mu t}K_{nc}(\mu z,\mu t;\mu t') = \mu^{-\Delta_+}\tilde{K}_{nc}(z,t;t')\ . \tag{6.72}$$

Finally, we can apply the rescaling on the three point function. Rescaling z,t,t',t_1,t_2 and t_3 by μ and using the same procedure applied to K_{nc}, one can show that:

$$\begin{gathered}\frac{\lambda\ell_0}{2\ell}\int_{\mathbb{R}\times\mathbb{R}_+}\frac{\cancel{\mu^2}dzdt}{\cancel{\mu^2}z^2}\left\{\int_{\mathbb{R}}\cancel{\mu}dt'\cancel{\mu^{-1}}\partial_{\tilde{z}}\left(\tilde{K}_{nc}(\mu\tilde{z},\mu t';\mu t_1)G_{nc}(\mu\tilde{z},\mu t';\mu z,\mu t)\right)\right|_{\mu\tilde{z}=\varepsilon}\times \\ \left.\left[\tilde{K}_{nc}(\mu z,\mu t;\mu t_2)\star_{\mu z,\mu t}\tilde{K}_{nc}(\mu z,\mu t;\mu t_3)\right] - \frac{1}{3}\tilde{K}_{nc}(\mu z,\mu t;\mu t_1)\star_{z,t}\tilde{K}_{nc}(\mu z,\mu t;\mu t_2)\star_{z,t}\tilde{K}_{nc}(\mu z,\mu t;\mu t_3)\right\} \\ = \langle\mathscr{O}(\mu t_1)\mathscr{O}(\mu t_2)\mathscr{O}(\mu t_3)\rangle^{(0)}\ ,\end{gathered}$$

with this, renaming the variables as we did before and reorganizing the terms using the previous results, we finally show that:

$$\begin{gathered}\frac{\lambda\ell_0}{\mu^{3\Delta_+}2\ell}\int_{\mathbb{R}\times\mathbb{R}_+}\frac{dzdt}{z^2}\left\{\int_{\mathbb{R}}dt'\partial_{z'}\left(\tilde{K}_{nc}(z',t';t_1)G_{nc}(z',t';z,t)\right)\right|_{z'=\varepsilon}\times\left[\tilde{K}_{nc}(z,t;t_2)\star_{z,t}\tilde{K}_{nc}(z,t;t_3)\right] - \\ \left.-\frac{1}{3}\tilde{K}_{nc}(z,t;t_1)\star_{z,t}\tilde{K}_{nc}(z,t;t_2)\star_{z,t}\tilde{K}_{nc}(z,t;t_3)\right\} = \langle\mathscr{O}(\mu t_1)\mathscr{O}(\mu t_2)\mathscr{O}(\mu t_3)\rangle^{(0)}\ ,\end{gathered}$$

$$\langle\mathscr{O}(\mu t_1)\mathscr{O}(\mu t_2)\mathscr{O}(\mu t_3)\rangle^{(0)} = \mu^{-3\Delta_+}\langle\mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3)\rangle^{(0)}\ . \tag{6.73}$$

This result shows that the three point function has the equivalent conformal scaling behaviour when compared to the commutative case.

Chapter 7

Conclusions

In our study of the relations between the commutative and non-commutative field theories we came to some interesting aspects. We explored the possibilitu of extending the AdS/CFT correspondence to the case of a non-commutative bulk by changing the gravity side. This approach makes sense and is motivated on the following fact: the non-commutative space time can be interpreted as a quasi-classical regime for a theory of quantum gravity because the correspondence assumes a duality between the full quantum gravity and CFT. Another fact is that the deformation leading to the non-commutative case preserves the *undeformed* $SO(2,1)$ conformal symmetry. To test these assumptions we first calculated the non-commutative corrections for the free particle case, and, as expected we found that the most relevant aspects of the commutative theory weren't lost. As demonstrated, most of the relevant results inside the commutative theory appeared in the non-commutative case as the standard ones corrected by some scalings. In the calculation of the 2- and 3- point correlation functions we struggled with some complications in the effective calculations, but, we managed to show that the overall effect caused by the non-commutative mapping was the addition of a re-scaling factor to their commutative counterparts, supporting the conclusions of [1]. Most of the calculation for the 2- point function were greatly simplified when compared to [1], however, to the 3- point correlator we were not able to find a closed expression. We managed to implicitly study the transformation properties of the correlator under conformal transformations which lead us to the conclusion that the main result should have the form of the commutative one multiplied by a re-scaling factor as the 2- point correlator.

$$< \mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3) >= \left(1+c\alpha^2\right) < \mathscr{O}(t_1)\mathscr{O}(t_2)\mathscr{O}(t_3) >^{(0)} + \mathscr{O}\left(\alpha^3\right) \ , \tag{7.1}$$

where the coefficient c should be calculated by an explicit evaluation of the terms in (6.71) up to the order α^2. Another remarkable fact obtained was that the non-commutative Killing cannot be obtained from the commutative ones in analogy to the Seiberg-Witten map for gauge theories, demonstrating the non-triviality of this result.

Some questions arise from this result. The first question ask if it is possible to assign to the effect of the non-commutativity on the correlations to some kind of renormalization of a boundary operator. To adress this issue we should compare the factors that multiply the 2- and 3- correlators and due to some techinical dificulties we weren't able to show this explicitly, but, it is a good subject for future works.

Another question to ask is if is possible to generalize this for AdS_{d+1}/CFT_d? We started with the $d = 1$ case because it is the only case where we can define the Poisson structure of the AdS space in a comprehensive way, for $d \geq 2$ it is not possible to do this, unless to the case AdS_4 which we plan to report to it in future papers. And finnaly one should try to quantize the Lorentzian case using other representations of the group $SU(1,1)$ and study what non-commutatity does to the structure of the correlators for this theory.

Chapter 8

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Appendix A

Appendix

A.1 Non-triviality of ncAdS

In this appendix we want to study to what extent the Killing vectors of AdS_2 can be mapped analogously to $ncAdS_2$ analysing if the main properties of the undeformed algebra are maintained after the transformation. Both sets of Killing vectors, commutative and noncommutative, satisfy the same undeformed $so(2,1)$ algebra expressed on the equation (3.30). The expressions for the Killing vectors in terms of the coordinates (z,t) are given in the equation (3.33), we define them for the canonical coordinates below

$$\begin{cases} K^0 = \partial_x \\ K^- = -\ell_0 e^x \partial_y \\ K^+ = \frac{1}{\ell_0} e^{-x} \left(2y\partial_x + (y^2 + \ell_0^2)\partial_y\right) \end{cases}, \tag{A.1}$$

The non-commutative Killing vectors were constructed on the section 4.3 and from the equation (4.88) we recall their equations

$$\begin{cases} K_\star^0 = \partial_x \equiv K^0 \\ K_\star^- = -\ell e^x \Delta_y \\ K_\star^+ = \frac{1}{\ell} e^{-x} \left(2y\partial_x S_y + (y^2 + \ell^2 + \frac{\alpha^2}{4}(1-\partial_x^2))\Delta_y\right) \end{cases}, \tag{A.2}$$

where Δ_y and S_y were defined in (4.85) and (4.78). Generally, $\ell = \ell(\alpha)$, such that $\ell(0) = \ell_0$.

One would ask if it is possible to find a similarity transformation between $(A.1)$ and $(A.2)$ that could map these two set of vectors in a good way. More specifically, we ask if there exists a non-degenerate map U such that

$$\begin{cases} U^{-1} K^\mu U = K_\star^\mu \\ U|_{\alpha=0} = \mathbb{1} \end{cases}. \tag{A.3}$$

If this were to be the case, then the two theories, would essentially be equivalent, as one would be able to map all the solutions of one theory to the ones of the other. One of the consequences would be the preservation of the conformal structure of the correlation functions after the mapping. The nonexistence of such a map would imply that the noncommutative theory is really a non-trivial result, generating a deformed set of solutions on the noncommutative realm. In the end of this appendix we will show the non-triviality of the $ncAdS_2$.

Note that if we add to K^+ a term proportional to $\ell_0 e^{-x}\partial_y \equiv -e^{-2x}K^-$ does not affect the $so(2,1)$ algebra. In order to obtain the correct equations of motion as the kernel of the Casimir operator one might define the Killing vectors using this shift term. But it is not needed to close the algebra, so it is more of a physical origin. One question that may appear is, does exist a similarity transformation that takes the commutative "shifted" operators to the non-commutative shifted ones? Towards this end we introduce the "shifted" Killing vectors:

$$\begin{cases} \tilde{K}^0 = K^0 = \partial_x \\ \tilde{K}^- = K^- = -\ell_0 e^x \partial_y \\ \tilde{K}^+ = K^+ - \ell_0 e^{-x}\partial_y \equiv \frac{1}{\ell_0} e^{-x}\left(2y\partial_x + y^2\partial_y\right) \end{cases} . \tag{A.4}$$

By "shifted" non-commutative generators we mean the following

$$\begin{cases} \tilde{K}^0_\star = K^0_\star \\ \tilde{K}^-_\star = K^-_\star \\ \tilde{K}^+_\star = \frac{1}{\ell} e^{-x}\left(2y\partial_x S_x + (y^2 - \frac{\alpha^2}{4}\partial_x^2)\Delta_y\right) + const \times e^{-x}\Delta_y \end{cases} . \tag{A.5}$$

Of course, one can immediately notice that the trivial similarity transformation $U_0 = \exp\left(-\ln\left(\frac{\ell}{\ell_0}\right)\partial_x\right)$ changes ℓ_0 to ℓ in (A.4), but this is not the case for the non-shifted generator (A.1) because ℓ_0 enters K^+ via a common factor. From now on, we will assume that this similarity transformation has been done and we will keep using the same notation swaping $\ell_0 \to \ell$. Another observation that can be made is that from $U^{-1}K^0U = K^0_\star$ it follows $\partial_x U = 0$ and it reduces the number of dependencies of $U = U(\partial_x, y, \partial_y)$

Instead of attempting to find an exact expression for U, since we are working on perturbations about α^2-order, it's wise to take this as our first step. To this order we have

$$\begin{cases} \tilde{K}^0_\star = \partial_x \\ \tilde{K}^-_\star = \tilde{K}^- + \alpha^2 \frac{\ell}{24} e^x \partial_x^3 + \mathcal{O}(\alpha^4) \\ \tilde{K}^+_\star = \tilde{K}^+ - \alpha^2 \frac{1}{4\ell} e^{-x}\left(\partial_y\partial_x^2 + y\partial_y^2\partial_x + \frac{1}{6}y^2\partial_y^3 + \kappa\partial_y\right) + \mathcal{O}(\alpha^4) \end{cases} . \tag{A.6}$$

Here the term proportional to the *unknown* constant κ is exactly the possible shift term. More specifically, the constant in (A.5) is equal to $-\frac{\alpha^2}{4\ell}\kappa$. We see that there is no α-linear term, so it is natural to suggest the following expansion for the map $\tilde{U}$ from $\tilde{K}^\mu$ to $\tilde{K}^\mu_\star = \tilde{U}^{-1}\tilde{K}^\mu\tilde{U}$

$$\tilde{U} = \mathbb{1} + \alpha^2\mathscr{G}(x, \partial_x, y, \partial_y) + \mathcal{O}(\alpha^4) . \tag{A.7}$$

Then we have the following conditions on $\mathscr{G}$: $\alpha^2[\tilde{K}^\mu, \mathscr{G}] = \tilde{K}^\mu_\star - \tilde{K}^\mu$ or in the components

$$\left[\tilde{K}^0,\mathscr{G}\right] = 0\;, \tag{A.8}$$

$$\left[\tilde{K}^-,\mathscr{G}\right] = \frac{\ell}{24}e^x\partial_x^3\;, \tag{A.9}$$

$$\left[\tilde{K}^+,\mathscr{G}\right] = -\frac{1}{4\ell}e^{-x}\left(\partial_y\partial_x^2 + y\partial_y^2\partial_x + \frac{1}{6}y^2\partial_y^3 + \kappa\partial_y\right)\;. \tag{A.10}$$

Let us analyze these conditions one by one.

1) The condition (A.8) is trivially satisfied since $\mathscr{G}$ does not depend on x.

2) The condition (A.9) is $[e^x\partial_y,\mathscr{G}] = -\frac{1}{24}e^x\partial_x^3$. We note that there is a trivial solution to it: $\mathscr{G}_0 = -\frac{1}{24}y\partial_y^3$, so, writing $\mathscr{G} = \mathscr{G}_0 + \tilde{\mathscr{G}}$, this is equivalent to $\left[e^x\partial_y,\tilde{\mathscr{G}}\right] = 0$. Because the full form of the non-commutative Killing vectors depends on derivatives with respect to x only up to ∂_x^2, it is possible to argue that $\tilde{\mathscr{G}}$ also does not involve terms with $\partial_x^{(k)}$ with $k \geq 3$, i.e. $\tilde{\mathscr{G}}$ takes the following form

$$\tilde{\mathscr{G}} = g_2(y,\partial_y)\partial_x^2 + g_1(y,\partial_y)\partial_x + g_0(y,\partial_y)\;. \tag{A.11}$$

Taking into account the independence of $\partial_x^{(k)}$ for different k, we conclude that g_2 is not a function of y and

$$g_1 = 2yg_2(\partial_y)\partial_y + \tilde{g}_1(\partial_y)\;\;;\qquad g_0 = y^2g_2(\partial_y)\partial_y^2 + y(g_2(\partial_y) + \tilde{g}_1(\partial_y))\partial_y + \tilde{g}_0\partial_y \tag{A.12}$$

In the last expression we indicated on what argument each function depends. Now it's clear that all the functions depends on ∂_y and we arrive at the following most general form for the candidate for the infinitesimal similarity transformation:

$$\begin{aligned}\mathscr{G} = -\frac{1}{24}y\partial_y^3 &+ g_2(\partial_y)\partial_x^2 + \Big(2yg_2(\partial_y)\partial_y + \tilde{g}_1(\partial_y)\Big)\partial_x + \\ &+ 2y^2g_2(\partial_y)\partial_y^2 + y\Big(g_2(\partial_y) + \tilde{g}_1(\partial_y)\Big)\partial_y + \tilde{g}_0(\partial_y)\;,\end{aligned} \tag{A.13}$$

where g_2 and $\tilde{g}_i$ are some arbitrary functions of the argument ∂_y.

3) We can first calculate $\left[\tilde{K}^+,\mathscr{G}_0\right]$

$$\left[\tilde{K}^+,\mathscr{G}_0\right] = \frac{1}{4\ell}e^{-x}\Big(y\partial_y^2\partial_x + \frac{5}{6}y^2\partial_y^3 + y\partial_y^2\Big) \tag{A.14}$$

Substituting this in (A.10) we need to find $\mathscr{G}$ in which this condition is satisfied

$$\left[e^{-x}(2y\partial_x + y^2\partial_y),\tilde{\mathscr{G}}\right] = -\frac{1}{4}e^{-x}(\partial_y\partial_x^2 + 2y\partial_y^2\partial_x + y^2\partial_y^3 + y\partial_y^2 + \kappa\partial_y) \tag{A.15}$$

Using the result of the previous step (A.13) in (A.10) and requiring that the term proportional to ∂_x^3 is absent on the RHS of (A.10) we immediately conclude that $g_2(\partial_y)$ is actually a constant, i.e. the whole dependence on it drops out. Continuing to compare the coefficients of $\partial_x^{(k)}$ for different $k = 0,1,2$, we arrive at the following result

$$g_2 = a\;,\; \tilde{g}_1 = \frac{1}{16}\partial_y^2 + b\;,\; \tilde{g}_0 = \frac{1}{2}\tilde{g}_1\;,\; \kappa = -\frac{1}{4}\;, \tag{A.16}$$

where a and b are some arbitrary constants, which do not contribute at this level. We still keep the dependence on a and b explicit to study the transformation of the shift term (see below).

This completes the prove of the perturbative (up to α^2-terms) equivalence of $\tilde{K}^\mu$ and $\tilde{K}^\mu_\star$ (with the very precise form of the generated shift term)

$$\begin{aligned}
&\tilde{U}^{-1}\tilde{K}^0\tilde{U} = K^0_\star = \partial_x \; , \\
&\tilde{U}^{-1}\tilde{K}^-\tilde{U} = \tilde{K}^-_\star + \mathscr{O}(\alpha^4) \; , \\
&\tilde{U}^{-1}\tilde{K}^+\tilde{U} = \tilde{K}^+_\star + \mathscr{O}(\alpha^4) = \frac{1}{\ell}e^{-x}\left(2y\,\partial_x S_x + (y^2 - \frac{\alpha^2}{4}\partial_x^2)\Delta_y\right) + \frac{\alpha^2}{16\ell}e^{-x}\Delta_y + \mathscr{O}(\alpha^4) \; ,
\end{aligned} \tag{A.17}$$

where

$$\begin{aligned}
\tilde{U} &= \mathbb{1} + \alpha^2\mathscr{G}(\partial_x, y, \partial_y) + \mathscr{O}(\alpha^4) = \mathbb{1} + \alpha^2\left(-\frac{1}{24}y\partial_y^3 + a(\partial_x^2 + 2y^2\partial_y^2 + 2y\partial_y\partial_x + y\partial_y) + \right. \\
&\left. + \frac{1}{32}(2y\partial_y + 2\partial_x + 1)\partial_y^2 + b(y\partial_y + \partial_x)\right) + \mathscr{O}(\alpha^4) \; .
\end{aligned} \tag{A.18}$$

Note, that as it was stressed above, (A.17) does not depend on the arbitrary a and b.

For the future use, we consider a more general choice for $\tilde{g}_0$: $\tilde{g}_0 = \frac{1}{2}(1+\lambda)\tilde{g}_1$. Of course this will produce some extra terms on the right hand side, but we will see how they are cancelled by the shift term.[1] So, we have (including the contribution from U_0)

$$\begin{aligned}
&\tilde{U} = \mathbb{1} + \alpha^2\mathscr{G}(\partial_x, y, \partial_y) + \mathscr{O}(\alpha^4) = \mathbb{1} + \alpha^2\left(-\frac{1}{24}y\partial_y^3 + a(\partial_x^2 + 2y^2\partial_y^2 + 2y\partial_y\partial_x + y\partial_y) + \right. \\
&\qquad\qquad \left. + \frac{1}{32}(2y\partial_y + 2\partial_x + 1 + \lambda)\partial_y^2 + b(y\partial_y + \partial_x) - \frac{1}{\ell_0^2}\frac{\ell_1}{\ell_0}\partial_x\right) + \mathscr{O}(\alpha^4) \; , \\
&\tilde{U}^{-1}\tilde{K}^0\tilde{U} = K^0_\star = \partial_x \; , \\
&\tilde{U}^{-1}\tilde{K}^-\tilde{U} = \tilde{K}^-_\star + \mathscr{O}(\alpha^4) \; , \\
&\tilde{U}^{-1}\tilde{K}^+\tilde{U} = \tilde{K}^+_\star + \mathscr{O}(\alpha^4) = \frac{1}{\ell}e^{-x}\left(2y\,\partial_x S_x + \left(y^2 - \frac{\alpha^2}{4}\partial_x^2\right)\Delta_y\right) + \\
&\qquad\qquad + \frac{\alpha^2}{16\ell_0}(1-\lambda)e^{-x}\Delta_y - \frac{\alpha^2\lambda}{8\ell_0}(\partial_y\partial_x + y\partial_y^2) + \mathscr{O}(\alpha^4) \; .
\end{aligned} \tag{A.19}$$

The problem with the shift term, $const \times e^{-x}\partial_y$, is immediately clear from the fact that neither expression in front of the constants a and b in (A.19) commutes with this term. So, as the consequence, we will produce terms explicitly depending on these constants. It is easy to obtain the perturbative form of the transformation of the shift term (we expand $\ell = \ell_0 + \frac{\alpha^2}{\ell_0^2}\ell_1 + \mathscr{O}(\frac{\alpha^4}{\ell_0^4})$)

$$U^{-1}U_0^{-1}(\ell_0 e^{-x}\,\partial_y)U_0 U = \ell e^{-x}\Delta_y + \alpha^2\ell_0 e^{-x}\left(4a(\partial_x + y\partial_y) + 2b - \frac{2\ell_1}{\ell_0^3} + \frac{1}{8}\partial_y^2\right)\partial_y + \mathscr{O}(\alpha^4) \; . \tag{A.20}$$

While the first term has a correct form (which, of course, remains correct after the expansion in α is done), the rest presents a correction (the difference between ℓ and ℓ_0 is of the next order in α.)

Combining (A.19) and (A.20) one can easily see that the choice $\lambda = 32a\ell_0^2$ and $b - a = \frac{1}{\ell_0^2}\left(\frac{\ell_1}{\ell_0} + \frac{3}{32}\right)$ (the separate values of a and b turn out to be irrelevant) almost does the mapping between the two sets of Killing vectors, (A.1) and (A.2) (we return to the "untilded" notation for U, because this is a map between K^μ and $K^\mu_\star$ as in (A.3))

[1] Changing g_2 or $\tilde{g}_1$ immediately will produce higher x-derivatives that will not be possible to compensate.

$$
\begin{aligned}
&U = \mathbb{1} + \alpha^2 G(\partial_x, y, \partial_y) + \mathscr{O}(\alpha^4) = \\
&\quad = \mathbb{1} + \alpha^2 \left(\frac{1}{96}(2y\partial_y + 6\partial_x + 3)\partial_y^2 + \frac{1}{\ell_0^2}\Big(\frac{\ell_1}{\ell_0} + \frac{3}{32}\Big)(y\partial_y + \partial_x) - \frac{1}{\ell_0^2}\frac{\ell_1}{\ell_0}\partial_x \right) + \mathscr{O}(\alpha^4) \ , \\
&U^{-1}K^0U = K^0_\star \ , \\
&U^{-1}K^-U = K^-_\star + \mathscr{O}(\alpha^4) \ , \\
&U^{-1}K^+U = K^+_\star + \alpha^2 \frac{\ell_0}{8} e^{-x}\partial_y^3 + \mathscr{O}(\alpha^4) \ .
\end{aligned}
\tag{A.21}
$$

To conclude, we can *almost* map the commutative Killing vectors to the noncommutative ones. The obstruction is the shift term, the presence of which leads to the appearance of the extra ∂_y^3-term. This is not only the proof of the “non-triviality” of $ncAdS_2$ but also serves as very convenient technical tool to simplify the perturbative analysis of the chapters 5 and 6.

A.2 Moyal star total divergence

Here we want to show that the commutative expression for the on-shell action (5.57) is valid in the non-commutative case to *all orders* in α. Towards this end, let us re-write (5.56) in terms of the non-commutative Killing vectors (A.2)

$$
S^{bdy}_{nc}[\Phi] = -\frac{1}{2\ell\alpha^2}\int dxdy\, [\mathscr{X}^\mu, \Phi \star [\mathscr{X}_\mu, \Phi]_\star]_\star = \frac{1}{2\ell}\iint_{\mathbb{R}^2} dxdy K^\mu_\star \left(\Phi \star K_{*\mu}\Phi\right) \ , \tag{A.22}
$$

where $K^\mu_\star$ are defined as $\alpha K^\mu_\star \Phi := i[\mathscr{X}^\mu, \Phi]_\star$.

In two dimensions, the Stokes theorem takes the form ($\omega = \omega_\mu dx^\mu$ is an arbitrary 1-form)

$$
\iint_V dxdy(\partial_x\omega_y - \partial_y\omega_x) = \int_{\partial V} \omega_x dx + \omega_y dy \ . \tag{A.23}
$$

Because the boundary of our space is located at $z = 0$ it is convenient to pass to Fefferman-Graham coordinates. Then $z = const$ corresponds to $x = const$ with $dy = \frac{\ell}{z}dt$ and for our case the Stokes formula takes the form

$$
\iint_{\mathbb{R}^2} dxdy(\partial_x\omega_y - \partial_y\omega_x) = \int_{-\infty}^{\infty} \left[\frac{\ell}{z}\omega_y\right]_{z=0} dt \ . \tag{A.24}
$$

This means that while studying the integrand of (A.22) we need to keep track only of the term of the form $\partial_x(\cdots)$. Also, because $\frac{\ell}{z}\omega_y$ is evaluated at $z = 0$, we only need to keep terms in ω_y up to $\mathscr{O}(z^2)$. This will allow us to arrive at the exact result. Using

$$
\begin{cases} [\Delta_y, y] = S_y \\ [S_y, y] = -\frac{\alpha^2}{4}\Delta_y \end{cases} \Rightarrow \ [\Delta_y, y^2] = 2yS_y - \frac{\alpha^2}{4}\Delta_y \ , \tag{A.25}
$$

where Δ_y and S_y are defined in (4.78) and (4.85) respectively, the Killings (A.2) take the form

$$\begin{cases} K_\star^0 = \partial_x \equiv K^0 \\ K_\star^- = -\Delta_y \ell\, e^x \\ K_\star^+ = \partial_x \frac{2}{\ell} e^{-x} y S_y + \Delta_y \frac{1}{\ell} e^{-x} \left(y^2 + \ell^2 - \frac{\alpha^2}{4} \partial_x^2 \right) \end{cases} , \tag{A.26}$$

where we moved all the relevant derivatives to the left (note that Δ_y has the form $\partial_y(\cdots)$). Then we have

$$\begin{aligned} K_\star^\mu \left(\Phi \star K_{*\mu} \Phi \right) &= K_\star^0 \left(\Phi \star K_\star^0 \Phi \right) - \frac{1}{2} K_\star^+ \left(\Phi \star K_\star^- \Phi \right) - \frac{1}{2} K_\star^- \left(\Phi \star K_\star^+ \Phi \right) = \\ &= \partial_x \left(\Phi \star K_\star^0 \Phi - \frac{1}{\ell} e^{-x} y S_y \left(\Phi \star K_\star^- \Phi \right) \right) - \partial_y(\cdots) \, . \end{aligned} \tag{A.27}$$

So we need to find the form, up to $\mathscr{O}(z^2)$, of the following expression

$$\Phi \star K_\star^0 \Phi - \frac{1}{\ell} e^{-x} y S_y \left(\Phi \star K_\star^- \Phi \right) \equiv -\Phi \star (z\partial_z + t\partial_t)\Phi + t\, S_t \left(\Phi \star \left(\frac{\ell}{z} \Delta_t \Phi \right) \right) , \tag{A.28}$$

where we passed to FG coordinates and $S_t = \cos\left(\frac{\alpha}{2\ell} z\partial_t\right)$ and $\Delta_t = \sin\frac{2}{\alpha}\left(\frac{\alpha}{2\ell} z\partial_t\right)$. Using these coordinates, the derivatives are given by

$$\begin{cases} \partial_x = -z\partial_z - t\partial_t \\ \partial_y = \frac{z}{\ell} \partial_t \end{cases} \tag{A.29}$$

it is obvious that the star-product

$$\star = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\alpha}{2} \right)^k \varepsilon^{i_1 j_1} \cdots \varepsilon^{i_k j_k} \overleftarrow{\partial_{i_1} \cdots \partial_{i_k}} \overrightarrow{\partial_{j_1} \cdots \partial_{j_k}} \, , \; (x^1, x^2) := (x, y) \tag{A.30}$$

cannot lower the degree of z. Moreover, every time we apply the derivative ∂_y, we raise the degree of z by 1. This, combined with the fact that

$$\frac{\ell}{z} \Delta_t = \partial_t + \mathscr{O}(z^2) \, , \;\; S_t = 1 + \mathscr{O}(z^2) \, ,$$

allows us to write

$$-\Phi \star (z\partial_z + t\partial_t)\Phi + t\, S_t \left(\Phi \star \left(\frac{\ell}{z} \Delta_t \Phi \right) \right) = -\Phi \star (z\partial_z + t\partial_t)\Phi + t\, (\Phi \star \partial_t \Phi) + \mathscr{O}(z^2) \, . \tag{A.31}$$

Using the explicit expression for the star-product (4.66), we see that it actually starts with the terms of the order of z^2 (also, see the discussion in the chapter 4 starting on the equation (4.58))

$$\star = 1 + \frac{i\alpha}{2} \left(\overleftarrow{\partial_t} z^2 \overrightarrow{\partial_z} - \overleftarrow{\partial_z} z^2 \overrightarrow{\partial_t} \right) + \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{i\alpha}{2} \right)^k \varepsilon^{i_1 j_1} \cdots \varepsilon^{i_k j_k} \overleftarrow{\partial_{i_1} \cdots \partial_{i_k}} \overrightarrow{\partial_{j_1} \cdots \partial_{j_k}} \, , \tag{A.32}$$

where the remaining sum is at least of the order of $\mathscr{O}(z^2)$. This finally allows us to write

$$-\Phi \star (z\partial_z + t\partial_t)\Phi + t\, S_t \left(\Phi \star \left(\frac{\ell}{z} \Delta_t \Phi \right) \right) = -z\Phi\partial_z\Phi + \mathscr{O}(z^2) \, . \tag{A.33}$$

Multiplying this by $\frac{\ell}{z}$, evaluating at $z = 0$, plugging into the boundary part of the action (A.22) and taking into account (A.24), we get (5.57) as an exact result.