Spectral Action Approach to Higher Derivative Gravity

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Abstract

We study the spectral action approach to higher derivative gravity for the case of pure gravity. Our goal is to establish a connection between the spectral action and higher derivative gravity, and show the usefulness of the same. The spectral action has been widely used in particle physics. However, its applications in the field of gravitation have remained obscure so far. In this dissertation we attempt to apply the spectral action approach motivated by non-commutative geometry to the higher derivative gravity and study the equations of motion coming from the gravitational actions containing higher derivatives, which are derived from asymptotic expansion using heat kernel techniques. We consider the case of heat kernel coefficient a_6 in a great detail and analyze it in two bases, namely Riemann one and Weyl. In particular, we construct the action based on a_6 in Riemann and Weyl dominated forms and calculate the equations of motion for the same. We apply these results to some black hole and cosmological solutions as well. A brief review on higher derivative gravity is also given to make the dissertation self-contained. We also discuss the spectral action approach with all of its details, which are necessary for our purpose. Moreover, the actions for the heat coefficients a_0, a_2 and a_4 and corresponding equations of motion are also evaluated.

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AUTHOR'S DECLARATION

I hereby declare that this dissertation is a record of bonafide project work carried out by me in accordance with the rules and regulations of the university for research degree programmes. I affirm that this work has not been submitted before for any other academic award. The work that has been used as a review or in any other way is indicated by a specific reference. Any of the views presented in this dissertation are that of the author.

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Contents

CHAPTER 1

INTRODUCTION

The conundrum of quantum gravity has remained unsolved so far. Although, many approaches have been introduced in order to tackle with this one of the most convoluted problems in theoretical physics, still there are not any considerable accomplishments. In this work we introduce the approach based on spectral action, which is motivated by non-commutative geometry. The spectral action has been widely used in the field of particle physics. However, its applications in the field of gravity have remained limited hitherto for some reasons. Even though, here we do not apply the spectral action approach explicitly to the problem of quantum gravity, but we set the stage to deal with the problem by studying its classical part. This study may provide us deeper insight and a stepping stone toward the solution of the most perplexing problem of quantum gravity.

Mainly we investigate if there is anything special about the spectral action approach. More precisely, we try to see if it produces any kind of cancellations that may shed some light on the hidden symmetries of theories under consideration. We consider the higher derivative gravity emanating from the spectral action principle and construct the gravitational actions containing higher derivatives of the metric. We exploit the heat kernel coefficients coming from asymptotic expansion of the trace of heat operator to devise these actions. We also calculate the equations of motion (EOM) for the same. Furthermore, we apply these EOM to evaluate some black hole and cosmological solutions.

The dissertation is organized as follows. Chapter 2 and 3 contain the review of higher derivative gravity and the spectral action approach, respectively. We consider mainly the case of pure gravity. Initially we discuss the core idea of higher derivative gravity, some problems associated with it and possible solutions. Later we establish a connection between the spectral action approach and higher derivative gravity. Next we present a detailed review of the spectral action approach and calculate the heat kernel coefficients explicitly to show that the higher derivative gravity arises quite naturally in the framework of spectral action principle. Although, we mainly use the results of heat kernel coefficients calculated on a base manifolds without boundaries, but we briefly discuss the general case as well i.e. base manifolds with boundaries and try to see how it modifies the heat kernel coefficients. Chapter 4 and 5 consist mainly of the original work, which is based on our studies of higher derivative gravity and the spectral action approach. In particular, first we simplify the existed form of an action for the heat kernel coefficient a_6 with higher derivatives (six) of the metric and formulate it in two bases, namely Riemann one and Weyl. We also calculate corresponding EOM by varying these two forms of the actions with respect to the covariant metric. Later on we analyze the Ricci flat (Riemann) and conformally flat (Weyl) solutions by applying these equations. Moreover, we briefly review the heat kernel coefficients a_0, a_2, a_4 and EOM for the same to make the dissertation complete in itself. Finally we conclude our work in ch. 6 by discussing the main outcomes and future aspects of our studies.

It is advisable for the reader to go through the notations and conventions given in appendix A before start reading the content of the dissertation. It will be also quite fruitful to review appendices B and C, where some useful formulae are given which have been used frequently in some of the derivations.

CHAPTER 2

A Brief Review of Higher Derivative Gravity

As the name suggests higher derivative theories contain higher derivatives, which are higher than the second order derivatives that appear in the standard theories. It is quite natural to expect such theories in different branches of physics due to some (quantum) corrections to the classical action of the theory, which require to add some terms involving higher derivatives. For example, the corrections to general relativity to make the theory renormalizable [16]; corrections in the case of cosmic strings [18–20], which are motivated by the prediction of the terms of the type of R^2 and higher in the framework of non-linear sigma models of string theory studied in [17] and few changes in the classical model of radiating electron [21]. The process of adding higher derivative terms in the original action comes with the price. It makes the original form of the theory behaving better purterbatively, but also gives rise to some problems such as more number of degrees of freedom compared to the normal (unperturbed) action, absence of ground energy state and negative energy. We shall consider this topic in more detail in the context of gravitational sector later. A detailed review on different classes of higher derivative theories can be found in [15]. Here we are mainly interested to study the class of higher derivative theories, which has to do with the corrections to general relativity [6], where the corrections are added to the standard form of the Lagrangian in the form of higher powers of the curvature and/or higher derivatives of the Riemann tensor. This is known as higher derivative gravity (HDG). It is crucial to note that such theories (especially with six derivatives) having complex massive poles behave as Lee-Wick theories [43, 44]. For such cases the super-renormalizable model of higher derivative quantum gravity was considered in [45]. Moreover, the multidimensional HDG with more number of degrees of freedom than the standard graviton field studied in [46] suggests that it is possible to make the theory finite in any dimension by introducing the local potential of the Riemann tensor. The Newtonian singularities in such theories (local HDG compatible with Lee-Wick theory), which are either renormalizable or super-renormalizable get evanesced, when the poles of the propagator are real and simple [47]. For some others studies on low energy effects in HDG models possessing real and complex massive poles we refer to [48].

There exists another approach to quantum gravity, which is contrary to the idea discussed above. It is known as Hořava-Lifshitz gravity (HLG) [22], where instead of adding the terms with higher derivatives in spacetime we include only the terms containing spatial (space) higher derivatives. Because of this reason it is quite natural to expect that it violates the Lorentz invariance, but HDG preserves the same, as it contains higher derivatives in spacetime. However, there is a benefit with the price being paid by breaking the Lorentz symmetry, that HLG does not have any problems with higher time derivatives, whereas, HDG suffers with such problems. As we shall see in the next chapter the spectral action approach to HDG depends on the choice of Dirac operator, in principle one may choose another form of the same than the standard one and may end up with different theory, such as HLG. There we shall consider briefly the relation between HLG and spectral action approach. However, concerning about this work we are mainly interested to study spectral action approach and its application to HDG. Therefore, we shall focus only on the aspects of spectral action that lead us to HDG. Now, in the next section we momentarily discuss the historical progress regarding HDG and some recent developments concerning quantum properties for the same. Later we take into account the main idea of HDG for the case of pure gravity, some problems related to the same and possible solutions. Finally, we conclude this chapter by explaining the connection of spectral action with HDG, which is to be considered in detail in the next chapter.

2.1 Historical Analysis of Higher Derivative Gravity

The standard form of the Einstein-Hilbert (EH) action involves the second order derivative of the metric, and obviously the EOM resulting from such an action has the same characteristics. In few year of publishing these results (more precisely general relativity in 1915), it was quite well understood that there might be some higher order derivative corrections to the standard form of EH action. It all started with an unsuccessful attempt to reconcile the gravity with electromagnetism [23, 24]. But due to the failure of this approach it was ruled out in later stages. However, some progress in the direction of the usefulness of HDG was seen in 1950, when Pais and Uhlenbeck [25] showed that it might be helpful to consider higher order corrections in context of quantum field theory in general. In particular by doing this it may help to tackle with the divergences in the theory and shed a light on the problem of quantum gravity. A remarkable work was done by Utiyama and DeWitt in 1962 [26] by studying the fact that singularities (especially the singularity of the type of $\log\infty$) of energy-momentum tensor can be removed by using the counter term coming from the Lagrangian which was basically quadratic in Riemann tensor i.e. with the four derivatives of the metric. They basically proved that it is possible to renormalize the divergences arising due to quantum corrections to the interactions of matter sector. This idea was put forward and strengthened by the work of t'Hooft and Veltman [9] in 1974, where they were able to absorb all the physical divergences in renormalization of a field for the case of pure gravity at one loop level. However, the use of improved energy-momentum tensor was unfruitful to remove the divergences in the case of gravitation interacting with scalar particles. Later on in 1977 Stelle [7] showed that by incorporating the correction terms proportional to $R_{\mu\nu}^2$ and R^2 in the standard (undeformed) form of the Lagrangian of gravity, it is possible to stabilize the divergences, consequently making the theory renormalizable, even with the matter part included. This astonishing work clearly indicates that it worth to incorporate more correction terms containing higher derivatives (higher than four derivatives) of the metric in the gravitational Lagrangian and study its variational consequences such as EOM, and beta functions in renormalization group. Getting inspired by this idea, we consider the deformed gravitational action (for the case of pure gravity) consisting of the terms, which comprise six derivatives of the metric [4] and calculate the EOM for such an action. We shall discuss more about this later on, when we explicitly study the gravitational action for the heat kernel coefficient a_6 and corresponding EOM.

Before we move on to the actual review of HDG, it would be quite interesting to take a look at some recent studies on quantum properties of HDG (concerning six and higher derivatives for the general case) such as super-renormalizability and scattering amplitudes. For example, super-renormalization for the case of action consisting of large number of higher derivatives of the metric was considered in [38], where the authors were able to show that the ultraviolet divergences are free from the choice of field reparametrization and the gauge fixing condition. Furthermore, quite recently the class of non-polynomial HDG was studied substantially in [39], where it was proven that in four dimension the extension of the theory turns out to be finite and more importantly all the beta functions get vanished even at one loop level. The generalization of these studies can be found in [40]. The scattering amplitudes for super-renormalizable gravitational theory was analyzed in [41], and it was shown that the scattering amplitudes for such theories are the same as that of Einstein gravity. The authors also managed to show that the four graviton scattering amplitudes in Weyl conformal gravity get evanesced (become zero), and these results turn out to be true for any number of external gravitons and in any dimension in general. The renormalization group for super-renormalizable theories was considered in [42] and the exact beta functions for the Newton constant derived by performing perturbative one loop calculations.

2.2 Higher Derivative Gravity, Problems and Solutions

We consider the case of pure gravity i.e. in absence of matter, where we pay attention to the geometrical part of the action only. There is a natural connection between HDG and the spectral action approach, which we shall take into account in the next section and detail analysis in the next chapter. Here we mainly discuss about the core idea of HDG and some problems related to the same. There are some possible solutions to the problems, and at the same time drawbacks arising from such solutions. However, as we mentioned in previous section, HDG may provide a clue toward possible solutions of the conundrum of quantum gravity by getting rid off the divergences at high energy level and consequently making the theory renormalizable in the framework of quantum field theory. This motivates us to study it more precisely in terms of heat kernel coefficients with the powerful tool known as spectral action approach, where HDG emanates in a quite natural way. Let us see below the conventional idea of HDG and how it gives rise to some serious problems.

We know that the EH action in $d = 4$ dimensional Euclidean space with cosmological constant included is given by,

$$S_{\text{EH}} = \int d^4x \, \sqrt{g} \, (\alpha_0 + \alpha_1 R), \tag{2.1}$$

where α_0 and α_1 are arbitrary numerical constants, and $S_{\text{const}} \equiv \int d^4x \, \alpha_0 \sqrt{g}$ represents the cosmological term. Here we see that the action (2.1) and resulting EOM (excluding cosmological term) from the same contain the second order derivative of the metric. One might think that there must be a similar action involving the higher derivatives of the metric, which act as correction terms in an undeformed action (2.1). This naive way of thinking is actually right in some sense, and it leads us to the action containing the four derivatives of the metric. As one may expect this deformed action consists of the terms such as $R^2_{\mu\nu\rho\sigma}$, $R^2_{\mu\nu}$ and R^2 (written in Feynman's notation, see the appendix A).

Now momentarily let us consider the Gauss-Bonnet (GB) term, see the appendix (B.5), which can also be written as [9],

$$\begin{aligned} R_{\mu\nu\alpha\beta}R_{\rho\sigma\gamma\delta}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} &= R_{\mu\nu\alpha\beta}R_{\rho\sigma\gamma\delta}\begin{vmatrix} \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma} & \delta_{\mu\delta} \\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma} & \delta_{\nu\delta} \\ \delta_{\rho\alpha} & \delta_{\rho\beta} & \delta_{\rho\gamma} & \delta_{\rho\delta} \\ \delta_{\sigma\alpha} & \delta_{\sigma\beta} & \delta_{\sigma\gamma} & \delta_{\sigma\delta} \end{vmatrix} = \\ &= 4\left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\sigma} - R^2\right) = 4\text{GB}. \end{aligned} \tag{2.2}$$

with $\epsilon^{\mu\nu\alpha\beta}$ being the standard Levi-Civita symbol. One can find the variation of the following term by using the variational method that we shall explain in detail in sect. the 5.1 and the table of variation (see the appendix C),

$$\begin{aligned} \delta\left(\sqrt{g}R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\delta}\eta^{\mu\nu\alpha\beta}\eta^{\rho\sigma\gamma\delta}\right) &= \delta\left(\frac{1}{\sqrt{g}}R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\delta}\right) = \\ &= -4\sqrt{g}\nabla_\mu\left(\nabla_\sigma h_{\nu\rho}R_{\alpha\beta\gamma\delta}\eta_{\mu\nu\alpha\beta}\eta_{\rho\sigma\gamma\delta}\right), \end{aligned} \tag{2.3}$$

where $\eta^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\alpha\beta}$. It means that the variation of GB is the total derivative and it should get vanished under the integral in $d = 4$. Moreover, the generalized Gauss-Bonnet theorem (Chern-Guass-Bonnet theorem) states that integral of the Pfaffian of the curvature 2-form of closed even dimensional Riemannian manifold equals to the Euler characteristic of the same.

Therefore, in $d = 4$ one may write,

$$\int d^4x\,\sqrt{g}\,\left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2\right) = \chi, \tag{2.4}$$

where χ stands for a number (topological invariant) known as Euler characteristic. Therefore, in $d = 4$ GB does not contribute to the gravitation action, and consequently EOM does not get affected by any of such terms. We shall consider a great advantage of the above results in a moment. At this point we know that the form of gravitational action (in $d = 4$) containing higher derivatives might be written in terms of arbitrary number coefficients as,

$$S_{\text{HD}-4} = \int d^4x\,\sqrt{g}\,\left(\alpha_2 R^2_{\mu\nu\rho\sigma} + \alpha_3 R^2_{\mu\nu} + \alpha_4 R^2\right), \tag{2.5}$$

where α_2, α_3 and α_4 are arbitrary numerical constants. Along with the correction terms the full form of the action is given by,

$$S = S_{\text{HD}-4} + S_{\text{EH}}. \tag{2.6}$$

Now in order to explain the main benefit of the above results for GB term we rewrite the action (2.5) in terms of the square Weyl tensor (B.4) and GB as follows [2],

$$S_{HD-4} = \int d^4x \, \sqrt{g} \Big(\alpha_5 C^2_{\mu\nu\rho\sigma} + \alpha_6 GB \Big), \tag{2.7}$$

where the numerical constants α_5 and α_6 are uniquely determined once we know α_2, α_3 and α_4 introduces in (2.5), the inverse is also true. Basically in the process of changing the basis to go from (2.5) to (2.7) the coefficient of the term R^2 gets vanished, which clearly explains the presence of only two terms in (2.7) instead of three. We shall point out this explicitely, when we reconsider the action (2.7) with the numerical values of the coefficients in ch. 4. We see that the second term in the above action becomes redundant due to the results (2.3) and (2.4), consequently it leaves only one term to vary to get the EOM. The desired form of such EOM can easily be derived by using the method to be considered in sect. 5.1. For this particular case one need to use (C.10) and simplify the resulting expression. More details on EOM coming from (2.5) and (2.7), and benefits of the same will be considered in chapter 5.

In a quite similar way one can go further and try to incorporate the terms with higher derivatives (higher than four derivatives) such as $R\Box R$, $R_{\mu\nu}\Box R_{\mu\nu}$ and so on. This way of adding the terms may lead to the action of the type S_{HD-6} involving six derivatives of the metric. However, eventually one may notice that this process of randomly adding the correction terms creates serious problem with the theory. It is quite important to note that these correction terms cannot to be added in randomly unorganized way that consequently lead to the unstable theory due to the well known result so called the theorem of Ostrogradsky [27]. In order to explain this we start with the Hamiltonian formalism of classical mechanics.

We know that the Euler-Lagrange equation for the standard form of Lagrangian $\mathscr{L} \equiv L(q,\dot{q})$ i.e. with one variable is given by,

$$\frac{\partial \mathscr{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}}, \tag{2.8}$$

where the dot represents the derivative with respect to time. By taking into account the non-degeneracy condition for the Lagrangian in (2.8) i.e. $\frac{\partial^2 \mathscr{L}}{\partial \dot{q}^2} \neq 0$ one may we rewrite it in the form of Newtonian equations of motion. Thus, one may find that the solutions to (2.8) require two independent variables, which are known as canonical coordinates. These coordinates are conventionally defined as follows,

$$Q \equiv q \qquad \text{and} \qquad P \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}}. \tag{2.9}$$

Now by taking the advantage of non-degeneracy we invert the relation given in (2.9) to find out the expression for $\dot{q}$, and by applying the Legendre transformation on a resulting expression we get,

$$\mathscr{H}(Q,P) = P\dot{q} - \mathscr{L}. \tag{2.10}$$

The resulting Hamiltonian equations are written as,

$$\dot{Q} = \frac{\partial \mathscr{H}}{\partial P} \qquad \text{and} \qquad \dot{P} = -\frac{\partial \mathscr{H}}{\partial Q}. \tag{2.11}$$

The generalization of the above results to N derivatives (eventually leading to $2N$ independent coordinates) yields [27],

$$\frac{\partial \mathscr{L}}{\partial q} + \sum_{i=1}^{N}\left(-\frac{d}{dt}\right)^i \frac{\partial \mathscr{L}}{\partial q^{(i)}} = 0, \tag{2.12}$$

where $q^{(i)}$ stands for the i^{th} order derivative of the canonical coordinate q with respect to time t. As we discussed above for the case of one variable, in a similar way eq. (2.12) gives us,

$$\mathscr{H} \;\; = \;\; \sum_{i=1}^{N} P_i q^{(i)} - \mathscr{L}, \tag{2.13}$$

which is known as the general N Ostrogradsky's Hamiltonian. Consequently, we have,

$$\dot{Q}_i = \frac{\partial \mathscr{H}}{\partial P_i} \qquad \text{and} \qquad \dot{P}_i = -\frac{\partial \mathscr{H}}{\partial Q_i}. \tag{2.14}$$

We note that the non-degeneracy of the Lagrangian i.e. in this case $\frac{\partial^2 \mathscr{L}}{\partial q^{(i)2}} \neq 0$ plays essential role to find out the Hamiltonian of the type of (2.13). Basically non-degeneracy condition ensures the alterations of phase space transformations corresponding to canonical coordinates, which lead us to the expression for $q^{(i)}$ and consequently to the Hamiltonian mentioned above. In other words generalized conjugate momentum expression can be inverted to find out the higher time derivative of the canonical coordinate that yields (2.13). This condition (non-degeneracy) is the core of the problem of Ostrogradsky instability. The instability coming from the potential energy is quite different, where the energy liberated as dynamical variable ends up as some special value (e.g. unstable equilibrium). But the Ostrogradsky instability is related to the instability problem with the kinetic energy arising from special dependence of dynamical variables on time.

Since the Hamiltonian corresponds to the total energy of the system and it also depends on conjugate momenta (kinetic energy), there exist positive and negative energy solutions of the same. For example, it can be shown for the case of higher derivative harmonic oscillator as a special case of the Hamiltonian (2.13) for $N = 2$ that the Ostrogradsky instability is inevitable due to negative energy solutions. For such a case the Lagrangian and Hamiltonian are given by the following expressions [27].

$$\mathscr{L} = -\frac{\epsilon m}{2\omega^2}\ddot{q}^2 + \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2, \tag{2.15}$$

where ϵ is a parameter representing the deviation of the system under consideration from the standard one. That is for $\epsilon = 0$ (2.15) coincides with the unperturbed Lagrangian of a simple harmonic oscillator. One can substitute the general solutions of the eq. (2.12) for $N = 2$ [27],

$$\begin{cases} q(t) = & C_\pm \cos(k_\pm t) + S_\pm \sin(k_\pm t), \\ P_1 = & m\dot{q} + \frac{\epsilon m}{\omega^2} q^{(3)}, \\ P_2 = & -\frac{\epsilon m}{\omega^2}\ddot{q}. \end{cases} \tag{2.16}$$

and (2.15) in the eq. (2.13) (for $N = 2$) and find the Hamiltonian of the system that we are interested in, which is given by [27],

$$\begin{aligned}\mathscr{H} &= \frac{\epsilon m}{\omega^2}\left(\dot{q}q^{(3)} - \frac{1}{2}\ddot{q}^2\right) + \frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2}q^2, \\ &= \frac{m}{2}(1-4\epsilon)^{1/2}\left[k_+^2\left(C_+^2 + S_+^2\right) - k_-^2\left(C_-^2 + S_-^2\right)\right],\end{aligned} \tag{2.17}$$

where $q^{(3)}$ represents the third order derivative of canonical coordinate in time and $k_\pm$ stands for the two frequencies corresponding to positive and negative energies, which are given by $k_\pm \equiv \omega\left(\frac{1\mp(1-4\epsilon)^{1/2}}{2\epsilon}\right)^{1/2}$. Moreover, $C_\pm$ and $S_\pm$ are the constants related to the positive and negative energy modes, which are written as functions of initial value data. The forms of these constants are not useful for the discussion of a point of our interest, but the interested reader can refer to [27].

As it is shown in [27], by analyzing the model described by (2.15), which yields (2.17) one may see that the energy of the system has a lower bound at zero for any constant value of the canonical coordinate q. However, it does not imply that the Ostrogradsky instability is avoidable in this case. In fact the negative energies are accomplished either by making $\ddot{q}$ more dominant than $q^{(3)}$ or simply by setting the large value of $q^{(3)}$ in (2.17) but keeping the overall term $\dot{q} + \frac{\epsilon q^{(3)}}{\omega^2}$ unchanged at the same time. Now once again coming back to the gravitational action (2.6), where in some sense the terms of the type of $R_{\mu\nu}^2$ with $\alpha_3 \neq 0$ given in (2.5) can lead us to the negative energy solutions (in a quite similar way as explained above in the case of harmonic oscillator) and give the unstable theory. Such a theory does not make any physical sense (see for example [28] when the system of self interacting particles carry both positive and negative energies).

In order to avoid such problems it is recommended (also explained in [31]) to spoil the non-degeneracy condition of the Lagrangian such that the theory becomes degenerate and give rise to the constraints. By doing this it may inflict the couplings between the canonical variables. For example, in our case it can be shown that by getting rid of the terms of the type of $R_{\mu\nu}^2$, that is by setting $\alpha_3 = 0$ one may find merely the positive energy solutions [29]. The generalization of this case, in particular for higher derivative models can be found in [30]. Along with these changes the action (2.6) eventuates as,

$$S = S_{f(R)} + S_{\text{const}}, \tag{2.18}$$

where $S_{f(R)} = \int \sqrt{g}\, \alpha_7 f(R)$ with α_7 being an arbitrary numerical constant. It is quite clear that such theories, so called $f(R)$ theories of gravity do not suffer with the problems of negative energy solutions due to absence of the terms of the type of $R^2_{\mu\nu}$ (though they do have their own problems, which are not the part of our discussion).

In spite of having all these problems with HDG, it remarkably turns out to be renormalizable [7, 9], even by incorporating the terms of the type of $R^2_{\mu\nu}$. Inclusion of more correction (higher derivative) terms in the standard theory may improve the dynamics of the theory by making the theory renormalizable (even at higher loop levels) and give a clue to the possible solution of the problem of quantum gravity (see some recent work [38–42] on super-renormalization). With this motivation in mind we study HDG in more depth (with six derivatives of the metric). More details on some other problems with HDG, solutions and more importantly quantum aspects of the same can be found in [31].

2.3 A Connection to the Spectral Action Approach

We shall see that the HDG arises quite naturally in the framework of spectral action principle and analyze it thoroughly in the next chapter. Here we present a brief preview of our detailed analysis that we shall consider later on and discuss the solution of the problem introduced in the previous section. In particular, the problem of finding the unknown numerical constants. We saw in the previous section that one can add the higher derivative terms in the standard form of the action and get HDG action. Apart from the problems that it may create, there is another hurdle that needs to be eradicated. One may note that the numerical constants which we introduced in gravitational actions for HDG are unknown and they multiply rapidly for higher orders. The spectral action approach provides a very accurate way to deal with this issue and at the same time it also takes care of the invariants that enter in the deformed gravitational actions.

As stated above we study the case of pure gravity in absence of matter. Therefore, geometrical sector of the gravity can be studied in full detail by considering the spectral action principle [2] as a special case (concerning about geometry). Let us consider the generalized form of the geometrical part of the action coming from asymptotic expansion of the trace of the heat operator,

$$\mathrm{Tr}\chi(\boldsymbol{L}) = \sum_{q=0}^{\infty} f_{2q} a_{2q}(\boldsymbol{L}), \tag{2.19}$$

where f_{2q} contain the complete information regarding the common numerical factor for corresponding a_{2q}, a_{2q} are the heat kernel coefficients and $\boldsymbol{L}$ is some generalized positively defined operator. We shall reconsider (2.19) in the next chapter and explain the derivation and meaning of the above equation in detail. For the moment being we consider its main features that basically provide the way to find the unknown numerical constants which we introduced in HDG actions.

Let us take into account the right hand side the eq. (2.19) and note that all the higher derivative terms of HDG action are encapsulated in a_{2q} (including the unknown numerical constants α_0 to α_6) and corresponding common factors (along with the scale factor Λ, which is to be considered in the next chapter) in f_{2q}. For example, a_0, a_2 and a_4 are equivalent to the cosmological constant introduced in (2.1), EH action (2.1) and HDG action (2.5) respectively. If one manages to find the complete forms of heat kernel coefficients a_0, a_2 and a_4, and common factors i.e. f_0, f_2 and f_4 then it will provide a very precise and accurate forms of the actions. That is what we are going to do in a moment. We shall also consider the case of the heat kernel coefficient a_6 and corresponding numerical constants (in particular how to find it) in a great detail. As we motioned above the spectral action approach not only solves the problem of finding the unknown constants, but it also helps to decide which terms (invariants) are supposed to be added to get the higher derivative terms in a quite organize way. In other words it excludes the possibility of adding the invariants in a haphazard manner and provide a very accurate way to incorporate them by using dimensional analysis (we shall study in the next chapter).

Moreover, the spectral action approach turns out to be quite useful to find out if there will be any higher order corrections to the standard action on a particular background. For example, in the case of $S_3 \times S_1$ the heat kernel coefficients a_4 and a_6 remarkably transpire to be zero [1]. We shall comment on this when we consider the action coming from a_6 in Weyl basis, which renders a bit easier way to see the cancellations due to conformal symmetry. But here these outcomes suggest that all the higher order terms a_{2q} get vanished on such background. It means there are not higher derivative corrections to the undeformed action for this background. Therewithal, positivity of the theory can be assured by taking the function $\chi \geq 0$ that gives the correct sign for the action written in Euclidean formalism, which may eradicate the possibility of having any kind of negative energy solutions. It would be also quite interesting to note that by quantizing unperturbed (EH) gravitational action (on shell) at higher loop levels, one may produce the higher derivative corrections (if the action is non-renormalizable) introduced in sect. 2.2 without actual numerical coefficients, which can easily be fixed in the framework of spectral action principle. Furthermore, α's given in the HDG actions can be related to the beta functions resulting from quantization.

The overall points discussed above provide us a very good reason to apply the spectral action approach to study HDG rigorously, which basically reinforces the idea of HDG by making it more precise and solid in terms of the values of numerical constants (of invariants of the actions) and invariants that need to be included in HDG actions with a great care. In addition to that the spectral action also determines if there will be any higher order corrections to the standard gravitational action as we mentioned above. As such the spectral action turns out to be the indispensable part of our studies, which basically makes the HDG well equipped by providing the necessary details, which the HDG lacks otherwise. So, in the next chapter we consider the spectral action approach coming from non-commutative geometry and analyze it thoroughly.

Chapter 3

The Spectral Action Approach

Our knowledge of the structure of spacetime is based on two main pillars of physics, which are basically general theory of relativity (GR) and the standard model (SM) of particle physics. The framework of GR depends on our understanding of Riemannian geometry, which works perfectly fine at the large scale structure. However, it crumbles at small scale, in particular at high energy level, where the quantum effects dominate. On the other hand the SM contains our comprehension of the spacetime geometry at very small scales. Thus, it is quite natural to look for a geometry coming out from the quantum world, where the real coordinates are replaced by the self adjoint operators in a Hilbert space, and such a geometry is known as non-commutative geometry (or spectral geometry) [5]. This geometry can be used to comprehend the relation between spacetime and SM [34], and merely gravitational aspects (GR and HDG) of theories (see for example [1] for both cases). The non-commutative geometry basically provides a way to deal with the spaces based on coordinates that do not commute with each other. Moreover, this geometry is spectral in nature. Since experimental data and our theoretical understanding of the nature from particle physics to astrophysics (SM to GR) is presented in the form of some sort of spectra, it worth to apply this new model of the geometry and study its results. However, due to the constraint for the compact space to be smooth manifold, such as Riemannian manifold and non-isometric properties of the same [32], it is quite difficult to reformulate it and find the quantum version of the same. The main tool leading to the non-commutative geometry is known as spectral triple (or non-commutative space). It comprises the complete information corresponding to geometrical and physical parts of the space as we shall see in a moment below. In particular, it is made up of some algebra $\mathscr{A}$, Hilbert space $\mathbb{H}$ on which this algebra is represented and the standard Dirac operator D. As such we denote the spectral triple by $(\mathscr{A}, \mathbb{H}, D)$. By choosing a particular spectral triple one may end up with different classes of theories in physics. After making a suitable choice one assumes that the physics of the system is described by the following action [33],

$$S = \mathrm{Tr}\, f\left(\frac{\mathbb{D}}{\Lambda}\right)^2 + \langle J\psi, \mathbb{D}\psi\rangle \equiv S_{\text{geom}} + S_{\text{matt}}, \tag{3.1}$$

where f is some cut-off function, $\mathbb{D}$ is generalized Dirac operator, Λ is some characteristic scale factor, J is real structure and ψ is the standard Dirac spinor. Here by taking J as a real structure we emphasize that we consider the real non-commutative space as it is explained in [5]. The spectral action principle studied in [2] lies in the core of spectral geometry approach to physics, which can be seen in the studies of some remarkable applications of the spectral action [1]. The spectral triple considered above plays essential role to devise such an effective (spectral) geometry.

Now at this point we know that by making a particular choice of the Dirac operator one can develop a proper framework for different theories. We stated at the very beginning of chapter 2 that HLG can be accomplished with an appropriate choice of Dirac operator. One of such studies can be found in [33], where the infrared action of HLG and matter coupled to the same is constructed by preserving the foliation diffeomorphism. However, as we specified before such theories do not respect the Lorentz symmetry. Concerning about this work we make our choice of the Dirac operator in such a way that it yields HDG. Moreover, we shall study merely S_{geom} (pure gravity) part of the action (3.1) in detail for our purpose.

We review the spectral action approach and its applications to the HDG. More precisely, we discuss and analyze the model studied in [1–4] and derive the heat kernel coefficients. In order to derive the heat kernel coefficients we mainly use the methods described in [8]. We consider the spectral action, which is principally based on non-commutative geometry. We begin with two important results associated with the standard Dirac operator, and derive the Lichnerowicz formula. Later on we consider the generalized form of the spectral action and by choosing the standard form of Dirac operator we boil it down to the particular case of our interest that leads us to HDG. In order to find the heat kernel coefficients, two methods are reviewed for the case of base manifolds without boundaries. These approaches were introduced by DeWitt and Gilkey. Since DeWitt approach (depends on recursive relations) is limited in some sense, we consider the powerful method devised by Gilkey, which is based on background manifolds. Finally, we conclude this chapter by briefly reviewing the case of base manifolds with boundaries. Below we start with few essential things and slowly move on to the core of this chapter. We write all the terms having summed over indices in Feynman's notation (all the indices downstairs), see the appendix A.

Before we embark ourselves on deriving the heat kernel coefficients we set the stage by calculating two important results, which will be used frequently in the following sections and chapters. Let us consider the Dirac operator given by,

$$D = \gamma_\mu \left[\left(\partial_\mu + \omega_\mu \right) \otimes \mathbb{1} + \mathbb{1} \otimes \left(-\frac{i}{2} g A_\mu \right) \right]. \tag{3.2}$$

where

$$\begin{cases} \gamma_\mu \equiv \gamma_\mu \otimes \mathbb{1} := e_{\mu a} \gamma_a, \\ \omega_\mu := \frac{1}{4} \omega_{\mu ab} \gamma_{ab}. \end{cases} \tag{3.3}$$

Here ω_μ is a spin connection associated with the tetrads $e_{\mu a}$ and γ_a are chosen in such a way that it satisfy the relation (A.2) in an Euclidean formalism i.e. $\{\gamma_a, \gamma_b\} = -2\delta_{ab}$ and γ_{ab} is defined by (A.1). We now introduce new notations and rewrite the eq. (3.2) as follows.

We define,

$$\omega_\mu := \omega_\mu \otimes \mathbb{1} \qquad \text{and} \qquad A_\mu := \mathbb{1} \otimes \mathscr{A}_\mu.$$

where $\mathscr{A}_\mu := -\frac{i}{2} g A_\mu$. Such that we get,

$$D = \gamma_\mu \left(\partial_\mu + \omega_\mu + A_\mu\right) \equiv \gamma_\mu \left(\partial_\mu + \tilde{\omega}_\mu\right). \tag{3.4}$$

where $\tilde{\omega}_\mu := \omega_\mu + A_\mu$.

We note that since $\otimes$ is over $C^\infty(\mathscr{M})$ we have $\partial_\mu := \partial_\mu(\mathbb{1} \otimes \mathbb{1})$, which has been used to get (3.4). Now, let us consider the following theorems, which are quite useful to derive our desired results. We shall not give the proof of these theorems here. However, in principle one can easily prove it with the given information.

Theorem 1 :

$$\left(g_{\mu\nu}\partial_\mu\partial_\nu + \mathbb{A}_\mu\partial_\mu + \mathbb{B}\right) \equiv \left(g_{\mu\nu}\nabla_{\mu_\omega}\nabla_{\nu_\omega} - \mathbb{E}\right), \tag{3.5}$$

where $\nabla_{\mu_\omega} := \nabla_\mu + \omega_\mu$

$$\begin{cases} \mathbb{A}_\mu = 2\omega_\mu - \Gamma_\mu, \\ \mathbb{B} = -\mathbb{E} + g_{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma_{\mu\nu\rho}\omega_\rho\right). \end{cases} \tag{3.6}$$

Here ∇_μ represents the usual Levi-Civita connection and ω_μ is spin connection. Moreover, we note that $\Gamma_\mu := g_{\rho\sigma}\Gamma_{\rho\sigma\mu}$ and $\Gamma_{\mu\nu\rho}\omega_\rho \equiv \Gamma^\rho_{\mu\nu}\omega_\rho$. Now, we quote the second theorem below, which is basically the main result that will lead us to the equation for an endomorphism $\mathbb{E}$.

Theorem 2 :

$$D^2 = -\left(g_{\mu\nu}\partial_\mu\partial_\nu + \mathbb{A}_\mu\partial_\mu + \mathbb{B}\right), \tag{3.7}$$

where

$$\begin{cases} \mathbb{A}_\mu = 2\tilde{\omega}_\mu - \Gamma_\mu, \\ \tilde{\omega}_\mu = \omega_\mu + A_\mu, \\ \mathbb{B} = \left(\partial_\mu\tilde{\omega}_\mu + \tilde{\omega}_\mu^2 - \Gamma_\mu\tilde{\omega}_\mu\right) + \frac{1}{4}R \otimes \mathbb{1} - \frac{1}{2}\gamma_{\mu\nu} \otimes F_{\mu\nu}. \end{cases} \tag{3.8}$$

Here R is the Ricci scalar associated with $g_{\mu\nu}$ and $F_{\mu\nu} := \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu + [\mathscr{A}_\mu, \mathscr{A}_\nu]$. Therefore, by the using the Theorem 1 (3.5) we may write,

$$D^2 = -\left(g_{\mu\nu}\nabla_{\mu_\omega}\nabla_{\nu_\omega} - \mathbb{E}\right), \tag{3.9}$$

where $\mathbb{E} = -\mathbb{B} + g_{\mu\nu}\left(\partial_\mu\tilde{\omega}_\nu + \tilde{\omega}_\mu\tilde{\omega}_\nu - \Gamma_{\mu\nu\rho}\tilde{\omega}_\rho\right)$. The last relation (3.9) is well known Lichnerowicz formula for the twisted spinor bundle $\epsilon := S \otimes W$, with S and W being the usual spinor bundles over the manifold $\mathscr{M}$ and some vector space, respectively. Finally by using the relation for an endomorphism $\mathbb{E}$ and (3.8) we get,

$$\mathbb{E} = -\frac{1}{4}\mathbb{R} \otimes \mathbb{1} + \frac{1}{2}\gamma_{\mu\nu} \otimes F_{\mu\nu}. \tag{3.10}$$

The above result is one of the equations that we are interested in. Now, in order to derive the second one we start with the standard form of the second Cartan structure equation. It is given by the following relation,

$$d\omega_{ab} + \omega_{ac} \wedge \omega_{cb} = R_{ab}, \tag{3.11}$$

which can esily be rewritten as,

$$\partial_\mu\omega_{\nu ab}\ \ dx_\mu \wedge dx_\nu + \omega_{\mu ac}\ \omega_{\nu cb}\ \ dx_\mu \wedge dx_\nu = \frac{1}{2}R_{\mu\nu ab}\ \ dx_\mu \wedge dx_\nu. \tag{3.12}$$

By using the anti-symmetrization with respect to μ and ν, one can write the eq. (3.12) as,

$$\partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega_{\mu ac}\ \omega_{\nu cb} - \omega_{\nu ac}\ \omega_{\mu cb} = R_{\mu\nu ab}. \tag{3.13}$$

We contract the eq. (3.13) with $\frac{1}{4}\gamma_{ab}$ on both sides and use the definition of ω_μ given in (3.3) such that (3.13) takes the following form,

$$\partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \frac{1}{4}\omega_{\mu ac}\omega_{\nu cb}\gamma_{ab} - \frac{1}{4}\omega_{\nu ac}\omega_{\mu cb}\gamma_{ab} = \frac{1}{4}R_{\mu\nu ab}\gamma_{ab}. \tag{3.14}$$

Now, let us consider the term $\omega_{\mu ac}\omega_{\nu cb}\gamma_{ab}$ and rewrite it as,

$$\omega_{\mu ac}\omega_{\nu cb}\gamma_{ab} = \frac{1}{4}\omega_{\mu ad}\omega_{\nu cb}\left(\delta_{dc}\gamma_{ab} - \delta_{db}\gamma_{ac} - \delta_{ac}\gamma_{db} + \delta_{ab}\gamma_{dc}\right). \tag{3.15}$$

We know that the generators of Lorentz group satisfy the following algebra,

$$[\Sigma_{ad},\Sigma_{cb}] = \delta_{dc}\Sigma_{ab} - \delta_{db}\Sigma_{ac} - \delta_{ac}\Sigma_{db} + \delta_{ab}\Sigma_{dc}. \tag{3.16}$$

By using the eq. (3.16) in (3.15) with $\Sigma_{ab} := \frac{1}{2}\gamma_{ab}$ we simplify it as,

$$\omega_{\mu ac}\omega_{\nu cb}\gamma_{ab} = \frac{1}{8}\omega_{\mu ad}\omega_{\nu cb}\left[\gamma_{ad},\gamma_{cb}\right]. \tag{3.17}$$

Therefore, by plugging (3.17) in (3.14) and simplifying the resulting equation finally we get,

$$\Omega_{\mu\nu} \equiv \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\,\omega_\nu - \omega_\nu\,\omega_\mu = \frac{1}{4}R_{\mu\nu ab}\gamma_{ab}. \tag{3.18}$$

The above equation is the second and the final one that we wanted to find. Now, in this dissertation we consider the case of pure gravity, for which (3.10) and (3.18) reduce to the forms as given by (A.4).

3.1 Prerequisites for the Heat Kernel Coefficients

In this section we study the heat kernel coefficients. But first and foremost we consider the very general expression representing the trace of an arbitrary function of some positive operator $\boldsymbol{L}$ on a Hilbert space. Let $\chi(\boldsymbol{L})$ be such a function, then we have,

$$\mathrm{Tr}\chi(\boldsymbol{L}). \tag{3.19}$$

Later we shall see the spectral action as a special case of (3.19), where we shall take $\boldsymbol{L} = -\left(\frac{D}{\Lambda}\right)^2$. Mainly we are interested to establish a relation between the expression (3.19) and the object so called heat kernel, which we shall introduce later in this section. Instead of being rigorous in deriving the result for (3.19) we briefly recapitulate the core idea behind this. A detailed derivation can be found in [our paper]. Let us consider the Mellin transform of the function χ given by,

$$\phi(s) = \int_0^\infty dx\, x^{s-1}\chi(x). \tag{3.20}$$

The inverse Mellin transform takes the form,

$$\chi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, x^{-s}\phi(s), \tag{3.21}$$

where c is the constant belonging to the fundamental strip $(0,+\infty)$. Now by using the spectral functional calculus and taking into account (3.21) we define a function of an operator $\boldsymbol{L}$ as follows,

$$\chi(\boldsymbol{L}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \boldsymbol{L}^{-s}\phi(s), \tag{3.22}$$

where $\phi(s)$ is the Mellin transform given by (3.20). Therefore, the trace of $\chi(\boldsymbol{L})$ takes the form,

$$\mathrm{Tr}\chi(\boldsymbol{L}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \zeta_{\boldsymbol{L}}(s)\phi(s), \tag{3.23}$$

where $\zeta_{\boldsymbol{L}}(s)$ is the generalized zeta function defined as $\zeta_{\boldsymbol{L}}(s) := \mathrm{Tr}\boldsymbol{L}^{-s}$. Now we consider the standard gamma function,

$$\Gamma(s) = \int_0^\infty dx\, x^{s-1}e^{-x} \quad \text{for} \quad \mathrm{Re}(s) > 0.$$

By formally changing the variable $x \to t\boldsymbol{L}$ and using the functional calculus for a positive operator we get,

$$\boldsymbol{L}^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-t\boldsymbol{L}}. \tag{3.24}$$

So, by taking the trace on both sides we may write,

$$\zeta_{\boldsymbol{L}}(s) \equiv \mathrm{Tr}\boldsymbol{L}^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\mathrm{Tr}\, e^{-t\boldsymbol{L}}, \tag{3.25}$$

where $\mathrm{Tr}\, e^{-t\boldsymbol{L}}$ is the trace of the heat operator and its asymptotic expansion for small t is given by [14][our paper],

$$\mathrm{Tr}\, e^{-t\boldsymbol{L}} \simeq \sum_{p\geq 0} t^{\frac{p-d}{m}} a_p(\boldsymbol{L}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, t^{-s}\Gamma(s)\zeta_{\boldsymbol{L}}(s), \tag{3.26}$$

where m is the order of operator $\boldsymbol{L}$, d is the dimension of the manifold $\mathscr{M}$ and a_p are the heat kernel coefficients for which we shall see the explicit relation with Seeley-DeWitt coefficients in the next section. Finally, by analyzing the poles of the expression $\zeta_{\boldsymbol{L}}(s)\phi(s)$ given in (3.23) and finding the same by some indirect method as described in [our paper] we get,

$$\mathrm{Tr}\chi(\boldsymbol{L}) = \phi(2)a_0(\boldsymbol{L}) + \phi(1)a_2(\boldsymbol{L}) + \sum_{s=0}^{\infty}(-1)^s\chi^s(0)a_{2(s+2)}(\boldsymbol{L}) \equiv \sum_{q=0}^{\infty} f_{2q}a_{2q}(\boldsymbol{L}), \tag{3.27}$$

where $f_0 = \phi(2)$, $f_2 = \phi(1)$, $f_{2(2+q)} = (-1)^q\chi^{(q)}(0)$, $q \geq 0$ and $\phi(s)$ is given by (3.20). Note that the infinite parameters (higher order heat kennel coefficients) in (3.27) get suppressed by the mass scale factor Λ for the particular choice of the operator $\boldsymbol{L} = -\left(\frac{D}{\Lambda}\right)^2$, for which (3.27) takes the form as $\mathrm{Tr}\chi(\boldsymbol{L}) = \sum_{q=0}^{\infty}\Lambda^{4-2q}f_{2q}a_{2q}$, that is basically the case of our interest. Now, if $\chi(0)$ is a cutoff function then we note that for $s = 1$, $\chi^{(1)}(0)$ gets vanished and remarkably yields the contribution due to a_6 to be zero. As such in general we have $\chi^{(s)}(0) = 0\ \forall\ s > 0$. It means there are no further contributions coming from heat kernel coefficients. However, in our studies we consider χ to be smooth cutoff function and analyze below the heat kernel coefficients thoroughly by using some quite useful techniques.

From now on we study the heat kernel coefficients explicitly, we mainly use the techniques and methods described in [8] to derive the same. Let us consider the heat equation given by,

$$(\partial_t + \boldsymbol{L})u(x;\ t) = 0, \tag{3.28}$$

where $\boldsymbol{L}$ is an elliptical second order differential operator acting on the sections of vector bundles over the Euclidean Riemannian manifold $\mathscr{M}$ and $t > 0$. The initial condition for the above equation is $u(x;\ 0) = f(x)$ with $f(x)$ being a function from L^2-space (Hilbert space of square integrable functions on $\mathscr{M}$). We find that the solution of the eq. (3.28) takes the form as $u(x;\ t) = \exp(-t\boldsymbol{L})f(x)$. Here $\exp(-t\boldsymbol{L})$ represents the heat operator as we mentioned earlier. With this information one can determine the heat kernel as,

$$u(x;\ t) = \int d^d y\, \boldsymbol{K}(x,\ y|t)\, f(y), \tag{3.29}$$

Therefore, we rewrite the heat equation (3.28) for the heat kernel as follows,

$$(\partial_t + \boldsymbol{L})\boldsymbol{K}(x, y|t) = 0. \tag{3.30}$$

where d represents an arbitrary number of dimensions and $\boldsymbol{K}(x,\ y|t)$ is the heat kernel with the initial condition $\boldsymbol{K}(x,\ y|0) = \delta^d(x,\ y)$. Here $\delta^d(x,\ y)$ is the kernel of the unit operator, which simply turns out to be the Dirac delta function $\delta^d(x-y)$ when the operator $\boldsymbol{L}$ is Laplacian on $\mathbb{R}^d$. For example, in the case of scalar Laplacian operator $\boldsymbol{L} = \Delta = -\partial_\mu^2$ on a torus T_d. Now, we expand the function f in a Fourier series as $f = \sum_k \alpha_k f_k(x)$, where f_k is a plane wave given by $f_k = (l_1 l_2 \ldots l_d)\exp(ik_\mu x_\mu)$ and $k_\mu = \frac{2\pi q_\mu}{l_\mu}$ with $\{q_\mu\} \in \mathbb{Z}^d$. Here k_μ and l_μ are the momenta and real numbers (radii), respectively. In this case the action of the heat operator takes the form as $e^{-t\Delta} : \alpha_k \to e^{-tk^2}\alpha_k$. We note that for $t > 0$ and $k_\mu \to \infty$ the exponential e^{-tk^2} converges and enhances the behavior of the Fourier coefficients α_k, ultimately it makes the function more smooth. Specifically, for $t > 0$ the heat operator always exists and maps $L^2 \to C^\infty$ and for this reason sometimes it is called the infinitely smoothing operator. Even the presence of some lower powers of momenta does not affect the existence of the heat operator as e^{-tk^2} prevails the universal contribution. Moreover, we note that the property of self adjointness of the operator $\boldsymbol{L}$ is not necessary for this purpose. The convergence of the heat operator for $t > 0$ and $k_\mu \to \infty$ forces the existence of the heat trace on the space L^2. Therefore, we write,

$$K(Q,\, \boldsymbol{L};\, t) = \mathrm{Tr}_{L^2}(Q\exp(-t\boldsymbol{L})), \tag{3.31}$$

where Q represents the partial differential operator. In our analysis we are interested for the cases when Q is a function i.e. zero order operator or when Q=1, for which we define $K(\boldsymbol{L};\, t) \equiv K(1,\, \boldsymbol{L};\, t)$. The equation for this spectral function $K(\boldsymbol{L};\, t)$ is given by,

$$K(\boldsymbol{L};\, t) = \sum_\lambda e^{-t\lambda}, \tag{3.32}$$

where λ represents the eigenvalues of the operator $\boldsymbol{L}$. Now, let us consider the Euclidean Riemannian manifold $\mathscr{M}$ of dimension d being either compact or it has a boundary, on which there exists the operator $\boldsymbol{L}$, which is an elliptical second order partial differential operator belonging to one of the classes of either $f|_{x=0,l} = 0$ (Dirichlet) or $\partial_x f|_{x=0,l} = 0$ (Neumann) strongly elliptical boundary conditions for the interval $[0, l]$.

The relation between the heat kernel and heat kernel coefficients is given by the following full asymptotic series for the function f as $t \to 0$ (first it was calculated in [14]),

$$K(f, \boldsymbol{L};\, t) = \mathrm{Tr}_{L^2}(f \exp(-t\boldsymbol{L})) \simeq \sum_{p=0}^{\infty} t^{\frac{p-d}{2}} a_p(f, \boldsymbol{L}), \tag{3.33}$$

where $a_p(f, \boldsymbol{L})$ are the heat kernel coefficients, which we already know from (3.26). As we shall see later on, in our case we consider $\boldsymbol{L} = -\left(\frac{D}{\Lambda}\right)^2 = -\frac{\Delta}{\Lambda^2}$ (see (3.1)) with Λ being some mass scale factor and D the standard Dirac operator such that the last relation takes the form as,

$$K\left(f, \boldsymbol{L};\, t'\right) = \mathrm{Tr}_{L^2}\left(f \exp\left(-t'\boldsymbol{L}\right)\right) \simeq \sum_{p=0}^{\infty} t'^{\frac{p-d}{2}} a_p(f, \boldsymbol{L}), \tag{3.34}$$

where $t \to t' = -\frac{t}{\Lambda^2}$ and Δ is the Laplacian $\boldsymbol{L} = \Delta = -\nabla^2 = -\nabla_\mu^2$, which reduces to $\Delta = -\partial_\mu^2$ in the case of flat spacetime. Now, we evaluate the series of the heat trace at $t \to 0$. In order to do this let us consider the Poisson summation formula given by,

$$\sum_{-\infty}^{\infty} h(2k\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dy\, h(y)\, e^{-iky}, \tag{3.35}$$

which is basically valid for any bounded function $h(y)$. In order to show the usefulness of the above formula we consider an asymptotic expansion of the sum of an exponential $\sum_{k=-\infty}^{\infty} e^{-tk^2}$ at $t \to 0$ and choose $h(y) = \exp\left(-\frac{1}{4\pi^2} t y^2\right)$ in (3.35) such that we have $h(2k\pi) = e^{-tk^2}$. With this particular choice the resulting equation takes the form as,

$$\sum_{k=-\infty}^{\infty} h(2k\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dy\, e^{-y\left((2\pi)^{-2} t y + ik\right)}. \tag{3.36}$$

One can easily calculate,

$$\int_{-\infty}^{\infty} dx\, e^{-px^2 + qx} = e^{\frac{q^2}{4p}} \int_{-\infty}^{\infty} dx\, e^{-p\left(x + \frac{q}{p}\right)^2} = \sqrt{\frac{\pi}{p}}\, e^{\frac{q^2}{4p}}. \tag{3.37}$$

Therefore, by comparing (3.36) and (3.37) we get,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{-y\left((2\pi)^{-2} t y + ik\right)} = \sqrt{\frac{\pi}{t}}\, e^{-\frac{k^2\pi^2}{t}}. \tag{3.38}$$

From (3.35), (3.36) and (3.38) along with given $h(y)$, we see that in the sum of an exponential e^{-tk^2} or on the right hand side of the eq. (3.35), all the terms are significantly small except at $k = 0$. Therefore, at $t \to 0$ we may write,

$$\sum_{k=-\infty}^{\infty} e^{-tk^2} \simeq \sqrt{\frac{\pi}{t}} + \mathcal{O}(e^{-\frac{1}{t}}), \tag{3.39}$$

where $\mathcal{O}(e^{-\frac{1}{t}})$ indicates the higher order terms, which get vanished exponentially for small t and are not relevant for our analysis. Now, one can find an asymptotic expansion of the heat kernel for the Laplacian $\Delta = -\partial_1^2 - \partial_2^2 - \ldots - \partial_d^2$ on d dimensional torus T_d by expanding $\exp\left(-t' k_\mu^2\right)$ at small t', where $k_\mu = \frac{2\pi q_\mu}{l_\mu}$ as we introduced above. This form of the expansion is given by [8],

$$K\left(\Delta; t'\right) = \sum_{q \in \mathbb{Z}^d} \exp\left(-t' \sum_\mu \frac{2\pi q_\mu^2}{l_\mu^2}\right) \simeq \frac{l_1 l_2 \ldots l_d}{(4\pi t')^{d/2}} + \mathcal{O}(e^{-\frac{1}{t}}). \tag{3.40}$$

Similarly, for the heat kernel on 2-sphere (S_2) and 3-sphere (S_3) we find [8],

$$K\left(\Delta_{S_2}; t'\right) \simeq \frac{1}{t'} + \frac{1}{3} + \frac{t'}{15} + \mathcal{O}(t'^2), \tag{3.41}$$

and

$$K\left(\Delta_{S_3}; t'\right) \simeq \frac{\sqrt{\pi}}{4}\left(\frac{1}{t'^{\frac{3}{2}}} + \frac{1}{t'^{\frac{1}{2}}} + \frac{t'^{\frac{1}{2}}}{2}\right) + \mathcal{O}(e^{-\frac{1}{t'}}). \tag{3.42}$$

We note that (3.40), (3.41) and (3.42) are of the form of (3.34). As we shall see below, the above results are quite useful to find the unknown numerical constants of the heat kernel coefficients for the general case. We begin with the DeWitt method, which is based on the recursion relations between the heat kernel coefficients. We note that here we consider our base manifolds without boundaries. Later on in the sect. 3.3 we shall also study the case of heat kernel coefficients on base manifolds with boundaries.

3.2 Heat Kernel Coefficients on Manifolds Without Boundaries

In this section we consider the base manifold namely the Euclidean Riemannian without boundaries and find the heat kernel coefficients by using two effective techniques known as DeWitt and Gilkey methods, which were introduced in 1965 and 1975 respectively.

3.2.1 DeWitt Method: Based on Recursive Relations

Let us consider the Laplacian $\boldsymbol{L} = -\partial_\mu^2$ on $\mathbb{R}^d$ with a flat unit metric, as such one can easily find the solution to the heat equation (3.30) by exploiting the relation (3.31) for

$Q = 1$, the Laplacian under consideration and $t \to t'$, which is given by the following flat space kernel [8],

$$\boldsymbol{K}(x, y|t') = \frac{1}{(4\pi t')^{d/2}} \exp\left(-\frac{(x-y)^2}{4t'}\right), \tag{3.43}$$

for which the initial condition takes the form same as that of the eq. (3.30), i.e. $\boldsymbol{K}(x,\ y|0) = \delta^d(x-y)$. From the eq. (3.43) we try to estimate the heat kernel for the Laplacian on a curved manifold without boundaries in the limit when $x \to y$ i.e. x is close to y and t' is very small, as such (3.43) remains valid up to curvature corrections. The DeWitt ansatz for this case takes the form as,

$$\boldsymbol{K}(x, y|t') \sim \frac{1}{(4\pi t')^{d/2}} \sqrt{\Delta_{VVM}(x,y)} \exp\left(-\frac{\sigma^2(x,y)}{4t'}\right) \sum_{p=0}^{\infty} b_p(x,y) t'^d. \tag{3.44}$$

In the above relation a biscalar determinant i.e. $\Delta_{VVM}(x,y)$ so called Van-Vleck-Morettee determinant is given by,

$$\Delta_{VVM}(x,y) = \frac{1}{\sqrt{g(x)g(y)}} \left| -\frac{1}{2}\partial_\mu \partial_\nu \sigma^2(x,y) \right|, \tag{3.45}$$

where $g(x) = |g_{\mu\nu}(x)|$ and $\sigma^2(x,y)$ is the geodesic distance between two close points x and y with coordinates x^μ and y^ν respectively. We assume that geodesics lines form a regular coordinate system close to x or y. We note that the above determinant (3.45) makes the expression given in the eq. (3.44) covariant and the coefficients b_p represent the corrections due to the curvature. It is quite discernible that when $x \to y$ (3.43) and (3.44) coincide.

Now, we proceed further and try to find the coefficients $b_p(x,y)$ for the scalar Laplacian under consideration. We perform our calculations in the Riemann normal coordinates or simply normal coordinates centered at point y such that $\boldsymbol{L} = -\nabla_\mu^2 = -\partial_\mu^2$. In normal coordinates the coordinates of the point x take the form as $x_\mu = s l_\mu$, with s being the length of geodesics connecting two points namely x and y, and l_μ a unit vector at point y, which is basically a tangent to the geodesic. Due to our assumption, since geodesic lines form a regular coordinate system, there is a single geodesic line between any two points. Moreover, by the definition of the normal coordinates, the metric at point y be as flat spacetime metric i.e. $\eta_{\mu\nu}$. Therefore, we have $\sigma^2(x,y) = x_\mu x_\nu \eta_{\mu\nu}$ such that $\Delta_{VVM}(x,y) = \Delta_{VVM}(x) \equiv \frac{1}{\sqrt{g(x)}}$. We note that in the normal coordinates the geodesic equation coincides with the equation of flat spacetime, that is we have $\frac{d^2x^\mu}{ds^2} = 0$. Consequently, in the normal coordinates DeWitt ansatz (3.44) reduces to,

$$\boldsymbol{K}(x,x|t') = \frac{1}{(4\pi t')^{d/2}} \sqrt{\Delta_{VVM}(x)} \exp\left(-\frac{x^2}{4t'}\right) \sum_{p=0}^{\infty} b_p(x) t'^p. \tag{3.46}$$

Here we see that as we need to perform the differentiation in the next step, the summed over index p is written in the standard Einstein's summation convention just like in the case of variations. However, here we put explicit summation sign as well, in order to emphasize that the summation is running over p from 0 to ∞. We now find out,

$$\partial_{t'} \boldsymbol{K}(x,x|t') = (4\pi t')^{-d/2} \sqrt{\Delta_{VVM}} \exp\left(-\frac{x^2}{4t'}\right) \sum_{p=0}^{\infty} b_p \left(\left(p - \frac{d}{2}\right) t'^{p-1} + \frac{x^2}{4} t'^{p-2}\right). \tag{3.47}$$

$$\begin{aligned} \boldsymbol{L}\boldsymbol{K}(x,x|t') = -\nabla_\mu^2 \boldsymbol{K}(x,x|t') \quad &= \quad (4\pi t')^{-d/2} \exp\left(-\frac{x^2}{4t'}\right) \sum_{p=0}^{\infty} t'^p \left[\sqrt{\Delta_{VVM}}\, b_p \times\right. \\ &\times \left(-\frac{\nabla^2 x^2}{4t'} + \frac{(\nabla x^2)^2}{16t'^2}\right) - \frac{1}{t'} x_\mu \nabla_\mu \left(\sqrt{\Delta_{VVM}}\, b_p\right) + \\ &\left. + \nabla^2 \left(\sqrt{\Delta_{VVM}}\, b_p\right)\right], \end{aligned} \tag{3.48}$$

where we have lowered all the summed over indices after performing the differentiation. This point of raising the indices to differentiate the expression and lower once it is done will be discussed in detail in ch. 5, particularly in order to explain the variation of the action and to deal with the EOM in general. So, by plugging results (3.47) and (3.48) into the heat kernel equation (3.30) or comparing (3.47) and (3.48) we get,

$$\begin{aligned} \sum_{p=0}^{\infty} b_p \left(\left(p - \frac{d}{2}\right) t'^{p-1} + \frac{x^2}{4} t'^{p-2}\right) \quad &= \quad \frac{1}{\sqrt{\Delta_{VVM}}} \sum_{p=0}^{\infty} t'^p \left[\sqrt{\Delta_{VVM}}\, b_p \left(-\frac{\nabla^2 x^2}{4t'} + \frac{(\nabla x^2)^2}{16t'^2}\right) - \right. \\ &\left. - \frac{1}{t'} x_\mu \nabla_\mu \left(\sqrt{\Delta_{VVM}}\, b_p\right) + \nabla^2 \left(\sqrt{\Delta_{VVM}}\, b_p\right)\right]. \end{aligned} \tag{3.49}$$

One can easily see that $(\nabla x^2)^2 = (\partial_\mu x^2)^2 = 4x^2$ and by taking the advantage of normal coordinates we may write $\nabla^2 x^2 = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g_{\mu\nu} \partial_\nu x^2\right)$. One can use $\Delta_{VVM}(x) = \frac{1}{\sqrt{g(x)}}$

in the last result and upon simplifying the resulting expression it produces $\nabla^2 x^2 = 2x_\mu \Delta_{VVM} \partial_\mu \Delta^{-1}_{VVM} + 2d$. We substitute these results in (3.49) and expand the same, after simplification it reduces to the following relation,

$$\sum_{p=0}^{\infty} p\, b_p\, t'^{p-1} = \sum_{p=0}^{\infty} \left(-t'^{p-1} x_\mu \partial_\mu b_p + \frac{1}{\sqrt{\Delta_{VVM}}} t'^p \nabla^2 \left(\sqrt{\Delta_{VVM}} b_p \right) \right). \tag{3.50}$$

By comparing the powers of t in (3.50) we get the recursion relations as follows,

$$(p+1)\, b_{p+1} + x_\mu \partial_\mu b_{p+1} = \frac{1}{\sqrt{\Delta_{VVM}}} \nabla^2 \left(\sqrt{\Delta_{VVM}} b_p \right). \tag{3.51}$$

$$x_\mu \partial_\mu b_0 = 0. \tag{3.52}$$

Since $K(x,y|0) = \delta^d(x-y)$ from (3.52) we find $b_0 = 1$. We substitute this value of the coefficient b_0 in (3.51) with $p = 0$. Therefore, we get,

$$b_1 + x_\mu \partial_\mu b_1 = \frac{1}{\sqrt{\Delta_{VVM}}} \nabla^2 \sqrt{\Delta_{VVM}}. \tag{3.53}$$

Since we are working in normal coordinates and $\frac{dx^\mu}{ds} = \frac{x^\mu}{s}$ one may find that the equations for the Levi-Civita connection turns out to be $\Gamma_{\sigma\mu\nu}(x) x_\mu x_\nu = 0$. Furthermore, the form of the metric is compatible with the Levi-Civita connection equation, and under the normal coordinates considered above it takes the form as follows,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma} x_\rho x_\sigma + \mathcal{O}(x^d), \tag{3.54}$$

where $\mathcal{O}(x^d)$ represents higher order terms. Since $\Delta_{VVM}(x) = \frac{1}{\sqrt{g(x)}}$, one can easily find that,

$$\begin{cases} |g_{\mu\nu}(x)| = & g(x) = 1 - \frac{1}{3} R_{\mu\nu} x_\mu x_\nu + \mathcal{O}(x^d), \\ \sqrt{\Delta_{VVM}} = & (g(x))^{-1/4} = 1 + \frac{1}{12} R_{\mu\nu} x_\mu x_\nu + \mathcal{O}(x^d). \end{cases}$$

In $x_\mu = 0$ limit we have,

$$\sqrt{\Delta_{VVM}} = 1 \quad \text{and} \quad \nabla^2 \sqrt{\Delta_{VVM}} = \frac{1}{6} R.$$

We note that the higher order terms get vanished in the limit $x_\mu = 0$ for the second derivative of VVM determinant. Now by substituting these results in (3.53) and taking care of the limit $x_\mu = 0$ one can easily find,

$$b_1(0) = \frac{1}{6}R. \tag{3.55}$$

This results lead us to the asymptotic expansion of the heat kernel at small t on a compact manifold without boundaries, which takes the form as given below,

$$\begin{aligned} \boldsymbol{K}(-\nabla^2;t') &= \int_{\mathcal{M}} d^dx\sqrt{g}\,K(x,x;t') \sim (4\pi t')^{-d/2}\int_{\mathcal{M}} d^dx\sqrt{g}\,(b_0+b_1t'+\ldots) = \\ &= (4\pi t')^{-d/2}\int_{\mathcal{M}} d^dx\sqrt{g}\left(1+\frac{1}{6}Rt'+\ldots\right). \end{aligned} \tag{3.56}$$

Therefore, by comparing (3.34) and (3.56) we find the heat kenrel coefficients as follows,

$$a_{2p} = (4\pi)^{-d/2}\int_{\mathcal{M}} d^dx\sqrt{g}\,b_p. \tag{3.57}$$

We note that the above equation relates the Seeley-DeWitt coefficients b_p with heat kernel coefficients introduced in (3.26). Finally, we use (3.57) and obtain some initial heat kernel coefficients below,

$$\begin{cases} a_0 = & (4\pi)^{-d/2}\int_{\mathcal{M}} d^dx\sqrt{g} = \frac{\mathrm{Vol}\mathcal{M}}{(4\pi)^{d/2}}, \\ a_2 = & (4\pi)^{-d/2}\int_{\mathcal{M}} d^dx\sqrt{g}\,\frac{1}{6}R. \end{cases} \tag{3.58}$$

We note that the heat kernel coefficients with odd p get vanished i.e. $a_{2p+1} = 0$ with $p = 0,1,2,\ldots$ and so on or $a_p = 0\ \forall$ odd p. Here we see that this method of finding the heat kernel coefficients by using the recursion relations is a bit cumbersome and makes it difficult to handle for higher order heat kernel coefficients. In order to deal with this problem Gilkey introduced the new approach in 1975, which is based on a background manifold to find the unknown number coefficients of preformed heat kernel coefficient expressions (by using dimensional analysis), as we shall see below in a moment.

3.2.2 Gilkey Method: Based on Background Manifolds

In order to take the advantage of this method first we review few prerequisites below and later on we move on to the detailed explanation of this powerful approach.

On the product manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ the heat kernel coefficients maybe written as [8],

$$a_k(x;\boldsymbol{L}) = \sum_{p+q=k} a_p(x_1;\boldsymbol{L}_1)a_q(x_2;\boldsymbol{L}_2). \tag{3.59}$$

where $\boldsymbol{L}$ is the Laplacian given by $\boldsymbol{L} = \boldsymbol{L}_1 \otimes 1 + 1 \otimes \boldsymbol{L}_2$.

Now, let us pay attention to the dimensional analysis in (3.34), which will directly help us to write the heat kernel coefficients. Let us assign the dimension -1 to the coordinate, $\dim(x_\mu) = -1$, then the dimension of the derivative is $+1$, $\dim(\partial_\mu) = 1$. Naturally the covariant derivative will also have the same dimension i.e. $\dim(\nabla_\mu) = 1$. We choose the dimension of f to be $\dim(f) = 0$ and keep the dimension of the metric also $\dim(g_{\mu\nu}) = 0$. Now, we note that the exponential appearing in (3.34) must be dimensionless. In order to satisfy this condition we take $\dim(t') = -\dim(\boldsymbol{L}) = -2$ for the second order operator. Let us consider the eq. (3.29) and note that since $\dim(x_\mu) = -1$, we have $\dim(d^d x) = -d$. By considering (3.29) and (3.34) we find that $\dim(d^d x) = -d$ cancels out with the $\dim(t'^{-d/2}) = d$ leaving the $\dim(t'^{p/2}) = -p$. This will be compensated by the dimension of the heat kernel coefficients a_p. Consequently, the overall term remains dimensionless. Therefore, any integrands appearing in the equations of the heat kernel coefficients must have the dimension p. It is quite easy to see that $\dim(E) = \dim(\Omega_{\mu\nu}) = \dim(R_{\mu\nu\rho\sigma}) = \dim(R_{\mu\nu}) = \dim(R) = 2$. This says that all the invariants that appear as the integrands of the heat kernel coefficients will have even dimensions. It implies that all the heat kernel coefficients with odd p should get vanished in the case of base manifolds having no boundaries. Thus, in general it gives us $a_{2p+1} = 0$ for $p = 0, 1, 2, \ldots$ so on or $a_p = 0 \; \forall$ odd p. Later on we shall see that, in the case of base manifolds with boundaries not only the coefficients with odd p survive but the coefficients with even p also get modified.

Now, one can write some initial heat kernel coefficients by using the dimensional analysis

Invariant/Scalar	Coeff.	Manifold(s)/Supporting equation
$f(x)$	c_0	T_4
E	c_1	$K\left(f, \boldsymbol{L}_0 - E; t'\right) = e^{t'E} K\left(f, \boldsymbol{L}_0; t'\right)$
R	c_2	S_2
$E_{;\mu\mu}$	c_3	Gauge fields and potentials
ER	c_4	same as c_1
E^2	c_5	
$R_{;\mu\mu}$	c_6	Conformal variations
R^2	c_7	$\mathscr{M}_1 \times \mathscr{M}_2$
$R^2_{\mu\nu}$	c_8	S_2 and S_3
$R^2_{\mu\nu\rho\sigma}$	c_9	
$\Omega^2_{\mu\nu}$	c_{10}	same as c_3

Table 3.1: Table of invariants (or scalars) and corresponding background manifold(s) and supporting equation for a_0, a_2 and a_4.

as follows [4, 8],

$$a_0(f, D) = (4\pi)^{-d/2} \int_{\mathscr{M}} d^d x \sqrt{g}\, \mathrm{Tr}(c_0 f), \tag{3.60}$$

$$a_2(f, D) = \frac{(4\pi)^{-d/2}}{6} \int_{\mathscr{M}} d^d x \sqrt{g}\, \mathrm{Tr}(f(c_1 E + c_2 R)), \tag{3.61}$$

$$\begin{aligned} a_4(f, D) \;&=\; \frac{(4\pi)^{-d/2}}{360} \int_{\mathscr{M}} d^d x \sqrt{g}\, \mathrm{Tr}\Big(f\left(c_3 E_{;\mu\mu} + c_4 ER + c_5 E^2 + c_6 R_{;\mu\mu} + \right. \\ &\left. + c_7 R^2 + c_8 R^2_{\mu\nu} + c_9 R^2_{\mu\nu\rho\sigma} + c_{10}\Omega^2_{\mu\nu}\right)\Big), \end{aligned} \tag{3.62}$$

where the coefficients $(4\pi)^{-d/2}$ are part of the definition, and $\frac{1}{6}$ and $\frac{1}{360}$ are quite arbitrary and have been written only for the matter of convenience. Moreover, c_0 to c_{10} are the unknown numerical constants to be determined by using the different manifolds and supporting equations as it is described in the table 3.1. We begin by finding the value of c_0. In order to do this we consider (3.34), (3.40) and (3.60), which take the following forms for $f = 1$, $p = 0$, $d = 4$ and the unit radii $l_d = 1$,

$$\begin{cases} K\left(\Delta_{T_4}; t'\right)\big|_{p=0} & = \frac{1}{t'^2} a_0\left(\Delta_{T_4}\right), \\ K\left(\Delta_{T_4}; t'\right) & = \frac{1}{16\pi^2 t'^2}, \\ a_0\left(\Delta_{T_4}\right) & = \frac{c_0}{16\pi^2}. \end{cases} \tag{3.63}$$

It is clear from (3.63) that we have,

$$c_0 = 1. \tag{3.64}$$

Similarly, one can derive the remaining coefficients. However, it requires to consider different backgrounds or supporting equations as mentioned above. We shall consider the unit spheres S_2 and S_3 to find the next set of coefficients. We know that the Riemann tensor, Ricci tensor and Ricci scalar on d-sphere (S_d) (having the radius r) are given by the equations $R_{\mu\nu\rho\sigma} = -\frac{1}{r^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$, $R_{\mu\nu} = \frac{d-1}{r^2}g_{\mu\nu}$ and $R = \frac{d(d-1)}{r^2}$, respectively. By using these relations one can easily find the following relations for the unit sphere (i.e. $r = 1$) $S_d : R^2 = d^2(d-1)^2$, $R^2_{\mu\nu} = d(d-1)^2$ and $R^2_{\mu\nu\rho\sigma} = 2d(d-1)$. We note that on S_2 and S_3 we have $E = 0$ and $\Omega_{\mu\nu} = 0$. Therefore, only the terms containing curvature contribute. Moreover, the volume of S_d can easily be found, which is given by the expression $\frac{2r^d\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$. So, we get Vol $S_2 = 4\pi$ and Vol $S_3 = 2\pi^2$ for the unit spheres S_2 and S_3 respectively.

Now, we evaluate (3.61) and (3.62) over the unit spheres S_2 and S_3, which produce the results (with f=1) as enumerated below,

$$a_2\left(\Delta_{S_2}\right) = \frac{1}{4\pi}\frac{c_2}{6}2(2-1)\int_{S_2} d^2x\sqrt{g} = \frac{c_2}{3}, \tag{3.65}$$

$$\begin{aligned} a_4\left(\Delta_{S_2}\right) &= \frac{1}{4\pi}\frac{1}{360}\left(c_7 2^2(2-1)^2 + c_8 2(2-1)^2 + c_9 2(2)(2-1)\right)\int_{S_2} d^2x\sqrt{g} = \\ &= \frac{1}{90}\left(c_7 + \frac{1}{2}c_8 + c_9\right), \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} a_4\left(\Delta_{S_3}\right) &= \frac{1}{(4\pi)^{\frac{3}{2}}}\frac{1}{360}\left(c_7 3^2(3-1)^2 + c_8 3(3-1)^2 + c_9 2(3)(3-1)\right)\int_{S_3} d^2x\sqrt{g} = \\ &= \frac{\sqrt{\pi}}{4}\frac{1}{30}\left(3c_7 + c_8 + c_9\right). \end{aligned} \tag{3.67}$$

We also note that for the unit spheres S_2 and S_3 the eq. (3.34) turns out to be as follows,

$$K\left(\Delta_{S_2}; t'\right) \simeq \sum_{p=0}^{\infty} t'^{\frac{p-2}{2}} a_p\left(\Delta_{S_2}\right), \tag{3.68}$$

and

$$K\left(\Delta_{S_3};t'\right) \simeq \sum_{p=0}^{\infty} t'^{\frac{p-3}{2}} a_p\left(\Delta_{S_3}\right). \tag{3.69}$$

Now, we rewrite the kernel for S_2 and heat coefficient a_2 by using (3.68) and (3.65) as,

$$K\left(\Delta_{S_2};t'\right)\big|_{p=2} \quad = a_2\left(\Delta_{S_2}\right) = \tfrac{c_2}{3}. \tag{3.70}$$

Therefore, by comparing the coefficients of t^0 in (3.41) and (3.70) we get,

$$c_2 = 1. \tag{3.71}$$

Once again we use (3.68) and (3.69), and re-write it for a_4 as follows,

$$K\left(\Delta_{S_2};t'\right)\big|_{p=4} = t' a_4\left(\Delta_{S_2}\right), \tag{3.72}$$

and

$$K\left(\Delta_{S_3};t'\right)\big|_{p=4} = t'^{\frac{1}{2}} a_4\left(\Delta_{S_3}\right). \tag{3.73}$$

Now, as we did before we rewrite (3.72) and (3.73) by using (3.66) and (3.67), such that the resulting equations take the following forms,

$$K\left(\Delta_{S_2};t'\right)\big|_{p=4} = t'\frac{1}{90}\left(c_7 + \frac{1}{2}c_8 + c_9\right), \tag{3.74}$$

and

$$K\left(\Delta_{S_3};t'\right)\big|_{p=4} = t'^{\frac{1}{2}}\frac{\sqrt{\pi}}{8}\frac{1}{30}\left(3c_7 + c_8 + c_9\right). \tag{3.75}$$

By comparing the coefficients of the terms t and $t^{\frac{1}{2}}$ given in (3.74) and (3.75) with the coefficients of corresponding terms in (3.41) and (3.42) we get,

$$\frac{1}{90}\left(c_7 + \frac{1}{2}c_8 + c_9\right) = \frac{1}{15} \Rightarrow 2c_7 + c_8 + 2c_9 = 12, \tag{3.76}$$

and

$$\frac{\sqrt{\pi}}{4}\frac{1}{30}\left(3c_7 + c_8 + c_9\right) = \frac{\sqrt{\pi}}{8} \Rightarrow 3c_7 + c_8 + c_9 = 15. \tag{3.77}$$

Now, we take $k=4$ in the eq. (3.59) and rewrite it as (with $R=R_1+R_2$),

$$\begin{cases} a_4\left(x;\,\Delta_{S_2}\right)=a_2\left(x_1;\,\Delta_{1S_2}\right)a_2\left(x_2;\,\Delta_{2S_2}\right)\Rightarrow \\ \Rightarrow \frac{1}{4\pi}\frac{1}{360}\int_{S_2}d^2x\sqrt{g}\left(c_7(R_1+2R_1\cdot R_2+R_2)+c_8R^2_{\mu\nu}+c_9R^2_{\mu\nu\rho\sigma}\right)= \\ =\left(\frac{1}{4\pi}\right)^2\left(\frac{c_2}{6}\right)^2\int_{S_2}d^2x_1d^2x_2\;\;gR_1\cdot R_2. \end{cases}$$

By comparing the terms containing only the square of the scalar curvature we get,

$$\begin{cases} \frac{1}{4\pi}\frac{1}{360}\int_{S_2}d^2x\sqrt{g}(2c_7R_1\cdot R_2)=\left(\frac{1}{4\pi}\right)^2\left(\frac{c_2}{6}\right)^2\int_{S_2}d^2x_1d^2x_2\;\;gR_1\cdot R_2\Rightarrow \\ \Rightarrow \frac{1}{4\pi}\frac{1}{360}2c_7 2^2(2-1)^2(4\pi)=\left(\frac{1}{4\pi}\right)^2\left(\frac{c_2}{6}\right)^2 2^2(2-1)^2(4\pi)^2. \end{cases}$$

These calculations yield,

$$c_7=5. \tag{3.78}$$

Therefore, equations (3.76) and (3.77) give us,

$$c_8=-2, \tag{3.79}$$

and

$$c_9=2. \tag{3.80}$$

Now, we consider the case when E is a constant and proportional to the unit matrix. We expand $K\left(f,\boldsymbol{L}_0-E;\,t'\right)=e^{t'E}K\left(f,\boldsymbol{L}_0;\,t'\right)$ in powers of t' as,

$$K\left(f,\boldsymbol{L}_0-E;\,t'\right)=\sum_{p=0}^{\infty}\left(t'^{\frac{p-d}{2}}a_p(f,\boldsymbol{L}_0)+Et'^{\frac{p-d+2}{2}}a_p(f,\boldsymbol{L}_0)+\frac{E^2}{2}t'^{\frac{p-d+4}{2}}a_p(f,\boldsymbol{L}_0)+\ldots\right).$$

We expand the above series and consider only the desired terms for our purpose such that we have,

$$K\left(f,\boldsymbol{L}_0-E;\,t'\right)\;\;=\ldots+E\;\;t'^{\frac{2-d}{2}}a_0(f,\boldsymbol{L}_0)+E\;\;t'^{\frac{4-d}{2}}a_2(f,\boldsymbol{L}_0)+\ldots+\frac{E^2}{2}\;\;t'^{\frac{4-d}{2}}a_0(f,\boldsymbol{L}_0)+\ldots$$

By using (3.60) and (3.61) we rewrite the above equation as,

$$\begin{aligned} K\left(f,\boldsymbol{L}_0-E;\,t'\right) \;\;=\;\; & \ldots+t'^2\frac{1}{(4\pi t')^{\frac{d}{2}}}\int_{\mathcal{M}}d^dx\;\;\sqrt{g}\mathrm{Tr}(Ec_0f)+ \\ & +t'^4\frac{1}{(4\pi t')^{\frac{d}{2}}}\frac{1}{6}\int_{\mathcal{M}}d^dx\;\;\sqrt{g}\mathrm{Tr}\left(f\left(c_1E^2+c_2ER\right)\right)+ \\ & +\ldots+\frac{t'^4}{2}\frac{1}{(4\pi t')^{\frac{d}{2}}}\int_{\mathcal{M}}d^dx\;\;\sqrt{g}\mathrm{Tr}\left(E^2c_0f\right)+\ldots \end{aligned} \tag{3.81}$$

Now, we consider the eq. (3.34) and expand it. In the resulting expression we take into account only the terms containing undifferentiated E as follows,

$$K(f,\boldsymbol{L};t') = \ldots + t'^2 \frac{1}{(4\pi t')^{\frac{d}{2}}} \frac{1}{6} \int_{\mathcal{M}} d^d x \sqrt{g}\, \mathrm{Tr}(f c_1 E) + t'^4 \frac{1}{(4\pi t')^{\frac{d}{2}}} \frac{1}{360} \times \times \int_{\mathcal{M}} d^d x \sqrt{g}\, \mathrm{Tr}\left(f\left(\ldots + c_4 E R + c_5 E^2 + \ldots\right)\right) + \ldots \tag{3.82}$$

By comparing (3.81) and (3.82) one can easily get,

$$\begin{cases} \frac{c_1}{6} = c_0 & \Rightarrow c_1 = 6, \\ \frac{c_4}{360} = \frac{c_2}{6} & \Rightarrow c_4 = 60, \\ \frac{c_5}{360} = \frac{c_0}{2} & \Rightarrow c_5 = 180. \end{cases} \tag{3.83}$$

Now, in the case of the part of heat expansion which depends on gauge fields and potentials, and does not involve any curvature terms, the kernel takes the form as [8],

$$K(f,\boldsymbol{L};t') = \frac{1}{(4\pi t')^{\frac{d}{2}}} \int_{T_d} d^d x\, f(x) \left[1 + t'E + t'^2 \left(\frac{1}{2}E^2 + \frac{1}{6}E_{;\mu\mu} + \frac{1}{12}\Omega^2_{\mu\nu}\right)\right]. \tag{3.84}$$

As we did before we compare (3.84) with (3.62) and take into account the eq. (3.34) such that it yield,

$$c_3 = 60 \qquad \text{and} \qquad c_{10} = 30. \tag{3.85}$$

We note that the eq. (3.84) also gives the values of c_0 (3.64), c_1 and c_5 (3.83). Still we are left with one unknown numerical constant, i.e. c_6, for which the calculations are bit non-trivial and involve the conformal variations. We shall not discuss the derivation of the same here, but it has been studies in [8]. It takes the value as follows,

$$c_6 = 12. \tag{3.86}$$

Therefore, finally we re-write (3.60), (3.61) and (3.62) with the known numerical constants and $f = 1$ as,

$$a_0(f,\boldsymbol{L}) = (4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1}) \int_{\mathcal{M}} d^d x \sqrt{g}, \tag{3.87}$$

$$a_2(f,\boldsymbol{L}) = \frac{(4\pi)^{-d/2}}{6} \int_{\mathcal{M}} d^d x \sqrt{g}\, \mathrm{Tr}(6E + R), \tag{3.88}$$

Invariant	Coeff.		Invariant	Coeff.		Invariant	Coeff.
$R_{ijij;kkll}$	$-18/7!$		$R_{ijku}R_{ilkp}R_{jlup}$	$-80/9\cdot 7!$		E^3	1/6
$R_{ijij;k}R_{ulul;k}$	17/7!		$\Omega_{ij;k}\Omega_{ij;k}$	1/45		$E\Omega_{ij}\Omega_{ij}$	1/30
$R_{ijik;l}R_{ujuk;l}$	$-2/7!$		$\Omega_{ij;j}\Omega_{ik;k}$	1/180		$\Omega_{ij}E\Omega_{ij}$	1/60
$R_{ijik;l}R_{ujul;k}$	$-4/7!$		$\Omega_{ij;kk}\Omega_{ij}$	1/60		$\Omega_{ij}\Omega_{ij}E$	1/30
$R_{ijku;l}R_{ijku;l}$	9/7!		$\Omega_{ij}\Omega_{ij;kk}$	1/60		$R_{ijij}E_{;kk}$	$-1/36$
$R_{ijij}R_{kuku;ll}$	28/7!		$\Omega_{ij}\Omega_{jk}\Omega_{ki}$	$-1/30$		$R_{ijik}E_{;jk}$	$-1/90$
$R_{ijik}R_{ujuk;ll}$	$-8/7!$		$R_{ijkl}\Omega_{ij}\Omega_{kl}$	$-1/60$		$R_{ijij;k}E_{;k}$	$-1/30$
$R_{ijik}R_{ujul;kl}$	24/7!		$R_{ijik}\Omega_{jl}\Omega_{kl}$	1/90		$E_{;j}\Omega_{ij;i}$	$-1/60$
$R_{ijkl}R_{ijkl;uu}$	12/7!		$R_{ijij}\Omega_{kl}\Omega_{kl}$	$-1/72$		$\Omega_{ij;i}E_{;j}$	1/60
$R_{ijij}R_{klkl}R_{pqpq}$	$-35/9\cdot 7!$		$R_{ijik}\Omega_{kl;lj}$	0		E^2R_{ijij}	$-1/12$
$R_{ijij}R_{klkp}R_{qlqp}$	$14/3\cdot 7!$		$R_{ijij;k}\Omega_{kl;l}$	0		$ER_{ijij;kk}$	$-1/30$
$R_{ijij}R_{klpq}R_{klpq}$	$-14/3\cdot 7!$		$R_{ijkl;l}\Omega_{ij;k}$	0		$ER_{ijij}R_{klkl}$	1/72
$R_{ijik}R_{julu}R_{kplp}$	$208/9\cdot 7!$		$E_{;iijj}$	1/60		$ER_{ijik}R_{ljlk}$	$-1/180$
$R_{ijik}R_{uplp}R_{jukl}$	$-64/3\cdot 7!$		$EE_{;ii}$	1/12		$ER_{ijkl}R_{ijkl}$	1/180
$R_{ijik}R_{julp}R_{kulp}$	$16/3\cdot 7!$		$E_{;ii}E$	1/12		-	-
$R_{ijku}R_{ijlp}R_{kulp}$	$-44/9\cdot 7!$		$E_{;i}E_{;i}$	1/12		-	-

Table 3.2: Table of invariants and corresponding numerical coefficients for a_6.

and

$$
\begin{aligned}
a_4(f,\boldsymbol{L}) &= \frac{(4\pi)^{-d/2}}{360}\int_{\mathcal{M}} d^dx\,\sqrt{g}\,\mathrm{Tr}\big(60E_{;\mu\mu}+60ER+180E^2+12R_{;\mu\mu}+ \\
&\quad +5R^2-2R^2_{\mu\nu}+2R^2_{\mu\nu\rho\sigma}+30\Omega^2_{\mu\nu}\big). \qquad (3.89)
\end{aligned}
$$

We note that the Gilkey method provides a very powerful way to deal with heat kernel coefficients. However, the only drawback with this approach is that number of independent invariants increase for the higher order heat kernel coefficients, and it requires more number of background manifolds to tackle with the same. By using this approach one can find the heat kernel coefficient a_6 as well, which has been already done by Gilkey in [3, 4]. We do not repeat the same here, but we just quote the table 3.2 [4] of 46 invariants and corresponding numerical coefficients appearing in the expression of a_6. One can find these numerical coefficients for a_6 in the same way as we did for a_4.

Now in order to explain the simplification of the terms given in the table 3.2 we take into account few terms that resemble each other and can easily be eradicated by combining with the similar term. Let us start with $\Omega_{ij;kk}\Omega_{ij}$ and $\Omega_{ij}\Omega_{ij;kk}$, we see that these two terms can be combined to yield one single term of such type, which reduces 46 terms by one. Similarly, one can go further and consider other terms such as $EE_{;ii}$ and $E_{;ii}E$; $E\Omega_{ij}\Omega_{ij}$, $\Omega_{ij}E\Omega_{ij}$ and $\Omega_{ij}\Omega_{ij}E$; $E_{;j}\Omega_{ij;i}$ and $\Omega_{ij;i}E_{;j}$ etc., which after combining to a single term in a respective group reduce 45 terms up to 40 terms. We note that the terms $E_{;j}\Omega_{ij;i}$ and $\Omega_{ij;i}E_{;j}$ having the same numerical constants get canceled out. Moreover, the terms which have the null i.e. 0 numerical constant do not contribute further, and consequently leave the total number of invariants 37 only. Finally these invariants can be used to construct the action for the heat kernel coefficient a_6. We shall see the complete form of such an action in the next chapter.

3.3 Heat Kernel Coefficients on Manifolds With Boundaries

As we mentioned in the previous section here we study the case of base manifolds with boundaries. Our main goal in this section is only to introduce a notion of heat kernel coefficients in the case of base manifolds having boundaries and see that by doing such, the coefficients a_p with odd p do not get vanished. That is contrary to the case considered in the previous section. For this purpose we briefly summarize the idea and quote few useful results. A detailed explanation on this topic can be found in [8].

We consider the usual form of scalar Laplacian on the interval $\mathcal{N} = [0, l]$ along with the Dirichlet or Neumann boundary conditions on the boundaries of some manifold so called half-space $\mathcal{M} = \mathbb{R}^{d-1} \times \mathbb{R}_+$. In this case the generalized form of heat kernel is given by [8],

$$K_{D,N}(x, y|t) = (4\pi t)^{-d/2}\left(e^{-\frac{(x-y)^2}{4t}} \mp e^{-\frac{(x-y')^2}{4t}}\right), \tag{3.90}$$

where $y' = (y^1, \ldots, y^{d-1}, -y^d)$, and (3.90) satisfies the heat kernel equation (3.30) for x and y inside the interval $\mathcal{N}$. Furthermore, it also satisfies the same heat equation for both of the boundary conditions namely Dirichlet and Neumann if either x or y is on the boundary. We note that when x and y both are near to the boundary one may write (3.90) in the form of (3.44) and apply DeWitt method to find out the recursion relations.

Now we write the generalized forms of the Dirichlet and Neumann boundary conditions that we quoted in the sect. 3.2 ($f|_{x=0,l} = 0$ (Dirichlet) or $\partial_x f|_{x=0,l} = 0$ (Neumann)),

$$f|_{\partial\mathscr{M}} = 0, \tag{3.91}$$

and

$$(\nabla_d + \mathscr{J}) f|_{\partial\mathscr{M}} = 0, \tag{3.92}$$

where $\mathscr{J}$ represents a matrix valued function on the boundary of a manifold under consideration, i.e. $\partial\mathscr{M}$. In this case the heat kernel coefficients consist of two parts namely bulk (the case of sect. 3.2) and boundary. By making the contribution coming from boundary part zero one can easily recover the results for the case considered in the sect. 3.2. Now, let us pay attention to the boundary $\partial\mathscr{M}$ and introduce the notion of extrinsic curvature, which basically characterizes the way how the boundary is embedded in the manifold $\mathscr{M}$ and it is given by,

$$H_{\mu\nu} = -w_{\mu\rho} w_{\nu\sigma} \eta_{\rho;\sigma}, \tag{3.93}$$

where $w_{\mu\nu} = \delta_{\mu\nu} - \eta_\mu \eta_\nu$ is the projector on the space tangent to the boundary and η_μ is the normal vector defined by the condition $\eta_\mu dx_\mu = 0$ or $\eta_\mu d y_\nu$=0 on the boundary, with an assumption that it is normalized i.e. $\eta_\mu \eta_\mu = 1$. We note that the extrinsic curvature defined in (3.93) is symmetric with respect to μ and ν and orthogonal to the normal vector i.e. $H_{\mu\nu}\eta_\nu = 0$.

Now, by doing the dimensional analysis of the invariants as we did in the sect. 3.2.2, we see that $\dim(H_{\mu\nu}) = 1$ and $\dim(\mathscr{L}) = 1$. Since the invariants are with odd dimensions, it implies that the heat kernel coefficients a_p with odd p do not get vanished. Consequently with this fact we write the structure of heat kernel coefficients as follows [8].

For Dirichlet boundary condition:

$$a_0(f,\boldsymbol{L}) = (4\pi)^{-d/2}\int_{\mathcal{M}} d^d x\,\sqrt{g}\,\mathrm{Tr}\,f, \tag{3.94}$$

$$a_1(f,\boldsymbol{L}) = (4\pi)^{-(d-1)/2}\int_{\partial\mathcal{M}} d^{d-1} x\,\sqrt{w}\,\mathrm{Tr}\,p_1(f), \tag{3.95}$$

$$\begin{aligned} a_2(f,\boldsymbol{L}) \;&=\; \frac{(4\pi)^{-d/2}}{6}\Big[\int_{\mathcal{M}} d^d x\,\sqrt{g}\,\mathrm{Tr}(f\,(6E+R)) + \\ &\quad + \int_{\partial\mathcal{M}} d^{d-1} x\,\sqrt{w}\,\mathrm{Tr}\big(p_2\,f\,H_{\mu\mu} + p_3\,f_{;d}\big)\Big]. \end{aligned} \tag{3.96}$$

For Neumann boundary condition:

$$a_0(f,\boldsymbol{L}) = (4\pi)^{-d/2}\int_{\mathcal{M}} d^d x\,\sqrt{g}\,\mathrm{Tr}\,f, \tag{3.97}$$

$$a_1(f,\boldsymbol{L}) = (4\pi)^{-(d-1)/2}\int_{\partial\mathcal{M}} d^{d-1} x\,\sqrt{w}\,\mathrm{Tr}\,q_1(f), \tag{3.98}$$

$$\begin{aligned} a_2(f,\boldsymbol{L}) \;&=\; \frac{(4\pi)^{-d/2}}{6}\Big[\int_{\mathcal{M}} d^d x\,\sqrt{g}\,\mathrm{Tr}(f\,(6E+R)) + \\ &\quad + \int_{\partial\mathcal{M}} d^{d-1} x\,\sqrt{w}\,\mathrm{Tr}\big(q_2\,f\,H_{\mu\mu} + q_3\,f_{;d} + q_4\,f\,\mathscr{L}\big)\Big]. \end{aligned} \tag{3.99}$$

From the above results we clearly see that, by introducing the boundary conditions on manifold not only the heat kernel coefficients a_p with odd p survive but the same with even p also get modified. Note that for the case under consideration in this section, the boundary term a_1 appearing in (3.95) and (3.98) is quite useful to study Gibbons-Hawking effect in classical and quantum gravity [58–60].

In this dissertation we consider the heat kernel coefficients on base manifolds without boundaries for further studies. Therefore, we work with the results derived in sect. 3.2 only.

CHAPTER 4

GRAVITATIONAL ACTIONS

In this chapter we discuss the gravitational action for the heat kernel coefficients a_0, a_2, a_4 and a_6. We study the action resulting from the heat kernel coefficient a_6 in detail. In particular, we considerably simplify the original form of the action for a_6 and reduce the number of terms in the action by significant amount. We construct the action for a_6 in two bases, namely Riemann and Weyl, which are useful to study the Ricci flat and conformally flat solutions respectively, as we shall see in the next chapter. Here we write all the formulae in terms of Feynman's notation (see the appendix A).

We commence with the deduction of the heat kernel coefficients for the case of pure gravity, for which we derived the generalized forms in the sect. 3.2.2. We enumerate below the heat kernel coefficients, which are boiled down to the case of pure gravity and studied over the base manifolds without boundaries in dimension $d = 4$. Let us start with the heat kernel coefficient a_0 (3.87), which is the cosmological constant given by,

$$a_0 \quad = \quad (4\pi)^{-2}\mathrm{Tr}(\mathbb{1})\int d^4x\,\sqrt{g}, \tag{4.1}$$

where we left $\mathrm{Tr}(\mathbb{1})$, which equals to 4 in $d = 4$. Now, by taking into account the case of pure gravity, i.e. (A.4) in (3.88) and (3.89), we have the standard form of the EH action and higher derivative action [1, 2] (containing four derivatives of the metric) as,

$$a_2 = -(4\pi)^{-2}\frac{\mathrm{Tr}(\mathbb{1})}{12}\int d^4x\,\sqrt{g}R, \tag{4.2}$$

$$a_4 = (4\pi)^{-2}\frac{\mathrm{Tr}(\mathbb{1})}{360}\int d^4x\,\sqrt{g}\left(5R^2-8R^2_{\mu\nu}-7R^2_{\mu\nu\rho\sigma}\right). \tag{4.3}$$

The action for the heat kernel coefficient a_4 (4.3) can be rewritten by using (B.4) and (B.5), which takes the form as follows [2],

$$a_4 = (4\pi)^{-2}\frac{\mathrm{Tr}(\mathbb{1})}{360}\int d^4x\,\sqrt{g}\left(-18C^2_{\mu\nu\rho\sigma}+11\mathrm{GB}\right). \tag{4.4}$$

We note that (2.7) coincides with (4.4), provided that $\alpha_5 = -18\,\mathrm{c.f.}$ and $\alpha_6 = 11\,\mathrm{c.f.}$ (see (2.7)) with c.f. being the common factor given in the above expression. One can easily deduce (4.4) from (4.3) by changing the basis and finding out corresponding coefficients (see e.g. (4.10)), where the coefficient for the term R^2 will get vanished. This action (4.4) is quite useful to deal with conformally flat metric as we shall in the sect. 5.3 in detail.

The equation given below represents the next heat kernel coefficient, namely a_6, which has been studied in [3, 4] (recall the table 3.2 and simplification of the terms discussed at the end of sect. 3.2.2). We shall use the following action to construct the Riemann and Weyl dominated actions, which are the essential parts of this project,

$$\begin{aligned} a_6(x) = {} & (4\pi)^{-d/2}\mathrm{Tr}\Big\{\frac{1}{7!}\big(-18R_{ijij;kkll}+17R_{ijij;k}R_{ulul;k}-2R_{ijik;l}R_{ujuk;l}- \\ & -4R_{ijik;l}R_{ujul;k}+9R_{ijku;l}R_{ijku;l}+28R_{ijij}R_{kuku;ll}-8R_{ijik}R_{ujuk;ll}+ \\ & +24R_{ijik}R_{ujul;kl}+12R_{ijkl}R_{ijkl;uu}\big)+\frac{1}{9\cdot 7!}\big(-35R_{ijij}R_{klkl}R_{pqpq}+ \\ & +42R_{ijij}R_{klkp}R_{qlqp}-42R_{ijij}R_{klpq}R_{klpq}+208R_{ijik}R_{julu}R_{kplp}- \\ & -192R_{ijik}R_{uplp}R_{jukl}+48R_{ijik}R_{julp}R_{kulp}-44R_{ijku}R_{ijlp}R_{kulp}- \\ & -80R_{ijku}R_{ilkp}R_{jlup}\big)+\frac{1}{360}\big(8\Omega_{ij;k}\Omega_{ij;k}+2\Omega_{ij;j}\Omega_{ik;k}+ \\ & +12\Omega_{ij}\Omega_{ij;kk}-12\Omega_{ij}\Omega_{jk}\Omega_{ki}-6R_{ijkl}\Omega_{ij}\Omega_{kl}+4R_{ijik}\Omega_{jl}\Omega_{kl}- \\ & -5R_{ijij}\Omega_{kl}\Omega_{kl}\big)+\frac{1}{360}\big(6E_{;iijj}+60EE_{;ii}+30E_{;i}E_{;i}+60E^3+ \\ & +30E\Omega_{ij}\Omega_{ij}-10R_{ijij}E_{;kk}-4R_{ijik}E_{;jk}-12R_{ijij;k}E_{;k}-30R_{ijij}E^2- \\ & -12R_{ijij;kk}E+5R_{ijij}R_{klkl}E-2R_{ijik}R_{ljlk}E+2R_{ijkl}R_{ijkl}E\big)\Big\}. \end{aligned} \tag{4.5}$$

4.1 Riemann Basis

Let us now simplify the eq. (4.5) to construct it in Riemann basis. We consider the terms under the trace and calculate the same as it is explained in an appendix D. After substituting those results in (4.5) we get,

$$
\begin{aligned}
a_6(x) \;=\; & (4\pi)^{-d/2}\mathrm{Tr}\Big\{\frac{1}{7!}\big(-18R_{ijij;kkll}+17R_{ijij;k}R_{ulul;k}-2R_{ijik;l}R_{ujuk;l}- \\
& -4R_{ijik;l}R_{ujul;k}+9R_{ijku;l}R_{ijku;l}+28R_{ijij}R_{kuku;ll}-8R_{ijik}R_{ujuk;ll}+ \\
& +24R_{ijik}R_{ujul;kl}+12R_{ijkl}R_{ijkl;uu}\big)+\frac{1}{9\cdot 7!}\big(-35R_{ijij}R_{klkl}R_{pqpq}+ \\
& +42R_{ijij}R_{klkp}R_{qlqp}-42R_{ijij}R_{klpq}R_{klpq}+208R_{ijik}R_{julu}R_{kplp}- \\
& -192R_{ijik}R_{uplp}R_{jukl}+48R_{ijik}R_{julp}R_{kulp}-44R_{ijku}R_{ijlp}R_{kulp}- \\
& -80R_{ijku}R_{ilkp}R_{jlup}\big)-\frac{1}{360}R^2_{ijab;k}-\frac{1}{720}R_{ia;b}R_{ia;b}+\frac{1}{720}R_{ia;b}R_{ib;a}- \\
& -\frac{1}{240}R_{ijab}R_{ijab;kk}-\frac{1}{240}R_{\mu\nu ab}R_{\nu\rho bc}R_{\rho\mu ac}+\frac{1}{480}R_{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma ab}+ \\
& +\frac{1}{720}R_{\mu\nu}R_{\mu\rho ab}R_{\nu\rho ab}-\frac{1}{576}RR^2_{\mu\nu ab}-\frac{1}{240}R_{;iijj}+\frac{1}{96}RR_{;ii}+\frac{1}{192}R^2_{;i}- \\
& -\frac{1}{384}R^3+\frac{1}{384}RR^2_{\mu\nu ab}-\frac{1}{144}RR_{;ii}-\frac{1}{360}R_{jk}R_{;jk}-\frac{1}{120}R^2_{;i}+ \\
& +\frac{1}{192}R^3-\frac{1}{120}RR_{;ii}-\frac{1}{288}R^3+\frac{1}{720}R^2_{\mu\nu}R-\frac{1}{720}R^2_{\mu\nu\rho\sigma}R\Big\}. \qquad (4.6)
\end{aligned}
$$

The above equation (4.6) can be simplified further. We start with the possible contractions of the indices. After doing the contractions, just for the purpose of convenience we rename the indices as follows: $i\to\mu$, $j\to\nu$, $k\to\rho$, $l\to\sigma$, $u\to\alpha$, $p\to\beta$, $q\to\gamma$, $a\to\delta$, $b\to\lambda$, $c\to\tau$. The next step is to do the simplification of the resulting equation and collect identical terms, so that the simplified form of the eq. (4.6) turns out to be the following,

$$
\begin{aligned}
a_6(x) \;=\; & (4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1})\Big\{\frac{1}{4032}R^2_{;\rho}-\frac{1}{560}R^2_{\nu\rho;\sigma}+\frac{1}{1680}R_{\nu\rho;\sigma}R_{\nu\sigma;\rho}-\frac{1}{1008}R^2_{\mu\nu\rho\alpha;\sigma}- \\
& -\frac{1}{630}R_{\nu\rho}R_{\nu\rho;\sigma\sigma}+\frac{1}{210}R_{\nu\rho}R_{\nu\sigma;\rho\sigma}-\frac{1}{560}R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma;\alpha\alpha}-\frac{1}{10368}R^3- \\
& -\frac{13}{2835}R_{\nu\rho}R_{\nu\sigma}R_{\rho\sigma}-\frac{4}{945}R_{\nu\rho}R_{\alpha\sigma}R_{\nu\alpha\rho\sigma}+\frac{101}{90720}R_{\mu\nu\rho\alpha}R_{\mu\nu\sigma\beta}R_{\rho\alpha\sigma\beta}+ \\
& +\frac{109}{45360}R_{\mu\nu\rho\alpha}R_{\mu\sigma\rho\beta}R_{\nu\sigma\alpha\beta}+\frac{1}{3024}R_{\mu\nu}R_{\mu\rho\delta\lambda}R_{\nu\rho\delta\lambda}-\frac{1}{1680}R_{;\mu\mu\nu\nu}+ \\
& +\frac{1}{1440}RR_{;\mu\mu}-\frac{1}{360}R_{\nu\rho}R_{;\nu\rho}+\frac{1}{2160}RR^2_{\mu\nu}+\frac{7}{17280}RR^2_{\mu\nu\rho\sigma}\Big\}. \qquad (4.7)
\end{aligned}
$$

Furthermore, we multiply both the sides of (4.7) by $\sqrt{g}$ and integrate with respect to x in an arbitrary dimension d and use the integration by parts i.e. $\int d^dx\sqrt{g}AB_{\mu;\mu} = -\int d^dx\sqrt{g}A_{;\mu}B_\mu$ such that the resulting equation reduces as follows,

$$
\begin{aligned}
\int d^dx\sqrt{g}a_6(x) \;=\; & \int d^dx\sqrt{g}(4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1})\Big\{-\frac{1}{4032}R\Box R+\frac{1}{560}R_{\nu\rho}\Box R_{\nu\rho}+\frac{1}{1008}R_{\mu\nu\rho\alpha}\Box R_{\mu\nu\rho\alpha}-\\
&-\frac{1}{630}R_{\nu\rho}\Box R_{\nu\rho}+\frac{1}{240}R_{\nu\rho}R_{\nu\sigma;\rho\sigma}-\frac{1}{560}R_{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma}-\frac{1}{10368}R^3-\\
&-\frac{13}{2835}R_{\nu\rho}R_{\nu\sigma}R_{\rho\sigma}-\frac{4}{945}R_{\nu\rho}R_{\alpha\sigma}R_{\nu\alpha\rho\sigma}+\frac{101}{90720}R_{\mu\nu\rho\alpha}R_{\mu\nu\sigma\beta}R_{\rho\alpha\sigma\beta}+\\
&+\frac{109}{45360}R_{\mu\nu\rho\alpha}R_{\mu\sigma\rho\beta}R_{\nu\sigma\alpha\beta}+\frac{1}{3024}R_{\mu\nu}R_{\mu\rho\delta\lambda}R_{\nu\rho\delta\lambda}+\frac{1}{1440}R\Box R-\\
&-\frac{1}{360}R_{\nu\rho}R_{;\nu\rho}+\frac{1}{2160}RR^2_{\mu\nu}+\frac{7}{17280}RR^2_{\mu\nu\rho\sigma}\Big\}.
\end{aligned}
\tag{4.8}
$$

Here we note that the term $\frac{1}{1680}R_{;\mu\mu\nu\nu}$ (given in eq. (4.7)) gets vanished under the integral, because it is a total derivative. Now consider the term $\frac{1}{240}R_{\nu\rho}R_{\nu\sigma;\rho\sigma}$, by using the equation (B.6) one can simplify it as,

$$
\int d^dx\sqrt{g}\frac{1}{240}R_{\nu\rho}R_{\nu\sigma;\rho\sigma}=\int d^dx\sqrt{g}\left(\frac{1}{960}R\Box R+\frac{1}{240}R_{\nu\rho}R_{\beta\sigma}R_{\rho\sigma\nu\beta}+\frac{1}{240}R_{\rho\beta}R_{\nu\rho}R_{\nu\beta}\right).
$$

In the process of simplifying the above term we used the eq. (B.11) and integrated by parts. Next we consider the term $-\frac{1}{360}R_{\nu\rho}R_{;\nu\rho}$ and integrate by parts then once again here also we apply eq. (B.11) on the resulting equation, which takes the following form after simplification,

$$
-\frac{1}{360}\int d^dx\sqrt{g}R_{\nu\rho}R_{;\nu\rho}=-\frac{1}{720}\int d^dx\sqrt{g}R\Box R.
$$

After plugging the above two results in the equation (4.8) and simplifying it we get the action with 11 terms,

$$
\begin{aligned}
\int d^dx\sqrt{g}a_6(x) \;=\; & \int d^dx\sqrt{g}(4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1})\Big\{\frac{1}{10080}R\Box R+\frac{1}{5040}R_{\nu\rho}\Box R_{\nu\rho}-\frac{1}{1260}R_{\mu\nu\rho\alpha}\Box R_{\mu\nu\rho\alpha}-\\
&-\frac{1}{10368}R^3+\frac{1}{2160}RR^2_{\mu\nu}-\frac{19}{45360}R_{\rho\beta}R_{\nu\rho}R_{\nu\beta}+\frac{7}{17280}RR^2_{\mu\nu\rho\sigma}-\\
&-\frac{1}{15120}R_{\nu\rho}R_{\beta\sigma}R_{\rho\sigma\nu\beta}+\frac{1}{3024}R_{\mu\nu}R_{\mu\rho\delta\lambda}R_{\nu\rho\delta\lambda}+\\
&+\frac{101}{90720}R_{\mu\nu\rho\alpha}R_{\mu\nu\sigma\beta}R_{\rho\alpha\sigma\beta}+\frac{109}{45360}R_{\mu\nu\rho\alpha}R_{\mu\sigma\rho\beta}R_{\nu\sigma\alpha\beta}\Big\}.
\end{aligned}
\tag{4.9}
$$

Now, in the next steps we shall focus on to eradicate the term $-\frac{1}{1260}R_{\mu\nu\rho\alpha}\Box R_{\mu\nu\rho\alpha}$. In order to do this we consider the first three terms of the above action and call it S,

$$S = a_1\, R\Box R + a_2\, R_{\nu\rho}\Box R_{\nu\rho} + a_3\, R_{\mu\nu\rho\alpha}\Box R_{\mu\nu\rho\alpha},$$

where a_1, a_2 and a_3 are the known numerical constants given in the action (4.9). Now, we change the basis of S in such a way that the result takes the form as,

$$S = b_1\, R\Box R + b_2\, C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha} + b_3\, \mathrm{GB}_1, \tag{4.10}$$

where b_1, b_2 and b_3 are the numerical constants to be determined. The terms $C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha}$ and GB_1 are given by the following expressions,

$$C_{\mu\nu\rho\sigma}\Box C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} - \frac{4}{d-2}R_{\mu\rho}\Box R_{\mu\rho} + \frac{2}{(d-2)(d-1)}R\Box R, \tag{4.11}$$

and

$$\mathrm{GB}_1 \;:=\; R_{\mu\nu\alpha\beta}\Box R_{\mu\nu\alpha\beta} - 4R_{\mu\nu}\Box R_{\mu\nu} + R\Box R. \tag{4.12}$$

Now, by performing some trivial calculations we find the unknown co-efficients and substitute S (4.10) back to the action (4.9), which lead us to,

$$\begin{aligned}\int d^dx\sqrt{g}a_6(x) = \int d^dx\sqrt{g}(4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1})\Big\{ & \frac{d-6}{6720(d-1)}R\Box R - \frac{d-2}{1344(d-3)}C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha} - \\ & -\frac{d-18}{20160(d-3)}\mathrm{GB}_1 - \frac{1}{10368}R^3 + \frac{1}{2160}RR^2_{\mu\nu} - \frac{19}{45360}R_{\rho\beta}R_{\nu\rho}R_{\nu\beta} + \\ & + \frac{7}{17280}RR^2_{\mu\nu\rho\sigma} - \frac{1}{15120}R_{\nu\rho}R_{\beta\sigma}R_{\rho\sigma\nu\beta} + \frac{1}{3024}R_{\mu\nu}R_{\mu\rho\delta\lambda}R_{\nu\rho\delta\lambda} + \\ & + \frac{101}{90720}R_{\mu\nu\rho\alpha}R_{\mu\nu\sigma\beta}R_{\rho\alpha\sigma\beta} + \frac{109}{45360}R_{\mu\nu\rho\alpha}R_{\mu\sigma\rho\beta}R_{\nu\sigma\alpha\beta}\Big\}.\end{aligned} \tag{4.13}$$

In order to find an expression for the term GB_1 under the integral, we start with the definition of the same (4.12). First we use the differential Bianchi identity to replace an appropriate term by differentiated Riemann tensors, then we exploit the relation (B.6) to make it useful for our purpose. After simplifying the outcomes, we perform the integration by parts under the spacetime integral and once again simplify the results, which yield the following expression for the term that we are interested in,

$$\begin{aligned}\int d^dx\sqrt{g}GB_1 = \int d^dx\sqrt{g}\big(& -4R_{\alpha\mu}R_{\beta\nu}R_{\mu\nu\alpha\beta} - 4R_{\alpha\mu}R_{\mu\beta}R_{\alpha\beta} + 4R_{\mu\nu\alpha\beta}R_{\gamma\beta\mu\sigma}R_{\sigma\nu\gamma\alpha} + \\ & + R_{\mu\nu\alpha\beta}R_{\alpha\beta\gamma\sigma}R_{\mu\nu\gamma\sigma} + 2R_{\alpha\sigma}R_{\mu\nu\alpha\beta}R_{\mu\nu\sigma\beta}\big),\end{aligned} \tag{4.14}$$

where in between the simplification one may need to use the cyclicity of the Riemann tensor. By substituting the integral of GB_1 (4.14) in the action (4.13) and simplifying it we get the following action with 10 terms,

$$\begin{aligned}
&\int d^d x \sqrt{g} a_6(x) = \\
= &\int d^d x \sqrt{g} (4\pi)^{-d/2} \mathrm{Tr}(\mathbb{1}) \Bigg\{ \frac{d-6}{6720(d-1)} R \Box R - \frac{d-2}{1344(d-3)} C_{\mu\nu\rho\alpha} \Box C_{\mu\nu\rho\alpha} - \\
&- \frac{1}{10368} R^3 + \frac{1}{2160} R R^2_{\mu\nu} + \frac{7}{17280} R R^2_{\mu\nu\rho\sigma} - \frac{2d+21}{9072(d-3)} R_{\alpha\mu} R_{\mu\beta} R_{\alpha\beta} + \\
&+ \frac{2d-51}{15120(d-3)} R_{\alpha\mu} R_{\beta\nu} R_{\mu\nu\alpha\beta} + \frac{7d+24}{30240(d-3)} R_{\alpha\sigma} R_{\mu\nu\alpha\beta} R_{\mu\nu\sigma\beta} + \\
&+ \frac{20d-33}{9072(d-3)} R_{\mu\nu\rho\alpha} R_{\mu\sigma\rho\beta} R_{\nu\sigma\alpha\beta} + \frac{193d-444}{181440(d-3)} R_{\mu\nu\alpha\beta} R_{\mu\nu\gamma\sigma} R_{\alpha\beta\gamma\sigma} \Bigg\}. \qquad (4.15)
\end{aligned}$$

Now, we re-write the eq. (4.15) in dimension $d = 4$ and use the identities given in the eq. (B.13) to get rid of some terms so that the above action reduces to the following form, which has only 8 terms. Basically we replace the terms $R_{\mu\nu\rho\alpha} R_{\mu\sigma\rho\beta} R_{\nu\sigma\alpha\beta}$ and $R_{\alpha\sigma} R_{\mu\nu\alpha\beta} R_{\mu\nu\sigma\beta}$ in the above action by using the identities in $d = 4$. Therefore, finally after simplifying the outcomes we get an elegant form of the action (4.15) in $d = 4$ as follows,

Riemann dominated action:

$$\begin{aligned}
\int d^4 x \sqrt{g} a_6(x) \;=\; &\frac{1}{4\pi^2} \int d^4 x \sqrt{g} \Bigg\{ \frac{1}{1120} R \Box R - \frac{1}{336} R_{\mu\nu} \Box R_{\mu\nu} + \frac{1}{126} R_{\mu\alpha} R_{\nu\beta} R_{\mu\nu\alpha\beta} - \\
&- \frac{43}{15120} R_{\alpha\mu} R_{\mu\beta} R_{\alpha\beta} - \frac{1}{1120} R^3 + \frac{13}{2016} R R^2_{\mu\nu} - \frac{1}{5040} R R^2_{\mu\nu\rho\sigma} - \\
&- \frac{1}{15120} R_{\mu\nu\rho\alpha} R_{\mu\nu\sigma\beta} R_{\rho\alpha\sigma\beta} \Bigg\}. \qquad (4.16)
\end{aligned}$$

It is quite easy to see that the terms given in the action (4.16), which are quadratic and cubic in either Ricci scalar or curvature will get vanished for the Ricci flat background. Therefore, one needs to vary only the last two terms, which makes easier to deal with the EOM for such cases. This is the main benefit to write two forms of the action, namely Riemann and Weyl dominated. We shall see the same thing with Weyl dominated action for the conformally flat background in the next section.

4.2 Weyl Basis

In this section we derive the Weyl dominated form of the action. This form of the action is very useful to study conformally flat backgrounds.

We rewrite the action (4.15) in terms of Weyl tensor. In order to do this we use the definition of Weyl tensor (B.3) and replace Riemann tensor with Weyl tensor. This will result into the following equation,

$$
\begin{aligned}
&\int d^dx\sqrt{g}a_6(x) = \\
= &\int d^dx\sqrt{g}(4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1})\Bigg\{\frac{d-6}{6720(d-1)}R\Box R - \frac{d-2}{1344(d-3)}C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha}- \\
&-\frac{1}{10368}R^3+\frac{1}{2160}RR^2_{\mu\nu}+\frac{7}{17280}R\left(C^2_{\mu\nu\rho\sigma}+\frac{4}{d-2}R^2_{\mu\rho}-\frac{2}{(d-2)(d-1)}R^2\right)- \\
&-\frac{2d+21}{9072(d-3)}R_{\alpha\mu}R_{\mu\beta}R_{\alpha\beta}+ \\
&+\frac{2d-51}{15120(d-3)}R_{\alpha\mu}R_{\beta\nu}\left(C_{\mu\nu\alpha\beta}-\frac{1}{d-2}4R_{[\mu[\alpha}g_{\nu]\beta]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\alpha}g_{\nu]\beta}R\right)+ \\
&+\frac{7d+24}{30240(d-3)}R_{\alpha\sigma}\left(C_{\mu\nu\alpha\beta}-\frac{1}{d-2}4R_{[\mu[\alpha}g_{\nu]\beta]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\alpha}g_{\nu]\beta}R\right)\times \\
&\times\left(C_{\mu\nu\sigma\beta}-\frac{1}{d-2}4R_{[\mu[\sigma}g_{\nu]\beta]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\sigma}g_{\nu]\beta}R\right)+ \\
&+\frac{20d-33}{9072(d-3)}\left(C_{\mu\nu\rho\alpha}-\frac{1}{d-2}4R_{[\mu[\rho}g_{\nu]\alpha]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\rho}g_{\nu]\alpha}R\right)\times \\
&\times\left(C_{\mu\sigma\rho\beta}-\frac{1}{d-2}4R_{[\mu[\rho}g_{\sigma]\beta]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\rho}g_{\sigma]\beta}R\right)\times \\
&\times\left(C_{\nu\sigma\alpha\beta}-\frac{1}{d-2}4R_{[\nu[\alpha}g_{\sigma]\beta]}+\frac{1}{(d-2)(d-1)}2g_{[\nu\alpha}g_{\sigma]\beta}R\right)+ \\
&+\frac{193d-444}{181440(d-3)}\left(C_{\mu\nu\alpha\beta}-\frac{1}{d-2}4R_{[\mu[\alpha}g_{\nu]\beta]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\alpha}g_{\nu]\beta}R\right)\times \\
&\times\left(C_{\mu\nu\gamma\sigma}-\frac{1}{d-2}4R_{[\mu[\gamma}g_{\nu]\sigma]}+\frac{1}{(d-2)(d-1)}2g_{[\mu\gamma}g_{\nu]\sigma}R\right)\times \\
&\times\left(C_{\alpha\beta\gamma\sigma}-\frac{1}{d-2}4R_{[\alpha[\gamma}g_{\beta]\sigma]}+\frac{1}{(d-2)(d-1)}2g_{[\alpha\gamma}g_{\beta]\sigma}R\right)\Bigg\}.
\end{aligned}
\tag{4.17}
$$

As it is explained below we split the eq. (4.17) into three parts and simplify it upto certain extent. We also use the fact that Weyl tensor is totally trace-free i.e $C_{\alpha\beta\alpha\gamma}=0$. Then we add the outcomes altogether and once again simplify it in such a way that it results into the final expression for the Weyl dominated action with 10 terms.

First of all we denote the terms given in first six lines of the eq. (4.17) by A_1, the terms with the coefficient $+\frac{20d-33}{9072(d-3)}$ by A_2 and the remaining terms that appear with the coefficient $+\frac{193d-444}{181440(d-3)}$ by A_3. Then we simplify it and use $C_{\alpha\beta\alpha\gamma}=0$ (trace-less Weyl tensor), which give us the following results,

$$\begin{aligned} A_1 \equiv\; & \frac{d-6}{6720(d-1)}R\Box R-\frac{d-2}{1344(d-3)}C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha}-\frac{1}{10368}R^3+\frac{1}{2160}RR^2_{\mu\nu}+ \\ & +\frac{7}{17280}\left(RC^2_{\mu\nu\rho\sigma}+\frac{4}{d-2}RR^2_{\mu\rho}-\frac{2}{(d-2)(d-1)}R^3\right)-\frac{2d+21}{9072(d-3)}R_{\alpha\mu}R_{\mu\beta}R_{\alpha\beta}+ \\ & +\frac{2d-51}{15120(d-3)}\left(R_{\alpha\mu}R_{\beta\nu}C_{\mu\nu\alpha\beta}+\frac{2}{d-2}R_{\alpha\mu}R_{\mu\nu}R_{\nu\alpha}+\frac{1}{(d-2)(d-1)}R^3-\right. \\ & \left.-\frac{2d-1}{(d-2)(d-1)}RR^2_{\mu\nu}\right)+\frac{7d+24}{30240(d-3)}\left(R_{\alpha\sigma}C_{\mu\nu\alpha\beta}C_{\mu\nu\sigma\beta}+\frac{4}{d-2}R_{\alpha\nu}R_{\mu\beta}C_{\mu\nu\alpha\beta}+\right. \\ & \left.+\frac{2(d-4)}{(d-2)^2}R_{\alpha\sigma}R_{\mu\alpha}R_{\mu\sigma}+\frac{2(d+1)}{(d-2)^2(d-1)}RR^2_{\mu\nu}-\frac{2}{(d-2)^2(d-1)}R^3\right), \end{aligned} \tag{4.18}$$

$$\begin{aligned} A_2 \equiv\; & \frac{20d-33}{9072(d-3)}\left(C_{\mu\nu\rho\alpha}C_{\mu\sigma\rho\beta}C_{\nu\sigma\alpha\beta}+\frac{6}{d-2}R_{\mu\alpha}C_{\mu\sigma\nu\beta}C_{\nu\sigma\alpha\beta}-\right. \\ & -\frac{3}{(d-2)(d-1)}RC_{\alpha\sigma\nu\beta}C_{\nu\sigma\alpha\beta}+\frac{3(d-4)}{(d-2)^2}R_{\nu\alpha}R_{\sigma\beta}C_{\nu\sigma\alpha\beta}+ \\ & \left.+\frac{2(3d-8)}{(d-2)^3}R_{\beta\rho}R_{\sigma\rho}R_{\sigma\beta}+\frac{2d^2-7d+4}{(d-2)^3(d-1)^2}R^3+\frac{-3d^2+6d+6}{(d-2)^3(d-1)}RR^2_{\nu\beta}\right), \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} A_3 \equiv\; & \frac{193d-444}{181440(d-3)}\left(C_{\mu\nu\alpha\beta}C_{\mu\nu\gamma\sigma}C_{\alpha\beta\gamma\sigma}-\frac{12}{d-2}R_{\mu\alpha}C_{\mu\beta\gamma\sigma}C_{\alpha\beta\gamma\sigma}+\right. \\ & +\frac{24}{(d-2)^2}R_{\mu\beta}R_{\alpha\sigma}C_{\mu\alpha\beta\sigma}+\frac{6}{(d-2)(d-1)}RC^2_{\alpha\beta\gamma\sigma}- \\ & \left.-\frac{8(d-4)}{(d-2)^3}R_{\mu\alpha}R_{\mu\gamma}R_{\alpha\gamma}-\frac{24}{(d-2)^3(d-1)}RR^2_{\mu\gamma}+\frac{4d}{(d-2)^3(d-1)^2}R^3\right). \end{aligned} \tag{4.20}$$

Now, we combine (4.18), (4.19) and (4.20), and simplify the resulting expression in such way that it yields the following action with 10 terms,

$$\begin{aligned}
&\int d^dx\sqrt{g}a_6(x) = \\
= \;& \int d^dx\sqrt{g}(4\pi)^{-d/2}\mathrm{Tr}(\mathbb{1})\Big\{\frac{d-6}{6720(d-1)}R\Box R - \frac{d-2}{1344(d-3)}C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha}+ \\
&+\frac{-35d^5+280d^4-1121d^3+2912d^2-3260d+1968}{362880(d-2)^3(d-1)^2}R^3+ \\
&+\frac{14d^4-57d^3+37d^2-288}{30240(d-2)^3(d-1)}RR^2_{\mu\nu}+\frac{-5d^3-21d^2+95d+120}{22680(d-2)^3}R_{\alpha\mu}R_{\mu\beta}R_{\alpha\beta}+ \\
&+\frac{49d^2-147d+470}{120960(d-2)(d-1)}RC^2_{\mu\nu\rho\sigma}+\frac{2d^2+33d+112}{15120(d-2)^2}R_{\alpha\mu}R_{\beta\nu}C_{\mu\nu\alpha\beta}+ \\
&+\frac{7d^2-176d+510}{30240(d-3)(d-2)}R_{\alpha\sigma}C_{\mu\nu\alpha\beta}C_{\mu\nu\sigma\beta}+\frac{20d-33}{9072(d-3)}C_{\mu\nu\rho\alpha}C_{\mu\sigma\rho\beta}C_{\nu\sigma\alpha\beta}+ \\
&+\frac{193d-444}{181440(d-3)}C_{\mu\nu\alpha\beta}C_{\mu\nu\gamma\sigma}C_{\alpha\beta\gamma\sigma}\Big\}. \qquad (4.21)
\end{aligned}$$

Now, as we did in the previous section we rewrite the eq. (4.21) in dimension $d=4$ by using the identities given in the eq. (B.13). We utilize these identities in $d=4$ for the terms $C_{\mu\nu\rho\alpha}C_{\mu\sigma\rho\beta}C_{\nu\sigma\alpha\beta}$ and $R_{\alpha\sigma}C_{\mu\nu\alpha\beta}C_{\mu\nu\sigma\beta}$ in (4.21), replace it by appropriate terms and simplify the results. So, we get the action with only 8 terms as given below.

Weyl dominated action:

$$\begin{aligned}
\int d^4x\sqrt{g}a_6(x) \;=\; & \frac{1}{4\pi^2}\int d^4x\sqrt{g}\Big\{-\frac{1}{10080}R\Box R-\frac{1}{672}C_{\mu\nu\rho\alpha}\Box C_{\mu\nu\rho\alpha}- \\
&-\frac{1}{68040}R^3+\frac{1}{3024}RR^2_{\mu\nu}-\frac{13}{15120}R_{\alpha\mu}R_{\mu\beta}R_{\alpha\beta}+\frac{1}{1728}RC^2_{\mu\nu\rho\sigma}+ \\
&+\frac{23}{5040}R_{\mu\alpha}R_{\nu\beta}C_{\mu\nu\alpha\beta}+\frac{19}{4320}C_{\mu\nu\alpha\beta}C_{\mu\nu\gamma\sigma}C_{\alpha\beta\gamma\sigma}\Big\}. \qquad (4.22)
\end{aligned}$$

As we discussed in the sect. 5.1 about the reduction of the terms of Riemann dominated action in the case of Ricci flat solution, here also we see that for the locally conformally flat background some terms containing C^n (Weyl) with degree $n\geq 2$ get vanished leaving the final equation with even less terms to vary. This makes it facile to get the EOM for such cases. We also note that the eq. (4.22) helps a lot and reduces the efforts by making it trivial to check that for the conformally flat background $S_3\times S_1$ the heat kernel coefficient a_6 will get vanished. This remarkable result has been already studied in [1] by the direct calculation of a_6 for this background.

CHAPTER 5

EQUATIONS OF MOTION

We consider the gravitational actions studied in previous chapter and vary the same. More specifically, by using the variational principle we derive the EOM for the action defined by the heat kernel coefficients a_0, a_2, a_4 and a_6. Here also we study the EOM for a_6 in a great detail. We derive the EOM for a_6 in Riemannian and Weyl (only for the conformally flat background) bases. We reduce the Riemann dominated EOM to the particular case of Ricci flat solution. We conlude this chapter by applying these two forms of EOM (Riemann and Weyl) to some black hole and cosmological solutions.

We explicitly derive the EOM by varying the Riemann (4.16) and Weyl (4.22) dominated forms of the action in sect. 5.1 and 5.2 respectively. Before we proceed to the main problem, we enumerate the EOM coming from the heat kernel coefficients a_0 (3.60), a_2 (3.61) and a_4 (3.62). The following set of EOM can easily be derived by using the techniques explained in detail in the next section (here we denote the variations of a_p by $E^{(p)}_{\mu\nu}$),

$$4\pi^2 E^{(0)}_{\mu\nu} = \frac{1}{2} g_{\mu\nu}, \tag{5.1}$$

$$4\pi^2 E^{(2)}_{\mu\nu} = \frac{1}{12}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right), \tag{5.2}$$

$$
\begin{aligned}
4\pi^2 E^{(4)}_{\mu\nu} \quad = \quad & -\frac{1}{144\pi^2} R R_{\mu\nu} - \frac{1}{120\pi^2}\nabla_\mu \nabla_\nu R - \frac{1}{240\pi^2} g_{\mu\nu}\Box R + \frac{1}{576\pi^2} g_{\mu\nu} R^2 + \\
& + \frac{11}{360\pi^2} R_{\alpha\mu\nu\beta} R_{\alpha\beta} + \frac{1}{40\pi^2}\Box R_{\mu\nu} - \frac{1}{360\pi^2} g_{\mu\nu} R^2_{\alpha\beta} + \frac{7}{720\pi^2} R_{\mu\beta\rho\sigma} R_{\nu\beta\rho\sigma} - \\
& - \frac{7}{360\pi^2} R_{\mu\alpha} R_{\nu\alpha} - \frac{7}{2880\pi^2} g_{\mu\nu} R^2_{\alpha\beta\rho\sigma}. \qquad (5.3)
\end{aligned}
$$

Note that the EOM (5.1), (5.2) and (5.3) are completely symmetric with respect to μ and ν. We would like to point out that the EOM deduced from the action (4.4) is the same as (5.3). Although we do not provide the explicit result here, but in principle one can easily derive it by exploiting either (2.3) or (2.4), which makes the action (4.4) easy to vary in $d = 4$ by leaving only one term $C^2_{\mu\nu\rho\sigma}$, and that can be done by using (C.10) and variational method explained in sect. 5.1 to produce the desired EOM. In order to see that (4.4) coincides with (5.3), it may require to replace all the Weyl tensors with the Riemann tensors by using the relations (B.3) and (B.4) in the resulting EOM coming from the action (4.4). Furthermore, we also observe that the EOM coming from the action (4.4) contains 9 terms and upon comparison with the eq. (5.3) gives the tensor identity.

More importantly we see that conformally flat solutions do not get affected by a_4. In other words EOM coming from a_4 does not contribute in the full EOM resulting from full action (including a_0, a_2 and a_6) for such solutions. It is quite easy to explain this with the help of the action of the form of (4.4). Note that the second term in this action gets vanished after doing the variation, because as we discussed before the variation of GB is the total derivative in $d = 4$ (2.3). Moreover, this is true also due to generalized GB theorem (2.4). The remaining term $C^2_{\mu\nu\rho\sigma}$ is trivially zero for the solutions under consideration as explained in sect. 4.2. We shall see example of such a solution in sect. 5.3.

Throughout this chapter the symmetrization with respect to the indices μ and ν is assumed (particularly for the final EOM) unless we specify it explicitly. In other words we do not use the symmetrization brackets "()" for the final answers of EOM, but we shall indicate wherever the equation is completely symmetric with respect to μ and ν, and does not require any symmetrizations, as we did above.

5.1 Riemann Basis

In this section we vary the Riemann dominated action (4.16) studied in the previous chapter. The motivation behind the study of this action and corresponding EOM is that it make easier to analyze Ricci flat solutions. We shall consider this point in detail when we scrutinize the EOM for some solutions.

Now, as we have mentioned in an appendix A that while doing the variations one has be cautious about summed over indices and it becomes quite necessary to use Einstein summation convention instead of Feynman's notation. Therefore, in order to illustrate this idea we give a brief explanation with an example of the derivation of EOM for one of the terms of the Riemann dominated action for a_6 (4.16).

In principle one can consider any of the terms of the action, but just to present the central idea of the method we take into account for example, the third term of the action a_6 (4.16), rewrite it in Einstein summation convention and vary it by using (C.1), (C.8) and (C.4). The resulting equation takes the form as,

$$\begin{aligned}
A &\equiv \int d^4x\delta\left(\sqrt{g}R^{\alpha\mu}R^{\beta\nu}R_{\mu\nu\alpha\beta}\right) = \\
&= \int d^4x\left(\delta\sqrt{g}R^{\mu\alpha}R^{\nu\beta}R_{\mu\nu\alpha\beta} + \sqrt{g}\delta R^{\mu\alpha}R^{\nu\beta}R_{\mu\nu\alpha\beta} + \sqrt{g}R^{\mu\alpha}\delta R^{\nu\beta}R_{\mu\nu\alpha\beta} + \right.\\
&\quad \left. + \sqrt{g}R^{\mu\alpha}R^{\nu\beta}\delta R_{\mu\nu\alpha\beta}\right) = \\
&= \frac{1}{2}\int d^4x\sqrt{g}hR^{\mu\alpha}R^{\nu\beta}R_{\mu\nu\alpha\beta} + \\
&\quad + \int d^4x\sqrt{g}\left(-2h^{(\mu\gamma}R^{\alpha)}{}_{\gamma} + \nabla^{\gamma}\nabla^{\mu}h^{\alpha}{}_{\gamma} - \frac{1}{2}\nabla^{\mu}\nabla^{\alpha}h - \frac{1}{2}\Box h^{\mu\alpha}\right)R^{\nu\beta}R_{\mu\nu\alpha\beta} + \\
&\quad + \int d^4x\sqrt{g}R^{\mu\alpha}\left(-2h^{(\nu\gamma}R^{\beta)}{}_{\gamma} + \nabla^{\gamma}\nabla^{\nu}h^{\beta}{}_{\gamma} - \frac{1}{2}\nabla^{\nu}\nabla^{\beta}h - \frac{1}{2}\Box h^{\nu\beta}\right)R_{\mu\nu\alpha\beta} + \\
&\quad + \int d^4x\sqrt{g}R^{\mu\alpha}R^{\nu\beta}\left(R_{\mu\nu[\alpha}{}^{\gamma}h_{\beta]\gamma} - 2\nabla_{[\mu}\nabla_{[\beta}h_{\nu]\alpha]}\right),
\end{aligned} \tag{5.4}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$ (C.1), and for the simplicity we neglected the numerical constants, which we shall recover in the final result. Now, the eq. (5.4) can safely be rewritten in terms of the Feynman's notation. Next we collect the identical terms and perform the integration by parts (by using Stoke's theorem) on the outcomes in order to make $h_{\alpha\beta}$ and h free from any derivatives, which basically lead us to the following expression,

$$\begin{aligned} A &= \frac{1}{2}\int d^4x\sqrt{g}hR_{\mu\alpha}R_{\nu\beta}R_{\mu\nu\alpha\beta} - 4\int d^4x\sqrt{g}h_{\nu\gamma}R_{\mu\alpha}R_{\beta\gamma}R_{\mu\nu\alpha\beta} + \\ &+2\int d^4x\sqrt{g}h_{\alpha\gamma}\nabla_\mu\nabla_\gamma\left(R_{\nu\beta}R_{\mu\nu\alpha\beta}\right) - \int d^4x\sqrt{g}h\nabla_\beta\nabla_\nu\left(R_{\mu\alpha}R_{\mu\nu\alpha\beta}\right) - \\ &-\int d^4x\sqrt{g}h_{\nu\beta}\Box\left(R_{\mu\alpha}R_{\mu\nu\alpha\beta}\right) + \int d^4x\sqrt{g}h_{\beta\gamma}R_{\mu\alpha}R_{\nu\beta}R_{\mu\nu\alpha\gamma} - \\ &-\int d^4x\sqrt{g}h_{\mu\beta}\nabla_\alpha\nabla_\nu\left(R_{\mu\alpha}R_{\nu\beta}\right) + \int d^4x\sqrt{g}h_{\nu\beta}\nabla_\alpha\nabla_\mu\left(R_{\mu\alpha}R_{\nu\beta}\right). \end{aligned} \tag{5.5}$$

The next step is to do the simplification of the eq. (5.5) and use the relation (B.6) to commute the covariant derivatives for some terms such that the resulting terms look more simple than how it was. Then one can use Bianchi identities (B.9) and (B.11) wherever it is possible in the outcomes and once again simplify it to yield,

$$\begin{aligned} A &= \int d^4x\sqrt{g}\delta g_{\mu\nu}\left[2R_{\alpha\gamma}R_{\mu\beta}R_{\nu\gamma\alpha\beta} + 2R_{\mu\beta\alpha\gamma}\nabla_\nu\nabla_\alpha R_{\gamma\beta} - 2\nabla_\mu R_{\alpha\beta}\nabla_\nu R_{\alpha\beta} + \right. \\ &+2\nabla_\nu R_{\alpha\beta}\nabla_\beta R_{\mu\alpha} + 2\nabla_\alpha R_{\gamma\beta}\nabla_\nu R_{\mu\beta\alpha\gamma} - 2R_{\alpha\beta}\nabla_\mu\nabla_\nu R_{\alpha\beta} + 2R_{\alpha\beta}\nabla_\nu\nabla_\beta R_{\mu\alpha} - \\ &-2R_{\gamma\beta}R_{\alpha\mu\nu\eta}R_{\alpha\gamma\eta\beta} - g_{\mu\nu}R_{\alpha\beta\gamma\eta}\nabla_\beta\nabla_\eta R_{\gamma\alpha} - 2g_{\mu\nu}\nabla_\beta R_{\gamma\alpha}\nabla_\gamma R_{\alpha\beta} + 2g_{\mu\nu}\nabla_\gamma R_{\alpha\beta}\nabla_\gamma R_{\alpha\beta} - \\ &-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\nabla_\alpha\nabla_\beta R - g_{\mu\nu}R_{\alpha\beta}R_{\alpha\gamma}R_{\gamma\beta} - \frac{1}{2}g_{\mu\nu}R_{\alpha\gamma}R_{\beta\eta}R_{\alpha\beta\gamma\eta} + g_{\mu\nu}R_{\alpha\beta}\Box R_{\alpha\beta} - \\ &-R_{\mu\alpha\nu\gamma}\Box R_{\alpha\gamma} - 2\nabla_\gamma R_{\alpha\beta}\nabla_\gamma R_{\mu\alpha\nu\beta} - R_{\alpha\beta}\Box R_{\mu\alpha\nu\beta} - R_{\mu\alpha}\nabla_\alpha\nabla_\nu R - R_{\mu\gamma}R_{\nu\beta}R_{\gamma\beta} - \\ &\left.-\nabla_\beta R_{\mu\alpha}\nabla_\alpha R_{\nu\beta} - \frac{1}{4}\nabla_\mu R\nabla_\nu R + \frac{1}{2}R_{\mu\nu}\Box R + \nabla_\alpha R\nabla_\alpha R_{\mu\nu} + R_{\alpha\beta}\nabla_\alpha\nabla_\beta R_{\mu\nu}\right]. \end{aligned} \tag{5.6}$$

Then by using the action principle we equate A to zero so that the bracketed terms in (5.6) can be used to get the final answer. Next we change the dummy indices and rewrite the variation of the third term with respect to the metric $g_{\mu\nu}$ as $E_3^{\mu\nu} = \frac{\delta(\sqrt{g}R^{\alpha\gamma}R^{\beta\delta}R_{\alpha\beta\gamma\delta})}{\sqrt{g}\delta g_{\mu\nu}}$. Now, it is easy to see that the bracketed terms in (5.6) give the final EOM $E_{3\mu\nu}$ (see below) that we are interested in. We note that we write the final answer for all EOM in terms of Feynman's notation, because in between while doing the variation we have lowered all the indices such that the final equation turns out to be with all indices downstairs. This process of deriving EOM can be applied to any of the terms of gravitational actions (including a_0, a_2 and a_4) in general. We enumerate the results below for the contributions coming from all the 8 terms given in the action (4.16) to the final EOM .

Here in order to avoid any confusions with the indices of EOM we change the dummy indices of the terms given in the action (4.16), which involve μ and ν. Moreover, initially in the final answer of EOM we write the indices μ and ν upstairs just to emphasize that the variations have been done with respect to the covariant metric $g_{\mu\nu}$, and later on we lower the same to follow our convention. As we mentioned before, here the symmetrizations with respect to the indices μ and ν are assumed.

$$\begin{aligned}
E_1^{\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{1120}\frac{\delta\left(\sqrt{g}R\Box R\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{1\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{1120}\left[-2R_{\mu\nu}R_{;\alpha\alpha}+2R_{;\alpha\alpha\mu\nu}-2g_{\mu\nu}R_{;\alpha\alpha\beta\beta}+R_{;\mu}R_{;\nu}-\right. \\
&\left.-\frac{1}{2}g_{\mu\nu}R^2_{;\alpha}\right],
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
E_2^{\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{336}\frac{\delta\left(\sqrt{g}R^{\alpha\beta}\Box R_{\alpha\beta}\right)}{\sqrt{g}\delta g_{\mu\nu}} = \\
&= \frac{1}{4\pi^2}\frac{1}{336}\frac{\delta\left(\sqrt{g}g^{\gamma\delta}\nabla_\gamma R^{\alpha\beta}\nabla_\delta R_{\alpha\beta}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{2\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{336}\left[-\frac{5}{2}g_{\mu\nu}R^2_{\alpha\beta;\gamma}-R_{;\alpha\alpha\mu\nu}-3R_{\nu\alpha;\mu}R_{;\alpha}-R_{\mu\alpha}R_{;\alpha\nu}-2R_{\alpha\beta;\mu}R_{\beta\nu;\alpha}-\right. \\
&-6R_{\gamma\beta}R_{\mu\beta;\gamma\nu}+R_{\alpha\beta;\mu}R_{\alpha\beta;\nu}+2R_{\alpha\beta}R_{\alpha\beta;\mu\nu}+4R_{\beta\alpha\nu\gamma;\mu}R_{\beta\gamma;\alpha}-4R_{\nu\gamma\alpha\beta}R_{\beta\gamma;\alpha\mu}+ \\
&+2R_{\alpha\mu\nu\beta}R_{\alpha\beta;\gamma\gamma}+\frac{1}{2}g_{\mu\nu}R_{;\alpha\alpha\beta\beta}+\frac{1}{2}g_{\mu\nu}R^2_{;\alpha}+2g_{\mu\nu}R_{\alpha\beta}R_{;\alpha\beta}-2g_{\mu\nu}R_{\sigma\alpha\rho\beta}R_{\alpha\rho}R_{\beta\sigma}+ \\
&+2g_{\mu\nu}R_{\alpha\gamma}R_{\alpha\beta}R_{\gamma\beta}-g_{\mu\nu}R_{\alpha\beta}R_{\alpha\beta;\gamma\gamma}+4g_{\mu\nu}R_{\gamma\alpha;\beta}R_{\beta\gamma;\alpha}+2g_{\mu\nu}R_{\alpha\rho\sigma\beta}R_{\rho\beta;\alpha\sigma}+ \\
&\left.+R_{\mu\nu;\alpha\alpha\beta\beta}+2R_{\gamma\mu\nu\beta}R_{\gamma\rho}R_{\beta\rho}-4R_{\rho\mu\sigma\beta}R_{\rho\beta}R_{\nu\sigma}+2R_{\nu\alpha}R_{\mu\beta}R_{\alpha\beta}\right],
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
E_3^{\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{126}\frac{\delta\left(\sqrt{g}R^{\alpha\gamma}R^{\beta\delta}R_{\alpha\beta\gamma\delta}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{3\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{126}\left[2R_{\alpha\gamma}R_{\mu\beta}R_{\nu\gamma\alpha\beta}+2R_{\mu\beta\alpha\gamma}R_{\gamma\beta;\alpha\nu}-2R_{\alpha\beta;\mu}R_{\alpha\beta;\nu}+\right. \\
&+2R_{\alpha\beta;\nu}R_{\mu\alpha;\beta}+2R_{\gamma\beta;\alpha}R_{\mu\beta\alpha\gamma;\nu}-2R_{\alpha\beta}R_{\alpha\beta;\mu\nu}+2R_{\alpha\beta}R_{\mu\alpha;\beta\nu}- \\
&-2R_{\gamma\beta}R_{\alpha\mu\nu\eta}R_{\alpha\gamma\eta\beta}-g_{\mu\nu}R_{\alpha\beta\gamma\eta}R_{\gamma\alpha;\eta\beta}-2g_{\mu\nu}R_{\gamma\alpha;\beta}R_{\alpha\beta;\gamma}+2g_{\mu\nu}R^2_{\alpha\beta;\gamma}- \\
&-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R_{;\alpha\beta}-g_{\mu\nu}R_{\alpha\beta}R_{\alpha\gamma}R_{\gamma\beta}-\frac{1}{2}g_{\mu\nu}R_{\alpha\gamma}R_{\beta\eta}R_{\alpha\beta\gamma\eta}+g_{\mu\nu}R_{\alpha\beta}R_{\alpha\beta;\gamma\gamma}- \\
&-R_{\mu\alpha\nu\gamma}R_{\alpha\gamma;\beta\beta}-2R_{\alpha\beta;\gamma}R_{\mu\alpha\nu\beta;\gamma}-R_{\alpha\beta}R_{\mu\alpha\nu\beta;\gamma\gamma}-R_{\mu\alpha}R_{;\nu\alpha}-R_{\mu\gamma}R_{\nu\beta}R_{\gamma\beta}- \\
&\left.-R_{\mu\alpha;\beta}R_{\nu\beta;\alpha}-\frac{1}{4}R_{;\mu}R_{;\nu}+\frac{1}{2}R_{\mu\nu}R_{;\alpha\alpha}+R_{;\alpha}R_{\mu\nu;\alpha}+R_{\alpha\beta}R_{\mu\nu;\alpha\beta}\right],
\end{aligned} \tag{5.9}$$

$$
\begin{aligned}
E_4^{\mu\nu} &= \frac{1}{4\pi^2}\frac{43}{15120}\frac{\delta\left(\sqrt{g}R^{\alpha}{}_{\gamma}R^{\gamma}{}_{\beta}R^{\beta}{}_{\alpha}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{4\mu\nu} &= \frac{1}{4\pi^2}\frac{43}{15120}\Big[-3R_{\beta\mu;\alpha}R_{\alpha\beta;\nu}-\frac{3}{2}R_{\mu\alpha}R_{;\nu\alpha}+3R_{\mu\beta}R_{\alpha\eta}R_{\alpha\nu\beta\eta}- \\
&-\frac{3}{2}R_{;\alpha}R_{\alpha\mu;\nu}-3R_{\alpha\beta}R_{\mu\beta;\nu\alpha}+\frac{3}{2}g_{\mu\nu}R_{\alpha\beta}R_{;\alpha\beta}+\frac{3}{8}g_{\mu\nu}R^2_{;\alpha}+\frac{3}{2}g_{\mu\nu}R_{\alpha\gamma;\beta}R_{\gamma\beta;\alpha}- \\
&-\frac{3}{2}g_{\mu\nu}R_{\alpha\gamma}R_{\eta\beta}R_{\beta\alpha\gamma\eta}+g_{\mu\nu}R_{\alpha\gamma}R_{\gamma\beta}R_{\alpha\beta}+3R_{\gamma\nu}R_{\mu\gamma;\alpha\alpha}+3R_{\mu\beta;\gamma}R_{\beta\nu;\gamma}\Big], \qquad (5.10)
\end{aligned}
$$

$$
\begin{aligned}
E_5^{\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{1120}\frac{\delta\left(\sqrt{g}R^3\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{5\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{1120}\Big[-\frac{1}{2}g_{\mu\nu}R^3+3R^2R_{\mu\nu}-6R_{;\mu}R_{;\nu}-6RR_{;\mu\nu}+6g_{\mu\nu}R^2_{;\alpha}+ \\
&+6g_{\mu\nu}RR_{;\alpha\alpha}\Big], \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
E_6^{\mu\nu} &= \frac{1}{4\pi^2}\frac{13}{2016}\frac{\delta\left(\sqrt{g}RR^{\alpha\beta}R_{\alpha\beta}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{6\mu\nu} &= \frac{1}{4\pi^2}\frac{13}{2016}\Big[\frac{1}{2}g_{\mu\nu}RR^2_{\alpha\beta}-2g_{\mu\nu}R^2_{\alpha\beta;\gamma}-2g_{\mu\nu}R_{\alpha\beta}R_{\alpha\beta;\gamma\gamma}- \\
&-g_{\mu\nu}R^2_{;\alpha}-g_{\mu\nu}R_{\alpha\beta}R_{;\alpha\beta}-\frac{1}{2}g_{\mu\nu}RR_{;\alpha\alpha}-R_{\mu\nu}R^2_{\alpha\beta}+2R_{\alpha\beta;\mu}R_{\alpha\beta;\nu}+ \\
&+2R_{\alpha\beta}R_{\alpha\beta;\mu\nu}+R_{;\mu}R_{;\nu}+2R_{\mu\alpha}R_{;\nu\alpha}+2R_{;\alpha}R_{\alpha\mu;\nu}+RR_{;\mu\nu}- \\
&-2RR_{\alpha\beta}R_{\alpha\mu\nu\beta}-R_{\mu\nu}R_{;\alpha\alpha}-2R_{;\alpha}R_{\mu\nu;\alpha}-RR_{\mu\nu;\alpha\alpha}\Big], \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
E_7^{\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{5040}\frac{\delta\left(\sqrt{g}RR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{7\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{5040}\Big[-\frac{1}{2}g_{\mu\nu}RR^2_{\alpha\beta\rho\sigma}+2g_{\mu\nu}R^2_{\alpha\beta\rho\sigma;\gamma}+2g_{\mu\nu}R_{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma;\gamma\gamma}+ \\
&+R_{\mu\nu}R^2_{\alpha\beta\rho\sigma}-2R_{\alpha\beta\rho\sigma;\mu}R_{\alpha\beta\rho\sigma;\nu}-2R_{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma;\mu\nu}+2RR_{\mu\beta\rho\sigma}R_{\nu\beta\rho\sigma}+ \\
&+8R_{;\alpha}R_{\mu\nu;\alpha}-8R_{;\alpha}R_{\alpha\mu;\nu}+4R_{\alpha\mu\nu\sigma}R_{;\sigma\alpha}-2RR_{;\mu\nu}-4RR_{\mu\alpha}R_{\nu\alpha}+ \\
&+4RR_{\sigma\alpha}R_{\sigma\mu\nu\alpha}+4RR_{\mu\nu;\alpha\alpha}\Big], \qquad (5.13)
\end{aligned}
$$

and

$$
\begin{aligned}
E_8^{\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{15120}\frac{\delta\left(\sqrt{g}R_{\alpha\beta}{}^{\gamma\delta}R_{\eta\lambda}{}^{\alpha\beta}R_{\gamma\delta}{}^{\eta\lambda}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{8\mu\nu} &= \frac{1}{4\pi^2}\frac{1}{15120}\Big[-\frac{1}{2}g_{\mu\nu}R_{\gamma\eta\rho\alpha}R_{\gamma\eta\sigma\beta}R_{\rho\alpha\sigma\beta}+3R_{\mu\alpha\sigma\beta}R_{\nu\alpha\gamma\eta}R_{\sigma\beta\gamma\eta}- \\
&-12R_{\mu\sigma;\beta}R_{\nu\sigma;\beta}+12R_{\mu\sigma;\beta}R_{\nu\beta;\sigma}+24R_{\mu\alpha\sigma\beta}R_{\nu\sigma;\beta\alpha}-6R_{\mu\alpha\sigma\beta;\rho}R_{\nu\rho\sigma\beta;\alpha}- \\
&-6R_{\rho\gamma}R_{\mu\gamma\sigma\beta}R_{\nu\rho\sigma\beta}+6R_{\mu\gamma\rho\alpha}R_{\nu\rho\sigma\beta}R_{\alpha\gamma\sigma\beta}+12R_{\mu\alpha\gamma\beta}R_{\nu\rho\sigma\beta}R_{\alpha\rho\sigma\gamma}\Big]. \qquad (5.14)
\end{aligned}
$$

We see that (5.7) and (5.11) are completely symmetric with respect to μ and ν. One can easily verify all the EOM given above by checking the divergence of each EOM separately, which should turn out to be zero due to the gauge symmetry of the theory. In the process of checking divergence it may require to use cyclicity for the Riemann tensor. Now, after collecting all the above 8 parts of EOM and simplifying the outcomes one can get the final EOM as given below, which has 61 terms.

$$\begin{aligned}
E^{(6)}_{\mu\nu} &= E_{1\mu\nu}+E_{2\mu\nu}+E_{3\mu\nu}+E_{4\mu\nu}+E_{5\mu\nu}+E_{6\mu\nu}+E_{7\mu\nu}+E_{8\mu\nu} \\
\Rightarrow 4\pi^2 E^{(6)}_{\mu\nu} &= \frac{13}{15120}g_{\mu\nu}R_{\alpha\beta}R_{\alpha\gamma}R_{\beta\gamma}-\frac{13}{2016}R_{\mu\nu}R^2_{\alpha\beta}-\frac{1}{504}R_{\mu\alpha}R_{\nu\beta}R_{\alpha\beta}+ \\
&+\frac{13}{4032}g_{\mu\nu}R^2_{\alpha\beta}R-\frac{1}{1260}R_{\mu\alpha}R_{\nu\alpha}R+\frac{3}{1120}R_{\mu\nu}R^2-\frac{1}{2240}g_{\mu\nu}R^3+ \\
&+\frac{1}{160}g_{\mu\nu}R_{\alpha\beta}R_{\gamma\delta}R_{\alpha\gamma\beta\delta}+\frac{1}{5040}R_{\mu\nu}R^2_{\alpha\beta\gamma\delta}-\frac{1}{10080}g_{\mu\nu}RR^2_{\alpha\beta\gamma\delta}- \\
&-\frac{1}{30240}g_{\mu\nu}R_{\alpha\beta\eta\lambda}R_{\alpha\beta\gamma\delta}R_{\gamma\delta\eta\lambda}+\frac{61}{5040}RR_{\mu\alpha\nu\beta}R_{\alpha\beta}+\frac{1}{84}R_{\mu\beta\alpha\gamma}R_{\nu\alpha}R_{\beta\gamma}- \\
&-\frac{1}{168}R_{\mu\beta\nu\gamma}R_{\alpha\gamma}R_{\alpha\beta}+\frac{1}{63}R_{\mu\gamma\nu\delta}R_{\alpha\beta}R_{\alpha\gamma\beta\delta}+\frac{1}{2520}RR_{\mu\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}+ \\
&+\frac{1}{5040}R_{\mu\alpha\beta\gamma}R_{\nu\alpha\delta\eta}R_{\beta\gamma\delta\eta}-\frac{41}{1680}R_{\mu\alpha}R_{\nu\beta\alpha\gamma}R_{\beta\gamma}-\frac{1}{2520}R_{\mu\alpha\gamma\delta}R_{\nu\beta\gamma\delta}R_{\alpha\beta}- \\
&-\frac{1}{2520}R_{\mu\alpha\beta\gamma}R_{\nu\beta\delta\eta}R_{\alpha\gamma\delta\eta}-\frac{1}{1260}R_{\mu\alpha\beta\gamma}R_{\nu\delta\beta\eta}R_{\alpha\delta\gamma\eta}-\frac{19}{3360}RR_{\mu\nu;\alpha\alpha}- \\
&-\frac{43}{10080}R_{\mu\nu}R_{;\alpha\alpha}+\frac{43}{20160}g_{\mu\nu}RR_{;\alpha\alpha}-\frac{17}{5040}R_{\mu\nu;\alpha}R_{;\alpha}+\frac{41}{40320}g_{\mu\nu}R^2_{;\alpha}+ \\
&+\frac{1}{126}R_{\mu\nu;\alpha\beta}R_{\alpha\beta}-\frac{1}{5040}g_{\mu\nu}R_{\alpha\beta}R_{;\alpha\beta}+\frac{43}{5040}R_{\mu\alpha;\beta\beta}R_{\nu\alpha}+\frac{1}{336}R_{\mu\nu;\alpha\alpha\beta\beta}- \\
&-\frac{1}{3360}g_{\mu\nu}R_{;\alpha\alpha\beta\beta}-\frac{43}{5040}R_{\mu\alpha;\nu\beta}R_{\alpha\beta}-\frac{1}{140}R_{\mu\alpha;\beta}R_{\nu\beta;\alpha}+\frac{13}{1680}R_{\mu\alpha;\beta}R_{\nu\alpha;\beta}- \\
&-\frac{1}{1260}R_{\mu\alpha\nu\beta}R_{;\alpha\beta}-\frac{1}{126}g_{\mu\nu}R_{\alpha\beta}R_{\alpha\beta;\gamma\gamma}-\frac{1}{72}R_{\mu\alpha\nu\beta}R_{\alpha\beta;\gamma\gamma}-\frac{1}{126}R_{\mu\alpha\nu\beta;\gamma\gamma}R_{\alpha\beta}+ \\
&+\frac{1}{3360}g_{\mu\nu}R_{\alpha\gamma;\beta}R_{\alpha\beta;\gamma}-\frac{1}{224}g_{\mu\nu}R^2_{\alpha\beta;\gamma}-\frac{1}{63}R_{\mu\alpha\nu\beta;\gamma}R_{\alpha\beta;\gamma}+\frac{1}{630}R_{\mu\gamma\alpha\beta}R_{\nu\alpha;\beta\gamma}- \\
&-\frac{1}{2520}R_{\mu\alpha\beta\gamma;\delta}R_{\nu\delta\beta\gamma;\alpha}-\frac{1}{504}g_{\mu\nu}R_{\alpha\gamma\beta\delta}R_{\alpha\beta;\gamma\delta}+\frac{1}{2520}g_{\mu\nu}R_{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta;\eta\eta}+ \\
&+\frac{1}{2520}g_{\mu\nu}R^2_{\alpha\beta\gamma\delta;\eta}-\frac{1}{168}R_{\alpha\beta;\mu}R_{\nu\alpha;\beta}+\frac{1}{420}R_{\mu\alpha;\nu}R_{;\alpha}+\frac{1}{84}R_{\mu\alpha\beta\gamma;\nu}R_{\alpha\beta;\gamma}+ \\
&+\frac{1}{84}R_{\alpha\beta;\gamma\mu}R_{\nu\alpha\beta\gamma}+\frac{1}{336}R_{\alpha\beta;\mu\nu}R_{\alpha\beta}-\frac{1}{2520}R_{\alpha\beta\gamma\delta;\mu\nu}R_{\alpha\beta\gamma\delta}+\frac{37}{5040}R_{\mu\alpha;\beta}R_{\alpha\beta;\nu}- \\
&-\frac{43}{10080}R_{\mu\alpha;\nu}R_{;\alpha}-\frac{1}{2520}R_{\alpha\beta\gamma\delta;\mu}R_{\alpha\beta\gamma\delta;\nu}-\frac{1}{63}R_{\mu\alpha\beta\gamma;\nu}R_{\alpha\beta;\gamma}-\frac{23}{10080}R_{\mu\alpha}R_{;\nu\alpha}- \\
&-\frac{1}{504}R_{\mu\alpha;\beta\nu}R_{\alpha\beta}-\frac{1}{63}R_{\mu\alpha\beta\gamma}R_{\alpha\beta;\gamma\nu}+\frac{1}{1440}RR_{;\mu\nu}-\frac{1}{840}R_{;\alpha\alpha\mu\nu}. \qquad (5.15)
\end{aligned}$$

The above EOM (5.15) is very useful to study the black hole solutions due to its dominance in Riemann tensor. In particular, as we discussed at the beginning of this section, this form of the EOM reduces a lot of calculations for the Ricci flat solutions for example, Schwarzchild black hole solution, which is not the exact solution for the EOM (5.15) coming from a_6 (4.16). We shall discuss more about this in a moment.

5.2 Weyl Basis

In this section we vary the action (4.22) under the locally conformally flat background such that the terms containing C^n (Weyl) with the degree $n \geq 2$ will get vanished due to the conformal symmetry. We note that most of the terms in this action (4.22), which survive under the condition of the background being locally conformally flat, resemble the terms given in the action (4.16) for which we calculated the EOM in the previous section. The only term which is remained to be varied is the seventh term of the Weyl dominated action. We calculate the EOM for the same by following the usual procedure discussed in sect. 5.1 (see also the points regarding indices discussed right above the list of EOM for all the 8 terms, which are equally valid here) as follows,

$$\begin{aligned}
E_7^{\mu\nu} &= \frac{1}{4\pi^2}\frac{23}{5040}\frac{\delta\left(\sqrt{g}R^{\alpha\gamma}R^{\beta\delta}C_{\alpha\beta\gamma\delta}\right)}{\sqrt{g}\delta g_{\mu\nu}} \\
\Rightarrow E_{7\mu\nu} &= \frac{1}{4\pi^2}\frac{23}{5040}\Big[g_{\mu\nu}R_{\alpha\beta}R_{\alpha\gamma}R_{\beta\gamma} + \frac{4}{3}R_{\mu\nu}R^2_{\alpha\beta} - 4R_{\mu\alpha}R_{\nu\beta}R_{\alpha\beta} - \\
&\quad -\frac{13}{12}g_{\mu\nu}R^2_{\alpha\beta}R + 3R_{\mu\alpha}R_{\nu\alpha}R - R_{\mu\nu}R^2 + \frac{1}{4}g_{\mu\nu}R^3 - \frac{1}{2}RR_{\mu\nu;\alpha\alpha} + \\
&\quad +\frac{1}{12}g_{\mu\nu}RR_{;\alpha\alpha} - \frac{1}{24}g_{\mu\nu}R^2_{;\alpha} + R_{\mu\nu;\alpha\beta}R_{\alpha\beta} + R_{\mu\alpha}R_{\nu\alpha;\beta\beta} - \\
&\quad -R_{\mu\alpha;\nu\beta}R_{\alpha\beta} - R_{\mu\alpha;\beta}R_{\nu\beta;\alpha} + R_{\mu\alpha;\beta}R_{\nu\alpha;\beta} - \frac{1}{3}g_{\mu\nu}R_{\alpha\beta}R_{\alpha\beta;\gamma\gamma} + \\
&\quad +\frac{1}{2}g_{\mu\nu}R_{\alpha\gamma;\beta}R_{\alpha\beta;\gamma} - \frac{1}{3}g_{\mu\nu}R^2_{\alpha\beta;\gamma} - R_{\alpha\beta;\mu}R_{\nu\alpha;\beta} + \frac{1}{2}R_{\mu\alpha;\nu}R_{;\alpha} + \\
&\quad +\frac{1}{3}R_{\alpha\beta;\mu\nu}R_{\alpha\beta} + \frac{1}{3}R_{\alpha\beta;\mu}R_{\alpha\beta;\nu} - \frac{1}{12}R_{;\mu}R_{;\nu} - \frac{1}{2}R_{\mu\alpha}R_{\nu\alpha} + \frac{1}{6}RR_{;\mu\nu}\Big].
\end{aligned} \tag{5.16}$$

Here we note that in order to get the final answer given in (5.16), we replaced all the Riemann tensors with Weyl tensors in the final equation resulting from the variation of the term and used the condition (locally conformally flat background) on the outcomes such that all the Weyl tensors became zero and led us to the above result. Now, by combining the eq. (5.16) with the results of EOM for the terms, which survive under the condition considered above, i.e. (5.7), (5.9), (5.10) and (5.11) we finally get the following EOM with 32 terms for the Weyl dominated action.

$$
\begin{aligned}
E^{(6)}_{\mu\nu} &= E_{1\mu\nu}+E_{3\mu\nu}+E_{4\mu\nu}+E_{5\mu\nu}+E_{7\mu\nu} \\
\Rightarrow 4\pi^2 E^{(6)}_{\mu\nu} &= \frac{29}{4320}g_{\mu\nu}R_{\alpha\beta}R_{\alpha\gamma}R_{\beta\gamma}+\frac{71}{10080}R_{\mu\nu}R^2_{\alpha\beta}-\frac{1}{48}R_{\mu\alpha}R_{\nu\beta}R_{\alpha\beta}- \\
&-\frac{5}{756}g_{\mu\nu}R^2_{\alpha\beta}R+\frac{9}{560}R_{\mu\alpha}R_{\nu\alpha}R-\frac{163}{30240}R_{\mu\nu}R^2+\frac{397}{272160}g_{\mu\nu}R^3- \\
&-\frac{79}{30240}RR_{\mu\nu;\alpha\alpha}-\frac{1}{7560}R_{\mu\nu}R_{;\alpha\alpha}+\frac{11}{36288}g_{\mu\nu}RR_{;\alpha\alpha}-\frac{1}{1512}R_{\mu\nu;\alpha}R_{;\alpha}- \\
&-\frac{11}{181440}g_{\mu\nu}R^2_{;\alpha}+\frac{23}{5040}R_{\mu\nu;\alpha\beta}R_{\alpha\beta}+\frac{29}{30240}g_{\mu\nu}R_{\alpha\beta}R_{;\alpha\beta}+ \\
&+\frac{1}{140}R_{\mu\alpha}R_{\nu\alpha;\beta\beta}+\frac{1}{5040}g_{\mu\nu}R_{;\alpha\alpha\beta\beta}-\frac{1}{140}R_{\mu\alpha;\nu\beta}R_{\alpha\beta}- \\
&-\frac{23}{5040}R_{\mu\alpha;\beta}R_{\nu\beta;\alpha}+\frac{1}{140}R_{\mu\alpha;\beta}R_{\nu\alpha;\beta}-\frac{11}{5040}g_{\mu\nu}R_{\alpha\beta}R_{\alpha\beta;\gamma\gamma}+ \\
&+\frac{1}{280}g_{\mu\nu}R_{\alpha\beta;\gamma}R_{\alpha\gamma;\beta}-\frac{11}{5040}g_{\mu\nu}R^2_{\alpha\beta;\gamma}-\frac{23}{5040}R_{\alpha\beta;\mu}R_{\nu\alpha;\beta}+ \\
&+\frac{89}{30240}R_{\mu\alpha;\nu}R_{;\alpha}+\frac{11}{5040}R_{\alpha\beta;\mu\nu}R_{\alpha\beta}-\frac{13}{5040}R_{\mu\alpha;\beta}R_{\alpha\beta;\nu}+ \\
&+\frac{11}{5040}R_{\alpha\beta;\mu}R_{\alpha\beta;\nu}-\frac{13}{10080}R_{\mu\alpha;\nu}R_{;\alpha}-\frac{43}{181440}R_{;\mu}R_{;\nu}- \\
&-\frac{11}{3780}R_{\mu\alpha}R_{;\nu\alpha}+\frac{13}{12960}R_{;\mu\nu}R-\frac{1}{5040}R_{;\alpha\alpha\mu\nu}.
\end{aligned} \tag{5.17}
$$

We note that in the derivation of the eq. (5.17) the condition of background being locally conformally flat must be taken into account. Therefore, one should be cautious while adding contributions coming from the remaining terms (excluding the term considered in (5.16)) to the final EOM (5.17), where all the Riemann tensors must be replaced by Weyl tensor. As such all the Weyl tensors will get vanished and yield the final answer given in (5.17). We shall assess the EOM for the conformally flat FLRW metric by using the eq. (5.17) in the next section.

5.3 Analysis of the Equations of Motion

Here we study the Ricci flat EOM as a special case of (5.1) (and including EOM for a_0, a_2 and a_4). Furthermore, we also analyze the EOM for the full action for two ansatzs, namely the Schwarzschild, and conformally flat FLRW of the form of,

$$g = a^2(t)\left(-dt^2 + dr^2 + r^2 d\Upsilon^2\right), \tag{5.18}$$

where $d\Upsilon^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. Let us consider the Ricci flat form of the eq. (5.15), which can easily be obtained by setting Ricci tensor and scalar to zero in (5.15) and by using some manipulations and applying (B.13) as,

$$\begin{aligned} 4\pi^2 E_{\mu\nu} \;\; = \;\; & \Lambda^4 f_0 \frac{1}{2} g_{\mu\nu} + \Lambda^{-2} f_6 \left\{ -\frac{1}{30240} g_{\mu\nu} R_{\alpha\beta\eta\lambda} R_{\alpha\beta\gamma\delta} R_{\gamma\delta\eta\lambda} - \frac{1}{1260} R_{\mu\alpha\beta\gamma} R_{\nu\delta\beta\eta} R_{\alpha\delta\gamma\eta} - \right. \\ & \left. - \frac{1}{2520} R_{\mu\alpha\beta\gamma;\delta} R_{\nu\delta\beta\gamma;\alpha} + \frac{1}{5040} \nabla_\lambda \nabla_\eta \left[\left(g_{\mu\nu} g_{\lambda\eta} - \delta_{\nu\lambda} \delta_{\mu\eta} \right) R^2_{\alpha\beta\gamma\delta} \right] \right\}, \end{aligned} \tag{5.19}$$

where Λ and f_{2q} (for $q \geq 0$) are the relative coefficients coming from the spectral action (3.27) $\left(\text{with } \boldsymbol{L} = -\left(\frac{D}{\Lambda}\right)^2 = -\frac{\Delta}{\Lambda^2}\right)$. We note that the EOM (5.19) is completely symmetric with respect to μ and ν, as such it does not require the symmetrization with respect to the same. Moreover, it is the full form of EOM coming from the heat kernel coefficients a_0 and a_6. The contributions coming from the heat kernel coefficient a_2 (or corresponding EOM) are trivially zero $\left(\text{because } R_{\mu\nu} = R = 0\right)$. Furthermore, the EOM for a_4 (5.3) gets simplified and gives only two terms for the Ricci flat case, these remaining two terms upon plugging the Schwarzschild ansatz produce zero. These points explain the complete form of the Ricci flat EOM (5.19). It can be applied to the Schwarzschild metric and checked whether it is an exact solution for the EOM for a_6. For this purpose one may neglect the first term (set $f_0 = 0$) in (5.19) and plug the metric in the same, which yields the equation given below. Now, as we discussed in sect. 5.1 and above, by plugging the standard form of Schwarzschild metric in Ricci flat EOM (5.19) one can easily get (here we adopt the mixed indices to present the results in simplest way),

$$\begin{cases} E^t{}_t = & \Lambda^{-2} f_6 \frac{M^2(-298M+135r)}{630r^9}, \\ E^r{}_r = & \Lambda^{-2} f_6 \frac{M^2(14M-9r)}{90r^9}, \\ E^\theta{}_\theta = & E^\phi{}_\phi = \Lambda^{-2} f_6 \frac{M^2(-442M+189r)}{630r^9}. \end{cases} \tag{5.20}$$

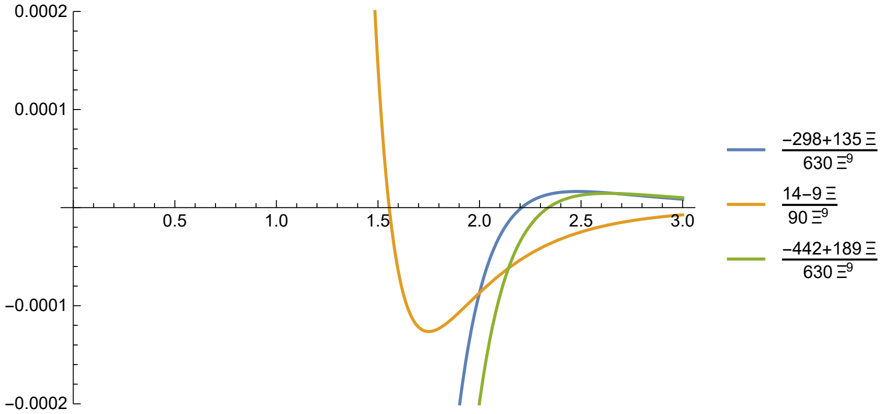

Figure 5.1: Graphs for the components of the EOM evaluated for Schwarzschild ansatz.

We use the components of EOM given in (5.20) and plot the graphs for three functions formulated with the reduced variable, namely $\Xi \equiv \frac{r}{M}$, which take the forms as shown in the Fig: 5.1. We also take into account zeros for the components considered in the above analysis, which are given by $\Xi = 2.2, 1.55$ and 2.34 for $E^t{}_t$, $E^r{}_r$ and $E^\theta{}_\theta = E^\phi{}_\phi$, respectively. We observe that energy density $E^t{}_t$ and azimuthal pressure $E^\theta{}_\theta = E^\phi{}_\phi$ are positive for significantly high radii. However, the radial pressure $E^r{}_r$ turns out to be negative for large radii as it changes its sign at $\Xi = 1.55$ (see Fig: 5.1). This is contrary to the ideal case, where all stress-energy sources should be positive. There lies the limitation of an effective source interpretation. In other words the effective source description cannot be applied inside the horizon (Schwarzschild radius), which is at $\Xi = 2$, where around this point the substantial changes occur as we explained above. This assessment and non-zero components of the EOM (5.20) imply that Schwarzschild metric is not the solution and it gets modified by higher derivative corrections.

Now, in a quite similar way as we did above, by using the conformally flat FLRW metric (5.18) either in (5.15) or (5.17) we find,

$$E^t{}_t = \frac{1}{2}\Lambda^4 f_0 - \frac{1}{4}\Lambda^2 f_2 \frac{a'^2}{a^4} + \Lambda^{-2} f_6 \left[\frac{a^{(3)2}}{560a^8} + \frac{a''^3}{63a^9} + \frac{137a'^6}{1008a^{12}} - \frac{a^{(4)}a''}{280a^8} + \right. \\ \left. + \frac{11a^{(3)}a'^3}{84a^{10}} - \frac{a^{(4)}a'^2}{35a^9} + \frac{a^{(5)}a'}{280a^8} - \frac{5a'^4 a''}{14a^{11}} + \frac{25a'^2 a''^2}{168a^{10}} - \frac{a^{(3)}a'a''}{21a^9} \right]. \tag{5.21}$$

and

$$E^r{}_r = E^\theta_\theta = E^\phi_\phi \;=\; \frac{1}{2}\Lambda^4 f_0 + \Lambda^2 f_2\left[-\frac{1}{6}\frac{a''}{a^3}+\frac{1}{12}\frac{a'^2}{a^4}\right]+\Lambda^{-2} f_6\left[-\frac{19a^{(3)2}}{1008a^8}+\frac{a^{(6)}}{840a^7}+ +\frac{17a''^3}{252a^9}-\frac{137a'^6}{336a^{12}}-\frac{73a^{(4)}a''}{2520a^8}-\frac{107a^{(3)}a'^3}{252a^{10}}+\frac{127a^{(4)}a'^2}{1260a^9}- -\frac{13a^{(5)}a'}{840a^8}+\frac{617a'^4a''}{504a^{11}}-\frac{415a'^2a''^2}{504a^{10}}+\frac{41a^{(3)}a'a''}{126a^9}\right]. \tag{5.22}$$

We recall the spectral expansion given in (3.27), i.e. $\mathrm{Tr}\chi(\boldsymbol{L})=\sum_{q=0}^{\infty} f_{2q}a_{2q}(\boldsymbol{L})$, where (as we saw in the sect. 3.1) χ, $\boldsymbol{L}$, f_{2q} and a_{2q} being an arbitrary function, positive definite operator, common factors and heat kernel coefficients, respectively. We reformulate the last expression for $\boldsymbol{L}=-\left(\frac{D}{\Lambda}\right)^2=-\frac{\Delta}{\Lambda^2}$ (note that Λ is some mass scale factor) and by expanding the outcomes rewrite it as $\sum_{q=0}^{\infty}\Lambda^{4-2q} f_{2q}a_{2q}=\left(\Lambda^4 f_0 a_0+\Lambda^2 f_2 a_2+\Lambda^0 f_4 a_4+\Lambda^{-2} f_6 a_6+\ldots\right)$. It means that the heat kernel coefficient a_0 should get multiplied by $\Lambda^4 f_0$, a_2 by $\Lambda^2 f_2$ and so on. This explains different powers of the mass scale factor Λ and appearance of the common factors f_{2q} in (5.19), (5.21) and (5.22). Moreover, we note that the EOM (5.3) coming from the heat kernel coefficient a_4 (4.3) does not contribute to the above EOM, because (5.3) produces zero for the conformally flat FLRW metric (5.18). Furthermore, recall also the other explanation based on the action (4.4), which we discussed at the beginning of this chapter. We saw that it is even possible to see at the level of action written in terms of Weyl and GB (4.4) that the EOM coming from a_4 does not contribute in the case of conformally flat solutions such as FLRW, because of (2.3) or (2.4) and Weyl tensor that is trivially zero on conformally flat background. This is the reason for absence of any of the terms with four derivatives of a scale factor $a(t)$ in (5.21) and (5.22).

At this point it is worthwhile to note the exact classical solutions studied in [37]. In their work the authors managed to find the exact solutions for the EOM coming from unitary and super-renormalizable non local theories of gravity for the general case. It was shown that metrics such as Schwarzschild, Kerr and AdS are the exact solutions of the EOM in Riemann basis. At the same time FLRW was also presented as an exact solution of the EOM written in Weyl basis.

Moreover, we would like to point out the work on infinite derivative gravity (being a subclass of general non-local theories), which turned out to be crucial, particularly to solve the problems of cosmological and black hole singularities. For example, the effects of higher derivative modifications on cosmology was considered in [52] and shown that the singularity can be averted by making the gravity weak at short distances. For this issue, it would be also nice to refer some historical work [49] on the isotropic cosmological systems without having singularity, also an anisotropic field of gravitation and the isotropization of the cosmological expansion concerning particle production analyzed in [50, 51]. Various generalization of ghost free theories of gravity were studied in [53] and the same for theories quadratic in curvature tensors along with the EOM (specifically a method to get it) was analyzed extensively in [54]. In further studies [55] it was proven that ghost free quadratic theories of gravity containing infinite derivatives do not allow the Schwarzschild ($1/r$) singularity to exist in the whole spacetime. The investigation done in [56] shows that ghost and singularity free theories of gravity (with infinite derivatives) do not have the Ricci flat solutions, because of the non-local gravitational interactions, which basically spoil the source, if it belongs to the spacetime. In addition to that in such theories the Riemann tensor turns out to be non trace-free contrary to the standard case, where it is traceless and does not coincide with the Weyl tensor. Such studies for the compact rotating object along with the ring singularity can be found in [57].

CHAPTER

6

CONCLUSION

We reviewed the spectral action approach and its applications that led us to HDG. More precisely, we considered the case of pure gravity and studied the HDG in the context of corrections to GR. We established a connection between the spectral action approach and HDG, and explained it in a great detail by deriving some results explicitely. We found that HDG emanates quite naturally in the framework of spectral action principle. We also reviewed the heat kernel coefficients a_0, a_2, a_4 and a_6 coming from asymptotic expansion of the trace of heat operator and calculated EOM for the respective actions. We studied the heat kernel coefficient a_6 with all of its classical features. More specifically, we constructed the action for a_6 in two bases, Riemann and Weyl, and calculated the EOM for the same by varying these actions.

There are two major applications of these results. One is to study the higher derivative corrections to Schwarzschild metric, which implies corrections to the Newton's law of gravitation in the weak-field limit. Basically we analyzed the Schwarzschild metric (5.20) for the EOM for a_6 (5.15) (Riemann dominated) or (5.19) (Ricci flat) and found peculiar behavior (Fig: 5.1) of non-zero components of EOM (5.20). These outcomes clearly indicates that Schwarzschild metric is not the solution for the theory based on a_6 and it gets modified by higher derivative corrections, so the classical law of gravitation as well. It would be quite intriguing to look for an exact solution for such an EOM with a big family of parameters (see e.g. [37] in general). As the second application, we applied the Weyl dominated EOM to the FLRW metric, which was calculated for the particular case of conformally flat background. We saw that FLRW metric (5.18) produces the same results (5.21) and (5.22) either we evaluate it by Riemann (5.15) or Weyl (5.17) dominated EOM.

On the other hand quantum aspects of these results may provide the answer to the question whether there is anything special about the spectral action. It particular, it would be quite interesting to see if the coefficients coming from the spectral action approach produce any kind of cancellations (in some cases it is possible to see even at classical level, see e.g. [1] for the case of $S_3 \times S_1$). One may try to find the beta functions for this model under consideration in the framework of quantum field theory and check the signs of beta functions that will explicitely depend on the coefficients coming from the spectral action and provide a clue whether the theory will get crumbled at high energy level or it will be renormalizable (see e.g. [7] for the case of four derivatives). In our investigation we found that the spectrum of the system studied in our work for a_6 possesses one pair of complex conjugate poles. These poles of the propagator are basically Lee-Wick pair so, we conclude that the theory based on a_6 falls under the category of Lee-Wick gravitational theory, for which some recent studies can be found in [45–48]. We may consider this aspect of our results in our future studies. It would be also nice to see whether the conformal background stays stable with quantum corrections.

The biggest breakthrough would be the direct quantization of the spectral action, but it is one of the most difficult problems and there has not been much progress in this direction. The main hurdle is to deal with the integral measure of the action and to find the correct form of the same. Moreover, it is also not clear whether the spectral action is directly quantizable in a way that usually works for the standard field theory. Some attempts to quantize the spectral action can be found in [35, 36].

APPENDIX

A

APPENDIX A: NOTATIONS AND CONVENTIONS

We work in a natural unit system i.e. $G = c = \hbar = k_B = 1$. We start with the Feynman's notation, which has been used in most of the formulae in this literature. Basically this notation means all the indices must be lowered, including summed over indices. For example, in this notation the standard Ricci tensor coming from the contraction of covariant Riemann tensor can be written as,

$$R_{ab} = g_{cd} R_{acbd}.$$

where the indices c and d are summed over.

The main goal to use this notation is to make the equations compact and easy to write. Furthermore, by doing this we also follow the same notations used by Gilkey [3, 4]. Just to give an example of the usefulness of this notation we quote some terms: $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma}$, $R^{\mu\nu}R_{\mu\nu} = R_{\mu\nu}R_{\mu\nu} = R^2_{\mu\nu}$ and so on. We also elucidate couple of terms explicitly to avoid misunderstanding with this notation: $\nabla^{\alpha}R^{\mu\nu}\nabla_{\alpha}R_{\mu\nu} = R_{\mu\nu;\alpha}R_{\mu\nu;\alpha} = \left(R_{\mu\nu;\alpha}\right)^2 = R^2_{\mu\nu;\alpha}$. Similarly, it is understood in the case of Riemann tensor as well. However, one needs to be careful while doing the variation of the action, where the summed over indices must be taken care. So, just for the purpose of doing variation we shall follow the Einstein summation convention i.e., one index should be upstairs and the other one downstairs for the summed over indices.

Throughout the literature s_{Ric} and s_{Riem} stand for the signs of Ricci and Riemann tensors respectively, which basically depend on the convention under consideration, and square brackets "[]" represent the anti-symmetrization of indices. The generalization of the formulae written in terms of s_{Ric} and s_{Riem} may help the reader to recollect the results and compare with the results coming from other conventions at any stage of calculations. As per our convention (same as [1]) $s_{\mathrm{Ric}} = -1$ and $s_{\mathrm{Riem}} = -1$. Moreover, the standard notation of the covariant derivative ";"(instead of ∇) has been used in most of the formulue to make it compact.

We define the commutation relation of gamma matrices as,

$$\gamma_{\mu\nu} := \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \tag{A.1}$$

Now, we note that the gamma matrices satisfy the following Clifford algebra,

$$\gamma_{\{\mu\nu\}} := \{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}\mathbb{1}. \tag{A.2}$$

where the sign convention is taken in such a way that it remains the same either in Euclidean or Minkowski formalism. In our convention Riemann tensor, Ricci tensor and Ricci scalar take the following forms,

$$\begin{cases} R_{\mu\nu\rho\sigma}V_\sigma & = s_{\mathrm{Riem}}\left[\nabla_\mu, \nabla_\nu\right]V_\rho, \\ R_{\mu\nu} & = s_{\mathrm{Ric}}\, g_{\rho\sigma}R_{\mu\rho\nu\sigma}, \\ R & = g_{\mu\nu}R_{\mu\nu}. \end{cases} \tag{A.3}$$

An endomorphism $\mathbb{E} \to E$ (for pure gravity) and tensor $\Omega_{\mu\nu}$, which have been used in the action a_6 for the case of pure gravity are given by (using (3.10) and (3.18)),

$$\begin{cases} E & = -\frac{R}{4}, \\ \Omega_{\mu\nu} & = \frac{1}{4}R_{\mu\nu ab}\gamma_{ab}. \end{cases} \tag{A.4}$$

APPENDIX B

APPENDIX B: USEFUL FORMULAE AND IDENTITIES

B.1 The Trace Identities of Gamma Matrices

The following set of trace identities of gamma matrices can easily be derived by using the eq. (A.2).

$$\begin{cases} \mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right) & = -g_{\mu\nu}\mathrm{Tr}(\mathbb{1}) = -4g_{\mu\nu}, \\ \mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) & = \mathrm{Tr}(\mathbb{1})\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right) = \\ & = 4\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right), \\ \mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_\beta\right) & = -\mathrm{Tr}(\mathbb{1})\left[g_{\mu\nu}\left(g_{\rho\sigma}g_{\alpha\beta} - g_{\rho\alpha}g_{\sigma\beta} + g_{\rho\beta}g_{\sigma\alpha}\right) - g_{\mu\rho}\left(g_{\nu\sigma}g_{\alpha\beta} - \right.\right. \\ & \left. -g_{\nu\alpha}g_{\sigma\beta} + g_{\nu\beta}g_{\sigma\alpha}\right) + g_{\mu\sigma}\left(g_{\nu\rho}g_{\alpha\beta} - g_{\nu\alpha}g_{\rho\beta} + g_{\nu\beta}g_{\rho\alpha}\right) - \\ & -g_{\mu\alpha}\left(g_{\nu\rho}g_{\sigma\beta} - g_{\nu\sigma}g_{\rho\beta} + g_{\nu\beta}g_{\rho\sigma}\right) + g_{\mu\beta}\left(g_{\nu\rho}g_{\sigma\alpha} - \right. \\ & \left.\left. -g_{\nu\sigma}g_{\rho\alpha} + g_{\nu\alpha}g_{\rho\sigma}\right)\right]. \end{cases} \tag{B.1}$$

We also present below the trace identities of the pair of gamma matrices, which can be deduced by using the results given in (B.1).

$$\begin{cases} \mathrm{Tr}\left(\gamma_{ab}\gamma_{cd}\right) & = -2\mathrm{Tr}(\mathbb{1})g_{a[c}g_{d]b}, \\ \mathrm{Tr}\left(\gamma_{ab}\gamma_{cd}\gamma_{ef}\right) & = -8\mathrm{Tr}(\mathbb{1})g_{[c[a}g_{b][e}g_{f]d]}. \end{cases} \tag{B.2}$$

B.2 Formulae for Curvature Tensors and Contracted Bianchi Identities

In the above set of euqations (B.2) γ_{ab} is defined by the eq. (A.1). The standard form of equation for the Weyl tensor in dimension d is given by,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{4s_{\text{Ric}}}{d-2} R_{[\mu[\rho} g_{\nu]\sigma]} + \frac{2s_{\text{Ric}}}{(d-2)(d-1)} g_{[\mu\rho} g_{\nu]\sigma} R. \tag{B.3}$$

We note that the co-efficients in the above equations (B.3) and (4.11) are very similar, provided that $s_{\text{Ric}} = 1$. With the given definition of Weyl tensor (B.3) the square of the same in dimension d takes the form as,

$$C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - \frac{4}{d-2} R^2_{\mu\nu} + \frac{2}{(d-2)(d-1)} R^2. \tag{B.4}$$

The Gauss-Bonnet term is given by [9],

$$\text{GB} := R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2. \tag{B.5}$$

Commutator of the covariant derivatives acting on generalized tensor can be written as,

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} &= \sum_{i=1}^{m} s_{\text{Riem}} R_{\mu\nu}{}^{\alpha_i}{}_{\sigma} T^{\alpha_1 \ldots \alpha_{i-1} \sigma \alpha_{i+1} \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} + \\ &\quad + \sum_{i=1}^{n} s_{\text{Riem}} R_{\mu\nu\beta_i}{}^{\sigma} T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_{i-1} \sigma \beta_{i+1} \ldots \beta_n}. \end{aligned} \tag{B.6}$$

Furthermore, commutator of the box operator and covariant derivative acting on an arbitrary tensor is given by,

$$[\Box, \nabla_\nu] = [\nabla_\mu \nabla_\mu, \nabla_\nu] = [\nabla_\mu, \nabla_\nu] \nabla_\mu + \nabla_\mu \left([\nabla_\mu, \nabla_\nu] \cdot \right). \tag{B.7}$$

The box operator acting on the product of two functions takes the form as,

$$\Box(fg) = \Box f g + 2\nabla_\mu f \nabla_\mu g + f \Box g. \tag{B.8}$$

Now, by doing the contraction of the first and third indices of Riemann tensor in the standard form of the 2^{nd} Bianchi identity or differential Bianchi identity we get the following contracted Bianchi identity,

$$R_{\mu\nu\rho\sigma;\mu} = s_{\text{Ric}} \left(R_{\nu\sigma;\rho} - R_{\nu\rho;\sigma} \right). \tag{B.9}$$

One can also contract the second and fourth indices of Riemann tensor to get,

$$R_{\mu\nu\rho\sigma;\nu} \;\;=\;\; s_{\text{Ric}}\left(R_{\mu\rho;\sigma} - R_{\mu\sigma;\rho}\right). \tag{B.10}$$

We note that equations (B.9) and (B.10) are the same.

It is quite easy to check this with the contraction for the other combinations. For example, if we write the 2^{nd} Bianchi identity as $R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\sigma\lambda;\rho} + R_{\mu\nu\lambda\rho;\sigma} = 0$ then one can check for the following pairs, $(\mu\sigma)$, $(\mu\lambda)$, $(\nu\rho)$, $(\nu\lambda)$. Naturally, the results of these combinations will coincide with the above results. Moreover, the remaining pairs $((\rho\lambda), (\rho\sigma), (\sigma\lambda))$ yield $0 = 0$.

Double contraction of the 2^{nd} Bianchi identity, that is the contraction of either equation (B.9) or (B.10) will lead us to,

$$R_{\mu\nu;\mu} \;\;=\;\; \frac{1}{2} R_{;\nu}. \tag{B.11}$$

B.3 Contracted Cubic Riemann Identities

The set of identities which we shall see in a moment below can easily be derived by using the equation $\delta^{pqrst}_{abcde} R^{ab}{}_{pq} R^{cd}{}_{rs} R^{e}{}_{t} = 0$ studied in [13] and doing some trivial manipulations. However, here we explain the derivation of the first identity explicitly that might be helpful to the reader. The remaining identities can be derived in a quite similar way.

Let us consider $R^{[ab}{}_{cd} R^{ef}{}_{ab} R^{c]d}{}_{ef}$ and rename the indices as $a \to 1$, $b \to 2$, $e \to 3$, $f \to 4$, $c \to 5$, where the square bracket "[]" emphasizes the anti-symmetrization of all five indices. This renaming will help us to do the permutations, which result into 120 terms. Initially these terms can be reduced upto 30 terms by using the anit-symmetrization of all five indices under the consideration then, one may use symmetries of the Riemann tensors and reduce it further up to 11 terms as follows,

$$
\begin{aligned}
&4\,(1,2,3,4,5)-8\,(1,2,3,5,4)-16\,(1,3,2,4,5)+16\,(1,3,2,5,4)-\\
&-16\,(1,3,4,5,2)-16\,(1,5,2,3,4)-8\,(1,5,3,4,2)+4\,(3,4,1,2,5)-\\
&-8\,(3,4,1,5,2)-8\,(3,5,1,2,4)+16\,(3,5,1,4,2).
\end{aligned}
$$

The above set of permutations can be applied to the upper indices of the product of three Riemann tensors mentioned above so that the simplification of the resulting equation leads us to,

$$
\begin{aligned}
R^{[ab}{}_{cd}R^{ef}{}_{ab}R^{c]d}{}_{ef} &= 4R_{abcd}R_{efab}R_{cdef}-16R_{aecd}R_{fcab}R_{bdef}+40s_{\mathrm{Ric}}R_{de}R_{abdc}R_{ceab}+\\
&+32R_{ca}R_{de}R_{aecd}+16s_{\mathrm{Ric}}R_{ad}R_{ea}R_{de}+4s_{\mathrm{Ric}}RR_{efcd}R_{cdef}-\\
&-8s_{\mathrm{Ric}}RR_{ed}R_{de}=0. \qquad \text{(B.12)}
\end{aligned}
$$

We note that on the left hand side of the above equation we have used the Einstein summation convention just for the purpose of doing permutation and the final answer is equation in the Feynman's notations. Moreover, the right hand side of the above equation turns out to be zero, because the identity has been derived in 5 dimensions so it has to be zero in 4 dimensions that we are interested in. Similarly, one can derive the remaining identities.

Whereas, the first identity requires some more calculations to make it useful for our purpose. In order to proceed further we make the list of five invariants of the product of three Riemann tensors as follows,

$$
\begin{cases}
I_1 = R_{abcd}R_{abef}R_{cdef},\\
I_2 = R_{abcd}R_{abef}R_{cedf},\\
I_3 = R_{abcd}R_{acef}R_{bedf},\\
I_4 = R_{abcd}R_{aecf}R_{bedf},\\
I_5 = R_{abcd}R_{aecf}R_{bfde}.
\end{cases}
$$

Let us consider the term $-16R_{aecd}R_{fcab}R_{bdef}$. By using the above relations one can derive the following set of equations,

$$\begin{cases} 4I_1 = 16\left(I_4 - R_{aecd}R_{fcab}R_{bdef}\right), \\ I_4 = R_{abcd}R_{aecf}R_{bdef} + I_5, \\ I_3 = I_4 - I_5, \\ I_2 = \frac{I_1}{2}, \\ I_2 = 2I_3. \end{cases}$$

Therefore, we can write $-16R_{aecd}R_{fcab}R_{bdef} = -16R_{abcd}R_{aecf}R_{bedf} + 4R_{abcd}R_{efab}R_{cdef}$ and use in the preliminary result of the first identity (B.12). After doing the simplification of the resulting expression it takes the form as it is given in the following equation. All the four identities are enumerated below.

$$\begin{cases} R^{[ab}{}_{cd}R^{ef}{}_{ab}R^{c]d}{}_{ef} & = 8R_{abcd}R_{efab}R_{cdef} - 16R_{abcd}R_{aecf}R_{bedf} + \\ & +40s_{\text{Ric}}R_{de}R_{abdc}R_{ceab} + 32R_{ca}R_{de}R_{aecd} + \\ & +16s_{\text{Ric}}R_{ad}R_{ea}R_{de} + 4s_{\text{Ric}}RR^2_{cdef} - 8s_{\text{Ric}}RR^2_{de} = 0, \\ R^{[ab}{}_{ce}R^{cd}{}_{ab}R_d{}^{e]} & = 16R_{de}R_{abce}R_{cdab} - 4RR^2_{abce} - 32R_{ac}R_{ba}R_{bc} + \\ & +32RR^2_{ae} + 32s_{\text{Ric}}R_{ac}R_{db}R_{dcab} - 4R^3 = 0, \\ C^{[ab}{}_{ce}C^{cd}{}_{ab}R_d{}^{e]} & = 16R_{dc}C_{abec}C_{edab} - 4RC^2_{abce} = 0, \\ C^{[ab}{}_{cd}C^{ef}{}_{ab}C^{c]d}{}_{ef} & = 8C_{abcd}C_{efab}C_{cdef} - 16C_{abcd}C_{aecf}C_{bedf} = 0. \end{cases} \tag{B.13}$$

APPENDIX C

Appendix C: A List of Variations

The following list of the variations is not complete. However, we provide with the necessary results which might be useful for the derivations which have been done in this dissertation.

$$g^{\mu\nu}h_{\mu\nu} \equiv h.$$

$$\begin{cases} \delta g_{\mu\nu} & = h_{\mu\nu}, \\ \delta g^{\mu}{}_{\nu} & = \delta g^{\mu}_{\mu} = \delta\delta^{\mu}{}_{\nu} = 0, \\ \delta g^{\mu\nu} & = -h^{\mu\nu}, \\ \delta \det g^{\mu\nu} & = -(\det g)^{-1}h, \\ \delta|\det g_{\mu\nu}| & = |\det g|h, \\ \delta\sqrt{|\det g_{\mu\nu}|} & = \frac{1}{2}\sqrt{|\det g|}h. \end{cases} \tag{C.1}$$

$$\delta\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(\nabla_{\rho}h_{\nu}{}^{\mu} + \nabla_{\nu}h^{\mu}{}_{\rho} - \nabla^{\mu}h_{\nu\rho}\right). \tag{C.2}$$

$$\delta R_{\mu\nu\rho}{}^{\sigma} = -R_{\mu\nu(\rho\alpha}h^{\sigma)\alpha} - 2s_{\text{Riem}}\nabla_{[\mu}\nabla_{[\rho}h_{\nu]}{}^{\sigma]}. \tag{C.3}$$

$$\delta R_{\mu\nu\rho\sigma} = R_{\mu\nu[\rho}{}^{\alpha} h_{\sigma]\alpha} + 2s_{\text{Riem}} \nabla_{[\mu} \nabla_{[\sigma} h_{\nu]\rho]}. \tag{C.4}$$

$$\delta R_{\mu\nu}{}^{\rho\sigma} = h^{[\rho}{}_{\beta} R_{\mu\nu}{}^{\sigma]\beta} + 2s_{\text{Riem}} \nabla_{[\mu} \nabla^{[\sigma} h_{\nu]}{}^{\rho]}. \tag{C.5}$$

$$\delta R^{\mu\nu\rho\sigma} = 2R^{[\nu\alpha\rho\sigma} h^{\mu]}{}_{\alpha} + R^{\mu\nu[\sigma\tau} h^{\rho]}{}_{\tau} + 2s_{\text{Riem}} \nabla^{[\mu} \nabla^{[\sigma} h^{\nu]\rho]}. \tag{C.6}$$

$$\delta R_{\mu\nu} = s_{\text{Ric}} s_{\text{Riem}} \left(\nabla^{\sigma} \nabla_{(\mu} h_{\nu)\sigma} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} h - \frac{1}{2} \Box h_{\mu\nu} \right). \tag{C.7}$$

$$\delta R^{\mu\nu} = -2h^{(\mu\alpha} R^{\nu)}{}_{\alpha} + s_{\text{Ric}} s_{\text{Riem}} \left(\nabla^{\sigma} \nabla^{\mu} h^{\nu}{}_{\sigma} - \frac{1}{2} \nabla^{\mu} \nabla^{\nu} h - \frac{1}{2} \Box h^{\mu\nu} \right). \tag{C.8}$$

$$\delta R = -R_{\mu\nu} h^{\mu\nu} + s_{\text{Ric}} s_{\text{Riem}} \left(\nabla^{\sigma} \nabla^{\nu} h_{\nu\sigma} - \Box h \right). \tag{C.9}$$

$$\begin{aligned} \delta C_{\mu\nu\rho\sigma} = \; & C_{\mu\nu[\rho\alpha} h_{\sigma]}{}^{\alpha} - \frac{2s_{\text{Ric}}}{d-2} R_{[\mu\alpha} g_{\nu][\rho} h_{\sigma]}{}^{\alpha} + 2s_{\text{Riem}} \nabla_{[\mu} \nabla_{[\sigma} h_{\nu]\rho]} - \\ & - \frac{4s_{\text{Riem}}}{d-2} g_{[\nu[\sigma} \left(\nabla^{\alpha} \nabla_{(\mu]} h_{\rho])\alpha} - \frac{1}{2} \nabla_{\mu]} \nabla_{\rho]} h - \frac{1}{2} \Box h_{\mu]\rho]} \right) - \\ & - \frac{2s_{\text{Ric}}}{d-2} R_{[\mu[\rho} h_{\nu]\sigma]} + \frac{2s_{\text{Ric}}}{(d-2)(d-1)} R g_{[\mu[\rho} h_{\nu]\sigma]} + \\ & + \frac{2s_{\text{Ric}}}{(d-2)(d-1)} g_{[\mu\rho} g_{\nu]\sigma} \left(-R_{\alpha\beta} h^{\alpha\beta} + s_{\text{Ric}} s_{\text{Riem}} \left(\nabla^{\alpha} \nabla^{\beta} h_{\beta\alpha} - \Box h \right) \right). \end{aligned} \tag{C.10}$$

$$\begin{aligned} \delta \nabla_{\mu} T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_m} = \; & \nabla_{\mu} \delta T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_m} + \\ & + \frac{1}{2} \sum_{i=1}^{m} (\nabla_{\rho} h_{\mu}{}^{\alpha_i} + \nabla_{\mu} h^{\alpha_i}{}_{\rho} - \nabla^{\alpha_i} h_{\mu\rho}) T^{\alpha_1 \ldots \alpha_{i-1} \rho \alpha_{i+1} \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} - \\ & - \frac{1}{2} \sum_{i=1}^{n} (\nabla_{\beta_i} h_{\mu}{}^{\rho} + \nabla_{\mu} h^{\rho}{}_{\beta_i} - \nabla^{\rho} h_{\mu\beta_i}) T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_{i-1} \rho \beta_{i+1} \ldots \beta_n}. \end{aligned} \tag{C.11}$$

APPENDIX

D

Appendix D: Calculating the Trace of the Action

Here we present explicit calculations for the following term of eq. (4.5) and the remaining terms can be calculated in a quite similar way. We note that the trace operator does not affect the Riemann tensor, because the trace is taken over the different space.

$$
\begin{aligned}
\mathrm{Tr}\left(\Omega_{ij;k}\Omega_{ij;k}\right) &= \frac{1}{(4)^2}\mathrm{Tr}\left((R_{ijab}\gamma_{ab})_{;k}(R_{ijcd}\gamma_{cd})_{;k}\right), \\
&= \frac{1}{16}\mathrm{Tr}\left(\gamma_{ab}\gamma_{cd}R_{ijab;k}R_{ijcd;k}\right), \\
&= \frac{1}{16}\mathrm{Tr}\left(\gamma_{ab}\gamma_{cd}\right)R_{ijab;k}R_{ijcd;k}, \\
&= \frac{1}{16}\left(-2\mathrm{Tr}(\mathbb{1})g_{a[c}g_{d]b}\right)R_{ijab;k}R_{ijcd;k}, \\
&= -\frac{2\mathrm{Tr}(\mathbb{1})}{16}R_{ijab;k}R_{ijab;k}, \\
&= -\frac{\mathrm{Tr}(\mathbb{1})}{8}R^2_{ijab;k}.
\end{aligned}
$$

In the above calculations we applied the trace operator on the left hand side of the equation and used the definition of Ω_{ij} given in the eq. (A.4). Moreover, we also used the eq. (B.2), and further steps involved just a simplification of the resulting equation which led us to the final answer given in the above calculations. Similarly, one can calculate the following terms, where it may require to use the eq. (B.9) in order to simplify some terms. Possible contraction of the indices must be taken care in the following results, where the eq. (A.3) might be useful.

$$\mathrm{Tr}\left(\Omega_{ij;j}\Omega_{ik;k}\right) = -\frac{\mathrm{Tr}(\mathbb{1})}{4}\left(R_{ia;b}R_{ia;b} - R_{ia;b}R_{ib;a}\right).$$

$$\mathrm{Tr}\left(\Omega_{ij}\Omega_{ij;kk}\right) = -\frac{\mathrm{Tr}(\mathbb{1})}{8}R_{ijab}R_{ijab;kk}.$$

$$\mathrm{Tr}\left(\Omega_{ij}\Omega_{jk}\Omega_{ki}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{8}R_{\mu\nu ab}R_{\nu\rho bc}R_{\rho\mu ac}.$$

$$R_{ijkl}\mathrm{Tr}\left(\Omega_{ij}\Omega_{kl}\right) = -\frac{\mathrm{Tr}(\mathbb{1})}{8}R_{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma ab}.$$

$$R_{ijik}\mathrm{Tr}\left(\Omega_{jl}\Omega_{kl}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{8}R_{\mu\nu}R_{\mu\rho ab}R_{\nu\rho ab}.$$

$$R_{ijij}\mathrm{Tr}\left(\Omega_{kl}^2\right) = \frac{\mathrm{Tr}(\mathbb{1})}{8}RR_{\mu\nu ab}^2.$$

$$\mathrm{Tr}\left(E_{;iijj}\right) = -\frac{\mathrm{Tr}(\mathbb{1})}{4}R_{;iijj}.$$

$$\mathrm{Tr}\left(EE_{;ii}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{16}RR_{;ii}.$$

$$\mathrm{Tr}\left(E_{;i}E_{;i}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{16}R_{;i}^2.$$

$$\mathrm{Tr}\left(E^3\right) = -\frac{\mathrm{Tr}(\mathbb{1})}{64}R^3.$$

$$\mathrm{Tr}\left(E\Omega_{ij}^2\right) = \frac{\mathrm{Tr}(\mathbb{1})}{32}RR_{\mu\nu ab}^2.$$

$$R_{ijij}\mathrm{Tr}\left(E_{;kk}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{4}RR_{;ii}.$$

$$R_{ijik}\mathrm{Tr}\left(E_{;jk}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{4} R_{jk} R_{;jk}.$$

$$R_{ijij;k}\mathrm{Tr}\left(E_{;k}\right) = \frac{\mathrm{Tr}(\mathbb{1})}{4} R_{;i}^2.$$

$$R_{ijij} E^2 \mathrm{Tr}(\mathbb{1}) \quad = \quad -\frac{\mathrm{Tr}(\mathbb{1})}{16} R^3.$$

$$R_{ijij;kk}\mathrm{Tr}(E) = \frac{\mathrm{Tr}(\mathbb{1})}{4} R R_{;ii}.$$

$$R_{ijij} R_{klkl} E \mathrm{Tr}(\mathbb{1}) \quad = \quad -\frac{\mathrm{Tr}(\mathbb{1})}{4} R^3.$$

$$R_{ijik} R_{ljlk} E \mathrm{Tr}(\mathbb{1}) \quad = \quad -\frac{\mathrm{Tr}(\mathbb{1})}{4} R R_{\mu\nu}^2.$$

$$R_{ijkl}^2 E \mathrm{Tr}(\mathbb{1}) \quad = \quad \frac{\mathrm{Tr}(\mathbb{1})}{4} R R_{\mu\nu\rho\sigma}^2.$$

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