Universidade de Brasília
Instituto de Física

ON THE MOTION OF FREE PARTICLES IN EXACT PLANE GRAVITATIONAL WAVES

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“On the motion of free particles in exact plane gravitational waves.”

Por

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Abstract

This work discusses the effect of exact plane gravitational waves (GWs), or pp-waves, on free particles. In order to do so, a review of the theory of General Relativity is given, followed by a study of the concept of GWs. Afterwards, the particle motion analysis is carried out by investigating the geodesic of the particles, and chaotic behavior is found. We evaluate numerically the angular momentum (its components and its modulus) per unit mass of the free particles. This evaluation shows interesting results, such as quasi-periodic variations of the angular momentum with respect to the width of the Gaussian that represents the gravitational pulse. Theoretical implications of the observed phenomenon are discussed in the conclusion.

Contents

Notation and Conventions

In this thesis we are going to be adopting the following notations and conventions:

- Index Notation

  This work adopts indexes to represent components of an object. In sections 1.1 and 1.2 Greek indexes, such as μ and ν, vary from 1 to n, and afterwards from 0 (the time component) to 3. Whenever Latin indexes are used, we assume the range to be in spatial components only, that is, 1 to 3.

  Furthermore, we differentiate upper and lower indexes. Upper indexes represent the contravariant components of an object, for example: x^μ represents the μ-th component of the contravariant vector x (it does not represent x to the μ-th power). Meanwhile, lower indexes are used for covariant components.

  In this notation, tensorial equations, which are independent of coordinates, have the same amount of non-repeated lower and upper indexes on the left and right hand sides. Equations with a free index are to be regarded as true for each possible value of it.

- Einstein Summation Convention

  We are going to use the so-called Einstein summation convention. When there are two indexes, one above and one below, we are going to assume that we are summing over the range of the indexes. For example:

$$V = v^\mu \partial_\mu = \sum_{\mu=0}^{3} v^\mu \partial_\mu = v^0\partial_0 + v^1\partial_1 + v^2\partial_2 + v^3\partial_3$$

- Metric Signature

  The metric signature will be chosen as $(-,+,+,+)$.

- Natural units

  We use the natural unit system, where the speed of light is unity, $c = 1$. In order to obtain familiar expressions for the Newtonian limit, we are not going to use the usual $G = 1$ until the discussion of pp-waves.

- $\mathscr{F}(M)$ - the set of all C^∞ functions $f : M \to \mathbb{R}$.

Introduction

Einstein's theory of gravity, General Relativity (GR), was developed between the years of 1907 and 1915. He expanded his previous works on Special Relativity to be able to discuss gravity. The elegance of the theory is to focus on gravity not as a force, but as a manifestation of the intricate curved geometry of spacetime. In addition to being elegant, it also solved concerning problems of Newtonian mechanics such as the precession of the perihelion of Mercury. Furthermore, this theory made predictions that along the years have been confirmed experimentally. That is the case for light deflections by massive astronomical objects, gravitational redshift and time delay in the trajectories of photons near the Sun.

Nevertheless, another consequence of General Relativity that gained expressive attention in the last decades is gravitational waves (GWs). These are supposed distortions of spacetime that propagate at the speed of light through the universe. Its existence and possible characteristics were discussed by notable authors such as Einstein himself [1], Henri Poincaré [2], Richard Feynman and many others.

Although the theoretical discussion on gravitational waves started right after the publication of General Relativity, the first experimental result was only obtained in 1975 (see [3]). It was an indirect result that showed that a binary system of stars was observed losing energy according to the rate predicted by the emission of GWs. Subsequently, many other teams of researchers designed their own GW experiments. These tests culminated in the direct detection made by LIGO (see [4]) using laser interferometers.

Therefore, the theoretical and experimental studies of gravitational waves are an important topic in physics currently. The present work introduces and investigates the theoretical aspects of gravitational waves, and studies in depth a special class of these phenomena: the plane-fronted gravitational waves with parallel rays, also known as pp-waves. This is a kind of exact GWs solution that yields interesting theoretical results, because they solve Einstein's equations of motion without approximations. Our main goal is to study free-particle dynamics in such spacetimes.

We have elaborated this work for someone with knowledge on multivariable calculus, Newtonian mechanics and Special Relativity. Experience with dynamical systems is desirable, but not necessary since the main points of the theory are summarized in Appendix B. We also assume some familiarity with advanced mathematical topics, such as variational calculus and partial differential equations. Therefore, no previous studies in differential geometry and in General Relativity are needed, but rather they are developed in summary at the begging of this thesis.

As mentioned, our focus is to analyze particle dynamics in an exact gravitational wave spacetime. In order to do so, this work is divided in three chapters: the first on General

Relativity, the second on Gravitational Waves and the third on the main analysis itself.

The first chapter intends to layout a fairly precise mathematical background on Differential Geometry, more specifically on Pseudo-Riemannian Geometry, and on General Relativity. The first section discusses the mathematical description of spacetime and its fields in terms of manifolds and tensors; the second, constructs the concept of curvature of spaces; and, the third, explains the main idea of General Relativity, i.e., how mass-energy generates the curvature. Since these are extensive topics, a less experienced reader can refer to [5], [6], [7], [8] for in-detail discussions.

The second chapter is on Gravitational Waves. First, we recommend that the reader refer to Appendix A where a deliberation about the Lie derivative, diffeomorphisms and gauge transformations in General Relativity is made. Subsequently, the first section of the chapter uses these mathematical structures in the development of the linearized theory of gravity. Also the first basic example of a gravitational wave is discussed: the approximate linearized wave. Finally, in the second section the exact plane wave solution for the full non-linear Einstein equation is presented.

After building all the required knowledge in chapters one and two, the third chapter studies particle motion in the plane gravitational wave spacetime. First, we present the equations governing particle movement, and some remarks are made. Later on, we numerically analyze trajectories of free-particles in this spacetime and we discuss it's chaotic behavior. Lastly, as a new contribution, variations on particle angular momentum are observed also by the means of numerical simulations and an interesting connection between pulse width and absorption of momentum is outlined.

1 General Relativity

The theory of General Relativity (GR) plays a central role in modern physics and is fundamental for the investigations of this work. As it was commented in *Introduction*, we assume that the reader is already familiar with Special Relativity (SR). Therefore, we expect that the one who reads this work is acquainted with dynamics in SR and Minkowski's mathematical formalism of this theory.

The first two sections of this chapter are devoted to Differential Geometry, more specifically Riemannian and pseudo-Riemannian geometries. Studying these topics is going to allow us to examine geometry in curved spaces, i.e., those where the parallel postulate of Euclid does not hold. The concept of curvature, which is discussed in the second section, makes it possible for us to study surfaces without having to rely on spaces with higher dimensions than the surface itself.

Building on the mathematics mentioned previously, we present the ideas and insights that led Einstein to his theory of gravity, i.e., General Relativity. Then, using two different methods, we obtain Einstein's Equation, which is the most important equation of GR, and will be used in the following chapters.

1.1 Manifolds and Tensors

We begin our discussion by defining the main subject of study of Riemannian Geometry: manifolds. They are the mathematical representations of spaces which locally approximate the Euclidean space. Formally[6]:

Definition 1.1.1. A *n-dimensional*, C^∞, *real manifold* M is a set, together with a collection of subsets $\{O_\alpha\}$ satisfying the following properties (see Figure 1):

1. Each $p \in M$ lies in at least one O_α, i.e., $\{O_\alpha\}$ is a cover of M;

2. For each α, there is a bijective map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$;

3. If any two sets O_α and O_β overlap, the sets $\psi_\alpha[O_\alpha \cap O_\beta]$ and $\psi_\beta[O_\alpha \cap O_\beta]$ are open. Furthermore, $\psi_\beta \circ \psi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ (infinitely continuous differentiable).

In the context of Pseudo-Riemannian Geometry, one may define the Lorentzian manifold, in which, loosely speaking, each open subset of the space is locally Minkowski. For a formal construction it is enough to substitute $\mathbb{R}^n$ by the appropriated dimension Minkowski spacetime in the above definition.

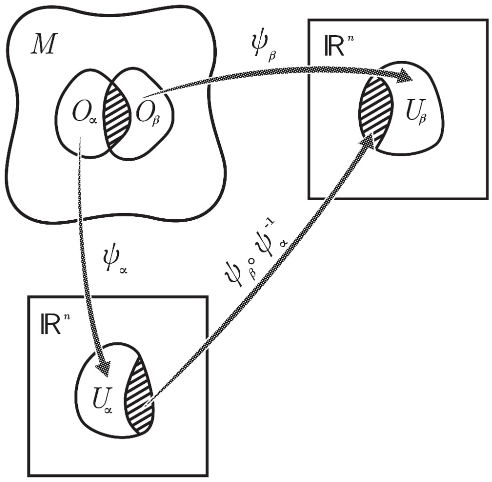

Figure 1 – Representation of a manifold and its maps.

Now that we have established the formal mathematical background on the spaces where GR is constructed, we are going to start adding structure to those spaces. The first aspect is that of differentiation and, at the same time, vector space construction. We use the following definition of a vector on a manifold:

Definition 1.1.2. A tangent vector, or contravariant vector, v at a point $p \in M$ is a map from $\mathscr{F}(M)$ to $\mathbb{R}$ which have the following two properties:

1. $v(af + bg) = av(f) + bv(g)$;
2. $v(fg) = f(p)v(g) + g(p)v(f)$.

for all $f, g \in \mathscr{F}$; $a, b \in \mathbb{R}$.

Defining the addition of vectors as:

$$(v + w)(f) = v(f) + w(f)$$

and the multiplication by a scalar as:

$$(av)(f) = av(f)$$

we have that the set of all vectors in $p \in M$ forms a vector space, which we are going to denote by T_pM.

We can introduce a base of T_pM by the means of the functions ψ as in the definition 1.1.1. Consider the vector X_μ given by:

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}\big(f \circ \psi^{-1}\big)\bigg|_{\psi(p)} \tag{1.1}$$

It can be shown[1] that if $v \in T_pM$, then:

$$v = v^\mu X_\mu$$

where:

$$v^\mu = v(x^\mu \circ \psi)$$

with x^μ being the μ-th canonical coordinate function (it chooses the μ-th component of a point $p \in \mathbb{R}^n$).

Finally, note that if you have a different set of coordinates, say x'^μ, we can relate the old basis vectors with the new ones using the chain rule. That is:

$$X_\mu = \frac{\partial x'^\nu}{\partial x^\mu} X'_\nu \tag{1.2}$$

therefore,

$$v = v^\mu X_\mu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} X'_\nu \equiv v'^\nu X'_\nu$$

This gives us the transformation law for the components of a contravariant vector:

$$v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} v^\mu \tag{1.3}$$

Note that definition 1.1.2 is only based on structures of the manifold itself. This means that we can discuss a particular space without needing to introduce a 'bigger' space. For example, using what we constructed previously, one can talk about a sphere without having to embed it in the Euclidean 3-space.

Now, we are going to define the dual vector space to T_pM. We have:

Definition 1.1.3. The *dual vector space* to T_pM is the space of linear maps $\omega : T_pM \to \mathbb{R}$, such that any of its elements are given by the basis $\{X^\mu\}$, which is defined to act on the basis of T_pM in the following way:

$$X^\mu(X_\nu) = \delta^\mu{}_\nu \tag{1.4}$$

This dual vector space is given the name of T_pM^* and its elements are denoted as dual vectors or covariant vectors. Any element $\omega \in T_pM^*$ can be written as:

$$\omega = \omega_\mu X^\mu$$

Using the defining property (1.4) and equation (1.2), we can verify how the components of a covariant vector behave under coordinate transformation, that is:

$$\omega'_\nu = \frac{\partial x^\nu}{\partial x'^\nu} \omega_\nu \tag{1.5}$$

[1] See [6] pages 15-16.

Analogously to definition 1.1.3, we could take the dual of the dual vector space, i.e., the maps between T_pM^* and $\mathbb{R}$. But we can associate each element of this 'double dual' space to the ones of the original space T_pM. This is done by requesting that $\tilde{v}(\omega) = \omega(v)$, where $\tilde{v} \in T_pM^{**}$, $\omega \in T_pM^*$ and $v \in T_pM$.

Now that we have constructed the tangent vector space to the point p of a manifold M and its dual, we can construct a general tensor at the point p. We define the *outer product*, $\otimes$, of a contravariant vector and a covariant vector to be a map from $T_pM \times T_pM^*$ to $\mathbb{R}$ by:

$$(v_1 \otimes w_1)(v_2, w_2) = v_1(w_2)w_1(v_2)$$

Moreover, we can think of a general map T from k copies of T_pM and l copies of T_pM^* to $\mathbb{R}$. Adding linearity to T, we can write it down decomposed in terms of the outer product of the basis of each space, that is:

$$T = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_l} \tag{1.6}$$

where we are using the following notation: $X_\mu \equiv \partial_\mu$ and $X^\mu \equiv dx^\mu$.

This map is called a (k, l) tensor. The components of this tensor change with the base as:

$$T'^{\mu'_1 \dots \mu'_k}{}_{\nu'_1 \dots \nu'_l} = \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \dots \frac{\partial x'^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x'^{\nu'_1}} \dots \frac{\partial x^{\nu_l}}{\partial x'^{\nu'_l}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \tag{1.7}$$

Notice that a contravariant vector can be thought of as a $(0, 1)$ tensor and a covariant vector as a $(1, 0)$ tensor. In fact, equations (1.3) and (1.5) are special cases of (1.7).

Now we have constructed all the objects we are going to need to further investigate the structures of the manifold. The last definition we need is the one of a tensor field:

Definition 1.1.4. A tensor field on M is a choice of a tensor at p for every point $p \in M$.

Note that we can use the above definition to construct vector and covector fields. A vector field can be defined by a curve on M, for example, for each $p \in M$ the chosen vector is the one that is tangent to the curve. Formally, given $\gamma : \mathbb{R} \to M$ (with parameter λ), then the associated tangent vector field will be given by:

$$v(f) = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial f}{\partial x^\mu}\bigg|_p \Rightarrow v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

From this construction, one could formulate a notion of infinitesimal distance using the tangent vector v and the parameter λ.

One special tensor that is worth introducing before moving on to the next section is the metric, g. It is a $(0, 2)$ tensor and, therefore, takes two vectors at a point p and produces a number. According to equation (1.6), we can decompose the metric tensor as:

$$g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$$

The number generated by the metric is associated with the internal product between two vectors and with the notion of infinitesimal distance along a curve. Therefore, one can write this 'infinitesimal distance squared' as:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

In the context of Riemannian Geometry, one requires that the metric is always positively defined, i.e. $g(v, w) > 0 \ \forall v, w \in T_pM \backslash \{0\}$, and non-degenerated, i.e. $g(v, w) = 0 \ \forall w \in T_pM \Rightarrow v = 0$. In the case of Pseudo-Riemannian Geometry, the positively defined condition is relaxed, but the non-degeneracy is kept. Since in both cases it is assumed that the metric is non-degenerated, an inverse map g^{-1} exists, such that $(g^{-1})^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$, where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta.

As mentioned in the beginning of the section, General Relativity is formulated in the mathematical structure of Pseudo-Riemannian Geometry. More specifically, it deals with four dimensional Lorentzian manifolds which are a class of pseudo-Riemannian manifolds have either $(+, -, -, -)$ or $(-, +, +, +)$ signature, i.e. the diagonalized metric has three positive elements and one negative or conversely. As commented in *Conventions*, this work will use $(-, +, +, +)$.

1.2 Curvature

In the previous section we introduced manifolds and tensors defined in points of the space. We are now interested in investigating how one may approach curvature in a manifold without embedding it in a higher dimensional Euclidean or Minkowski space. In order to do so, we are going to introduce the concepts of parallel transport and differentiation.

Derivatives are generators of infinitesimal translations; and, as already discussed, vectors tangent to curves are also related to derivatives. All of these relations guide us to try and define an *appropriate* derivative in a manifold, which would lead us to a method of producing parallel transport.

The candidate of a derivative ∇ on a manifold should satisfy:

1. *Linearity*: $\nabla(aT + bS) = a\nabla(T) + b\nabla(S)$;

2. *Leibeniz Rule*: $\nabla(TS) = \nabla(T)S + T\nabla(S)$.

where a and b are numbers and T and S are tensors. Also note that the outer product $\otimes$ was omitted, and hereafter we will continue to do the same.

One could ask why it is not sufficient to use the partial derivative as our derivative in the manifold, defined as:

$$\nabla T = \frac{\partial}{\partial x^\lambda} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \partial_{\mu_1} \dots \partial_{\mu_k} dx^{\nu_1} \dots dx^{\nu_l} dx^\lambda$$

The answer is that the derivative defined above is not a true tensor (unless T is a $(0,0)$ tensor, that is, a function f) and, therefore, depends on the choice of the coordinates. Since we are looking for physical relevant objects, which obviously do not change with coordinate transformation, the partial derivative is not the one we seek.

Before looking for the appropriate derivative ∇ we can define what we mean by parallel transport:

Definition 1.2.1. Given a curve $\gamma : M \to \mathbb{R}$ (with parameter λ) and a derivative ∇ in M, we say that a tensor T is parallel transported along this curve if:

$$\nabla_{\dot{\gamma}} T = 0$$

or, equivalently, in components:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \nabla_\mu T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = 0 \tag{1.8}$$

This rather abstract definition is based on the fact that we are trying to define a directional derivative for the given ∇. One could compare (1.8) with the familiar:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial}{\partial x^\mu} f = 0$$

which is to say that the function is constant along the curve.

Now, let us move on to finally choosing an *appropriate* derivative. We are going to request that the internal product of two vectors is maintained if both are parallel transported along the same curve. That is, if v and w are two vectors and t is the tangent vector to a curve γ, then:

$$\begin{aligned} t^\lambda \nabla_\lambda (g_{\mu\nu} v^\mu w^\nu) &= 0 \\ t^\lambda (v^\mu w^\nu \nabla_\lambda g_{\mu\nu} + g_{\mu\nu} v^\mu \nabla_\lambda w^\nu + g_{\mu\nu} w^\nu \nabla_\lambda v^\mu) &= 0 \\ t^\lambda v^\mu w^\nu \nabla_\lambda g_{\mu\nu} + g_{\mu\nu} v^\mu t^\lambda \nabla_\lambda w^\nu + g_{\mu\nu} w^\nu t^\lambda \nabla_\lambda v^\mu &= 0 \\ t^\lambda v^\mu w^\nu \nabla_\lambda g_{\mu\nu} &= 0 \end{aligned} \tag{1.9}$$

where in the second line we used Leibeniz rule and in the fourth, $t^\lambda \nabla_\lambda v^\mu = 0$, that is the vectors were parallel transported.

In order for equation (1.9) to be valid for any vectors and any curves, we must have $\nabla_\lambda g_{\mu\nu} = 0$. This is the mathematical expression of the requirement that the internal product of two parallel transported vectors do not change. It is called *metric compatibility.*

To finally construct the derivative in the manifold, consider the derivative of a vector field as a guide. Let:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda} v^\lambda \tag{1.10}$$

where the term $\Gamma^\nu{}_{\mu\lambda} v^\lambda$ was introduced to adjust the transformation of the partial derivative of a vector. Now lets consider the derivative of a covariant vector:

$$\nabla_\mu w_\nu = \partial_\mu w_\nu + \tilde{\Gamma}^\lambda{}_{\mu\nu} w_\lambda \tag{1.11}$$

where again $\tilde{\Gamma}^\lambda{}_{\mu\nu} w_\lambda$ is a correction term. We want to compare $\Gamma^\nu{}_{\mu\lambda}$ with $\tilde{\Gamma}^\lambda{}_{\mu\nu}$, this is done by noticing that $\nabla_\mu(v^\lambda w_\lambda) = \partial_\mu(v^\lambda w_\lambda)$. Using Leibeniz rule and equations (1.10) and (1.11), we obtain:

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = -\Gamma^\lambda{}_{\mu\nu}$$

Therefore, for a general tensor we need only to combine the above results with the outer product operation. In particular, we can write the derivative of the metric and impose metric compatibility:

$$\nabla_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda{}_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\sigma\nu} g_{\mu\lambda} = 0 \tag{1.12}$$

Rearranging the indexes, we also have the following equations:

$$\nabla_\mu g_{\nu\sigma} = \partial_\mu g_{\nu\sigma} - \Gamma^\lambda{}_{\mu\nu} g_{\lambda\sigma} - \Gamma^\lambda{}_{\mu\sigma} g_{\nu\lambda} = 0 \tag{1.13}$$

$$\nabla_\nu g_{\sigma\mu} = \partial_\nu g_{\sigma\mu} - \Gamma^\lambda{}_{\nu\sigma} g_{\lambda\mu} - \Gamma^\lambda{}_{\nu\mu} g_{\sigma\lambda} = 0 \tag{1.14}$$

Solving equations (1.12)-(1.14) for $\Gamma^\lambda{}_{\sigma\mu}$ and imposing that $\Gamma^\lambda{}_{\sigma\mu} = \Gamma^\lambda{}_{\mu\sigma}$ (this imposition does not hold for other theoretical formulations of gravitation, such as teleparallelism), we arrive at the following definition:

Definition 1.2.2. Given a metric g and a coordinate system x^μ, the Christoffel connection or Christoffel symbols are given by:

$$\Gamma^\lambda{}_{\sigma\mu} = \frac{1}{2} g^{\lambda\nu}(\partial_\sigma g_{\nu\mu} + \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu}) \tag{1.15}$$

With this result, we have constructed a derivative operator on our manifold. This operator is called *covariant derivative* and it is given by (1.10), in the case where it is applied to a contravariant vector.

With the covariant derivative constructed, we can discuss *geodesics*. These are one of the most important curves on a manifold and any of its main characteristics can be used as its defining property. We are going to use the quality which states that geodesics are curves such that tangent vectors are parallel transported along themselves. This leads us to the following definition:

Definition 1.2.3. Given a curve $\gamma : \Re \to M$, parametrized by $x^\mu(\lambda)$, it is called a geodesic if its tangent vector is parallel transported along itself. That is:

$$T^\nu \nabla_\nu T^\mu = 0$$

where T is the tangent vector to the curve. Using the given parametrization we can rewrite the above equation as:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\nu\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \tag{1.16}$$

In the case of flat spacetime, equations (1.16) are of a straight line. To easily see this, one could choose Cartesian coordinates (where $\Gamma^\mu{}_{\nu\sigma} = 0$) and the equations would simply be $\mathrm{d}^2 x^\mu / \mathrm{d}\lambda^2 = 0$. It can also be shown that the geodesics are the curves which maximize the length (or the proper-time, in Lorentzian manifolds) between two points (or events)[2]. Furthermore, as a postulate of the theory, free particles move along geodesics.

Now we already have a notion of parallel transport and of a derivative in our manifold. We are ready to discuss curvature. This concept is quite easy to grasp intuitively, but at the same time a hard one to formalize. Different authors begin the discussion either by considering parallel transport of a vector in a infinitesimal loop, the failure of geodesics to remain parallel, or the non-commutativity of the covariant derivative. All these forms end up equivalent, so we are going to follow the last one, as it was done by S. Carroll (see [7]).

Consider a vector field V, the commutator of the covariant derivative applied to this field in components is given by:

$$[\nabla_\mu, \nabla_\nu] V^\lambda = \nabla_\mu \left(\nabla_\nu V^\lambda \right) - \nabla_\nu \left(\nabla_\mu V^\lambda \right)$$

Using (1.10) twice on the right side of the above equation, we have:

$$[\nabla_\mu, \nabla_\nu] V^\lambda = \left(\partial_\mu \Gamma^\lambda{}_{\nu\sigma} - \partial_\nu \Gamma^\lambda{}_{\mu\sigma} + \Gamma^\lambda{}_{\mu\rho} \Gamma^\rho{}_{\nu\sigma} - \Gamma^\lambda{}_{\nu\rho} \Gamma^\rho{}_{\mu\sigma} \right) V^\sigma$$

This result show us by how much the covariant derivatives fail to commute. It is a measure of the 'non flatness' of the space. We therefore define:

Definition 1.2.4. Given a connection Γ, the *curvature tensor* or *Riemann tensor* R is given in components by:

$$R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda{}_{\nu\sigma} - \partial_\nu \Gamma^\lambda{}_{\mu\sigma} + \Gamma^\lambda{}_{\mu\rho} \Gamma^\rho{}_{\nu\sigma} - \Gamma^\lambda{}_{\nu\rho} \Gamma^\rho{}_{\mu\sigma} \tag{1.17}$$

One might ask if the above definition of the curvature tensor is in fact a tensor, since it is built upon manifestly non-tensorial elements. The answer is that it is a true tensor, this

[2] See, for example, [6] pages 43-45.

can be seen by deriving the transformation law of the Christoffel Symbols using equation (1.15), and then analyzing the transformation of $R^{\lambda}{}_{\sigma\mu\nu}$ by equation (1.17). The negative terms end up canceling out all non-tensorial parts of the transformation law.

The fact that the Riemann tensor describes the curvature is best shown in the following proposition:

Proposition 1.2.1. A given manifold (or Lorentzian manifold) is flat, i.e., there is a coordinate system in which the metric components are constant throughout the manifold, if and only if the Riemann tensor is identically zero.

The first part of the proof of this proposition is trivial: since the metric components are constant everywhere we have that the Christoffel Symbols and its derivatives are zero and, therefore, equation (1.17) gives us an identically vanishing Riemann tensor. The second part of the proof is more complicated and does not belong to the scope of this work, but it can be seen in the references[3].

Now we turn our attention to the four most important properties of the Riemann tensor. They are:

Proposition 1.2.2. The Riemann curvature tensor satisfies:

1. $R_{\mu\nu\sigma\lambda} = -R_{\nu\mu\sigma\lambda}$
2. $R_{\mu\nu\sigma\lambda} = -R_{\mu\nu\lambda\sigma}$
3. $R_{\mu\nu\sigma\lambda} + R_{\mu\sigma\lambda\nu} + R_{\mu\lambda\nu\sigma} = 0$
4. $\nabla_{\alpha} R_{\mu\nu\sigma\lambda} + \nabla_{\mu} R_{\nu\alpha\sigma\lambda} + \nabla_{\nu} R_{\alpha\mu\sigma\lambda} = 0$

All these properties are direct consequences of the equation (1.17).

We now define the Ricci tensor and scalar.

Definition 1.2.5. Given the Riemann curvature tensor, the *Ricci tensor* is defined by:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^{\lambda}{}_{\nu\mu} - \partial_{\nu}\Gamma^{\lambda}{}_{\lambda\mu} + \Gamma^{\lambda}{}_{\lambda\rho}\Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\rho}{}_{\lambda\mu} \tag{1.18}$$

and the *Ricci scalar* or *curvature scalar*, by:

$$R = R^{\mu}{}_{\mu} = g^{\mu\lambda} R_{\lambda\mu}$$

One of the most important characteristics of the Ricci tensor is that it is symmetric. We can see that by calculating its symmetric part by (1.18) and observing the properties 1 to 3 of the Riemann tensor. At the end the symmetric part is going to be equal to the tensor itself.

[3] See [7] pages 124-126.

Now we turn our focus to property 4 of proposition 1.2.2, which is called the second Bianchi identity (whereas property 3 is the first). Taking the following contraction (and using metric compatibility of the covariant derivative):

$$\begin{aligned} g^{\nu\lambda}g^{\alpha\sigma}\left(\nabla_\alpha R_{\mu\nu\sigma\lambda}+\nabla_\mu R_{\nu\alpha\sigma\lambda}+\nabla_\nu R_{\alpha\mu\sigma\lambda}\right)&=0\\ \nabla^\sigma R_{\mu\sigma}-\nabla_\mu R+\nabla^\lambda R_{\mu\lambda}&=0\\ 2\nabla^\sigma R_{\mu\sigma}-\nabla^\sigma(g_{\mu\sigma}R)&=0 \end{aligned} \tag{1.19}$$

This leads us to define the following (covariantly) conserved tensor:

Definition 1.2.6. Given a Riemann tensor and, therefore, a Ricci tensor and scalar, the *Einstein tensor* is defined as:

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$$

By the symmetry of the metric and of the Ricci tensor, the Einstein tensor also has this property. As stated before, by equation (1.19), the newly defined tensor is conserved along the manifold. The Einstein tensor is going to be of fundamental importance in the next section.

1.3 Einstein's Field Equations

Several approaches can be used to motivate Einstein's Field Equations. However, in the end they should be treated as a postulate of the theory, not something derivable from other assumptions. Nevertheless, some motivation can be presented.

There are two main paths to Einstein's equation, and we are going to present both in this work. First, similar to Einstein's original idea, we relate the mass distribution with a second order derivative of a 'gravitational potential'. Afterwards, we introduce the idea attributed to Hilbert, which is to follow Hamilton's principle of least action.

In Newtonian gravity, the gravitational potential Φ obeys Poisson's equation:

$$\nabla^2\Phi=4\pi G\rho \tag{1.20}$$

Einstein's insight was to treat gravity not as a force but as an effect of a curved spacetime. This comes from the *Equivalence Principle* which states that there is no local experiment that can be used to detect gravity. Therefore, in equation (1.20), Φ must be substituted by some quantity related to the metric. The right-hand side is readily replaced by the energy momentum tensor, as in *Special Relativity*.

To try and equate a curvature tensor to the energy momentum tensor, which is a $(0,2)$ tensor, one might suggest:

$$R^{\lambda}{}_{\mu\lambda\nu} = R_{\mu\nu} = \kappa T_{\mu\nu}$$

where κ is a constant. Yet this equation has a problem, the energy momentum is conserved $\nabla^{\mu} T_{\mu\nu} = 0$ and the Ricci tensor is not, in general. Nonetheless, definition 1.2.6 shows us that Einstein's tensor is a geometrical quantity that is conserved through out the manifold. Therefore, this leads us to the correct equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.21}$$

Taking the trace of the above equation, we get:

$$R = -\kappa G T$$

using this result back in (1.21), we get the equivalent:

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) \tag{1.22}$$

This formula is especially useful for the vacuum case, $T_{\mu\nu} = T = 0$, where obviously we are going to have $R_{\mu\nu} = 0$.

The constant κ should be chosen to match experimental results. Nevertheless, we can define it in order for the above equation to correspond with Newton's theory of gravitation for the limit of static metric, weak field and slow moving particles.

The Newtonian limit means: particles should move according to Newton's second law and the gravitational field of a configuration of mass should obey Poisson's equation.

Let us analyze the first assertion. We already discussed that free particles should move along geodesics. In General Relativity, we recognize gravity as a manifestation of the curvature of spacetime. Therefore, particles moving due to the effect of gravity are free. Since we are considering a 'weak field', we take the metric to be that of flat spacetime plus a perturbation, i.e.:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.23}$$

where $|h_{\mu\nu}| << 1$ [4]. In order to calculate the inverse metric with respect to this decomposition, we impose that the second order terms in h are ignored:

$$\begin{aligned} g^{\mu\lambda} g_{\lambda\nu} &= \delta^{\mu}{}_{\nu} \\ (\eta^{\mu\lambda} + \tilde{h}^{\mu\lambda})(\eta_{\lambda\nu} + h_{\lambda\nu}) &= \delta^{\mu}{}_{\nu} \\ \delta^{\mu}{}_{\nu} + \tilde{h}^{\mu}{}_{\nu} + h^{\mu}{}_{\nu} &= \delta^{\mu}{}_{\nu} \\ \tilde{h}^{\mu}{}_{\nu} = -h^{\mu}{}_{\nu} &\Rightarrow \tilde{h}^{\mu\nu} = -h^{\mu\nu} \end{aligned}$$

[4] We are using 1 as a reference to the canonical form of $\eta_{\mu\nu}$, that is diagonal$(-1, 1, 1, 1)$. Since the main interest is to obtain κ, it is sufficient to do the calculations in one specific system of coordinates. For simplicity, we chose Cartesian coordinates.

Hence,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{1.24}$$

Notice that in order to not generate terms of higher order, the indexes of h are raised and lowered by the flat metric. Now, with equations (1.23) and (1.24), we can calculate the Christoffel symbols:

$$\Gamma^{\lambda}{}_{\sigma\mu} = \frac{1}{2}\eta^{\lambda\nu}(\partial_\sigma h_{\nu\mu} + \partial_\mu h_{\nu\sigma} - \partial_\nu h_{\sigma\mu}) \tag{1.25}$$

The constraint that the particles move slowly can be written as:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} << \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

Therefore, the only important term of the geodesic equation is $\Gamma^{\mu}{}_{00}$. The static field hypothesis states that $\Gamma^{0}{}_{00} = 0$, since (1.25) only has time derivatives. This implies:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0$$

thus we can reparametrize $x(\tau)$ to $x(t)$. The other geodesic equations are:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} + \Gamma^{i}{}_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0 \Rightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\Gamma^{i}{}_{00} = \frac{1}{2}\partial_i h_{00}$$

Finally if we identify $h_{00} = -2\Phi$, where Φ is the Newtonian potential, we recover:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial_i \Phi$$

which is the correct equation for the motion of a particle.

Now we need to obtain Poisson's equation from (1.22) using the same assumptions. The energy-momentum tensor for an energy-mass distribution in which the particles move slowly (the so-called 'dust' approximation) is:

$$T_{\mu\nu} = \rho U_\mu U_\nu$$

Working on the distribution frame of reference, $U_\mu = (-1, 0, 0, 0)$, therefore, the only non-vanishing component of the energy-momentum tensor is $T_{00} = \rho$ and $T = -\rho$, also in the linear approximation. Finally, equation (1.22) states:

$$R_{00} = \frac{1}{2}\kappa\rho$$

The Ricci tensor component above is given by:

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = \partial_\lambda \Gamma^{\lambda}{}_{00} - \partial_0 \Gamma^{\lambda}{}_{\lambda 0} + \Gamma^{\lambda}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{00} - \Gamma^{\lambda}{}_{0\sigma}\Gamma^{\sigma}{}_{\lambda 0}$$

Notice that in the right-hand side the only relevant term is the first one. The second vanishes by the assumption of field is static; and, the third and fourth are second order in the perturbation h. Calculating the first term by (1.25):

$$R_{00} = -\frac{1}{2}\eta^{\lambda\sigma}\partial_\lambda\partial_\sigma h_{00} = -\frac{1}{2}\nabla^2 h_{00} = \nabla^2\Phi$$

Therefore,

$$\nabla^2\Phi = \frac{1}{2}\kappa\rho$$

Choosing $\kappa = 8\pi G$ we recover Poisson's equation. Finally, we arrive to Einstein's equation:

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{1.26}$$

Now we want to derive the same equation above by using the principle of least action. As shown in the previous chapter, the only geometrical relevant scalar that arises naturally in our calculations is the Ricci scalar. Thus, our first guess for an action would be:

$$S_G = \int R\sqrt{-g}\,\mathrm{d}^4x \tag{1.27}$$

where the term $\sqrt{-g}$ was added to ensure that the integration runs over the manifold volume, and G reminds us that this is a geometric action. Now we need to vary the above action with respect to the metric. First, let us take the variation of R:

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}$$

In order to vary the Ricci tensor, one should use its definition (1.18); and, the variation of the Riemann tensor should be as in (1.17). The result is:

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho{}_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho{}_{\mu\rho}$$

Notice that the above expression is well defined, since the variation of a connection, in this case the Christoffel one, can be shown as a tensor. This will not contribute to the variation of the action, because they originate surface terms that we set to zero.

The variation in $\sqrt{-g}$ is given by:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\,\delta g$$

To calculate the variation of g with respect to $g^{\mu\nu}$ we use the identity $\ln(\det(A)) = \mathrm{tr}(\ln(A))$, which is a corollary of Jacobi's formula:

$$\begin{aligned}\delta \ln(g) &= \delta\, \mathrm{tr}(\ln(g^{\mu\nu}))\\ \frac{1}{g}\delta g &= g_{\mu\nu}\,\delta g^{\mu\nu}\\ \delta g &= g g_{\mu\nu}\,\delta g^{\mu\nu}\end{aligned}$$

Therefore,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}$$

Then the variation of the action (1.27) is:

$$\delta S_G = \int \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\sqrt{-g}\,\delta g^{\mu\nu}\,\mathrm{d}^4x$$

By the definition of functional derivative, we have that:

$$\frac{\delta S_G}{\delta g^{\mu\nu}} = \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)$$

Thus, in order for the variation of the action, with respect to the (inverse) metric, to be zero, we need to have:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

Which is the Einstein field equation for the vacuum, $T_{\mu\nu} = 0$. In order to account for energy and matter (and obtain equation (1.26)), we must add a term S_M to the total action:

$$S = S_G + \alpha_M S_M$$

where α_M is a constant. Then,

$$\begin{aligned}\frac{\delta S}{\delta g^{\mu\nu}} &= \frac{\delta S_G}{\delta g^{\mu\nu}} + \alpha_M\frac{\delta S_M}{\delta g^{\mu\nu}} = 0\\ \frac{\delta S}{\delta g^{\mu\nu}} &= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \alpha_M\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \qquad (1.28)\end{aligned}$$

Defining the energy-momentum tensor to be:

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\alpha_M}{8\pi G}\frac{\delta S_M}{\delta g^{\mu\nu}} \qquad (1.29)$$

and α_M to be $16\pi G$ [5] we obtain Einstein's field equations (1.26) using (1.28). Notice that in the definition (1.29) we already took into consideration the Newtonian limit discussed previously.

[5] We introduced this constant (and did not directly compute it in the definition of the energy-momentum tensor), because doing so makes it easier to analyze how this construction of energy-momentum tensor relates to those of Klein-Gordan or Maxwell theories. See appendix E of [6].

2 Gravitational Waves

The theory of General Relativity (GR), developed by Einstein in 1915, is one of the greatest achievements im physics. Since its publication, experimental tests have confirmed several predictions of this theory. The astonishing detection of the first gravitational wave (GWs), by LIGO in September 2015[4], proved what some consider to be one of the most subtle and important predictions of GR. Subsequent discoveries by LIGO and VIRGO were made, and new possibilities of research in this field have continued to grow.

In this chapter we are going to discuss the linearized theory of gravity and exact plane wave solutions. The linearized theory was responsible for the predictions of gravitational waves and it plays a comparing role in the analysis of exact plane gravitational waves, the main topic of this work. These exact waves are the main interest of this study and will continue to be in chapter 3.

2.1 Linearized Gravitational Waves

In deriving the Newtonian limit in section 1.3, we made three requirements: static metric, slow moving particles and weak fields. In this section we are going to drop the first two conditions and keep the third. Apart from making our discussion more general, this will allow us to study the motion of photons and time related effects such as gravitational waves.

Again, the assumption of weak field is translated to the decomposition:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

where $|h_{\mu\nu}| << 1$. We have:

$$\begin{aligned}
\Gamma^{\lambda}{}_{\sigma\mu} &= \frac{1}{2}(\eta^{\lambda\nu} - h^{\lambda\nu})\left[\partial_\sigma(\eta_{\nu\mu} + h_{\nu\mu}) + \partial_\mu(\eta_{\nu\sigma} + h_{\nu\sigma}) - \partial_\nu(\eta_{\sigma\mu} + h_{\sigma\mu})\right] \\
&= \frac{1}{2}\eta^{\lambda\nu}\left(\partial_\sigma\eta_{\nu\mu} + \partial_\mu\eta_{\nu\sigma} - \partial_\nu\eta_{\sigma\mu}\right) + \frac{1}{2}\eta^{\lambda\nu}\left(\partial_\sigma h_{\nu\mu} + \partial_\mu h_{\nu\sigma} - \partial_\nu h_{\sigma\mu}\right) \\
&- \frac{1}{2}h^{\lambda\nu}\left(\partial_\sigma\eta_{\nu\mu} + \partial_\mu\eta_{\nu\sigma} - \partial_\nu\eta_{\sigma\mu}\right) \\
&= {}^{0}\Gamma^{\lambda}{}_{\sigma\mu} + \frac{1}{2}\eta^{\lambda\nu}\left(\nabla_\sigma h_{\nu\mu} + \nabla_\mu h_{\nu\sigma} - \nabla_\nu h_{\sigma\mu}\right)
\end{aligned}$$

where ${}^{0}\Gamma^{\lambda}{}_{\sigma\mu}$ and ∇ refer to the flat metric $\eta_{\mu\nu}$.

Now, with no further assumption, we compute the Ricci tensor by equation (1.18). Notice that since the Christoffel Symbols are already first order in the perturbation, the Γ^2 terms of the Ricci tensor are not going to be considered. Therefore,

$$\begin{aligned} R_{\mu\nu} &= {}^0R_{\mu\nu} + \frac{1}{2}\left(\nabla_\lambda\nabla_\mu h^\lambda{}_\nu + \nabla_\lambda\nabla_\nu h^\lambda{}_\mu - \Box h_{\mu\nu} - \nabla_\nu\nabla_\mu h\right) \\ &= \frac{1}{2}\left(\nabla^\lambda\nabla_\mu h_{\lambda\nu} + \nabla^\lambda\nabla_\nu h_{\lambda\mu} - \Box h_{\mu\nu} - \nabla_\nu\nabla_\mu h\right) \end{aligned} \tag{2.2}$$

where ${}^0R_{\mu\nu}$ is the Ricci tensor associated with the flat metric, h is the trace of $h_{\mu\nu}$ and $\Box$ is the d'Alembertian operator defined by $\Box \equiv \nabla_\lambda\nabla_\lambda$. Now, taking the trace of the above formula, we get the Ricci scalar:

$$R = \eta^{\mu\nu}R_{\mu\nu} = \nabla_\lambda\nabla_\sigma h^{\lambda\sigma} - \Box h \tag{2.3}$$

With equations (2.2) and (2.3) we can write the Einstein tensor as:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}\left(\nabla^\lambda\nabla_\mu h_{\lambda\nu} + \nabla^\lambda\nabla_\nu h_{\lambda\mu} - \nabla_\nu\nabla_\mu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\nabla_\lambda\nabla_\sigma h^{\lambda\sigma} + \eta_{\mu\nu}\Box h\right) \end{aligned} \tag{2.4}$$

We can simplify by using the *trace reversed perturbation*, that is:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.5}$$

Notice that the name actually makes sense, since $\bar{h} = -h$. We can do all our computations with this new tensor, since knowing it is equivalent to knowing the original perturbation. This comes4 from the fact that we can revert relation (2.5) to obtain:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

Then, equation (2.4) becomes:

$$G_{\mu\nu} = \frac{1}{2}\left(\nabla^\lambda\nabla_\mu\bar{h}_{\nu\lambda} + \nabla^\lambda\nabla_\nu\bar{h}_{\mu\lambda} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\nabla_\lambda\nabla_\sigma\bar{h}^{\lambda\sigma}\right) \tag{2.6}$$

The above expression does not seem to help our attempt to simplify the theory. But before we plug in (2.4) into Einstein's field equations and solve it, we are going to use the gauge freedom as discussed in appendix A.

The gauge transformation for the perturbation (2.1) is given by (A.10):

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \tag{2.7}$$

where ξ is a vector field associated with our gauge freedom. In trace reverse, we can write:

$$\begin{aligned} \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' \\ &= h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu - \frac{1}{2}\eta_{\mu\nu}\left(h + 2\nabla_\lambda\xi^\lambda\right) \\ &= \bar{h}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu - \eta_{\mu\nu}\nabla_\lambda\xi^\lambda \end{aligned} \tag{2.8}$$

Therefore, the trace reverse gauge transformation is the same as the original perturbation minus the term $\eta_{\mu\nu}\nabla_\lambda\xi^\lambda$.

Now, we can simplify the expression (2.6) by choosing an adapted gauge. Choosing ξ to satisfy:

$$\Box\xi_\mu = -\nabla^\nu\bar{h}_{\mu\nu} \tag{2.9}$$

then, equation (2.8) states:

$$\nabla^\nu\bar{h}'_{\mu\nu} = 0 \tag{2.10}$$

This is known as Lorenz gauge and it implies that, after the gauge transformation, the linearized Einstein tensor (2.6) is:

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} \tag{2.11}$$

where the prime has been emitted in order to make the notation more clear.

With (2.11) we can now write the linearized Einstein's equations to be:

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{2.12}$$

This is the most import formul for the linearized Einstein theory with the Minkowski background. We are going to solve this equation with and without sources.

2.1.1 Vacuum Solution — The Transverse Traceless Gauge

First let us consider when $T_{\mu\nu} = 0$. Equation (2.12) becomes:

$$\Box\bar{h}_{\mu\nu} = 0 \tag{2.13}$$

we immediately recognize this to be the wave equation. It describes the propagation of a gravitational wave very distant from the source. Before solving this equation, we are going to analyze our gauge choice. Notice that the Lorenz gauge does not uniquely define the vector field ξ_μ, therefore, more gauge transformations of the form (2.7) can be performed provided that we do not modify (2.9). Choosing:

$$2\partial_\mu\xi^\mu = -h$$

equation (2.7) implies that $h' = 0$. Hence, with this additional gauge $\bar{h}_{\mu\nu} = h_{\mu\nu}$, i.e., $h_{\mu\nu}$ is traceless. Furthermore we can choose,

$$\partial_0\xi_\mu + \partial_\mu\xi_0 = -h_{0\mu}$$

which, by equation (2.7), implies $h'_{0\mu} = 0$. This together with the Lorenz gauge and the linearized Einstein equation result in $h_{00} = 0$ (good behavior at infinity is assumed).

With this we finalize the so-called *traceless transverse gauge* or *radiation gauge* given by:

$$\Box\xi_\mu = -\partial^\nu \bar{h}_{\mu\nu} \tag{2.14}$$

$$2\partial_\mu \xi^\mu = -h \tag{2.15}$$

$$\partial_0 \xi_\mu + \partial_\mu \xi_0 = -h_{0\mu} \tag{2.16}$$

Solving for ξ, these equations together with (2.7) imply the following properties of our perturbation:

$$\partial^\nu h_{\mu\nu} = 0 \tag{2.17}$$

$$h = 0 \tag{2.18}$$

$$h_{0\mu} = 0 \tag{2.19}$$

Now, with all the gauge freedom used, we turn our attention to equation (2.13). Plane waves are solutions to this equation, so:

$$h_{\mu\nu} = C_{\mu\nu} \exp\big(ik_\lambda x^\lambda\big) \tag{2.20}$$

where $C_{\mu\nu}$ is a constant symmetric tensor and k_λ is a constant vector. By the end of our calculations we are going to be interested in the real part of the above solution. The wave equation states that:

$$k^\mu k_\mu = 0 \tag{2.21}$$

which in some sense tells us that these waves travel at the speed of light.

Let us now analyze the restrictions of $C_{\mu\nu}$ imposed by (2.17)-(2.19):

$$k^\mu C_{\mu\nu} = 0 \tag{2.22}$$

$$\eta^{\mu\nu} C_{\mu\nu} = 0 \tag{2.23}$$

$$C_{0\mu} = 0 \tag{2.24}$$

The above equations are comprised of 8 independent restrictions to $C_{\mu\nu}$. These limitations together with the fact that it is a symmetric 4 by 4 matrix, tell us that $C_{\mu\nu}$ has only two independent terms which are interpreted as the two possible polarization modes. Hence any linearized gravitational wave can be written as a linear superposition of these modes.

2.1.2 Matter Solutions — Green's Function

Now we return to equation (2.12) with the matter term different from 0. We are not going to assume the transverse traceless gauge.

Consider the following[1]:

$$\Box_x G(x^\sigma - y^\sigma) = -16\pi G \delta^{(4)}(x^\sigma - y^\sigma) \tag{2.25}$$

where the subscript x was used to show that the d'Alembertian is with respect to the coordinates x^σ, and $\delta^{(4)}(x^\sigma - y^\sigma)$ is the Dirac delta, defined by the following properties:

$$\delta^{(4)}(x^\sigma - y^\sigma) = 0, \text{ if } x^\sigma \neq y^\sigma \tag{2.26}$$

$$\int_{\text{All Space}} \delta^{(4)}(x^\sigma - y^\sigma) \mathrm{d}^4 y = 1 \tag{2.27}$$

$$\int_{\text{All Space}} f(y^\sigma)\delta^{(4)}(x^\sigma - y^\sigma) \mathrm{d}^4 y = f(y^\sigma) \tag{2.28}$$

where $f(x^\sigma)$ is an arbitrary C^∞ function. Multiplying both sides of (2.25) by $T_{\mu\nu}(y^\sigma)$ and integrating with respect to y^σ in all the region of the manifold:

$$\int T_{\mu\nu}(y^\sigma)\Box_x G(x^\sigma - y^\sigma)\,\mathrm{d}^4 y = -16\pi G \int T_{\mu\nu}(y^\sigma)\delta^{(4)}(x^\sigma - y^\sigma)\,\mathrm{d}^4 y$$

Since $T_{\mu\nu}(y^\sigma)$ is dependent of x^σ, we can place it inside the differential operator $\Box_x$; and, since the operator is linear, we can also place the integral inside. This leaves us with:

$$\Box_x \left(\int T_{\mu\nu}(y^\sigma) G(x^\sigma - y^\sigma)\,\mathrm{d}^4 y \right) = -16\pi G \int T_{\mu\nu}(y^\sigma)\delta^{(4)}(x^\sigma - y^\sigma)\,\mathrm{d}^4 y$$

but according to the defining property (2.28), the right hand side is equal to $T_{\mu\nu}(x^\sigma)$, hence:

$$\Box_x \left(\int T_{\mu\nu}(y^\sigma) G(x^\sigma - y^\sigma)\,\mathrm{d}^4 y \right) = -16\pi G T_{\mu\nu}$$

Comparing this equation with equation (2.12), we have that our sought-after solution is:

$$\bar{h}_{\mu\nu} = \int T_{\mu\nu}(y^\sigma) G(x^\sigma - y^\sigma)\,\mathrm{d}^4 y \tag{2.29}$$

Therefore, we have reduced our problem of finding $\bar{h}_{\mu\nu}$ in equation (2.12) to that of finding the function $G(x^\mu - y^\mu)$ in equation (2.25). This function is known as *Green's Function.* The solution of this reminiscent problem is exactly the same encountered in the standard derivations of electromagnetic waves and goes beyond the scope of the present

[1] The reader should be careful to not confuse G, the Newtonian constant of gravitation, with $G(x^\sigma - y^\sigma)$, the Green's Function.

work. The derivation of the Green's function for this problem can be encountered at references such as [9]. The solution is:

$$G(x^\sigma - y^\sigma) = \frac{4G}{|\mathbf{x}-\mathbf{y}|}\left[\delta(|\mathbf{x}-\mathbf{y}| - (x^0 - y^0)) - \delta(|\mathbf{x}-\mathbf{y}| + (x^0 - y^0))\right] \tag{2.30}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the spatial part of x^σ and y^σ, respectively.

One restriction must be imposed on (2.30), the one of causality. In the case we are studying, we do not expect the term $\delta(|\mathbf{x}-\mathbf{y}| + (x^0 - y^0))$ to be physical. It would imply a disturbance on the metric being propagated at a speed greater than that of the light, which is against the postulates of relativity. Therefore, we must have only the denominated retarded Green function G_R:

$$G_R(x^\sigma - y^\sigma) = \frac{4G}{|\mathbf{x}-\mathbf{y}|}\delta(|\mathbf{x}-\mathbf{y}| - (x^0 - y^0)) \tag{2.31}$$

Applying this to (2.29), we have:

$$\bar{h}_{\mu\nu} = 4G\int \frac{T_{\mu\nu}(y^\sigma)}{|\mathbf{x}-\mathbf{y}|}\delta(|\mathbf{x}-\mathbf{y}| - (x^0 - y^0))\,\mathrm{d}^4 y$$

integrating over y^0:

$$\bar{h}_{\mu\nu} = 4G\int \frac{T_{\mu\nu}(t_R, \mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}^3 y \tag{2.32}$$

where t_R is the retarded time given by:

$$t_R = y^0 - |\mathbf{x}-\mathbf{y}|$$

Finally, the sought-after solution to equation (2.12) is (2.32) for any $T_{\mu\sigma}$. With this expression further investigations can be made such as multipole expansion and numerical solutions.

2.2 Exact Plane Waves

Differently to what was done in the previous section, we are now going to investigate a specific exact solution to Einstein's equation (1.26) first given by Brinkmann[10] in 1925. The solution we are going to analyze belongs to the general class of solutions called *plane-fronted gravitational waves with parallel rays*, or simply *pp-waves*. In formal treatments such as [11], this class of solutions are characterized by the existence of a covariantly constant (therefore, a Killing type) null vector field k throughout the spacetime, that is:

$$\nabla_\mu k^\nu = 0 \tag{2.33}$$

It can be shown (see [11]) that this condition enables us to write the general line element:

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x,y,u)du^2 \tag{2.34}$$

that upon some conditions on the function $H(x,y,u)$ are the exact solutions of Einstein's equation. The coordinates in the above line element, the so-called Brinkmann coordinates, relate to the Cartesian ones by:

$$u = \frac{1}{\sqrt{2}}(z-t) \quad v = \frac{1}{\sqrt{2}}(z+t) \tag{2.35}$$

where u and v are called the retarded and advanced time, respectively. The specific solution we are going to study is the *vacuum pp-waves*, where we impose that (2.34) satisfies Einstein's equation for $T_{\mu\nu} \equiv 0$.

As a final remark, notice that if the function $H(x,y,u)$ is identically zero, the line element becomes:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + 2dudv \\ &= dx^2 + dy^2 + (dz-dt)(dz+dt) \\ &= -dt^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

which is the Minkowski spacetime metric.

Let us now show that the metric (2.34) indeed allows us to choose a k such that condition (2.33) is satisfied and finds what conditions we should impose on the function $H(x,y,u)$ in order to have a solution to Einstein's equation. First, the matrix representation of the metric is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & H(x,y,u) & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the inverse metric:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -H(x,y,u) \end{pmatrix}$$

We calculate the Christoffel symbols by (1.15). The non-vanishing elements are:

$$\begin{aligned}
&{\Gamma^x}_{uu} = -\frac{1}{2}\frac{\partial H}{\partial x} && {\Gamma^y}_{uu} = -\frac{1}{2}\frac{\partial H}{\partial y} \\
&{\Gamma^v}_{uu} = \frac{1}{2}\frac{\partial H}{\partial u} && {\Gamma^v}_{xu} = {\Gamma^v}_{ux} = \frac{1}{2}\frac{\partial H}{\partial x} \\
&{\Gamma^v}_{yu} = {\Gamma^v}_{uy} = \frac{1}{2}\frac{\partial H}{\partial y}
\end{aligned} \tag{2.36}$$

The lack of any Christoffel symbol of the form ${\Gamma^\nu}_{\mu v}$ indicates that the vector $k = \partial_v$ is the sought-after null covariantly constant vector. In fact:

$$\begin{aligned}
\nabla_\mu k^\nu &= \partial_\mu k^\nu + {\Gamma^\nu}_{\mu\lambda} k^\lambda \\
&= \partial_\mu {\delta^\nu}_v + {\Gamma^\nu}_{\mu\lambda} {\delta^\lambda}_v \\
&= {\Gamma^\nu}_{\mu v} \equiv 0
\end{aligned}$$

We then calculate the Riemann tensor. By (1.17):

$${R^\lambda}_{\sigma\mu\nu} = \partial_\mu {\Gamma^\lambda}_{\nu\sigma} - \partial_\nu {\Gamma^\lambda}_{\mu\sigma} + {\Gamma^\lambda}_{\mu\rho}{\Gamma^\rho}_{\nu\sigma} - {\Gamma^\lambda}_{\nu\rho}{\Gamma^\rho}_{\mu\sigma}$$

by a series of logical steps, we conclude that all the terms ${\Gamma^\lambda}_{\mu\rho}{\Gamma^\rho}_{\nu\sigma} - {\Gamma^\lambda}_{\nu\rho}{\Gamma^\rho}_{\mu\sigma}$ are equal to 0, therefore, for our present choice of coordinates, the Riemann tensor is given by:

$${R^\lambda}_{\sigma\mu\nu} = \partial_\mu {\Gamma^\lambda}_{\nu\sigma} - \partial_\nu {\Gamma^\lambda}_{\mu\sigma} \tag{2.37}$$

then, the non-zero components are:

$$\begin{aligned}
&{R^x}_{uux} = -{R^x}_{uxu} = \frac{1}{2}\frac{\partial^2 H}{\partial x^2} && {R^x}_{uuy} = -{R^y}_{uyu} = \frac{1}{2}\frac{\partial^2 H}{\partial y \partial x} \\
&{R^y}_{uux} = -{R^y}_{uxu} = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y} && {R^y}_{uuy} = -{R^y}_{uyu} = \frac{1}{2}\frac{\partial^2 H}{\partial y^2} \\
&{R^v}_{xxu} = -{R^v}_{xux} = \frac{1}{2}\frac{\partial^2 H}{\partial x^2} && {R^v}_{xyu} = -{R^v}_{xuy} = \frac{1}{2}\frac{\partial^2 H}{\partial y \partial x} \\
&{R^v}_{yxu} = -{R^v}_{yux} = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y} && {R^v}_{yyu} = -{R^v}_{yuy} = \frac{1}{2}\frac{\partial^2 H}{\partial y^2}
\end{aligned}$$

The Ricci tensor is obtained by contracting the indexes λ and μ of (2.37), that is:

$$R_{\mu\nu} = \partial_\lambda {\Gamma^\lambda}_{\mu\nu} - \partial_\nu {\Gamma^\lambda}_{\mu\sigma}$$

but notice that the term $\partial_\nu {\Gamma^\lambda}_{\mu\sigma}$ must be zero. We then have:

$$R_{\mu\nu} = \partial_\lambda {\Gamma^\lambda}_{\mu\nu}$$

Therefore, the only non-zero component of the Ricci tensor is:

$$R_{uu} = -\frac{1}{2}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) \tag{2.38}$$

Finally, the Ricci scalar is:

$$R = R^{\mu}{}_{\mu} = R_{\mu\nu} g^{\mu\nu} = R_{uu} g^{uu} = 0$$

Therefore, Einstein's equation (1.26) in vacuum is:

$$R_{uu} = 0$$

using (2.38),

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \equiv \nabla^2_{x,y} H = 0 \tag{2.39}$$

Hence, the only condition for the metric (2.34) to be an exact solution of the vacuum Einstein's equation is that the function $H(u, x, y)$ is harmonic in the variables x and y. Notice that condition (2.39) does not impose any restriction on the functional dependence of H with respect to u, which is a normal feature of wave solutions.

3 Free Particles on pp-Wave Spacetimes

In this chapter we are going to study particle motion in vacuum pp-wave spacetimes as defined in the previous chapter. For this, we are going to derive the geodesic equations governing this movement and analyze the Einstein equation (2.39). Then, we will integrate the equations of motion numerically using *Mathematica* [12] and show typical trajectories for some choices of function H.

Afterwards, we will evaluate the variation of angular momentum of free particles due to the passage of the wave. This analysis is done by checking how the variation of this dynamical quantity is affected by the parameters of the wave and by the initial conditions of the particle.

3.1 Analytical Investigation

As discussed in the previously, the line element of the vacuum pp-wave solution is given by (2.34):

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x,y,u)du^2 \tag{3.1}$$

The Christoffel symbols are given by (2.36) and, therefore, the geodesic equation (1.16) reads:

$$\ddot{x} - \frac{1}{2}\frac{\partial H}{\partial x}\dot{u}^2 = 0 \tag{3.2}$$

$$\ddot{y} - \frac{1}{2}\frac{\partial H}{\partial y}\dot{u}^2 = 0 \tag{3.3}$$

$$\ddot{u} = 0 \tag{3.4}$$

$$\ddot{v} + \frac{\partial H}{\partial x}\dot{x}\dot{u} + \frac{\partial H}{\partial y}\dot{y}\dot{u} + \frac{1}{2}\frac{\partial H}{\partial u}\dot{u}^2 = 0 \tag{3.5}$$

Notice that equation (3.4) is easily solved. Its solution is: $u = K_1 s + K_2$ (where s is the geodesic parameter and K_1 and K_2 are constants). Since we are going to analyze only the motion of time-like particles, we are allowed to reparametrize the geodesics with the coordinate u.

Using the definitions of advanced and retarded time (2.35), it is possible to rewrite the above equations to Cartesian coordinates as follows:

$$\ddot{x} - \frac{1}{2}\frac{\partial H}{\partial x} = 0 \tag{3.6}$$

$$\ddot{y} - \frac{1}{2}\frac{\partial H}{\partial y} = 0 \tag{3.7}$$

$$\ddot{z} + \frac{1}{2\sqrt{2}}\frac{\partial H}{\partial u} + \frac{1}{\sqrt{2}}\frac{\partial H}{\partial x}\dot{x} + \frac{1}{\sqrt{2}}\frac{\partial H}{\partial y}\dot{y} = 0 \tag{3.8}$$

Additionally, we can still present the geodesic equation in cylindrical coordinates:

$$\ddot{\rho} - \rho\dot{\phi}^2 - \frac{1}{2}\frac{\partial H}{\partial \rho} = 0 \tag{3.9}$$

$$\ddot{\phi} + 2\frac{\dot{\rho}\dot{\phi}}{\rho} - \frac{1}{2\rho^2}\frac{\partial H}{\partial \phi} = 0 \tag{3.10}$$

$$\ddot{z} + \frac{1}{2\sqrt{2}}\frac{\partial H}{\partial u} + \frac{1}{\sqrt{2}}\frac{\partial H}{\partial \rho}\dot{\rho} + \frac{1}{\sqrt{2}}\frac{\partial H}{\partial \phi}\dot{\phi} = 0 \tag{3.11}$$

The above form of the equations are going to be useful when investigating the angular momentum of free particles. Notice that in any coordinate system the geodesic equations are highly dependent on the functional form of $H(u, x, y)$, therefore, no analytical solutions are known for arbitrary function H.

Choosing the function $H(u, x, y)$ is the same as choosing the pp-wave. Nevertheless, as discussed previously, our function must satisfy Einstein's equation in the form of (2.39). However, since this imposes a condition only on the xy dependence of H, we are free to opt for the u dependence.

Let us divide the functional dependence of H as follows:

$$H(u, x, y) = A(u)h(x, y)$$

where $A(u)$ is called the amplitude of the pp-wave. This decomposition is standard in the study of partial differential equations. It was proven (see [13]) that this method yields the general solution for the Laplace's equation. Now, Einstein's equation is written as:

$$\nabla^2 h(x, y) = 0 \tag{3.12}$$

This is a two dimensional Laplace's equation, its solutions (see, for example, [13]) are given by the real and imaginary part of $(x + iy)^n$. Since the expression in Cartesian coordinates can be extensive, we write it in polar coordinates:

$$\begin{aligned}(x + iy)^n &= (\rho e^{i\phi})^n = \rho^n e^{in\phi} \\ &= \rho^n \cos(n\phi) + i\rho^n \sin(n\phi)\end{aligned}$$

Therefore, the solutions to (3.12) are in the form of $\rho^n \cos(n\phi)$ or $\rho^n \sin(n\phi)$, or any other linear combination, since the equation is linear. For a specific n the sine and cosine solutions are regarded as two distinct polarizations. Furthermore, for $n = 2$ the pp-wave is denoted as *homogeneous*, and for $n > 2$ as *non-homogeneous*.

On the other hand, no specific class of functions are naturally selected for the amplitude $A(u)$. Nevertheless, in this work we are going to choose Gaussians, both normalized and non-normalized. These functions serve well to represent pulses of waves without having to deal with any discontinuities. The general form of the normalized Gaussian is:

$$A(u) = \frac{1}{\lambda\sqrt{\pi}} e^{-u^2/\lambda^2} \tag{3.13}$$

Normalized Gaussains are helpful to study the limit $\lambda \to 0$, where the expression (3.13) tends to a Dirac delta, as defined by the properties (2.26)-(2.28).

3.2 Chaotic and Non-Chaotic Motion

In this section we discuss free particle motion in pp-wave spacetimes. In order to do so, we need to make a choice on what function H to use. As it was commented in the previous section, the function H can be classified as homogeneous or non-homogeneous. For clarity, we are going to favor the homogeneous pp-wave $2\rho^2 \cos(2\phi)$ and the non-homogeneous $-(2/3)\rho^2 \cos(3\phi)$[1]. These functions are representatives of their kind, other polarizations are going to behave similarly, and $n = 3$ contains the main characteristics of non-homogeneous pp-waves.

Furthermore, notice that in the geodesic formula, the variable z does not enter in the equations for x or y (or for ρ and ϕ in cylindrical coordinates). For this reason and for the simplicity of bi-dimensional plots, we are going to focus our attention on the motion in the xy plane.

3.2.1 Homogeneous pp-Waves — Non-Chaotic Motion

The first case we are going to analyze is that of the homogeneous pp-wave $2\rho^2 \cos(2\phi)$. In order to obtain prior knowledge on how motion in this space behaves, we are going to first choose a constant amplitude function. So, let:

$$H(u, \rho, \phi) = H(\rho, \phi) = 2\rho^2 \cos(2\phi)$$

or alternatively in Cartesian coordinates:

$$H(u, x, y) = H(x, y) = x^2 - y^2 \tag{3.14}$$

[1] The constants 2 and $-2/3$ have been chosen for mathematical simplicity.

Since we are dealing with second powers only, the Cartesian coordinates are simpler to use. With the function (3.14) and the geodesic equations (3.6) and (3.7) read:

$$\ddot{x} = x$$
$$\ddot{y} = -y$$

The above system of differential equations allow an analytical solution given by:

$$\begin{aligned} x &= x_0 \cosh(u) + v_{0x} \sinh(u) \\ y &= y_0 \cos(u) + v_{0y} \sin(u) \end{aligned} \tag{3.15}$$

where x_0 and y_0 are the particle's x and y coordinates at $u = 0$ and v_{0x} and v_{0y} – its velocity components.

In order to show the general behavior of the motion defined by the solution (3.15), the trajectories of a ring of particles, a unit distance away from the center, were plotted in Figure 2.

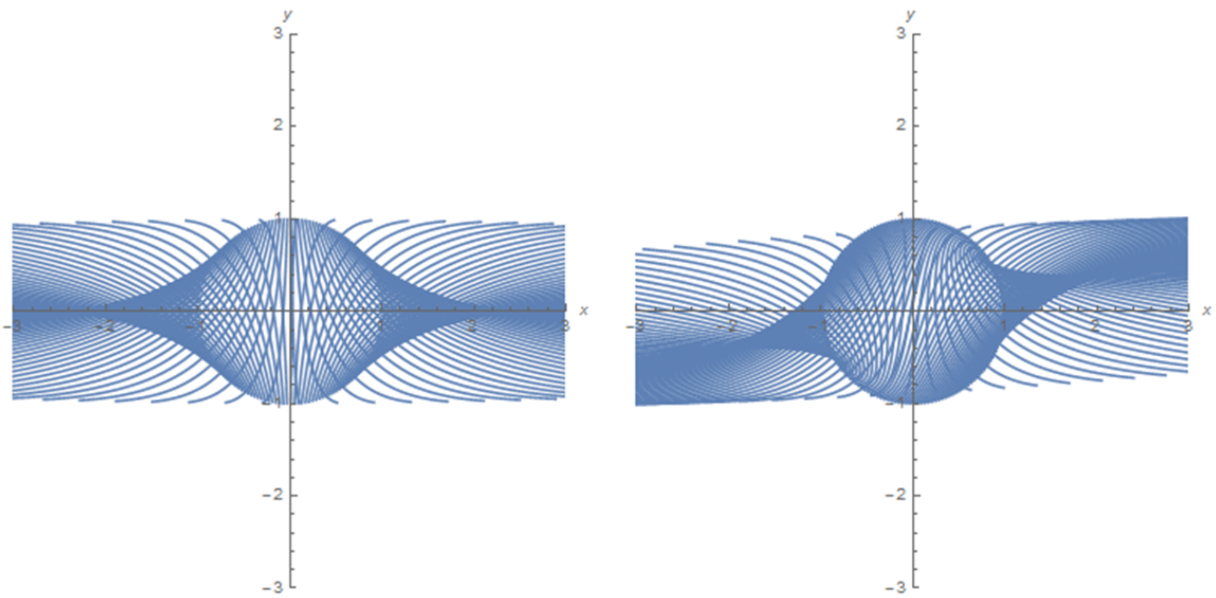

Figure 2 – Trajectory of a ring of particles. In the left image, all particles start with no velocity. In the right image, $\dot{\rho}(0) = -0.2$ and $\dot{\phi}(0) = 0.4$.

We can clearly see from the image the possible two outcomes for the particles. Furthermore, with expression (3.15) in hands we can determine what initial conditions x_0, y_0, v_{0x}, v_{0y} yield certain asymptotic behavior as $u \to \infty$.

Nevertheless, the function (3.14) is not a physically reasonable choice for a gravitational wave due to its non-limited nature at spatial infinity. Therefore, we now choose a more reasonable H function given by:

$$H(u, x, y) = (x^2 - y^2)e^{-u^2/\lambda^2} \tag{3.16}$$

for some value of λ. The relevant geodesic equations are:

$$\ddot{x} = xe^{-u^2/\lambda^2}$$
$$\ddot{y} = -ye^{-u^2/\lambda^2}$$

Although the above equations have a simple form, no exact analytical solution is known. Therefore, we use numerical methods to plot the particle trajectories. Figure 3 shows results for certain values of lambda.

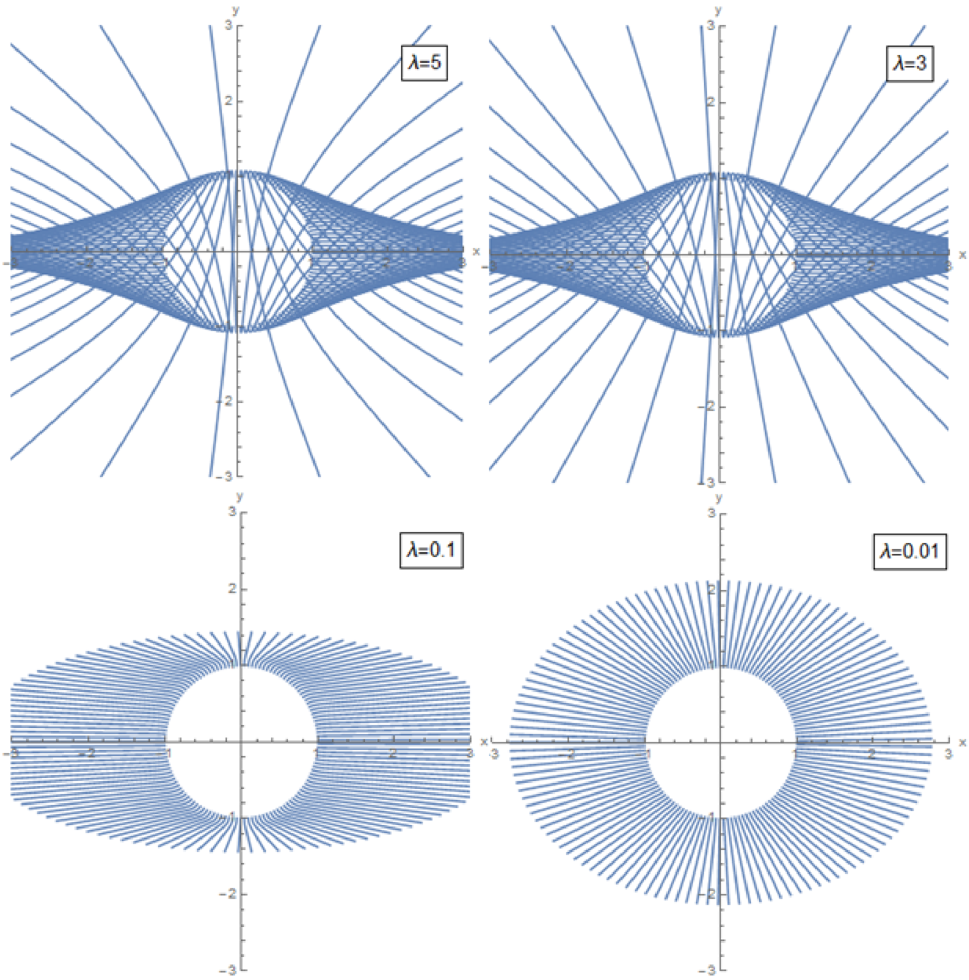

Figure 3 – Trajectory of a ring of particles for various values of lambda.

As one would expect, we notice that the particles start moving in straight lines after the passage of the wave. This is the case because, as shown previously, in the limit $H \to 0$ we recover Minkowski space.

3.2.2 Non-Homogeneous pp-Waves — Chaotic Motion

We consider now the motion of the particles in the non-homogeneous case. This analysis was deeply investigated by J. Podolský and K. Veselý in two main papers (see [14] and

[15]). Here we present a review of its main findings and expand their work in the basin of attraction of the non-homogeneous pp-waves.

Again, we start by considering an H function of constant amplitude:

$$H(u,\rho,\phi) = H(\rho,\phi) = -\frac{2}{3}\rho^3\cos(3\phi) \tag{3.17}$$

and, therefore, the equations of motion (3.9) and (3.10) read:

$$\ddot{\rho} - \rho\dot{\phi}^2 + \rho^2\cos(3\phi) = 0$$
$$\ddot{\phi} + 2\frac{\dot{\rho}\dot{\phi}}{\rho} - \rho\sin(3\phi) = 0$$

These equations do not have any analytical solution. We again plot the evolution of a ring of particles, at $\rho = 1$, in this spacetime in Figure 4.

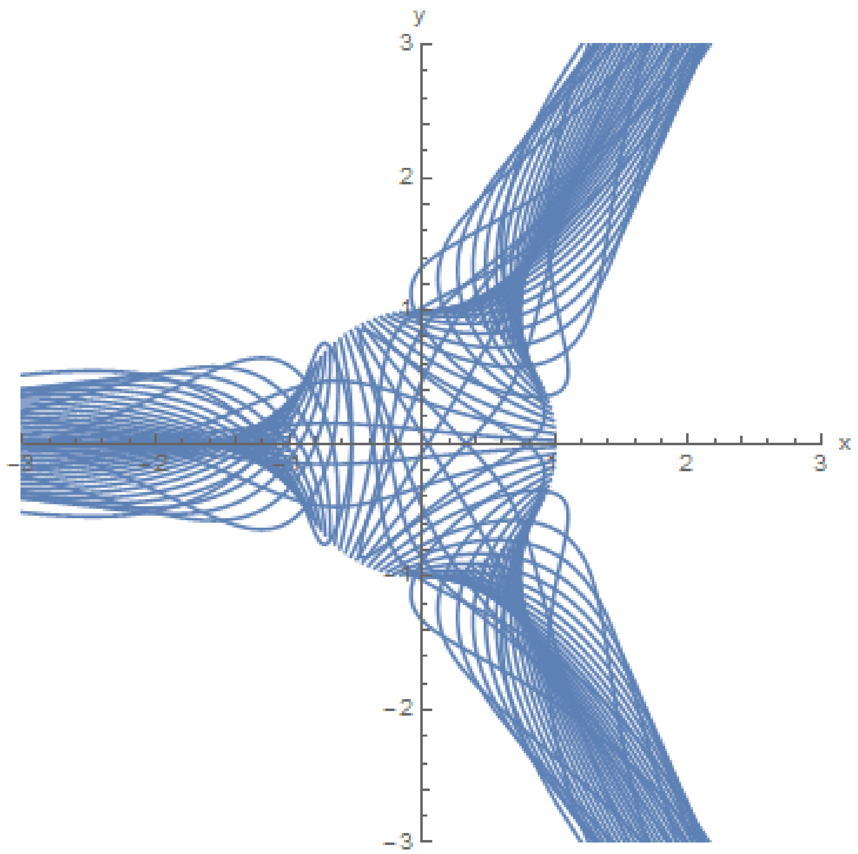

Figure 4 – Trajectory of a ring of particles. No initial velocities were given.

The result that can be seen in the cited figure is that of particles escaping to infinity by three possible channels: one centered at $\phi = \pi$, another at $\phi = -\pi/3$ and the last at $\phi = \pi/3$. They are going to be denoted as channel 1, 2 and 3, respectively. Figure 5 shows us the channel a particle with initial angular position ϕ_0 went to.

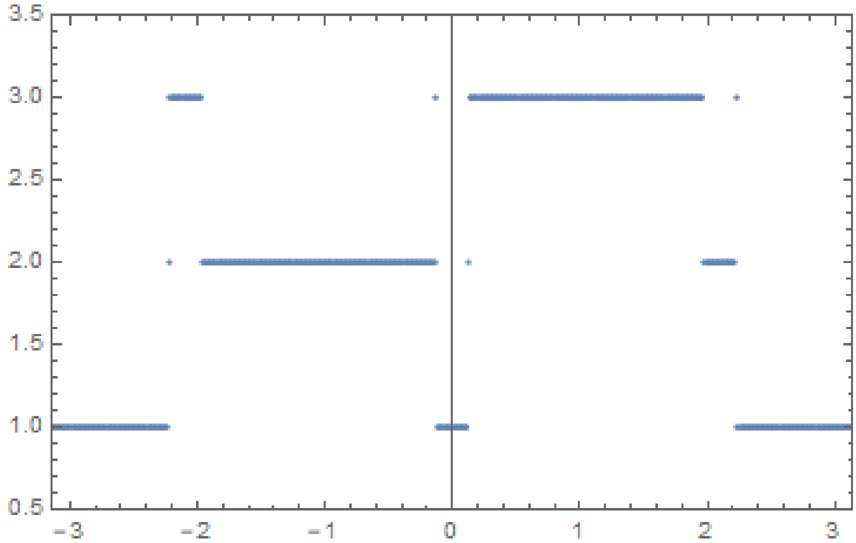

Figure 5 – Channel by initial angle.

In certain locations of the plot it is unclear which channels the particles use to escape to infinity. In fact, this system presents a fractal structure (see [14]) and therefore, we have indications of a chaotic system, as it was discussed in Appendix B. This fact was proven by [16].

To further support and generalize the observation that geodesics of the non-homogeneous pp-wave with constant amplitude are chaotic, we present Figure 6. In this set of figures, we have initial conditions varying on the axis and each final state is characterized by a color[2]. We notice that the boundaries of 'predictable' regions have different behavior than its surroundings, they also have fractal structure (see [14]).

Now, we focus our attention on more realistic non-homogeneous plane waves, i.e., with an amplitude that is not constant. Choosing the amplitude to be a Gaussian, we have:

$$H(u,x,y) = -\frac{2}{3}\rho^3\cos(3\phi)e^{-u^2/\lambda^2} \tag{3.18}$$

with geodesic equations:

$$\begin{aligned} \ddot{\rho} - \rho\dot{\phi}^2 + \rho^2\cos(3\phi)e^{-u^2/\lambda^2} &= 0 \\ \ddot{\phi} + 2\frac{\dot{\rho}\dot{\phi}}{\rho} - \rho\sin(3\phi)e^{-u^2/\lambda^2} &= 0 \end{aligned}$$

As it was done before with the homogeneous case, we show in Figure 7 several trajectories for different values of λ. We notice that we obtain a similar result to the one encountered previously, that is: after a certain characteristic time, the particles enter the flat region of space and start to follow straight lines. However, this smearing effect has

[2] Channel 1 is represented by blue 1, channel 2 by green and channel 3 by yellow.

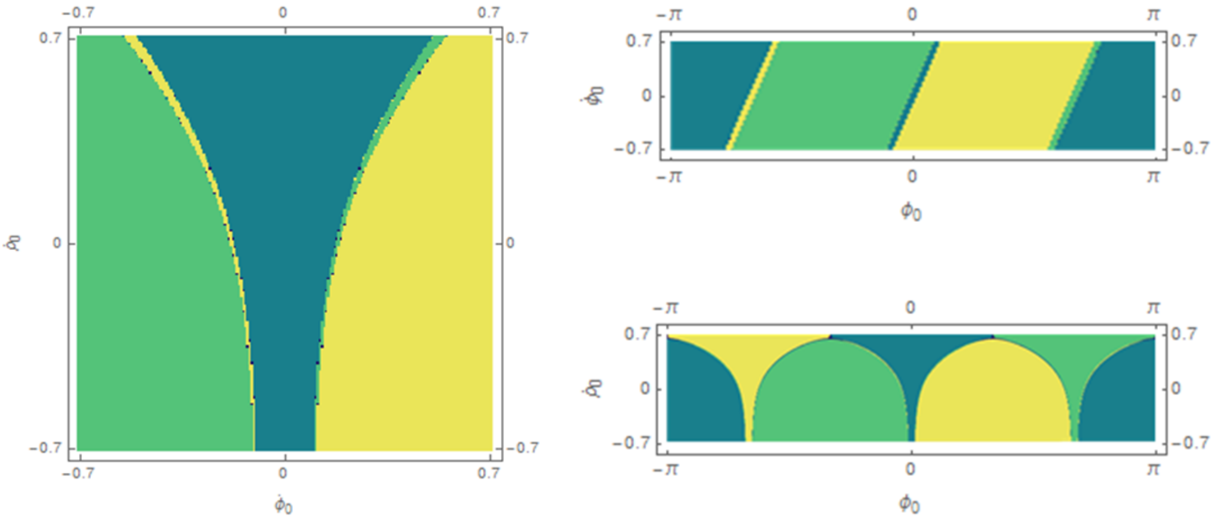

Figure 6 – Initial conditions on the axis and channel represented by colors. In the left graph we set $\phi_0 = 0$, the one in the top right corner uses $\dot{\rho} = 1/\sqrt{2}$, and the one in the bottom right corner, $\phi - 0 = 0$.

an important role when dealing with non-homogeneous waves. As lambda gets smaller, the channels start to get less defined and the chaotic effects fade away.

Furthermore, we see that for small λ not the whole structure disappears, but rather we have a new pattern forming. In fact, it has been proven (see [15]) that as we approach the impulsive case, i.e. $\lambda \to 0$, one obtains an analytical solution that is not chaotic.

3.3 Angular Momentum

The focus of our analysis is the behavior of particles moving slowly before and after their interaction with the gravitational wave. Therefore, we are going to take the classical approximation of the angular momentum (per unit mass). This is further supported by the fact that an experiment's measurement is normally made in a Newtonian regime. Hence:

$$\mathbf{M} = \mathbf{r} \times \mathbf{v}$$

Using cylindrical coordinates, we can write the components of the classical angular momentum as:

$$M_\rho = \frac{1}{\sqrt{2}} \rho z \dot{\phi} \tag{3.19}$$

$$M_\phi = -\frac{1}{\sqrt{2}} \left(z\dot{\rho} - \rho \dot{z} \right) \tag{3.20}$$

$$M_z = -\frac{1}{\sqrt{2}} \rho^2 \dot{\phi} \tag{3.21}$$

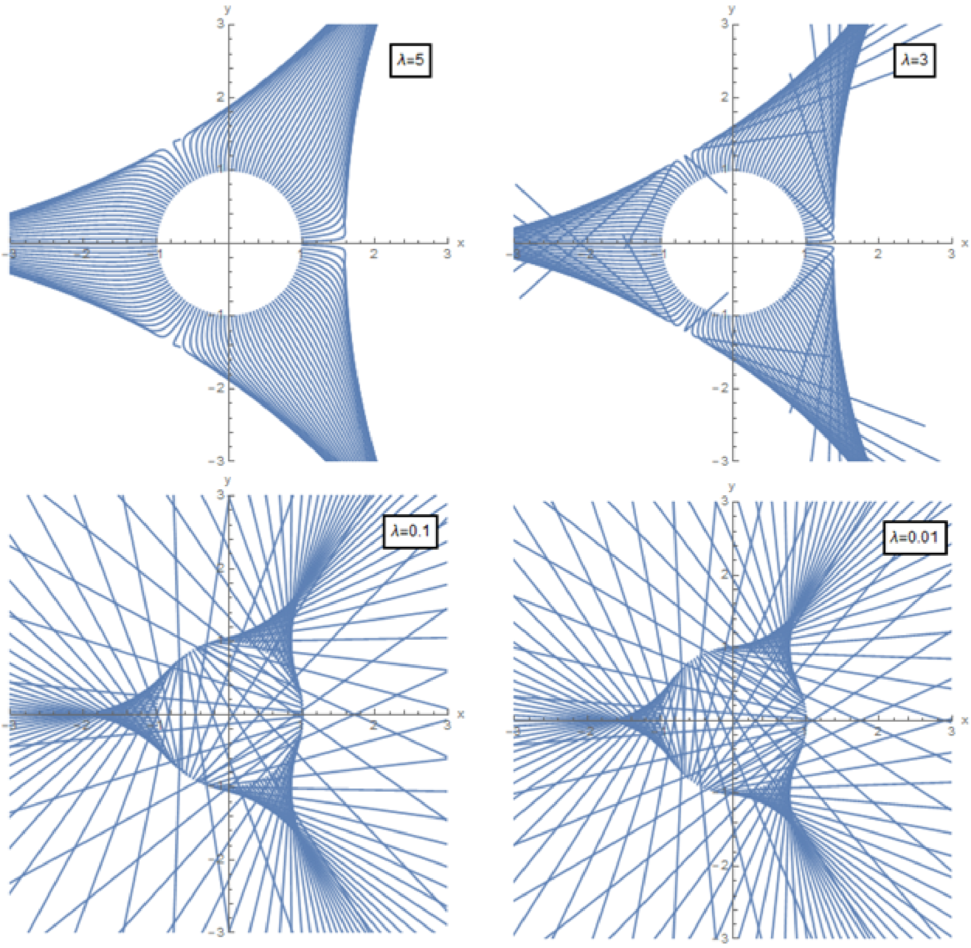

Figure 7 – Trajectory of a ring of particles for various values of lambda.

where the dot represents the total derivative with respect to u. The magnitude M is therefore given by:

$$M = \frac{1}{\sqrt{2}} \left((\rho z \dot{\phi})^2 + (z\dot{\rho} - \rho \dot{z})^2 + \rho^4 \dot{\phi}^2 \right)^{1/2} \tag{3.22}$$

The software *Mathematica* [12] was used to integrate the equations of motion (3.9)-(3.11) and to calculate the angular momentum of the free particles by (3.19)-(3.21) and (3.22). The initial state is given at $u = 6\mathrm{FWHM}$ and the final state at $u = -6\mathrm{FWHM}$, where $\mathrm{FWHM} = \sqrt{\ln(2)}\lambda$ is the full width at half maximum of the Gaussian[3].

Our main goal is to study how dynamical quantities vary for free particles in pp-wave spacetimes. The analysis of the variation of the kinetic energy was done by [17]. We further their discussion by examining the behavior of the angular momentum.

[3] Notice that the signs are correct since we are considering retarded time. Also, the value 6FWHM yields a multiplicative factor of the order of 10^{-11} which is enough for a flat-space time approximation.

3.3.1 Initial Conditions: Regions of Angular Momentum Transfer

To begin investigating the effects of the initial conditions on the angular momentum of free particles, first we must choose a standard $h(\rho, \phi)$ function and a Gaussian width λ. In the previous section we have shown that the non-homogeneous wave has a chaotic behavior. Therefore, in order to deal with less complications, we focus on the homogeneous wave:

$$H = -\rho^2 \cos(2\phi) \frac{e^{-u^2/\lambda^2}}{\lambda\sqrt{\pi}} \tag{3.23}$$

with $\lambda = 7$.

An example of the variation of a particle's angular momentum during the passage of a wave with these parameters is shown on Figure 8.

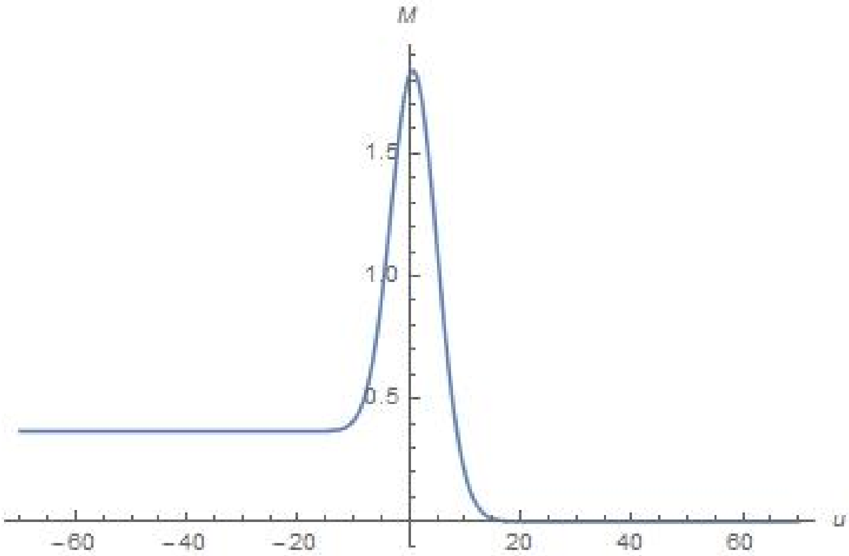

Figure 8 – Angular momentum of a free particle by u, where $\rho(6\text{FWHM}) = 0.6$, $\dot{\rho}(6\text{FWHM}) = 0.2$ and the other initial conditions are zero.

This is already an interesting result, since it shows a net transfer of angular momentum between the particle and the wave, but it does not show the overall effect of the initial conditions on the gain or loss of this quantity. For this another graph was made. In Figure 9 we see a plot where on the axis are the values of $\rho(0)$ and $\dot{\rho}(0)$ and in color is the variation of the angular momentum. All other initial conditions were set to zero.

Figure 9 shows us that if the particle has a negative velocity, the wave tends to absorb angular momentum from it, and vice-versa. As it was shown in [17], this works in the opposite manner when compared to the kinetic energy of particles.

3.3.2 Gaussian Width

We have already seen that the angular momentum of the particles changes with the passage of the wave. Now, we are interested in knowing if this variation is dependent on

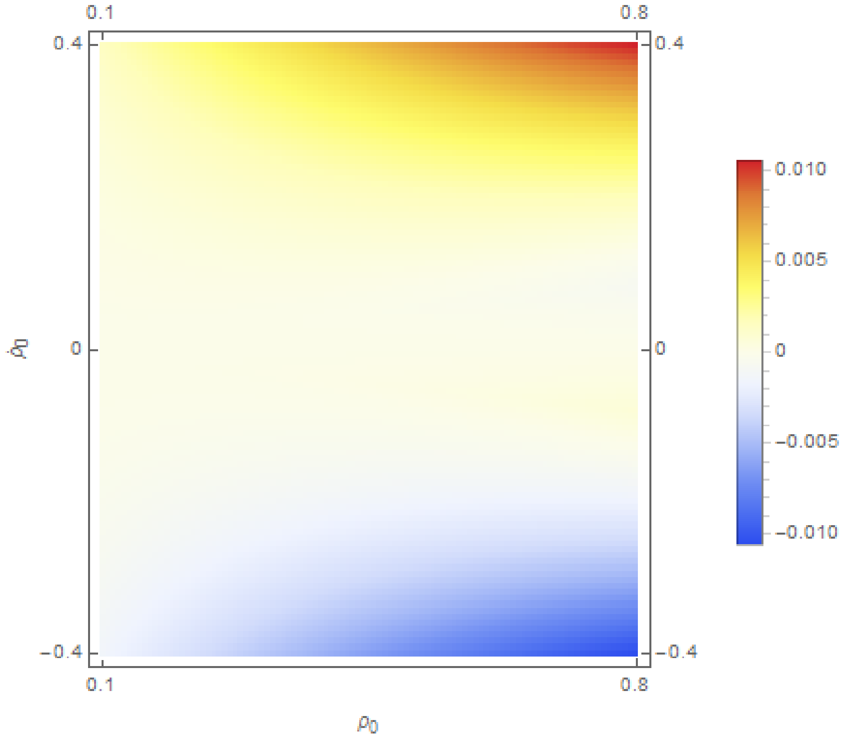

Figure 9 – Angular momentum variation in color with the initial condition on the axis.

the width of the Gaussian or not.

In order to be able to draw comparisons with the reference [18], we have set the initial conditions at $u = 0$. That is, we are imposing the positions and velocities when they are at the peak of the wave. This is a choice of convenience rather than a physical one. Furthermore, we select the non-normalized form of (3.23), that is:

$$H = -\rho^2 \cos(2\phi) e^{-u^2/\lambda^2} \tag{3.24}$$

The initial conditions are set to $\rho(0) = 0.6$, $\dot{\rho}(0) = 0.2$ and the remaining, zero. Then the variation of the angular momentum is computed before and after the passage of the wave. The results are given in Figure 10.

We have obtained certain values of lambda that are responsible for maximum transfer of angular momentum, what might suggest a possible quantum manifestation in a purely classical discussion. By interpolation of this data we get the following behavior:

$$\Delta M = ax \sin(b\lambda + c)$$

where $a = 0.00926303$, $b = 2.49892$ and $c = 5.72529$. Therefore, the maximum transfers occur at:

$$b\lambda + c = \left(\frac{1}{2} + 2n\right)\pi$$

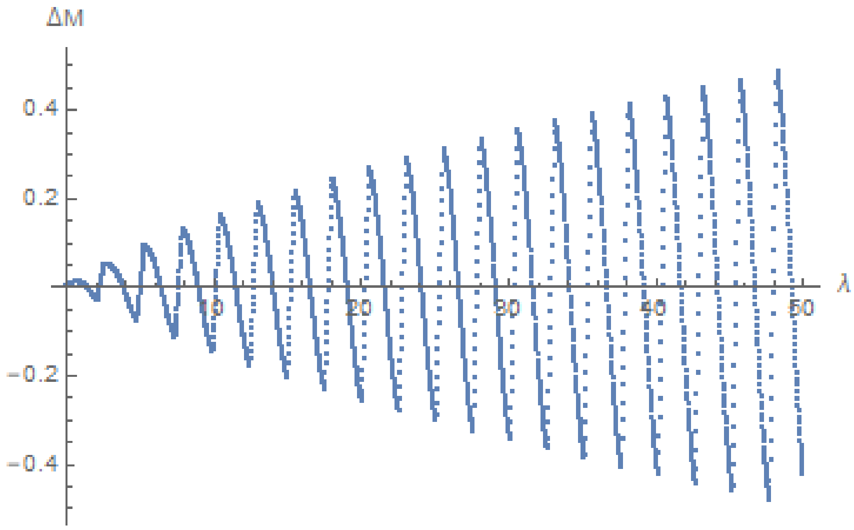

Figure 10 – Variation of the angular momentum depending on the width of the Gaussian.

Comparing this interpolation with the one done for the kinetic energy given in [17][4], we conclude that the frequency parameter b is the same for both phenomena. Furthermore, we notice that the peaks of the angular momentum and kinetic energy are offset by an amount close to $\pi/2$, which implies that there is a trade off between them. All of this evidence reinforces the idea of a possible manifestation of quantum physics.

Now, we would like to verify how the angular momentum of free particles changes for the λs of maximum transfer. Figure 11 shows the evolution of the angular momentum for several maximum lambdas.

We notice that for higher values of n, we obtain more peaks of transfers and, therefore, more interactions with the wave. Nevertheless, we must recall that any analysis of the angular momentum given by (3.22) is better justified when $H \to 0$.

Despite the interesting results obtained previously, the function (3.24) does not yield the impulsive limit as $\lambda \to 0$. Using equation (3.23) we generated the plot shown in Figure 12. We notice that an oscillatory pattern still emerges.

Let us now choose the other possible polarization of the homogeneous wave:

$$H = -\rho^2 \sin(2\phi) e^{-u^2/\lambda^2} \tag{3.25}$$

Figure 13 also shows how the variation of the angular momentum depends on λ.

We notice that for this polarization we do not obtain the same oscillatory behavior as for the cosine one. Using this choice the angular momentum rapidly goes to infinity and a plot for higher lambda yields bad computational results.

[4] There is an error in the fitted curve, it should read 'b' whereas in the article the authors write '$b/100$'.

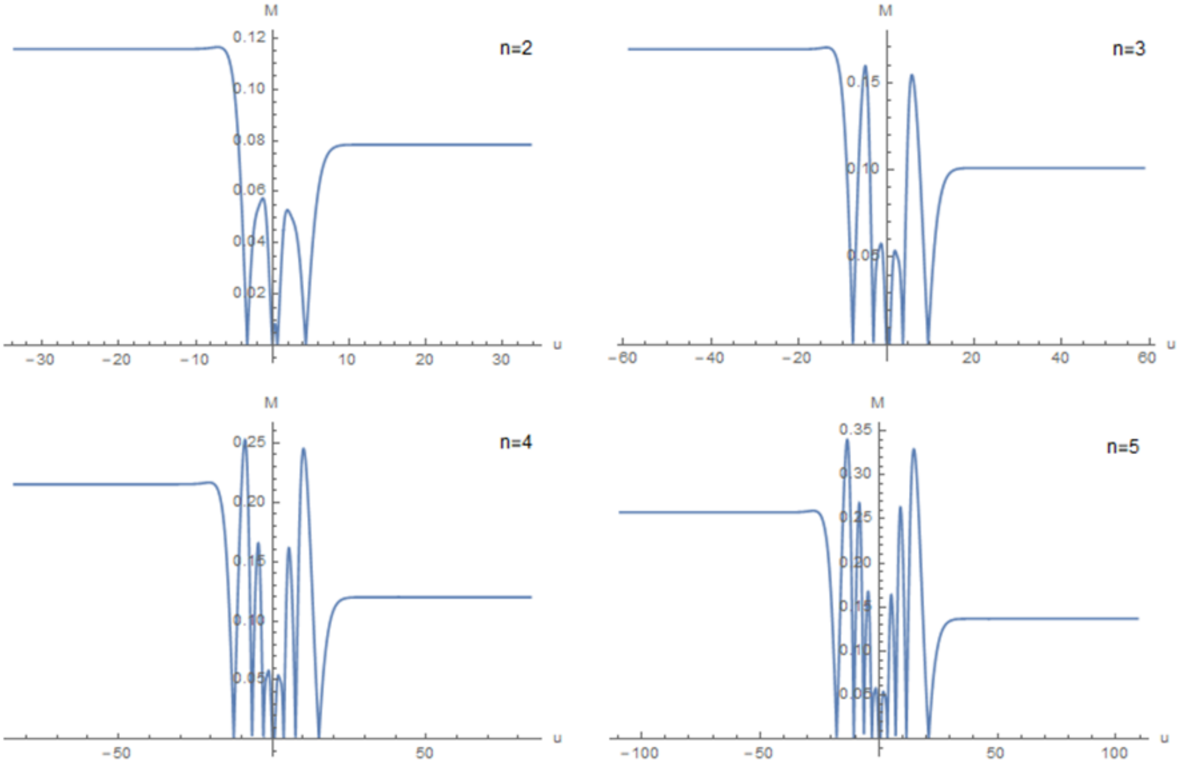

Figure 11 – Variation of the angular momentum depending for lambdas of maximum transfers.

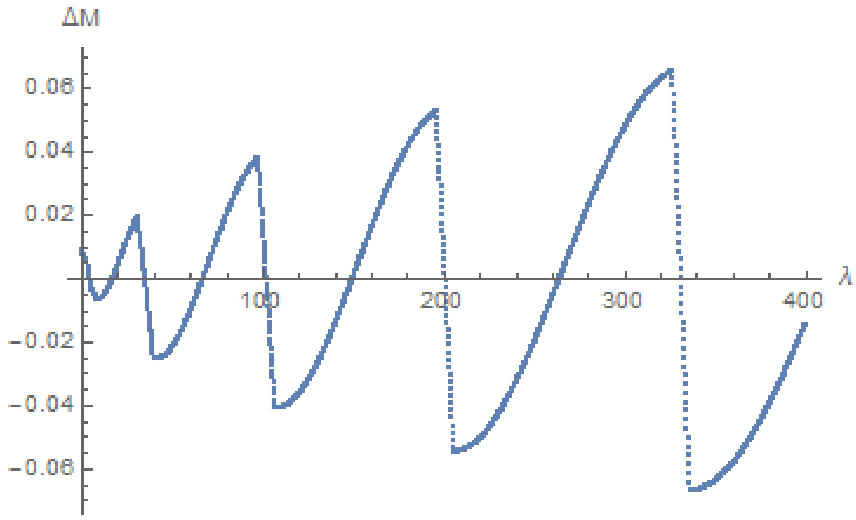

Figure 12 – Variation of the angular momentum for a normalized Gaussian.

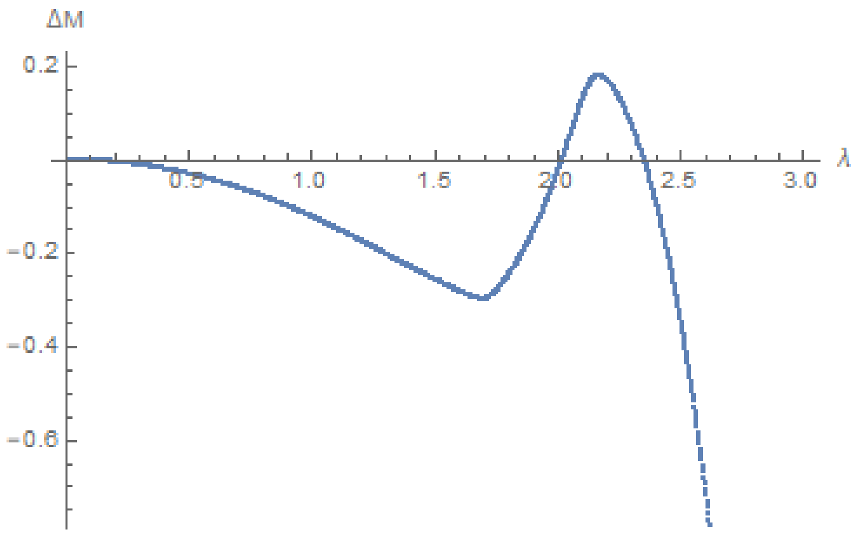

Figure 13 – Sine polarization of the homogeneous pp-wave.

Conclusion

The study of gravitational waves (GWs) dates back to the year after the publication of General Relativity, but with the experimental results of the last decades, more attention has recently been driven to the topic. It is in this context that the analysis of exact GW solutions to the full non-linear Einstein's equations gathers the interest of many researches.

In this work we investigated the motion of free-particles in such spacetimes. In order to do so, in chapter one we summarized the main results of Pseudo-Riemannian Geometry and General Relativity. Subsequently, we introduced two types of gravitational waves: the linearized GW and the *plane-fronted gravitational waves with parallel rays*, also known as *pp-waves*. The latter was the one in which particle motion was studied.

First, we studied the characteristics of the trajectories of the particles in the xy-plane, and we concluded that the movement can be either chaotic or non-chaotic. For homogeneous pp-waves, we have non-chaotic motion regardless of the nature of the amplitude of the wave. In fact, for H functions independent of u, we were able to solve the geodesic equations analytically.

In the case of non-homogeneous waves, we found chaotic motion for waves independent of u. For this particular situation, we showed in Figure 6 that the chaoticity is present for the various possible combinations of initial conditions. Nevertheless, this characteristic was gradually eliminated as we chose the amplitude to be normalized Gaussians with decreasing width. These results are in agreement with the research conducted by J. Podolský and K. Veselý (see [14] and [15]).

Later on, we examined how homogeneous pp-waves can alter the dynamics of free-particles, more specifically, their angular momentum. In the beginning we held a brief discussion on how the initial conditions ρ and $\dot{\rho}$ affect angular momentum transfer for a fixed λ. The result was Figure 9 which shows that positive values of the velocity yield a transfer from the wave to the particles, and for negative values, the opposite is true.

Subsequently, we analyzed how the variation of angular momentum depends on the width of the Guassian pulse. This was the main investigation of this work. We concluded that there are peaks of maximum transfer for the cosine polarization. Furthermore, these peaks occur with the same frequency as those of the kinetic energy (see [17]), but with an offset close to $\pi/2$. This result might suggest a quantum characteristic of this system, even in a purely classical formulation.

Additionally, we have shown (see Figure 12) that we still have peaks of angular momentum transfer even for amplitudes molded by normalized Gaussians. Also, we demonstrated that the cited behaviors do not occur for the sine polarization (see 13).

In conclusion, this work analyzed the motion of timelike free-particles in various spacetimes representing pp-waves, focusing on the trajectories and angular momentum of these objects. Further studies could systematize how all the possible choices of initial conditions affect the absorption of angular momentum. Another possibility for future endeavors is to repeat the dynamical analysis presented here for non-homogeneous pp-waves and study how the chaoticity of these waves modifies the present investigation.

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APPENDIX A – Lie Derivative, Diffeomorphisms and Gauge Transformations

Before constructing the linear theory of gravity, we are going to introduce in this section some necessary mathematical concepts. Here we motivate and define the notions of *diffeomorphism*, *pull-back*, *push-forward* and *Lie derivative* in order to construct the gauge transformations of the linearized Einstein equation (1.26), which we are going to obtain in the next section.

Consider a bijective map $\phi : M \to N$ which is C^∞. Such a map is called a *diffeomorphism* if it has an inverse $\phi^{-1} : N \to M$ which is also C^∞. We are going to demonstrate that general relativity is invariant under these maps and how this allows us to make gauge transformations.

At first, diffeomorphisms seem to map only points of one manifold to the other, but we can use its mathematical structure to build map tensors between them.

Consider $p \in M$, $q \in N$, $f_M \in \mathscr{F}(M)$, $f_N \in \mathscr{F}(N)$, $v_M \in T_pM$, $v_N \in T_pN$, $\omega_M \in T_pM^*$ and $\omega_N \in T_pN^*$, as shown in figure A. We want to construct the maps ϕ_* which 'push-forward' objects of M to N and maps ϕ^* which 'pull-back' objects of N to M. This is obtained by:

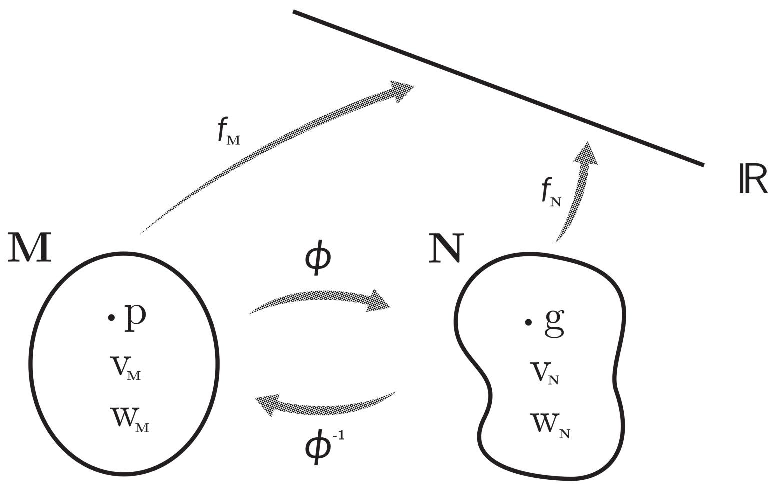

Figure 14 – Maps between manifolds.

$$(\phi^* f_N)(p) = f_N(\phi(p)) \tag{A.1}$$
$$(\phi_* f_M)(q) = f_M(\phi^{-1}(q)) \tag{A.2}$$
$$(\phi^* v_N)(f_M) = v_N(\phi_* f_M) \tag{A.3}$$
$$(\phi_* v_M)(f_N) = v_M(\phi^* f_N) \tag{A.4}$$
$$(\phi^* \omega_N)(v_M) = \omega_N(\phi_* v_M) \tag{A.5}$$
$$(\phi_* \omega_M)(v_N) = \omega_M(\phi^* v_N) \tag{A.6}$$

Notice that in all the above equations the right-hand side is well defined. Therefore, given a diffeomorphism ϕ and its inverse, by equations A.1-A.6 we can move functions, vectors and dual vectors from one manifold to the other. We can now compute the push-forward of a general (k, l) tensor by:

$$(\phi_* T)(v_1, ..., v_k, \omega_1, ..., \omega_l) = T(\phi^* v_1, ..., \phi^* v_k, \phi^* \omega_1, ..., g f^* \omega_l) \tag{A.7}$$

and analogously, the pull-back.

Now, to help us understand what is happening, let us introduce coordinates x^μ on M and y^ν on N. In this basis, (A.4) becomes:

$$\begin{aligned}(\phi_* v_M)^\alpha \partial_\alpha f_N &= v_M^\lambda \partial_\lambda (f_N(\phi(p))) \\ &= v_M^\lambda \frac{\partial y^\alpha}{\partial x^\lambda} \partial_\alpha f_N\end{aligned}$$

since, the function f_N is arbitrary:

$$(\phi_* v_M)^\alpha = \frac{\partial y^\alpha}{\partial x^\lambda} v_M^\lambda$$

Analogously for (A.6):

$$(\phi_* \omega_M)_\alpha = \frac{\partial y^\lambda}{\partial x^\alpha} \omega_\lambda$$

Hence, for (A.7):

$$(\phi_* T)^{\mu'_1 \dots \mu'_k}{}_{\nu'_1 \dots \nu'_l} = \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial y^{\nu'_l}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

We notice that this is nothing more than the transformation law (1.7). Therefore, any theory that is invariant under change of coordinate (what is reasonable to ask of a physical theory) is also invariant under a diffeomorphism.

In fact, we can think of the existence of a diffeomorphism between M and N as the definition of the two manifolds being the same. Also, this map can be interpreted as an "active" coordinate transformation, in the same sense of alias and alibi transformations.

In general, any physical theory that describes spacetime as a manifold and physical quantities as tensor fields on it should be invariant under diffeomorphisms. That is, given $\phi : M \to N$ and (M, T), then $(N, \phi_* T)$ describes the same spacetime and its characteristics.

We now consider the case of *one parameter family of diffeomorphisms*. Given a vector field V^μ, we can find its *integral curves* $x^\mu(\lambda)$ by:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = V^\mu$$

with the initial condition $x^\mu(0) = p$.

Then we define the *one parameter family of diffeomorphisms* $\phi_\lambda : \Re \times M \to M$ to be given by: $\phi_\lambda(p)$ is the point lying at parameter λ along the integral curve of V starting at the point p. Notice that by this definition $\phi_t \circ \phi_s = \phi_{t+s}$, where t and s are real numbers.

With this machinery in hands, we can define a fundamental derivative in the manifold. Given a vector field V, together with its associated one parameter family diffeomorphism ϕ_λ, and a tensor T; the Lie derivative $\mathcal{L}_V T$ at the point $p \in M$ of this tensor with respect to V is:

$$\mathcal{L}_V T = \lim_{\lambda \to 0} \frac{\phi^*_\lambda(T(\phi_\lambda(p))) - T(p)}{\lambda} \tag{A.8}$$

From this definition and from the linearity of tensors, it is clear that the Lie derivative has the ordinary properties of a derivative, i.e., linearity and Leibniz rule. It can also been shown (see [5]) that given a derivative operator ∇, this definition implies:

$$\mathcal{L}_V T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = V^\lambda \nabla_\lambda T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} - \sum_i T^{\mu_1 \ldots \lambda \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \nabla_\lambda V^{\mu_i} + \sum_j T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \lambda \ldots \nu_l} \nabla_{\nu_j} V^\lambda$$

specifically for the metric tensor:

$$\mathcal{L}_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \tag{A.9}$$

where, in this case, ∇ is the derivative operator related to the metric $g_{\mu\nu}$. Vectors V for which $\mathcal{L}_V g_{\mu\nu} = 0$ are called *Killing vectors*.

With this formalism, we can discuss gauge freedom in general relativity. Let us consider a perturbation around a known metric $G_{\mu\nu}$ on a manifold M:

$$g_{\mu\nu}(\lambda) = G_{\mu\nu} + \lambda h_{\mu\nu}(\lambda)$$

When λ is assumed to be small we have the linearized theory and only linear terms in this parameter are considered. Notice:

$$G_{\mu\nu} = g_{\mu\nu}(0) \qquad\qquad h_{\mu\nu} \equiv h_{\mu\nu}(0) = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda} g_{\mu\nu}\right|_{\lambda=0}$$

Consider now an arbitrary one parameter family of diffeomorphisms $\phi_\lambda : M \to M$. We already know that $(M, \phi_\lambda^* g_{\mu\nu})$ represents the same spacetime. Also in the linear approximation:

$$h'_{\mu\nu} \equiv h_{\mu\nu}(0) = \frac{\mathrm{d}}{\mathrm{d}\lambda}(\phi_\lambda^* g_{\mu\nu})\bigg|_{\lambda=0}$$

We want to relate h_μ with $h'_{\mu\nu}$, i.e., we seek the expression for the gauge transformation of the perturbation h. Notice that by (A.8), we have:

$$\begin{aligned}
\mathcal{L}_V g_{\mu\nu} &= \lim_{\lambda\to 0} \frac{\phi_\lambda^* g_{\mu\nu}(\lambda) - g_{\mu\nu}(\lambda)}{\lambda} \\
&= \lim_{\lambda\to 0} \frac{\phi_\lambda^* g_{\mu\nu}(\lambda) - \phi_0^* g_{\mu\nu}(0) + \phi_0^* g_{\mu\nu}(0) - g_{\mu\nu}(\lambda)}{\lambda} \\
&= \lim_{\lambda\to 0} \frac{\phi_\lambda^* g_{\mu\nu}(\lambda) - \phi_0^* g_{\mu\nu}(0)}{\lambda} - \lim_{\lambda\to 0} \frac{g_{\mu\nu}(\lambda) - g_{\mu\nu}(0)}{\lambda} \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}(\phi_\lambda^* g_{\mu\nu})\bigg|_{\lambda=0} - \frac{\mathrm{d}}{\mathrm{d}\lambda} g_{\mu\nu}\bigg|_{\lambda=0} \\
&= h'_{\mu\nu} - h_{\mu\nu}
\end{aligned}$$

where V is the vector field associated with ϕ_λ. Therefore:

$$h'_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_V g_{\mu\nu}$$

Using equation (A.9) we arrive at the desired expression for the gauge transformation of the perturbation h:

$$h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu V_\nu + \nabla_\nu V_\mu \tag{A.10}$$

APPENDIX B – Chaos Theory

The main subject of study of *Chaos Theory* are deterministic dynamical systems that are unpredictable and highly dependent on initial conditions. This is the case for several systems such as the three body problem, the double pendulum and, most importantly, the motion in pp-wave spacetimes, the latter is verified in the third chapter of this work. In this appendix we intend to state the main definitions of chaos theory and investigate some of its implications. One can find in-depth discussions in books [19] and [20].

Before characterizing a *dynamical system*, we must define: periodic points, dense sets, transitivity and sensitive dependence on initial conditions.

Definition B.1 Periodic Point

The point at which a particle starts and returns, after a finite time evolution, is defined as periodic.

This notion is a continuous generalization of the standard definition of periodic point for *maps* (where the dynamical system evolves in finite timesteps). An example would be any point of a harmonic oscillator; whereas, the only periodic point of a falling object is the ground.

Definition B.2 Dense Set

Let X be a set and Y a subset of X. Y is said to be dense in X if for every $x \in X$ there is an element $y \in Y$ arbitrarily close to x.

It is worth mentioning that in the above definition we used the expression 'close' vaguely. The concept of proximity of two points in a mathematical space has to be previously given by a certain distance function. For example, one could use either the Euclidean norm or the maximum metric[1] for the Euclidean space.

A classical case of a dense set are rational numbers in real numbers. To prove this is true, consider the following: if x is rational, just take $y = x$. Otherwise, if x is irrational, we can take y to be the truncated decimal representation of x with enough digits to satisfy the 'closeness' condition.

Definition B.3 Transitivity

A dynamical system is transitive if for all pairs of x and y and for any $\epsilon > 0$, there is a point z within ϵ of x such that a trajectory that passes by z also passes within a distance ϵ of y.

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[1] In $\mathbb{R}^2$ this is the metric that defines the distance d between two points to be $d = \max(|x_1 - x_2|, |y_1 - y_2|)$. And equivalently for higher dimensions.

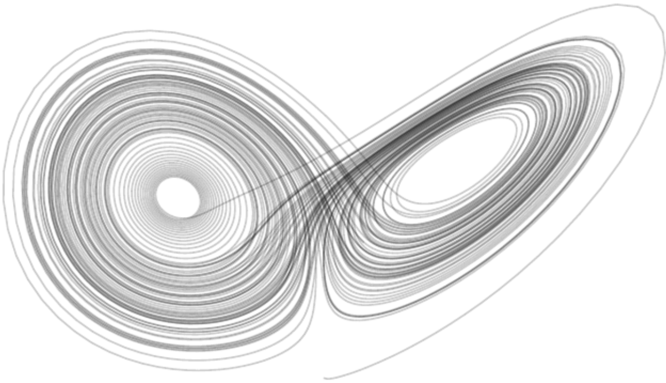

Figure 15 – The Lorenz attractor.

Definition B.4 Sensitive Dependence on Initial Conditions

A system depends sensitively on the initial conditions if there is a $\beta > 0$ *such that for any point* x *and for any* $\epsilon > 0$*, there is a point* y *within* ϵ *of* x *that will be at least a distance* β *from* x *after a certain amount of time.*

This is the most well known property of chaotic systems, in fact, it was this property that initiated the whole field of study. Intuitively, the sensitive dependence on initial conditions means that we can not predict the exact behavior of certain dynamical systems even with them being completely deterministic.

Now, with all the previous definitions we can state what it means mathematically for a dynamical system to be chaotic.

Definition B.4 Chaotic System

A system is said to be chaotic if:

1. *It depends sensitively on the initial conditions;*
2. *It is transitive;*
3. *The set of periodic points is dense.*

The former definition of a chaotic system is mathematically precise. Nevertheless, we encounter great difficulty in proving that a specific physical system is chaotic by this definition. However, dynamical systems have, for the most part, characteristics that indicate chaoticity.

One of these characteristics is the presence of *attractors*. In chaotic systems, one cannot predict exactly what is going to happen to a particle for a specific time. Nonetheless, when we analyze the long time behavior of orbits we observe, most of the time, that the particle (or a set of particles) describes a pattern called *attractor*. For example, Figure 15 shows the famous form found by Lorenz in his weather studies.

However, if this attractor was a simple geometric shape, there would not be a sensitive dependence on initial conditions. For this reason, chaotic systems have fractal[2] attractors in general. The complexity of such objects indicates the presence of the defining properties of Definition B.4. In Figure 15 the fractal structure is present for some sections of the graph.

[2] Fractals are geometrical objects that, in simple terms, stay intricate as one 'zooms' into any part of the whole. In many cases, this characteristic is obtained by self-similarities. Examples of fractals are: the Mandelbrot, Julia and Cantor sets.