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Research Article

On Integer Numbers with Locally Smallest Order of Appearance in the Fibonacci Sequence

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Let F_n be the nth Fibonacci number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k. For instance, for all $n = F_m \geq 5$, we have $z(n-1) > z(n) < z(n+1)$. In this paper, we will construct infinitely many natural numbers satisfying the previous inequalities and which do not belong to the Fibonacci sequence.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots. \tag{1.1}$$

The Fibonacci numbers are well known for possessing wonderful and amazing properties (consult [1] together with its very extensive annotated bibliography for additional references and history). In 1963, the Fibonacci Association was created to provide enthusiasts an opportunity to share ideas about these intriguing numbers and their applications. Also, in the issues of *The Fibonacci Quarterly,* we can find many new facts, applications, and relationships about Fibonacci numbers.

Let n be a positive integer number, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For instance, every positive integer divides some Fibonacci number, that is, $z(n) < \infty$ for all $n \geq 1$. The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [2, page 300]. Also, it is a simple matter to prove that $z(F_n - 1) > z(F_n) < z(F_n + 1)$, for $n \geq 5$. In fact, if $z(F_m + \epsilon) = j_\epsilon$ with $\epsilon \in \{\pm 1\}$, then $F_m + \epsilon$ divides F_{j_ϵ}, for some $j \geq 5$ and thus $F_{j_\epsilon} = u(F_m + \epsilon)$ with $u \geq 2$. Therefore, the inequality $F_{j_\epsilon} \geq 2F_m + 2\epsilon > F_m$ gives $z(F_m + \epsilon) = j_\epsilon > m = z(F_m)$. So the order of appearance of a Fibonacci number is locally smallest in this sense. On the other hand, there are integers n for which $z(n)$ is locally smallest but which are not Fibonacci numbers, for example, $n = 11, 17, 24, 26, 29, 36, 38, 41, 44, 48, \ldots$. So, a natural question arises: are there infinitely many natural numbers n that do not belong to the Fibonacci sequence and such that $z(n-1) > z(n) < z(n+1)$?

In this note, we give an affirmative answer to this question by proving the following.

Theorem 1.1. *Given an integer $k \geq 3$, the number $N_m := F_{mk}/F_k$ has order of appearance mk, for all $m \geq 5$. In particular, it is not a Fibonacci number. Moreover, one has*

$$z(N_m - 1) > z(N_m) < z(N_m + 1), \tag{1.2}$$

for all sufficiently large m.

2. Proof of Theorem 1.1

We recall that the problem of the existence of infinitely many prime numbers in the Fibonacci sequence remains open; however, several results on the prime factors of a Fibonacci number are known. For instance, a *primitive divisor* p of F_n is a prime factor of F_n that does not divide $\prod_{j=1}^{n-1} F_j$. In particular, $z(p) = n$. It is known that a primitive divisor p of F_n exists whenever $n \geq 13$. The above statement is usually referred to the *Primitive Divisor Theorem* (see [3] for the most general version).

Now, we are ready to deal with the proof of the theorem.

Since N_m divides F_{mk}, then $z(N_m) \leq mk$. On the other hand, if N_m divides F_j, then we get the relation

$$F_k F_j = t F_{mk}, \tag{2.1}$$

where t is a positive integer number. Since $mk \geq 15$, the Primitive Divisor Theorem implies that $j \geq mk$. Therefore, $z(N_m) \geq mk$ yielding $z(N_m) = mk$. Now, if N_m is a Fibonacci number, say F_t, we get $t = z(N_m) = mk$ which leads to an absurdity as $F_k = 1$ (keep in mind that $k \geq 3$). Therefore, N_m is not a Fibonacci number, for all $m \geq 5$.

Now, it suffices to prove that $z(N_m + \epsilon) > mk = z(N_m)$, or equivalently, if $N_m \pm 1$ divides F_j, then $j > mk$, for all sufficiently large m, where $\epsilon \in \{\pm 1\}$.

Let u be a positive integer number such that $F_j = u(N_m + \epsilon)$. If $u \geq F_k + 1$, we have

$$F_j \geq \left(1 + \frac{1}{F_k}\right) F_{mk} + \epsilon(F_k + 1) > F_{mk}, \tag{2.2}$$

where in the last inequality above, we used the fact that $F_{mk} > F_m F_k > \epsilon(F_k + 1)F_k$, for $m > k \geq 3$. Thus, $j > mk$ as desired. For finishing the proof, it suffices to show that there exist only finitely many pairs (k, j) of positive integers, such that

$$\frac{F_j}{N_m + \epsilon} = u \in \{1, \ldots, F_k\}, \tag{2.3}$$

or equivalently,

$$uF_{mk} - F_k F_j = -\epsilon u F_k. \tag{2.4}$$

Towards a contradiction, suppose that (2.4) have infinitely many solutions (u_n, m_n, j_n) with $u_n \in \{1, \dots, F_k\}$ and $n \geq 1$. Hence, $(m_n)_n$ and $(j_n)_n$ are unbounded sequences. Since $(u_n)_n$ is bounded, we can assume, without loss of generality, that u_n is a constant, say u, for all sufficiently large n (by the reordering of indexes if necessary). Now, by (2.4), we get

$$\lim_{n\to\infty} \frac{F_{m_n k}}{F_{j_n}} = \frac{F_k}{u}. \tag{2.5}$$

On the other hand, the well-known Binet's formula

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2}, \tag{2.6}$$

leads to

$$\frac{F_{m_n k}}{F_{j_n}} = \frac{\alpha^{m_n k - j_n} - (-1)^{m_n k} \alpha^{-m_n k - j_n}}{1 - (-1)^{j_n} \alpha^{-2j_n}}. \tag{2.7}$$

Thus,

$$\lim_{n\to\infty} \frac{F_{m_n k}}{F_{j_n}} = \lim_{n\to\infty} \alpha^{m_n k - j_n}. \tag{2.8}$$

Combining (2.5) and (2.8), we get

$$\lim_{n\to\infty} \alpha^{m_n k - j_n} = \frac{F_k}{u}. \tag{2.9}$$

Since $m_n k - j_n$ is an integer and $|\alpha| > 1$, we have that $m_n k - j_n$ must be constant with respect to n, say t, for all n sufficiently large. Therefore, (2.9) yields the relation $\alpha^t = F_k / u \in \mathbb{Q}$ and so $t = 0$ (because α^t is irrational for all nonzero rational number). But, this leads to (by (2.4))

$$\epsilon F_k^2 = \epsilon u F_k = F_k F_{m_n k} - F_k F_{m_n k} = 0, \tag{2.10}$$

which is absurd. This completes the proof of the theorem.

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References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, NY, USA, 2001.

[2] E. Lucas, "Theorie des fonctions numeriques simplement periodiques," *American Journal of Mathematics*, vol. 1, no. 4, pp. 289–321, 1878.

[3] Yu. Bilu, G. Hanrot, and P. M. Voutier, "Existence of primitive divisors of Lucas and Lehmer numbers," *Journal für die Reine und Angewandte Mathematik*, vol. 539, pp. 75–122, 2001.