Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Obtaining and breaking uniqueness of positive solutions to strong singular problems

por

Lais Moreira dos Santos

Brasília

2018

Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Obtaining and breaking uniqueness of positive solutions to strong singular problems

por

Lais Moreira dos Santos †

sob orientação do

Prof. Dr. Carlos Alberto Pereira dos Santos

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos necessários para obtenção do título de Doutora em Matemática.

† A autora contou com apoio financeiro CAPES durante a realização deste trabalho.

Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Obtaining and breaking uniqueness of positive solutions to strong singular problems

por

Lais Moreira dos Santos †

Tese de Doutorado submetida ao Programa de Pós-Graduação em Matemática da Universidade de Brasília, PPGMat – UnB, como parte dos requisitos necessários para obtenção do título de

Doutora em Matemática.

Área de Concentração: Análise

Examinada e aprovada por:

Prof. Dr. Carlos Alberto Pereira dos Santos (MAT – UnB)

Orientador

Prof. Dr. Grey Ercole (MAT – UFMG)

Prof. Dr. Marcos Leandro Mendes Carvalho (IME – UFG)

Prof. Dr. João Marcos Bezerra do Ó (MAt – UnB)

Prof. Dr. Jiazheng Zhou (MAT – UnB)

Brasília, 24 de agosto de 2018

†A autora contou com apoio financeiro CAPES durante a realização deste trabalho.

Agradecimentos

Aos meus pais Vera e Jairo, meus irmãos Danilo e Murilo, aos meus avós Therezinha e Claudomicio, dedico esse trabalho e luta. Em vocês encontro o meu porto seguro, de onde parto, mas sempre volto a ancorar. Obrigada por toda confiaça, amor e sacrifício. Amo vocês!

Ao meu orientador, Prof. Carlos Alberto dos Santos, agradeço por todos esses anos de orientação, desde o mestrado até aqui, sempre me guiando com dedicação e sabedoria.

Aos professores Grey Ercole, Marcos Leandro Carvalho, João Marcos do Ó e Jiazheng Zhou, agradeço por aceitarem compor minha banca de defesa e pelas valiosas sugestões dadas.

Aos professores e funcionários do Departamento de Matemática da UnB, toda minha gratidão por terem contribuído com minha formação e por todo apoio prestado.

Aos meus grandes e queridos amigos Eliana, Wildes, Éder, Iran, Oscar e Camila, que me acompanham desde o ínicio dessa jornada, me sinto honrada por ter sempre vocês ao meu lado. Tantas aventuras e presepadas tornaram esse período memorável. De toda alegria que sinto por estar concluindo mais essa fase, também tenho o pesar de não ter mais vocês em meu cotidiano. Agradeço a vocês por cada um dos momentos especiais que vivemos.

Ao Bhavin, meu companheiro da vida, que veio de tão longe para me iluminar. A você todo amor e gratidão.

Meus sinceros agradecimentos a todas as pessoas incríveis que passaram de maneira tão tocante em minha vida, em especial Alan, Ari, Keidna, Teló, Daiane, Ju, Pawan, Hiuri, Martino, Lucimeire, Yerko, Martinha, Solange, Cacau, Deusinha, Claudinha, Bruno, Dióscoros, Cid, Amanda, Valter, Elaine, Ricardo, Thiago, John, Genildo, Natalia, Karla, Gustavo, Nara, Marcelo e Jaqueline.

À CAPES pelo apoio financeiro.

Resumo

Neste trabalho, estudamos unicidade, multiplicidade e também existência de *continuum* de soluções positivas, no sentido *loc*, para problemas elípticos quasilineares em domínios limitados do $\mathbb{R}^N$ $(N \geqslant 2)$, envolvendo operadores tanto homogêneos quanto não-homogêneos, perturbados por um termo de reação fortemente singular em ambos os casos local e não-local.

A partir de informações sobre existência e unicidade de soluções positivas para problemas locais singulares, nós mostramos como quebrar essa unicidade, seja por introduzir um termo não-local ou por considerar perturbações apropriadas deste problema singular. Nossa abordagem é baseada em técnicas de bifurcação, princípio de comparação para sub e supersoluções no sentido *loc* e Teorema do Passo da Montanha para funcionais de Szulkin.

Palavras-chave: Não-linearidades fortemente singulares, Princípio de comparação para $W^{1,p}_{\text{loc}}(\Omega)$-sub e supersoluções, Unicidade, Problemas não-locais, *Continuum* de soluções, Funcionais de Szulkin.

Abstract

In this work, we study uniqueness, multiplicity and also existence of *continuum* of positive solutions in *loc*-sense both for quasilinear elliptic problems on bounded domains in $\mathbb{R}^N$ $(N \geqslant 2)$ involving homogeneous operators and non-homogeneous ones perturbed by strongly-singular reaction terms both for local and non-local cases.

From information about existence and uniqueness of positive solutions for local singular problems, we show how to break this uniqueness by either introducing non-local terms or considering appropriate perturbations of this singular problem. Our approach is based on bifurcation techniques, comparison principle for sub and super-solutions in *loc*-sense and Mountain Pass Theorem for Szulkin functionals.

Keywords: Strongly-singular nonlinearities, Comparison principle for $W^{1,p}_{\mathrm{loc}}(\Omega)$-sub and supersolutions, Uniqueness, Non-local problems, *Continuum* of solutions, Szulkin functionals.

CONTENTS

INTRODUCTION

In this thesis, we present a study on the issues related to non-existence, existence and multiplicity of positive solutions to the following class of problems

$$-A\Big(x,\int_\Omega g(x,u,\nabla u)dx\Big)\mathcal{L}u = f_{\lambda,\mu}(x,u) \text{ in } \Omega,\ u>0 \text{ in } \Omega \text{ and } u=0 \text{ on } \partial\Omega, \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\mathcal{L}$ is a quasilinear operator and $t \mapsto f_{\lambda,\mu}(x,t)$ may have singular behavior at $t=0$. We are mainly interested in the case when $f_{\lambda,\mu}$ is strongly singular at $t=0$.

The class (1) includes, in particular, the problem

$$-\Delta_p u = a(x)u^{-\delta} + b(x)u^{\beta} \text{ in } \Omega,\ u>0 \text{ in } \ \Omega, \quad u=0 \text{ on } \partial\Omega \qquad (2)$$

with $\beta \in \mathbb{R}$ and $\delta > 0$.

Although (2) has been much studied in recent years, up to now there are no results in literature about existence and uniqueness of $W^{1,p}_{\text{loc}}(\Omega)$- solutions in the case where δ can assume any positive value, $0 < \beta < p-1$ and $a, b \not\equiv 0$. Some attempts have been done in recent years for particular cases. For example, in 2016 Canino, Sciunzi and Trombetta [14] proved that, when a and Ω satisfy suitable conditions and $b \equiv 0$, (2) admits a unique $W^{1,p}_{\text{loc}}(\Omega)$-solution.

In this work, in addition to establish a uniqueness result for (2), we show how this uniqueness can be broken, either by introducing non-local terms or by considering appropriate perturbations of the singular term. According to the specificities of A and $f_{\lambda,\mu}$, sub-supersolution, bifurcation and non-smooth analysis techniques were employed. Next, we present precisely what was developed in each chapter.

In Chapter 1, we study in detail the following problem (which encompasses (2) by taking $\alpha = 1$)

$$(L_\alpha) \qquad \begin{cases} -\Delta_p u = \alpha\Big(a(x)u^{-\delta} + b(x)u^{\beta}\Big) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

with respect to existence and uniqueness of solutions in $W^{1,p}_{\mathrm{loc}}(\Omega)$ sense. In this direction, by using domain perturbation techniques and penalty arguments, we refine the proofs of existence of solutions found in [8], [14] and [46] to include both more general potentials a, b and a bigger range of p values. The more delicate issue is the uniqueness of solutions in $W^{1,p}_{loc}(\Omega)$ for the problem (L_α). The main results in [13] and [14] treated about this. In [13], by exploring the linearity of the Laplacian operator, the authors showed uniqueness of solutions to (L_α) with $p = 2$, $b = 0$ and $a \in L^1(\Omega)$, while in [14] the problem (L_α) with $b = 0$ was treated with some restrictions either on the potential a or on the geometry of the domain.

In what follows, we present the result obtained by us in this direction. Despite the next result being so classical, it is new even for the Laplacian operator both in generality of the potentials a and b and principally by the uniqueness of solution in the $W^{1,p}_{loc}(\Omega)$ setting for very singular nonlinearities perturbed by $(p-1)$-sublinear ones.

After the remarkable paper of Mckenna [40], in 1991, we know that a solution of the problem (L_α), with $a = 1$, $b = 0$ and $p = 2$, still lies in $H^1_0(\Omega)$ if, and only if, $0 < \delta < 3$. Thus, for stronger singularities, we need a more general concept of zero-boundary condition and solution. Therefore, before stating our first result, let us clarify what we mean by the Dirichlet boundary condition and solution for (L_α).

Definition 0.0.1 *We say that $u \leqslant 0$ on $\partial\Omega$ if $(u-\epsilon)^+ \in W_0^{1,p}(\Omega)$ for every $\epsilon > 0$ given. Furthermore, $u \geqslant 0$ if $-u \leqslant 0$ and $u = 0$ on $\partial\Omega$ if u is non-negative and non-positive on $\partial\Omega$.*

Next, we give a notion of $W^{1,p}_{\text{loc}}(\Omega)$- solution for the problem (L_α).

Definition 0.0.2 *We say u is a $W^{1,p}_{loc}(\Omega)$-solution for (L_α) if $u > 0$ in Ω (that is, for each $\Theta \subset\subset \Omega$ given there exists a positive constant c_Θ such that $u \geqslant c_\Theta > 0$ in Θ) and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \alpha \int_\Omega \Big(a(x)u^{-\delta} + b(x)u^{\beta}\Big)\varphi dx,$$

for all $\varphi \in C_c^\infty(\Omega)$.

Theorem 0.0.3 *Assume $0 \leqslant b \in L^{(\frac{p^*}{\beta+1})'}(\Omega)$ and $0 \leqslant a$ in Ω. If one of the assumptions below holds*

(h_1): $0 < \delta < 1$ and $a \in L^{(\frac{p^*}{1-\delta})'}(\Omega)$;

(h_2): $\delta \geqslant 1$ and $a \in L^1(\Omega)$,

then, for each $\alpha > 0$ given, there exists a solution $u = u_\alpha \in W^{1,p}_{loc}(\Omega)$ of the problem (L_α). Moreover, if $\delta \leqslant 1$ then $u \in W_0^{1,p}(\Omega)$. Besides, the solution is **unique** *if $a + b > 0$ in Ω.*

It is worth mentioning that the Theorem 0.0.3, in addition playing a fundamental role in the next chapters of this thesis, also has an intrinsic importance. Indeed, our result of uniqueness for the local problem (L_α) improves the main theorems of [13] and [14] by:

(i) removing any requirement about the geometry of the domain,

(ii) permitting a perturbation of the very singular term by a sublinear one,

(ii) including more general potentials a and b.

In Chapter 2, we approach the following non-local quasilinear λ-problem

$$(P_1) \qquad \begin{cases} -\Big(\displaystyle\int_\Omega g(x,u)dx\Big)^r \Delta_p u = \lambda\Big(a(x)u^{-\delta} + b(x)u^\beta\Big) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

obtained from (1) by considering $A(x,t) = t^r, g(x,t,\vec{v}) = g(x,t)$ and $f_{\lambda,\mu}(x,t) = \lambda\big(a(x)t^{-\delta} + b(x)t^\beta\big)$, where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a smooth bounded domain, $-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < N$, $\delta > 0$, $0 < \beta < p-1$, $\lambda > 0$ being a real parameter and $a, b, g \geqslant 0$ are appropriate functions.

Problem (P_1) is non-local due to the presence of the term $\Big(\displaystyle\int_\Omega g(x,u)dx\Big)^r$, which implies that equation in (P_1) is no longer a pointwise equality. In general, the presence of such terms gives rise to some additional difficulties in approaching this kind of problems by classical arguments. For example, many non-local problems are non-variational, in the sense that techniques of variational methods can not be applied in a direct way.

The non-local problems have been extensively studied in recent years and their applications arise in various contexts, for example, in the study of systems of particles in thermodynamical equilibrium via gravitational potential ([4], [36]), 2-D fully turbulent behavior of real flow [11], thermal runaway in Ohmic heating ([6], [15]), physics of plasmas, thermo-electric flow in a conductor [39], gravitational equilibrium of polytropic stars [35], modeling of cell aggregation through interaction with a chemical [59] and population behavior [18].

In [16], it was investigated that the equation

$$\frac{du}{dt} - A\Big(\int_\Omega u\Big)\Delta u = f \tag{3}$$

describes the behavior of a population subject to some kind of spreading. In this case, u and A represent the population density and the diffusion coefficient, respectively. When A is a constant, the above model does not take into account that the phenomena of crowding and isolation can change the dynamics of the migration. Therefore, in a closer model to the reality, the coefficient A is supposed to depend on the entire population in the domain Ω as in (3).

The literature about non-local problems with autonomous non-local term is vast (see, for example, [5], [10], [17], [20] and [26]), but up to this date there is no result in the direction of the p-Laplacian operator, when $p \neq 2$, in the context of $W^{1,p}_{loc}(\Omega)$-solutions to singular ones. About related problems with weak singularities ($0 < \delta < 1$) for Laplacian operator, we quote the works [3, 61, 62], which show the existence of positive solutions to non-local singular problems. We remark that the problems in above references are treated in the context of classical solutions, except in [3], where the weak solution lies in $H^1_0(\Omega)$.

Although García-Melián and Lis [30] have not studied neither a singular problem nor a Dirichlet boundary condition problem, we are going to highlight their techniques to study (P_1). They showed existence of solution to the blow-up problem

$$\Big(1 + \frac{1}{|\Omega|}\int_\Omega g(u)dx\Big)\Delta u = \lambda f(u) \text{ in } \Omega,\ u > 0 \text{ in } \Omega,\quad u = \infty \text{ on } \partial\Omega, \tag{4}$$

where $f : [0, \infty) \to (0, \infty)$ is an appropriate continuous function, by decoupling (4) in the system

$$\begin{cases} \Delta u = \alpha f(u) \ \text{ in } \ \Omega,\ \ u = \infty \ \text{ on } \ \partial\Omega \\ \alpha = \lambda\Big(1 + \frac{1}{|\Omega|}\int_\Omega g(u)dx\Big)^{-1} \end{cases} \tag{5}$$

and studying the behavior of the pair (α, u), solution of (5).

García-Melián and Lis's strategy inspired us to obtain branches of bifurcation in $(0, \infty) \times \|\cdot\|_\infty$ for the problem (P_1). By using a new Comparison Principle for $W^{1,p}_{loc}(\Omega)$-sub and supersolutions, which we prove in Chapter 1, we explore the α-behavior of the pair (α, u_α) in the $(0, \infty) \times W^{1,p}_{loc}(\Omega)$-topology, where u_α is the only solution of (L_α). Taking advantage of this approach, we present a complete picture of the bifurcation diagram of Problem (P_1). In particular, we show how the presence of the non-local term changes the structure of the bifurcation of the local problem (see problem (L_α) above), that emanates from $(0, 0)$ and bifurcates from infinity at infinity.

Before stating the next result, we make it clear that in this context the Dirichlet boundary condition is understood as in Definition 0.0.1 and solution is defined as follows.

Definition 0.0.4 *We say u is a $W^{1,p}_{loc}(\Omega)$-solution for (P_1) if $u > 0$ in Ω (for each $\Theta \subset\subset \Omega$ given there exists a positive constant c_Θ such that $u \geqslant c_\Theta > 0$ in Θ) and*

$$\Big(\int_\Omega g(x,u)dx\Big)^r \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \lambda \int_\Omega \Big(a(x)u^{-\delta} + b(x)u^\beta\Big)\varphi dx,$$

for all $\varphi \in C_c^\infty(\Omega)$.

Let us also fix the following assumptions

(h_3): $a, b \in L^m(\Omega)$ for some $m > N/p$,

(h_4): $a, b \in L^m(\Omega)$ for some $m > N$

and denote by

$$\Sigma = \{(\lambda, u) \in (0,\infty) \times C(\overline{\Omega}) : u \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega}) \text{ is a solution of } (P_1)\}.$$

Thus, by considering

(g_∞): $\lim\limits_{t\to\infty} g(x,t)t^{\theta_\infty} = g_\infty(x) > 0$ uniformly in $\overline{\Omega}$, for some $\theta_\infty \in \mathbb{R}$ and $g_\infty \in C(\overline{\Omega})$,

(g'_∞): $\lim\limits_{t\to\infty} g(x,t)t^{\theta_\infty} = +\infty$ uniformly in $\overline{\Omega}$, for some $\theta_\infty \in \mathbb{R}$,

(g_0): $\lim\limits_{t\to 0^+} g(x,t)t^{\theta_0} = g_0(x) > 0$ uniformly in $\overline{\Omega}$, for some $\theta_0 \in \mathbb{R}$ and $g_0 \in C(\overline{\Omega})$,

(g'_0): $\lim\limits_{t\to 0^+} g(x,t)t^{\theta_0} = \infty$ uniformly in $\overline{\Omega}$, for some $\theta_0 \in \mathbb{R}$,

we have the following.

Theorem 0.0.5 *Assume $\delta > 0$ and $0 < \beta < p-1$ hold. If:*

1) $g \in C(\overline{\Omega} \times [0,\infty), (0,\infty))$ *and in addition*

a) (h_3), (g_∞) *and* $\theta_\infty r < p-1-\beta$ *hold, then* (P_1) *admits at least one solution in* Σ*, for each* $\lambda > 0$ *given. Besides this, the same conclusion remains true if* $\{r < 0$ *and* $g_\infty \equiv 0$ *in* $(g_\infty)\}$ *or* $\{(g'_\infty)$ *and* $r \geqslant 0\}$ *holds.*

b) (h_4), (g_∞), $\theta_\infty r > p-1-\beta$ *and* $\theta_\infty < 1$ *hold, then there exists* $0 < \lambda^* < \infty$ *such that* (P_1) *admits at least two* $W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$*-solutions for each* $\lambda \in (0, \lambda^*)$ *given, at least one solution for* $\lambda = \lambda^*$ *and no solution for* $\lambda > \lambda^*$*. Furthermore, if* $\{r \geqslant 0$ *and* $g_\infty \equiv 0$ *in* $(g_\infty)\}$ *or* $\{(g'_\infty)$ *and* $r < 0\}$ *holds, then the same conclusion is valid.*

2) $g \in C(\overline{\Omega} \times (0,\infty), (0,\infty))$, (h_4) *is satisfied and additionally*

a) (g_∞), (g_0), $\theta_\infty r < p-1-\beta$, $\theta_0 r > p-1+\delta$ *and* $\theta_0 < 1$ *hold, then there exists a* $0 < \lambda^* < \infty$ *such that* (P_1) *admits at least two* $W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$*-solutions for* $\lambda > \lambda^*$*, at least one for* $\lambda = \lambda^*$ *and no solutions for* $0 < \lambda < \lambda^*$*. Moreover, the conclusion is the same if we assume either* $\{r > 0, (g'_0)$ *and* $(g'_\infty)\}$ *or* $\{r < 0, (g_0), (g_\infty)$ *and* $g_0 \equiv g_\infty \equiv 0\}$*.*

b) $\theta_\infty r > p-1-\beta$, $\theta_0 r > p-1+\delta$ *and* $\theta_\infty, \theta_0 < 1$ *hold, then* (P_1) *admits at least one* $W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$*-solution for each* $\lambda > 0$ *given. In this case, the conclusion remains true if we assume either* $\{r > 0, (g'_0)$ *and* (g_∞) *with* $g_\infty \equiv 0\}$ *or* $\{r < 0, (g'_\infty)$, *and* (g_0) *with* $g_0 \equiv 0\}$*.*

Moreover, in all the cases Σ *is the continuum of solutions given by a curve which:*

(i) *emanates from* 0 *at* $\lambda = 0$ *and bifurcates from infinity at* $\lambda = \infty$ *in the case* $1-a)$ *(see Fig. 1),*

(ii) *emanates from* 0 *at* $\lambda = 0$ *and bifurcates from infinity at* $\lambda = 0$ *in the case* $1-b)$ *(see Fig. 2),*

(iii) *emanates from* 0 *at* $\lambda = \infty$ *and bifurcates from infinity at* $\lambda = \infty$ *in the case* $2-a)$ *(see Fig. 3),*

(iv) *emanates from* 0 *at* $\lambda = \infty$ *and bifurcates from infinity at* $\lambda = 0$ *in the case* $2-b)$ *(see Fig. 4),* We draw below the $(0,\infty)\times\|\cdot\|_\infty$-diagram of $W^{1,p}_{loc}(\Omega)\cap C(\overline{\Omega})$-solutions obtained from the Theorem 0.0.5

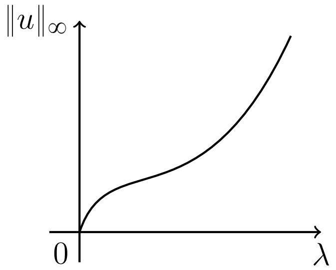

Fig. 1

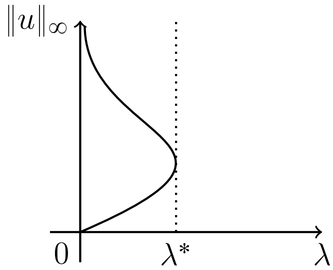

Fig. 2

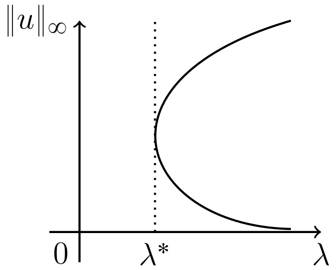

Fig. 3

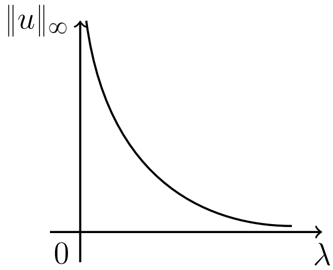

Fig. 4

Next, we list some of the main contributions of study of (P_1) to the literature:

i) singular problems of the type (P_1) involving the p-Laplacian operator with δ taking any positive value and potentials a and b being unbounded, have not been considered in the literature up to now,

ii) the non-local term in (P_1) is not required essentially to be bounded from below by positive constant or from above, in fact, it may be singular at $t=0$. See for instance [29], [61] and references therein.

In the Chapter 3, we study existence, multiplicity and non-existence of positive $W^{1,p}_{\mathrm{loc}}(\Omega)$-solutions for the following non-autonomous and non-local λ-problem

$$(P_2)\begin{cases} -A\Big(x,\displaystyle\int_\Omega u^\gamma dx\Big)\Delta_p u=\lambda f(x,u) \text{ in } \Omega,\\ u>0 \text{ in } \Omega,\quad u=0 \text{ on } \partial\Omega,\end{cases}$$

obtained by doing $g(x,t,\vec{v})=t^\gamma$ in (1), where again $\Omega\subset\mathbb{R}^N(N\geqslant 2)$ is a smooth bounded domain, $p \in (1, N), \lambda > 0$ is a real parameter, $A \in C(\overline{\Omega} \times [0, \infty), (0, \infty))$ and $f \in C(\overline{\Omega} \times (0, \infty), (0, \infty))$ can be strongly (very) singular at $u = 0$.

Once again, due to the lack of variational structure, non-local problems such as (P_2) are treated, in general, through topological methods. A recurrent argument in the treatment of autonomous non-local problems is, just like the one done by García-Melián and Lis, to relate the non-local problem to a local problem and thereon to study the behavior of the associated local problem. This type of argument, in general, can not be applied for non-autonomous and non-local problems. There are few papers on the non-autonomous case, see [19], [53], [29] and references therein. In particular, we refer to [29] where the problem (P_2) is treated via bifurcation theory with $p = 2$ and $f(x, u) = u^{\beta}$, for $0 < \beta < 1$.

In this chapter, since A is a non-autonomous function and no monotonicity is posed on the quotient $t \mapsto f(x, t)/t^{p-1}$, the same strategy as in Chapter 2 can not be applied anymore. In [21], Rabinowitz et. al. studied semilinear local singular problems in the context of classical solutions. We inspire our approach on their ideas to obtain an unbounded ϵ-limit connected component of positive solutions from ϵ-unbounded *continuum* of positive solutions for a ϵ-perturbed problems. For qualitative properties about this *continuum*, we were inspired by the ideas from Figueiredo-Sousa et. al. [29], where a semilinear non-local problem was treated with non-singular (sublinear) growth. The strategies from both of the above papers do not work in our approach, principally by the lack of the linearity of the p-Laplacian operator and by the singularity in the Sobolev spaces setting. To overcome these difficulties, we approached (P_2) in an indirect way, since no functional equation can be directly associated to (P_2), by combining penalization arguments, a-priori estimates and a Comparison Principle for $W^{1,p}_{\mathrm{loc}}(\Omega)$-sub and supersolutions, which will be proved in the first chapter.

Before stating the main results of this chapter, we need to mimic the Definition 0.0.4 for the solution of (P_2).

Definition 0.0.6 *We say u is a $W^{1,p}_{\text{loc}}(\Omega)$-solution for (P_2) if $u > 0$ in Ω, that is, for each $\Theta \subset\subset \Omega$ given there exists a positive constant c_Θ such that $u \geqslant c_\Theta > 0$ in Θ, $u^\gamma \in L^1(\Omega)$ and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx = \lambda \int_\Omega \frac{f(x,u)}{A\left(x, \int_\Omega u^\gamma dx\right)} \varphi dx \quad \textit{for all } \varphi \in C_c^\infty(\Omega). \tag{6}$$

Henceforth, we will always assume that $f \in C(\overline{\Omega} \times (0,\infty), (0,\infty))$. Let us set some hypotheses that we need in the next theorem.

(A_0) $A \in C(\overline{\Omega} \times \mathbb{R})$ satisfies $A(x,t) > 0$ for all $t \geqslant 0$ and $x \in \overline{\Omega}$,

(f_0) $\lim\limits_{t \to 0^+} \dfrac{f(x,t)}{t^{p-1}} = \infty$ uniformly in $\overline{\Omega}$,

(f_∞) $\lim\limits_{t \to \infty} \dfrac{f(x,t)}{t^{p-1}} = 0$ uniformly in $\overline{\Omega}$.

Our first result in Chapter 3 can be stated as follows.

Theorem 0.0.7 *Suppose that $\gamma \geqslant 0$, (A_0) and (f_0) hold. Then, there exists an unbounded continuum $\Sigma \subset \mathbb{R} \times C(\overline{\Omega})$ of positive solutions of the problem (P_2) that emanates from $(0,0)$. In addition, if (f_∞) holds and $A(x,t) \geqslant a_0$ in $\overline{\Omega} \times \mathbb{R}^+$ for some $a_0 > 0$, then $Proj_{\mathbb{R}}\Sigma = (0,\infty)$.*

Below, we present more qualitative information about the *continuum* Σ by relating the non-local and nonlinear terms. In this case, we need to consider certain additional conditions:

(A_∞) $\lim\limits_{t \to \infty} A(x,t)t^\theta = a_\infty(x) \geqslant 0$ uniformly in $\overline{\Omega}$, for some $a_\infty \in C(\overline{\Omega})$,

(A'_∞) $\lim\limits_{t \to \infty} A(x,t)t^\theta = \infty$ uniformly in $\overline{\Omega}$,

(f_1) $\lim\limits_{t \to \infty} \dfrac{f(x,t)}{t^\beta} = c_\infty(x) > 0$ uniformly in $\overline{\Omega}$, for some $-\infty < \beta < p-1$ and $c_\infty \in C(\overline{\Omega})$,

(f_2) $\lim\limits_{t \to 0^+} \dfrac{f(x,t)}{t^\delta} = c_0(x) > 0$ uniformly in $\overline{\Omega}$, for some $-\infty < \delta < p-1$ and $c_0 \in C(\overline{\Omega})$.

Theorem 0.0.8 *Assume (A_0) and that f satisfies (f_1) and (f_2) with $\delta \leqslant \beta$. If* *a)* $\gamma > 0$ *and either* $\{\theta\gamma = p - 1 - \beta$ *and* $(A'_\infty)\}$ *or* $\{\theta\gamma < p - 1 - \beta$ *and* (A_∞) *with* $a_\infty > 0$ *in* $\overline{\Omega}\}$ *hold, then* $Proj_{\mathbb{R}}\Sigma = (0, \infty)$ *(see Fig. 5),*

b) $\gamma > 0$, $\theta\gamma \geqslant p - 1 - \beta$ *and* (A_∞) *hold, then* $Proj_{\mathbb{R}}\Sigma \subset (0, \lambda^*)$ *for some* $0 < \lambda^* < \infty$. *Furthermore, if*

  i) $a_\infty > 0$ *in* $\overline{\Omega}$ *and* $\theta\gamma = p - 1 - \beta$, *then* $\lambda = 0$ *can not be a bifurcation point from* ∞ *(see Fig. 6 or 7);*

  ii) $a_\infty = 0$ *in* $\overline{\Omega}$, *then* $\lambda = 0$ *is a bifurcation point from* ∞ *(see Fig. 8);*

c) $-1 < \gamma < 0$, $\theta\gamma \geqslant p - 1 - \delta$ *and either* (A'_∞) *or* (A_∞) *with* $0 < a_\infty$ *hold, then* (P_2) *does not admit positive solution for* $\lambda > 0$ *small.*

Summarizing the above information, we have the following diagrams.

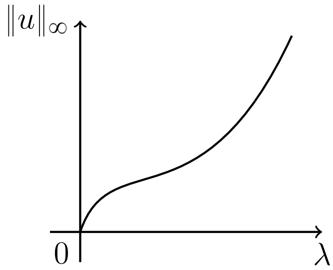

Fig. 5

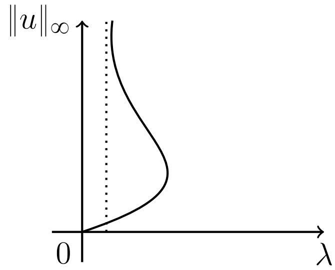

Fig. 6

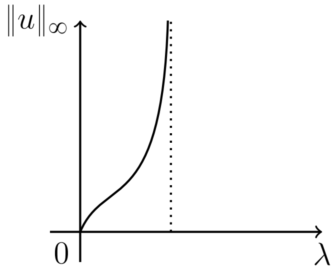

Fig. 7

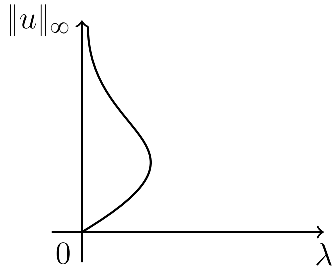

Fig. 8

In the above item (c), we stated that the problem (P_2) has no solution for $\lambda > 0$ close to 0 when the non-local term is also singular. We note that the issue about existence of solution is not possible to treat with the same arguments anymore, as in the proof of Theorem 0.0.7. However, when the non-local term is autonomous, we are also able to prove the global existence of $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$-solutions.

More precisely, we have the following result.

Theorem 0.0.9 *Assume that* (f_1), (f_2) *with* $\delta \leqslant \beta$, (A_0) *and either* (A_∞) *with* $a_\infty > 0$ *or* (A'_∞) *hold. If* $\theta\gamma > p-1-\delta$ *and* $-1 < \gamma < 0$, *then there exists a* $\lambda^* > 0$ *such that the problem*

$$\begin{cases} -A\Big(\displaystyle\int_\Omega u^\gamma dx\Big)\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{7}$$

admits at least one $W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$*-solution for* $\lambda \geqslant \lambda^*$ *and no solution for* $\lambda < \lambda^*$.

By taking advantage of the ideas explored in the proofs of the above Theorems, we were able to consider non-autonomous Kirchhoff-type problems as well. For sake of the clarity, we study just a classical Kirchhoff model. Precisely, we consider

$$(Q_1) \quad \begin{cases} -M\Big(x, \|\nabla u\|_p^p\Big)\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where M, modeled as non-homogeneous Kirchhoff term, satisfies:

(M_0) $M(x,t) = a(x) + b(x)t^\gamma$; $a, b \in C(\overline{\Omega})$; $a(x) \geqslant \underline{a}$; $b(x) \geqslant 0$ in $\overline{\Omega}$

and

(Γ_0) either $\gamma > 0$ if $-1 \leqslant \delta < p-1$ or $0 < \gamma < \frac{p-1-\delta}{-\delta-1}$ if $-\frac{2p-1}{p-1} \leqslant \delta < -1$.

Theorem 0.0.10 *Assume that* (f_2), (M_0) *and* (Γ_0) *hold. Then there exists an unbounded continuum* $\Sigma \subset \mathbb{R}^+ \times C(\overline{\Omega})$ *of solutions of* (Q_1) *which emanates from* $(0,0)$. *Furthermore, if* (f_∞) *holds then* $Proj_{\mathbb{R}^+}\Sigma = (0,\infty)$. *Moreover, if* $\gamma < 1$ *then* Σ *is unbounded vertically as well.*

We remark that there are few articles dealing with Kirchhoff type problems with singular nonlinearity. In this direction, we found some results in [41] and [42] for weak singularities, that permitted them to approach by variational methods. Recently, in 2018, Agarwal, O'Regal and Yan [60] studied a Kirchhoff-type problem with nonlinearity of the form $f(x,u) = K(x)u^{-\delta}$, for $\delta > 0$, in the context of the

Laplacian operator. They used principally sub-supersolution techniques to get existence and uniqueness of classical solution.

It is worth mentioning that, as far as we know, non-autonomous and non-local quasilinear problems with very singular nonlinearities have not yet been considered in the literature, and the same is true for Kirchhoff-type problems. Our results contribute to the literature principally by:

i) Theorem 0.0.7 being new even in the context of local problems (and for $p = 2$), by guaranteeing the existence of a *continuum* of solutions for a strongly-singular problem. Moreover, the conclusion that this *continuum* is horizontally unbounded is obtained without any boundedness condition on f, as required in Theorem 1.9 and Corollary 1.10 proved by Rabinowitz et. al. in [21],

ii) Theorem 0.0.8 proving the principal results of Suárez et. al. [29] in the context of strongly-singular problems as well,

iii) Theorem 0.0.9 including singularity also in the non-local term and obtaining global existence of solutions in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ setting. This situation was not yet considered in the literature,

iv) Theorem 0.0.10 including non-autonomous Kirchhoff terms and capturing the same sharp power for existence of solutions still in $W_0^{1,p}(\Omega)$ for the associated local problem.

In the previous theorem, due to the techniques employed, the homogeneity of the operator was important for the multiplicities results established. Moreover, both Theorem 0.0.5 and Theorem 0.0.8 were directly or indirectly influenced by the existence of solution for the strong singular problem (L_α).

In the last chapter, our main goal is still to show multiplicity of positive solutions for a quasilinear problem when the operator is no longer homogeneous. In the same sense of the previous results, the next ones are still linked to the existence of solution to a singular local problem. More precisely, we deal with the following quasilinear problem involving the Φ-Laplacian operator

$$(Q_{\lambda,\mu})\quad \begin{cases} -M\Big(\displaystyle\int \Phi(|\nabla u|)dx\Big)\Delta_\Phi u = \lambda f(x,u) + \mu b(x)u^{-\delta} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\lambda, \mu > 0$ are real parameters, $M \in C([0,\infty),[0,\infty))$, $f : \Omega \times [0,\infty) \to [0,\infty)$ is a Carathéodory function not identically zero, $b : \Omega \to \mathbb{R}$ is a positive function that belongs to an appropriate Lebesgue space, $0 < \delta$ depends on the summability of b and $-\Delta_\Phi u = -div(a(|\nabla u|)\nabla u)$ is the Φ-Laplacian operator, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a N-function of the form $\Phi(t) = \displaystyle\int_0^{|t|} \phi(s)ds$, with $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = \begin{cases} a(|t|)t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Inspired by [28] and using non-smooth analysis techniques, we prove how the presence of the superlinear perturbation (at $t = 0$) and the Kirchhoff term break the uniqueness of the solution for the purely singular problem. In fact, we have established that under appropriate conditions on f, λ and μ, the existence of three different $W_0^{1,\Phi}(\Omega)$-solutions to the problem $(Q_{\lambda,\mu})$ is guaranteed and this is strictly related to the existence of $W_0^{1,\Phi}(\Omega)$-solution to the problem

$$(S)\quad \begin{cases} -\Delta_\Phi u = b(x)u^{-\delta} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

Results of this kind for singular problems have already been obtained in [28] and [63], but with more restrictive conditions on the operator, potential b and singularity. In the context of non-singular problems, several works (see [12], [45], [27], [52], [63] and [64]) dealt with this issue.

The main difficulty found in our study, is the lack of variational structure to approach the problem $(Q_{\lambda,\mu})$. Notice that the functional naturally associated to $(Q_{\lambda,\mu})$ is $I : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$, given by

$$I(u) = \hat{M}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) - \lambda \int_\Omega F(x,u)dx - \frac{\mu}{1-\delta}\int_\Omega b(x)u^{1-\delta}dx,$$

where $\hat{M}(t) = \displaystyle\int_0^t M(s)ds$ and $F(x,t) = \displaystyle\int_0^t f(x,s)ds$.

When $0 < \delta < 1$, the functional I is well defined in $W_0^{1,\Phi}(\Omega)$ and, although it is not differentiable, we can apply, in indirect way, techniques of variational methods to study $(Q_{\lambda,\mu})$. However, when $1 \leqslant \delta$ and less than certain sharp value, the functional I is well defined only in a subset of $W_0^{1,\Phi}(\Omega)$ and when δ extrapolates this sharp value, the functional I is not well defined in the whole $W_0^{1,\Phi}(\Omega)$.

By taking advantage of the technique presented by Ricceri in [50] and Faraci-Smyrlis in [28], we show a necessary and sufficient condition for the existence of three different solutions of $(Q_{\lambda,\mu})$ in $W_0^{1,\Phi}(\Omega)$ for suitable values of λ and μ. Before stating the result obtained, let us define what we mean by solution in this context.

Definition 0.0.11 *A function $u \in W_0^{1,\Phi}(\Omega)$ is a weak solution for problem $(Q_{\lambda,\mu})$ if $u > 0$ a.e in Ω, $bu^{-\delta}\varphi \in L^1(\Omega)$ and*

$$M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big)\int_\Omega a(|\nabla u|)\nabla u \nabla\varphi dx = \int_\Omega [\lambda f(x,u) + \mu b u^{-\delta}]\varphi dx,$$

for all $\varphi \in W_0^{1,\Phi}(\Omega)$.

Throughout this chapter, we assume that Φ is an N-function, given as above, satisfying:

(ϕ_0): $a \in C^1((0,\infty),(0,\infty))$ and ϕ is an increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$;

(ϕ_1): $0 < a_- := \inf\limits_{t>0} \dfrac{t\phi'(t)}{\phi(t)} \leqslant \sup\limits_{t>0} \dfrac{t\phi'(t)}{\phi(t)} := a_+ < \infty$.

and denote by $\phi_- = a_- + 1$ and $\phi_+ = a_+ + 1$.

Now, let us denote by Φ^* the function whose inverse is given by $(\Phi^*)^{-1}(t) = \displaystyle\int_0^t \Phi^{-1}(s)s^{-1-1/N}ds$. In order Φ^* to be a N-function, we require that

$$\int_0^1 \Phi^{-1}(s)s^{-1-1/N}ds < \infty \quad \text{and} \quad \int_1^\infty \Phi^{-1}(s)s^{-1-1/N}ds = \infty.$$

In this case, we assume that Φ^* is the N-function represented by ϕ^*, namely, $\Phi^*(t) = \displaystyle\int_0^{|t|} \phi^*(s)ds$. Thus, let us also consider the following assumption:

(ϕ_2): $\phi_+ < \phi_-^* := \inf\limits_{t>0} \dfrac{t\phi^*(t)}{\Phi^*(t)}$.

With respect to M, we suppose

(M): $M(t) \geqslant m_0 t^{\alpha-1}$ for all $t \geqslant 0$ with $1 \leqslant \alpha < \frac{\phi_-^*}{\phi_+}$.

About the potential b, let us assume

$$(b) \ : \begin{cases} b \in L^{(\frac{\phi_-^*}{1-\delta})'}(\Omega) \ \text{ if } 0 < \delta < 1; \\ b \in L^q(\Omega) \text{ for some } q > 1 \ \text{ if } \delta = 1; \\ b \in L^1(\Omega) \ \text{ if } \delta > 1 \end{cases}$$

and about the nonlinearity f, we consider $f(x,t) = 0$ a.e in Ω for all $t \leqslant 0$ and

(f_1'): there exists an odd increasing homeomorphism h from $\mathbb{R}$ to $\mathbb{R}$ and nonnegative constants a_1 and a_2 such that

$$f(x,t) \leqslant a_1 + a_2 h(|t|), \quad \forall t \in \mathbb{R} \text{ and } \forall x \in \overline{\Omega}$$

and $H << \Phi_*$, where $H(t) = \displaystyle\int_0^{|t|} h(s)ds$. We also assume the following condition on H:

$$1 < h_- := \inf_{t>0} \frac{th(t)}{H(t)} \leqslant \sup_{t>0} \frac{th(t)}{H(t)} := h_+ < \infty; \tag{8}$$

(f_2'): $\lim\limits_{t\to 0^+} \dfrac{\sup\limits_{\overline{\Omega}} F(x,t)}{t^{\alpha\phi_+}} = 0$;

(f_3'): $\lim\limits_{t\to\infty} \dfrac{\sup\limits_{\overline{\Omega}} F(x,t)}{t^{\alpha\phi_-}} = 0.$

After having established all these hypotheses, we are in position to state the main result of the last chapter.

Theorem 0.0.12 *Suppose that* $(\phi_0), (\phi_1), (\phi_2), (M)$, (b), $(f_1') - (f_3')$ *hold. Assume* $\delta > 1$ *and*

$$\lambda^* = \inf\left\{ \frac{\hat{M}\Big(\int_\Omega \Phi(|\nabla u|)\Big)}{\int_\Omega F(x,u)dx} \ : \ u \in W_0^{1,\Phi}(\Omega) \ \textit{and} \ \int_\Omega F(x,u)dx > 0 \right\}.$$

Then, the following are equivalent:

i) there exists $0 < u_0 \in W_0^{1,\Phi}(\Omega)$ *such that* $\int_\Omega b u_0^{1-\delta} dx < \infty$;

ii) the problem

$$(S): \qquad -\Delta_\Phi u = b(x)u^{-\delta} \ \ \textit{in } \Omega, \ u > 0 \ \ \textit{in } \Omega \ \ \textit{and} \ \ u = 0 \ \textit{on } \partial\Omega$$

admits (unique) weak solution;

iii) for each $\lambda > \lambda^*$ *there exists* $\mu_\lambda > 0$ *such that for* $\mu \in (0, \mu_\lambda]$ *the problem* $(Q_{\lambda,\mu})$ *admits at least three weak solutions.*

Corollary 0.0.13 *Replacing* $\delta > 1$ *with* $\delta \leqslant 1$ *and assuming the hypotheses of above theorem, the claims* $i) - iii)$ *remain true independent of each other.*

As a consequence of $i) \Longrightarrow ii)$ in the previous theorem, we have the following corollary, which relates δ to the summability of b and gives us a range of δ-values for which the existence of solution for (S) is still assured.

Corollary 0.0.14 *Assume that* (ϕ_0) *and* (ϕ_1) *hold. If* $b \in L^q(\Omega)$ *for some* $1 < q$ *and*

$$1 < \delta < 1 + \frac{\phi_+'}{q'} := \delta_q, \tag{9}$$

then (S) *admits a* $W_0^{1,\Phi}(\Omega)$*-solution.*

In particular, as a consequence of Theorem 0.0.12 and Corollary 0.0.14, we have the following.

Corollary 0.0.15 *Assume that* $(\phi_0), (\phi_1), (M)$ *and* $(f_1') - (f_3')$ *hold. If* $b \in L^q(\Omega)$ *for some* $1 < q$ *and* δ *satisfies (9), then for each* $\lambda > \lambda^*$ *there exists* $\mu_\lambda > 0$ *such that for* $\mu \in (0, \mu_\lambda]$ *the problem* $(Q_{\lambda,\mu})$ *admits at least three weak solutions.*

It is worth mentioning that the above theorem is more general than the related results present in the literature both by the presence of the Kirchhoff term and by the generality of the potential b, singularity δ and operator. Let us summarize some contributions of the above results to the literature:

i) Theorem 0.0.12 establishes necessary and sufficient conditions for the existence of multiple solutions for $(Q_{\lambda,\mu})$ and the existence of $W_0^{1,\Phi}(\Omega)$ solution for (S);

ii) Theorem 0.0.12 extends the result of Faraci et.al [28] by considering non-homogeneous operator, more general conditions on potential and singularity and including a Kirchhoff term;

iii) In the proof of Theorem 0.0.12, we have also extended the result of Yijing [54] to a non-homogeneous operator;

iv) Corollary 0.0.14 gives us an explicit range for δ, in which the existence of a solution in $W_0^{1,\Phi}(\Omega)$ is still guaranteed. In particular, when $\Phi(t) = |t|^p/p$ and $b_0 \leqslant b(x) \in L^\infty(\Omega)$ for some constant $b_0 > 0$, the value δ_q coincides with the sharp values obtained in [32] and [40];

v) Corollary 0.0.15 complements the principal result in [28] by expliciting a range to δ, which leads to the multiplicity result, namely,

$$0 < \delta < \frac{p(N-1)}{N(p-1)} = \delta_{(p*)'}.$$

This thesis has the following structure. In Chapter 1, we prove the existence and uniqueness of $W_{loc}^{1,p}(\Omega)$-solutions to the strongly singular problem (L_α) inspired on ideas of [14] and [23]. To prove the uniqueness, a comparison principle for $W^{1,p}_{loc}(\Omega)$-sub and supersolutions is established.

In the Chapter 2, by exploring the uniqueness of $W^{1,p}_{loc}(\Omega)$-solutions to Problem (L_α), appropriate test functions together with a result of Boccardo and Murat [7], we are able to prove that the operator $T : (0, \infty) \to W^{1,p}_{loc}(\Omega)$ (see (2.1) below) is well-defined and continuous. By using this fact, in the last section of the chapter 2 we conclude the proof of Theorem 0.0.5.

In Chapter 3, we present in the first section the proof of Theorem 0.0.7. The qualitative study of the *continuum* obtained in the first section, will be done in section 3, as well as the proof of Theorem 0.0.9. We conclude the section 3, by studying the degenerate case in problem (P_2). In the last section we prove Theorem 0.0.10.

In Chapter 4, we present in the first section basic concepts and facts about Orlicz-Sobolev spaces. In the second section, we show the necessary tools related to non-smooth analysis, which will be necessary to prove the main theorem of this chapter. In the last section, we conclude the proof of Theorem 0.0.12.

NOTATION

- $C, C_1, C_2, \cdots$ denote positive constants.
- For $1 < p$, we denote by p' the conjugate of p satisfying $1/p + 1/p' = 1$.
- $\mathbb{R}^N$ denote the $N-$dimensional Euclidean Space.
- $B_R(x_0)$ is the open ball centered at x_0 and with radius $R > 0$.
- $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.
- $\partial\Omega$ is the boundary of Ω.
- $d(x) = dist(x, \partial\Omega) = \inf\limits_{y \in \partial\Omega} |x - y|$.
- If $A \subset \mathbb{R}^N$ is Lebesgue measurable, then $|A|$ denote the Lebesgue measure of A.
- If $A \subset \mathbb{R}^N$, we denote $\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in A. \end{cases}$
- supp $f = \overline{\{x \in \Omega \ : \ f(x) \neq 0\}}$ denote the support of the function $f : \Omega \to \mathbb{R}$.
- $A \subset\subset \Omega$ means that $\overline{A} \subset \Omega$.
- By $u_n \to u$ we mean that u_n converges strongly to u.
- By $u_n \rightharpoonup u$ we mean that u_n converges weakly to u.

- $L^p(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable $: \displaystyle\int_\Omega |u|^p dx < \infty\}$ endowed with the norm $\|u\|_p = \Big(\displaystyle\int_\Omega |u|^p dx\Big)^{1/p}$.
- $L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable $:$ $\mathrm{esssup}_{x\in\Omega}|u(x)| < \infty\}$ endowed with the norm $\|u\|_\infty = \mathrm{esssup}_{x\in\Omega}|u(x)|$.
- $L^\infty_{\mathrm{loc}}(\Omega) = \{u \in L^\infty(K)$ for all compact $K \subset \Omega\}$.
- $W_0^{1,p}(\Omega)$ is the usual Sobolev Space endowed with the norm $\|\nabla u\|_p$.
- $u_{x_i}(x) = \frac{\partial u(x)}{\partial x_i}$.
- For $1 < p < N$, we denote by $p* = Np/(N-p)$ the critical exponent for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.
- $W^{1,p}_{\mathrm{loc}}(\Omega) = \{u : \Omega \to \mathbb{R}\ :\ u \in W^{1,p}(K)$ for all compact $K \subset \Omega\}$.
- $C(\Omega)$ denote the space of continuous functions in Ω.
- $C^k_0(\overline{\Omega}) = \{u \in C^k(\overline{\Omega})\ :\ u|_{\partial\Omega} = 0\}$.
- $C^k_c(\Omega) = \{u \in C^k(\Omega)\ :\ \mathrm{supp}\ u \subset \Omega$ is compact$\}$.
- $C^{k,\alpha}(\Omega)$ is the space of functions whose k-th derivatives are α- Hölder continuous.

CHAPTER 1

EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A VERY SINGULAR LOCAL PROBLEM

We reserve this first chapter to deal with existence and uniqueness of $W^{1,p}_{\text{loc}}(\Omega)$-solution for

$$(L_\alpha) \qquad \begin{cases} -\Delta_p u = \alpha\Big(a(x)u^{-\delta} + b(x)u^{\beta}\Big) \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \quad u > 0 \text{ on } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a smooth bounded domain, $1 < p < N$, $\delta > 0$, $0 < \beta < p-1$, $\lambda > 0$ is a real parameter and $a, b \geqslant 0$ are appropriate functions.

For convenience of the reader, let us restate the main result of this chapter.

Theorem 0.0.3 *Assume $0 \leqslant b \in L^{(\frac{p^*}{\beta+1})'}(\Omega)$ and $0 \leqslant a$ in Ω. If one of the assumptions below holds*

(h_1): $0 < \delta < 1$ and $a \in L^{(\frac{p^*}{1-\delta})'}(\Omega)$;

(h_2): $\delta \geqslant 1$ and $a \in L^1(\Omega)$,

then, for each $\alpha > 0$ given, there exists a solution $u = u_\alpha \in W^{1,p}_{loc}(\Omega)$ of the problem (L_α). Moreover, if $\delta \leqslant 1$ then $u \in W^{1,p}_0(\Omega)$. Besides, the solution is **unique** *if $a + b > 0$ in Ω.*

Since our arguments are independent of α, let us simply consider (L_1).

In the first section, we prove existence of $W^{1,p}_{\rm loc}(\Omega)$-solution for (L_1). Although existence results for strongly singular problems have already been established in [8] and [46], the techniques therein are not directly applicable in our case. In [8], the estimates obtained by the authors can not be proved here due to the presence of the sublinear term. On the other hand, our nonlinearity does not satisfy the hypothesis (f_3) in [46]. However, by combining domain perturbation technique of [46] with penalization arguments of [8], we were able to prove existence in our case too.

The uniqueness is a more delicate issue. Since we are allowing δ to assume any positive value, we can not expect our solutions to belong to $W^{1,p}_0(\Omega)$. In this case, the solution obtained can not be tested in the problem, which makes it unfeasible to use classical arguments to prove the uniqueness asserted.

In the second section, by using truncation technique and the construction of a function, with suitable decay and compact support defined in an appropriate subset of Ω, we were able to establish a Comparison Principle for $W^{1,p}_{\rm loc}$−sub and supersolutions of (L_1) without requiring any further hypothesis of regularity in potentials a and b. As a consequence of this Comparison Principle, the uniqueness follows in a direct way.

1.1 Existence of $W^{1,p}_{\rm loc}(\Omega)$-solutions

In this section, let us prove the existence as stated in Theorem 0.0.3. For this, we will consider the following auxiliary problem:

$$\begin{cases} -\Delta_p u = \frac{a_n(x)}{(u+\frac{1}{n})^\delta} + b_n(x)u^\beta \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \quad u > 0 \text{ on } \Omega \end{cases} \tag{1.1}$$

where $a_n(x) = \min\{a(x), n\}$ and $b_n(x) = \{b(x), n\}$, with $n \in \mathbb{N}$.

Lemma 1.1.1 *For each $n \in \mathbb{N}$, the problem (1.1) admits a solution $u_n \in W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. Furthermore, for each compact set $\Theta \subset\subset \Omega$ there exists $c_\Theta > 0$ such that*

$u_n \geqslant c_\Theta > 0$ *in* Θ, *for all* $n \in \mathbb{N}$.

Proof: For each $v \in L^p(\Omega)$, we claim that there exists a unique function $\omega \in W_0^{1,p}(\Omega)$ such that

$$-\Delta_p \omega = \frac{a_n(x)}{(|v| + \frac{1}{n})^\delta} + b_n(x)|v|^\beta. \tag{1.2}$$

In fact, consider the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(\omega) = \frac{1}{p}\int_\Omega |\nabla \omega|^p dx - \int_\Omega \frac{a_n(x)}{(|v| + \frac{1}{n})^\delta}\omega dx - \int_\Omega b_n(x)|v|^\beta \omega dx.$$

We can easily verify that J is differentiable, strictly convex and coercive. Hence J admits a unique critical point, that is, (1.2) admits a solution.

Denoting by $S : L^p(\Omega) \to L^p(\Omega)$ the operator, which associates to each $v \in L^p(\Omega)$ the unique solution $w = S(v) \in L^p(\Omega)$ of (1.2), one can prove that S is a continuous and compact operator. Furthermore, if $\omega = \lambda S(\omega)$ for some $\lambda \in (0,1]$ and $\omega \in W_0^{1,p}(\Omega)$, then by Poincaré's and Hölder inequalities

$$\begin{aligned} \|\omega\|_p{}^p &\leqslant C\lambda^p \int_\Omega |\nabla S(\omega)|^p dx = C\lambda^p \int_\Omega \Big[\frac{a_n}{(\frac{1}{n} + |\omega|)^\delta} S(\omega) + b_n(x)|\omega|^\beta S(\omega)\Big] dx \\ &\leqslant C\lambda^{p-1}\int_\Omega \Big(n^{1+\delta}|\omega| + n|\omega|^{\beta+1}\Big)dx \leqslant C\Big(\|\omega\|_p + \|\omega\|_p^{\beta+1}\Big), \end{aligned}$$

where $C > 0$ is a cumulative constant.

Thus, by the previous inequality, there exists a positive constant R, independent of λ and ω, such that $\|\omega\|_p \leqslant R$. So, by the Schaefer Fixed Point Theorem (see Theorem **A.1.1** in Appendix), there exists a $u_n \in W_0^{1,p}(\Omega)$ such that $S(u_n) = u_n$.

Note that, $a_n(|t| + \frac{1}{n})^{-\delta} + b_n|t|^\beta \leqslant C_n(1 + |t|^\beta)$. Thus, since $\beta < p - 1$ we have $u_n \in L^\infty(\Omega)$, which by Theorem **A.2.1** in Appendix implies $u_n \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Furthermore, $a_n(|u_n| + \frac{1}{n})^{-\delta} + b_n|u_n|^\beta \geqslant 0$ allows to conclude $u_n \geqslant 0$, which by Theorem **A.1.2** lead us to $u_n > 0$ in Ω. Therefore, u_n is a positive solution of (1.1).

Besides, suppose that $\tilde{u}_1$ is a solution of

$$-\Delta_p u = \frac{a_1(x)}{(1+u)^\delta} \text{ in } \Omega,\ u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \Omega. \tag{1.3}$$

By taking $(\tilde{u}_1 - u_n)^+ \in W_0^{1,p}(\Omega)$ as a test function in (1.1) and in (1.3) and using that $-\Delta_p$ is strictly monotonic, we get

$$\begin{aligned} 0 &\leqslant \int_\Omega \Big(|\nabla \tilde{u}_1|^{p-2}\nabla \tilde{u}_1 - |\nabla u_n|^{p-2}\nabla u_n\Big)\nabla(\tilde{u}_1 - u_n)^+ dx \\ &\leqslant \int_\Omega a_1\Big[\frac{1}{(1+\tilde{u}_1)^\delta} - \frac{1}{(1+u_n)^\delta}\Big](\tilde{u}_1 - u_n)^+ dx \leqslant 0, \end{aligned}$$

which leads to $(\tilde{u}_1 - u_n)^+ = 0$, that is, $\tilde{u}_1 \leqslant u_n$ in Ω.

Finally, once again by Theorem **A.2.1**, we conclude that $\tilde{u}_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. As a consequence, using this and the positivity of $\tilde{u}_1$ in Ω, the last part of the Lemma follows.

■

***Proof of Theorem 0.0.3 (Existence-Conclusion)*:** Consider a sequence (Ω_k) of smooth open sets in Ω such that $\Omega_k \subset \Omega_{k+1}$, $\bigcup_k \Omega_k = \Omega$ and define $\delta_k = \inf_{\Omega_k} \tilde{u}_1 > 0$, where $\tilde{u}_1$ is the solution of (1.3). Take $\varphi = (u_n - \delta_1)^+$ as a test function in (1.1), where u_n is a solution of (1.1) obtained in Lemma 1.1.1. If (h_1) holds, then using Hölder's inequality and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we have

$$\begin{aligned} &\int_{u_n>\delta_1} |\nabla u_n|^p dx = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - \delta_1)^+ dx \\ = &\int_\Omega \frac{a_n}{u_n^\delta}(u_n - \delta_1)^+ dx + \int_\Omega b_n u_n^\beta (u_n - \delta_1)^+ dx \\ \leqslant &\int_{u_n>\delta_1} \Big[a(u_n - \delta_1 + \delta_1)^{1-\delta} + b(u_n - \delta_1 + \delta_1)^{\beta+1}\Big] dx \\ \leqslant\ & C\Big[1 + \int_\Omega a[(u_n-\delta_1)^+]^{1-\delta} + b[(u_n-\delta_1)^+]^{\beta+1} dx\Big] \\ \leqslant\ & C\Big[1 + \|a\|_{(\frac{p*}{1-\delta})'}\Big(\int_\Omega [(u_n-\delta_1)^+]^{p^*} dx\Big)^{\frac{1-\delta}{p*}} + \|b\|_{(\frac{p*}{\beta+1})'}\Big(\int_\Omega [(u_n-\delta_1)^+]^{p^*} dx\Big)^{\frac{\beta+1}{p*}}\Big] \\ \leqslant\ & C\Big[1 + \Big(\int_{u_n>\delta_1} |\nabla u_n|^p dx\Big)^{\frac{1-\delta}{p}} + \Big(\int_{u_n>\delta_1} |\nabla u_n|^p dx\Big)^{\frac{\beta+1}{p}}\Big]. \end{aligned}$$

If (h_2) holds, proceeding in a similar way, we get

$$\begin{aligned}
\int_{u_n>\delta_1} |\nabla u_n|^p dx &\leqslant \int_{u_n>\delta_1} \Big(\frac{a}{u_n^{\delta-1}} + b u_n^{\beta+1}\Big) dx \\
&\leqslant C\Big[1 + \delta_1^{1-\delta} \int_\Omega a dx + \|b\|_{(\frac{p^*}{\beta+1})'} \Big(\int_\Omega [(u_n-\delta_1)^+]^{p^*} dx\Big)^{\frac{\beta+1}{p^*}} \\
&\leqslant C\Big[1 + \Big(\int_{u_n>\delta_1} |\nabla u_n|^p dx\Big)^{\frac{\beta+1}{p}}\Big].
\end{aligned}$$

Therefore, $\displaystyle\int_{\Omega_1} |\nabla u_n|^p dx$ will be bounded in any case. In addition, since $(u_n - \delta_1)^+ \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}
\int_{\Omega_1} {u_n}^p dx &\leqslant \int_{u_n>\delta_1} {u_n}^p dx \leqslant C\Big[1 + \int_\Omega (u_n-\delta_1)^{+p} dx\Big] \\
&\leqslant C\Big[1 + \int_{u_n>\delta_1} |\nabla u_n|^p dx\Big] \leqslant C.
\end{aligned}$$

Thus, we conclude that (u_n) is bounded in $W^{1,p}(\Omega_1)$.

Since Ω_k is smooth for all $k \in \mathbb{N}$, there exists $u_{\Omega_1} \in W^{1,p}(\Omega_1)$ and a subsequence $(u_{n_j^1})$ of (u_n) such that

$$\begin{cases}
u_{n_j^1} \rightharpoonup u_{\Omega_1} \text{ weakly in } W^{1,p}(\Omega_1) \\
\text{and strongly in } L^q(\Omega_1) \text{ for } 1 \leqslant q < p^* \\
u_{n_j^1} \to u_{\Omega_1} \ \ a.e \text{ in } \Omega_1.
\end{cases}$$

Proceeding as above, we can obtain subsequences $(u_{n_j^k})$ of (u_n), where $(u_{n_j^{k+1}}) \subset (u_{n_j^k})$, and functions $u_{\Omega_k} \in W^{1,p}(\Omega_k)$ such that

$$\begin{cases}
u_{n_j^k} \rightharpoonup u_{\Omega_k} \text{ weakly in } W^{1,p}(\Omega_k) \text{ and strongly in } L^p(\Omega_k) \text{ for } 1 \leqslant q < p^*, \\
u_{n_j^k} \to u_{\Omega_k} \ a.e \text{ in } \Omega_k.
\end{cases}$$

By construction, $u_{\Omega_{k+1}}\Big|_{\Omega_k} = u_{\Omega_k}$. Defining

$$u = \begin{cases}
u_{\Omega_1} \text{ in } \Omega_1, \\
u_{\Omega_{k+1}} \text{ in } \Omega_{k+1} \backslash \Omega_k,
\end{cases}$$

we get $u \in W^{1,p}_{loc}(\Omega)$. Further, by following close arguments as done in [46], we are able to show that u is a positive solution of (L_1). Indeed:

i) Given $\varphi \in C_c^\infty(\Omega)$, we can fix $k_1 \geqslant 1$ such that supp $\varphi \subset \Omega_{k_1}$. In this case, considering the subsequence $(u_{n_j^{k_1}})$ we have

$$\int_\Omega |\nabla u_{n_j^{k_1}}|^{p-2}\nabla u_{n_j^{k_1}}\nabla\varphi dx = \int_\Omega \Big[\frac{a_{n_j^{k_1}}\varphi}{(u_{n_j^{k_1}}+\frac{1}{n_j^{k_1}})^\delta} + b_{n_j^{k_1}}u^\beta_{n_j^{k_1}}\Big]\varphi dx.$$

As we have seen, $u_{n_j^{k_1}} \to u$ a.e in Ω_{k_1}. Moreover, when $\beta \geqslant 1$ we can easily verify that $1 < \frac{p^*\beta}{\beta+1} < p^*$. In this case, it follows from the compact embedding of $W_0^{1,p}(\Omega_{k_1})$ in the Lebesgue space $L^{\frac{p^*\beta}{\beta+1}}(\Omega_{k_1})$ and Theorem **A.1.3** that, up to subsequence, $u_{n_j^{k_1}} \leqslant h$ for some $h \in L^{\frac{p^*\beta}{\beta+1}}(\Omega_{k_1})$, which gives $|b_{n_j^{k_1}}u^\beta_{n_j^{k_1}}\varphi| \leqslant Cbh^\beta \in L^1(\Omega_{k_1})$.

On the other hand, when $\beta < 1$, by using the compact embedding of $W_0^{1,p}(\Omega_{k_1})$ into $L^{\frac{p^*}{\beta+1}}(\Omega_{k_1})$, Theorem **A.1.3** and Lemma 1.1.1, we get $|b_{n_j^{k_1}}u^\beta_{n_j^{k_1}}\varphi| = bu_{n_j^{k_1}}u^{\beta-1}_{n_j^{k_1}}|\varphi| \leqslant bh(\inf_{\Omega_{k_1}}\tilde{u}_1)^{\beta-1}|\varphi| \in L^1(\Omega_{k_1})$ for some $h \in L^{\frac{p^*}{\beta+1}}(\Omega_{k_1})$.

In any case, it follows from the Dominated Convergence Theorem that

$$\int_\Omega \Big[\frac{a_{n_j^{k_1}}}{(u_{n_j^{k_1}}+\frac{1}{n_j^{k_1}})^\delta} + b_{n_j^{k_1}}u^\beta_{n_j^{k_1}}\Big]\varphi dx \to \int_\Omega \Big(\frac{a}{u^\delta}+bu^\beta\Big)\varphi dx \text{ as } j\to\infty. \quad (1.4)$$

Moreover, once again using Lemma 1.1.1 and defining by $\Theta :=$ supp φ, we get

$$\begin{aligned}\int_\Omega \Big[\frac{a_{n_j^{k_1}}}{(u_{n_j^{k_1}}+\frac{1}{n_j^{k_1}})^\delta} + b_{n_j^{k_1}}u^\beta_{n_j^{k_1}}\Big]\varphi dx &\leqslant \|\varphi\|_\infty\Big(\int_\Omega \frac{a}{\inf_\Theta \tilde{u}_1^\delta}dx\Big) \\ &+ C_\Theta\|\varphi\|_\infty\|b\|_{(\frac{p^*}{1+\beta})'}\Big(\int_{\Omega_{k_1}} u^{p^*}_{n_j^{k_1}}dx\Big)^{\frac{\beta}{p^*}} \\ &\leqslant C_\Theta\|\varphi\|_\infty\Big[1+\Big(\int_{\Omega_{k_1}}|\nabla u_{n_j^{k_1}}|^p dx\Big)^{\frac{\beta}{p}}\Big] \\ &\leqslant C_\Theta\|\varphi\|_\infty,\end{aligned}$$

for some cumulative constant C_Θ, where in the last inequality we use the boundedness of $\left(\int_{\Omega_{k_1}} |\nabla u_{n_j^{k_1}}|^p\right)^{\frac{\beta}{p}}$. Hence, it follows immediately from Theorem **A.1.4** that $\nabla u_{n_j^{k_1}} \to \nabla u$ strongly in $(L^q(\Omega_{k_1}))^N$, for any $q < p$. In particular, $\nabla u_{n_j^{k_1}} \to \nabla u$ a.e in Ω_{k_1} and $|\nabla u_{n_j^{k_1}}|^{p-1} \leqslant h$, for some $h \in L^1(\Omega_{k_1})$. In this way, we conclude

$$\int_\Omega |\nabla u_{n_j^{k_1}}|^{p-2}\nabla u_{n_j^{k_1}}\nabla\varphi dx \to \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx \text{ as } j \to \infty,$$

which together with (1.4) leads to

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx = \int_\Omega \left(\frac{a}{u^\delta} + bu^\beta\right)\varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

ii) Fixe $\epsilon > 0$. By taking $(u_n - \epsilon)^+$ as a test function in (1.1) and proceeding as in the proof of the item$-i$), we can show that $(u_n - \epsilon)^+$ is a bounded sequence in $W_0^{1,p}(\Omega)$.

Thus, there exists $v \in W_0^{1,p}(\Omega)$ such that $(u_n - \epsilon)^+$ converges weakly in $W_0^{1,p}(\Omega)$ to some $v \in W_0^{1,p}(\Omega)$, up to subsequence. However, we have proved in item$-i$) that $u_n \to u$ a.e in Ω, so $v = (u-\epsilon)^+ \in W_0^{1,p}(\Omega)$.

By i) and ii), we conclude $u \in W^{1,p}_{loc}(\Omega)$ is a solution of (L_1) and satisfies the considered boundary condition.

To finish the proof, let us note that when $\delta \leqslant 1$, by taking u_n as test function in (1.1) and following similar arguments as above, one can conclude that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Therefore, u defined as above belongs to $W_0^{1,p}(\Omega)$. ■

1.2 Comparison principle for sub and supersolutions in $W^{1,p}_{loc}(\Omega)$

Now, we are going to prove a Comparison Principle for $W^{1,p}_{\mathrm{loc}}(\Omega)-$sub and supersolutions, whereof will follow the uniqueness stated in Theorem 0.0.3. Besides this result being important in itself, will be a fundamental tool in the other chapters of this thesis.

Before stating the main result of this section, let us define subsolution and supersolution to the problem

$$(L_1) \quad \begin{cases} -\Delta_p u = a(x)u^{-\delta} + b(x)u^{\beta} \text{ in } \Omega, \\ u > 0 \text{ in } \partial\Omega, \quad u > 0 \text{ on } \Omega. \end{cases}$$

Definition 1.2.1 *A function $\underline{v} \in W^{1,p}_{\text{loc}}(\Omega)$ is a subsolution of (L_1) if:*

i) there is a positive constant c_Θ such that $\underline{v} \geqslant c_\Theta$ in Θ for each $\Theta \subset\subset \Omega$ given;

ii) the inequality

$$\int_\Omega |\nabla \underline{v}|^{p-2}\nabla\underline{v}\nabla\varphi dx \leqslant \int_\Omega \Big(\frac{a(x)}{\underline{v}^\delta} + b(x)\underline{v}^\beta\Big)\varphi dx \tag{1.5}$$

holds for all $0 \leqslant \varphi \in C^\infty_c(\Omega)$.

A function $\overline{v} \in W^{1,p}_{\text{loc}}(\Omega)$ satisfying i) and the reversed inequality in (1.5), is called a supersolution of (L_1).

Theorem 1.2.2 ($W^{1,p}_{loc}(\Omega)$-Comparison Principle) *Suppose $b \in L^{(\frac{p^*}{\beta+1})'}(\Omega)$ and $a + b > 0$ in Ω. Assume that one of the following holds*

(h_1): $0 < \delta < 1$ and $a \in L^{(\frac{p^*}{1-\delta})'}(\Omega)$;

(h_2'): $\delta > 1$ and $a \in L^1(\Omega)$,

(h_2''): $\delta = 1$ and $a \in L^s(\Omega)$ for some $s > 1$.

If $\underline{v}, \overline{v} \in W^{1,p}_{loc}(\Omega)$ are subsolution and supersolution of (L_1), respectively, with $\underline{v} \leqslant 0$ in $\partial\Omega$, then $\underline{v} \leqslant \overline{v}$ a.e. in Ω. In addition, if $\underline{v}$, $\overline{v} \in W^{1,p}_0(\Omega)$ and (1.5) is satisfied for all $0 \leqslant \varphi \in W^{1,p}_0(\Omega)$, then the same conclusion holds even for $a \in L^1(\Omega)$ in (h_2'').

To prove Theorem 1.2.2, let us consider for each $\epsilon > 0$ given, the functional $J_\epsilon : W^{1,p}_0(\Omega) \to \mathbb{R}$ defined by

$$J_\epsilon(\omega) = \frac{1}{p}\int_\Omega |\nabla\omega|^p dx - \int_\Omega F_\epsilon(x,\omega)dx,$$

where $F_\epsilon(x,\omega) = \displaystyle\int_0^\omega f_\epsilon(x,s)ds$ with

$$f_\epsilon(x,s) = \begin{cases} a(x)(s+\epsilon)^{-\delta} + b(x)(s+\epsilon)^\beta & \text{if } \ s \geqslant 0 \\ a(x)\epsilon^{-\delta} + b(x)\epsilon^\beta & \text{if } \ s < 0. \end{cases}$$

Also denote by $\mathcal{C}$ the convex and closed set

$$\mathcal{C} = \{\omega \in W_0^{1,p}(\Omega) : 0 \leqslant \omega \leqslant \overline{v}\},$$

where $\overline{v} \in W^{1,p}_{loc}(\Omega)$ is a supersolution to the problem (L_1).

Lemma 1.2.3 *If $b \in L^{(\frac{p^*}{\beta+1})'}(\Omega)$ and one of the hypotheses (h_1), (h_2') or (h_2'') holds, then the functional J_ϵ is coercive and weakly lower semicontinuous on $\mathcal{C}$.*

Proof: Set $\omega \in \mathcal{C}$. First, we note that if (h_2'') holds, then there exists a $C_\epsilon > 0$ such that $\ln|z+\epsilon| \leqslant C_\epsilon(z+\epsilon)^t$ for all $z \geqslant 0$ and for a fix $t = \min\{p^*/s', p-1\} > 0$. Thus, by using either this fact, (h_1) or (h_2') and Sobolev embedding, we obtain

$$J_\epsilon(\omega) \geqslant \begin{cases} \frac{1}{p}\|\nabla\omega\|_p^p - C\Big[\|a\|_{(\frac{p*}{1-\delta})'}\|\omega\|_{p*}^{1-\delta} + \|b\|_{(\frac{p*}{1+\beta})'}\|\omega\|_{p*}^{\beta+1} + 1\Big] & \text{if } \ 0 < \delta < 1, \\ \frac{1}{p}\|\nabla\omega\|_p^p - C\Big[\|a\|_s\|\omega\|_{p*}^t + \|b\|_{(\frac{p*}{1+\beta})'}\|\omega\|_{p*}^{\beta+1} + 1\Big] & \text{if } \ \delta = 1, \\ \frac{1}{p}\|\nabla\omega\|_p^p - C\Big[\|b\|_{(\frac{p*}{1+\beta})'}\|\omega\|_{p*}^{\beta+1} + 1\Big] & \text{if } \ \delta > 1 \end{cases}$$

which leads to the coerciveness of J_ϵ in all the cases.

Next, let us show that J_ϵ is weakly lower semicontinuous on $\mathcal{C}$. Let $(\omega_n) \subset \mathcal{C}$ such that $\omega_n \rightharpoonup \omega$ in $W_0^{1,p}(\Omega)$.

Suppose first that $0 < \delta < 1$ and consider a positive constant C_1 such that $\displaystyle\left(\int_\Omega (\omega_n+\epsilon)^{p^*}dx\right)^{\frac{1-\delta}{p^*}} \leqslant C_1$. We claim that

$$\int_\Omega\int_0^{\omega_n} a(x)(s+\epsilon)^{-\delta}dsdx \longrightarrow \int_\Omega\int_0^\omega a(x)(s+\epsilon)^{-\delta}dsdx \text{ as } n \to \infty. \tag{1.6}$$

In fact, since $a \in L^{\left(\frac{p^*}{1-\delta}\right)'}(\Omega)$ it follows from the absolute continuity of the Lebesgue integral that for given $\epsilon' > 0$, there exists $\delta' > 0$ such that

$$\int_A a(x)^{\frac{p^*}{p^*+\delta-1}} dx \leqslant \left(\frac{\epsilon'}{C_1}\right)^{\frac{p^*}{p^*+\delta-1}},$$

for all measurable subset A of Ω such that $|A| < \delta'$. Thus,

$$\int_A a(x)(\omega_n + \epsilon)^{1-\delta} dx \leqslant \left(\int_A a(x)^{\frac{p^*}{p^*+\delta-1}} dx\right)^{\frac{p^*+\delta-1}{p^*}} \left(\int_\Omega (\omega_n + \epsilon)^{p^*} dx\right)^{\frac{1-\delta}{p^*}} \leqslant \epsilon',$$

that is, (ω_n) has uniformly absolutely continuous integrals over Ω. If $\delta = 1$, we can redo the above arguments. Hence, in both cases our claim follows by applying Vitali's Convergence Theorem (see Theorem **A.1.5**).

In the case $\delta > 1$, the convergence (1.6) follows from the classical Lebesgue's Theorem.

Following close arguments as above, we obtain

$$\int_\Omega \int_0^{\omega_n} b(x)(s+\epsilon)^\beta ds dx \to \int_\Omega \int_0^{\omega} b(x)(s+\epsilon)^\beta ds dx \text{ as } n \to \infty.$$

as well. This finishes the proof of the Lemma. ∎

Since $\mathcal{C}$ is convex and closed in the $W_0^{1,p}(\Omega)$-topology, it follows from Lemma 1.2.3 that there exists a $\omega_0 \in \mathcal{C}$ such that

$$J_\epsilon(\omega_0) = \inf_{\omega \in \mathcal{C}} J_\epsilon(\omega).$$

Lemma 1.2.4 *For all $\varphi \geqslant 0$ in $C_c^\infty(\Omega)$, we have*

$$\int_\Omega |\nabla \omega_0|^{p-2} \nabla \omega_0 \nabla \varphi dx \geqslant \int_\Omega \Big[a(\omega_0+\epsilon)^{-\delta} + b(\omega_0+\epsilon)^\beta\Big]\varphi dx.$$

Proof: First, given a non-negative $\varphi \in C_c^\infty(\Omega)$, for each $t > 0$ let us define $v_t := \min\{\omega_0 + t\varphi, \overline{v}\}$ and $\omega_t := (\omega_0 + t\varphi - \overline{v})^+$. As $\omega_0 \leqslant \overline{v}$, we conclude that $v_t = \omega_0$ and $\omega_t = 0$ in $\Omega \backslash \text{supp } \varphi$. Moreover, since $\overline{v} \in W^{1,p}(\text{supp } \varphi)$ and $0 \leqslant v_t \leqslant \overline{v}$, we have $v_t \in \mathcal{C}$. Besides, since $\overline{v} > 0$ (see definition 1.2.1), we can find a $t > 0$ small enough such that $t\varphi \leqslant 2\overline{v} - \omega_0$, that is, $\omega_t \in \mathcal{C}$ as well.

We define $\sigma : [0,1] \to \mathbb{R}$ by $\sigma(s) = J_\epsilon\Big(sv_t + (1-s)\omega_0\Big)$. Then

$$\begin{aligned} 0 &\leqslant \lim_{s\to 0^+} \frac{\sigma(s)-\sigma(0)}{s} = \lim_{s\to 0^+} \frac{J_\epsilon\Big(sv_t + (1-s)\omega_0\Big) - J_\epsilon(0)}{s} \\ &= \int_\Omega |\nabla\omega_0|^{p-2}\nabla\omega_0\nabla(v_t-\omega_0)dx - \int_\Omega a(x)(\omega_0+\epsilon)^{-\delta}(v_t-\omega_0)dx \\ &\quad - \int_\Omega b(x)(\omega_0+\epsilon)^{\beta}(v_t-\omega_0)dx. \end{aligned}$$

Hence, using $v_t - \omega_0 = t\varphi - \omega_t$ and the previous inequality, we get

$$\begin{aligned} 0 &\leqslant t\int_\Omega \Big[|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\varphi - a(x)(\omega_0+\epsilon)^{-\delta}\varphi - b(x)(\omega_0+\epsilon)^{\beta}\varphi\Big]dx \\ &\quad - \int_\Omega \Big[|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\omega_t - a(x)(\omega_0+\epsilon)^{-\delta}\omega_t - b(x)(\omega_0+\epsilon)^{\beta}\omega_t\Big]dx. \end{aligned} \tag{1.7}$$

However, since $\overline{v}$ is a supersolution of (L_1) and $0 \leqslant \omega_t \in W^{1,p}_0(\Omega) \cap L^\infty_{loc}(\Omega)$ (note that $\omega_t \leqslant t\varphi$), by the classical density arguments one obtains

$$\int_\Omega |\nabla\overline{v}|^{p-2}\nabla\overline{v}\nabla\omega_t dx \geqslant \int_\Omega \Big(a(x)\overline{v}^{-\delta} + b(x)\overline{v}^{\beta}\Big)\omega_t dx. \tag{1.8}$$

Dividing both the sides of (1.7) by $t > 0$ and using (1.8), we get

$$\begin{aligned} 0 &\leqslant \int_\Omega \Big[|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\varphi - a(x)(\omega_0+\epsilon)^{-\delta}\varphi - b(x)(\omega_0+\epsilon)^{\beta}\Big]dx \\ &\quad + \frac{1}{t}\int_\Omega \Big(|\nabla\overline{v}|^{p-2}\nabla\overline{v} - |\nabla\omega_0|^{p-2}\nabla\omega_0\Big)\nabla\omega_t dx \\ &\quad + \frac{1}{t}\int_\Omega \Big[a(x)\Big((\omega_0+\epsilon)^{-\delta} - \overline{v}^{-\delta}\Big) + b(x)\Big((\omega_0+\epsilon)^{\beta} - \overline{v}^{\beta}\Big)\Big]\omega_t dx. \end{aligned} \tag{1.9}$$

Let us estimate now the last two integrals in (1.9). First, by using $\omega_t \to 0$ a.e in Ω as $t \to 0^+$, the limit $|\text{supp } \omega_t| \xrightarrow{t\to 0^+} 0$ and the monotonicity of the p-Laplacian operator, we obtain

$$
\begin{aligned}
&\frac{1}{t}\int_{\Omega}\Big(|\nabla\omega_0|^{p-2}\nabla\omega_0-|\nabla\overline{v}|^{p-2}\nabla\overline{v}\Big)\nabla\omega_t dx\\
=\;&\frac{1}{t}\int_{\text{supp}\omega_t}\Big(|\nabla\omega_0|^{p-2}\nabla\omega_0-|\nabla\overline{v}|^{p-2}\nabla\overline{v}\Big)\nabla(\omega_0-\overline{v})dx\\
&+\int_{\text{supp }\omega_t}\Big(|\nabla\omega_0|^{p-2}\nabla\omega_0-|\nabla\overline{v}|^{p-2}\nabla\overline{v}\Big)\nabla\varphi dx\\
\geqslant\;&\int_{\text{supp }\omega_t}\Big(|\nabla\omega_0|^{p-2}\nabla\omega_0-|\nabla\overline{v}|^{p-2}\nabla\overline{v}\Big)\nabla\varphi dx\to 0 \text{ as } t\to 0.
\end{aligned}
$$

To last integral, noting that $\omega_0\leqslant\overline{v}$, we have

$$
\begin{aligned}
&\frac{1}{t}\int_{\text{supp }\omega_t}\Big[a(x)\Big(\overline{v}^{-\delta}-(\omega_0+\epsilon)^{-\delta}\Big)+b(x)\Big(\overline{v}^{\beta}-(\omega_0+\epsilon)^{\beta}\Big)\Big]\omega_t dx\\
\geqslant\;&-\int_{\text{supp }\omega_t}\Big[a(x)\Big|\overline{v}^{-\delta}-(\omega_0+\epsilon)^{-\delta}\Big|+b(x)\Big|\overline{v}^{\beta}-(\omega_0+\epsilon)^{\beta}\Big|\Big]\varphi dx\to 0 \text{ as } t\to 0.
\end{aligned}
$$

Hence, by using these information in (1.9), we conclude the proof.

∎

***Proof of Theorem 1.2.2-Conclusion*:** Let us set

$$\mathcal{O}_\epsilon := \{x\in\Omega : \underline{v}(x) > \omega_0(x)+\epsilon\} \text{ and } \mathcal{O}^n_\epsilon = \mathcal{O}_\epsilon\cap\{x\in\Omega : \underline{v}(x) < n\}$$

for given $\epsilon>0$ and $n\in\mathbb{N}$. Thus, $\mathcal{O}_\epsilon=\bigcup_{n\in\mathbb{N}}\mathcal{O}^n_\epsilon$.

Assume that $|\mathcal{O}_\epsilon|>0$, for some $\epsilon>0$. Then, it is clear that $|\mathcal{O}^n_\epsilon|>0$ for all $n\geqslant n_0'$ for some $n_0'\in\mathbb{N}$, because $\mathcal{O}^n_\epsilon\subset\mathcal{O}^{n+1}_\epsilon$. Let us fix one of this n. We claim that there exists a ball $B_R(x_0)\subset\subset\Omega$ such that $|B_R(x_0)\cap\mathcal{O}^n_\epsilon|>0$. Indeed, from the compactness of $\overline{\Omega}$, we can find an open set $B\subset\mathbb{R}^N$ such that $|B\cap\mathcal{O}^n_\epsilon|>0$. Denote this measure by $|B\cap\mathcal{O}^n_\epsilon|=2\delta'>0$. If $B\cap\partial\Omega\neq\varnothing$, set $\Omega_{\epsilon_0}=\{x\in\Omega : dist(x,\partial\Omega)<\epsilon_0\}$, where $\epsilon_0>0$ is taken in such a way that $|B\cap\Omega_{\epsilon_0}|<\delta'$. In this case, $|\overline{B\cap\Omega^C_{\epsilon_0}}\cap\mathcal{O}^n_\epsilon|>\delta'$. So our claim follows from the fact that $\overline{B\cap\Omega^C_{\epsilon_0}}$ is a compact set.

Set $\phi\in C^\infty_c(\Omega,[0,1])$ such that supp $\phi\subset B_{R+r}(x_0)$, $\phi=1$ in $B_R(x_0)$ and $|\nabla\phi|\leqslant Cr^{-\tau}$ in $B_{R+r}(x_0)\backslash B_R(x_0)$ for an appropriate $\tau>0$, which will be determined later. Thus, it is a consequence of this construction that $0 \neq \varphi_1, \varphi_2 \in L^\infty_c(\Omega)$, where

$$\varphi_1 := \phi\Big[v_n^p - (\omega_0+\epsilon)^p\Big]^+ v_n^{1-p} \ \text{ and } \ \varphi_2 := \phi\Big[v_n^p - (\omega_0+\epsilon)^p\Big]^+ (\omega_0+\epsilon)^{1-p},$$

with $v_n := \min\{\underline{v}, n\}$.

Hence,

$$\begin{aligned}\nabla\varphi_1 &= \phi\Big[\nabla v_n - p\frac{(\omega_0+\epsilon)^{p-1}}{v_n^{p-1}}\nabla(\omega_0+\epsilon) + (p-1)\frac{(\omega_0+\epsilon)^p}{v_n^p}\nabla v_n\Big]\chi_{[v_n \geqslant \omega_0+\epsilon]} \\ &\quad + \Big[\frac{v_n^p - (\omega_0+\epsilon)^p}{v_n^{p-1}}\Big]^+\nabla\phi\end{aligned}$$

and

$$\begin{aligned}\nabla\varphi_2 &= \phi\Big[\frac{p v_n^{p-1}}{(\omega_0+\epsilon)^{p-1}}\nabla v_n - \nabla(\omega_0+\epsilon) - (p-1)\frac{v_n^p}{(\omega_0+\epsilon)^p}\nabla(\omega_0+\epsilon)\Big]\chi_{[v_n \geqslant \omega_0+\epsilon]} \\ &\quad + \Big[\frac{v_n^p - (\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\nabla\phi,\end{aligned}$$

which leads to $|\nabla\varphi_1|, |\nabla\varphi_2| \in L^p(\Omega)$, because $0 < c_\Theta \leqslant v_n \leqslant n$ in $\Theta = \operatorname{supp}\phi$.

Since $\varphi_1, \varphi_2 \geqslant 0$ and $\varphi_1, \varphi_2 \in W^{1,p}_0(\Omega) \cap L^\infty_c(\Omega)$, we get by density arguments that

$$\int_\Omega |\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\varphi_1 dx \leqslant \int_\Omega \Big(a(x)\underline{v}^{-\delta} + b(x)\underline{v}^\beta\Big)\varphi_1 dx$$

and

$$\int_\Omega |\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\varphi_2 dx \geqslant \int_\Omega \Big[a(x)(\omega_0+\epsilon)^{-\delta} + b(x)(\omega_0+\epsilon)^\beta\Big]\varphi_2 dx$$

hold, where ω_0 is as in Lemma 1.2.4.

Therefore, by calculating and using the above inequalities, we obtain

$$\begin{aligned}\int_\Omega \Big(a(x)\underline{v}^{-\delta} + b(x)\underline{v}^\beta\Big)\varphi_1 dx &\geqslant \int_{[\underline{v}\leqslant n]} |\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\Big[\frac{\underline{v}^p - (\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\phi dx \\ &\quad + \int_\Omega |\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\phi\Big[\frac{v_n^p - (\omega_0+\epsilon)^p}{v_n^{p-1}}\Big]^+ dx \\ &\quad - p\int_{[\underline{v}>n]} |\nabla\underline{v}|^{p-2}\nabla\underline{v}\Big[\frac{(\omega_0+\epsilon)^{p-1}\nabla(\omega_0+\epsilon)}{n^{p-1}}\Big]\chi_{[\omega_0+\epsilon<n]}\phi dx\end{aligned}$$

and

$$\int_\Omega\Big[a(x)(\omega_0+\epsilon)^{-\delta}+b(x)(\omega_0+\epsilon)^{\beta}\Big]\varphi_2 dx \leqslant \int_\Omega |\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\phi\Big[\frac{v_n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+dx$$
$$+\int_{[\underline{v}\leqslant n]}|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\phi dx+\int_{[\underline{v}>n]}|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\phi dx.$$

Hence, by combining the previous inequalities we have

$$\begin{aligned}
&\int_\Omega|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\phi\Big[\frac{v_n^p-(\omega_0+\epsilon)^p}{v_n^{p-1}}\Big]^+dx+\int_{[\underline{v}\leqslant n]}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\phi dx\\
&-p\int_{[\underline{v}>n]}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\Big[\frac{(\omega_0+\epsilon)^{p-1}\nabla(\omega_0+\epsilon)}{n^{p-1}}\Big]\chi_{[\omega_0+\epsilon<n]}\phi dx\\
&-\int_\Omega|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\phi\Big[\frac{v_n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+dx\\
&-\int_{[\underline{v}\leqslant n]}|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\phi dx\\
&-\int_{[\underline{v}>n]}|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\phi dx\\
=\;&\int_\Omega|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\varphi_1dx-\int_\Omega|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\varphi_2dx\\
\leqslant\;&\int_\Omega a(x)\Big[\frac{\underline{v}^{-\delta}}{v_n^{p-1}}-\frac{(\omega_0+\epsilon)^{-\delta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\phi dx\\
&+\int_\Omega b(x)\Big[\frac{\underline{v}^{\beta}}{v_n^{p-1}}-\frac{(\omega_0+\epsilon)^{\beta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\phi dx.
\end{aligned}$$

Since

$$-\int_{[\underline{v}>n]}|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]\phi dx=\int_{[\underline{v}>n]}|\nabla\omega_0|^p\Big[1+\frac{n^p(p-1)}{(\omega_0+\epsilon)^p}\Big]dx\geqslant 0,$$

by using the previous inequalities and the classical Picone's Identity (see Theorem **A.1.6**), we get

$$\begin{aligned}
0\;\leqslant\;&\int_{[\underline{v}\leqslant n]}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\phi dx\\
&-\int_{[\underline{v}\leqslant n]}|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\phi dx\\
\leqslant\;&\frac{p}{n^{p-1}}\int_{[\underline{v}>n]}|\nabla\underline{v}|^{p-1}|\nabla\omega_0|(\omega_0+\epsilon)^{p-1}\chi_{[\omega_0+\epsilon<n]}\phi dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{B_{R+r}\backslash B_R} |\nabla \underline{v}|^{p-1}|\nabla\phi|\Big[\frac{v_n^p-(\omega_0+\epsilon)^p}{v_n^{p-1}}\Big]^+dx\\
&+\int_{B_{R+r}\backslash B_R} |\nabla \omega_0|^{p-1}|\nabla\phi|\Big[\frac{v_n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+dx\\
&+\int_\Omega a(x)\Big[\frac{\underline{v}^{-\delta}}{v_n^{p-1}}-\frac{(\omega_0+\epsilon)^{-\delta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\phi dx\\
&+\int_\Omega b(x)\Big[\frac{\underline{v}^{\beta}}{v_n^{p-1}}-\frac{(\omega_0+\epsilon)^{\beta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\phi dx. \qquad (1.10)
\end{aligned}
$$

Next, let us estimate the integrals in (1.10).

For the last two integrals, we can deduce by the assumption $a+b>0$, the inequality $\underline{v}^{-\delta}\leqslant v_n^{-\delta}$ $(n\in\mathbb{N})$ and Lebesgue's Theorem, that

$$
\begin{aligned}
-4\epsilon' \;>\; &\int_\Omega a(x)\Big[\frac{\underline{v}^{-\delta}}{v_{n_0}^{p-1}}-\frac{(\omega_0+\epsilon)^{-\delta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_{n_0}^p-(\omega_0+\epsilon)^p]^+\phi dx\\
&+\int_\Omega b\Big(\frac{\underline{v}^{\beta}}{v_{n_0}^{p-1}}-\frac{(\omega_0+\epsilon)^{\beta}}{(\omega_0+\epsilon)^{p-1}}\Big)\Big[v_{n_0}^p-(\omega_0+\epsilon)^p\Big]^+\phi dx,
\end{aligned}
$$

holds for some $\epsilon'>0$ and $n_0>1$ large.

Now, let us consider the first integral in the second line. We claim that $|[\underline{v}>n]|\overset{n\to\infty}{\longrightarrow}0$. Indeed, otherwise would exists $\delta'>0$ and a subsequence $\mathbb{N}'\subset\mathbb{N}$ such that $|[(\underline{v}-\epsilon)^+>n-\epsilon]|=|[\underline{v}>n]|>\delta'$, for all $n\in\mathbb{N}'$. By using that $(\underline{v}-\epsilon)^+\in W_0^1(\Omega)$, we would have

$$
(n-\epsilon)\delta'<\int_{[(\underline{v}-\epsilon)^+>n-\epsilon]}(\underline{v}-\epsilon)^+dx\leqslant\int_\Omega(\underline{v}-\epsilon)^+dx\leqslant C\|\nabla(\underline{v}-\epsilon)^+\|_p<\infty,\ \forall n\in\mathbb{N}',
$$

which is absurd. Therefore, as $|[\underline{v}>n]|\overset{n\to\infty}{\longrightarrow}0$ and n_0 was taking sufficiently large, we obtain

$$
\begin{aligned}
\Big|p\int_{[\underline{v}>n_0]}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\Big[\frac{(\omega_0+\epsilon)^{p-1}\nabla(\omega_0+\epsilon)}{{n_0}^{p-1}}\Big]\chi_{[\omega_0+\epsilon<n_0]}\phi dx\Big| &\leqslant\Big(\int_{[\underline{v}>n_0]}|\nabla\underline{v}|^p\phi^p\Big)^{\frac{p-1}{p}}\|\nabla\omega_0\|_p\\
&\leqslant\epsilon'.
\end{aligned}
$$

To estimate the first integral on $B_{R+r}\backslash B_R$, we note that the choice of ϕ

leads to

$$\begin{aligned}
\int_{B_{R+r}\setminus B_R} |\nabla \underline{v}|^{p-1}|\nabla\phi|\Big[\frac{v_{n_0}^p-(\omega_0+\epsilon)^p}{v_{n_0}^{p-1}}\Big]^+ dx &\leqslant \int_{B_{R+r}\setminus B_R} |\nabla \underline{v}|^{p-1}|\nabla\phi| n_0 dx\\
&\leqslant Cn_0\|\nabla\phi\|_{L^p(B_{R+r}\setminus B_R)}\\
&\leqslant Cn_0 r^{-\tau}|B_{R+r}\setminus B_R|^{\frac{1}{p}} \leqslant C_1 n_0 r^{-\tau+\frac{1}{p}}.
\end{aligned}$$

By taking a $\tau < 1/p$, we can choose $r > 0$ sufficiently small such that

$$\int_{B_{R+r}\setminus B_R} |\nabla \underline{v}|^{p-1}|\nabla\phi|\Big[\frac{v_{n_0}^p-(\omega_0+\epsilon)^p}{v_{n_0} n^{p-1}}\Big]^+ dx < \epsilon'.$$

In a similar way, we can infer

$$\int_{B_{R+\delta}\setminus B_R} |\nabla \omega_0|^{p-1}|\nabla\phi|\Big[\frac{v_{n_0}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+ dx < \epsilon'$$

as well.

Hence, getting back to the inequality (1.10) and using the above information, we get

$$\begin{aligned}
0 \leqslant & \int_{[\underline{v}\leqslant n_0]} |\nabla \underline{v}|^{p-2}\nabla\underline{v}\nabla\Big(\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big)\phi dx\\
&-\int_{[\underline{v}\leqslant n_0]} |\nabla \omega_0|^{p-2}\nabla\omega_0\nabla\Big(\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big)\phi dx < 0,
\end{aligned}$$

which is an absurd. Therefore $|\mathcal{O}_\epsilon^n| = 0$ for all n, which implies $|\mathcal{O}_\epsilon| = 0$ and so $\underline{v} \leqslant \omega_0+\epsilon \leqslant \overline{v}+\epsilon$ a.e in Ω for all $\epsilon > 0$, whence $\underline{v} \leqslant \overline{v}$.

To finish the proof, let us assume that $\underline{v}$, $\overline{v} \in W_0^{1,p}(\Omega)$ and (1.5) is satisfied for all $0 \leqslant \varphi \in W_0^{1,p}(\Omega)$. If we suppose $(\underline{v}-\overline{v})^+ \neq 0$, then by defining $\underline{v}_n^\epsilon(x) := \min\{\underline{v}(x)+\epsilon, n\}$, $\overline{v}_n^\epsilon(x) := \min\{\overline{v}(x)+\epsilon, n\}$ and the test functions

$$\varphi_1 = \Big[(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p\Big]^+ (\underline{v}_n^\epsilon)^{1-p} \text{ and } \varphi_2 = \Big[(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p\Big]^+ (\overline{v}_n^\epsilon)^{1-p},$$

we obtain

$$\begin{aligned}
&\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]}\Big(-|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\overline{v}\frac{(\overline{v}+\epsilon)^{p-1}p}{n^{p-1}}+|\nabla\overline{v}|^p+\frac{(p-1)n^p|\nabla\overline{v}|^p}{(\overline{v}+\epsilon)^p}\Big)dx\\
&+\int_{[\overline{v}+\epsilon\leqslant\underline{v}+\epsilon\leqslant n]}\Big(|\nabla\underline{v}|^p-p\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^{p-1}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\overline{v}+(p-1)\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^p|\nabla\underline{v}|^p\\
&\qquad+|\nabla\overline{v}|^p-p\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^{p-1}|\nabla\overline{v}|^{p-2}\nabla\overline{v}\nabla\underline{v}+(p-1)\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^p|\nabla\overline{v}|^p\Big)dx\\
=&\int_\Omega|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\varphi_1dx-\int_\Omega|\nabla\overline{v}|^{p-2}\nabla\overline{v}\nabla\varphi_2dx\\
\leqslant&\int_\Omega a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}^\epsilon_n)^{p-1}}-\frac{\overline{v}^{-\delta}}{(\overline{v}^\epsilon_n)^{p-1}}\Big][(\underline{v}^\epsilon_n)^p-(\overline{v}^\epsilon_n)^p]^+dx\\
&+\int_\Omega b\Big[\frac{\underline{v}^{\beta}}{(\underline{v}^\epsilon_n)^{p-1}}-\frac{\overline{v}^{\beta}}{(\overline{v}^\epsilon_n)^{p-1}}\Big][(\underline{v}^\epsilon_n)^p-(\overline{v}^\epsilon_n)^p]^+dx.
\end{aligned}$$

Denoting by

$$\begin{aligned}
I=&\int_{[\overline{v}+\epsilon\leqslant\underline{v}+\epsilon\leqslant n]}\Big(|\nabla\underline{v}|^p-p\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^{p-1}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\overline{v}+(p-1)\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^p|\nabla\underline{v}|^p\\
&\qquad+|\nabla\overline{v}|^p-p\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^{p-1}|\nabla\overline{v}|^{p-2}\nabla\overline{v}\nabla\underline{v}+(p-1)\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^p|\nabla\overline{v}|^p\Big)dx
\end{aligned}$$

and using the previous inequality along with the Picone's Identity (Theorem **A.1.6**), we have

$$\begin{aligned}
0\leqslant I\leqslant&\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]}|\nabla\underline{v}|^{p-1}|\nabla\overline{v}|dx+\int_\Omega a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}^\epsilon_n)^{p-1}}-\frac{\overline{v}^{-\delta}}{(\overline{v}^\epsilon_n)^{p-1}}\Big][(\underline{v}^\epsilon_n)^p-(\overline{v}^\epsilon_n)^p]^+dx\\
&+\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]}b\Big[\frac{\underline{v}^\beta}{n^{p-1}}-\frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][n^p-(\overline{v}+\epsilon)^p]dx\\
&+\int_{[\overline{v}+\epsilon\leqslant\underline{v}+\epsilon\leqslant n]}b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}}-\frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p-(\overline{v}+\epsilon)^p]dx. \quad (1.11)
\end{aligned}$$

Let us consider each one of the integrals in (1.11).

First, note that the Dominated Convergence Theorem implies that

$$\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]}|\nabla\underline{v}|^{p-1}|\nabla\overline{v}|dx\to 0 \text{ as } n\to\infty. \quad (1.12)$$

By manipulating the second integral in (1.11), we obtain

$$\int_\Omega a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}_n^\epsilon)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}_n^\epsilon)^{p-1}}\Big][(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p]^+ dx \leqslant 0 \tag{1.13}$$

for all $n \in \mathbb{N}$ and $\epsilon > 0$. By Dominated Convergence Theorem once again, we also get

$$\begin{aligned}&\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]} b\Big[\frac{\underline{v}^\beta}{n^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][n^p - (\overline{v}+\epsilon)^p]dx \\ \leqslant &\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]} b\Big[\underline{v}^\beta(\underline{v}+\epsilon) + \overline{v}^\beta(\overline{v}+\epsilon)\Big]dx \to 0 \text{ as } n \to \infty. \end{aligned} \tag{1.14}$$

For the last integral, since

$$b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]^+ \leqslant \Big[\underline{v}^\beta(\underline{v}+\epsilon) + \overline{v}^\beta(\overline{v}+\epsilon)\Big] \in L^1(\Omega),$$

it follows from the Fatou's Lemma that

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_{[\overline{v}+\epsilon\leqslant\underline{v}+\epsilon\leqslant n]} b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]dx \\ \leqslant &\int_{[\overline{v}+\epsilon\leqslant\underline{v}+\epsilon\leqslant n]} b\Big[\frac{\underline{v}^\beta}{\underline{v}^{p-1}} - \frac{\overline{v}^\beta}{\overline{v}^{p-1}}\Big][\underline{v}^p - \overline{v}^p]dx \leqslant 0, \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{1.15}$$

Hence, going back to (1.11) and using (1.12), (1.13), (1.14) and (1.15), we get

$$\begin{aligned}0 \leqslant \limsup_{\epsilon\to 0^+}\liminf_{n\to\infty} I \leqslant \limsup_{\epsilon\to 0^+}\liminf_{n\to\infty}\Big(&\int_\Omega a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}_n^\epsilon)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}_n^\epsilon)^{p-1}}\Big][(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p]^+ dx \\ &+ \int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leqslant n]} b\Big[\frac{\underline{v}^\beta}{n^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][n^p - (\overline{v}+\epsilon)^p]dx \\ &+ \int_{[\overline{v}+\epsilon\leqslant\underline{v}+\epsilon\leqslant n]} b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]dx\Big).\end{aligned}$$

Since we are assuming that $(\underline{v} - \overline{v})^+ \neq 0$ and $a + b > 0$, it follows from the previous inequality that

$$0 \leqslant \limsup_{\epsilon\to 0^+}\liminf_{n\to\infty} I < 0,$$

which is absurd. Therefore, $(\underline{v} - \overline{v})^+ = 0$ and this ends the proof. ∎

Following the proof of the above Theorem, we have the next result.

Corollary 1.2.5 *Suppose that $-\infty < \theta_1 \leqslant \theta_2 < p-1$, $a_1+a_2 > 0$ in Ω and $a_1 \leqslant a_2$ in Ω hold. Assume that the pair (θ_i, a_i) satisfies one of the following conditions:*

- $-1 < \theta_i < p-1$ *and* $a_i \in L^{(\frac{p^*}{p^*-1-\theta_i})}(\Omega)$,
- $\theta_i < -1$ *and* $a_i \in L^1(\Omega)$,
- $\theta_i = -1$ *and* $a_i \in L^s(\Omega)$ *for some* $s > 1$

for $i \in \{1, 2\}$.

If $\underline{v}, \overline{v} \in W^{1,p}_{\text{loc}}(\Omega)$ are subsolution and supersolution, respectively, of

$$\begin{cases} -\Delta_p u = a_1(x)u^{\theta_1}\chi_{[u<a]} + a_2(x)u^{\theta_2}\chi_{[u\geqslant a]} \text{ in } \Omega, \\ u > 0 \text{ in } \partial\Omega, \quad u > 0 \text{ on } \Omega, \end{cases}$$

with $\underline{v} \leqslant 0$ in $\partial\Omega$ and $0 \leqslant a < 1$, then $\underline{v} \leqslant \overline{v}$ a.e. in Ω.

Proof: It is sufficient to revisit the proof of Theorem 1.2.2 and observe that, under the contradictory assumption $|[(u^p - v^p)^+\phi > 0]| > 0$, we also obtain

$$\begin{aligned} &\int_{[u\geqslant v]} \Big[\frac{a_1(x)u^{\theta_1}\chi_{[u<a]} + a_2(x)u^{\theta_2}\chi_{[u\geqslant a]}}{u^{p-1}}\Big](u^p - v^p)\varphi dx \\ - &\int_{[u\geqslant v]} \Big[\frac{a_1(x)v^{\theta_1}\chi_{[v<a]} + a_2(x)v^{\theta_2}\chi_{[v\geqslant a]}}{v^{p-1}}\Big](u^p - v^p)\varphi dx < 0, \end{aligned}$$

which leads us to a similar contradiction, as in the proof of Theorem 1.2.2.

∎

***Proof of Theorem 0.0.3 (Uniqueness)*:** In any case, by the Theorem 1.2.2 we get $u \leqslant v$ and $v \leqslant u$, which results in $u = v$. ∎

CHAPTER 2

BREAKING THE UNIQUENESS OF SOLUTIONS OF A VERY SINGULAR PROBLEM BY NON-LOCAL TERMS

In this chapter, we are going to study

$$(P_1) \qquad \begin{cases} -\Big(\displaystyle\int_\Omega g(x,u)dx\Big)^r \Delta_p u = \lambda\Big(a(x)u^{-\delta} + b(x)u^{\beta}\Big) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a smooth bounded domain, $1 < p < N$, $\delta > 0$, $0 < \beta < p-1$, $\lambda > 0$ is a real parameter and $a, b, g \geqslant 0$ are appropriate functions.

As we saw in the previous chapter, in the local case ($r = 0$), the problem (P_1) admits a unique solution. However, as we shall see shortly, by introducing the non-local term, this behavior may change completely. In fact, we will see that there are situations in which global multiplicity is guaranteed.

This chapter has the following structure. In the first section, by exploring the uniqueness of $W^{1,p}_{\text{loc}}(\Omega)$-solutions to Problem (L_α), we will prove how these solutions behave with respect to the parameter α. It enables us to prove that the operator $T : (0, \infty) \to W^{1,p}_{\text{loc}}(\Omega)$ (see (2.1) below) is well-defined and continuous. In Section 2.2, we conclude the proof of Theorem 0.0.5.

Below, let us rewrite the hypotheses that will be considered in this chapter.

(h_3): $a, b \in L^m(\Omega)$ for some $m > N/p$,

(h_4): $a, b \in L^m(\Omega)$ for some $m > N$

(g_∞): $\lim_{t\to\infty} g(x,t)t^{\theta_\infty} = g_\infty(x) > 0$ uniformly in $\overline{\Omega}$, for some $\theta_\infty \in \mathbb{R}$ and $g_\infty \in C(\overline{\Omega})$,

(g'_∞): $\lim_{t\to\infty} g(x,t)t^{\theta_\infty} = +\infty$ uniformly in $\overline{\Omega}$, for some $\theta_\infty \in \mathbb{R}$,

(g_0): $\lim_{t\to 0^+} g(x,t)t^{\theta_0} = g_0(x) > 0$ uniformly in $\overline{\Omega}$, for some $\theta_0 \in \mathbb{R}$ and $g_0 \in C(\overline{\Omega})$,

(g'_0): $\lim_{t\to 0^+} g(x,t)t^{\theta_0} = \infty$ uniformly in $\overline{\Omega}$, for some $\theta_0 \in \mathbb{R}$

2.1 $W^{1,p}_{loc}(\Omega)$-continuity and a α-behavior for a solution application

Throughout this section, we are going to assume the hypotheses of Theorem 0.0.3. Thus, it is well-defined the solution application $T : (0,\infty) \to W^{1,p}_{loc}(\Omega)$ given by

$$T(\alpha) = u_\alpha, \tag{2.1}$$

where $u_\alpha \in W^{1,p}_{loc}(\Omega)$ is the unique solution of Problem (L_α) given by Theorem 0.0.3.

Besides, the Proposition below it is an immediate consequence of Theorem 1.2.2.

Proposition 2.1.1 *The application T is non-decreasing.*

Next, let us prove that T is a "$W^{1,p}_{loc}(\Omega)$-continuous application", i.e.

if $\alpha_n \to \alpha$ in $\mathbb{R}$, then $T(\alpha_n) \to T(\alpha)$ in $W^{1,p}(U)$ for each $U \subset\subset \Omega$ given.

In what follows, $\Phi_{H_1} \in W^{1,p}_0(\Omega)$ will denote the positive normalized eigenfunction associated to

$$-\Delta_p \Phi_{H_1} = \lambda_1 H_1(x) {\Phi_{H_1}}^{p-1} \text{ in } \Omega, \quad \Phi_{H_1}|_{\partial\Omega} = 0 \tag{2.2}$$

where $H_1(x) := \min\{a(x), b(x)\} \geqslant 0$ and $\lambda_1 > 0$ is the first eigenvalue of (2.2) (see [22] for more details about (2.2)). If (h_3) is satisfied, then by [38] one can conclude that $\Phi_{H_1} \in C(\overline{\Omega})$. Moreover, if (h_4) holds, then Φ_{H_1} belongs to the interior of the positive cone in $C^1_0(\overline{\Omega})$ (see Theorem **A.2.2**) and hence for some positive constant C, one has

$$Cd(x) \leqslant \Phi_{H_1}(x) \text{ in } \Omega, \tag{2.3}$$

where $d(x)$ stands for the distance between $x \in \Omega$ and the boundary $\partial\Omega$.

Similarly, defining $H_2(x) := \max\{a(x), b(x)\} \geqslant 0$ and denoting the unique positive solution of

$$-\Delta_p u = H_2(x) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

by $e_{H_2} \in W^{1,p}_0(\Omega)$, it follows from (h_3) and [38] that $e_{H_2} \in C(\overline{\Omega})$.

Lemma 2.1.2 **($T(\alpha)$-behavior for small $\alpha > 0$)** *Suppose that (h_3) is satisfied. Then $T(\alpha) \in [\underline{u}_\alpha, \overline{u}_\alpha]$ for all $\alpha \in (0, 1]$, where $\underline{u}_\alpha := m_1\alpha^\tau\Phi_{H_1}$ and $\overline{u}_\alpha := m_2\alpha^\tau e^t_{H_2}$, with $\tau = \frac{1}{p-1+\delta}$, $t = \frac{p-1}{p-1+\delta}$ and m_1, m_2 appropriate positive constants independent of α. In particular, $T(\alpha) \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for all $\alpha \in (0, 1]$.*

Proof: Let $\alpha > 0$. Since $\tau = \frac{1}{p-1+\delta}$ holds, by fixing $m_1 = \left(1/\lambda_1\|\Phi^{1/\tau}_{H_1}\|_\infty\right)^\tau$ we have $m_1^{1/\tau}\sup_{\overline{\Omega}}\Phi^{1/\tau}_{H_1}\lambda_1 H_1(x) \leqslant a(x)$ in Ω. Thus,

$$-\Delta_p\underline{u}_\alpha \leqslant \lambda_1\alpha^{\tau(p-1)}H_1(x)\sup_{\overline{\Omega}}\Phi^{p-1}_{H_1} \leqslant \frac{\alpha^{1-\tau\delta}}{\sup\limits_{\overline{\Omega}}\Phi^\delta_{H_1}}a(x) = \alpha\frac{a(x)}{\underline{u}^\delta_\alpha} \leqslant \alpha\Big(\frac{a(x)}{\underline{u}^\delta_\alpha} + b(x)\underline{u}^\beta_\alpha\Big)$$

holds true.

To the supersolution, define $\overline{u}_\alpha = m_2\alpha^\tau e^t_{H_2}$, where $t = \frac{p-1}{p-1+\delta}$, $\tau = \frac{1}{p-1+\delta}$ and m_2 will be chosen later. Hence, by using $0 < t < 1$ we obtain

$$\begin{aligned}\int_\Omega |\nabla\overline{u}_\alpha|^{p-2}\nabla\overline{u}_\alpha\nabla\varphi dx &\geqslant \int_\Omega |\nabla e_{H_2}|^{p-2}\nabla e_{H_2}\nabla\Big[\varphi(\alpha^\tau m_2 e^{t-1}_{H_2}t)^{p-1}\Big]dx \\ &= \int_\Omega H_2(x)\Big[\varphi(\alpha^\tau m_2 e^{t-1}_{H_2}t)^{p-1}\Big]dx\end{aligned}$$

for all $\varphi \geqslant 0$ in $C^\infty_c(\Omega)$.

To verify that $\overline{u}_\alpha$ is a supersolution for (L_α) for $\alpha \in (0,1]$, by the previous inequality it is enough to show that

$$(\alpha^\tau m_2 t)^{p-1} \geqslant \alpha \max\{1, \|e_{H_2}^{t(\beta+\delta)}\|_\infty\}(m_2^{-\delta}\alpha^{-\tau\delta} + m_2^\beta \alpha^{\tau\beta}),$$

for some m_2 appropriate. Therefore, if we take

$$m_2 = \max\Big\{1, \Big(\frac{3\max\{1, \|e_{H_2}^{t(\beta+\delta)}\|_\infty\}}{t^{p-1}}\Big)^{1/(p-1-\beta)}\Big\},$$

since $\alpha \in (0,1]$ the previous inequality holds. Hence, for this choice of m_2 and $\alpha \in (0,1]$, $\overline{u}_\alpha$ is a supersolution for (L_α).

As u_α is simultaneously a sub and supersolution to (L_α), the inclusion $T(\alpha) \subset [\underline{u}_\alpha, \overline{u}_\alpha]$ is a consequence of the comparison principle proved in Theorem 1.2.2.

Finally, it follows from the hypothesis (h_3), the fact that $T(\alpha) \in [\underline{u}_\alpha, \overline{u}_\alpha]$ and Corollary 8.1 in [38] that $u_\alpha \in C(\Omega)$ for $\alpha \in (0,1]$. As $\underline{u}_\alpha$ and $\overline{u}_\alpha \in C(\overline{\Omega})$ and $\underline{u}_\alpha|_{\partial\Omega} = \overline{u}_\alpha|_{\partial\Omega} = 0$, the required regularity follows.

■

Following close arguments as done above, we can prove the next Lemma.

Lemma 2.1.3 ($T(\alpha)$-behavior for large $\alpha > 0$) *Suppose that (h_3) is satisfied. Then $T(\alpha) \in [\underline{u}_\alpha, \overline{u}_\alpha]$ for all $\alpha \in (1,\infty)$, where $\underline{u}_\alpha := m_1\alpha^\tau\Phi_{H_1}$ and $\overline{u}_\alpha := m_2\alpha^\tau e^t_{H_2}$, with $\tau = \frac{1}{p-1-\beta}$, $t = \frac{p-1}{p-1+\delta}$ and m_1, m_2 appropriate positive constants independent of α. In particular, $T(\alpha) \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for all $\alpha > 1$.*

After the above Lemmas, we obtain that

$$T((0,\infty)) \subset W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$$

when (h_3) holds. Now, we are in position to prove the continuity of T.

Lemma 2.1.4 *Suppose (h_3) holds. Then T is a continuous application in the $W^{1,p}_{loc}(\Omega)$ topology as well as in $C(\overline{\Omega})$.*

Proof: First let us prove the continuity of T in the $W^{1,p}_{loc}(\Omega)$-topology. Consider $\alpha_n \to \alpha > 0$ in $\mathbb{R}$. Then, it follows from Lemmas 2.1.2, 2.1.3 and monotonicity established in the Proposition 2.1.1, that there exist $0 < \underline{\alpha} < 1$ and $\overline{\alpha} > 1$ such that

$$\underline{\alpha}^{1/(p-1+\delta)} m_1 \Phi_{H_1} \leqslant u_{\underline{\alpha}} \leqslant u_{\alpha_n} \leqslant u_{\overline{\alpha}} \leqslant \overline{\alpha}^{1/(p-1-\beta)} m_2 e^t_{H_2} \ \text{ in } \ \Omega, \ \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Take an open set $U \subset\subset \Omega$ and $\xi \in C^\infty_c(\Omega)$ such that $0 \leqslant \xi \leqslant 1$ and $\xi = 1$ in U. By using $u_{\alpha_n}\xi^p$ as a test functions in (L_{α_n}), we obtain

$$\begin{aligned} &\int_\Omega |\nabla u_{\alpha_n}|^p \xi^p dx + p\int_\Omega |\nabla u_{\alpha_n}|^{p-2} \nabla u_{\alpha_n} \nabla \xi u_{\alpha_n} \xi^{p-1} dx \\ = \ &\alpha_n \int_\Omega \Big[a(x) u_{\alpha_n}^{-\delta+1} + b(x) u_{\alpha_n}^{\beta+1}\Big] \xi^p dx. \end{aligned} \tag{2.5}$$

Thus, it follows from the boundedness of (u_n) in $L^\infty(\Omega)$ and Young's inequality that

$$\begin{aligned} \int_\Omega |\nabla u_{\alpha_n}|^{p-2} \nabla u_{\alpha_n} \nabla \xi u_{\alpha_n} \xi^{p-1} dx &\leqslant \int_\Omega |\nabla u_{\alpha_n}|^{p-1} |\nabla \xi| u_{\alpha_n} \xi^{p-1} dx \\ &\leqslant \epsilon \int_\Omega \left(|\nabla u_{\alpha_n}|^{p-1} \xi^{p-1}\right)^{\frac{p}{p-1}} dx + C(\epsilon) \int_\Omega u^p_{\alpha_n} |\nabla \xi|^p dx \\ &\leqslant \epsilon \int_\Omega \xi^p |\nabla u_{\alpha_n}|^p dx + C(\epsilon), \end{aligned} \tag{2.6}$$

where $C(\epsilon)$ is a cumulative positive constant.

Hence, by using (2.4) and (2.6) in (2.5), we obtain

$$\int_U |\nabla u_{\alpha_n}|^p dx \leqslant \int_\Omega |\nabla u_{\alpha_n}|^p \xi^p dx \leqslant C(\epsilon),$$

which implies that (u_{α_n}) is bounded in $W^{1,p}_{loc}(\Omega)$. So, there exists $u \in W^{1,p}_{loc}(\Omega)$ such that

$$\begin{cases} u_{\alpha_n} \rightharpoonup u \ \text{ in } \ W^{1,p}(U) \\ u_{\alpha_n} \to u \ \text{ in } \ L^q(U) \ \text{ for all } \ 1 \leqslant q < p^* \\ u_{\alpha_n}(x) \to u(x) \ \text{ a.e in } \Omega, \end{cases} \tag{2.7}$$

for each $U \subset\subset \Omega$ given.

By using (2.4) and applying Theorem **A.1.4**, we obtain

$$\nabla u_{\alpha_n} \to \nabla u, \ \text{ in } (L^q(\Omega))^N \text{ for any } \ q < p.$$

As a consequence, for each $\varphi \in C_c^\infty(\Omega)$ we get

$$\int_\Omega (|\nabla u_{\alpha_n}|^{p-2}\nabla u_{\alpha_n} - |\nabla u(x)|)^{p-2}\nabla u)\nabla\varphi dx \to 0.$$

Moreover, if Θ denote the support of φ, we have

$$\left|\left(\frac{a}{u^\delta_{\alpha_n}} + bu^\beta_{\alpha_n}\right)\varphi\right| \leqslant \left(\frac{a}{(\underline{\alpha}^{1/(p-1+\delta)}m_1 \inf\limits_\Theta \Phi_{H_1})^\delta} + b\overline{\alpha}^{\beta/(p-1-\beta)}m_2 \sup_\Theta e^{t\beta}_{H_2}\right)\|\varphi\|_\infty \in L^1(\Theta),$$

whence using the Dominated Convergence Theorem, we get

$$\alpha_n \int_\Omega \left(\frac{a}{u^\delta_{\alpha_n}} + bu^\beta_{\alpha_n}\right)\varphi dx \to \alpha \int_\Omega \left(\frac{a}{u^\delta} + bu^\beta\right)\varphi dx \text{ as } n \to \infty.$$

Hence, we conclude that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \alpha \int_\Omega \left(\frac{a}{u^\delta} + bu^\beta\right)\varphi, \quad \forall\ \varphi \in C_c^\infty(\Omega).$$

Since u_{α_n} satisfy (2.4), we have $\underline{\alpha}^{1/(p-1+\delta)}m_1\Phi_{H_1} \leqslant u \leqslant \overline{\alpha}^{1/(p-1-\beta)}m_2 e^t_{H_2}$. Thus, as Φ_{H_1} and $e_{H_2} \in C(\overline{\Omega})$ we obtain $0 \leqslant (u-\epsilon)^+ \leqslant (\overline{\alpha}^{1/(p-1-\beta)}m_2 e^t_{H_2} - \epsilon)^+$, that is, $(u-\epsilon)^+ \in W^{1,p}_0(\Omega)$ for each $\epsilon > 0$. Therefore, u satisfies the boundary condition of Definition 0.0.1. Hence, by applying the uniqueness of $W^{1,p}_{loc}(\Omega)$-solutions to Problem (L_α) claimed in Theorem 0.0.3, we have $u = u_\alpha$.

For the $C(\overline{\Omega})$-continuity, it follows from (2.4) and [38] that the sequence

(u_{α_n}) is bounded in $C^{\alpha}(\Theta)$ for some $\alpha \in (0,1)$ and in each compact $\Theta \subset \Omega$. So it follows from Arzelà-Ascoli's Theorem and (2.7), that $u_{\alpha_n} \to u$ in $C(\Omega)$. Furthermore, by using (2.4), we obtain $u \in C(\overline{\Omega})$ and $u_{\alpha_n} \to u$ in $C(\overline{\Omega})$.

■

2.2 Existence and multiplicity of $W^{1,p}_{loc}(\Omega)$-solutions for a non-local problem

Now we are able to prove Theorem 0.0.5. Before that, we will introduce the applications $G : D(G) \subset W^{1,p}_{loc}(\Omega) \to [0,\infty)$ and $H : (0,\infty) \to (0,\infty)$ defined by

$$G(u) = \Big(\int_{\Omega} g(x,u)dx\Big)^{r} \quad \text{and} \quad H(\alpha) = \alpha G(T(\alpha)),$$

where $D(G) = \{0 \leqslant u \in W^{1,p}_{loc}(\Omega) : G(u) < \infty\}$.

In addition, let us consider the system

$$\begin{cases} \displaystyle\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \alpha \int_{\Omega} \Big(a(x)u^{-\delta} + b(x)u^{\beta}\Big)\varphi dx \\ \alpha G(u) = \lambda, \end{cases} \tag{2.8}$$

remind that

$$\Sigma = \{(\lambda, u) \in (0,\infty) \times C(\overline{\Omega}) : u \in W^{1,p}_{loc}(\Omega) \text{ is solution of } (P_1)\}$$

and set

$$\Sigma' = \{(H(\alpha), u_{\alpha}) \in (0,\infty) \times C(\overline{\Omega}) : \alpha \in (0,\infty) \text{ and } u_{\alpha} \in W^{1,p}_{loc}(\Omega) \text{ is a solution of } (L_{\alpha})\}.$$

As a consequence of Lemma 2.1.4, we can prove the next result.

Lemma 2.2.1 *Suppose one of the following item holds:*

(*i*) (h_3) *is satisfied and* $g \in C(\overline{\Omega} \times [0,\infty), (0,\infty))$*;*

(ii) (h_4) *is satisfied,* $g \in C(\overline{\Omega} \times (0,\infty),(0,\infty))$ *and* $\lim_{t\to 0^+} g(x,t)t^{\theta_0} = g_0(x) \geqslant 0$ *uniformly in* $\overline{\Omega}$*, for some* $g_0 \in C(\overline{\Omega})$ *and* $0 < \theta_0 < 1$*.*

Then $T\big((0,\infty)\big) \subset D(G)$ *and, in particular,* H *is well-defined. Besides this,* H *is a continuous function.*

Proof: Take $\alpha > 0$. It follows from Lemmas 2.1.2, 2.1.3 and the monotonicity established in Proposition 2.1.1, we can find $0 < \underline{\alpha} = \underline{\alpha}(\alpha) < 1$ and $\overline{\alpha} = \overline{\alpha}(\alpha) > 1$ such that

$$\underline{\alpha}^{1/(p-1+\delta)} m_1 \Phi_{H_1} \leqslant u_\alpha \leqslant \overline{\alpha}^{1/(p-1-\beta)} m_2 e^t_{H_2} \text{ in } \Omega, \tag{2.9}$$

where m_1 and m_2 are given in Lemma 2.1.2 and Lemma 2.1.3, respectively.

First, let us assume (ii) holds. So, by choosing an $\epsilon, t_0 > 0$ sufficiently small such that $\overline{\alpha}^{1/(p-1-\beta)} m_2 e^t_{H_2} < t_0$ for all $x \in \Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < \epsilon\}$, we obtain from (2.3), (2.9) and hypothesis (ii), that $0 < g(x,u_\alpha) \leqslant Cd(x)^{-\theta_0}$ in Ω_ϵ for some positive constant C. Since $\theta_0 < 1$, it follows from [40] and the previous inequality that $g(x,u_\alpha) \in L^1(\Omega_\epsilon)$, which proves that H is well-defined in this case.

About the case (i), the result follows directly from the fact that e_{H_2} is a bounded function. So, in both cases, we showed that $T(\alpha) \in D(G)$ for each $\alpha > 0$ given.

To show the continuity, consider $\alpha_n \to \alpha > 0$. By an analogous argument as in first part, we can conclude that in any case there exists a $h(x) \in L^1(\Omega)$ such that $g(x,u_{\alpha_n}) \leqslant h(x)$, for all $x \in \Omega$ and $n \in \mathbb{N}$. Thus, the continuity follows from the Lemma 2.1.4 and Convergence Dominated Theorem.

■

After this Lemma, by using the uniqueness claimed in Theorem 0.0.3, we obtain the next one.

Lemma 2.2.2 *Let* $\lambda > 0$*. Then Problem* (P_1) *admits a* $W^{1,p}_{loc}(\Omega)$*-solution if, and only if, there exist* $(\alpha,u) = (\alpha_\lambda,u_\lambda) \in (0,\infty) \times W^{1,p}_{loc}(\Omega)$ *solution of* (2.8)*. In particular, Problem* (P_1) *admits a* $W^{1,p}_{loc}(\Omega)$*-solution if, and only if,* $\lambda \in H((0,\infty))$*.*

As a rereading of the above Lemma and a consequence of Lemma 2.1.4, we conclude that

$$\Sigma = \{(H(\alpha), u_\alpha) \in (0,\infty)\times C(\overline{\Omega}) : \alpha \in (0,\infty) \text{ and } u_\alpha \in W^{1,p}_{loc}(\Omega) \text{ is a solution of } (L_\alpha)\}$$

is the *continuum* of solutions to Problem (P_1) given by a curve.

Now, let us recall the Theorem 0.0.5 and prove it.

Theorem 0.0.5 *Assume $\delta > 0$ and $0 < \beta < p-1$ hold. If:*

1) *$g \in C(\overline{\Omega} \times [0,\infty), (0,\infty))$ and in addition*

   a) *(h_3), (g_∞) and $\theta_\infty r < p-1-\beta$ hold, then (P_1) admits at least one solution in Σ, for each $\lambda > 0$ given. Besides this, the same conclusion remains true if $\{r < 0$ and $g_\infty \equiv 0$ in $(g_\infty)\}$ or $\{(g'_\infty)$ and $r \geqslant 0\}$ holds.*

   b) *(h_4), (g_∞), $\theta_\infty r > p-1-\beta$ and $\theta_\infty < 1$ hold, then there exists $0 < \lambda^* < \infty$ such that (P_1) admits at least two $W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$-solutions for each $\lambda \in (0,\lambda^*)$ given, at least one solution for $\lambda = \lambda^*$ and no solution for $\lambda > \lambda^*$. Furthermore, if $\{r \geqslant 0$ and $g_\infty \equiv 0$ in $(g_\infty)\}$ or $\{(g'_\infty)$ and $r < 0\}$ holds, then the same conclusion is valid.*

2) *$g \in C(\overline{\Omega} \times (0,\infty), (0,\infty))$, (h_4) is satisfied and additionally*

   a) *(g_∞), (g_0), $\theta_\infty r < p-1-\beta$, $\theta_0 r > p-1+\delta$ and $\theta_0 < 1$ hold, then there exists a $0 < \lambda^* < \infty$ such that (P_1) admits at least two $W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$-solutions for $\lambda > \lambda^*$, at least one for $\lambda = \lambda^*$ and no solutions for $0 < \lambda < \lambda^*$. Moreover, the conclusion is the same if we assume either $\{r > 0, (g'_0)$ and $(g'_\infty)\}$ or $\{r < 0, (g_0), (g_\infty)$ and $g_0 \equiv g_\infty \equiv 0\}$.*

   b) *$\theta_\infty r > p-1-\beta$, $\theta_0 r > p-1+\delta$ and $\theta_\infty, \theta_0 < 1$ hold, then (P_1) admits at least one $W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$-solution for each $\lambda > 0$ given. In this case, the conclusion remains true if we assume either $\{r > 0, (g'_0)$ and (g_∞) with $g_\infty \equiv 0\}$ or $\{r < 0, (g'_\infty)$, and (g_0) with $g_0 \equiv 0\}$.*

Moreover, in all the cases Σ is the continuum of solutions given by a curve which:

(*i*) *emanates from* 0 *at* $\lambda = 0$ *and bifurcates from infinity at* $\lambda = \infty$ *in the case* $1 - a$),

(*ii*) *emanates from* 0 *at* $\lambda = 0$ *and bifurcates from infinity at* $\lambda = 0$ *in the case* $1 - b$),

(*iii*) *emanates from* 0 *at* $\lambda = \infty$ *and bifurcates from infinity at* $\lambda = \infty$ *in the case* $2 - a$),

(*iv*) *emanates from* 0 *at* $\lambda = \infty$ *and bifurcates from infinity at* $\lambda = 0$ *in the case* $2 - b$).

***Proof of Theorem 0.0.5-Completed* :** Since the additional part in each item follows analogously, we will prove only the first part in each one of them.

1-a) Firstly, note that by the continuity of g and Lemma 2.1.2, we get $\lim_{\alpha\to 0} H(\alpha) = 0$. We will split the proof in two cases:

i) *case* 1*:* $r \geqslant 0$. By taking $U \subset\subset \Omega$ and using (g_∞) together with Lemma 2.1.3, we obtain

$$\int_\Omega g(x, u_\alpha)dx \geqslant \int_U g(x, u_\alpha)dx \geqslant C\alpha^{-\theta_\infty/(p-1-\beta)}$$

for all α sufficiently large. Since $\theta_\infty r < p - 1 - \beta$, we get

$$H(\alpha) = \alpha\left(\int_\Omega g(x, u_\alpha)dx\right)^r \geqslant C\alpha^{1-r\theta_\infty/(p-1-\beta)} \to \infty \text{ as } \alpha \to \infty.$$

ii) *case* 2*:* $r < 0$. Consider the case $\theta_\infty \geqslant 0$. By the hypothesis (g_∞) and the continuity of g, we obtain $\int_\Omega g(x, u_\alpha)dx \leqslant C$, that is, $H(\alpha) \geqslant C^r\alpha \to \infty$ as $\alpha \to \infty$.

Analogously, when $\theta_\infty < 0$, we obtain by the Lemma 2.1.3 and the hypothesis (g_∞) that

$$H(\alpha) \geqslant C\alpha\Big(1 + \alpha^{-\theta_\infty/(p-1-\beta)}\Big)^r = C\Big(\alpha^{\frac{1}{r}} + \alpha^{\frac{1}{r}-\frac{\theta_\infty}{p-1-\beta}}\Big)^r,$$

showing that $H(\alpha) \to \infty$ as $\alpha \to \infty$ because $\theta_\infty r < p-1-\beta$. Hence, in all cases we have $H(\alpha) \to 0$ as $\alpha \to 0$ and $H(\alpha) \to \infty$ as $\alpha \to \infty$. Since H is continuous (see Lemma 2.2.1), our claim follows.

To finish the proof, it just remains to show the behavior of the *continuum* Σ at $\lambda = 0$ and $\lambda = \infty$. For $\lambda = 0$, let us take $\epsilon > 0$ and define $\delta = \inf\limits_{[\epsilon,\infty)} H(\alpha)$. Since $H(\alpha) \to \infty$ as $\alpha \to \infty$, it follows from the Lemma 2.2.1 that $\delta > 0$ and $(0,\delta) \subset H((0,\epsilon))$, that is, for each $\lambda_n \in (0,\delta)$, there exists an $\alpha_n \in (0,\epsilon)$ such that $H(\alpha_n) = \lambda_n$. Thus, if $\lambda_n \to 0$, then $\alpha_n \to 0$, which implies by the Lemma 2.1.2 that $\|u_{\alpha_n}\|_\infty \to 0$.

For $\lambda = \infty$, define $m = \max\limits_{[0,M]} H(\alpha)$ for each $M > 0$ given. Then $m < \infty$ and $(m,\infty) \subset H((M,\infty))$, that is, for each $\lambda_n \in (m,\infty)$, there exists $\alpha_n \in (M,\infty)$ such that $\lambda_n = H(\alpha_n)$. Hence, if $\lambda_n \to \infty$, then $\alpha_n \to \infty$ and so by using Lemma 2.1.3, we obtain that $\|u_{\alpha_n}\|_\infty \to \infty$. See picture Fig. 1.

1-*b*) Initially, suppose that $r > 0$. In this case $\theta_\infty > 0$, because we are assuming $\theta_\infty r > p-1-\beta > 0$.

By the hypothesis (g_∞) and continuity of g in $\overline{\Omega} \times [0,\infty)$, we obtain $g(x,t) \leqslant C_1 t^{-\theta_\infty}$ for all $t > 0$ and for some $C_1 > 0$.

Since we are assuming (h_4), we have $Cd(x) \leqslant \Phi_{H_1}(x)$ in Ω, which together with Lemma 2.1.3 leads to $\displaystyle\int_\Omega g(x,u_\alpha)dx \leqslant C_2\alpha^{-\theta_\infty/(p-1-\beta)}$ for $\alpha > 1$. Thus, as we are assuming $\theta_\infty r > p-1-\beta$, we obtain

$$H(\alpha) \leqslant C_3\alpha^{1-\theta_\infty r/(p-1-\beta)} \to 0 \text{ as } \alpha \to \infty.$$

Let us now consider the case when $r < 0$. In this case, by our hypothesis on θ_∞ and r, we necessarily have $\theta_\infty < 0$. Hence, proceeding analogously as above, we can prove $H(\alpha) \leqslant C\alpha^{1-\theta_\infty r/(p-1-\beta)} \to 0$ as $\alpha \to \infty$.

In any case, as we have proved, we obtain $\lim\limits_{\alpha\to\infty} H(\alpha) = 0$. On the other hand, $H(\alpha) \to 0$ as $\alpha \to 0$. Therefore, by taking $\lambda^* = \sup\limits_{\mathbb{R}^+} H(\alpha)$, the result follows.

Next, let us study the behavior of Σ. Letting $(\lambda, u) \in \Sigma$, it is clear that $\lambda \leqslant \lambda^*$. Since $\lim_{\alpha\to 0} H(\alpha) = \lim_{\alpha\to\infty} H(\alpha) = 0$, we get $(0,\delta) \subset H((0,\epsilon)) \cap H((M,\infty))$ for each $\epsilon > 0$ small and $M > 0$ large, where $0 < \delta = \min_{[\epsilon,M]} H(\alpha)$. Thus, for each $\lambda_n \in (0,\delta)$ there exists $\alpha_n^1 \in (0,\epsilon)$ and $\alpha_n^2 \in (M,\infty)$ such that $\lambda_n = H(\alpha_n^1) = H(\alpha_n^2)$. So, $\lambda_n \to 0$ imply $\alpha_n^1 \to 0$ and $\alpha_n^2 \to \infty$, which lead us to conclude that $\|u_{\alpha_n^1}\|_\infty \to 0$ and $\|u_{\alpha_n^2}\|_\infty \to \infty$ after to use Lemmas 2.1.2 and 2.1.3 again. See Fig. 2.

2-a) Initially assume $r > 0$. In this case $\theta_0 > 0$, because $\theta_0 r > p - 1 + \delta > 0$. Then, by using the hypothesis (g_0), Lemma 2.1.2 and taking $U \subset\subset \Omega$, we get

$$H(\alpha) \geqslant C\alpha \left(\int_U \frac{1}{\alpha^{\theta_0/(p-1+\delta)} e_{H_2}(x)^{t\theta_0}} dx\right)^r = C\alpha^{1-r\theta_0/(p-1+\delta)} \tag{2.10}$$

for some $C > 0$ cumulative constant and $\alpha > 0$ small enough. As $\theta_0 r > p - 1 + \delta > 0$, we obtain from (2.10) that $H(\alpha) \to +\infty$ as $\alpha \to 0^+$. In the same way, when $r, \theta_0 < 0$, by the hypothesis (g_0) and Lemma 2.1.2 we obtain $H(\alpha) \to +\infty$ as $\alpha \to 0^+$.

On the other hand, by following the same idea as in the proof of the item $1 - a)$, we can verify that $H(\alpha) \to \infty$ as $\alpha \to \infty$. Thus, by considering $\lambda^* = \inf_{\alpha\in\mathbb{R}^+} H(\alpha)$, the result follows.

2-b) By the same argument as in the proof the items $1 - b)$ and $2 - a)$, we can verify that $H(\alpha) \overset{\alpha\to\infty}{\longrightarrow} 0$ and $H(\alpha) \overset{\alpha\to 0^+}{\longrightarrow} \infty$, whence the result follows again. These ends the proof of Theorem 0.0.5.

Similarly to the cases $1 - a)$ and $1 - b)$, we are able to verify that the *continuum* Σ behaves as in the figures Fig.3 (item $2-a)$) and Fig. 4 (item $2-b)$), respectively. ■

CHAPTER 3

CONTINUUMS OF POSITIVE SOLUTIONS FOR NON-AUTONOMOUS NON-LOCAL STRONGLY-SINGULAR PROBLEMS

In this chapter, we show the existence of *continuums* of positive solutions for the following non-local quasilinear problem

$$(P_2)\begin{cases} -A\Big(x, \displaystyle\int_\Omega u^\gamma dx\Big)\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a smooth bounded domain, $p \in (1, N), \lambda > 0$ is a real parameter, $A \in C(\overline{\Omega} \times [0,\infty), (0,\infty))$ and $f \in C(\overline{\Omega} \times (0,\infty), (0,\infty))$ can be strongly (very) singular at $u = 0$.

We approach this problem by applying the Bifurcation Theory to the corresponding ϵ-perturbed problems and using a comparison principle for $W^{1,p}_{\mathrm{loc}}(\Omega)$-sub and supersolutions (see Theorem 1.2.2) to obtain qualitative properties of the ϵ-*continuum* limit. Moreover, this technique empowers us to study existence of a *continuum* of positive solutions to the following strongly-singular and non-homogeneous Kirchhoff problem

$$(Q_1) \qquad \begin{cases} -M\Big(x, \|\nabla u\|_p^p)\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a smooth bounded domain, $p \in (1, N), \lambda > 0$ is a real parameter, $M \in C(\overline{\Omega} \times [0,\infty), (0,\infty))$ and $f \in C(\overline{\Omega} \times (0,\infty), (0,\infty))$.

It is worth mentioning that in Chapter 2, since a monotonicity condition on $f(x,t)/t^{p-1}$ was assumed, a uniqueness result was shown and as a consequence of this, the analysis of the behavior of the *continuum* was done by studying the parameter-solution application. Here, the same strategy can not be applied any-more, because A is a non-autonomous function and no monotonicity is posed on the quotient $f(x,t)/t^{p-1}$.

This chapter follows the following structure. In the first section, we present the proof of Theorem 0.0.7. In Section 3.2, we establish the fundamental tools to study the behavior of Σ. The qualitative study of the *continuum* obtained in the first section will be done in Section 3.3, as well the proof of Theorem 0.0.9. We conclude the Section 3.3, by studying the degenerate case in problem (P_2). In the last section we prove Theorem 0.0.10. For convenience, all the results mentioned will be restated in their corresponding sections. However, for completeness, below we recall once again all the assumptions required throughout this chapter.

(A_0) $A \in C(\overline{\Omega} \times \mathbb{R})$ satisfies $A(x,t) > 0$ for all $t \geqslant 0$ and $x \in \overline{\Omega}$,

(A_∞) $\lim\limits_{t\to\infty} A(x,t)t^\theta = a_\infty(x) \geqslant 0$ uniformly in $\overline{\Omega}$, for some $a_\infty \in C(\overline{\Omega})$,

(A'_∞) $\lim\limits_{t\to\infty} A(x,t)t^\theta = \infty$ uniformly in $\overline{\Omega}$,

(f_∞) $\lim\limits_{t\to\infty} \dfrac{f(x,t)}{t^{p-1}} = 0$ uniformly in $\overline{\Omega}$,

(f_0) $\lim\limits_{t\to 0^+} \dfrac{f(x,t)}{t^{p-1}} = \infty$ uniformly in $\overline{\Omega}$,

(f_1) $\lim\limits_{t\to\infty} \dfrac{f(x,t)}{t^\beta} = c_\infty(x) > 0$ uniformly in $\overline{\Omega}$, for some $-\infty < \beta < p-1$ and $c_\infty \in C(\overline{\Omega})$,

(f_2) $\displaystyle\lim_{t\to 0^+} \frac{f(x,t)}{t^\delta} = c_0(x) > 0$ uniformly in $\overline{\Omega}$, for some $-\infty < \delta < p-1$ and $c_0 \in C(\overline{\Omega})$,

(M_0) $M(x,t) = a(x) + b(x)t^\gamma$, $a, b \in C(\overline{\Omega}), a(x) \geqslant \underline{a}$ and $b(x) \geqslant 0$ in $\overline{\Omega}$,

(Γ_0) $\gamma > 0$ if $-1 \leqslant \delta < p-1$ and $0 < \gamma < \frac{p-1-\delta}{-\delta-1}$ if $-\frac{2p-1}{p-1} \leqslant \delta < -1$.

3.1 Existence of a *continuum* of $W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$-solutions

Throughout this section, we will denote by $e_1 \in C^1_0(\overline{\Omega})$ the unique positive solution of

$$-\Delta_p u = 1 \ \text{ in } \Omega,\ u|_{\partial\Omega} = 0$$

and by $\Phi_1 \in C^1_0(\overline{\Omega})$ the first positive normalized eigenfunction associated to the first positive eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$, that is,

$$-\Delta_p \Phi_1 = \lambda_1 \Phi_1^{p-1} \ \text{ in } \Omega,\ \Phi_1|_{\partial\Omega} = 0.$$

For each $\epsilon > 0$ given, let us introduce the following ϵ-perturbed problem

$$(P_\epsilon) \qquad \begin{cases} -A\Big(x, \displaystyle\int_\Omega u^\gamma dx\Big)\Delta_p u = \lambda f(x, u+\epsilon) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \end{cases}$$

and show that (P_ϵ) admits an unbounded ϵ-*continuum* of positive solutions by using the Rabinowitz Global Bifurcation Theorem (see Theorem **A.1.7** in Appendix).

Lemma 3.1.1 *Suppose that $\gamma \geqslant 0$ and (A_0) hold. Then, there exists an unbounded continuum $\Sigma_\epsilon \subset \mathbb{R}^+ \times C(\overline{\Omega})$ of positive solutions of (P_ϵ) that emanates from $(0,0)$, for each $\epsilon > 0$ given.*

Proof: It follows from the classical theory of existence and regularity for elliptic equations and hypothesis (A_0) that the problem

$$-A\Big(x, \int_\Omega |v|^\gamma dx\Big)\Delta_p u = \lambda f(x, |v| + \epsilon) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3.1}$$

admits a unique solution $u \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$ and for each $(\lambda, v) \in \mathbb{R}^+ \times C(\overline{\Omega})$ (see Theorem **A.2.1**). Thus, the operator $T : \mathbb{R}^+ \times C(\overline{\Omega}) \to C(\overline{\Omega})$, which associates each pair $(\lambda, v) \in \mathbb{R}^+ \times C(\overline{\Omega})$ to the only weak solution of (3.1), is well-defined.

It is classical to show that T is a compact operator, using Arzelà-Ascoli's Theorem. Hence, we are able to apply Theorem **A.1.7** to get an unbounded ϵ-*continuum* $\Sigma_\epsilon \subset \mathbb{R}^+ \times C(\overline{\Omega})$ of solutions of

$$T(\lambda, u) = u. \tag{3.2}$$

Moreover, as by the definition $T(0, v) = 0$ and if $T(\lambda, 0) = 0$ implies $\lambda = 0$, we can conclude that $\Sigma_\epsilon \backslash \{(0,0)\}$ is formed by nontrivial solutions of (3.2).

Finally, using that $0 < f(x, |v| + \epsilon)/A\Big(x, \int_\Omega |v|^\gamma\Big) \in L^\infty(\Omega)$ for each given $v \in C(\overline{\Omega})$ and classical strong maximum principle (see Theorem **A.1.2**), we obtain that $T((\mathbb{R}^+\backslash\{0\}) \times C(\overline{\Omega})) \subset C(\overline{\Omega})_+$, where $C(\overline{\Omega})_+ = \{u \in C(\overline{\Omega}) : u > 0 \text{ in } \Omega\}$. Therefore, Σ_ϵ is a ϵ-*continuum* of positive solutions of (P_ϵ), for each $\epsilon > 0$ given. This ends the proof. ■

As a consequence of the result we just proved, for every $\epsilon > 0$ and for each bounded open set $U \subset \mathbb{R} \times C(\overline{\Omega})$ containing $(0,0)$, there exists a pair $(\lambda_\epsilon, u_\epsilon) \in \Sigma_\epsilon \cap \partial U$. An essential argument in our approach is to show that if $\epsilon_n \to 0^+$ and $\lambda_n \to \lambda$, then $\lambda > 0$ and $\{u_{\epsilon_n}\}$ converges in $C(\overline{\Omega})$ to a function $u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$, where (λ, u) is a solution of (P_2).

To prove this, let us begin with the following result which is motivated by the arguments of Crandall, Rabinowitz and Tartar [21].

Lemma 3.1.2 *Admit that* (A_0) *and* (f_0) *hold. Let* $U \subset \mathbb{R} \times C(\overline{\Omega})$ *be a bounded* *open set containing* $(0,0)$*, a positive constant* K *and a pair* $(\lambda_\epsilon, u_\epsilon) \in \Big((0,\infty) \times (C(\overline{\Omega}) \cap W_0^{1,p}(\Omega))\Big) \cap \partial U$ *of solution of* (P_ϵ) *satisfying* $\lambda_\epsilon \leqslant K$ *and* $u_\epsilon \leqslant K$ *in* $\overline{\Omega}$*. Then, there exist constants* $\mathcal{K}_1 = \mathcal{K}_1(K,U) > 0$*,* $\mathcal{K}_2 = \mathcal{K}_2(k,K) > 0$ *and* $\epsilon_0 > 0$ *such that*

$$\lambda_\epsilon^{\frac{1}{p-1}} \mathcal{K}_1(K,U)\Phi_1 \leqslant u_\epsilon \leqslant k + \lambda_\epsilon^{\frac{1}{p-1}} \mathcal{K}_2(k,K)^{\frac{1}{p-1}} e_1 \quad \text{in } \Omega, \tag{3.3}$$

for each $k \in (0,K]$ *fixed and for all* $0 < \epsilon < \epsilon_0$*.*

Proof: Let $K > 0$ as above. Besides this, define $0 < a_K = \min\limits_{\overline{\Omega}\times[0,\ |\Omega|K^\gamma]} A(x,t)$ and

$$\mathcal{K}_2(k,K) = \max\Big\{\frac{f(x,t)}{a_K} :\ x \in \overline{\Omega} \ \text{ and } k \leqslant t \leqslant K+1\Big\},$$

where k is a fixed number on $(0,K]$. Thus, $\mathcal{K}_2(k,\cdot)$ is non-decreasing for each k fixed.

To show the second inequality in (3.3), let us consider the open set $\mathcal{O}_k = \{x \in \Omega : u_\epsilon > k\}$. Then, it follows from the definition of $\mathcal{K}_2$ that

$$\begin{aligned} -\Delta_p\Big(k + \lambda_\epsilon^{\frac{1}{p-1}} \mathcal{K}_2(k,K)^{\frac{1}{p-1}} e_1\Big) &= \lambda_\epsilon \mathcal{K}_2(k,K) \geqslant \frac{\lambda_\epsilon}{a_K} f(x, u_\epsilon + \epsilon) \\ &\geqslant \frac{\lambda_\epsilon}{A\Big(x, \int_\Omega u_\epsilon^\gamma\Big)} f(x, u_\epsilon+\epsilon) = -\Delta_p u_\epsilon \ \text{ in } \mathcal{O}_k. \end{aligned}$$

Since $k + \lambda_\epsilon^{\frac{1}{p-1}} \mathcal{K}_2(k,K)^{\frac{1}{p-1}} e_1 - u_\epsilon = \lambda_\epsilon^{\frac{1}{p-1}} \mathcal{K}_2(k,K)^{\frac{1}{p-1}} e_1 \geqslant 0$ on $\partial\mathcal{O}_k$, the second inequality in (3.3) is valid in $\mathcal{O}_k$ by classical comparison principle. Now, using the above fact together with the definition of $\mathcal{O}_k$, we conclude that $u_\epsilon \leqslant k + \lambda_\epsilon^{\frac{1}{p-1}} \mathcal{K}_2(k,K)^{\frac{1}{p-1}} e_1$ in Ω.

Now, we are going to prove the first inequality in (3.3). Let us denote by $\delta' = \mathrm{dist}(\partial U, (0,0)) > 0$. We claim that

$$\lambda_\epsilon > C_* := \min\Big\{\frac{1}{\mathcal{K}_2(\delta'/4, K)}\Big(\frac{\delta'}{4\|e_1\|_\infty}\Big)^{p-1}, \frac{\delta'}{4}\Big\}.$$

In fact, otherwise by taking $k = \delta'/4$ in the second inequality in (3.3), we conclude that $(\lambda_\epsilon, u_\epsilon) \in B_{3\delta'/4}(0,0) \subset \mathbb{R} \times C(\overline{\Omega})$, which is an absurd as $(\lambda_\epsilon, u_\epsilon) \in \partial U$.

Now, by defining $\underline{u}_\epsilon = \lambda_\epsilon^{\frac{1}{p-1}}\mathcal{K}_1(K,U)\Phi_1$, where $\mathcal{K}_1(K,U)$ will be chosen later, it follows from Picone's inequality (Theorem **A.1.6**), hypothesis (A_0) and the fact that $(\lambda_\epsilon, u_\epsilon)$ is a solution of (P_ϵ), that

$$\begin{aligned}
0 &\leqslant \int_\Omega |\nabla \underline{u}_\epsilon|^{p-2}\nabla\underline{u}_\epsilon\nabla\Big(\frac{(\underline{u}_\epsilon+\epsilon)^p-(u_\epsilon+\epsilon)^p}{(\underline{u}_\epsilon+\epsilon)^{p-1}}\Big)^+dx\\
&\quad -|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\nabla\Big(\frac{(\underline{u}_\epsilon+\epsilon)^p-(u_\epsilon+\epsilon)^p}{(u_\epsilon+\epsilon)^{p-1}}\Big)^+dx\\
&\leqslant \lambda_\epsilon\int_\Omega\Big[\frac{\lambda_1\mathcal{K}_1^{p-1}\Phi_1^{p-1}}{(\lambda_\epsilon^{1/(p-1)}\mathcal{K}_1\Phi_1+\epsilon)^{p-1}}-\frac{f(x,u_\epsilon+\epsilon)}{(u_\epsilon+\epsilon)^{p-1}A_K}\Big]\Big((\underline{u}_\epsilon+\epsilon)^p-(u_\epsilon+\epsilon)^p\Big)^+dx\\
&\leqslant \lambda_\epsilon\int_\Omega\Big[\frac{\lambda_1}{\lambda_\epsilon}-\frac{f(x,u_\epsilon+\epsilon)}{(u_\epsilon+\epsilon)^{p-1}A_K}\Big]\Big((\underline{u}_\epsilon+\epsilon)^p-(u_\epsilon+\epsilon)^p\Big)^+dx, \qquad (3.4)
\end{aligned}$$

where $A_K = \max_{\overline{\Omega}\times[0,K^\gamma|\Omega|]} A$.

To complete the proof, let us argue by contradiction. First, let us fix $\tilde{K} > \big(\lambda_1 A_K\big)/C_*$ and conclude from hypothesis (f_0) that there exists $a > 0$ small enough such that $f(x,t) \geqslant \tilde{K}t^{p-1}$, for all $x \in \Omega$ and $0 < t < a$. Hence, by choosing $\mathcal{K}_1(K,U) = a/\Big(4K^{\frac{1}{p-1}}\|\Phi_1\|_\infty\Big)$, we claim that $[\underline{u}_\epsilon > u_\epsilon]$ has zero measure for every $\epsilon < \epsilon_0 := a/4$ given. Otherwise, if we assume $|[\underline{u}_\epsilon > u_\epsilon]| > 0$ for some $0 < \epsilon < \epsilon_0$, we get

$$u_\epsilon + \epsilon \leqslant \underline{u}_\epsilon + \epsilon < \frac{a}{2} \text{ on } [\underline{u}_\epsilon > u_\epsilon].$$

Therefore, by going back to (3.4) and using $\lambda_1/\lambda_\epsilon \leqslant \lambda_1/C_*$, we have

$$\begin{aligned}
0 &\leqslant \lambda_\epsilon\int_\Omega\Big[\frac{\lambda_1}{\lambda_\epsilon}-\frac{f(x,u_\epsilon+\epsilon)}{(u_\epsilon+\epsilon)^{p-1}A_K}\Big]\Big((\underline{u}_\epsilon+\epsilon)^p-(u_\epsilon+\epsilon)^p\Big)^+dx\\
&\leqslant \lambda_\epsilon\int_\Omega\Big[\frac{\lambda_1}{C_*}-\frac{\tilde{K}(u_\epsilon+\epsilon)^{p-1}}{(u_\epsilon+\epsilon)^{p-1}A_K}\Big]\Big((\underline{u}_\epsilon+\epsilon)^p-(u_\epsilon+\epsilon)^p\Big)^+dx < 0,
\end{aligned}$$

which is an absurd. Hence, $\lambda_\epsilon^{\frac{1}{p-1}}\mathcal{K}_1(K,U)\Phi_1 \leqslant u_\epsilon$ in Ω for all $0 < \epsilon < \epsilon_0$, as we claimed.

■

Theorem 0.0.7 *Suppose that $\gamma \geqslant 0$, (A_0) and (f_0) hold. Then, there exists an unbounded continuum $\Sigma \subset \mathbb{R}\times C(\overline{\Omega})$ of positive solutions of the problem (P_2) that* *emanates from* $(0,0)$. *In additional, if* (f_∞) *holds and* $A(x,t) \geqslant a_0$ *in* $\overline{\Omega} \times \mathbb{R}^+$ *for some* $a_0 > 0$, *then* $Proj_{\mathbb{R}}\Sigma = (0,\infty)$.

Proof:

For each $i \in \mathbb{N}$ given, define

$$\mathcal{F}_i = \Big\{(\lambda, u) \in \mathbb{R}^+ \times C(\overline{\Omega}) \text{ that solves } (P_2) : \frac{\lambda^{\frac{1}{p-1}}}{i}\Phi_1(x) \leqslant u(x) \leqslant k + \lambda^{\frac{1}{p-1}}\mathcal{K}_2(k,i)^{\frac{1}{p-1}}e_1(x)$$

$$\text{in } \Omega \text{ for each } k \in (0,i]\Big\},$$

where $\mathcal{K}_2(k,i)$ was introduced in the Lemma 3.1.2.

To end the proof, it suffices to set

$$\mathcal{F} = \bigcup_{i\in\mathbb{N}} \mathcal{F}_i \cup \{(0,0)\} \subset \mathbb{R}^+ \times C(\overline{\Omega}) \tag{3.5}$$

and prove that there is an unbounded connected component $\Sigma \subset \mathcal{F}$. By Theorem 2 in [56] (see also [58]), the existence of Σ is a consequence of the following two claims:

Claim 1: For each $U \subset \mathbb{R} \times C(\overline{\Omega})$ bounded neighborhood of $(0,0)$ in $\mathbb{R} \times C(\overline{\Omega})$, there is a solution $(\lambda, u) \in \partial U \cap \mathcal{F}$.

Claim 2: Closed and bounded (in $\mathbb{R} \times C(\overline{\Omega})$) subsets of $\mathcal{F}$ are compact.

Let us prove each of the above claims one by one.

Proof of Claim 1: Consider $U \subset \mathbb{R} \times C(\overline{\Omega})$ be a bounded neighborhood of $(0,0)$ in $\mathbb{R} \times C(\overline{\Omega})$ and a sequence $\epsilon_n \to 0^+$. By the Lemma 3.1.1, there exists $(\lambda_n, u_n) = (\lambda_{\epsilon_n}, u_{\epsilon_n}) \in \partial U \cap \Big((0,\infty) \times W^{1,p}_0(\Omega)\Big)$ a solution of (P_{ϵ_n}), for each $n \in \mathbb{N}$. Moreover, as U is a bounded set, we can find a positive constant $K > 0$ such that $0 \leqslant \lambda_n \leqslant K$ and $0 \leqslant u_n \leqslant K$ in Ω. Thus, by the Lemma 3.1.2, we obtain

$$\lambda_n^{\frac{1}{p-1}}\mathcal{K}_1(K,U)\Phi_1 \leqslant u_n \leqslant k + \lambda_n^{\frac{1}{p-1}}\mathcal{K}_2(k,K)^{\frac{1}{p-1}}e_1 \ \text{ in } \Omega, \tag{3.6}$$

for all $n \in \mathbb{N}$ sufficiently large and for each $k \in (0,K]$ given.

Suppose that $\lambda_n \to \lambda \geqslant 0$. If $\lambda = 0$, we conclude by (3.6) that $u_n \to 0$

in $C(\overline{\Omega})$, that is, $(\lambda_n, u_n) \to (0,0)$ in $\mathbb{R} \times C(\overline{\Omega})$. Since $(\lambda_n, u_n) \in \partial U$ and U is a bounded neighborhood of $(0,0)$, we obtain a contradiction. Therefore $\lambda > 0$, which implies that $0 < \lambda - \delta' < \lambda_n < \lambda + \delta'$ for n sufficiently large and some $\delta' > 0$.

Consider a sequence (Ω_l) of open sets in Ω such that $\Omega_l \subset \Omega_{l+1}$ and $\bigcup_l \Omega_l = \Omega$ and define $\delta_l = \min\limits_{\overline{\Omega}_l}(\lambda - \delta')^{\frac{1}{p-1}}\mathcal{K}_1(K,U)\Phi_1$, for each $l \in \mathbb{N}$. Taking $\varphi = (u_n - \delta_1)^+$ as a test function in (P_{ϵ_n}), using (3.6) and the hypothesis (A_0), we obtain

$$\int_{[u_n \geqslant \delta_1]} |\nabla u_n|^p dx = \lambda_n \int_{[u_n \geqslant \delta_1]} \frac{f(x, u_n + \epsilon_n)}{A\left(x, \int_\Omega u_n^\gamma\right)}(u_n - \delta_1)^+ dx \leqslant C_1,$$

where $C_1 > 0$ is a real constant independent of n. Thus, it follows from the previous inequality that $\{u_n\}$ is bounded in $W^{1,p}(\Omega_1)$. Hence, there exists $u_{\Omega_1} \in W^{1,p}(\Omega_1)$ and a subsequence $\{u_{n_j^1}\}$ of $\{u_n\}$ such that

$$\begin{cases} u_{n_j^1} \rightharpoonup u_{\Omega_1} \text{ weakly in } W^{1,p}(\Omega_1) \text{ and strongly in } L^q(\Omega_1) \text{ for } 1 \leqslant q < p^* \\ u_{n_j^1} \to u_{\Omega_1} \ \ a.e. \text{ in } \ \Omega_1. \end{cases}$$

Proceeding as above, we can obtain subsequences $\{u_{n_j^l}\}$ of $\{u_n\}$, with $\{u_{n_j^{l+1}}\} \subset \{u_{n_j^l}\}$, and functions $u_{\Omega_l} \in W^{1,p}(\Omega_l)$ such that

$$\begin{cases} u_{n_j^l} \rightharpoonup u_{\Omega_l}, \text{ weakly in } \ W^{1,p}(\Omega_l) \text{ and strongly in } L^p(\Omega_l) \text{ for } 1 \leqslant q < p^* \\ u_{n_j^l} \to u_{\Omega_l} \ \ a.e. \text{ in } \Omega_l. \end{cases}$$

By construction, we have $u_{\Omega_{l+1}}\Big|_{\Omega_l} = u_{\Omega_l}$. Hence, by defining

$$u = \begin{cases} u_{\Omega_1} \ \text{ in } \ \Omega_1, \\ u_{\Omega_{l+1}} \ \text{ in } \ \Omega_{l+1} \backslash \Omega_l, \end{cases}$$

we obtain that $u \in W^{1,p}_{\text{loc}}(\Omega)$ and satisfies (3.6). In particular, by choosing $i > K$

large enough and using that $\mathcal{K}_2(k, \cdot)$ is non-decreasing, we have that

$$\frac{\lambda^{\frac{1}{p-1}}}{i}\Phi_1(x) \leqslant u(x) \leqslant k + \lambda^{\frac{1}{p-1}}\mathcal{K}_2(k,i)^{\frac{1}{p-1}}e_1(x) \tag{3.7}$$

holds for each $k \in (0, i]$.

Furthermore, we claim that (λ, u) is a solution for (P_2). Indeed, by taking $\varphi \in C_c^\infty(\Omega)$ and using Theorem **A.1.4**, we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \to \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx, \tag{3.8}$$

up to a subsequence. On the other side, by using the continuity of f, the inequality (3.6) and the hypothesis (A_0), we obtain from Lebesgue Dominated Convergence Theorem that

$$\lambda_n \int_\Omega \frac{f(x, u_n + \epsilon_n)}{A\Big(x, \int_\Omega u_n^\gamma\Big)}\varphi dx \to \lambda \int_\Omega \frac{f(x,u)}{A\Big(x, \int_\Omega u^\gamma\Big)}\varphi dx. \tag{3.9}$$

Thus, from (3.8) and (3.9) it is evident that (λ, u) satisfies (6). Also, by (3.7) we obtain that $u > 0$ (in the sense of Definition 0.0.6). To verify that u satisfies the boundary condition (see Definition 0.0.1), it suffices to note that the arguments used above lead us to the fact that the sequence $(u_n - \epsilon)^+$ is bounded in $W_0^{1,p}(\Omega)$ as well. Therefore, $(u - \epsilon)^+ \in W_0^{1,p}(\Omega)$ for each $\epsilon > 0$ given.

Finally, by the continuity of f, hypothesis (A_0) and (3.6), we obtain from Theorem **A.2.3** and Arzelà-Ascoli's Theorem that $u \in C(\Omega)$ and $u_n \to u$ in $C(\Theta)$, for each compact set $\Theta \subset \Omega$ given. Thus, by using this fact and (3.6), we obtain that $(\lambda_n, u_n) \to (\lambda, u)$ in $\mathbb{R} \times C(\overline{\Omega})$, which on combining with (3.7) implies that $(\lambda, u) \in \partial U \cap \mathcal{F}_i \subset \partial U \cap \mathcal{F}$, as required.

Proof of Claim 2: Let $\{(\lambda_n, u_n)\} \subset \mathcal{F}$ be a bounded sequence (in $\mathbb{R} \times C(\overline{\Omega})$). We aim to prove that $\{(\lambda_n, u_n)\}$ admits a subsequence that converges to some element of $\mathcal{F}$.

Initially, let us suppose that finitely many terms of $\{(\lambda_n, u_n)\}$ belongs to $\mathbb{R} \times$

$C(\overline{\Omega})\backslash B_{\delta'}(0,0)$, for each $\delta' > 0$ given. In this case, $(0,0)$ would be an accumulation point of the sequence and our claim will hold. Otherwise, let us assume that infinitely many terms of $\{(\lambda_n, u_n)\}$ belongs to $\mathbb{R} \times C(\overline{\Omega})\backslash B_{\delta'}(0,0)$, for some $\delta' > 0$. Since $\{(\lambda_n, u_n)\}$ is bounded by a constant $K > 0$, the second inequality in (3.3) is true. Apart from this, since $\|(\lambda_n, u_n)\|_{\mathbb{R}\times C(\overline{\Omega})} \geqslant \delta'$ (just for the subsequence in our assumption), the first inequality in (3.3) holds true as well. Hence, by fixing $i \in \mathbb{N}$ sufficiently large, we get that $\{(\lambda_n, u_n)\} \subset \mathcal{F}_i$ for that subsequence.

Let us fix such subsequence. By the boundedness of $\{\lambda_n\} \subset \mathbb{R}$ and $(\lambda_n, u_n) \subset \mathcal{F}_i \cap \Big((\mathbb{R} \times C(\overline{\Omega}))\backslash B_{\delta'}(0,0)\Big)$, it follows that $\lambda_n \to \lambda > 0$, up to subsequence. As a consequence of this, we get

$$\frac{\lambda^{1/(p-1)}}{2i}\Phi_1 \leqslant u_n \leqslant K \text{ in } \Omega \tag{3.10}$$

for $n \in \mathbb{N}$ large enough.

Let $U \subset\subset \Omega$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in U with $U \subset \Theta := \text{supp } \varphi$. Thus, by (3.10), we have a uniform bound of $(f(x, u_n))$ on $\Theta \times [k, K]$, where $k := \min_\Theta \frac{\lambda^{1/(p-1)}}{2i}\Phi_1 > 0$. Hence, using this information together with boundedness of (λ_n, u_n) in $\mathbb{R} \times C(\overline{\Omega})$), Hölder's inequality and the hypothesis (A_0), we have

$$\begin{aligned}
&\frac{1}{2^p}\int_\Theta |\nabla(\varphi u_n)|^p dx = \frac{1}{2^p}\int_\Theta |\nabla\varphi u_n + \nabla u_n \varphi|^p dx \leqslant \int_\Theta |\nabla\varphi|^p {u_n}^p dx + \int_\Theta |\nabla u_n|^p \varphi^p dx \\
&\leqslant C_1\int_\Theta |\nabla\varphi|^p dx + \int_\Theta |\nabla u_n|^{p-2}\nabla u_n \nabla u_n \varphi^p dx - \int_\Theta |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi(p\varphi^{p-1}u_n) dx \\
&\leqslant C_1\int_\Theta |\nabla\varphi|^p dx + \lambda_n\int_\Theta \frac{f(x,u_n)u_n}{A\Big(x, \displaystyle\int_\Omega u_n^\gamma\Big)}\varphi^p dx + C_2\int_\Theta |\nabla u_n|^{p-1}|\nabla\varphi|\varphi^{p-1}u_n dx \\
&\leqslant C_3\Big[1 + \Big(\int_\Theta |\varphi\nabla u_n|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_\Theta (|u_n\nabla\varphi|)^p dx\Big)^{\frac{1}{p}}\Big] \qquad \text{(using } (A_0)) \\
&\leqslant C_4\Big[1 + \Big(\int_\Theta |\nabla(\varphi u_n)|^p dx\Big)^{\frac{p-1}{p}}\Big],
\end{aligned}$$

where C_4 is a positive constant, independent of n. Thus, $\{\varphi u_n\}$ is bounded in $W_0^{1,p}(\Theta)$ and as a consequence of this, $\{u_n\}$ is bounded in $W^{1,p}(U)$. By using the arbitrariness of U and proceeding as in the proof of the Claim 1, we obtain a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } \quad W^{1,p}(U) \ \text{ for each } \ U \subset\subset \Omega, \\ u_n \to u \text{ in } C(\overline{\Omega}), \\ \frac{\lambda^{\frac{1}{p-1}}}{i}\Phi_1(x) \leqslant u(x) \leqslant k + \lambda^{\frac{1}{p-1}}\mathcal{K}_2(k,i)^{\frac{1}{p-1}}e_1(x) \ \text{ in } \Omega \text{ for all } k \in (0,i] \end{cases} \tag{3.11}$$

for i as fixed before.

From the last inequality in (3.11), it follows that $(u-\epsilon)^+ \in W^{1,p}_0(\Omega)$ for each $\epsilon > 0$ given, as noted in Claim 1. Hence, to complete the proof of the existence of the *continuum*, we just need to show that (λ, u) satisfies the equation in (P_2), that is, (6). Since (λ_n, u_n) solves (P_{ϵ_n}), it follows from density arguments, (3.10) and (3.11) that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\Big(\varphi(u_n-u)\Big)dx = \lambda_n \int_\Omega \frac{f(x,u_n)}{A\Big(x, \int_\Omega u_n^\gamma\Big)}\varphi(u_n-u)dx \to 0 \tag{3.12}$$

for all $\varphi \in C_c^\infty(\Omega)$.

Since $\{u_n\}$ is a bounded sequence in $W^{1,p}_{\mathrm{loc}}(\Omega)$, we obtain

$$\Big|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi(u_n-u)dx\Big| \leqslant C\|u_n-u\|_p \to 0 \tag{3.13}$$

by using the Hölder's inequality. Therefore, it follows from (3.12) and (3.13) that

$$\int_\Omega \varphi\Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\Big)\nabla(u_n-u)dx \to 0,$$

up to subsequence, which implies that $\nabla u_n \to \nabla u$ *a.e.* in Ω.

Thus, proceeding as in proof of the Claim 1, we obtain that $(\lambda, u) \in \mathcal{F}_i \subset \mathcal{F}$, which concludes the proof of the existence of an unbounded *continuum* of positive solutions for (P_2).

In order to finish the proof of later part of the Theorem 0.0.7, let us assume (f_∞) and $A(x,t) > a_0$ in $\overline{\Omega} \times \mathbb{R}^+$ holds for some $a_0 > 0$. Assume by contradiction that $Proj_{\mathbb{R}}\Sigma \subset [0, \lambda^*]$ for some $0 < \lambda^* < \infty$, that is, $0 \leqslant \lambda \leqslant \lambda^*$ whenever

$(\lambda, u) \in \Sigma$. Hence, by taking $R > 0$ and $\epsilon_n = 1/n$ $(n \in \mathbb{N})$, we obtain by Lemma 3.1.1 that there exists $(\lambda_n, u_n) = (\lambda_{n,R}, u_{n,R}) \in \Sigma_n \cap \partial B_R(0,0)$, where Σ_n is the unbounded ϵ_n-*continuum* of positive solutions of (P_{ϵ_n}) .

We claim that there exists $R_0 > 0$ such that $\lambda_n \geqslant \lambda^* + 1$ for all $n \in \mathbb{N}$ and $R > R_0$. Otherwise, we can find a sequence $R_l \to \infty$ and a subsequence $\{u_{n_l}\}$ satisfying

$$\|u_{n_l}\|_\infty = R_l - \lambda_{n_l} \geqslant R_l - \lambda^* - 1. \tag{3.14}$$

However, by Lemma 3.1.2 we have $\|u_{n_l}\|_\infty \leqslant 1 + \mathcal{K}_2(1, R_l)^{1/(p-1)}(\lambda^* + 1)^{1/(p-1)}\|e_1\|_\infty$, where $\mathcal{K}_2(1, R_l) = \max\left\{\frac{f(x,t)}{a_{R_l}} : x \in \overline{\Omega} \text{ and } 1 \leqslant t \leqslant R_l + 1\right\}$ with $a_{R_l} = \min_{\overline{\Omega}\times[0,R_l^\gamma|\Omega|]} A \geqslant a_0$ by our assumption. Hence, it follows from the hypothesis (f_∞) that for each $\epsilon > 0$ there exists a positive constant C^1_ϵ such that $\mathcal{K}_2(1, R_l) \leqslant C^1_\epsilon + \frac{\epsilon}{a_0} R_l^{p-1}$ holds for all $l \in \mathbb{N}$ sufficiently large. As a consequence of these information, we obtain

$$\|u_{n_l}\|_\infty \leqslant 1 + \left(C^1_\epsilon + \frac{\epsilon}{a_0} R_l^{p-1}\right)^{1/(p-1)} (\lambda^* + 1)^{1/(p-1)}\|e_1\|_\infty \leqslant C^2_\epsilon + C_2\epsilon^{1/(p-1)} R_l, \tag{3.15}$$

for l large enough and for some positive constants C^2_ϵ and C_2, where C_2 is independent of ϵ.

Let $\epsilon > 0$ be such that $1 - \epsilon^{1/(p-1)}C_2 > 0$. Since $R_l \to \infty$, we can take a l large enough such that $R_l > C_2^{\epsilon+\lambda^*+1}/(1 - \epsilon^{1/(p-1)}C_2)$. Thus, by going back to (3.15), we obtain for such l that $\|u_{n_l}\|_\infty \leqslant C^2_\epsilon + C_2\epsilon^{1/(p-1)}R_l < R_l - \lambda^* - 1$ holds, but this contradicts (3.14).

Therefore, by fixing $R > R_0 > 0$ and proceeding as in the proof of the Claim 1, we obtain that $(\lambda_n, u_n) = (\lambda_{n,R}, u_{n,R})$ converges in $\mathbb{R} \times C(\overline{\Omega})$ to a pair $(\lambda, u) \in \Sigma \cap \partial B_R(0,0)$, which implies that $\lambda \geqslant \lambda^* + 1$, but this is not possible by the contrary hypothesis of $Proj_{\mathbb{R}^+}\Sigma \subset [0, \lambda^*]$. This ends the proof. ■

3.2 $W^{1,p}_{\rm loc}(\Omega)$-behavior to a parameter for $(p-1)$-sublinear problems

Let us present some results which are important in itself and are required to overcome some obstacles on the strategies of Rabinowitz [48] and Figueiredo-Sousa [29], in order to approach non-autonomous non-local singular problems involving p-Laplacian operator in the setting of $W^{1,p}_{\rm loc}(\Omega)$-solutions.

The next Lemma brings out an important parametric behavior of the solution of $(p-1)$-sublinear problem. This result is crucial in our approach.

Lemma 3.2.1 *Assume that* (f_1) *and* (f_2) *are satisfied with* $c_0, c_\infty > 0$ *in* $\overline{\Omega}$ *and* $\delta \leqslant \beta$. *Then, there exist* $\alpha_0, \alpha_\infty, m_1, m_2 > 0$ *such that any positive solution* $u \in W^{1,p}_{\rm loc}(\Omega)$ *of*

$$-\Delta_p u = \alpha f(x,u) \quad in\ \Omega, \quad u|_{\partial\Omega} = 0, \tag{3.16}$$

(see definition 0.0.6 *with* $A \equiv 1$*) satisfies*

$$\alpha^\tau m_1 \Phi_1 \leqslant u \leqslant \alpha^\tau m_2 e_1^t \ \ in\ \overline{\Omega}, \tag{3.17}$$

where $t = \min\{1, (p-1)/(p-1-\delta)\}$,

a) $\tau = 1/(p-1-\delta)$ *for all* $\alpha \in (0, \alpha_0)$ *and* *b)* $\tau = 1/(p-1-\beta)$ *for all* $\alpha > \alpha_\infty$.

Proof: Let $u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ be a solution of (3.16). It follows from (f_1) and (f_2) that there exist constants $m, M > 0$ such that

$$m\Big(u^\delta \chi_{[u<a]} + u^\beta \chi_{[u \geqslant a]}\Big) \leqslant f(x,u) \leqslant M\Big(u^\delta + u^\beta\Big)$$

holds for some $0 < a < 1$ small enough, that is, $u \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ is a subsolution for

$$-\Delta_p u = \alpha M\Big(u^\delta + u^\beta\Big) \tag{3.18}$$

and a supersolution for

$$-\Delta_p u = \alpha m\Big(u^\delta \chi_{[u<a]} + u^\beta \chi_{[u\geqslant a]}\Big). \tag{3.19}$$

Now, we build a positive supersolution for (3.18) and a positive subsolution for (3.19), as required by Theorem 1.2.2. First, let us define $\overline{u}_\alpha = m_2 \alpha^\tau e_1^t$, $\alpha > 0$, with $t = \min\{1, (p-1)/(p-1-\delta)\}$ and $\tau, m_2 > 0$ being constants independent of α, to be chosen later. Thus, using that $0 < t \leqslant 1$, we have

$$\int_\Omega |\nabla \overline{u}_\alpha|^{p-2}\nabla\overline{u}_\alpha \nabla\varphi dx \geqslant \int_\Omega |\nabla e|^{p-2}\nabla e\nabla\Big[\varphi(\alpha^\tau m_2 e^{t-1}t)^{p-1}\Big]dx = \int_\Omega \varphi(\alpha^\tau m_2 e^{t-1}t)^{p-1}dx$$

for each $0 \leqslant \varphi \in C_c^\infty(\Omega)$ given.

To verify that $\overline{u}_\alpha$ is a supersolution for (3.18), it is enough to show that

$$(\alpha^\tau m_2 t)^{p-1} \geqslant \alpha M \max\{1, \|e_1^{t(\beta-\delta)}\|_\infty\}\Big(m_2^\delta \alpha^{\tau\delta} + m_2^\beta \alpha^{\tau\beta}\Big) \tag{3.20}$$

holds, for some appropriately chosen $\tau, m_2 > 0$.

To do this, let us fix $m_2 = \max\left\{1, \left(\frac{3M\max\{1,\|e_1^{t(\beta-\delta)}\|_\infty\}}{t^{p-1}}\right)^{1/(p-1-\beta)}\right\}$ and consider two cases on the size of α. If $\alpha < 1$, we obtain that the inequality (3.20) holds by choosing $\tau = 1/(p-1-\delta)$, while for $\alpha \geqslant 1$ we obtain (3.20) by taking $\tau = 1/(p-1-\beta)$. Therefore, in both the cases $\overline{u}_\alpha$ is a supersolution for (3.18) for every $\alpha > 0$.

Next, we build a subsolution for (3.19) as follows. Setting $\underline{u}_\alpha = \alpha^\tau m_1 \Phi_1$, $\alpha > 0$, we have that $\underline{u}_\alpha$ will be a subsolution for (3.19) if

$$(m_1\alpha^\tau)^{(p-1)}\lambda_1\Phi_1^{p-1} \leqslant \alpha m\Big(m_1^\delta\alpha^{\tau\delta}\Phi_1^\delta\chi_{[m_1\alpha^\tau\phi_1<a]} + m_1^\beta\alpha^{\tau\beta}\Phi_1^\beta\chi_{[m_1\alpha^\tau\Phi_1\geqslant a]}\Big) \tag{3.21}$$

is satisfied, for some $\tau, m_1 > 0$ independent of α.

Again, let us consider two cases on α. First, let $0 < \alpha < \lambda_1 a^{p-1-\delta}/m$. By taking $\tau = 1/(p-1-\delta)$ and $m_1 = \Big(m/\lambda_1\|\Phi_1^{1/\tau}\|_\infty\Big)^\tau = m^{1/(p-1-\delta)}/(\|\Phi_1\|_\infty\lambda_1^{1/(p-1-\delta)})$, the inequality (3.21) holds. On the other hand, for $\alpha \geqslant \lambda_1 a^{p-1-\delta}/m$, let us take $\tau = 1/(p-1-\beta)$ and $m_1 = \Big(m/\lambda_1\|\Phi_1^{1/\tau}\|_\infty\Big)^\tau = m^{1/(p-1-\beta)}/(\|\Phi_1\|_\infty\lambda_1^{1/(p-1-\beta)})$ to obtain the inequality (3.21) again. Therefore, in both the cases, we have that $\underline{u}_\alpha$ is a subsolution of (3.19) for each $\alpha > 0$ given.

Fix

$$\alpha_0 = \min\left\{1, \frac{\lambda_1 a^{p-1-\delta}}{m}\right\} \text{ and } \alpha_\infty = \max\left\{1, \frac{\lambda_1 a^{p-1-\delta}}{m}\right\}.$$

Now, using u as a subsolution of (3.18) and $\overline{u}_\alpha = \alpha^\tau m_2 e_1^t$ as a supersolution of (3.18), for $\tau = 1/(p-1-\delta)$ and $\alpha < \alpha_0$, together with Theorem 1.2.2, we get the second inequality in the item$-a$).

Moreover, using u as a supersolution of (3.19) and $\underline{u}_\alpha = \alpha^\tau m_1 \Phi_1$ as a subsolution of (3.19), for $\tau = 1/(p-1-\delta)$ and $\alpha < \alpha_0$, together with Corollary 1.2.5, we get the first inequality in item$-a$).

Similarly, for $\alpha > \alpha_\infty$ and $\tau = 1/(p-1-\beta)$, arguing as before we get both the inequalities in item$-b$).

■

As immediate consequence of the proof of the previous Lemma, we have the following Corollary.

Corollary 3.2.2 *Assume that $-\infty < \delta \leqslant \beta < p-1$. If there exist $M, m > 0$ and $0 < u, v \in W^{1,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ such that:*

(*i*) *the inequality*

$$-\Delta_p u \leqslant \alpha M(u^\delta + u^\beta) \text{ in } \Omega \text{ and } u \leqslant 0 \text{ on } \partial\Omega \tag{3.22}$$

holds, then u satisfies the second inequality in (3.17), for some m_2 independent of $\alpha > 0$, where τ is given in the items $a) - b)$ of the Lemma 3.2.1. In particular, if u satisfies $-\Delta_p u \leqslant L(u^\delta + u^\beta)$ for some $L > 0$ and $u \leqslant 0$ on $\partial\Omega$, then $\|u\|_\infty \leqslant C(L)$,

(*ii*) *the inequality*

$$-\Delta_p v \geqslant \alpha m(v^\delta \chi_{[v<a]} + v^\beta \chi_{[v \geqslant a]}) \text{ in } \Omega \tag{3.23}$$

holds for some $0 < a < 1$*, then* v *satisfies the first inequality in (3.17), for some* m_1 *independent of* $\alpha > 0$*, where* τ *is given in the items* $a) - b)$ *of the Lemma 3.2.1.*

Proof: It remains only to prove the particular case in item $-i)$. Without loss of generality, we can assume that $L > \alpha_\infty$. Thus, by identifying $\alpha = L$ and $M = 1$ in (3.22), it follows from the first part of the proof of the above Lemma that $u \leqslant m_2 L^{1/(p-1-\beta)} e_1^t$, where $m_2 = \max\left\{1, \left(\frac{3\max\{1, \|e_1^{t(\beta-\delta)}\|_\infty\}}{t^{p-1}}\}\right)^{1/(p-1-\beta)}\right\}$. Therefore, $\|u\|_\infty \leqslant m_2 L^{1/(p-1-\beta)} \|e_1^t\|_\infty := C(L)$.

■

3.3 Qualitative information of the *continuum*

In this section, we prove Theorems 0.0.8 and 0.0.9. We also prove an existence and non-existence result for the degenerate problem (i.e. $A(x, 0) = 0$ in Ω) in Theorem 3.3.2. We begin with Theorem 0.0.8.

Theorem 0.0.8 *Assume* (A_0) *and that* f *satisfies* (f_1) *and* (f_2) *with* $\delta \leqslant \beta$*. If*

a) $\gamma > 0$ *and either* $\{\theta\gamma = p - 1 - \beta$ *and* $(A'_\infty)\}$ *or* $\{\theta\gamma < p - 1 - \beta$ *and* (A_∞) *with* $a_\infty > 0$ *in* $\overline{\Omega}\}$ *hold, then* $Proj_{\mathbb{R}}\Sigma = (0, \infty)$ *(see Fig. 5),*

b) $\gamma > 0$*,* $\theta\gamma \geqslant p - 1 - \beta$ *and* (A_∞) *hold, then* $Proj_{\mathbb{R}}\Sigma \subset (0, \lambda^*)$ *for some* $0 < \lambda^* < \infty$*. Furthermore, if*

i) $a_\infty > 0$ *in* $\overline{\Omega}$ *and* $\theta\gamma = p - 1 - \beta$*, then* $\lambda = 0$ *can not be a bifurcation point from* ∞ *(see Fig. 6 or 7);*

ii) $a_\infty = 0$ *in* $\overline{\Omega}$*, then* $\lambda = 0$ *is a bifurcation point from* ∞ *(see Fig. 8);*

c) $-1 < \gamma < 0$*,* $\theta\gamma \geqslant p - 1 - \delta$ *and either* (A'_∞) *or* (A_∞) *with* $0 < a_\infty$ *hold, then* (P_2) *does not admit positive solution for* $\lambda > 0$ *small.*

Proof: First, we note that under the hypotheses (A_0) and (f_2), we are able to apply Theorem 0.0.7 to guarantee the existence of an unbounded *continuum* Σ of positive $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$-solutions for (P_2).

a) Let us prove just the case $\{\theta\gamma = p-1-\beta$ and $(A'_\infty)\}$, because the other one is similar. Assume by contradiction that Σ is horizontally bounded. Then, there exists a sequence $(\lambda_n, u_n) \subset \Sigma$ and $0 < \lambda^* < \infty$ such that $\lambda_n \leqslant \lambda^*$ and $\|u_n\|_\infty \to \infty$. We claim that $\int_\Omega u_n^\gamma dx \to \infty$. Otherwise, it would follow from (A_0), (f_1) and (f_2) that

$$-\Delta_p u_n \leqslant L\Big(u_n^\delta + u_n^\beta\Big)$$

holds, up to a subsequence, for some $L > 0$ independent of n. Using this information and Corollary 3.2.2$-i)$, we obtain $\|u_n\|_\infty \leqslant C(L)$ but this contradicts the fact that $\|u_n\|_\infty \to \infty$.

Now, for $t = \min\{1, (p-1)/(p-1-\delta)\}$, fix $m_2 \in (0,\ \min\{1, (\int_\Omega e_1^{t\gamma} dx)^{-1/\gamma}\})$ and $C_1 > 0$ such that

$$\frac{\lambda^*}{C_1} \leqslant \frac{m_2^{p-1-\delta} t^{p-1}}{2\max\{1, \|e_1\|_\infty^{t|\beta-\delta|}\}}. \tag{3.24}$$

First, we note that as a consequence of $\int_\Omega u_n^\gamma dx \to \infty$ and the hypothesis (A'_∞), for n large we have $A\Big(x, \int_\Omega u_n^\gamma dx\Big)\Big(\int_\Omega u_n^\gamma dx\Big)^\theta \geqslant C_1 > 0$ which leads us to

$$-\Delta_p u_n = \frac{\lambda_n (\int_\Omega u_n^\gamma dx)^\theta f(x, u_n)}{A\Big(x, \int_\Omega u_n^\gamma dx\Big)\Big(\int_\Omega u_n^\gamma dx\Big)^\theta} \leqslant \frac{\lambda^*}{C_1}\tilde{\lambda}_n\Big(u_n^\delta + u_n^\beta\Big),$$

where $\tilde{\lambda}_n = \Big(\int_\Omega u_n^\gamma dx\Big)^\theta$.

Next, let us define $\overline{u}_n = m_2 \tilde{\lambda}_n^\tau e_1^t$, with $\tau = (p-1-\beta)^{-1}$. By proceeding as in the proof of Lemma 3.2.1$-b)$ and using (3.24), we have

$$-\Delta_p \overline{u}_n \geqslant \frac{\lambda^*}{C_1}\tilde{\lambda}_n\Big(\overline{u}_n^\delta + \overline{u}_n^\beta\Big)$$

for n sufficiently large.

Therefore, by Theorem 1.2.2 we obtain $u_n \leqslant m_2\Big(\int_\Omega u_n^\gamma\Big)^{\theta\tau} e_1^t$, which results in

$$\int_\Omega u_n^\gamma dx \leqslant \Big(\int_\Omega u_n^\gamma dx\Big)^{\theta\tau\gamma} m_2^\gamma \int_\Omega e_1^{t\gamma} dx.$$

As $\theta\gamma = p-1-\beta$, it follows from the previous inequality that $1 \leqslant m_2^\gamma \int_\Omega e_1^{t\gamma} dx$, but this is a contradiction by our choice of $m_2 < (\int_\Omega e_1^{t\gamma} dx)^{-1/\gamma}$.

b) Assume that there exists a sequence (λ_n, u_n) of solutions of (P_2) such that $\lambda_n \to \infty$. We claim that $\int_\Omega u_n^\gamma dx \to \infty$. Otherwise, by the hypotheses (f_1) and (f_2) there exist constants $C_1 > 0$ and $0 < a < 1$ such that

$$-\Delta_p u_n \geqslant C_1\lambda_n\Big(u_n^\delta \chi_{[u_n<a]} + u_n^\beta \chi_{[u_n\geqslant a]}\Big) \tag{3.25}$$

holds, up to a subsequence. Thus, we obtain from (3.25) and Corollary 3.2.2$-ii$) that $\lambda_n^\tau m_1\phi_1 \leqslant u_n$ for some $m_1 > 0$ independent of n, $\tau = (p-1-\beta)^{-1}$ and n large enough. Hence, from this we get $C \geqslant \int_\Omega u_n^\gamma dx \geqslant \lambda_n^{\tau\gamma}\int_\Omega \Phi_1^\gamma dx \to \infty$, which is a contradiction.

From the above claim and the hypothesis $0 \leqslant a_\infty < \infty$ on $\overline{\Omega}$, we obtain

$$A\Big(x, \int_\Omega u_n^\gamma dx\Big)\Big(\int_\Omega u_n^\gamma dx\Big)^\theta \leqslant C_2$$

for some constant $C_2 > 0$ and, as a consequence of this, we have

$$-\Delta_p u_n \geqslant C_3\lambda_n\Big(\int_\Omega u_n^\gamma dx\Big)^\theta\Big(u_n^\delta \chi_{[u_n<a]} + u_n^\beta \chi_{[u_n\geqslant a]}\Big)$$

for some $C_3 > 0$ independent of n.

Now, by taking $m = C_3$ and $\alpha = \lambda_n\Big(\int_\Omega u_n^\gamma dx\Big)^\theta$ in (3.23), it follows from Corollary 3.2.2$-ii$) that $\lambda_n^\tau\Big(\int_\Omega u_n^\gamma dx\Big)^{\tau\theta} m_1\Phi_1 \leqslant u_n$, for some $m_1 > 0$ independent of n, $\tau = (p-1-\beta)^{-1}$ and n sufficiently large. Thus, we conclude that $\lambda_n^{\gamma\tau} \leqslant C_4\Big(\displaystyle\int_\Omega u_n^\gamma dx\Big)^{1-\tau\theta\gamma} = C_4$ for some $C_4 > 0$, where in the last equality we used $\tau\theta\gamma = 1$. But this is a contradiction, since $\gamma\tau > 0$ and

$\lambda_n \to \infty$.

Below, let us prove the items $i) - ii)$.

i) Assume that there exists a sequence $(\lambda_n, u_n) \subset \Sigma$ such that $\lambda_n \to 0$ and $\|u_n\|_\infty \to \infty$. In the same way as proved in the item $-a)$ above, we get $\int_\Omega u_n^\gamma dx \to \infty$. Using this fact and the hypothesis $a_\infty > 0$ in $\overline{\Omega}$, we obtain

$$-\Delta_p u_n \leqslant C_1 \lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta (u_n^\delta + u_n^\beta),$$

which implies that $\lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta \to \infty$. If not, we would have $C_1 \lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta \leqslant C_2$ for some C_2 large, hence by Corollary 3.2.2$-i)$ we get $\|u_n\|_\infty \leqslant C(C_2)$. However, this is a contradiction because we are supposing that $\|u_n\|_\infty \to \infty$.

Therefore, by taking $M = C_1$ and $\alpha = \lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta$ in (3.22) and applying Corollary 3.2.2$-i)$, we get $u_n \leqslant m_2 \lambda_n^\tau \Big(\int_\Omega u_n^\gamma dx \Big)^{\tau\theta} e_1^t$ for some m_2 independent of n, $\tau = (p - 1 - \beta)^{-1}$ and n large enough, which lead us to conclude that $1 = \Big(\int_\Omega u_n^\gamma dx \Big)^{1 - \tau\theta\gamma} \leqslant C \lambda_n^{\tau\gamma} \to 0$ by the choice of θ, which is impossible.

ii) Assume that there exists a sequence $(\lambda_n, u_n) \subset \Sigma$ such that $\lambda_n \to \lambda^* > 0$ and $\|u_n\|_\infty \to \infty$. Then, by the same idea as used to prove the item $-a)$ above, we have that $\int_\Omega u_n^\gamma dx \to \infty$. Thus, for a given $\epsilon > 0$, we obtain from the hypothesis $a_\infty \equiv 0$ that $0 < \lambda^*/2 < \lambda_n$ and $A\Big(x, \int_\Omega u_n^\gamma dx\Big) \Big(\int_\Omega u_n^\gamma dx \Big)^\theta < \epsilon$ for all n as large as required. From this we obtain that $-\Delta_p u_n \geqslant \frac{\lambda^* C_1}{2\epsilon} \Big(\int_\Omega u_n^\gamma dx \Big)^\theta (u_n^\delta \chi_{[u_n \leqslant a]} + u_n^\beta \chi_{[u_n > a]})$, for some C_1 independent of n and $\epsilon > 0$.

Hence, taking $m = C_1$ and $\alpha = \frac{\lambda^*}{2\epsilon} \Big(\int_\Omega u_n^\gamma dx \Big)^\theta$ in (3.23), we get by the Corollary 3.2.2$-ii)$ that $\Big(\frac{\lambda^*}{2\epsilon}\Big)^\tau \Big(\int_\Omega u_n^\gamma dx \Big)^{\theta\tau} m_1 \Phi_1 \leqslant u_n$ for some m_1 independent of n, $\tau = (p - 1 - \beta)^{-1}$ and n large. As a consequence of this information and by $\theta\gamma \geqslant p - 1 - \beta$, we obtain $1 \geqslant \Big(\int_\Omega u_n^\gamma dx \Big)^{1 - \tau\gamma\theta} \geqslant \frac{C}{\epsilon^\tau}$, which is an absurd for $\epsilon > 0$ small enough, as C is independent of ϵ.

c) Assume that there exists a pair (λ_n, u_n) which solves (P) with $\lambda_n \to 0^+$. Then $\int_\Omega u_n^\gamma dx \to \infty$ must occur, otherwise

$$-\Delta_p u_n \leqslant C_1 \lambda_n \left(u_n^\delta + u_n^\beta \right)$$

holds, up to subsequence. By taking $M = C_1$ and $\alpha = \lambda_n$ in (3.22), we get by Corollary 3.2.2$-i)$ that $u_n \leqslant m_2 \lambda_n^\tau e_1^t$ for some m_2 independent of n, $\tau = (p-1-\delta)^{-1}$ and t as defined before. As a consequence of this fact and $-1 < \gamma < 0$, we have $C \geqslant \int_\Omega u_n^\gamma dx \geqslant m_2^\gamma \lambda_n^{\gamma\tau} \int_\Omega e_1^{t\gamma} dx \to \infty$, which is an absurd. Therefore, $\int_\Omega u_n^\gamma dx \to \infty$ which implies $\lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta \to 0$, since $\theta < 0$.

Hence, by using this information together with the hypothesis on A, we obtain

$$-\Delta_p u_n \leqslant C_2 \lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta (u_n^\delta + u_n^\beta)$$

for some C_2 independent of n.

Next, by fixing $M = C_2$ and $\alpha = \lambda_n \Big(\int_\Omega u_n^\gamma dx \Big)^\theta$ in (3.22), we obtain by Corollary 3.2.2$-i)$ that $u_n \leqslant m_2 \lambda_n^\tau \Big(\int_\Omega u_n^\gamma dx \Big)^{\theta\tau} e_1^t$ for $\tau = (p-1-\delta)^{-1}$, for some $m_2 > 0$ independent of n and for n appropriately large. Therefore, for the choice of θ, we have $C_3 \geqslant C_3 \Big(\int_\Omega u_n^\gamma dx \Big)^{1-\tau\theta\gamma} \geqslant \lambda_n^{\tau\gamma} \to \infty$ for some $C_3 > 0$, which leads us to a contradiction again.

This ends the proof of Theorem. ■

To prove Theorem 0.0.9, let us take advantage of Theorem 0.0.7 to get an unbounded *continuum* Σ_0 of positive $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$-solutions of

$$\begin{cases} -\Delta_p u = \alpha f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

with $Proj_{\mathbb{R}^+} \Sigma_0 = (0, \infty)$. This allows us to define an appropriated map H_λ on Σ_0 such that its zeros are connected with the solutions of (7). More precisely, a pair

$(\lambda, u) \in (0,\infty) \times W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a solution of (7) if and only if $(\alpha, u) \in \Sigma_0$ with $\alpha = \lambda\Big[A\Big(\int_\Omega u^\gamma dx\Big)\Big]^{-1}$, which is equivalent to the pair $(\alpha, u) \in \Sigma_0$ being a zero of the map

$$H_\lambda(\alpha, u) = \alpha - \lambda\Big[A\Big(\int_\Omega u^\gamma dx\Big)\Big]^{-1} = \Big(\Psi(\alpha, u) - \lambda\Big)\Big[A\Big(\int_\Omega u^\gamma dx\Big)\Big]^{-1}, \ (\alpha, u) \in \Sigma_0,$$

where $\Psi(\alpha, u) = \alpha A\Big(\int_\Omega u^\gamma dx\Big)$.

Now, we prove the next proposition, which assists us to prove a global existence result for (7).

Proposition 3.3.1 *Assume that* $-1 < \gamma < 0$ *and* (A_0)*. If*

$$\limsup_{\substack{\alpha\to 0^+ \\ (\alpha,u)\in\Sigma_0}} \Psi(\alpha, u) = \infty \quad \textit{and} \quad \limsup_{\substack{\alpha\to \infty \\ (\alpha,u)\in\Sigma_0}} \Psi(\alpha, u) = \infty \tag{3.26}$$

hold, then there exists a $\lambda^* > 0$ *such that* (7) *has at least one solution for each* $\lambda \in [\lambda^*, \infty)$ *and no solution for* $\lambda < \lambda^*$.

Proof: As revealed in the proofs of the Claim 1 and Claim 2 of Theorem 0.0.7, we have $\Sigma_0 \subset \mathcal{F}$, where $\mathcal{F}$ is defined at (3.5). As a consequence, we conclude that the function Ψ (as above) is well-defined and continuous on Σ_0. Let us define

$$\lambda^* = \inf\{\Psi(\alpha, u) : (\alpha, u) \in \Sigma_0\}.$$

First, we claim that $\lambda^* > 0$. If not, there exists a sequence $\{(\alpha_n, u_n)\} \subset \Sigma_0$ such that $\alpha_n A\Big(\int_\Omega u_n^\gamma dx\Big) \to 0$, which implies by (3.26) that there are positive constants C_1 and C_2 satisfying $C_1 \leqslant \alpha_n \leqslant C_2$. It follows from this fact and Corollary 3.2.2$-ii)$ that $C_3\Phi_1 \leqslant u_n$ in Ω, for some positive constant C_3 independent of n, which results in $A\Big(\int_\Omega u_n^\gamma dx\Big) \geqslant C_4 > 0$. As a consequence of this fact and $C_1 \leqslant \alpha_n \leqslant C_2$, we have $C_5 \leqslant \alpha_n A\Big(\int_\Omega u_n^\gamma dx\Big)$ for some $C_5 > 0$, but this contradicts the fact that $\alpha_n A\Big(\int_\Omega u_n^\gamma dx\Big) \to 0$.

Next, let us set $\lambda > \lambda^*$. By definition of λ^*, we can find a pair $(\alpha^*, u^*) \in \Sigma_0$ satisfying $\lambda^* < \Psi(\alpha^*, u^*) < \lambda$. On the other hand, it follows from (3.26) that there exists $(\alpha^{**}, u^{**}) \in \Sigma_0$ such that $\Psi(\alpha^{**}, u^{**}) > \lambda$. In particular, we have proven that $H_\lambda(\alpha^*, u^*) < 0$ and $H_\lambda(\alpha^{**}, u^{**}) > 0$. Thus, by Theorem **A.1.8** we get the existence of at least one zero of H_λ in Σ_0.

Now, we prove that (7) admits at least one solution to $\lambda = \lambda^*$. For this, it is enough to show that there is a pair $(\alpha, u) \in \Sigma_0$ such that $\Psi(\alpha, u) = \lambda_*$. However, by the definition of λ^*, we can find a sequence $(\alpha_n, u_n) \subset \Sigma_0$ satisfying $\Psi(\alpha_n, u_n) \to \lambda^*$. Using the hypothesis (3.26), we again conclude that $C_1 \leqslant \alpha_n \leqslant C_2$, up to subsequence, for some positive constants C_1 and C_2. Thus, following the same argumentation of the proof of the Theorem 0.0.7, we obtain that $(\alpha_n, u_n) \to (\alpha, u) \in \Sigma_0$ in $\mathbb{R} \times C(\overline{\Omega})$. As Ψ is a continuous application in Σ_0, we get $\Psi(\alpha, u) = \lambda^*$ as we wanted.

Finally, the non-existence of solutions to $\lambda < \lambda^*$ is a consequence of the definition of λ^*. This ends the proof.

■

Through the previous proposition, we are able to prove the Theorem 0.0.9.

***Proof of Theorem 0.0.9-Completion*:** It suffices to verify the hypotheses at (3.26) and apply the above Proposition. To begin with, we prove the first limit at (3.26). We recall that by Lemma 3.2.1$-a$), the inequality $u \leqslant \alpha^\tau m_2 e_1^t$ holds true whenever $(\alpha, u) \in \Sigma_0$ with $\alpha < \alpha_0$, for some $m_2 > 0$ independent of α, $\tau = 1/(p-1-\delta)$ and $t = (p-1)/(p-1-\delta)$. By using this inequality and $\gamma < 0$, we get

$$\limsup_{\substack{\alpha \to 0^+ \\ (\alpha,u) \in \Sigma_0}} \int_\Omega u^\gamma = \infty. \tag{3.27}$$

Thus, as either (A'_∞) or (A_∞) with $0 < a_\infty$ holds, it follows from (3.27) that

$$\Psi(\alpha, u) = \alpha A\Big(\int_\Omega u^\gamma dx\Big) \geqslant C_1 \alpha \Big(\int_\Omega u^\gamma dx\Big)^{-\theta} \geqslant C\alpha^{1-\tau\theta\gamma}$$

for α small. Since $\theta\gamma > p-1-\delta$, we get

$$\limsup_{\substack{\alpha\to 0^+ \\ (\alpha,u)\in\Sigma_0}} \Psi(\alpha,u) = \infty.$$

Now, let us prove the second limit at (3.26). By Lemma 3.2.1$-b$), we know that $\alpha^\tau m_1\Phi_1 \leqslant u$ for some $m_1 > 0$ independent of α and for $\tau = 1/(p-1-\beta)$, whenever $(\alpha,u)\in\Sigma_0$ with $\alpha > \alpha_\infty$. As a result, since $\gamma < 0$, we have

$$\limsup_{\substack{\alpha\to\infty \\ (\alpha,u)\in\Sigma_0}} \int_\Omega u^\gamma = 0. \tag{3.28}$$

Therefore, by continuity and positivity of A at $t=0$ and (3.28), we obtain

$$\limsup_{\substack{\alpha\to\infty \\ (\alpha,u)\in\Sigma_0}} \Psi(\alpha,u) = \infty.$$

This ends the proof. ∎

Again, let us be benefited by our tools and follow the strategy of [29] to approach the problem (P_2) for the degenerate case, that is, when $A(x,0) = 0$. This procedure allows us to complement the results in [29] both to p-Laplacian operator, with $1 < p < \infty$, and strongly-singular non-linearities.

Theorem 3.3.2 (Degenerate case: A(x, 0) = 0) *Assume that $\gamma > 0$ and f satisfies (f_1), (f_2) with $\delta \leqslant \beta$. If $A \in C(\overline{\Omega}\times[0,\infty),[0,\infty))$ with $A(x,0) = 0$ in Ω, $\theta\gamma = p-1-\beta$ and:*

a) *(A'_∞) holds, then (P_2) has at least one solution for each $\lambda > 0$.*

b) *(A_∞) holds with $0 < a_\infty$ in $\overline{\Omega}$, then (P_2) has at least one solution for λ small and no solution for λ large.*

Proof: For each $n \in \mathbb{N}$, consider

$$(P_{1/n}) \begin{cases} -A_n\Big(x, \int_\Omega u^\gamma dx\Big)\Delta_p u = \lambda f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $A_n(x,t) = A(x,t) + 1/n$. Since $\lim_{t\to\infty} A_n(x,t)t^\theta = \infty$, with $\theta\gamma = p-1-\beta$, it follows from the item $a)$ of Theorem 0.0.8 that $(P_{1/n})$ has at least one solution for each $\lambda > 0$. Thus, given a $\lambda > 0$, denote by u_n one such solution of $(P_{1/n})$. From this, let us prove the items $a)$ and $b)$ above.

a) The proof of this item is a consequence of the following claims:

$$i) \int_\Omega u_n^\gamma dx \nrightarrow 0 \qquad \text{and} \qquad ii) \int_\Omega u_n^\gamma dx \nrightarrow \infty. \tag{3.29}$$

Let us prove the first claim in (3.29). Suppose by contradiction, that $\int_\Omega u_n^\gamma dx \to 0$. Since $A(x,0) = 0$ and A is a continuous function, for given $C > 0$ sufficiently large there exists $n_0 \in \mathbb{N}$ such that $A_n\Big(x, \int_\Omega u_n^\gamma dx\Big) < 1/C$ for all $n > n_0$. Thus, we get $-\Delta_p u_n \geqslant \lambda C f(x,u_n)$, which implies by Corollary 3.2.2$-ii)$ that $u_n \geqslant (\lambda C)^\tau m_1 \Phi_1$ for n large, where $\tau = (p-1-\beta)^{-1}$. Hence, from this inequality we get $0 < (\lambda C)^{\tau\gamma} m_1^\gamma \int_\Omega \Phi_1^\gamma dx \leqslant \int_\Omega u_n^\gamma dx \to 0$, which is an absurd.

Now we will prove the second claim in (3.29). Again, suppose by contradiction that $\int_\Omega u_n^\gamma dx \to \infty$. From (A'_∞), for each $C > 0$ large enough , we have $A\Big(x, \int_\Omega u_n^\gamma dx\Big)\Big(\int_\Omega u_n^\gamma dx\Big)^\theta > C$ for all n big enough. In this case, we obtain $-\Delta_p u_n \leqslant \frac{\lambda}{C}\Big(\int_\Omega u_n^\gamma dx\Big)^\theta f(x,u_n)$, which by the Corollary 3.2.2$-i)$ and simple calculations implies

$$\Big(\int_\Omega u_n^\gamma dx\Big)^{1-\tau\theta\gamma} \leqslant \Big(\frac{\lambda}{C}\Big)^\tau m_2^\gamma, \tag{3.30}$$

where $\tau = (p-1-\beta)^{-1}$. As $\theta\gamma = p-1-\beta$ and $C > 0$ was taken large enough, the inequality (3.30) results into $1 \leqslant \Big(\frac{\lambda}{C}\Big)^\tau m_2^\tau < 1$. This is an absurd and from this the Claim in $ii)$ is proved.

Observe that from claims in $i) - ii)$, we get $0 < C_1 \leqslant \int_\Omega u_n^\gamma dx \leqslant C_2$, for some positive constants C_1 and C_2. Thus, proceeding as in the proof of the Claim 2 in Theorem 0.0.7, we can show that u_n converge in $W^{1,p}_{\text{loc}}(\Omega)$ for some $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, which is a solution of (P_2). It concludes the proof of item$-a)$.

b) As in the item$-a)$, the proof here follows from the following assertions:

$$i) \int_\Omega u_n^\gamma dx \nrightarrow 0 \qquad \text{and} \qquad ii) \int_\Omega u_n^\gamma dx \nrightarrow \infty, \text{for each } \lambda > 0 \text{ small.} \quad (3.31)$$

The proof of the first Claim in (3.31) is the same as in item$-a)$.

Let us prove $ii)$. As $a_\infty > 0$ in $\overline{\Omega}$, defining $C = (\inf_\Omega a_\infty)/2$, there exists $t_0 > 0$ such that $A(x,t)t^\theta \geqslant C > 0$ for all $t > t_0$. Thus, if we suppose that $\int_\Omega u_n^\gamma dx \to \infty$, we obtain $-\Delta_p u_n \leqslant \frac{\lambda}{C}\Big(\int_\Omega u_n^\gamma dx\Big)^\theta f(x,u_n)$, which again by Corollary 3.2.2$-i)$ implies in $u_n \leqslant \Big(\frac{\lambda}{C}\Big)^\tau \Big(\int_\Omega u_n^\gamma dx\Big)^{\theta\tau} m_2 e_1^t$ for some $m_2 > 0$, $\tau = (p-1-\beta)^{-1}$, $t = (p-1)/(p-1-\delta)$ and n appropriately large. As a consequence of this, we obtain $\Big(\int_\Omega u_n^\gamma dx\Big)^{1-\theta\gamma\tau} \leqslant \Big(\frac{\lambda}{C}\Big)^{\gamma\tau} m_2^\gamma \int_\Omega e_1^{t\gamma} dx$. Since $\theta\gamma = p-1-\beta$, we get by the last inequality that $1 \leqslant \Big(\frac{\lambda}{C}\Big)^{\gamma\tau} m_2^\gamma \int_\Omega e_1^{t\gamma} dx$. However this is a contradiction for $\lambda < C\Big(m_2^\gamma \int_\Omega e_1^{t\gamma} dx\Big)^{-1/\gamma\tau} = \lambda^*$. Therefore, $\int_\Omega u_n^\gamma dx \nrightarrow \infty$ for $0 < \lambda < \lambda^*$.

From $i) - ii)$, by the same argument as in item$-a)$ we conclude that (P_2) admits at least one positive solutions for $0 < \lambda < \lambda^*$. To justify that (P_2) does not have solution for λ large, just follow the same argument as in item $b)$ of Theorem 0.0.8, using $\theta\gamma = p-1-\beta$.

This proves the Theorem.

■

3.4 A strongly-singular non-autonomous Kirchhoff problem

In this section, we prove Theorem 0.0.10 which deals with a non-autonomous Kirchhoff problem, defined in (Q_1), with strongly-singular nonlinearity.

The proof of Theorem 0.0.10 follows the same steps of Theorem 0.0.7 with small adaptations. Recall that in the proof of Lemma 3.1.2 we used that $\|u_\epsilon\|_\gamma \leqslant C$ for some C independent of ϵ, where $(\lambda_\epsilon, u_\epsilon)$ is a solution of perturbed problem (P_ϵ) and belongs to the boundary of an open bounded set containing $(0,0)$. Here, due to the presence of $\|\nabla u\|_p$ in the Kirchhoff term, we need a similar estimate on $\|\nabla u_\epsilon\|_p$, which is crucial in our argument. To avoid repetition, we present a sketch of each step while giving attention to the notable points. Corresponding to (Q_1), we introduce the following perturbed problem

$$(Q_\epsilon) \qquad \begin{cases} -M\Big(x, \|\nabla u\|_p^p\Big)\Delta_p u = \lambda f(x, u+\epsilon) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

About (Q_ϵ), we have the following result.

Lemma 3.4.1 *Suppose that $\gamma > 0$ and M satisfies (M_0). Then, for each $\epsilon > 0$ there exists an unbounded ϵ-continuum $\Sigma_\epsilon \subset \mathbb{R}^+ \times C(\overline{\Omega})$ of positive solutions of (Q_ϵ) emanating from $(0,0)$.*

Proof: Consider for each $\lambda, R > 0$ and $v \in C(\overline{\Omega})$, the auxiliary problem

$$\begin{cases} -M(x, R)\Delta_p u = \lambda f(x, |v|+\epsilon) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.32}$$

As $M(x,t) = a(x) + b(x)t^\gamma$ with $a(x) \geqslant \underline{a} > 0$ and f is continuous, (3.32) admits a unique solution $u_R \in C^{1,\alpha}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$, for some $\alpha \in (0,1)$. Thus

$$\int_\Omega |\nabla u_R|^p dx = \int_\Omega \frac{\lambda f(x, |v|+\epsilon)u_R}{M(x,R)} dx.$$

Define $h : \mathbb{R}^+ \to \mathbb{R}^+$ by $h(R) = \displaystyle\int_\Omega \frac{\lambda f(x,|v|+\epsilon)u_R}{M(x,R)}dx$. Note that h is continuous and $h(0) > 0$. Moreover, observe that h is non-increasing. Indeed, if $R_1 < R_2$ then

$$-\Delta_p u_{R_2} = \frac{\lambda f(x,|v|+\epsilon)}{M(x,R_2)} \leqslant \frac{\lambda f(x,|v|+\epsilon)}{M(x,R_1)} = -\Delta_p u_{R_1}.$$

Also, as $u_{R_1}|_{\partial\Omega} = u_{R_2}|_{\partial\Omega}$, from classical comparison principle, we have $u_{R_2} \leqslant u_{R_1}$ and as a consequence we conclude that $h(R_2) \leqslant h(R_1)$. Thus, there exists a unique solution (say $\tilde{R}$) of $h(R) = R$, that is,

$$\tilde{R} = \int_\Omega \frac{\lambda f(x,|v|+\epsilon)u_{\tilde{R}}}{M(x,\tilde{R})}dx = \int_\Omega |\nabla u_{\tilde{R}}|^p dx.$$

Hence, $u_{\tilde{R}}$ is a solution of

$$\begin{cases} -M\Big(x, \|\nabla u\|_p^p)\Delta_p u = \lambda f(x,|v|+\epsilon) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.33}$$

We claim that (3.33) has a unique solution. In fact, suppose that $u \neq w \in W_0^{1,p}(\Omega)$ are two solutions of (3.33). If $\int_\Omega |\nabla u|^p dx = \int_\Omega |\nabla w|^p dx$, then $u = w$ in Ω. On the other hand, if $R_1 = \int_\Omega |\nabla u|^p dx < \int_\Omega |\nabla w|^p dx = R_2$, we have $u_{R_2} \leqslant u_{R_1}$ and as a consequence

$$R_2 = \int_\Omega |\nabla w|^p dx = \int_\Omega \frac{f(x,|v|+\epsilon)u_{R_2}}{M(x,R_2)}dx \leqslant \int_\Omega \frac{f(x,|v|+\epsilon)u_{R_1}}{M(x,R_1)}dx = \int_\Omega |\nabla u|^p dx = R_1.$$

Therefore, in any case we get a contradiction, which proves that (3.33) has only one solution. Now, we consider the operator $T : \mathbb{R}^+ \times C(\overline{\Omega}) \to C(\overline{\Omega})$ which associates each pair $(\lambda, v) \in \mathbb{R}^+ \times C(\overline{\Omega})$ to the only solution of (3.33). Since $M(x,t) \geqslant \underline{a} > 0 \in \Omega$, the rest of the proof follows from Lemma 3.1.1, in a similar way.

■

In order to study the limit behavior of the components Σ_ϵ, we prove the following Lemma.

Lemma 3.4.2 *Suppose (f_2), (M_0) and (Γ_0) holds. Let $U \subset \mathbb{R}\times C(\overline{\Omega})$ be a bounded open set containing $(0,0)$ and $(\lambda_\epsilon, u_\epsilon)$ be a solution of (Q_ϵ) such that $(\lambda_\epsilon, u_\epsilon) \in \Sigma_\epsilon \cap \Big((0,\infty)\times W_0^{1,p}(\Omega))\Big) \cap \partial U$. Then, for some positive constant $C(U)$, independent of ϵ, we have $\|\nabla u_\epsilon\|_p \leqslant C(U)$.*

Proof: Consider $(\lambda_\epsilon, u_\epsilon) \in \Sigma_\epsilon \cap \partial U$, then $\lambda_\epsilon \leqslant K, \|u_\epsilon\|_\infty \leqslant K$ for some positive constant K depending only on U. Taking u_ϵ as a test function in (Q_ϵ) and using (f_2) we get

$$\|\nabla u_\epsilon\|_p^p \leqslant C_1\lambda_\epsilon\Big(\int_\Omega (u_\epsilon+\epsilon)^{\delta+1}dx + 1\Big). \tag{3.34}$$

If $\delta \geqslant -1$, then by (3.34) the required boundedness follows trivially from the fact that $\lambda_\epsilon \leqslant K, \|u_\epsilon\|_\infty \leqslant K$. Now, suppose that $\delta \in \Big(-\frac{2p-1}{p-1}, -1\Big)$. As $\|u_\epsilon\|_\infty \leqslant K$, by the continuity of f we can find a $C_2 > 0$ independent of ϵ such that $f(u_\epsilon+\epsilon) \geqslant C_2(u_\epsilon+\epsilon)^\delta$. Thus, $u_\epsilon+\epsilon$ is a supersolution of

$$-\Delta_p u = \frac{\lambda_\epsilon C_2 u^\delta}{\max\limits_{\overline{\Omega}} a + \max\limits_{\overline{\Omega}} b\|\nabla u_\epsilon\|_p^{\gamma p}}. \tag{3.35}$$

On the other hand, take $\underline{u} = s\Phi_1^{\frac{p}{p-1-\delta}}$, where $s>0$ will be fixed later, then a simple calculation shows that

$$\begin{aligned} -\Delta_p \underline{u} &= \Big(\frac{sp}{p-1-\delta}\Big)^{p-1}\Phi_1^{\frac{\delta p}{p-1-\delta}}\Big[\frac{(-\delta-1)(p-1)}{p-1-\delta}|\nabla\Phi_1|^p + \lambda_1\Phi_1^p\Big] \\ &\leqslant C_3\Big(\frac{sp}{p-1-\delta}\Big)^{p-1}\Phi_1^{\frac{\delta p}{p-1-\delta}} = C_3 s^{p-1-\delta}\Big(\frac{p}{p-1-\delta}\Big)^{p-1}\underline{u}^\delta, \end{aligned}$$

where $C_3 = \max\limits_{\overline{\Omega}}\Big[\frac{(-\delta-1)(p-1)}{p-1-\delta}|\nabla\Phi_1|^p + \lambda_1\Phi_1^p\Big]$. Therefore, if we choose

$$s = C_4\Big(\frac{\lambda_\epsilon}{\max\limits_{\overline{\Omega}} a + \max\limits_{\overline{\Omega}} b\|\nabla u_\epsilon\|_p^{\gamma p}}\Big)^{\frac{1}{p-1-\delta}},$$

where $C_4 = \Big[\frac{C_2(p-1-\delta)^{p-1}}{C_3 p^{p-1}}\Big]^{\frac{1}{p-1-\delta}}$, then $\underline{u}$ is a subsolution of (3.35) and by the Theorem 1.2.2 we get

$$u_\epsilon + \epsilon \geqslant C_4\Big(\frac{\lambda_\epsilon}{\max\limits_{\overline{\Omega}} a + \max\limits_{\overline{\Omega}} b\|\nabla u_\epsilon\|_p^{\gamma p}}\Big)^{\frac{1}{p-1-\delta}}\Phi_1^{\frac{p}{p-1-\delta}}. \tag{3.36}$$

Now, coming back to (3.34) and using (3.36) together with $\delta \in \Big(-\frac{2p-1}{p-1}, -1\Big)$, we obtain

$$\|\nabla u_\epsilon\|_p^p \leqslant C_5\Big(1 + \|\nabla u_\epsilon\|_p^{-\frac{\gamma p(\delta+1)}{p-1-\delta}}\Big).$$

Since $\gamma < \frac{p-1-\delta}{-1-\delta}$, it follows from the last inequality that $\|\nabla u_\epsilon\|_p \leqslant C(U)$, where $C(U)$ is independent of ϵ.

■

In the light of above result, we prove the following Lemma, similar to Lemma 3.1.2. We highlight only the principal points in the proof.

Lemma 3.4.3 *Admit that f, M and γ satisfy (f_2), (M_0) and (Γ_0), respectively. Let $U \subset \mathbb{R} \times C(\overline{\Omega})$ be a bounded open set containing $(0,0)$ and a pair $(\lambda_\epsilon, u_\epsilon) \in \Sigma_\epsilon \cap \Big((0,\infty) \times (C(\overline{\Omega}) \cap W_0^{1,p}(\Omega))\Big) \cap \partial U$ be a solution of (Q_ϵ) satisfying $\lambda_\epsilon \leqslant K$, $\|u_\epsilon\|_\infty \leqslant K$. Then, there are positive constants $\mathcal{K}_1 = \mathcal{K}_1(K,U)$, $\mathcal{K}_2 = \mathcal{K}_2(k,K)$ and $\epsilon_0 > 0$ such that*

$$\lambda_\epsilon^{\frac{1}{p-1}}\mathcal{K}_1(K,U)\Phi_1 \leqslant u_\epsilon \leqslant k + \lambda_\epsilon^{\frac{1}{p-1}}\mathcal{K}_2(k,K)^{\frac{1}{p-1}}e_1 \quad \text{in } \Omega \tag{3.37}$$

for each $k \in (0,K]$ fixed and for all $0 < \epsilon < \epsilon_0$.

Proof: Define $\mathcal{K}_2(k,K) = \max\Big\{\frac{f(x,t)}{\underline{a}}\ x \in \overline{\Omega} :\ k \leqslant t \leqslant K+1\Big\}$, where $k \in (0,K]$. For this constant, a second inequality in (3.37) holds.

To obtain the first inequality, we must proceed as in the proof of the first inequality in Lemma 3.1.2. To get the constant $\mathcal{K}_1(K,U)$, in (3.4) we choose $A'_U := \max\{M(x,t) : x \in \overline{\Omega} \text{ and } 0 \leqslant t \leqslant C(U)^p\}$ instead of A_K, where $C(U)$ is given in the Lemma 3.4.2.

■

Now we are ready to prove the Theorem 0.0.10.

Theorem 0.0.10 *Assume that* (f_2)*,* (M_0) *and* (Γ_0) *hold. Then there exists an unbounded continuum* $\Sigma \subset \mathbb{R}^+ \times C(\overline{\Omega})$ *of solutions of* (Q_1) *which emanates from* $(0,0)$*. Furthermore, if* (f_∞) *holds then* $Proj_{\mathbb{R}^+}\Sigma = (0,\infty)$*. Moreover, if* $\gamma < 1$ *then* Σ *is unbounded vertically as well.*

Proof: Suppose that $\epsilon_n \to 0^+$ and denote by $\Sigma_n \subset \mathbb{R}^+ \times C(\overline{\Omega})$ the component associated with the problem (Q_{ϵ_n}). Let $U \subset \mathbb{R} \times C(\overline{\Omega})$ be an open neighborhood of $(0,0)$. As Σ_n is unbounded, there exists $(\lambda_n, u_n) \in \Sigma_n \cap \partial U$ and $K > 0$ such that $\lambda_n \leqslant K, \|u_n\|_\infty \leqslant K$. Moreover, from Lemma 3.4.2 we can assume, without loss of generality, that $\|\nabla u_n\|_p^p \leqslant K$ and from Lemma 3.4.3 that $\lambda_n \to \lambda > 0^+$, up to a subsequence. As a consequence, for $\delta' > 0$ small there exists $n_0 \in \mathbb{N}$ such that $0 < \lambda - \delta' < \lambda_n < \lambda + \delta'$ for all $n \geqslant n_0$, which implies again by the Lemma 3.4.3 that

$$(\lambda - \delta')^{1/(p-1)}\mathcal{K}_1(K,U)\Phi_1 \leqslant u_n \leqslant k + (\lambda+\delta')^{1/(p-1)}\mathcal{K}_2(k,K)^{1/(p-1)}e_1 \ \text{ in } \Omega, \quad (3.38)$$

for each $k \in (0, K]$.

From Lemma 3.4.2, $\{u_n\}$ being bounded in $W_0^{1,p}(\Omega)$, there exists $u = u_\lambda \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ weakly. Proceeding as in the proof of Theorem 0.0.7, we conclude by (3.38) that u satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx = \lambda \int_\Omega \frac{f(x,u)}{M(x, \|\nabla u\|_p^p)}\varphi dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega). \qquad (3.39)$$

Let us prove that (3.39) holds also for $\varphi \in W_0^{1,p}(\Omega)$. For this, take $\varphi \in W_0^{1,p}(\Omega)$. Then, by the density results, there exists a sequence $\{\varphi_n\} \in C_c^\infty(\Omega)$ such that $\varphi_n \to \varphi$ in $W_0^{1,p}(\Omega)$. Now, for each $\epsilon > 0$ the function $\phi = \sqrt{\epsilon^2 + |\varphi_n - \varphi_k|^2} + \epsilon \in C_c^1(\Omega)$ and hence taking ϕ as a test function in (3.39), we obtain

$$\lambda \int_\Omega \frac{f(x,u)}{M(x, \|\nabla u\|_p^p)}\Big(\sqrt{\epsilon^2 + |\varphi_n - \varphi_k|^2} - \epsilon\Big)dx \ = \ \int_\Omega |\nabla u|^{p-2}\nabla u \frac{|\varphi_n - \varphi_k|\nabla(\varphi_n - \varphi_k)}{\sqrt{\epsilon^2 + |\varphi_n - \varphi_k|^2}}dx$$

$$
\begin{aligned}
&\leqslant \int_\Omega |\nabla u|^{p-1}|\nabla(\varphi_n - \varphi_k)|dx \\
&\leqslant C\|\nabla u\|_p^{p-1}\|\nabla(\varphi_n - \varphi_k)\|_p.
\end{aligned}
$$

Applying the Fatou's Lemma, we obtain by the previous inequality

$$
\begin{aligned}
\lambda \int_\Omega \frac{f(x,u)}{M(x,\|\nabla u\|_p^p)}|\varphi_n - \varphi_k|dx &\leqslant \liminf_{\epsilon\to 0^+} \lambda \int_\Omega \frac{f(x,u)}{M(x,\|\nabla u\|_p^p)}\left(\sqrt{\epsilon^2 + |\varphi_n - \varphi_k|^2} - \epsilon\right)dx \\
&\leqslant C\|\nabla u\|_p^{p-1}\|\nabla(\varphi_n - \varphi_k)\|_p.
\end{aligned}
$$

Letting $n, k \to \infty$ in the previous inequality we obtain

$$
\lambda \int_\Omega \frac{f(x,u)}{M(x,\|\nabla u\|_p^p)}|\varphi_n - \varphi_k|dx \to 0.
$$

Thus, we have

$$
\int_\Omega \frac{f(x,u)}{M(x,\|\nabla u\|_p^p)}\varphi_n dx \longrightarrow \lambda \int_\Omega \frac{f(x,u)}{M(x,\|\nabla u\|_p^p)}\varphi dx \text{ as } n \to \infty. \tag{3.40}
$$

By the classical density arguments, we also have

$$
\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi_n dx \longrightarrow \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx \text{ as } n \to \infty. \tag{3.41}
$$

Therefore, joining (3.40) and (3.41) we obtain that $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ is solution of (Q) and satisfies (3.38).

Now, if we consider $\mathcal{F}$ as in the proof of Theorem 0.0.7, then in a similar way we can show that closed and bounded (in $\mathbb{R} \times C(\overline{\Omega})$) subsets of $\mathcal{F}$ are compacts and this ends the proof of existence of the unbounded *continuum* Σ.

The proof of $Proj_{\mathbb{R}^+}\Sigma = (0, \infty)$ if (f_∞) holds, is the same as done in the proof of Theorem 0.0.7.

Now, suppose that there exists a constant $C > 0$, independent of λ and u, such that $\|u\|_\infty \leqslant C$ whenever $(\lambda, u) \in \Sigma$. Then, let us take $(\lambda, u) \in \Sigma$ with $\lambda > 1$. So u satisfies

$$
-\Delta_p u \geqslant \frac{\lambda C_1}{\max\limits_{\overline{\Omega}} a(x) + \max\limits_{\overline{\Omega}} b(x)\|\nabla u\|_p^{p\gamma}} u^\delta.
$$

Besides this, for $\epsilon > 0$ small $\underline{u} = \Big(\frac{\epsilon\lambda}{\max_{\overline{\Omega}} a(x) + \max_{\overline{\Omega}} b(x)\|\nabla u\|_p^{p\gamma}}\Big)^{1/(p-1-\delta)}\Phi_1^{\frac{p}{p-1-\delta}}$ satisfies

$$-\Delta_p \underline{u} \leqslant \frac{\lambda C_1}{\max_{\overline{\Omega}} a(x) + \max_{\overline{\Omega}} b(x)\|\nabla u\|_p^{p\gamma}}\underline{u}^{\delta},$$

and so we get by Theorem 1.2.2 that $u \geqslant \underline{u}$. Taking u as a test function in (Q) and using $\lambda > 1$, $u \geqslant \underline{u}$ and $\|u\|_\infty \leqslant C$, we obtain that

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^p dx \leqslant C_1\lambda \quad \text{if } \delta \geqslant -1 \\ \displaystyle\int_\Omega |\nabla u|^p dx \leqslant C\lambda^{\frac{p}{p-1-\delta}}(\|\nabla u\|_p^{\frac{p(-\delta-1)\gamma}{p-1-\delta}} + 1) \quad \text{if } -\dfrac{2p-1}{p-1} < \delta < -1. \end{cases} \tag{3.42}$$

Without loss of generality, let us assume that $\|\nabla u\|_p > 1$, otherwise we would get

$$C \geqslant u \geqslant \underline{u} \geqslant \Big(\frac{\epsilon\lambda}{\max_{\overline{\Omega}} a(x) + \max_{\overline{\Omega}} b(x)}\Big)^{1/(p-1-\delta)}\Phi_1^{\frac{p}{p-1-\delta}} \text{ for all } \lambda > 0.$$

Then, coming back to (3.42) and using $\|\nabla u\|_p > 1$, we obtain for $-\frac{2p-1}{p-1} < \delta < -1$ that $\|\nabla u\|_p \leqslant C\lambda^{\frac{1}{p+(\delta+1)(\gamma-1)}}$. Thus, as $u \geqslant \underline{u}$ we have

$$u \geqslant C\Big(\frac{\lambda}{1 + \lambda^{\frac{p\gamma}{p+(\delta+1)(\gamma-1)}}}\Big)^{1/(p-1-\delta)}\Phi_1^{\frac{p}{p-1-\delta}}. \tag{3.43}$$

Also, when $\delta \geqslant -1$, by (3.42) we get

$$u \geqslant C\Big(\frac{\lambda}{1 + \lambda^{\gamma}}\Big)^{1/(p-1-\delta)}\Phi_1^{\frac{p}{p-1-\delta}}. \tag{3.44}$$

Then, from (3.43) and (3.44) with $\gamma < 1$, it follows that $\|u\|_\infty \to \infty$ as $\lambda \to \infty$, contradicting the fact that $\|u\|_\infty \leqslant C$. ■

CHAPTER 4

THREE SOLUTIONS TO A STRONGLY-SINGULAR QUASILINEAR KIRCHHOFF PROBLEM

4.1 Orlicz–Sobolev setting

In order to study the problem $(Q_{\lambda,\mu})$, let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz-Sobolev spaces, useful for what follows. For more information about this issue, we refer [1], [34], [37] and [49].

4.1.1 Orlicz spaces

Definition 4.1.1 *We say that* $\Phi : \mathbb{R} \to [0,\infty)$ *is an* N*-function (or Young function), if* $\Phi(t) = \displaystyle\int_0^{|t|} \phi(s)ds$ *where* $\phi : [0,\infty) \to [0,\infty)$ *has the following properties:*

i) $\phi(0) = 0$*;*

ii) $\phi(s) > 0$ *for* $s > 0$*;*

iii) ϕ *is right-continuous to any* $s \geqslant 0$*, that is,* $\displaystyle\lim_{t\to s^+} \phi(t) = \phi(s)$;

iv) ϕ *is nondecreasing in* $[0,\infty)$*;*

v) $\displaystyle\lim_{s\to\infty} \phi(s) = \infty$.

Example 4.1.2 *Examples of N-functions:*

- $\Phi_1(t) = |t|^p/p, \; p \in (1, \infty)$;
- $\Phi_2(t) = e^{t^2} - 1$;
- $\Phi_3(t) = (1 + |t|)ln(1 + |t|) - |t|$.

Definition 4.1.3 *(Complementary N-function) Let* Φ *be a N-function. Then*

$$\tilde{\Phi}(t) := \sup_{s \geqslant 0}\{st - \Phi(s)\}$$

is called the complementary N-function of Φ.

We list some useful properties of the Young functions below.

Proposition 4.1.4 *Let* Φ *be a N-function and* $\tilde{\Phi}$ *the complementary N-function of* Φ. *The following statements are true:*

i) $\Phi(t) < t\phi(t)$ *for all* $t > 0$;

ii) $\tilde{\Phi}(\phi(t)) \leqslant \Phi(2t)$;

iii) (Young inequality) $ts \leqslant \Phi(t) + \tilde{\Phi}(s)$, *for all* $t, s \in \mathbb{R}$.

Now, let us introduce the class of N-functions appropriate for the proposed study. Consider $a : (0, \infty) \to (0, \infty)$, with $a \in C^1(0, \infty)$, such that $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \begin{cases} a(|t|)t & \text{if } \; t \neq 0 \\ 0 \; \text{ if } \; t = 0 \end{cases}$$

is an increasing homeomorphism from $\mathbb{R}$ onto itself, with inverse denoted by $\phi^{-1} : \mathbb{R} \to \mathbb{R}$. From ϕ, we can define

$$\Phi(t) = \int_0^{|t|} \phi(s)ds.$$

In this case, the N-function represented by ϕ^{-1}, that is,

$$\tilde{\Phi}(t) = \int_0^{|t|} \phi^{-1}(s)ds,$$

is the Young functions complementary to Φ.

Throughout this chapter, we assume the following condition on Φ

(ϕ_1): $0 < a_- := \inf\limits_{t>0} \dfrac{t\phi'(t)}{\phi(t)} \leqslant \sup\limits_{t>0} \dfrac{t\phi'(t)}{\phi(t)} := a_+ < \infty.$

and denote $\phi_- = a_- + 1$ and $\phi_+ = a_+ + 1$.

The Orlicz class defined by the N-function Φ is the set

$$\mathcal{L}^{\Phi}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \ : \ \int_\Omega \Phi(|u(x)|)dx < \infty\Big\}$$

and the Orlicz space $L^{\Phi}(\Omega)$ is then defined as the linear hull of the set $\mathcal{L}^{\Phi}(\Omega)$. However, under the condition (ϕ_1), the Orlicz space $L^{\Phi}(\Omega)$ coincides with the Orlicz class $\mathcal{L}^{\Phi}(\Omega)$.

The space $L^{\Phi}(\Omega)$ endowed with the Luxemburg norm, defined by

$$\|u\|_\Phi := \inf\Big\{\alpha > 0 \ : \ \int_\Omega \Phi\Big(\frac{|u(x)|}{\alpha}\Big)dx \leqslant 1\Big\},$$

is a Banach space and since (ϕ_1) is satisfied, $L^{\Phi}(\Omega)$ is also reflexive and separable space. Moreover, if $u \in L^{\Phi}(\Omega)$ and $v \in L^{\tilde{\Phi}}(\Omega)$, then

$$\int_\Omega uvdx \leqslant 2\|u\|_\Phi \|v\|_{\tilde{\Phi}} \qquad \text{(Hölder inequality)}$$

The next proposition gives us an alternative way to verify convergence in $L^{\Phi}(\Omega)$.

Proposition 4.1.5 *Let Φ be a N-function satisfying (ϕ_1). Then, $u_n \to u$ in $L^{\Phi}(\Omega)$ if and only if*

$$\int_\Omega \Phi(|u_n(x) - u(x)|)dx \to 0.$$

Proposition 4.1.6 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, Φ a N-function satisfying*

(ϕ_1) and $(u_n) \subset L^\Phi(\Omega)$ such that $u_n \to u$ in $L^\Phi(\Omega)$. Then there exists a subsequence (u_{n_k}) and a function $h \in L^\Phi(\Omega)$ satisfying:

- *$u_{n_k}(x) \to u(x)$ a.e in Ω;*
- *$|u_{n_k}(x)| \leqslant h(x)$ a.e in Ω.*

The inclusion between Orlicz spaces are generalized in following way.

Definition 4.1.7 *Let Φ_1 and Φ_2 be N-functions. We say that Φ_1 dominates Φ_2, and write $\Phi_2 \prec \Phi_1$, if there exists positive constants α and t_0 such that $\Phi_2(t) \leqslant \Phi_1(\alpha t)$, for all $t \geqslant t_0$. We say that Φ_2 increases essentially more slowly than Φ_1 $(\Phi_2 << \Phi_1)$, if $\lim_{t\to\infty} \frac{\Phi_2(\alpha t)}{\Phi_1(t)} = 0$ for all $\alpha > 0$.*

Proposition 4.1.8 *Let $\Omega \subset \mathbb{R}^N$ a bounded domain, Φ_1 and Φ_2 be N-functions. Then $L^{\Phi_1}(\Omega) \hookrightarrow L^{\Phi_2}(\Omega)$ if, and only if, $\Phi_2 \prec \Phi_1$.*

4.1.2 Orlicz-Sobolev spaces

We denote by $W^{1,\Phi}(\Omega)$ the Orlicz-Sobolev space corresponding to Φ defined by

$$W^{1,\Phi}(\Omega) = \Big\{u \in L^\Phi(\Omega) \ : \ u_{x_i} \in L^\Phi(\Omega), \ i = 1, \cdots, N\Big\}.$$

This is a Banach spaces with respect to the norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \|\nabla u\|_\Phi$$

and again, since we are assuming (ϕ_1), the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is reflexive and separable.

Denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$, that is, $W_0^{1,\Phi}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\Phi}}$.

Proposition 4.1.9 *(Poincaré Inequality) Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Then, there exists a positive constant S such that*

$$\|u\|_\Phi \leqslant S\|\nabla u\|_\Phi, \qquad \forall u \in W_0^{1,\Phi}(\Omega).$$

By Proposition 4.1.9, we know that $\|u\|_{1,\Phi}$ and $\|\nabla u\|_\Phi$ are equivalent norms on $W_0^{1,\Phi}(\Omega)$. We will use $\|\nabla u\|_\Phi$ to replace $\|u\|_{1,\Phi}$ in the following discussions.

Now, let us introduce the Orlicz-Sobolev conjugate Φ_* of Φ, whose inverse is given by

$$\Phi_*^{-1}(t) := \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}ds, \quad \text{for } t > 0,$$

where we are supposing that

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}ds < \infty \quad \text{and} \quad \int_1^\infty \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}}ds = +\infty. \tag{4.1}$$

In the case $\Phi(t) = |t|/p$, (4.1) holds if and only if $N > p$.

Proposition 4.1.10 *Let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain and Φ a N-function satisfying (4.1) and (ϕ_1). Then*

- $W^{1,\Phi}(\Omega) \overset{cont}{\hookrightarrow} L^{\Phi_*}(\Omega)$;
- $W^{1,\Phi}(\Omega) \overset{comp}{\hookrightarrow} L^{\Psi}(\Omega)$ *whenever* $\Psi << \Phi_*$;
- $W^{1,\phi_+}(\Omega) \hookrightarrow W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\phi_-}(\Omega)$.

4.1.3 Consequences of condition (ϕ_1)

We reserve this section to present some essential consequences of the hypothesis (ϕ_1). Throughout this section, let us denote $\phi_- = a_- + 1$ and $\phi_+ = a_+ + 1$.

Lemma 4.1.11 *Suppose that Φ is a N-function satisfying (ϕ_1), with complementary N-function given by $\tilde{\Phi}$. Then*

i) $\min\{t^{\phi_-}, t^{\phi_+}\}\Phi(s) \leqslant \Phi(ts) \leqslant \max\{t^{\phi_-}, t^{\phi_+}\}\Phi(s)$, $\forall s, t > 0$;

ii) $\min\{\|u\|_\Phi^{\phi_-}, \|u\|_\Phi^{\phi_+}\} \leqslant \int_\Omega \Phi(u)dx \leqslant \max\{\|u\|_\Phi^{\phi_-}, \|u\|_\Phi^{\phi_+}\}$, $\forall u \in L^\Phi(\Omega)$;

iii) $\min\{\|\nabla u\|_\Phi^{\phi_-}, \|\nabla u\|_\Phi^{\phi_+}\} \leqslant \int_\Omega \Phi(|\nabla u|)dx \leqslant \max\{\|\nabla u\|_\Phi^{\phi_-}, \|\nabla u\|_\Phi^{\phi_+}\}$, $\forall u \in W_0^{1,\Phi}(\Omega)$;

iv) $\min\{t^{\frac{\phi_-}{\phi_- - 1}}, t^{\frac{\phi_+}{\phi_+ - 1}}\}\tilde{\Phi}(s) \leqslant \tilde{\Phi}(ts) \leqslant \max\{t^{\frac{\phi_-}{\phi_- - 1}}, t^{\frac{\phi_+}{\phi_+ - 1}}\}\tilde{\Phi}(s)$, $\forall s, t > 0$;

$v)$ $\min\{\|u\|_{\tilde{\Phi}}^{\frac{\phi_-}{\phi_- -1}}, \|u\|_{\tilde{\Phi}}^{\frac{\phi_+}{\phi_+ -1}}\} \leqslant \int_\Omega \tilde{\Phi}(u)dx \leqslant \max\{\|u\|_{\tilde{\Phi}}^{\frac{\phi_-}{\phi_- -1}}, \|u\|_{\tilde{\Phi}}^{\frac{\phi_+}{\phi_+ -1}}\}$, $\forall u \in L^{\tilde{\Phi}}(\Omega)$.

As a consequence of item$-i)$ of the Lemma above, we have the following result.

Lemma 4.1.12 *Let Φ be a N-function satisfying (ϕ_1). Then, there exists $C > 0$ such that*

$$\Phi(s+t) \leqslant C\Big(\Phi(s) + \Phi(t)\Big), \quad \textit{for all } s,t > 0.$$

Lemma 4.1.13 *If Φ is a N-function satisfying (ϕ_1), then:*

$i)$ $a_- - 1 = \inf\limits_{t>0} \dfrac{ta'(t)}{a(t)} \leqslant \sup\limits_{t>0} \dfrac{ta'(t)}{a(t)} = a_+ - 1 < \infty;$

$ii)$ $\min\{t^{a_- -1}, t^{a_+ -1}\}a(s) \leqslant a(st) \leqslant \max\{t^{a_- -1}, t^{a_+ -1}\}a(s)$, *for all* $s,t > 0$.

It is well known that the Sobolev spaces $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ are uniformly convex, so

$$\text{“if } u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } \int_\Omega |\nabla u_n|^p dx \to \int_\Omega |\nabla u|^p dx, \text{ then } u_n \to u \text{ in } W_0^{1,p}(\Omega).\text{”} \tag{4.2}$$

Next, we show that if Φ satisfies (ϕ_1), then property (4.2) remains valid. Before presenting this result, we need to introduce the following concepts.

Definition 4.1.14 *A N-function Φ is said to be uniformly convex, if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|s-t| \leqslant \epsilon \max\{s,t\} \quad \textit{or} \quad \Phi\Big(\Big|\frac{s+t}{2}\Big|\Big) \leqslant (1-\delta)\frac{\Phi(s)+\Phi(t)}{2},$$

for all $s,t \geqslant 0$.

An alternative way to verify that an N-function is uniformly convex, is given by the following proposition, which is proved in Proposition 6, page 284 in [49].

Proposition 4.1.15 *Let Φ be a N-function. Then Φ is uniformly convex if, and only if, for each $\epsilon > 0$, there exists constants $K_\epsilon > 1$ and $s(\epsilon) > 0$ such that*

$$\Phi'((1+\epsilon)s) \geqslant K_\epsilon \Phi'(xs), \quad \textit{for all } s \geqslant s(\epsilon).$$

Lemma 4.1.16 *If Φ is a N-function satisfying (ϕ_1), then Φ is uniformly convex.*

Proof: Given $\epsilon > 0$, by Lemma 4.1.13 we have

$$\frac{\Phi'((1+\epsilon)s)}{\Phi'(s)} = \frac{\phi((1+\epsilon)s)}{\phi(s)} = \frac{a((1+\epsilon)s)(1+\epsilon)s}{a(s)s} \geqslant (1+\epsilon)^{a_-} > 1, \quad \text{for all } s > 0,$$

so the result follows directly from the above proposition. ∎

Proposition 4.1.17 *Let Φ be a N-function satisfying (ϕ_1). If $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ and $\int_\Omega \Phi(|\nabla u_n|)dx \to \int_\Omega \Phi(|\nabla u|)dx$, then $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$.*

Proof: By using the hypothesis (ϕ_1), we obtain from the lemma above that Φ is uniformly convex. Thus, the result follows from Theorem 2.4.11 and Lemma 2.4.17 in [25]. ∎

Finally, let us introduce the functional $\mathcal{P} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{P}(u) = \int_\Omega \Phi(|\nabla u|)dx. \tag{4.3}$$

The next Lemma lists some properties of $\mathcal{P}$.

Lemma 4.1.18 *Suppose Φ satisfies (ϕ_0) and (ϕ_1), then the following statements are true:*

i) $\mathcal{P} \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *and*

$$\langle \mathcal{P}'(u), \varphi \rangle = \int_\Omega a(|\nabla u|)\nabla u \nabla\varphi dx \quad \text{for all } \varphi \in W_0^{1,\Phi}(\Omega);$$

ii) $\mathcal{P}$ *is sequentially weakly lower semicontinuous, that is, if* $u_n \rightharpoonup u$ *in* $W_0^{1,\Phi}(\Omega)$ *then* $\liminf_{n\to\infty} \mathcal{P}(u_n) \geqslant \mathcal{P}(u)$;

iii) $\mathcal{P}'$ *is strictly monotone, i.e,*

$$\langle \mathcal{P}'(u) - \mathcal{P}'(v), u - v \rangle > 0, \quad \forall\ u, v \in W_0^{1,\Phi}(\Omega),\ u \neq v;$$

iv) $\mathcal{P}'$ *is of type* (S_+)*, that is,*

$$\text{“ if } u_n \rightharpoonup u \text{ and } \lim_{n\to\infty}\sup\langle \mathcal{P}'(u_n), u_n - u\rangle \leqslant 0, \text{ then } u_n \to u \text{ in } W_0^{1,\Phi}(\Omega)\text{”}.$$

4.2 Preliminary results in the setting of non-smooth analysis in Orlicz Sobolev spaces

In this section, we present some preliminary results which will assist us in the proof of the main result of this chapter.

We start by presenting some concepts and facts of non-smooth analysis that will be important for what follows. For more information on this subject, we request the reader to refer [55].

Let $W_0^{1,\Phi}(\Omega)$ be the Orlicz-Sobolev space associated to Φ and $\Psi_2 : W_0^{1,\Phi}(\Omega) \to (-\infty, \infty]$ a convex, lower semicontinuous and proper ($\Psi_2 \not\equiv +\infty$) functional. The set $Dom(\Psi_2) = \{u \in W_0^{1,\Phi}(\Omega) \ : \ \Psi_2(u) < \infty\}$ is called the effective domain of Ψ_2.

Definition 4.2.1 *Consider* $I = \Psi_1 + \Psi_2$ *with* $\Psi_1 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *and* $\Psi_2 : W_0^{1,\Phi}(\Omega) \to (-\infty, \infty]$ *a convex, lower semicontinuous and proper functional. A point* $u \in Dom(\Psi_2)$ *is said to be a critical point of* I *if*

$$\langle \Psi_1'(u), v - u\rangle + \Psi_2(v) - \Psi_2(u) \geqslant 0, \ \forall v \in W_0^{1,\Phi}(\Omega).$$

In this context, the (PS) condition is understood in the following sense.

Definition 4.2.2 *We say that* I *satisfies the Palais-Smale condition if the following holds:*

“ If (u_n) *is a sequence such that* $I(u_n) \to c \in \mathbb{R}$ *and*

$$\langle \Psi_1'(u_n), v - u_n\rangle + \Psi_2(v) - \Psi_2(u_n) \geqslant -\epsilon_n \|\nabla(v - u_n)\|_\Phi, \ \forall v \in W_0^{1,\Phi}(\Omega)$$

where $\epsilon_n \to 0^+$*, then* (u_n) *possesses a convergent subsequence. ”*

The proof of Theorem 0.0.12 stems from the following result due to Szulkin.

Theorem A *Suppose that* $I : W_0^{1,\Phi}(\Omega) \to (-\infty, \infty]$ *is defined by* $I = \Psi_1 + \Psi_2$, *where* $\Psi_1 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *and* $\Psi_2 : W_0^{1,\Phi}(\Omega) \to (-\infty, \infty]$ *is a convex, lower semicontinuous and proper functional. If* I *satisfies* (PS) *and admits two local minima, then it has at least three critical points.*

Proof: See Corollary 3.3 in [55]. ■

In order to apply the above theorem to get our result, let us first construct the appropriate functional setting.

Thus, consider $\hat{M}(t) := \displaystyle\int_0^t M(s)ds$ and $J_1(u) := \hat{M}(\mathcal{P}(u))$, where $\mathcal{P}$ was defined in (4.3). In addition, let $J_2 : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ be given by $J_2(u) = \displaystyle\int_\Omega F(x,u)dx$, where $F(x,t) = \displaystyle\int_0^t f(x,s)ds$.

Before announcing the next lemma, let us recall the hypothesis (f_1'), which is given by

(f_1'): there exists an odd increasing homeomorphism h from $\mathbb{R}$ to $\mathbb{R}$ and nonnegative constants a_1 and a_2 such that

$$f(x,t) \leqslant a_1 + a_2 h(|t|), \quad \forall t \in \mathbb{R} \text{ and } \forall x \in \overline{\Omega}$$

and $H << \Phi_*$, where $H(t) := \displaystyle\int_0^{|t|} h(s)ds$ satisfies

$$1 < h_- := \inf_{t>0} \frac{th(t)}{H(t)} \leqslant \sup_{t>0} \frac{th(t)}{H(t)} := h_+ < \infty. \tag{8}$$

Lemma 4.2.3 *Suppose* (ϕ_0), (ϕ_1) *and* (f_1') *holds. Then:*

i) $J_1 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *and*

$$\langle J_1'(u), \varphi \rangle = M(\mathcal{P}(u)) \int_\Omega a(|\nabla u|)\nabla u \nabla\varphi dx \quad \forall \varphi \in W_0^{1,\Phi}(\Omega);$$

ii) $J_2 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *and*

$$\langle J_2'(u), \varphi \rangle = \int_\Omega f(x,u)\varphi dx \quad \forall \varphi \in W_0^{1,\Phi}(\Omega);$$

iii) J_1' *is of type* (S_+)*, that is,*

$$\text{“}\textit{if } u_n \rightharpoonup u \textit{ and } \lim_{n\to\infty} \sup \langle J_1'(u_n), u_n - u \rangle \leqslant 0, \textit{ then } u_n \to u \textit{ in } W_0^{1,\Phi}(\Omega)\text{”};$$

iv) *If* $u_n \rightharpoonup u$ *in* $W_0^{1,\Phi}(\Omega)$*, then* $\langle J_2'(u_n), u_n - u \rangle \to 0$*;*

iv) *If* $u_n \rightharpoonup u$ *in* $W_0^{1,\Phi}(\Omega)$*, then* $J_2(u_n) \to J_2(u)$*;*

v) J_1 *is sequentially weakly lower semicontinuous in* $W_0^{1,\Phi}(\Omega)$*.*

Proof:

i) This assertion follows directly from Lemma 4.1.18$-i$) and chain rule.

ii) Noting that

$$\frac{1}{t}[F(x, u + t\varphi) - F(x,u)] \to f(x,u)\varphi \text{ as } t \to 0 \text{ for all } x \in \Omega$$

and by (f_1')

$$\begin{aligned}\left|\frac{F(x,u+t\varphi) - F(x,u)}{t}\right| &\leqslant \int_0^1 f(x, u + st\varphi)|\varphi| ds \leqslant a_1|\varphi| + a_2 h(|u| + |\varphi|)|\varphi| \\ &\leqslant \ a_1|\varphi| + a_2 h_+ H(|u| + |\varphi|) \in L^1(\Omega)\end{aligned}$$

holds, the result follows from dominated convergence.

iii) If $u_n \rightharpoonup u$, then $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$ and, passing to a subsequence, if necessary, we may assume that $\mathcal{P}(u_n) \to t_0$ for some $t_0 \geqslant 0$. If $t_0 = 0$, then $u_n \to 0$ and, except if $u = 0$ a.e in Ω, it leads us to a contradiction because $u_n \to u$ a.e in Ω. On the other hand, if $t_0 > 0$, then it follows from the continuity of the function M that $M\Big(\mathcal{P}(u_n)\Big) \to M(t_0) > 0$, whence for our assumptions we get

$$\lim_{n\to\infty} \sup\langle \mathcal{P}'(u_n), u_n - u\rangle \leqslant 0,$$

which by Lemma 4.1.18$-iv)$ implies in $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. Repeating the same argument, we conclude every subsequence of (u_n) admits a subsequence converging to u. Therefore $u_n \to u$, as desired.

iv) Suppose that $u_n \rightharpoonup u$. Then, $u_n \to u$ a.e in Ω and, since (f_1') is satisfied, we have

$$\begin{aligned} |f(x,u_n)(u_n - u)| &\leqslant a_1|u_n - u| + a_2 h(|u_n - u| + |u|)|u_n - u| \\ &\leqslant C(|u_n - u| + H(|u_n - u| + |u|)). \end{aligned}$$

Hence, by using the compact embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^H(\Omega)$, the last inequality and Theorem **A.1.3**, we get $|f(x,u_n)(u_n - u)| \leqslant g(x)$, for some $g \in L^1(\Omega)$. Thus,once again by dominated convergence we get the result.

v) As in the previous item, by using (f_1') and dominated convergence the result follows.

vi) If $u_n \rightharpoonup u$, then by Lemma 4.1.18$-ii)$ one has $\lim_{n\to\infty}\inf \mathcal{P}(u_n) \geqslant \mathcal{P}(u)$. Moreover, as $\hat{M}$ is a continuous and increasing function in $[0,\infty)$, then

$$\lim_{n\to\infty}\inf \hat{M}\Big(\mathcal{P}(u_n)\Big) \geqslant \hat{M}\Big(\lim_{n\to\infty}\inf \mathcal{P}(u_n)\Big) \geqslant \hat{M}\Big(\mathcal{P}(u)\Big).$$

■

Defining $\Psi_1 : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ by

$$\Psi_1(u) = J_1(u) - \lambda J_2(u),$$

as a direct consequence of the previous Lemma, we can derive the following properties to Ψ_1.

Lemma 4.2.4 *If* (ϕ_0), (ϕ_1) *and* (f_1') *hold. Then the functional* $\Psi_1 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *is sequentially weakly lower semicontinuous and* Ψ_1' *is of type* (S_+).

Proof: From items $i) - ii)$ in the above lemma, we conclude $\Psi_1 \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$. By $iii) - iv)$, one has Ψ_1' is of type (S_+). The last part is obtained by using $v) - vi)$. ■

Next, let us assume

$$(b) \ : \begin{cases} b \in L^{(\frac{\phi_-^*}{1-\delta})'}(\Omega) \ \text{ if } 0 < \delta < 1; \\ b \in L^q(\Omega) \text{ for some } q > 1 \ \text{ if } \delta = 1; \\ b \in L^1(\Omega) \ \text{ if } \delta > 1. \end{cases}$$

and define $G : \Omega \times \mathbb{R} \to (-\infty, \infty]$ by

$$\text{a) } G(x,t) = \begin{cases} \frac{-b(x)t^{1-\delta}}{1-\delta} \ \text{ if } x \in \Omega \text{ and } t \geqslant 0 \\ +\infty \ \text{ if } x \in \Omega \text{ and } t < 0 \end{cases} \qquad \text{if } 0 < \delta < 1;$$

$$\text{b) } G(x,t) = \begin{cases} -b(x)\ln(t) \ \text{ if } x \in \Omega \text{ and } t > 0 \\ +\infty \ \text{ if } x \in \Omega \text{ and } t \leqslant 0 \end{cases} \qquad \text{if } \delta = 1;$$

$$\text{c) } G(x,t) = \begin{cases} \frac{b(x)t^{1-\delta}}{\delta-1} \ \text{ if } x \in \Omega \text{ and } t > 0 \\ +\infty \ \text{ if } x \in \Omega \text{ and } t \leqslant 0 \end{cases} \qquad \text{if } \delta > 1.$$

Note that, when $0 < \delta < 1$, we have $bu^{1-\delta} \in L^1(\Omega)$ for all $0 \leqslant u$ in $W_0^{1,\Phi}(\Omega)$, because in this situation we are supposing $b \in L^{(\frac{\phi_-^*}{1-\delta})'}(\Omega)$. In the case $\delta > 1$, one has $G(x,u) \geqslant 0$ in Ω, $\forall u \geqslant 0$ in $W_0^{1,\Phi}(\Omega)$. Finally, when $\delta = 1$, let us decompose G as $G(x,u) = -b(x)\ln(u).\chi_{[0<u<1]} - b(x)\ln(u).\chi_{[u \geqslant 1]}$ and fix $s \in \left(0, \frac{\phi_-^*}{q'}\right)$, where q' is the conjugate of q. Since $\ln(t)/t^s \to 0$ as $t \to +\infty$, we can find $C > 0$ such that $\ln(t) \leqslant Ct^s$ for all $t \geqslant 1$. By using this fact, we obtain

$$G(x,u) \geqslant -b(x)\ln(u).\chi_{[u \geqslant 1]} \geqslant -Cb(x)u^s\chi_{[u \geqslant 1]},$$

in which $bu^s \in L^1(\Omega)$ by our choice of s.

Combining all the information, we can conclude that in any case $\int_\Omega G(x,u)dx \neq -\infty$ for all $u > 0$ in $W_0^{1,\Phi}(\Omega)$.

Now, we can define the functional $\Psi_2 : W_0^{1,\Phi}(\Omega) \to (-\infty, \infty]$ given by

$$\Psi_2(u) = \begin{cases} \displaystyle\int_\Omega G(x,u)dx & \text{if } G(\cdot, u(\cdot)) \in L^1(\Omega) \\ +\infty & \text{if } G(\cdot, u(\cdot)) \notin L^1(\Omega). \end{cases}$$

Regarding Ψ_2, we have the following result.

Lemma 4.2.5 *Assume that* (b) *holds. If either* $\delta > 1$ *and*

$$(S): \qquad -\Delta_\Phi u = b(x)u^{-\delta} \ \text{ in } \Omega, \ u > 0 \ \text{ in } \Omega \ \text{ and } \ u = 0 \ \text{on } \partial\Omega$$

admits solution in $W_0^{1,\Phi}(\Omega)$ *or* $\delta \leqslant 1$, *then* Ψ_2 *is proper, convex and sequentially weakly lower semicontinuous.*

Proof: First, let us prove that Ψ_2 is proper, that is, the effective domain $Dom(\Psi_2)$ of Ψ_2 is non-empty. In fact, when $0 < \delta < 1$, every non-negative function $u \in W_0^{1,\Phi}(\Omega)$ belongs to $Dom(\Psi_2)$. If $\delta > 1$ and $u_0 \in W_0^{1,\Phi}(\Omega)$ is a solution of (S), then $u_0 \in Dom(\Psi_2)$, which proves that $Dom(\Psi_2) \neq \varnothing$ in this cases.

For the case $\delta = 1$ define the problem

$$-\Delta_{\phi_+} u = \frac{b(x)}{u^{s+1}} \ \text{ in } \Omega, \ \ u > 0 \ \text{ in } \Omega \ \text{ and } \ u|_{\partial\Omega} = 0, \tag{4.4}$$

where $0 < s < \frac{\phi_+(q-1)}{q(\phi_+-1)}$, in which q is given by (b).

From Corollary 0.0.14, which will be proved later, for chosen s the problem (4.4) admits a solution $0 < u_0 \in W_0^{1,\phi_+}(\Omega) \hookrightarrow W_0^{1,\Phi}(\Omega)$ (see Proposition 4.1.10). Now, let us prove that $u_0 \in Dom(\Psi_2)$. Indeed, by using that $\ln(t) \leqslant C_1 t^s$ for all $t \geqslant 1$ and for some positive constants C_1, one has

$$\begin{aligned} \int_\Omega G(x,u_0)dx &= \int_\Omega G(x,u_0)\chi_{[0<u_0<1]}dx + \int_\Omega G(x,u_0)\chi_{[u_0\geqslant 1]}dx \\ &= \int_\Omega b \ln\left(\frac{1}{u_0}\right)\chi_{[0<u_0<1]}dx - \int_\Omega b\ln(u_0)\chi_{[u_0\geqslant 1]}dx \end{aligned}$$

$$\leqslant \ C_1 \int_\Omega \frac{b}{u_0^s}\chi_{[0<u_0<1]}dx \leqslant C_1 \int_\Omega |\nabla u_0|^{\phi_+}dx < \infty.$$

Therefore, $u_0 \in Dom(\Psi_2)$.

Next, let us verify that Ψ_2 is convex. For this, it suffices note that for every $x \in \Omega$, $G(x,\cdot) \in C^1(0,\infty)$ and $G'(x,t) = -b(x)t^{-\delta}$, which is increasing in $t > 0$.

Finally, to show that Ψ_2 is sequentially weakly lower semicontinuous, let us take $u_n \rightharpoonup u$. When $0 < \delta < 1$, by using the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\phi_-^*}(\Omega)$ and proceeding exactly as in the proof of Lemma 1.2.3, we get the result. For the case $\delta > 1$, the claim follows directly from Fatou's Lemma.

In the last case, when $\delta = 1$, we observe that by Fatou's Lemma

$$-\int_\Omega b(x)\ln(u)\chi_{[u\leqslant 1]}dx \leqslant \liminf_{n\to\infty} -\int_\Omega b(x)\ln(u_n)\chi_{[u_n\leqslant 1]}dx.$$

On the other hand, by fixing $s \in \left(0, \frac{\phi_-^*}{q'}\right)$, we have $\ln(t) \leqslant Ct^s$ for all $t \geqslant 1$ and for some $C > 0$. So by using this information together with the compact embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^\tau(\Omega)$ for all $1 < \tau < \phi_-^*$, we get by dominated convergence

$$-\int_\Omega b(x)\ln(u_n)\chi_{[u_n\geqslant 1]}dx \to \int_\Omega b(x)\ln(u)\chi_{[u\geqslant 1]}dx \quad \text{as } n \to \infty.$$

Therefore, from this two information, the result follows also for the case $\delta = 1$. ■

Now, we can define the appropriate functional to apply the Theorem A. Let $I : W_0^{1,\Phi}(\Omega) \to (-\infty,\infty]$ be the functional given by

$$I(u) = \Psi_1(u) + \mu\Psi_2(u).$$

By the Lemma 4.2.4 we have $\Psi_1 \in C^1(W_0^{1,\Phi}(\Omega),\mathbb{R})$. Moreover, by Lemma 4.2.5, we know that Ψ_2 is proper, convex and lower semicontinuous.

Next, we will prove that I satisfies (PS). Let us recall the following hypotheses before stating the next lemma:

(ϕ_2): $\phi_+ < \phi_-^* := \inf_{t>0} \frac{t\phi^*(t)}{\Phi^*(t)}$;

(M): $M(t) \geqslant m_0 t^{\alpha-1}$, for all $t \geqslant 0$, with $1 \leqslant \alpha < \frac{\phi_-^*}{\phi_+}$;

(f_3'): $\lim_{t\to\infty} \frac{\sup_{\overline{\Omega}} F(x,t)}{t^{\alpha\phi_-}} = 0.$

Lemma 4.2.6 *Suppose* $(\phi_0) - (\phi_2)$, (M), (b), (f_1'), (f_3') *hold. In addition, when* $\delta > 1$, *assume* (S) *admits a* $W_0^{1,\Phi}(\Omega)-$*solution as well. Then* I *satisfies the* (PS) *condition.*

Proof: Let $(u_n) \subset W_0^{1,\Phi}(\Omega)$ and $(\epsilon_n) \subset (0,\infty)$ be sequences such that $I(u_n) \to c \in \mathbb{R}$, $\epsilon_n \to 0$ and

$$\langle \Psi_1'(u_n), \varphi - u_n\rangle + \mu\Big(\Psi_2(\varphi) - \Psi_2(u_n)\Big) \geqslant -\epsilon_n \|\nabla(u_n - \varphi)\|_\Phi, \text{ for all } \varphi \in W_0^{1,\Phi}(\Omega), n \geqslant 1. \tag{4.5}$$

First, let us show that (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. For this, is enough prove that I is coercive. In fact, by (M) and Lemma 4.1.11, we obtain

$$\hat{M}\Big(\mathcal{P}(u)\Big) \geqslant \frac{m_0}{\alpha}\|\nabla u\|_\Phi^{\alpha\phi_-}, \quad \text{for all } u \in W_0^{1,\Phi}(\Omega) \text{ with } \|\nabla u\|_\Phi \geqslant 1.$$

Moreover, it follows from (f_1') and (f_3') that for given $\epsilon > 0$ small enough, there exists $C_1 > 0$ such that $F(x,t) \leqslant C_1 + \epsilon|t|^{\alpha\phi_-}$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

Therefore, by the above informations and the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\alpha\phi_-}(\Omega)$, which follows from the hypothesis (ϕ_2), we conclude

$$\begin{aligned} \Psi_1(u) &\geqslant \frac{m_0}{\alpha}\|\nabla u\|_\Phi^{\alpha\phi_-} - \lambda\Big(C_1|\Omega| + \epsilon\int_\Omega |u|^{\alpha\phi_-} ds\Big) \\ &\geqslant \frac{m_0}{\alpha}\|\nabla u\|_\Phi^{\alpha\phi_-} - \lambda\Big(C_1|\Omega| + \epsilon C_2\|\nabla u\|_\Phi^{\alpha\phi_-}\Big), \end{aligned}$$

whence, by taking $\epsilon > 0$ small enough, the previous inequality leads us to

$$\Psi_1(u) \geqslant C_3\Big(\|\nabla u\|_\Phi^{\alpha\phi_-} - 1\Big) \tag{4.6}$$

for some $C_3 > 0$.

Besides this, it follows from the inclusion $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\phi_-^*}(\Omega)$ that

$\Psi_2(u) \geqslant -C_4\|b\|_{(\frac{\phi_-^*}{1-\delta})'}\|\nabla u\|_\Phi^{1-\delta}$, for $0 < \delta < 1$. For $\delta > 1$ we have $\Psi_2(u) \geqslant 0$ and for $\delta = 1$, by taking $s \in \Big(0, \min\{\frac{\phi_-^*}{q'}, \alpha\phi_-\}\Big)$, we obtain

$$\Psi_2(u) \geqslant -\int_\Omega b(x)\ln(u)\chi_{[u\geqslant 1]}dx \geqslant -C_5\|b\|_q\|u\|_{sq'}^s \geqslant -C_5\|b\|_q\|\nabla u\|_\Phi^s$$

for some cumulative constant $C_5 > 0$, where the last inequality follows from the fact that $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{sq'}(\Omega)$.

Considering all of these information together with (4.6), we get

$$I(u) \geqslant \begin{cases} C\Big(\|\nabla u\|_\Phi^{\alpha\phi_-} - \|b\|_{(\frac{\phi_-^*}{1-\delta})'}\|\nabla u\|_\Phi^{1-\delta} - 1\Big) & \text{if } 0 < \delta < 1 \\ C\Big(\|\nabla u\|_\Phi^{\alpha\phi_-} - \|b\|_q\|\nabla u\|_\Phi^s\Big) & \text{if } \delta = 1 \\ C\|\nabla u\|_\Phi^{\alpha\phi_-} & \text{if } \delta > 1, \end{cases}$$

whence $I(u) \to \infty$ as $\|\nabla u\|_\Phi \to \infty$, that is, I is coercive.

Since I is coercive and $I(u_n) \to c$, we conclude (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. As a consequence, passing to a subsequence, if necessary, we may assume that $u_n \rightharpoonup u$. By Lemma 4.2.4 and 4.2.5, one gets I sequentially weakly lower semicontinuous, so $I(u) \leqslant \liminf_{n\to\infty} I(u_n) = c < \infty$. Therefore, $\Psi_2(u) < \infty$ and by taking $\varphi = u$ in (4.5) we obtain

$$\langle \Psi_1'(u_n), u_n - u\rangle \leqslant \mu\Big(\Psi_2(u) - \Psi_2(u_n)\Big) + \epsilon_n\|\nabla(u_n - u)\|_\Phi \quad \text{for } n \in \mathbb{N}.$$

Thus, once again by using the fact that Ψ_2 is a lower semicontinuous functional and last inequality, we get $\limsup_{n\to\infty}\langle \Psi_1'(u_n), u_n - u\rangle \leqslant 0$, which by Lemma 4.2.4 implies $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. This concludes the Lemma. ■

Henceforth, our aim is to show that I has two local minima, as required by Theorem A. The next proposition allows us to accomplish this task.

Proposition 4.2.7 *Assume $(\phi_0)-(\phi_2)$, (M), (f_1') and (f_3'). Then any strict local minimum of the functional $\Psi_1 = J_1 - \lambda J_2$ in the strong topology of $W_0^{1,\Phi}(\Omega)$ is so in the weak topology.*

Proof: We just need to verify that, under these assumptions, the conditions of Theorem **A.1.11** in Appendix are met.

First, note that $W_0^{1,\Phi}(\Omega)$ is reflexive and separable by (ϕ_1). Besides this, it follows from Lemma 4.2.3 that J_1 and J_2 are sequentially weakly lower semicontinuous and by (4.6) the functional Ψ_1 is also coercive. Therefore, to conclude the proof, we need only to check that $J_1 \in \mathcal{W}_{W_0^{1,\Phi}}$, that is,

" if $u_n \rightharpoonup u$ and $\liminf_{n\to\infty} J_1(u_n) \leqslant J_1(u)$, then (u_n) has a subsequence converging strongly to u."

Suppose $u_n \rightharpoonup u$ and $\liminf_{n\to\infty} J_1(u_n) \leqslant J_1(u)$. Since the functional J_1 is sequentially weakly lower semicontinuous, there exists a subsequence of (u_n), still denoted by (u_n) such that $\lim_{n\to\infty} J_1(u_n) = J_1(u)$. As $t \mapsto \hat{M}(t)$ is continuous and strictly increasing in $t \geqslant 0$, the previous limit give us $\lim_{n\to\infty} \mathcal{P}(u_n) = \mathcal{P}(u)$. Therefore, as $\lim_{n\to\infty} \mathcal{P}(u_n) = \mathcal{P}(u)$ and $u_n \rightharpoonup u$, it follows from Proposition 4.1.17 that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. ■

Let us show that I admits two local minima in $W_0^{1,\Phi}(\Omega)$ for suitable values of λ and μ. Before, let us recall that

$$(f_2'):\ \lim_{t\to 0^+} \frac{\sup_{\overline{\Omega}} F(x,t)}{t^{\alpha\phi_+}} = 0.$$

Lemma 4.2.8 *Suppose $(\phi_0)-(\phi_2)$, (b), (M) and $(f_1')-(f_3')$ hold. In addition, when $\delta > 1$, assume (S) admits a $W_0^{1,\Phi}(\Omega)$-solution. Then I has two local minima.*

Proof: Fix $\lambda > \lambda^*$. Since Ψ_1 is lower semicontinuous and coercive (see Lemma 4.2.4 and (4.6)), there exists $u_0 \in W_0^{1,\Phi}(\Omega)$ a global minimum of Ψ_1 in $W_0^{1,\Phi}(\Omega)$. Moreover, as $\lambda > \lambda^*$, by using the definition of λ^*, we obtain $\Psi_1(u_0) = J_1(u_0) - \lambda J_2(u_0) < 0$.

Let us denote by C the constant of the immersion $W_0,^{1,\Phi}(\Omega) \hookrightarrow L^{\alpha\phi_+}(\Omega)$, that is, $\|u\|_{\alpha\phi_+} \leqslant C\|\nabla u\|_\Phi$ for all $u \in W_0^{1,\Phi}(\Omega)$. So by taking $\epsilon < \frac{m_0 C^{\alpha\phi_+}}{\lambda\alpha}$, it follows from (f_2') and (f_3') that $F(x,t) \leqslant \epsilon t^{\alpha\phi_+}$ for all $t \in (0,m) \cup (M,\infty)$, for some $m > 0$ small enough and $M > 0$ large enough.

On the other hand, if we suppose that $\|\nabla u\|_\Phi < \epsilon'$, then

$$m^{\alpha\phi_+}\Big|[m \leqslant u \leqslant M]\Big| \leqslant \Big(\int_{[m\leqslant u\leqslant M]} u^{\alpha\phi_+}dx\Big)^{1/\alpha\phi^+} \leqslant \|u\|_{\alpha\phi_+} \leqslant C\|\nabla u\|_\Phi \leqslant C\epsilon',$$

which implies in $\Big|[m \leqslant u \leqslant M]\Big| \leqslant C\epsilon'/m^{\alpha\phi_+}$.

Therefore, since (f_1') is also satisfied, if $\epsilon' > 0$ is enough small we have

$$\begin{aligned}\int_\Omega F(x,u)dx &= \int_{[u<m]} F(x,u)dx + \int_{[u>M]} F(x,u)dx + \int_{[m\leqslant u\leqslant M]} F(x,u)dx\\ &\leqslant \epsilon\int_{\Omega\backslash[m\leqslant u\leqslant M]} u^{\alpha\phi_+}dx + \int_{[m\leqslant u\leqslant M]} F(x,u)dx\\ &\leqslant \epsilon\int_{\Omega\backslash[m\leqslant u\leqslant M]} u^{\alpha\phi_+}dx + \sup_{m\leqslant t\leqslant M} F(x,t)\frac{C\epsilon'}{m^{\alpha\phi_+}} \leqslant \epsilon\int_\Omega u^{\alpha\phi_+}dx,\end{aligned}$$

that is, $J_2(u) \leqslant \epsilon\|u\|_{\alpha\phi_+}^{\alpha\phi^+}$. By using this fact, hypothesis (M) and Lemma 4.1.11, we obtain

$$\begin{aligned}\Psi_1(u) &= \hat{M}\Big(\mathcal{P}(u)\Big) - \lambda\int_\Omega F(x,u)dx \geqslant \frac{m_0}{\alpha}\mathcal{P}(u)^\alpha - \epsilon\lambda\|u\|_{\alpha\phi_+}^{\alpha\phi_+}\\ &\geqslant \frac{m_0}{\alpha}\|\nabla u\|_\Phi^{\alpha\phi_+} - \lambda\epsilon\|u\|_{\alpha\phi_+}^{\alpha\phi_+} \geqslant \frac{m_0C^{\alpha\phi_+}}{\alpha}\|u\|_{\alpha\phi_+}^{\alpha\phi_+} - \lambda\epsilon\|u\|_{\alpha\phi_+}^{\alpha\phi_+} > 0 = \Psi_1(0)\end{aligned}$$

whenever $\|\nabla u\|_\Phi < \epsilon'$. Hence, 0 is a strict local minimum of Ψ_1 in the strong topology, which by Proposition 4.2.7 leads us to conclude that 0 is a strict minimum of Ψ_1 in the weak topology as well, i.e, there exists a weak neighborhood V_w of 0 such that

$$0 = \Psi_1(0) < \Psi_1(u) \ \text{ for all } u \in V_w\backslash\{0\}.$$

Next, since Ψ_1 is lower semicontinuous and coercive, $\Psi_1^{-1}((-\infty,\tau])$ is weakly compact for every $\tau \in \mathbb{R}$. In particular, $\Psi_1^{-1}((-\infty,0])$ is weakly compact. Thus, by defining the disjoint weak compact sets $A_1 = \{0\}$ and $A_2 = \Psi_1^{-1}((-\infty,0])\backslash V_w = \Psi_1^{-1}((-\infty,0])\backslash\{0\}$, we have

$$\bigcap_{n=1}^{\infty}\Psi_1^{-1}((-\infty,1/n]) = \Psi_1^{-1}((-\infty,0]) = A_1 \cup A_2.$$

Hence, by using Theorem **A.1.12** in appendix, we can find $n_0 \in \mathbb{N}$ and disjoint weakly compact sets $B_i \supseteq A_i, i = 1, 2$, such that

$$\Psi_1^{-1}((-\infty, 1/n_0]) = B_1 \cup B_2,$$

where $0 \in B_1$ and $u_0 \in B_2$ (remember that $\Psi_1(u_0) < 0$). Since B_1 and B_2 are disjoint weakly compact sets, we can find C_1 and C_2 disjoint weakly open sets, and therefore open with respect to the strong topology also, such that $B_i \subset C_i, i = 1, 2$. Through these sets, we can define

$$D_i := \{u \in C_i \ : \ \Psi_1(u) < 1/n_0\} \subset B_i, \quad i = 1, 2,$$

where $0 \in D_1$ and $u_0 \in D_2$. In addition, as Ψ_1 is a continuous operator, the sets D_1 and D_2 are also open in the strong topology. Since $0 \in D_1$ and $u_0 \in D_2$, by taking $\hat{u} \in Dom(\Psi_2)$ (remember that, under our hypotheses, Ψ_2 is proper) and using that $t \mapsto G(x, \cdot)$ is increasing, we conclude for ϵ enough small, $\epsilon\hat{u} \in D_1$ and $u_0 + \epsilon\hat{u} \in D_2$, that is, $D_i \cap Dom(\Psi_2) \neq \varnothing$, $i = 1, 2$.

Therefore, as Ψ_2 is sequentially weakly lower semicontinuous and B_i is sequentially weakly compact sets, the infimum of Ψ_2 on $B_i, i = 1, 2$, is attained. In this way, we can define

$$\alpha_i = \inf\left\{\frac{\Psi_2(u) - \min\limits_{B_i}\Psi_2}{n_0^{-1} - \Psi_1(u)} \ : \ u \in D_i \cap Dom(\Psi_2)\right\}, \quad i = 1, 2.$$

Thus, by taking $\mu_\lambda > 0$ such that $\mu_\lambda = 1/\max\{\alpha_1, \alpha_2\}$, for $\mu \in (0, \mu_\lambda)$ we have $1/\mu > \alpha_i$ for $i = 1, 2$, which by the definition of α_i gives

$$\frac{1}{\mu} > \frac{\Psi_2(\omega_i) - \min\limits_{B_i}\Psi_2}{n_0^{-1} - \Psi_1(\omega_i)},$$

for some $\omega_i \in D_i \cap Dom(\Psi_2)$, i.e

$$\frac{1}{n_0} > I(\omega_i) - \mu \min_{B_i} \Psi_2, \quad i = 1, 2. \tag{4.7}$$

On the other hand, again by using that B_i is sequentially weakly compact and I sequentially weakly lower semicontinuous, we can find $\tilde{\omega}_i \in B_i \cap Dom(\Psi_2)$, $i = 1, 2$, such that

$$\min_{B_i} I = I(\tilde{\omega}_i), \quad i = 1, 2.$$

By contradiction, assume $\Psi_1(\tilde{\omega}_i) \geqslant 1/n_0$ for some $i = 1, 2$. Then, by (4.7) we obtain

$$\min_{B_i} I = I(\tilde{\omega}_i) \geqslant \frac{1}{n_0} + \mu\Psi_2(\tilde{\omega}_i) \geqslant \frac{1}{n_0} + \mu \min_{B_i} \Psi_2 > I(\omega_i), \quad i = 1, 2,$$

which is absurd. Therefore, $\Psi_1(\tilde{\omega}_i) < 1/n_0$, that is, $\tilde{\omega}_i \in D_i$ for $i = 1, 2$.

Finally, by using that D_i is an open set, $\tilde{\omega}_i \in D_i$ and $D_1 \cap D_2 = \varnothing$, we conclude that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are distinct local minima of I. ■

Corollary 4.2.9 *Suppose* $(\phi_0) - (\phi_2), (b), (M)$ *and* $(f_1') - (f_3')$ *hold. In addition, when* $\delta > 1$, *assume* (S) *admits a* $W_0^{1,\Phi}(\Omega)$*-solution. Then* I *has three critical points.*

Proof: It follows directly from Theorem A, Lemma 4.2.6 and Lemma 4.2.8. ■

4.3 Necessary and sufficient condition to a multiplicity result

In this section, we will present the proof of the main result of this chapter.

In addition to obtaining necessary and sufficient conditions for multiplicity of solutions for $(Q_{\lambda,\mu})$, the proof we will exhibit also brings us relevant information about the pure singular problem obtained by taking $f \equiv 0$ and $M \equiv 1$ in $(Q_{\lambda,\mu})$.

Theorem 0.0.12 *Suppose that* $(\phi_0), (\phi_1), (\phi_2), (M)$, (b), $(f_1') - (f_3')$ *hold. Assume* $\delta > 1$ *and*

$$\lambda^* = \inf \left\{ \frac{\hat{M}\Big(\displaystyle\int_\Omega \Phi(|\nabla u|)\Big)}{\displaystyle\int_\Omega F(x,u)dx} \ : \ u \in W_0^{1,\Phi}(\Omega) \ \text{and} \ \int_\Omega F(x,u)dx > 0 \right\}.$$

Then, the following are equivalent:

i) *there exists* $0 < u_0 \in W_0^{1,\Phi}(\Omega)$ *such that* $\displaystyle\int_\Omega b u_0^{1-\delta} dx < \infty$;

ii) *the problem*

$$(S): \qquad -\Delta_\Phi u = b(x)u^{-\delta} \ \ in\ \Omega,\ \ u > 0 \ \ in\ \Omega \ \ and \ \ u = 0 \ on\ \partial\Omega$$

admits (unique) weak solution;

iii) *for each* $\lambda > \lambda^*$ *there exists* $\mu_\lambda > 0$ *such that for* $\mu \in (0, \mu_\lambda]$ *the problem* $(Q_{\lambda,\mu})$ *admits at least three weak solutions.*

Proof:

$(i \Longrightarrow ii)$
If $\displaystyle\int_\Omega b u_0^{1-\delta} dx < \infty$ for some $0 < u_0 \in W_0^{1,\Phi}(\Omega)$, then

$$\mathcal{A} := \Big\{ u \in W_0^{1,\Phi}(\Omega) \ : \ \int_\Omega b(x)|u|^{1-\delta} dx < \infty \Big\}$$

is non-empty.

Let us define the following sets

$$\mathcal{N} := \Big\{ u \in W_0^{1,\Phi}(\Omega) : \int_\Omega \Big(a(|\nabla u|)|\nabla u|^2 - b(x)|u|^{1-\delta} \Big) dx \geqslant 0 \Big\},$$

$$\mathcal{N}^* := \Big\{ u \in W_0^{1,\Phi}(\Omega) : \int_\Omega \Big(a(|\nabla u|)|\nabla u|^2 - b(x)|u|^{1-\delta} \Big) dx = 0 \Big\}$$

and the functional $J : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ given by

$$J(u) = \int_\Omega \Phi(|\nabla u|) dx + \frac{1}{\delta - 1} \int_\Omega b(x)|u|^{1-\delta} dx.$$

Henceforth, we will prove that J admits a minimum in $\mathcal{N}$ and that this minimum is the sought solution. To ensure this, we need to establish the following claims.

Claim 1: $\mathcal{N}^*$ and $\mathcal{N}$ are non-empty sets and $\mathcal{N}$ is an unbounded set.

To prove this assertion, let us take $u \in \mathcal{A}$ and define the function

$$\sigma(t) := J(tu) = \int_\Omega \Phi(t|\nabla u|)dx + \frac{t^{1-\delta}}{\delta - 1}\int_\Omega b(x)|u|^{1-\delta}dx, \ t > 0.$$

Then, for $t > 0$ one has

$$\sigma'(t) = \int_\Omega \phi(t|\nabla u|)|\nabla u|dx - t^{-\delta}b(x)|u|^{1-\delta}dx$$

and

$$\sigma''(t) = \int_\Omega \phi'(t|\nabla u|)|\nabla u|^2dx + \delta t^{-\delta-1}\int_\Omega b(x)|u|^{1-\delta}dx$$

By using (ϕ_1) and Lemma 4.1.11$-i)$, we get

$$\int_\Omega \phi(t|\nabla u|)|\nabla u|dx \geqslant \frac{\phi_-}{t}\int_\Omega \Phi(t|\nabla u|)dx \geqslant \phi_- \min\{t^{\phi_- -1}, t^{\phi_+ -1}\}\int_\Omega \Phi(|\nabla u|)dx \tag{4.8}$$

and

$$\int_\Omega \phi(t|\nabla u|)|\nabla u|dx \leqslant \frac{\phi_+}{t}\int_\Omega \Phi(t|\nabla u|)dx \leqslant \phi_- \max\{t^{\phi_- -1}, t^{\phi_+ -1}\}\int_\Omega \Phi(|\nabla u|)dx. \tag{4.9}$$

Thus, as a consequence of (4.8) we conclude that $\sigma'(t) \to \infty$ as $t \to \infty$ and by (4.9) we obtain $\sigma'(t) \to -\infty$ as $t \to 0^+$. On the other hand, since ϕ is increasing, we get $\sigma''(t) > 0$ for all $t > 0$, which implies σ' is also an increasing function. Therefore, joining all these information, we conclude that there exists a unique $t_* = t_*(u)$ (which is a global minimum of σ) such that $\sigma'(t_*) = 0$ and $\sigma'(t) \geqslant 0$ for all $t > 0$ large enough, that is,

$$\int_\Omega \phi(t_*|\nabla u|)t_*|\nabla u|dx - t_*^{1-\delta}\int_\Omega b(x)|u|^{1-\delta}dx = 0$$

and

$$\int_\Omega \phi(t|\nabla u|)t|\nabla u|dx - t^{1-\delta}\int_\Omega b(x)|u|^{1-\delta}dx \geqslant 0, \quad \text{for all } t > 0 \text{ large}.$$

Thus, $t_* u \in \mathcal{N}^*$ and $\mathcal{N}$ is unbounded and our claim is proved.

Claim 2: $\mathcal{N}$ is a closed set.

Indeed, suppose $(u_n) \subset \mathcal{N}$ and $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. By Lemma 4.1.18$-i$) we obtain $\langle \mathcal{P}'(u_n), u_n\rangle \to \langle \mathcal{P}'(u), u\rangle$ as $n \to \infty$. Besides this, by Fatou's Lemma $\int_\Omega b(x)|u|^{1-\delta}dx \leqslant \liminf_{n\to\infty}\int_\Omega b(x)|u_n|^{1-\delta}dx$. Thus, by taking advantage of this information and using $\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|dx - \int_\Omega b(x)|u_n|^{1-\delta}dx \geqslant 0$ for all $n \in \mathbb{N}$, we conclude that $u \in \mathcal{N}$.

Claim 3: 0 is not an accumulation point of $\mathcal{N}$.

Assume on the contrary that there exists $(u_n) \subset \mathcal{N}$ such that $u_n \to 0$ in $W_0^{1,\Phi}(\Omega)$. Then, by Theorem **A.1.9** and since $\delta > 1$, one has

$$\begin{aligned} \infty > C \ &> \ \phi_+\int_\Omega \Phi(|\nabla u_n|)dx \geqslant \int_\Omega b(x)|u_n|^{1-\delta}dx \\ &\geqslant \ \Big(\int_\Omega b(x)^{1/\delta}dx\Big)^{\delta}\Big(\int_\Omega |u_n|dx\Big)^{1-\delta} \to \infty \text{ as } n \to \infty \end{aligned}$$

which is clearly impossible. Hence, there is $C_1 > 0$ such that $\|\nabla u\|_\Phi \geqslant C_1$ for all $u \in \mathcal{N}$.

Claim 4: J is a coercive and lower semi-continuous functional.

Note that, by Lemma 4.1.11$-iii$) we have $J(u) \to \infty$ as $\|\nabla u\|_\Phi \to \infty$, that is, J is a coercive functional. Moreover, by Lemma 4.1.18$-ii$) and Fatou's Lemma, we conclude that J is lower semi-continuous, which proves the Claim 4.

From the Claims 1, 2 and 4, the assumptions of the Ekeland Variational Principle (see Theorem **A.1.10** in appendix) are assured. So, taking the minimizing sequence $(u_n) \subset \mathcal{N}$ corresponding to $\inf_{\mathcal{N}} J$, the following conditions are satisfied:

i) $J(u_n) \leqslant \inf\limits_{\mathcal{N}} J + \dfrac{1}{n}$;

ii) $J(u_n) \leqslant J(w) + \frac{1}{n}\|\nabla(u_n - w)\|_\Phi, \ \forall w \in \mathcal{N}$.

As $J(|u_n|) = J(u_n)$, we can assume $u_n \geqslant 0$. Moreover, if we suppose $u_n = 0$ in a measurable set $\Omega_0 \subset \Omega$, with $|\Omega_0| > 0$, then since $u_n \in \mathcal{N}$ and $b(x) > 0$ a.e in Ω, we obtain again by Theorem **A.1.9**

$$\infty > \phi_+ \int_\Omega \Phi(|\nabla u_n|)dx \geqslant \int_{\Omega_0} b(x)u_n^{1-\delta} \geqslant \Big(\int_{\Omega_0} b(x)^{1/\delta}dx\Big)^\delta \Big(\int_{\Omega_0} |u_n|dx\Big)^{1-\delta} = \infty,$$

which is an absurd. Thus, $u_n(x) > 0$ a.e in Ω.

Since $J(u_n) \to \inf\limits_{\mathcal{N}} J \geqslant 0$, we have

$$\min\{\|\nabla u_n\|_\Phi^{\phi_-}, \|\nabla u_n\|_\Phi^{\phi_+}\} \leqslant \int_\Omega \Phi(|\nabla u_n|)dx \leqslant \epsilon + \inf_{\mathcal{N}} J$$

for all n large enough. Hence, $\|\nabla u_n\|_\Phi \leqslant C_2$ for suitable constant C_2 and this implies, by Proposition 4.1.10, that (up to subsequence)

$$\begin{cases} u_n \rightharpoonup u_* \text{ in } W_0^{1,\Phi}(\Omega); \\ u_n \to u_* \text{ strongly in } L^G(\Omega) \text{ for all N-function } \ G << \Phi_*; \\ u_n \to u_* \text{ a.e in } \Omega \end{cases}$$

for some $u_* \in W_0^{1,\Phi}(\Omega)$.

By the Fatou's Lemma one has $\inf_{\mathcal{N}} J \geqslant J(u_*)$, which implies $\displaystyle\int_\Omega b(x)u_*^{1-\delta}dx < \infty$, that is, $u_* \in \mathcal{A}$. Thus, it follows from the proof of Claim 1 that $t_*u_* \in \mathcal{N}^*$, where $t_* = t_*(u_*)$ is the minimum of $t \mapsto J(tu_*)$. Therefore,

$$\inf_{\mathcal{N}} J = \lim_{n\to\infty} \inf J(u_n) \geqslant J(u_*) \geqslant J(t_*u_*) \geqslant \inf_{\mathcal{N}^*} J \geqslant \inf_{\mathcal{N}} J$$

which results in $J(u_*) = \inf\limits_{\mathcal{N}} J$, that is

$$\int_\Omega \Phi(|\nabla u_n|)dx + \frac{1}{\delta-1}\int_\Omega b(x)|u_n|^{1-\delta}dx \overset{n\to\infty}{\longrightarrow} \int_\Omega \Phi(|\nabla u_*|)dx + \frac{1}{\delta-1}\int_\Omega b(x)|u_*|^{1-\delta}dx. \tag{4.10}$$

On the another side, by Fatou's Lemma and Lemma 4.1.18$-ii$), one has

$$\int_\Omega \Phi(|\nabla u_*|)dx \leqslant \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)dx$$

and

$$\int_\Omega b(x)|u_*|^{1-\delta}dx \leqslant \liminf_{n\to\infty} \int_\Omega b(x)|u_n|^{1-\delta}dx,$$

whence joining this information with (4.10), we conclude that

$$\lim_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)dx = \int_\Omega \Phi(|\nabla_* u|)dx,$$

which by Proposition 4.1.17 implies

$$u_n \to u_* \text{ in } W_0^{1,\Phi}(\Omega). \tag{4.11}$$

Next, we will prove that u_* is a solution of (S). The proof will be split into two cases.

Case 1: Infinite terms of (u_n) belong to $\mathcal{N}\backslash\mathcal{N}^*$.

In this case, by fixing $0 \leqslant \varphi \in W_0^{1,\Phi}(\Omega)$, as $u_n \in \mathcal{N}\backslash\mathcal{N}^*$, we obtain

$$\int_\Omega b(x)(u_n + t\varphi)^{1-\delta}dx \leqslant \int_\Omega b(x)u_n^{1-\delta}dx < \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|dx \quad \text{for all } t > 0,$$

so $(u_n + t\varphi) \in \mathcal{N}$ for $t > 0$ enough small. Thus, by item$-ii$) above we have $J(u_n) - J(u_n + t\varphi) \leqslant n^{-1}\|t\nabla\varphi\|_\Phi$ and from this we get

$$\frac{1}{n}\|\nabla\varphi\|_\Phi + \int_\Omega \frac{\Phi(|\nabla u_n + t\nabla\varphi|) - \Phi(|\nabla u_n|)}{t}dx \geqslant \frac{1}{\delta-1}\int_\Omega b(x)\Big[\frac{u_n^{1-\delta} - (u_n + t\varphi)^{1-\delta}}{t}\Big]dx,$$

which together with Fatou's Lemma leads us to conclude that

$$\begin{aligned}
\frac{1}{n}\|\nabla\varphi\|_{\Phi}+\int_{\Omega}a(|\nabla u_n|)\nabla u_n\nabla\varphi dx &= \frac{1}{n}\|\nabla\varphi\|_{\Phi}+\liminf_{t\to 0^+}\int_{\Omega}\frac{\Phi(|\nabla u_n+t\nabla\varphi|)-\Phi(|\nabla u_n|)}{t}dx\\
&\geqslant \frac{1}{\delta-1}\liminf_{t\to 0^+}\int_{\Omega}b(x)\Big[\frac{u_n^{1-\delta}-(u_n+t\varphi)^{1-\delta}}{t}\Big]dx\\
&\geqslant \int_{\Omega}b(x)u_n^{-\delta}\varphi dx. \qquad (4.12)
\end{aligned}$$

Once again by Fatou's Lemma, (4.12), the convergence in (4.11) and Lemma 4.1.18$-i$), letting n tend to infinity we obtain

$$\int_{\Omega}a(|\nabla u_*|)\nabla u_*\nabla\varphi dx \geqslant \int_{\Omega}b(x)u_*^{-\delta}\varphi dx. \qquad (4.13)$$

Case 2: Finite terms of (u_n) belong to $\mathcal{N}\backslash\mathcal{N}^*$.

In this case, infinite terms of (u_n) belong to $\mathcal{N}^*$. By fixing again $0\leqslant\varphi\in W_0^{1,\Phi}(\Omega)$, we obtain $\int_{\Omega}b(x)(u_n+t\varphi)^{1-\delta}dx\leqslant\int_{\Omega}\phi(|\nabla u_n|)|\nabla u_n|dx<\infty$ for each $t\geqslant 0$, which implies $(u_n+t\varphi)\in\mathcal{A}$. Once again by the proof of Claim 1, there exists unique $f_{n,\varphi}(t)>0$ such that $f_{n,\varphi}(t)(u_n+t\varphi)\in\mathcal{N}^*$, that is

$$\int_{\Omega}\phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|dx-f_{n,\varphi}^{1-\delta}(t)\int_{\Omega}b(x)|u_n+t\nabla\varphi|^{1-\delta}dx=0. \qquad (4.14)$$

Since we are supposing that $u_n\in\mathcal{N}^*$, we get $f_{n,\varphi}(0)=1$. Besides this, we claim that f_n is a continuous function in $[0,\infty)$. Indeed, if $0\leqslant t_k\to t$, then by (4.14), hypothesis (ϕ_1) and Lemma 4.1.11$-i$) we have

$$\phi_-\min\left\{f_{n,\varphi}^{\phi_-+\delta-1}(t_k),f_{n,\varphi}^{\phi_++\delta-1}(t_k)\right\}\frac{\int_{\Omega}\Phi(|\nabla u_n+t_k\nabla\varphi|)dx}{\int_{\Omega}b(x)|u_n+t_k\varphi|^{1-\delta}dx}\leqslant 1,$$

and

$$\phi_- \max\left\{f_{n,\varphi}^{\phi_-+\delta-1}(t_k), f_{n,\varphi}^{\phi_++\delta-1}(t_k)\right\}\frac{\displaystyle\int_\Omega \Phi(|\nabla u_n + t_k\nabla\varphi|)dx}{\displaystyle\int_\Omega b(x)|u_n+t_k\varphi|^{1-\delta}dx} \geqslant 1, \tag{4.15}$$

hence $f_{n,\varphi}(t_k)$ is a bounded sequence, so $f_{n,\varphi}(t_k) \to s \geqslant 0$ up to subsequence. Note that, from (4.15) and dominated convergence, it follows that $s \neq 0$, which again by dominated convergence and (4.14) results in $s(u_n + t\varphi) \in \mathcal{N}^*$, whence $s = f_{n,\varphi}(t)$ and this concludes the proof of our assertion.

Next, let us define $s_{n,\varphi} := \lim\limits_{t\to 0^+} (f_{n,\varphi}(t) - 1)/t \in [-\infty, \infty]$. If this limit does not exist, we can replace $t \to 0^+$ by $t_k \to 0^+$ as $k \to \infty$, for some suitable sequence (t_k).

By using (4.14) and $f_{n,\varphi}(0) = 1$, one has

$$\begin{aligned} 0 \;=\; & \int_\Omega \Big[\frac{a(f_{n,\varphi}(t)|\nabla u_n + t\nabla\varphi|)f_{n,\varphi}^2(t)|\nabla u_n + t\nabla\varphi|^2 - a(|\nabla u_n|)|\nabla u_n|^2}{t}\Big]dx \\ & - \int_\Omega b(x)\Big[\frac{f_{n,\varphi}(t)^{1-\delta}|u_n+t\varphi|^{1-\delta} - |u_n|^{1-\delta}}{t}\Big]dx \end{aligned}$$

so, by taking $t \to 0^+$ in the previous equality we obtain

$$\begin{aligned} 0 = & \int a'(|\nabla u_n|)\Big[s_{n,\varphi}|\nabla u_n| + \frac{\nabla u_n\nabla\varphi}{|\nabla u_n|}\Big]|\nabla u_n|^2dx + 2\int_\Omega a(|\nabla u_n|)[s_{n,\varphi}|\nabla u_n|^2 + \nabla u_n\nabla\varphi]dx \\ & +(\delta-1)\int_\Omega [b(x)s_{n,\varphi}u_n^{1-\delta} + b(x)u_n^{-\delta}\varphi]dx = s_{n,\varphi}\int_\Omega\Big(a'(|\nabla u_n|)|\nabla u_n| + a(|\nabla u_n|)\Big)|\nabla u_n|^2dx \\ & +s_{n,\varphi}\int_\Omega \Big(a(|\nabla u_n|)|\nabla u_n|^2 + (\delta-1)b(x)u_n^{1-\delta}\Big)dx + 2\int_\Omega a(|\nabla u_n|)\nabla u_n\nabla\varphi dx \\ & + \int_\Omega \Big(a'(|\nabla u_n|)|\nabla u_n|\nabla u_n\nabla\varphi + (\delta-1)b(x)u_n^{-\delta}\varphi\Big)dx. \end{aligned}$$

Therefore, by Lemma 4.1.13 and the previous equality, we get

$$\begin{aligned} & s_{n,\varphi}\overbrace{\int_\Omega \Big(a'(|\nabla u_n|)|\nabla u_n| + a(|\nabla u_n|)\Big)|\nabla u_n|^2dx}^{\geqslant 0} + s_{n,\varphi}\delta\overbrace{\int_\Omega b(x)u_n^{1-\delta}dx}^{\geqslant 0} \\ \leqslant\; & C\int_\Omega \phi(|\nabla u_n|)|\nabla\varphi|dx \leqslant C\int_\Omega \Big(\tilde{\Phi}(\phi(|\nabla u_n|)) + \Phi(|\nabla\varphi|)\Big)dx \quad \text{(Prop. 4.1.4-iii)} \end{aligned}$$

$$\leqslant \quad C\int_\Omega \Big(\Phi(|\nabla u_n|)+\Phi(|\nabla\varphi|)\Big)dx, \qquad \text{(Prop. 4.1.4-ii)}$$

for some cumulative constant $C>0$, whence using the boundedness of (u_n) in $W_0^{1,\Phi}(\Omega)$ and the last inequality, we conclude that $s_{n,\varphi}\neq +\infty$ and, in addition, $s_{n,\varphi}\leqslant C_3$, for some $C_3>0$.

Now, we will prove that $s_{n,\varphi}\neq -\infty$ and $s_{n,\varphi}$ is bounded below by a constant independent of n.

Suppose by contradiction that $s_{n,\varphi}=-\infty$. In this case, for $t>0$ enough small $f_{n,\varphi}(t)<1$. Thus, again by Theorem **A.1.10** one has

$$\begin{aligned}
&\frac{1-f_{n,\varphi}(t)}{n}\|\nabla u_n\|_\Phi+\frac{tf_{n,\varphi}(t)}{n}\|\nabla\varphi\|_\Phi \geqslant \frac{1}{n}\|\nabla u_n(1-f_{n,\varphi}(t))-tf_{n,\varphi}(t)\nabla\varphi\|_\Phi\\
\geqslant\quad & J(u_n)-J(f_{n,\varphi}(t)(u_n+t\varphi))=\int_\Omega \Phi(|\nabla u_n|)dx+\frac{1}{\delta-1}\int_\Omega b(x)|u_n|^{1-\delta}dx\\
&-\int_\Omega \Phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)dx-\frac{f_{n,\varphi}^{1-\delta}(t)}{\delta-1}\int_\Omega b(x)|u_n+t\varphi|^{1-\delta}dx.
\end{aligned}$$

So rearranging the terms and dividing the previous inequality by $t>0$, we get

$$\begin{aligned}
f_{n,\varphi}(t)\frac{\|\nabla\varphi\|_\Phi}{n}\geqslant\ & \frac{f_{n,\varphi}(t)-1}{t}\frac{\|\nabla u_n\|_\Phi}{n}+\int_\Omega\frac{\Phi(|\nabla u_n|)-\Phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)}{t}dx\\
&-\frac{1}{\delta-1}\int_\Omega b(x)\frac{f_{n,\varphi}^{1-\delta}(t)|u_n+t\varphi|^{1-\delta}-u_n^{1-\delta}}{t}dx\\
=\ & \frac{f_{n,\varphi}(t)-1}{t}\frac{\|\nabla u_n\|_\Phi}{n}+\int_\Omega\frac{\Phi(|\nabla u_n|)-\Phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)}{t}dx\\
&-\frac{1}{\delta-1}\int_\Omega\frac{\phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|-\phi(|\nabla u_n|)|\nabla u_n|}{t}dx,
\end{aligned}\tag{4.16}$$

where the last equality was obtained using $f_{n,\varphi}(t)(u_n+t\varphi)\in\mathcal{N}^*$. Thus, as $t\mapsto\phi(t)t\ \ (t>0)$ is increasing and $0<f_{n,\varphi}(t)<1$, one has

$$\phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|\leqslant\phi(|\nabla u_n+t\nabla\varphi|)|\nabla u_n+t\nabla\varphi|,$$

which in turn using (4.16) gives us

$$
\begin{aligned}
f_{n,\varphi}(t)\frac{\|\nabla\varphi\|_\Phi}{n} \geqslant\ & \frac{f_{n,\varphi}(t)-1}{t}\frac{\|\nabla u_n\|_\Phi}{n} + \int_\Omega \frac{\Phi(|\nabla u_n|)-\Phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|))}{t}dx \\
& -\frac{1}{\delta-1}\int_\Omega \frac{\phi(|\nabla u_n+t\nabla\varphi|)|\nabla u_n+t\nabla\varphi|-\phi(|\nabla u_n|)|\nabla u_n|}{t}dx.
\end{aligned}
$$

Taking $t\to 0^+$ and using Lemma 4.1.13, we get

$$
\begin{aligned}
\frac{\|\nabla\varphi\|_\Phi}{n} \geqslant\ & s_{n,\varphi}\frac{\|\nabla u_n\|_\Phi}{n} - \int_\Omega \phi(|\nabla u_n|)\Big[s_{n,\varphi}|\nabla u_n| + \frac{\nabla u_n\nabla\varphi}{|\nabla u_n|}\Big]dx \\
& -\frac{1}{\delta-1}\int_\Omega \Big[a'(|\nabla u_n|)|\nabla u_n| + 2a(|\nabla u_n|)\Big]\nabla u_n\nabla\varphi dx \\
=\ & s_{n,\varphi}\Big(\frac{\|\nabla u_n\|_\Phi}{n} - \int_\Omega a(|\nabla u_n|)|\nabla u_n|^2dx\Big) \\
& -\Big(1+\frac{a^+ +1}{\delta-1}\Big)\int_\Omega a(|\nabla u_n|)|\nabla u_n||\nabla\varphi|dx. \qquad (4.17)
\end{aligned}
$$

Clearly, the left hand side of (4.17) goes to zero as $n\to\infty$. On the other hand, since $\phi_-\min\{\|\nabla u_n\|_\Phi^{\phi_-},\|\nabla u_n\|_\Phi^{\phi_+}\}\leqslant\int_\Omega a(|\nabla u_n|)|\nabla u_n|^2dx\leqslant\phi_+\max\{\|\nabla u_n\|_\Phi^{\phi_-},\|\nabla u_n\|_\Phi^{\phi_+}\}$, $\|\nabla u_n\|_\Phi\leqslant C_2$ and $s_{n,\varphi}\to-\infty$, using the Claim 3 above we can conclude that the right hand side of (4.17) tends to infinity as $n\to\infty$, which is absurd. Thus $s_{n,\varphi}\neq-\infty$ and $s_{n,\varphi}\geqslant C_4$ for some $C_4\in\mathbb{R}$ independent of n. Therefore, putting together all the information obtained so far, we conclude that $|s_{n,\varphi}|\leqslant C_5$, for some $C_5>0$ independent of n.

Now, we will prove that (4.13) also holds true in this case. For this, we will use the Theorem **A.1.10** one more time to get

$$
\begin{aligned}
& |1-f_{n,\varphi}(t)|\frac{\|\nabla u_n\|_\Phi}{n} + \frac{tf_{n,\varphi}(t)\|\nabla\varphi\|_\Phi}{n} \geqslant J(u_n)-J(f_{n,\varphi}(t)(u_n+t\varphi)) \\
=\ & \int_\Omega \Phi(|\nabla u_n|)dx + \frac{1}{\delta-1}\int_\Omega b(x)u_n^{1-\delta}dx - \int_\Omega \Phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)dx \\
& -\frac{f_{n,\varphi}^{1-\delta}(t)}{\delta-1}\int_\Omega b(x)(u_n+t\varphi)^{1-\delta}dx + \frac{1}{\delta-1}\int_\Omega b(x)(u_n+t\varphi)^{1-\delta}dx \\
& -\frac{1}{\delta-1}\int_\Omega b(x)(u_n+t\varphi)^{1-\delta}dx.
\end{aligned}
$$

Rearranging the terms and dividing both sides of the previous inequality by $t>0$, we still obtain

$$\frac{1}{n}\Big(\frac{|1-f_{n,\varphi}(t)|}{t}\|\nabla u_n\|_{\Phi}+f_{n,\varphi}(t)\|\nabla\varphi\|_{\Phi}\Big) \geqslant -\int_{\Omega}\frac{\Phi(f_{n,\varphi}(t)|\nabla u_n+t\nabla\varphi|)-\Phi(|\nabla u_n|)}{t}dx$$
$$-\frac{1}{\delta-1}\int_{\Omega}\frac{b(x)(u_n+t\varphi)^{1-\delta}-b(x)u_n^{1-\delta}}{t}dx+\frac{1}{\delta-1}\int_{\Omega}b(x)(u_n+t\varphi)^{1-\delta}\frac{1-f_{n,\varphi}^{1-\delta}(t)}{t}dx,$$

which by taking $t\to 0^+$ leads us to

$$\begin{aligned}\frac{1}{n}\Big(|s_{n,\varphi}|\|\nabla u_n\|_{\Phi}+\|\nabla\varphi\|_{\Phi}\Big) &\geqslant -\int_{\Omega}\phi(|\nabla u_n|)\Big[s_{n,\varphi}|\nabla u_n|+\frac{\nabla u_n\nabla\varphi}{|\nabla u_n|}\Big]dx\\ &\quad+\int_{\Omega}b(x)u_n^{-\delta}\varphi dx+s_{n,\varphi}\int_{\Omega}b(x)u_n^{1-\delta}dx\\ &= -\int_{\Omega}a(|\nabla u_n|)\nabla u_n\nabla\varphi dx+\int_{\Omega}b(x)u_n^{-\delta}\varphi dx,\end{aligned}$$

where in the last equality we use that $u_n\in\mathcal{N}^*$. So, by Fatou's Lemma, (4.11) and using that $|s_{n,\varphi}|\leqslant C_5$, we get again

$$\int_{\Omega}a(|\nabla u_*|)\nabla u_*\nabla\varphi dx\geqslant\int_{\Omega}b(x)u_*^{-\delta}\varphi dx.$$

Therefore, in any case

$$\int_{\Omega}a(|\nabla u_*|)\nabla u_*\nabla\varphi dx\geqslant\int_{\Omega}b(x)u_*^{-\delta}\varphi dx. \tag{4.18}$$

By replacing φ in (4.18) by u_*, we conclude that $u_*\in\mathcal{N}$. Besides this, as $\inf_{\mathcal{N}}J=J(u_*)$, then $t_*(u_*)=1$ (see Claim 1 above) and consequently $u_*\in\mathcal{N}^*$ and $u_*>0$ in Ω.

Now, we can prove that $u_*\in W_0^{1,\Phi}(\Omega)$ is the desired solution of (S). For this, let us fix $\epsilon>0$ and $\phi\in W_0^{1,\Phi}(\Omega)$. By taking $(u_*+\epsilon\phi)^+\in W_0^{1,\Phi}(\Omega)$ as a test function in (4.18), we obtain

$$\begin{aligned}0 &\leqslant \int_{[u_*+\epsilon\phi\geqslant 0]}\Big(a(|\nabla u_*|)\nabla u_*\nabla(u_*+\epsilon\phi)-b(x)u_*^{-\delta}(u_*+\epsilon\phi)\Big)dx\\ &= \int_{\Omega}-\int_{[u_*+\epsilon\phi<0]}\Big(a(|\nabla u_*|)\nabla u_*\nabla(u_*+\epsilon\phi)-b(x)u_*^{-\delta}(u_*+\epsilon\phi)\Big)dx\end{aligned}$$

$$
\begin{aligned}
&= \epsilon \int_\Omega \Big(a(|\nabla u_*|)\nabla u_* \nabla \phi - b(x) u_*^{-\delta} \phi \Big) dx \\
&\quad - \int_{[u_* + \epsilon\phi < 0]} \Big(a(|\nabla u_*|)\nabla u_* \nabla (u_* + \epsilon\phi) - b(x) u_*^{-\delta} (u_* + \epsilon\phi) \Big) dx \quad (u_* \in \mathcal{N}^*) \\
&\leqslant \epsilon \int_\Omega \Big(a(|\nabla u_*|)\nabla u_* \nabla \phi - b(x) u_*^{-\delta} \phi \Big) dx - \epsilon \int_{[u_* + \epsilon\varphi < 0]} a(|\nabla u_*|)\nabla u_* \nabla \phi dx \\
&\quad - \int_{[u_* + \epsilon\varphi < 0]} a(|\nabla u_*|)|\nabla u_*|^2 dx + \int_{[u_* + \epsilon\phi < 0]} b(x) u_*^{-\delta}(u_* + \epsilon\phi) dx \\
&\leqslant \epsilon \int_\Omega \Big(a(|\nabla u_*|)\nabla u_* \nabla \phi - b(x) u_*^{-\delta} \phi \Big) dx - \epsilon \int_{[u_* + \epsilon\varphi < 0]} a(|\nabla u_*|)\nabla u_* \nabla \phi dx,
\end{aligned}
$$

which dividing both sides by $\epsilon > 0$ and taking $\epsilon \to 0^+$, gives

$$\int_\Omega a(|\nabla u_*|)\nabla u_* \nabla \phi dx \geqslant \int_\Omega b(x) u_*^{-\delta} \phi dx.$$

By the arbitrariness of $\phi \in W_0^{1,\Phi}(\Omega)$, we conclude that u_* is $W_0^{1,\Phi}(\Omega)$-solution of (S).

$(ii \Longrightarrow iii)$

By Corollary 4.2.9, to conclude the desired result it suffices to show that any critical point of I is a solution of $(Q_{\lambda,\mu})$.

Let $u \in Dom(\Psi_2)$ be a critical point of I, that is,

$$\langle \Psi_1'(u), v - u \rangle + \mu\Big(\Psi_2(v) - \Psi_2(u)\Big) \geqslant 0,\ \forall v \in W_0^{1,\Phi}(\Omega). \tag{4.19}$$

Since $u \in Dom(\Psi_2)$, we have $\int_\Omega |G(x,u)| dx < \infty$, which implies $G(\cdot, u(\cdot))$ finite almost everywhere. Therefore, by the definition of G when $\delta > 1$, we necessarily have $u > 0$ a.e in Ω.

Next, we will prove that $bu^{-\delta}\varphi \in L^1(\Omega)$ and

$$M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)\nabla u \nabla \varphi dx - \lambda \int_\Omega f(x,u)\varphi dx - \mu \int_\Omega b u^{-\delta}\varphi dx \geqslant 0,$$

for all $0 \leqslant \varphi \in W_0^{1,\Phi}(\Omega)$.

So let us take $0 \leqslant \varphi \in W_0^{1,\Phi}(\Omega)$. Putting $v = u + t\varphi$ in (4.19), one has

$$\mu \int_\Omega \Big[\frac{G(x,u) - G(x,u+t\varphi)}{t}\Big] dx \leqslant M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)\nabla u \nabla \varphi dx - \lambda \int_\Omega f(x,u)\varphi dx.$$

By using that $G(x,\cdot)$ is a decreasing function once again and applying Fatou's Lemma with $t \to 0^+$ in the previous inequality, we get

$$\mu \int_\Omega b(x)u^{-\delta}\varphi dx \leqslant M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)\nabla u \nabla \varphi dx - \lambda \int_\Omega f(x,u)\varphi dx. \qquad (4.20)$$

Therefore, $bu^{-\delta}\varphi \in L^1(\Omega)$.

Next, we will prove that the integral equality in Definition 0.0.11 holds.

So for $t \in (0,1)$, let us set $v = (1-t)u$ in (4.19) to obtain

$$0 \leqslant -M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)|\nabla u|^2 dx + \lambda \int_\Omega f(x,u)u dx + \mu \int_\Omega \frac{G(x,u-tu) - G(x,u)}{t} dx.$$

Thus, for some $\tau = \tau(t) \in (0,t)$ we have

$$0 \leqslant -M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)|\nabla u|^2 dx + \lambda \int_\Omega f(x,u)u dx + \mu(1-\tau)^{-\delta} \int_\Omega b(x)u^{-\delta+1} dx,$$

which by passing the limit as $t \to 0^+$ give us

$$0 \leqslant -M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)|\nabla u|^2 dx + \lambda \int_\Omega f(x,u)u dx + \mu \int_\Omega b(x)u^{-\delta+1} dx. \quad (4.21)$$

Putting $\varphi = u$ in (4.20) and combining this with (4.21), we conclude

$$0 = -M\Big(\mathcal{P}(u)\Big) \int_\Omega a(|\nabla u|)|\nabla u|^2 dx + \lambda \int_\Omega f(x,u)u dx + \mu \int_\Omega b(x)u^{-\delta+1} dx. \quad (4.22)$$

Therefore, by fixing $\varphi \in W_0^{1,\Phi}(\Omega)$, as $0 \leqslant (u+\epsilon\varphi)^+ \in W_0^{1,\Phi}(\Omega)$, taking $(u+\epsilon\varphi)^+$ as a test function in (4.20) and using (4.22), we obtain

$$\begin{aligned}
0 &\leqslant M\Big(\mathcal{P}(u)\Big)\int_\Omega a(|\nabla u|)\nabla u\nabla(u+\epsilon\varphi)^+dx-\lambda\int_\Omega f(x,u)(u+\epsilon\varphi)^+dx\\
&\quad -\mu\int_\Omega b(x)u^{-\delta}(u+\epsilon\varphi)^+dx\\
&= \epsilon\Big[M\Big(\mathcal{P}(u)\Big)\int_\Omega a(|\nabla u|)\nabla u\nabla\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega b(x)u^{-\delta}\varphi dx\Big]\\
&\quad -\Big[M\Big(\mathcal{P}(u)\Big)\int_{[u+\epsilon\varphi<0]} a(|\nabla u|)\nabla u\nabla(u+\epsilon\varphi)dx-\lambda\int_{[u+\epsilon\varphi<0]} f(x,u)(u+\epsilon\varphi)dx\\
&\quad -\mu\int_{[u+\epsilon\varphi<0]} b(x)u^{-\delta}(u+\epsilon\varphi)dx\Big]\\
&\leqslant \epsilon\Big[M\Big(\mathcal{P}(u)\Big)\int_\Omega a(|\nabla u|)\nabla u\nabla\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega b(x)u^{-\delta}\varphi dx\Big]\\
&\quad -\epsilon M\Big(\mathcal{P}(u)\Big)\int_{[u+\epsilon\varphi<0]} a(|\nabla u|)\nabla u\nabla\varphi dx.
\end{aligned}$$

By noting that $\int_{[u+\epsilon\varphi<0]} a(|\nabla u|)\nabla u\nabla\varphi dx \to 0$ as $\epsilon\to 0^+$, let us divide the previous inequality by ϵ and take the limit as $\epsilon\to 0^+$ to get

$$0\leqslant M\Big(\mathcal{P}(u)\Big)\int_\Omega a(|\nabla u|)\nabla u\nabla\varphi dx-\lambda\int_\Omega f(x,u)\varphi dx-\mu\int_\Omega b(x)u^{-\delta}\varphi dx.$$

Since φ was chosen arbitrarily, we conclude that the integral equality in Definition 0.0.11 is satisfied and this conclude the proof of the implication ($ii \Longrightarrow iii$).

($iii \Longrightarrow i$)

If $0<u_0\in W_0^{1,\Phi}(\Omega)$ is a solution of $(Q_{\lambda,\mu})$, then in particular $u_0\in Dom(\Psi_2)$, that is, $\int_\Omega bu_0^{1-\delta}dx<\infty$. ■

Corollary 0.0.13 *Replacing $\delta>1$ with $\delta\leqslant 1$ and assuming the hypotheses of above theorem, the claims $i)-iii)$ remains true independent of each other.*

Proof: Let us assume $\delta\leqslant 1$ and prove that the claims $i)-iii)$ holds true.

Since we are assuming (b), by Lemma 4.2.5 we have $Dom(\Psi_2)\neq\varnothing$, which prove the item $i)$. The proof of $ii)$, is similar to that done in the Theorem 0.0.3, so we will omit it.

For the last item, Corollary 4.2.9 guarantees the existence of three critical points for I. Thus, it remains to show that any critical point of I is solution of $(Q_{\lambda,\mu})$.

Then, let $u \in W_0^{1,\Phi}(\Omega)$ be a critical point of I. The integral equality in Definition 0.0.11 and the fact that $bu^{-\delta}\varphi \in L^1(\Omega)$ for all $\varphi \in W_0^{1,\Phi}(\Omega)$, follows from exactly the same argument made in the proof of implication $(ii \implies iii)$ above.

To conclude the proof, we just need to check that $u > 0$ in Ω. Indeed, since $u \in Dom(\Psi_2)$, we have $\int_\Omega |G(x,u)|dx < \infty$, which results in $G(\cdot, u(\cdot))$ finite almost everywhere. Therefore, by the definition of G, we necessarily have $u \geqslant 0$ a.e in Ω when $0 < \delta < 1$ and $u > 0$ a.e in Ω when $\delta = 1$.

Now, let us exclude the possibility of u to be zero in a set of positive measure when $0 < \delta < 1$. For this, consider $0 < \delta < 1$ and suppose that $u = 0$ in Ω_0, for some $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. By taking $0 < \varphi \in W_0^{1,\Phi}(\Omega)$, $\epsilon > 0$ small enough and replacing v by $u + \epsilon\varphi$ in (4.19), we get

$$\begin{aligned} 0 \quad &\leqslant \quad \epsilon M\Big(\mathcal{P}(u)\Big)\int_\Omega a(|\nabla u|)\nabla u \nabla\varphi dx - \epsilon\lambda\int_\Omega f(x,u)\varphi dx \\ &\quad +\mu\int_{\Omega_0} G(x,\epsilon\varphi)dx + \mu\int_{\Omega\setminus\Omega_0}\Big[G(x,u+\epsilon\varphi) - G(x,u)\Big]dx, \end{aligned}$$

whence, by using $G(x.\cdot)$ is a decreasing function in $[0,\infty)$ and dividing both the sides of the previous inequality by ϵ, we have

$$\begin{aligned} 0 \quad &\leqslant \quad M\Big(\mathcal{P}(u)\Big)\int_\Omega a(|\nabla u|)\nabla u \nabla\varphi dx - \lambda\int_\Omega f(x,u)\varphi dx \\ &\quad -\frac{\mu\epsilon^{-\delta}}{1-\delta}\int_{\Omega_0} b(x)\varphi^{1-\delta}dx \to -\infty \quad \text{as } \epsilon \to 0^+ \end{aligned}$$

which is absurd. Therefore, $u > 0$ a.e in Ω and this ends the proof. ■

In [40], Lazer and Mckenna has proven that, when $0 < b_0 \leqslant b \in L^\infty(\Omega)$ and $\Phi(t) = |t|^p/p$ in (S), with $p = 2$, then (S) admits solution in $H_0^1(\Omega)$ if and only if $\delta < 3$. Mohammed also proved in [44] that in the case where $\Phi(t) = |t|^p/p$ $(p > 1)$, the sharp power is given by $(2p-1)/(p-1)$. Through the Theorem 0.0.12, we are able to show the existence of $\delta_q > 1$, which depends on the summability $L^q(\Omega)$ of b, such that for $\delta < \delta_q$ the existence of solution in $W_0^{1,\Phi}(\Omega)$ to (S) is still ensured, and this is the content of the next corollary.

Corollary 0.0.14 *Assume that* (ϕ_0) *and* (ϕ_1) *hold. If* $b \in L^q(\Omega)$ *for some* $1 < q$ *and*

$$1 < \delta < 1 + \frac{\phi'_+}{q'} := \delta_q, \tag{9}$$

then (S) *admits a* $W_0^{1,\Phi}(\Omega)$*-solution.*

Proof: By implication $(i \Longrightarrow ii)$ in Theorem 0.0.12, it suffices to show that there exists $u_0 \in W_0^{1,\Phi}(\Omega)$ such that $\int_\Omega b u_0^{1-\delta} dx < \infty$. Let us construct such u_0.

First, since Ω is a smooth domain, we can find $\epsilon > 0$ sufficiently small such that $d \in C^2(\overline{\Omega_{2\epsilon}})$ and $|\nabla d(x)| = 1$ in $\Omega_{2\epsilon}$, where $d(x) := dist(x, \partial\Omega)$ and $\Omega_{2\epsilon} = \{x \in \Omega \ : \ d(x) < 2\epsilon\}$. So, by fixing this $\epsilon > 0$, let us define

$$u_0(x) = \begin{cases} d(x)^\theta \ \text{ if } d(x) < \epsilon \\ \epsilon^\theta + \displaystyle\int_\epsilon^{d(x)} \theta\epsilon^{\theta-1}\Big(\frac{2\epsilon - t}{\epsilon}\Big)^{2/(\phi_- - 1)} dt \quad \text{if } \epsilon \leqslant d(x) < 2\epsilon \\ \epsilon^\theta + \displaystyle\int_\epsilon^{2\epsilon} \theta\epsilon^{\theta-1}\Big(\frac{2\epsilon - t}{\epsilon}\Big)^{2/(\phi_- - 1)} dt \quad \text{if } \epsilon \leqslant d(x) < 2\epsilon \end{cases}$$

where $0 < \theta < 1$ will be chosen later.

A simple calculation gives us

$$\nabla u_0(x) = \begin{cases} \theta d(x)^{\theta-1}\nabla d(x) \ \text{ if } d(x) < \epsilon \\ \theta\epsilon^{\theta-1}\Big(\frac{2\epsilon - d(x)}{\epsilon}\Big)^{2/(\phi_- - 1)}\nabla d(x) \quad \text{if } \epsilon \leqslant d(x) < 2\epsilon \\ 0 \quad \text{if } \epsilon \leqslant d(x) < 2\epsilon. \end{cases}$$

In order to $u_0 \in W_0^{1,\Phi}(\Omega)$, it is enough that

$$\int_{\Omega_\epsilon} \Phi(\theta d(x)^{\theta-1}|\nabla d(x)|)dx < \infty. \tag{4.23}$$

But, by Lemma 4.1.11$-i$) we have

$$\int_{\Omega_\epsilon} \Phi(\theta d(x)^{\theta-1}|\nabla d(x)|)dx = \int_{\Omega_\epsilon} \Phi(\theta d(x)^{\theta-1})dx \overset{\theta<1}{\leqslant} C\int_{\Omega_\epsilon} d(x)^{(\theta-1)\phi_+}dx.$$

Thus, if $(\theta-1)\phi_+ > -1$, by [40] we get $\displaystyle\int_{\Omega_\epsilon} d(x)^{(\theta-1)\phi_+}dx < \infty$ and consequently (4.23) will be satisfied.

On the other hand, in order to $\displaystyle\int_{\Omega} b(x)u_0(x)^{1-\delta}dx < \infty$, it is enough that

$$\int_{\Omega_\epsilon} b(x)d(x)^{\theta(1-\delta)}dx < \infty. \tag{4.24}$$

Since we are assuming $b \in L^q(\Omega)$, if $\displaystyle\int_{\Omega_\epsilon} d(x)^{\theta(1-\delta)q'}dx < \infty$ holds, then (4.24) will occur. So once again by [40], if $\frac{\theta q(1-\delta)}{q-1} > -1$, the condition (4.24) will be satisfied.

Therefore, if $1-\frac{1}{\phi_+} < \frac{q-1}{q(\delta-1)}$, that is, $0<\delta<\frac{q(2\phi_+-1)-\phi_+}{q(\phi_+-1)}$, by taking $\theta \in \left(1-\frac{1}{\phi_+}, \min\left\{1, \frac{q-1}{q(\delta-1)}\right\}\right)$ the function u_0, defined as above, satisfies the condition of item i) in Theorem 0.0.12, which finishes the proof. ■

Corollary 0.0.15 *Assume that $(\phi_0), (\phi_1), (\phi_2), (M)$ and $(f_1')-(f_3')$ hold. If $b \in L^q(\Omega)$ for some $1<q$ and δ satisfies (9), then for each $\lambda > \lambda^*$ there exists $\mu_\lambda > 0$ such that for $\mu \in (0,\mu_\lambda]$ the problem $(Q_{\lambda,\mu})$ admits at least three weak solutions.*

Proof: It follows from the previous corollary and Theorem 0.0.12. ■

Remark 4.3.1 *Although we do not know if $\delta_q = 1+\frac{\phi_+'}{q'}$ is the sharp value for the existence of solution in $W_0^{1,\Phi}(\Omega)$, we observe that $1+\frac{\phi_+'}{q'} \to \frac{2\phi_+-1}{\phi_+-1}$ as $q\to\infty$, which reobtains the sharp values obtained by [40] and [44], for the cases $\Phi(t)=|t|^2/2$ and $\Phi(t)=|t|^p/p$ $(p>1)$, respectively.*

APPENDIX

1 General Results

Theorem A.1.1 ***(Schaeffer's Fixed Point Theorem):*** *Let X be a Banach space and $S : X \to X$ be a continuous and compact mapping. If the set*

$$\{x \in X : x = \lambda S(x) \text{ for some } \lambda \in [0,1]\}$$

is bounded, then S has a fixed point.

Proof: See Theorem 11.3 in [31]. ■

Theorem A.1.2 ***(See [57]):*** *Let $u \in C^1(\Omega)$ be a nonnegative function that satisfies $-\Delta_p u \geqslant 0$ a.e in Ω. If u does not vanish identically it is positive everywhere in Ω.*

Theorem A.1.3 *Let (u_n) be a sequence in $L^p(\Omega)$ and let $u \in L^p(\Omega)$ be such that $\|u_n - u\|_p \to 0$. Then, there exists a subsequence (u_{n_k}) and a function $h \in L^p(\Omega)$ such that:*

i) $u_{n_k}(x) \to u(x)$ *a.e on* Ω,

ii) $|u_{n_k}(x)| \leqslant h(x)$ *for all* k, *a.e on* Ω.

Proof: See Theorem 4.9 in [9]. ∎

Theorem A.1.4 ***(See [7]):*** *Consider de equation*

$$-\Delta_p u_n = g_n \text{ in } \mathcal{D}'(\Omega)$$

and assume that $u_n \rightharpoonup u$ *weakly in* $W^{1,p}(\Omega)$*, strongly in* $L^p_{\mathrm{loc}}(\Omega)$ *and a.e in* Ω*. Moreover, assume that* $g_n \in W^{-1,p'}(\Omega)$ $(p' = p/(p-1))$ *and* g_n *is bounded in the space of Radon measures, i.e.*

$$\left|\int_\Omega \varphi d g_n\right| \leqslant C_\Theta \|\varphi\|_\infty, \text{ for any } \varphi \in \mathcal{D}(\Omega) \text{ with supp } \varphi \subset \Theta,$$

where C_Θ *is a constant which depends on the compact set* Θ*. Then*

$$Du_n \to Du \text{ strongly in } \Big(L^q(\Omega)\Big)^N, \text{ for any } q < p.$$

Theorem A.1.5 ***(Vitali):*** *Let* μ *be a finite positive measure on a measure space* X*. A sequence* $\{u_n\} \in L^1(\mu)$ *is said to have uniformly absolutely continuous integrals if to each* $\epsilon > 0$ *there is corresponds* $\delta > 0$ *such that* $\mu(E) < \delta$ *implies*

$$\left|\int_E u_n d\mu\right| < \epsilon \qquad (n = 1, 2, 3, \cdots).$$

If $\{u_n\}$ *has uniformly absolutely continuous integrals and if* $u_n(x) \to u(x)$ *a.e., then* $u \in L^1(\mu)$ *and*

$$\lim_{n\to\infty} \int_X u_n d\mu = \int_X u d\mu.$$

Theorem A.1.6 ***(Picone's Identity):*** *Let* $v > 0$ *and* $u \geqslant 0$ *be weakly differentiable. Denote*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla u \cdot \nabla v.$$

Then $L(u,v) \geqslant 0$ *and* $L(u,v) = 0$ *a.e on* Ω *if and only if* $u = \alpha v$ *for some constant* α *in each component of* Ω*.*

Proof: See [2]. ■

Theorem A.1.7 *Let X be a Banach space and $T : \mathbb{R}^+ \times X \to X$ a compact map such that $T(0, u) = 0$ for all $u \in X$. Then the equation $u = T(\lambda, u)$ possesses an unbounded continuum $\Sigma \subset \mathbb{R}^+ \times X$ of solutions with $(0, 0) \in \Sigma$.*

Proof: See Theorem 3.2 in [48]. ■

Theorem A.1.8 ***(Bolzano's Theorem):*** *Let X be a Banach space and let h be a continuous function in a continuum $\Sigma_0 \subset [0, \infty) \times X$ and suppose that there exist $(\alpha_1, u_1), (\alpha_2, u_2) \in \Sigma_0$ such that $h(\alpha_1, u_1) \cdot h(\alpha_2, u_2) < 0$. Then there exists $(\alpha, u) \in \Sigma_0$ such that $h(\alpha, u) = 0$.*

Theorem A.1.9 ***(Reverse Hölder inequality):*** *Assume that $p \in (0, \infty)$ and $\Omega \subset \mathbb{R}^N$ is a subset with finite measure. If f and g are measurable functions such that $g(x) \neq 0$ a.e in Ω, then*

$$\|fg\|_1 \geqslant \|f\|_{1/p}\|g\|_{-1/(p-1)}.$$

Theorem A.1.10 ***(Ekeland's Variational Principle):*** *Let $(\mathcal{M}, d)$ be a complete metric space and J a lower semicontinuos functional (s.c.i) bounded below in $\mathcal{M}$. If $c = \inf_{\mathcal{M}} J$, then for each $\epsilon > 0$ there exists $u_\epsilon \in \mathcal{M}$ such that*

$$\begin{cases} c \leqslant J(u_\epsilon) \leqslant c + \epsilon \\ J(u) - J(u_\epsilon) + \epsilon d(u, u_\epsilon) > 0 \quad \text{for all } u \in \mathcal{M}, u \neq u_\epsilon. \end{cases}$$

Proof: See Lemma 6.8 in [33], page 162. ■

Theorem A.1.11 *If X is a real Banach space, we denote by $\mathcal{W}_X$ the class of all functionals $J : X \to \mathbb{R}$ possessing the following property: "if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\lim_{n\to\infty} \inf J(u_n) \leqslant J(u)$, then $\{u_n\}$ has a subsequence converging strongly to u".*

Let X be a reflexive and separable real Banach space, let $J, I : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals with J_1 belonging to $\mathcal{W}_X$. Assume that

$$\lim_{\|x\|\to+\infty}(J(x)+I(x)) = +\infty.$$

Then, any strict local minimum of the functional $J+I$ in the strong topology is so in the weak topology.

Proof: See Theorem C in [51]. ∎

Theorem A.1.12 *Let X be a Hausdorff topological space and $\{\Theta_n\}$ be a sequence of nonempty compact subsets of X such that $\Theta_{n+1} \subseteq \Theta_n$ for all $n \in \mathbb{N}$ and*

$$\bigcap_{n=1}^{\infty}\Theta_n = D_1 \cup D_2, \quad D_1 \cap D_2 = \varnothing,$$

where D_1, D_2 are nonempty and compact. Then, there exist $n_0 \in \mathbb{N}$ and C_1, C_2 nonempty compact sets such that

$$\Theta_{n_0} = C_1 \cup C_2, \quad C_1 \cap C_2 = \varnothing, \quad D_1 \subseteq C_1, \quad D_2 \subseteq C_2.$$

Proof: See [28]. ∎

2 Regularity

Theorem A.2.1 ***(See [43]):*** *Consider*

$$(R)\begin{cases} div\mathcal{A}(x,u,\nabla u) + \mathcal{B}(x,u,\nabla u) = 0 \ in \ \Omega, \\ u = 0 \ on \ \partial\Omega, \end{cases}$$

with $(a^{ij}) = (\partial\mathcal{A}^i/\partial p_j)$.

Let $\alpha, \lambda, \Lambda, M_0$ be positive constants with $\alpha \leqslant 1$ and $\Lambda \geqslant \lambda$, κ a nonnegative constant, Ω be a domain in $\mathbb{R}^N$ with $C^{1,\alpha}$ boundary. Suppose $\mathcal{A}$ and $\mathcal{B}$ satisfy the *following conditions:*

(H_1) $a^{ij}(x,z,p)\xi_i\xi_j \geqslant \lambda(\kappa+|p|)^m|\xi|^2$,

(H_2) $|a^{ij}(x,z,p)| \leqslant \Lambda(\kappa+|p|)^m$,

(H_3) $|\mathcal{A}(x,z,p)-\mathcal{A}(y,w,p)| \leqslant \Lambda(1+|p|)^{m+1}[|x-y|^\alpha+|z-w|^\alpha]$,

(H_4) $|B(x,z,p)| \leqslant \Lambda(1+|p|)^{m+2}$

for all $(x,z,p) \in \partial\Omega \times [-M_0, M_0] \times \mathbb{R}^N$, *all* (y,w) *in* $\Omega \times [-M_0, M_0]$ *and all* $\xi \in \mathbb{R}^N$. *If* u *is a bounded weak solutions of (R) with* $|u| \leqslant M_0$ *in* Ω, *then there is a positive constant* $\beta = \beta(\alpha, \Lambda/\lambda, m, N)$ *such that* $u \in C^{1,\beta}(\overline{\Omega})$, *moreover* $|u|_{1,\beta} \leqslant C(\alpha, \Lambda/\lambda, m, M_0, N, \Omega)$.

Theorem A.2.2 ***(See [47]):*** *Consider*

$$(R)\begin{cases} -\Delta_p u = g(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Suppose $g \in L^m(\Omega)$ *for some* $m > N$. *Then (R) has a unique weak solution* $u \in C_0^1(\overline{\Omega})$. *If in addition* $g \geqslant 0$ *is nontrivial, then*

$$u > 0 \text{ in } \Omega, \quad \partial u/\partial \nu \text{ on } \partial\Omega,$$

where ν *is the interior unit normal on* $\partial\Omega$.

Theorem A.2.3 ***(See Theorem 2 in [24]):*** *Consider*

$$(R) \qquad -\Delta_p u + \mathcal{B}(x,u,\nabla u) = 0 \text{ in } \mathcal{D}'(\Omega),$$

where $p > 1$ *and* $|\mathcal{B}(x,u,\nabla u)| \leqslant C(|\nabla u|^p + \psi(x))$ *for some* $\psi \in L^p_{\mathrm{loc}}(\Omega)$ *with* $q > p'N$. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *be a local weak solution of (R). Consider* Ω' *a subdomain of* Ω *such that* $\overline{\Omega}' \subset \Omega$ *and let* $M = \text{ess sup}_{\Omega'}|u|$. *Then* $x \mapsto \nabla u(x)$ *is locally Hölder continuous in* Ω', *i.e, for every compact* $\Theta \subset \Omega'$, *there exist constants*

C_1 *and* $\alpha \in (0,1)$, *depending only upon* C, p, N, M *and* $dist(\Theta, \partial\Omega')$, *such that*

$$|u_{x_i}(x) - u_{x_1}(y)| \leqslant C_1|x-y|^{\alpha}, \;\; x, y \in \Theta; \;\; i = 1, 2, \cdots N.$$

BIBLIOGRAPHY

[1] R. A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 65.

[2] W. Allegretto and Y. X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal. **32** (1998), no. 7, 819–830.

[3] C. O. Alves and D. Covei, *Existence of solution for a class of non-local elliptic problem via sub-supersolution method*, Nonlinear Anal. Real World Appl. **23** (2015), 1–8.

[4] J. J. Aly, *Thermodynamics of a two-dimensional self-gravitating system*, Phys. Rev. E (3) **49** (1994), no. 5, part A, 3771–3783.

[5] D. Arcoya, T. Leonori, and A. Primo, *Existence of solutions for semilinear nonlocal elliptic problems via a Bolzano theorem*, Acta Appl. Math. **127** (2013), 87–104.

[6] J. W. Bebernes and A. A. Lacey, *Global existence and finite-time blow-up for a class of nonlocal parabolic problems*, Adv. Differential Equations **2** (1997), no. 6, 927–953.

[7] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. **19** (1992), no. 6, 581–597.

[8] L. Boccardo and L. Orsina, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations **37** (2010), no. 3-4, 363–380.

[9] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[10] H. Bueno, G. Ercole, W. Ferreira, and A. Zumpano, *Existence and multiplicity of positive solutions for the p-Laplacian with nonlocal coefficient*, J. Math. Anal. Appl. **343** (2008), no. 1, 151–158.

[11] E. Caglioti, P.-L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*, Comm. Math. Phys. **143** (1992), no. 3, 501–525.

[12] F. Cammaroto and L. Vilasi, *Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator*, Nonlinear Anal. **74** (2011), no. 5, 1841–1852.

[13] A. Canino and B. Sciunzi, *A uniqueness result for some singular semilinear elliptic equations*, Commun. Contemp. Math. **18** (2016), no. 6, 1550084, 9.

[14] A. Canino, B. Sciunzi, and A. Trombetta, *Existence and uniqueness for p-Laplace equations involving singular nonlinearities*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 2, Art. 8, 18.

[15] J. A. Carrillo, *On a nonlocal elliptic equation with decreasing nonlinearity arising in plasma physics and heat conduction*, Nonlinear Anal. **32** (1998), no. 1, 97–115.

[16] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 7 (Athens, 1996), vol. 30, 1997, pp. 4619–4627.

[17] F. J. S. A. Corrêa, *On positive solutions of nonlocal and nonvariational elliptic problems*, Nonlinear Anal. **59** (2004), no. 7, 1147–1155.

[18] F. J. S. A. Corrêa, M. Delgado, and A. Suárez, *Some non-local problems with nonlinear diffusion*, Math. Comput. Modelling **54** (2011), no. 9-10, 2293–2305.

[19] F. J. S. A Corrêa and G. M. Figueiredo, *A variational approach for a nonlocal and nonvariational elliptic problem*, J. Integral Equations Appl. **22** (2010), no. 4, 549–557.

[20] F. J. S. A. Corrêa, S. D. B. Menezes, and J. Ferreira, *On a class of problems involving a nonlocal operator*, Appl. Math. Comput. **147** (2004), no. 2, 475–489.

[21] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), no. 2, 193–222.

[22] M. Cuesta, *Eigenvalue problems for the p-Laplacian with indefinite weights*, Electron. J. Differential Equations (2001), No. 33, 9.

[23] J. I. Díaz and J. E. Saá, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 12, 521–524.

[24] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.

[25] L. Diening, P. Harjulehto, P. Hasto, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011.

[26] Y. Du, *Bifurcation from infinity in a class of nonlocal elliptic problems*, Differential Integral Equations **15** (2002), no. 5, 587–606.

[27] F. Fang and Z. Tan, *Existence of three solutions for quasilinear elliptic equations: an Orlicz-Sobolev space setting*, Acta Math. Appl. Sin. Engl. Ser. **33** (2017), no. 2, 287–296.

[28] F. Faraci and G. Smyrlis, *Three solutions for a class of higher dimensional singular problems*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 4, Art. 45, 14.

[29] T. S. Figueiredo-Sousa, C. Morales-Rodrigo, and A. Suárez, *A non-local non-autonomous diffusion problem: linear and sublinear cases*, Z. Angew. Math. Phys. **68** (2017), no. 5, Art. 108, 20.

[30] J. García-Melián and J. C. Sabina de Lis, *A boundary blow-up problem with a nonlocal reaction*, Nonlinear Anal. **75** (2012), no. 5, 2774–2792.

[31] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[32] M. Guedda and L. Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879–902.

[33] O. Kavian, *Introduction a la theorie des points critiques et applications aux problemes elliptiques*, Mathematiques et Applications (Berlin) [Mathematics et Applications], vol. 13, Springer-Verlag, Paris, 1993.

[34] M. A. Krasnosels'kii and J. B. Rutickii, *Convex functions and Orlicz spaces*, Translated from the first Russian edition by Leo F. Boron, P. Noordhoff Ltd., Groningen, 1961.

[35] A. Krzywicki and T. Nadzieja, *Some results concerning the Poisson-Boltzmann equation*, Zastos. Mat. **21** (1991), no. 2, 265–272.

[36] ______, *Nonlocal elliptic problems*, Evolution equations: existence, regularity and singularities (Warsaw, 1998), Banach Center Publ., vol. 52, Polish Acad. Sci. Inst. Math., Warsaw, 2000, pp. 147–152.

[37] A. Kufner, O. John, and S. Fŭcík, *Function spaces*, Noordhoff International Publishing, Leyden; Academia, Prague, 1977, Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[38] T. Kuusi and G. Mingione, *Universal potential estimates*, J. Funct. Anal. **262** (2012), no. 10, 4205–4269.

[39] A. A. Lacey, *Diffusion models with blow-up*, J. Comput. Appl. Math. **97** (1998), no. 1-2, 39–49.

[40] A. C. Lazer and P. J. McKenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 721–730.

[41] C. Lei, J. Liao, and C. Tang, *Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents*, J. Math. Anal. Appl. **421** (2015), no. 1, 521–538.

[42] J. Liao, P. Zhang, J. Liu, and C. Tang, *Existence and multiplicity of positive solutions for a class of Kirchhoff type problems with singularity*, J. Math. Anal. Appl. **430** (2015), no. 2, 1124–1148.

[43] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[44] A. Mohammed, *Positive solutions of the p-Laplace equation with singular nonlinearity*, J. Math. Anal. Appl. **352** (2009), no. 1, 234–245.

[45] T. C. Nguyen, *Three solutions for a class of nonlocal problems in Orlicz-Sobolev spaces*, J. Korean Math. Soc. **50** (2013), no. 6, 1257–1269.

[46] K. Perera and E. A. B. Silva, *On singular p-Laplacian problems*, Differential Integral Equations **20** (2007), no. 1, 105–120.

[47] K. Perera and Zhitao. Zhang, *Multiple positive solutions of singular p-Laplacian problems by variational methods*, Bound. Value Probl. (2005), no. 3, 377–382.

[48] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Analysis **7** (1971), 487–513.

[49] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker, Inc., New York, 1991.

[50] B. Ricceri, *Sublevel sets and global minima of coercive functionals and local minima of their perturbations*, J. Nonlinear Convex Anal. **5** (2004), no. 2, 157–168.

[51] ———, *A further three critical points theorem*, Nonlinear Anal. **71** (2009), no. 9, 4151–4157.

[52] ———, *On an elliptic Kirchhoff-type problem depending on two parameters*, J. Global Optim. **46** (2010), no. 4, 543–549.

[53] G. C. G. Santos, G. M. Figueiredo, and L. S. Tavares, *A sub-supersolution method for a class of nonlocal problems involving the $p(x)$-Laplacian operator and applications*, Acta Appl. Math. **153** (2018), 171–187.

[54] Y. Sun, *Compatibility phenomena in singular problems*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), no. 6, 1321–1330.

[55] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), no. 2, 77–109.

[56] M. Väth, *Global solution branches and a topological implicit function theorem*, Ann. Mat. Pura Appl. (4) **186** (2007), no. 2, 199–227.

[57] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.

[58] G. T. Whyburn, *Topological analysis*, Princeton Mathematical Series. No. 23, Princeton University Press, Princeton, N. J., 1958.

[59] G. Wolansky, *A critical parabolic estimate and application to nonlocal equations arising in chemotaxis*, Appl. Anal. **66** (1997), no. 3-4, 291–321.

[60] B. Yan, D. O'Regan, and R. P. Agarwal, *The existence of positive solutions for Kirchhoff-type problems via the sub-supersolution method*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **26** (2018), no. 1, 5–42.

[61] B. Yan and Q. Ren, *Existence, uniqueness and multiplicity of positive solutions for some nonlocal singular elliptic problems*, Electron. J. Differential Equations (2017), Paper No. 138, 21.

[62] B. Yan and D. Wang, *The multiplicity of positive solutions for a class of nonlocal elliptic problem*, J. Math. Anal. Appl. **442** (2016), no. 1, 72–102.

[63] L. Zhao, Y. He, and P. Zhao, *The existence of three positive solutions of a singular p-Laplacian problem*, Nonlinear Anal. **74** (2011), no. 16, 5745–5753.

[64] L. Zhao and P. Zhao, *The existence of three solutions for p-Laplacian problems with critical and supercritical growth*, Rocky Mountain J. Math. **44** (2014), no. 4, 1383–1397.